The de Rham Complex on $\mathbb{R}^n$

Question. What is dx?

Differential Forms

Ex. Let $a, b, c \in \mathbb{R}^3$. Then $\det[a\ b\ c]$ is 3-linear (linear in each of the 3 arguments a, b, c) and alternating

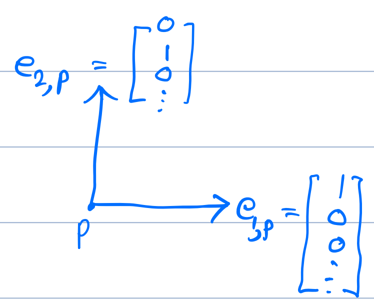
$$\det[b\ a\ c] = -\det[a\ b\ c].$$

Def. A k-tensor on a vector space V is a k-linear function $\alpha : V \times \cdots \times V \to \mathbb{R}$. The tensor α is alternating if for any $\sigma \in S_k = \{\text{permutations of } 1, \ldots, k\}$,

$$\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma)\, \alpha(v_1, \ldots, v_k).$$

A 0-tensor has no argument. We define a 0-tensor to be a constant.

Notation. $A_k(V) = \{\text{alternating } k\text{-tensors on } V\}$.



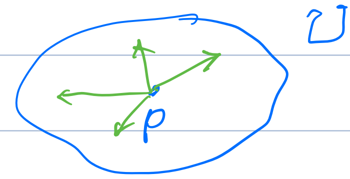
vector $v = \langle v^1, \ldots, v^n \rangle = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \sum_{i=1}^{n} v^i e_{i,p}$

point $p = (p^1, \ldots, p^n)$

In this lecture, U will be an open subset of $\mathbb{R}^n$ and $p \in U$.

T_pU = tangent space of U at p

$= \{ \langle v^1, \dots, v^n \rangle \mid v^i \in \mathbb{R} \} \simeq \mathbb{R}^n$.



Def. A k-form on U is the assignment to each $p \in U$ of an alternating k-tensor ω_p on T_pU:

$$\omega_p : T_pU \times \cdots \times T_pU \to \mathbb{R},$$

Thus, a k-form is a function $\omega : U \longrightarrow \coprod_{p \in U} A_k(T_pU)$ s.t. $\omega_p \in A_k(T_pU)$.

Vectors as Derivations at a Point

If $v \in T_pU$, let $D_v : C^\infty(U) \to \mathbb{R}$ be the directional derivative at

For $v \in T_pU$ and $f \in C^\infty(U) := \{ C^\infty \text{ functions on } U \}$,

$$vf := D_v f = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = \frac{d}{dt}\Big|_{t=0} f(p+tv)$$

$$= \sum \frac{\partial f}{\partial x^i}(p+tv) \frac{d}{dt}(p^i + tv^i)\Big|_{t=0} \quad \text{(chain rule)}$$

$$= \sum_i \frac{\partial f}{\partial x^i}(p) v^i = \left(\sum v^i \frac{\partial}{\partial x^i}\Big|_p \right) f.$$

Thus,

$$vf = \boxed{\left(\sum v^i e_{i,p} \right) f = \left(\sum v^i \frac{\partial}{\partial x^i}\Big|_p \right) f.}$$

Because of this formula, we will write

$$\boxed{e_{i,p} = \frac{\partial}{\partial x^i}\Big|_p}$$

The Differential of a Function

Def. If $f \in C^{\infty}(U)$, then df is the 1-form on U defined by: for $p \in U$ and $v \in T_pU$,

$$(df)_p(v) = vf.$$

Ex. Let $x^1, \dots, x^n$ be the standard coordinates in $\mathbb{R}^n$, $p \in U$, and $v = \sum v^i e_{i,p} \in T_pU$. Then dx^j is the 1-form s.t.

$$(dx^j)_p(v) = v(x^j) = D_v x^j$$

$$= \left(\sum_i v^i \frac{\partial}{\partial x^i}\bigg|_p\right) x^j = \sum_i v^i \delta_i^j = v^j.$$

Ex. In $\mathbb{R}^3$ with coordinates x, y, z, the symbol dx stands for the 1-form that picks out the x-coordinate v^1 of a vector $v = \langle v^1, v^2, v^3 \rangle \in T_p(\mathbb{R}^3)$ at any point $p \in \mathbb{R}^3$.

Prop. For U open in $\mathbb{R}^n$ and $p \in U$, the cotangent space $T_p^*U := (T_pU)^\vee$ = dual of tangent space has basis $(dx^1)_p, \dots, (dx^n)_p$.

Pf. A basis for T_pU is $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$.

Since $(dx^i)_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p x^i = \delta_j^i$,

$(dx^1)_p, \dots, (dx^n)_p$ is dual to $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$. $\square$

So any 1-form ω on U can be written uniquely $\omega = \sum a_i\, dx^i$ for some function a_i on U.

Def. A 1-form $\omega = \sum a_i dx^i$ on U is smooth or C^∞ if all $a_i : U \to \mathbb{R}$ are C^∞.

Th. $\boxed{df = \sum \frac{\partial f}{\partial x^i} dx^i}$ on $U \subset \mathbb{R}^n$.

Pf. Suppose $df = \sum a_i dx^i$. Applying both sides to $\partial/\partial x^j$ gives

$$\underset{\displaystyle \frac{\partial f}{\partial x^j}}{\overset{\|}{(df)\left(\frac{\partial}{\partial x^j}\right)}} = \sum_i a_i dx^i\left(\frac{\partial}{\partial x^j}\right) = a_j.$$

Hence, $df = \sum \frac{\partial f}{\partial x^i} dx^i$. $\square$

Wedge Product

Def. Let V be a vector space. If $\alpha \in A_k(V)$ and $\beta \in A_\ell(V)$, then $\alpha \wedge \beta \in A_{k+\ell}(V)$ is defined by:

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

Th. If $\alpha^1, \dots, \alpha^n$ is a basis for $A_1(V) = V^\vee$, then a basis for $A_k(V)$ is $\alpha^I = \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$, where $I = (i_1, \dots, i_k)$ runs through all $1 \le i_1 < \dots < i_k \le n$.

Every k-form ω on U can be written uniquely as a linear combination

$$\omega = \sum_{1 \le i_1 < \dots < i_k \le n} a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} := \sum_I a_I dx^I.$$

Def. A k-form $\omega = \sum a_I \, dx^I$ on U is C^∞ if all a_I are C^∞ on U.

Notation. $\Omega^k(U) = \{ C^\infty \text{ } k\text{-forms on } U \}$.

$\Omega^0(U) = C^\infty(U)$

The Exterior Derivative

Def. If $\omega = \sum a_I \, dx^I \in \Omega^k(U)$, define $d\omega \in \Omega^{k+1}(U)$ by

$$d\omega = \sum da_I \wedge dx^I.$$

Th. $d : \Omega^*(U) \to \Omega^{*+1}(U)$

satisfies

(i) d is an antiderivation of degree 1 :

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau ;$$

(ii) $d^2 = d \circ d = 0$;

(iii) d on $\Omega^0(U) = C^\infty(U)$ is the differential.

De Rham Cohomology

We have a sequence of vector spaces and linear map

$$0 \to \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \Omega^2(U) \xrightarrow{d^2} \cdots \to \Omega^n(U) \to 0$$

s.t. $d^k \circ d^{k-1} = 0$ called the de Rham complex.

Since $d^k \circ d^{k-1} = 0$, $\operatorname{im} d^{k-1} \subset \ker d^k$.

Def. $H^k(U) =$ kth de Rham cohomology of U

$$:= \frac{\ker d^k}{\operatorname{im} d^{k-1}} := \frac{Z^k(U)}{B^k(U)}$$

(Z is from German "Zyklus" for "cycle";
B standards for "Boundary.")

Example. Cohomology of $\mathbb{R} = \mathbb{R}^1$

$\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R}) = \{ C^\infty \ f : \mathbb{R} \to \mathbb{R} \}$

$\Omega^1(\mathbb{R}) = \{ f(x)\,dx \mid f \in C^\infty(\mathbb{R}) \}$.

$$0 \to \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \to 0$$

$$f \mapsto f'(x)\,dx.$$

$Z^0(\mathbb{R}) = \ker d = \{ f \in C^\infty(\mathbb{R}) \mid f' = 0 \} = \{ \text{const functions} \} = \mathbb{R}$.

$B^0(\mathbb{R}) = 0$ because the previous map is 0.

Hence, $\boxed{H^0(\mathbb{R}) = \frac{Z^0(\mathbb{R})}{B^0(\mathbb{R})} = \frac{\mathbb{R}}{0} = \mathbb{R}}$.

$Z^1(\mathbb{R}) = \ker 0 = \Omega^1(\mathbb{R}) = \{ g(x)\,dx \mid g \in C^\infty(U) \}$

$B^1(\mathbb{R}) = \operatorname{im} d = \{ df \mid f \in C^\infty(\mathbb{R}) \} = \{ f'(x)\,dx \mid f \in C^\infty(U) \}$.

Given any $g \in C^\infty(\mathbb{R})$, define

$$f(x) = \int_0^x g(u)\,du$$

By the fundamental theorem of calculus $f'(x) = g(x)$. Thus,
$Z^1(\mathbb{R}) \subset B^1(\mathbb{R})$, so $Z^1(\mathbb{R}) = B^1(\mathbb{R})$ and $\boxed{H^1(\mathbb{R}) = \frac{Z^1}{B^1} = 0}$.