CH.2 Continuous-Time Signals and Systems

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1 Signal transformation

1.1 basic transformation

The transformations of signal $x(t)$ can be summarized as

Table 1: Basic signal transformations

	Time transform (horizontally)	Amplitude transform (vertically)
Shifting (b>0)	$y(t) = x(t-b)$, shift right by b	$y(t) = x(t) - b$, move down by b
	$y(t) = x(t+b)$, shift left by b	$y(t) = x(t) + b$, move up by b
Scaling (a>1)	$y(t) = x(at)$, compress horizontally by a	$y(t) = ax(t)$, stretch vertically by a
	$y(t) = x\left(\frac{1}{a}t\right)$, expand horizontally by a	$y(t) = \frac{1}{a}x(t)$, compress vertically by a

Please be aware of the wired behaviour of Time transform.

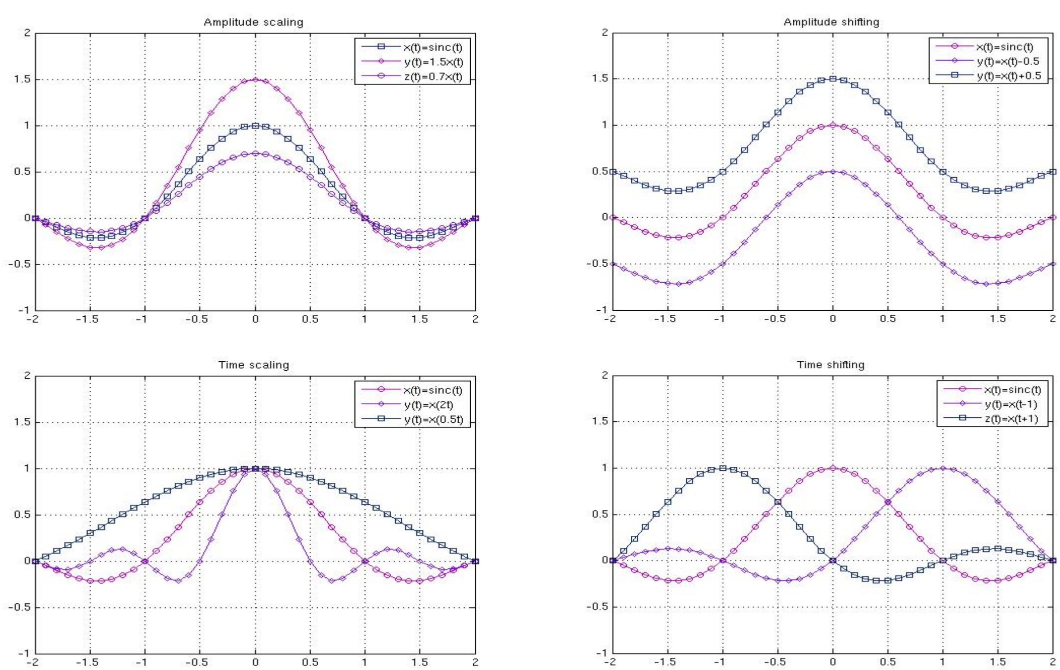


Figure 1: Illustration of Signal transformation

1.2 How to sketch the transformed signal $y(t) = ax(bt - c) + d$

Order means a lot when we combine all the transforms together. I personally prefer the following order: time shifting, time scaling, time reversing, amplitude scaling and amplitude shifting, which is straightforward. Refer to the following example, given $x(t)$, the sketch of $y(t) = 3x(-0.5t + 1) + 2$ can be obtained.

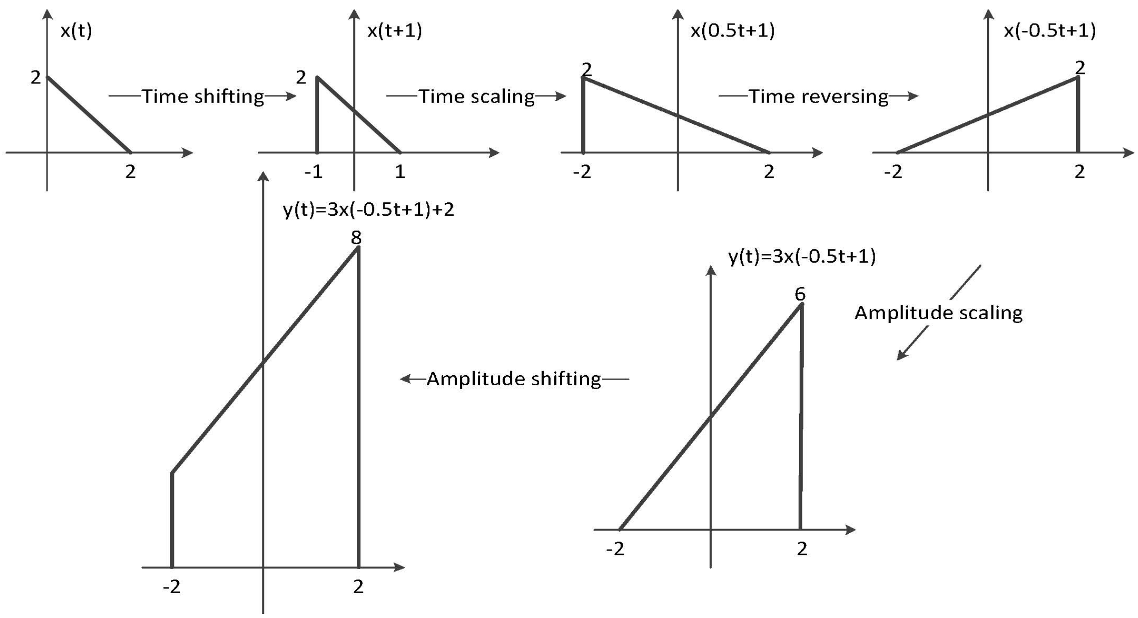


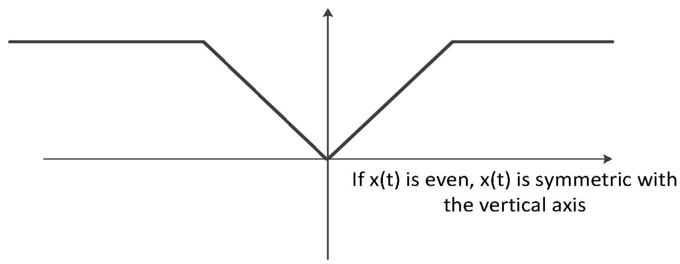
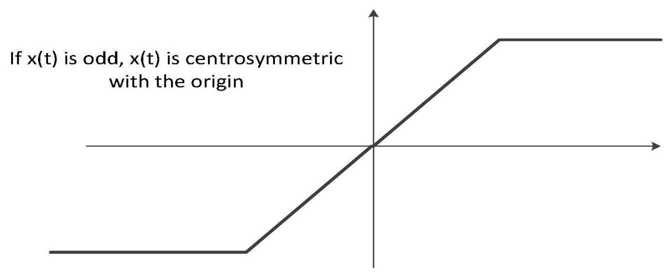
Figure 2: Illustration of obtaining the sketch of transformed signal

2 Signal property

2.1 even or odd

Table 2 shows the definition of even/odd function and its geometrical property.

Table 2: Even or odd function

	Definition	Geometrical property
$x(t)$ is even	$x(t) = x(-t)$	
$x(t)$ is odd	$x(t) = -x(-t)$	

About the even/odd functions, we should also know

Table 3: More on even/odd functions

	Calculation	Theorem
Summation	Even+Even=Even; Odd+Odd=Odd; Even+Odd=Neither	Any function can be represented by the summation of one even function and one odd function, i.e. $x(t) = x_e(t) + x_o(t)$, where $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ and $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$
Multiplication	EvenEven=Even; OddOdd=Even; EvenOdd=Odd	

2.2 periodic or aperiodic

Definition: $x(t) = x(t+T)$. The signal has a pattern and the pattern repeats. The minimum time duration required for the pattern to repeat is referred as **minimum period**: T_m. For any integer value N, if x(t) is periodic with period T_m, it is also periodic with period NT_m. Frequency is another measure to indicate or define periodicity. It characterize how fast the pattern repeats: $f = 1/T$ times/sec. Angular frequency $w = 2\pi f$.
Theorem 2.1: Is the sum of two periodic function still periodic? Suppose that $x_1(t)$ and $x_2(t)$ are periodic signals with fundamental periods T_1 and T_2, respectively. Then, the sum $y(t) = x_1(t) + x_2(t)$ is a periodic signal if and only if the ratio $T1/T2$ is a rational number.
Theorem 2.2: Is one period long enough to include all the frequency component? The answer is YES. The whole patter can not start repeating until the slowest (i.e. the component with lowest frequency) caught up. This is the fundamental of Fourier analysis in Ch. 4.

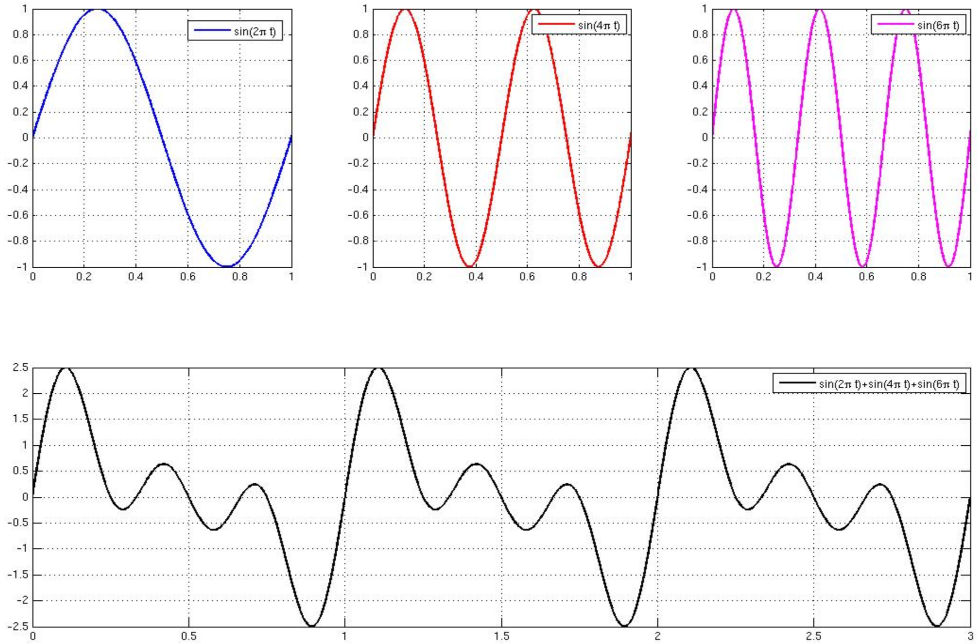


Figure 3: Illustration of one period many frequencies

2.3 causal or anticausal

The property of causal/anticausal is used to define the support set of a function. We can see how it works from following example

Example 2.6. Suppose that we have a signal $x(t)$ with the following properties:

$$
\begin{aligned}
&x(t-3)\text{is even,}\\
&x(t)=t+5, -5\le t\le -3,\\
&x(t-1)\text{is anticausal, and} x(t-5)\text{is causal.}
\end{aligned}
$$

Find $x(t)$ for all t.

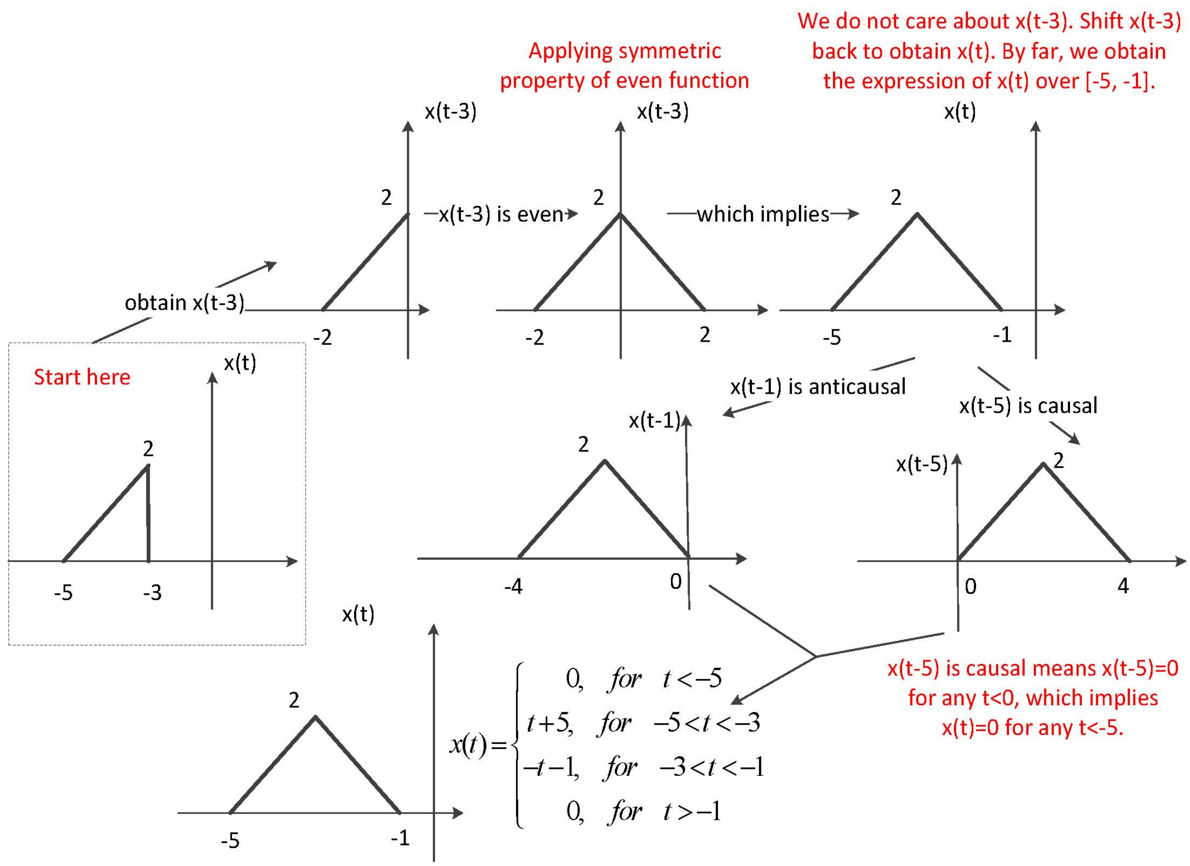
3 Elementary signals

3.1 unite-rectangular pulse

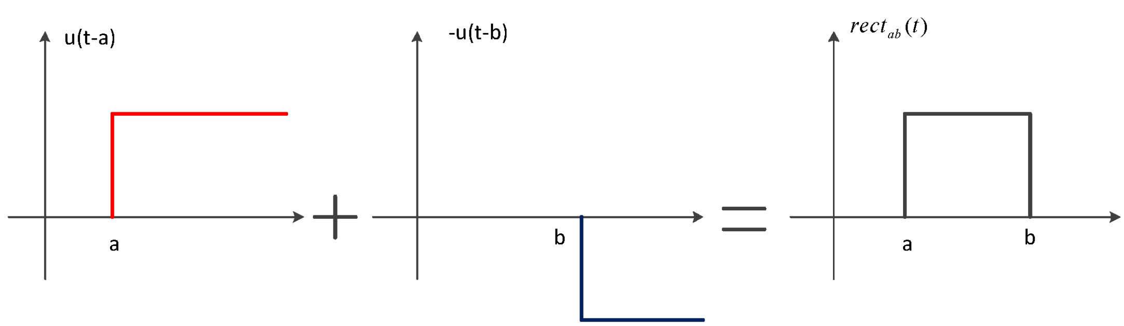
The definition of unite-rectangular pulse

$$
\text{rect}(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}
$$

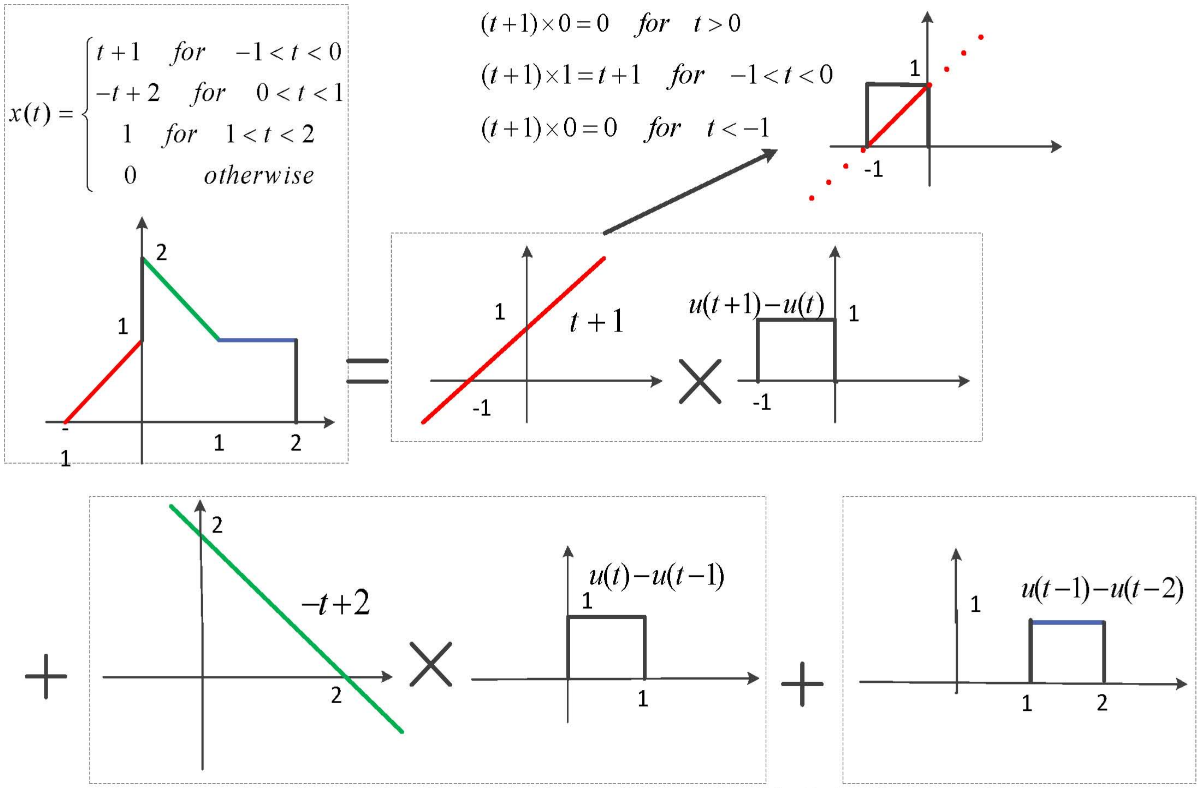
How to construct an arbitrary rectangular pulse



$$rect_{ab}(t) = \begin{cases} 1 & for \quad a < t < b \\ 0 & otherwise \end{cases} = u(t-a) - u(t-b)$$



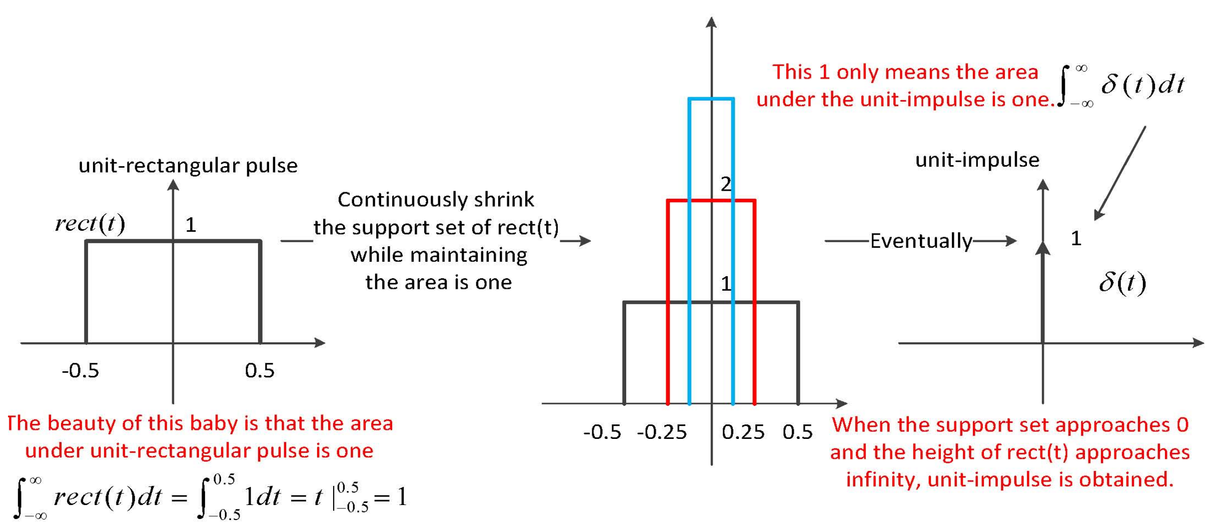
Signal representation using elementary signals. Rectangular pulse acts as a filter, or a window intuitively. Anything captured by the window can pass through without distortion. Otherwise, it will be eliminated. Please see the following example



$$x(t) = (t+1)[u(t+1)-u(t)] + (-t+2)[u(t)-u(t-1)] + [u(t+1)-u(t+2)]$$

3.2 unite-impulse function

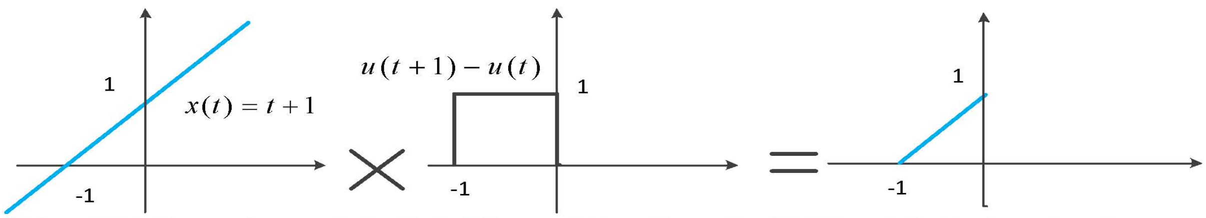
I have to say I understand your pain. $\delta(t)$ was the pain in my ass when I was an undergraduate student. But do not freak out. Let me help make your worst enemy your best friend. $\delta(t)$ is just a very very special case of unit-rectangular pulse.



Hence, $\delta(t)$ is rectangular pulse with support set $t = 0$ rather than a interval. That is it!

3.2.1 property

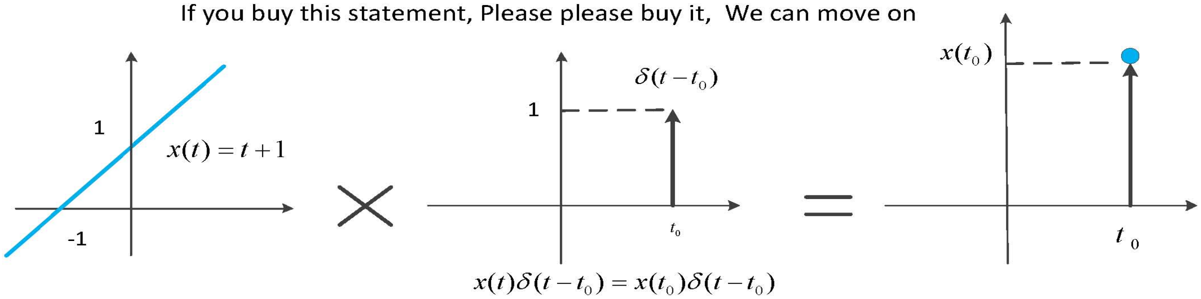
Equivalence property. I like calling this proper sampling property, which is more intuitive. Since $\delta(t)$ is a special case of $\text{rect}(t)$, you may understand it better by analogy.



This multiplication can be regarded as that x(t) is sampled over the region [-1, 0] by a shifted rectangular pulse.

Or alternatively, use the corresponding values of x(t) over the region [-1, 0] to scale a shifted rectangular pulse.

If you buy this statement, Please please buy it, We can move on



This multiplication here is actually sampling x(t) at a particular point t_0 by a shifted unit-impulse function

Or alternatively, use x(t) at t_0 , which is x(t_0), to scale a shifted unit-impulse function

Integral property. Basically, this is a very simple property. If the range of integration includes the support set (i.e. the particular point that $\delta(t)$ is defined), the integration equals one. Otherwise, the integration equals zero since the integration is taken over nothing.

Example. Calculate the following integration

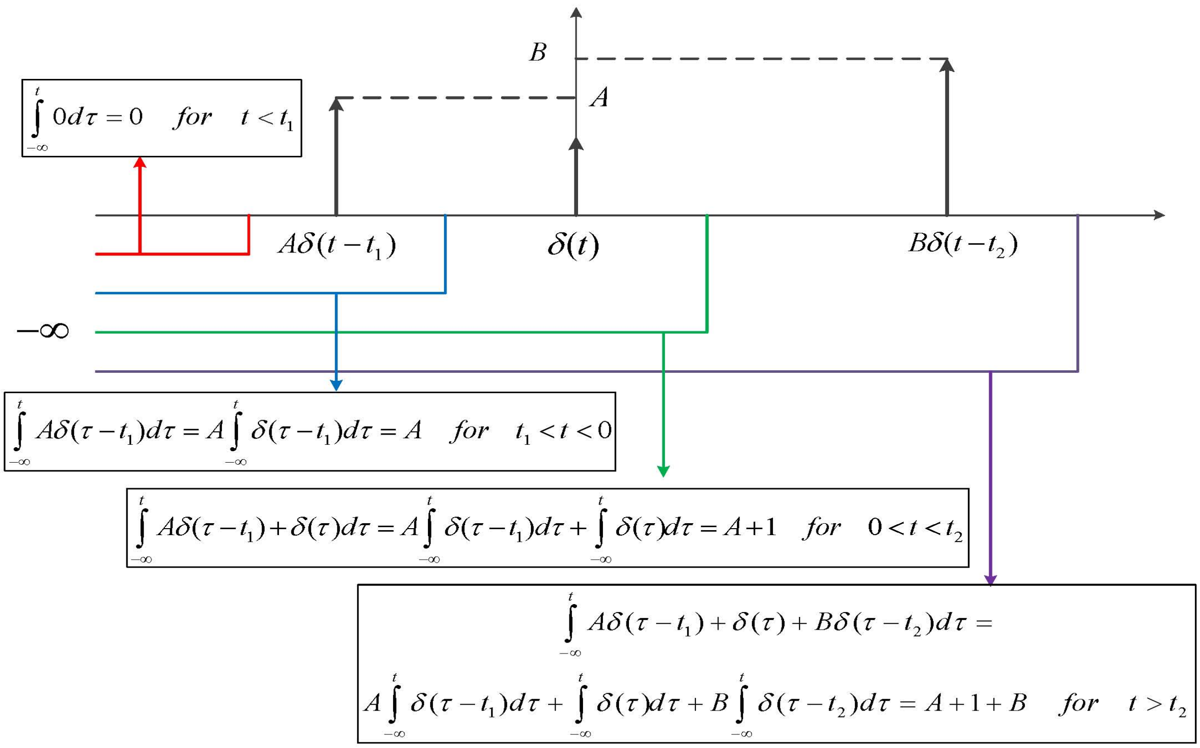
$$\int_{-\infty}^{t} A\delta(\tau - t_1) + \delta(\tau) + B\delta(\tau - t_2) d\tau$$

where A and B are constants, and $t_1 < 0 < t_2$.

Solution

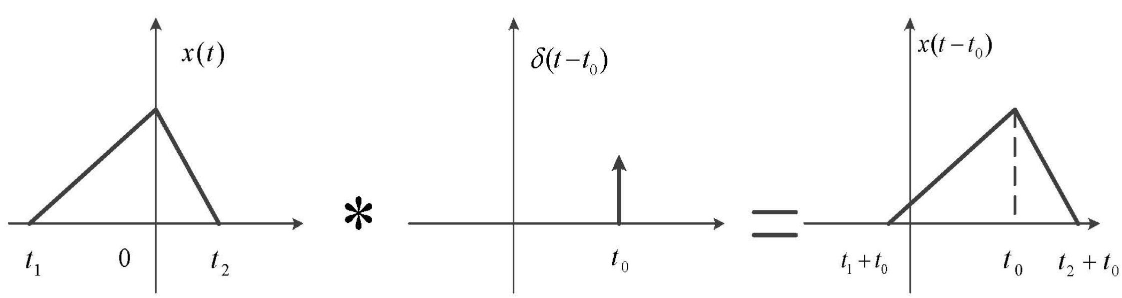
$$\int_{-\infty}^{t} A\delta(\tau - t_1) + \delta(\tau) + B\delta(\tau - t_2) d\tau = \begin{cases} 0, & t \leq t_1 \\ A, & t_1 \leq t \leq 0 \\ A + 1, & 0 \leq t \leq t_2 \\ A + 1 + B, & t \geq t_2 \end{cases}$$

How to solve such integration? The only thing you have to do is check the range of integration and the support set of your impulse functions.



convolution property. We can somehow call this property shifting property. It is not significant in this chapter. But it is pretty important in the coming chapters.

$$x(t) * \delta(t - t_0) = x(t - t_0)$$



4 System

The system operates on input $x(t)$ to produce output $y(t)$. One can regard the output as the function of a function (i.e. $y(t)$ is a function of $x(t)$, which is also a function).

$$y(t) = \mathcal{F}[x(t)]$$

4.1 property of system

Table 4: System property

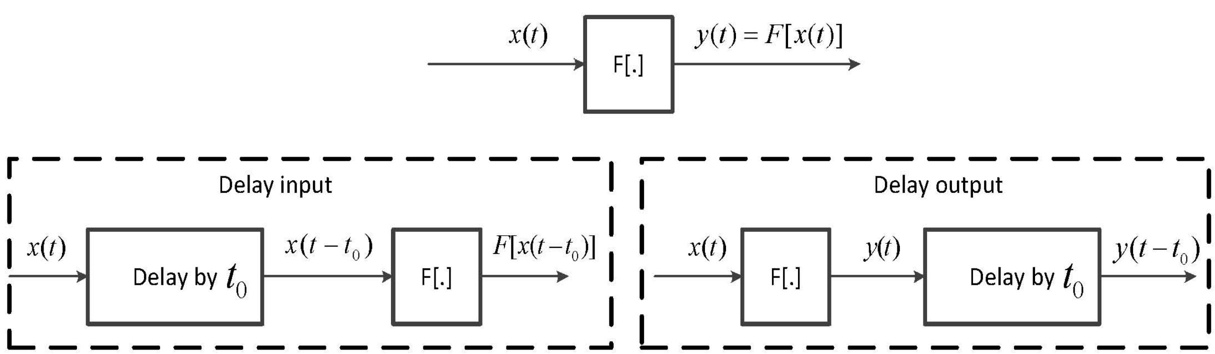
	one-word Definition
Memory	For a memoryless system, current output only depends on the current input
Causality	For a system to be causal, future input can not be used to produce current output
Invertibility	For a system to be invertible, only one possible input corresponds to a given output
Stability	For a system to be BIBO stable, a bounded input always results in bounded output
Time invariance	For a system to be TI, time shift in the input results in an identical time shift in the output
Linearity	Additivity: sum of the inputs produces the sum of the corresponding outputs Homogeneity: scaling the input leads to the identical scaling in the output

I will skip memory, causality, invertibility and stability since they are relatively simple. In your mid-term or final, check these properties by definition!!.

4.1.1 time invariance

This property actually compare the effects of two different operations. If the effects are identical, the system is time invariant.

What kind of operations? Suppose that the the operation of the system is defined as $\mathcal{F}[\cdot]$, given an input $x(t)$, the corresponding output is $y(t) = \mathcal{F}[x(t)]$. As such, the different operations are defined as:



Time Invariant: effects of delaying input and those of delaying output are identical

$$F[x(t-t_0)] = y(t-t_0)$$

Problem.2.13.(a):

$$y(t) = \mathcal{F}[x(t)] = \frac{d[x(t)]}{dt} \tag{1}$$

Firstly, make the delayed input pass through the system (Replacing $x(t)$ by $x(t-t_0)$), we have

$$\mathcal{F}[x(t-t_0)] = \frac{d[x(t-t_0)]}{dt} = \frac{d[x(t-t_0)]}{d(t-t_0)} \times \frac{d(t-t_0)}{dt} = \frac{d[x(s)]}{d(s)} \times 1 = \frac{d[x(s)]}{ds}$$

Secondly, delay the output $y(t)$ by t_0 (Replacing t by t_0),

$$y(t-t_0) = \frac{d[x(t-t_0)]}{d(t-t_0)} = \frac{d[x(s)]}{ds}$$

$\mathcal{F}[x(t-t_0)] = y(t-t_0)$, this is a time-invariant system.

Problem.2.13.(f):

$$y(t) = \mathcal{F}[x(t)] = \int_{-\infty}^{2t} x(\tau)d\tau \tag{2}$$

Firstly, make the delayed input pass through the system (Replacing $x(t)$ by $x(t-t_0)$), we have

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{2t} x(\tau - t_0)d\tau$$

Applying variable replacement, denote $s = \tau - t_0$

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{2t-t_0} x(s)ds$$

Secondly, delay the output $y(t)$ by t_0(Replacing t by t_0),

$$y(t-t_0) = \int_{-\infty}^{2(t-t_0)} x(\tau)d\tau = \int_{-\infty}^{2t-2t_0} x(\tau)d\tau$$

$\mathcal{F}[x(t-t_0)] \neq y(t-t_0)$, this is a time-variant system.