NCTS 5053 Lecture 2 2/22/2024

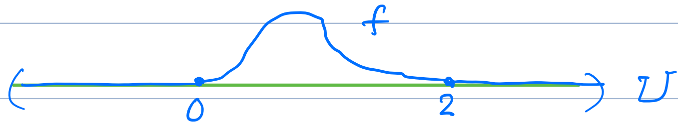
Compact Supports, Manifolds

Def. A C^∞ form ω is closed if $d\omega = 0$; it is exact if $\omega = d\tau$ for some $\tau \in \Omega^{k-1}(M)$.

Compact Supports

Let U be an open subset of $\mathbb{R}^n$.

Ex.



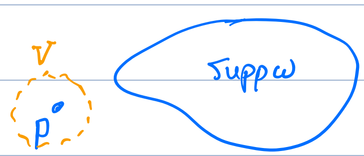
f is nonzero on $(0,2)$.
Supp f is its closure $[0,2]$.

Def. The zero set of a k-form ω on U is
$$Z(\omega) = \{ p \in U \mid \omega_p = 0 \}$$
The support of ω is
$$\operatorname{supp}\omega = \operatorname{cl}\{ p \in U \mid \omega_p \neq 0 \}$$
$$= \operatorname{cl}(U \setminus Z(\omega)) = \operatorname{cl}(Z(\omega)^c),$$
where $(\ )^c$ stands for the complement.

Let $\Omega_c^k(U) = \{ C^\infty$ k-forms on U with compact support $\}$.

Prop. (d is support-decreasing). For $\omega \in \Omega^k(U)$,
$$\operatorname{supp}(d\omega) \subset \operatorname{supp}\omega.$$

Pf.



Suppose $p \notin \operatorname{supp}\omega$. Since $\operatorname{supp}\omega$ is closed, $\exists$ open nbd V of p

disjoint from Supp ω. Then $\omega \equiv 0$ on V, So $d\omega \equiv 0$ on V.
Therefore, $p \notin \text{supp}\, d\omega$. We have proven

$$(\text{supp}\,\omega)^c \subset (\text{supp}\, d\omega)^c.$$

Taking complement gives $\text{supp}\, d\omega \subset \text{Supp}\, \omega$, □

Cor. If $\omega \in \Omega_c^k(U)$ has compact supp, so does $d\omega$.

Pf. $\text{supp}\, d\omega$ is a closed subset of the compact set $\text{supp}\, \omega$.

We therefore obtain a differential complex

$$\Omega_c^*(U): 0 \to \Omega_c^0(U) \xrightarrow{d} \Omega_c^1(U) \xrightarrow{d} \cdots \to \Omega_c^n(U) \to 0,$$

the deRham cx with compact supp of U.

Def. $H_c^*(U)$ is the cohomology of this cx.

Degree Zero

A k-tensor has k variables. A 0-tensor has no variable..

Def. A 0-tensor on a vector space V is a constant.

Thus, $A_0(V) = \mathbb{R}$.

A 0-form on U assigns to each point of U a 0-tensor (constant) Hence, $\boxed{\text{0-form} = \text{function.}}$ $\Rightarrow$ $\boxed{\Omega^0(U) = C^\infty(U)}$

Example. $H_c^*(\mathbb{R})$

$$0 \to \Omega_c^0(\mathbb{R}) \xrightarrow{d} \Omega_c^1(\mathbb{R}) \longrightarrow 0$$

$Z_c^0(\mathbb{R}) = \{ f \in C_c^\infty(\mathbb{R}) \mid df = 0 \}$.

$df = f'(x)\, dx = 0 \Rightarrow f'(x) = 0 \Rightarrow f = \text{const on } \mathbb{R}$.

f does not have compact supp.

Hence, $Z_c^0(\mathbb{R}) = 0$. $\Rightarrow$ $\boxed{H_c^0(\mathbb{R}) = \dfrac{Z_c^0(\mathbb{R})}{B_c^0(\mathbb{R})} = \dfrac{0}{0} = 0,}$

Next we compute $H^1_c(\mathbb{R})$.

$$Z^1_c(\mathbb{R}) = \Omega^1_c(\mathbb{R}) = \{ g(x)\,dx \mid g \in C^\infty_c(\mathbb{R}) \}.$$

$$B^1_c(\mathbb{R}) = \{ df = f'(x)\,dx \mid f \in C^\infty_c(\mathbb{R}) \}.$$

If $g(x) = f'(x)$, then $\displaystyle\int_{-\infty}^{\infty} g(u)\,du = \int_{-\infty}^{\infty} f'(u)\,du$

$\displaystyle = f(u)\Big]_{-\infty}^{\infty} = 0$ since f has cpt support.

$$\boxed{\text{The integral of an exact form with compact supp is } 0.}$$

Define $\displaystyle\int_{-\infty}^{\infty} : Z^1_c(\mathbb{R}) \to \mathbb{R}$, $\quad g(x)\,dx \mapsto \displaystyle\int_{-\infty}^{\infty} g(x)\,dx$

We have shown that $\boxed{B^c_1(\mathbb{R}) \subset \ker \int_{-\infty}^{\infty}}$.

We now prove the reverse inclusion.

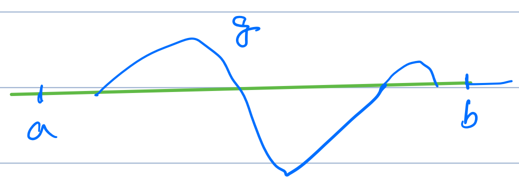
Lemma. $\ker \int_{-\infty}^{\infty} \subset B^1_c(\mathbb{R})$.

Proof. Suppose $\int_{-\infty}^{\infty} g(x)\,dx = 0$. Define $f(x) = \displaystyle\int_{\infty}^{x} g(u)\,du$.

By the fund. th. of calculus, $f'(x) = g(x)$. It remains to show that f has compact support.

Since supp g is compact, supp $g \subset [a, b]$ for some $a < b \in \mathbb{R}$. For $x < a$, $f(x) = \displaystyle\int_{-\infty}^{x} g(u)\,du = \int_{-\infty}^{x} 0 = 0$.

For $x > b$, $f(x) = \displaystyle\int_{-\infty}^{x} g(u)\,du = \int_{-\infty}^{\infty} g(u)\,du = 0$ by hypothesis.



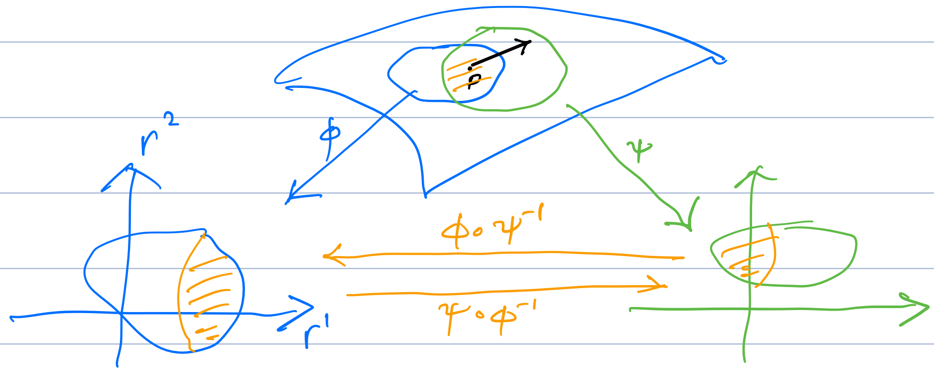
Hence, supp $f \subset [a,b]$. As a closed subset of a compact set, supp f is compact. Thus, $g(x)\,dx \in B^1_c(\mathbb{R})$. $\square$

The map $\int_{-\infty}^{\infty} : Z^1_c(\mathbb{R}) \to \mathbb{R}$ is surjective with kernel $B^1_c(\mathbb{R})$.

By the 1st. structure th of linear algebra,

$$\boxed{H^1_c(\mathbb{R}) =} \; Z^1_c(\mathbb{R}) / B^1_c(\mathbb{R}) = Z^1_c(\mathbb{R}) \Big/ \ker \int_{-\infty}^{\infty} = \operatorname{im} \int_{-\infty}^{\infty} = \boxed{\mathbb{R}.}$$

Manifolds



Def. A topological space M is <u>locally Euclidean</u> of dim n if every point $p \in M$ has a nbd U that is homeomorphic to an open subset of $\mathbb{R}^n$ via a homeomorphism
$\phi : U \to \phi(U) \subset \mathbb{R}^n$.

(U, ϕ) = chart

Two charts (U, ϕ) and (V, ψ) are <u>C^∞-compatible</u> if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$

and

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞.

An <u>atlas</u> is a collection of C^∞-compatible charts $\{(U_\alpha, \phi_\alpha)\}$ that cover M.

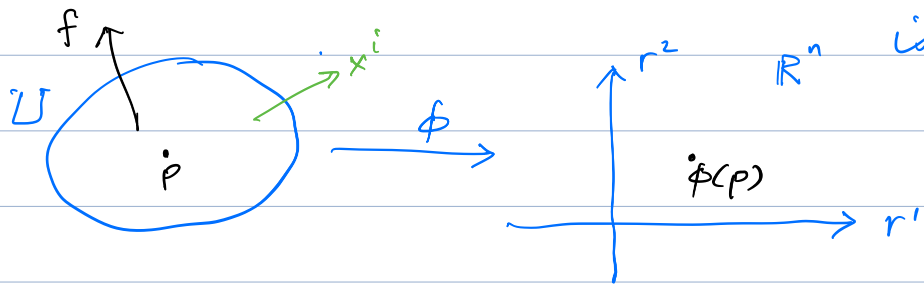
A <u>topological manifold</u> is a locally Euclidean, Hausdorff, and 2nd countable topological space.

A <u>C^∞</u> or <u>smooth manifold</u> is a topological manifold together with a maximal atlas.

Tangent Space

Let $r^1, \dots, r^n$ be the standard coordinates on $\mathbb{R}^n$. If (U, ϕ) is a chart, set

$$x^i = r^i \circ \phi.$$



Def. $\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial\, f\circ\phi^{-1}}{\partial r^i}(\phi(p))$

Def. tangent space T_pM = vector space spanned by $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$.

Differential Forms

Def. A k-form ω on a manifold M is the assignment to each $p\in M$ of an alternating k-tensor ω_p on T_pM.

In a chart $(U,\phi) = (U, x^1,\dots,x^n)$, a k-form ω is uniquely

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \sum_I a_I\, dx^I.$$

Def. A k-form ω on a manifold M is C^∞ if $\exists$ an atlas $\mathcal{U}$ s.t. on each chart $(U, x^1,\dots,x^n)\in\mathcal{U}$, the coef a_I in $\omega = \sum a_I\, dx^I$ are all C^∞.

Notation $\Omega^k(M) = \{ C^\infty\ k\text{-forms on } M\}$

$\Omega^k_c(M) = \{ C^\infty\ k\text{-forms with compact supp on } M\}$.

We can define $H^*(M)$, $H^*_c(M)$ as on $\mathbb{R}^n$.