# CH.3 Continuous-Time Linear Time-Invariant System

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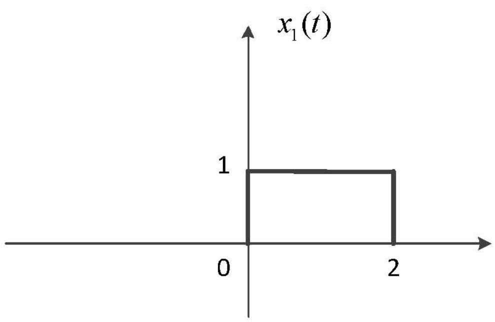
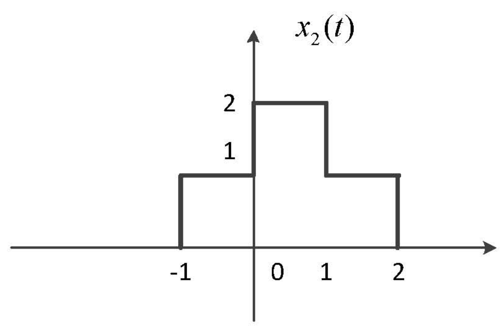
## 1 LTI System Characterization

### 1.1 what does LTI mean?

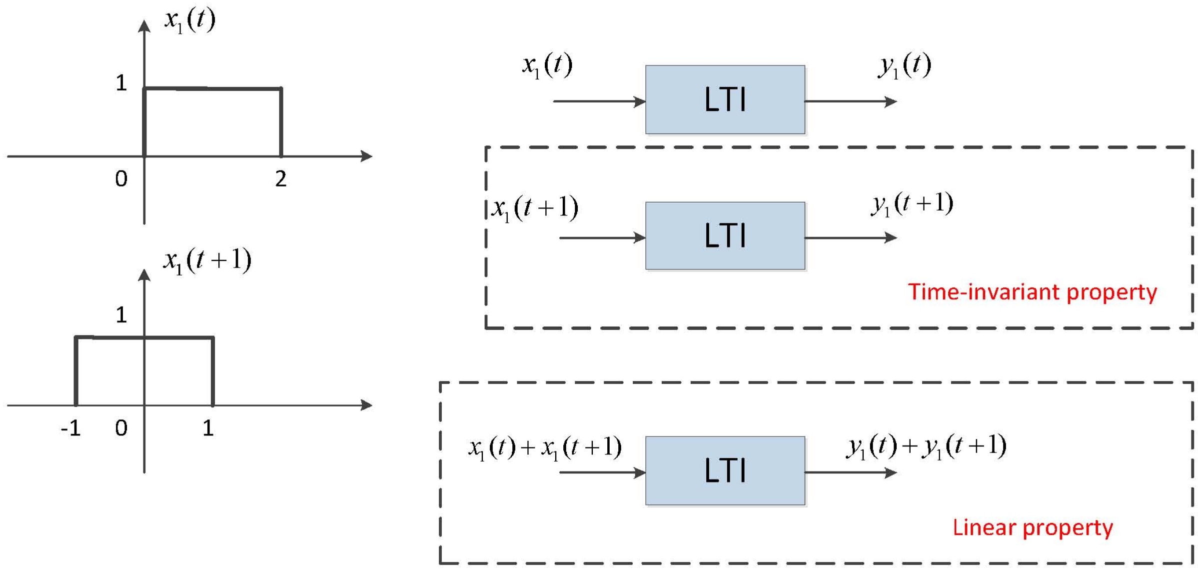
In Ch.2, the properties of the system are investigated. We are particularly interested in linear time-invariant (LTI) system since it is easy and general in practical. LTI systems are extensively used in signal processing. In fact, they form the foundation of the whole field. That is why it is significant to fully understand what LTI means. To clarify that, let us go through some examples together.

**Example 1**: $y_1(t)$ is the corresponding output of a LTI system when $x_1(t)$ is applied as the input. Represent the output $y_2(t)$ by $y_1(t)$, where $y_2(t)$ is the output of same LTI system to the input $x_2(t)$. The sketch of $x_1(t)$ and $x_2(t)$ are revealed below




**Solution**: If you take a careful look, $x_2(t) = x_1(t) + x_1(t+1)$.




## 1.2 Impulse Response of LTI System

Just refresh your memory, the definition of impulse function is

$$\delta(t) = \begin{cases} 1, & t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

If you apply $\delta(t)$ to the system, the output is so called impulse response, normally shown by $h(t)$. As the name shown, $h(t)$ is the response to $\delta(t)$. If the system is LTI, $h(t)$ can fully identify the system. This is super important. So far, the explain is beyond your knowledge, Since you guys know nothing about the frequency representation of a signal. I will come back and explain this when we finish chapter 5, Fourier transform. How to obtain the impulse response? Replacing $x(t)$ by $\delta(t)$. Here, you need to carefully apply the properties of $\delta(t)$. As a quick reminder, the properties of $\delta(t)$ are equivalence property, integral property and convolution property, respectively.

**Example**: Find the impulse response of the LTI system characterized by each of the equations below. In each case, the input and output of the system are denoted as $x(t)$ and $y(t)$, respectively.

(a).

$$y(t) = \int_{-\infty}^{t} x(3\tau)d\tau$$

(b).

$$y(t) = \int_{-\infty}^{\infty} x(\tau + 4)e^{t+\tau}u(t + \tau)d\tau$$

**Solution** (a)The impulse response is given by

$$h(t) = \int_{-\infty}^{t} \delta(3\tau)d\tau = \frac{1}{3}\int_{-\infty}^{t} \delta(\tau)d\tau = \frac{1}{3}u(t)$$

(b)The impulse response is given by

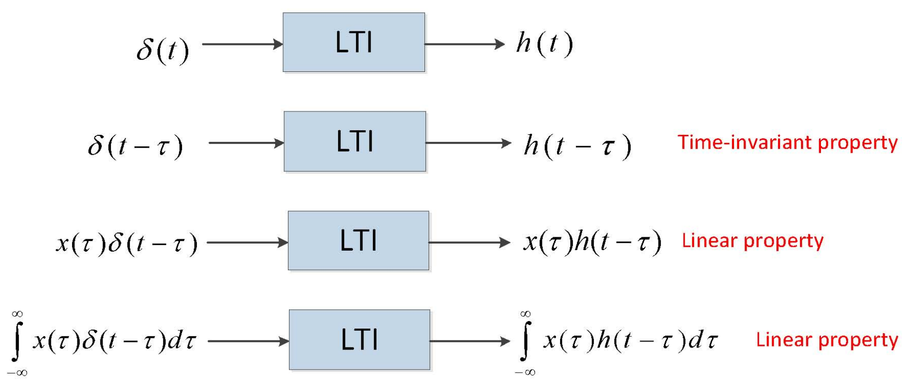
$$h(t) = \int_{-\infty}^{\infty} \delta(\tau + 4) e^{t+\tau} u(t + \tau) d\tau$$

Here, the integral is taken w.r.t $\tau$ and $t$ is assumed to be constant. Applying equivalence property, $h(t)$ can be written as

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} \delta(\tau + 4) e^{t-4} u(t - 4) d\tau \\ &= e^{t-4} u(t-4) \int_{-\infty}^{\infty} \delta(\tau + 4) d\tau \\ &= e^{t-4} u(t-4) \end{aligned}$$

# 2 Continuous-time Convolution

Convolution means complicated detail literally. So, again, I can understand your pain. However, we have to talk about convolution, it characterise the input and output relation of a LTI system. But, how come? please refer to the figure below.




Look at the LHS, it implies that the output of a LTI system, given any arbitrary input $x(t)$ can be calculated by $\int_{-\infty}^{\infty} x(\tau)h(\tau - t)d\tau$, usually denoted by $x(t) * h(t)$.

The LHS of last step shows us $x(t) = \int_{\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$, which implies, given any arbitrary input $x(t)$, the output of a LTI system can be calculated by $\int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau$. The integration is named convolution. This is the reason that we have to investigate and calculate convolution even though it tortures us. But convolution is just complicated. It is not difficult at all. You need to fully understand the procedure and practise.

## 2.1 Property of convolution

Any property of convolution comes from the property of integration. When you solve the problem like proving the property of convolution or proving the assertions on convolution is true, please write the integral definition of convolution first.

**Example**: Consider the convolution $y(t) = x(t) * h(t)$. Assuming that the convolution y(t) exists, prove that each of the following assertions is true:

(a) If $x(t)$ is periodic then $y(t)$ is periodic.

(b) If $x(t)$ is even and $h(t)$ is odd, then $y(t)$ is odd.

**Solution**

(a). By definition

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Since $x(t)$ is periodic, $x(t) = x(t+T)$. We obtain

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau+T)h(t-\tau)d\tau$$

Applying variable replacement, denote $s = \tau + T$, which yields

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(s)h(t+T-s)ds = y(t+T)$$

(b). By definition

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Applying variable replacement, denote $s = -\tau$, which yields

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{+\infty}^{-\infty} x(-s)h(t+s)d(-s) \\ &= \int_{-\infty}^{\infty} x(-s)h(t+s)ds \end{aligned}$$

Because $x(t)$ is even and $h(t)$ is odd, $x(-s) = x(s)$ and $h(s+t) = -h(-s-t)$. We obtain

$$y(t) = x(t) * h(t) = -\int_{-\infty}^{\infty} x(s)h(-t-s)ds = -y(-t)$$

Hence, $y(t)$ is odd.

## 2.2 Calculate convolution

Firstly, do not freak out. Solving convolution is actually solving convolution integration. Here is the systematic process in calculating convolution

1. Plot $x(\tau)$ and $h(t-\tau)$, or the other way around.
2. Find the critical points, before and after which the the mathematical expression is different, in both $x(\tau)$ and $h(t-\tau)$.
3. Compare the critical points and find different scenarios.
4. Form the integral for each scenario.

Refer to the following example. Let us solve convolution together step by step.
**Example**:$x(t)$ is given by

$$x(t) = \begin{cases} 1-t, & 0 \leq t < 1; \\ 1+t, & -1 \leq t \leq 0. \end{cases}$$

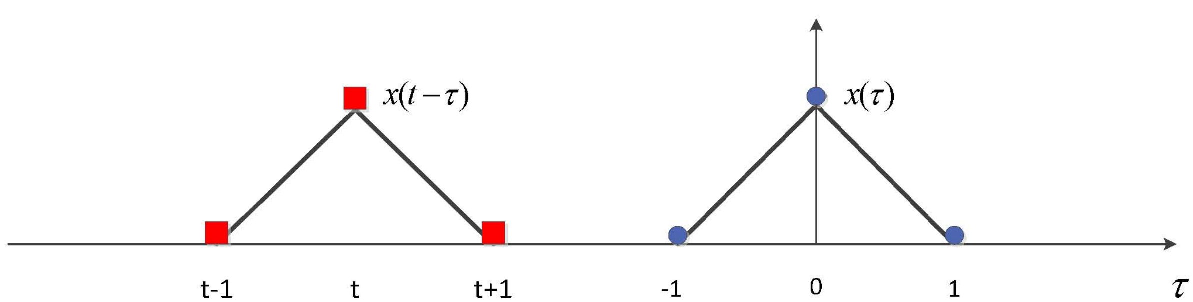
Calculate the convolution $x(t) * x(t)$
**Solution**

$$y(t) = x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$$

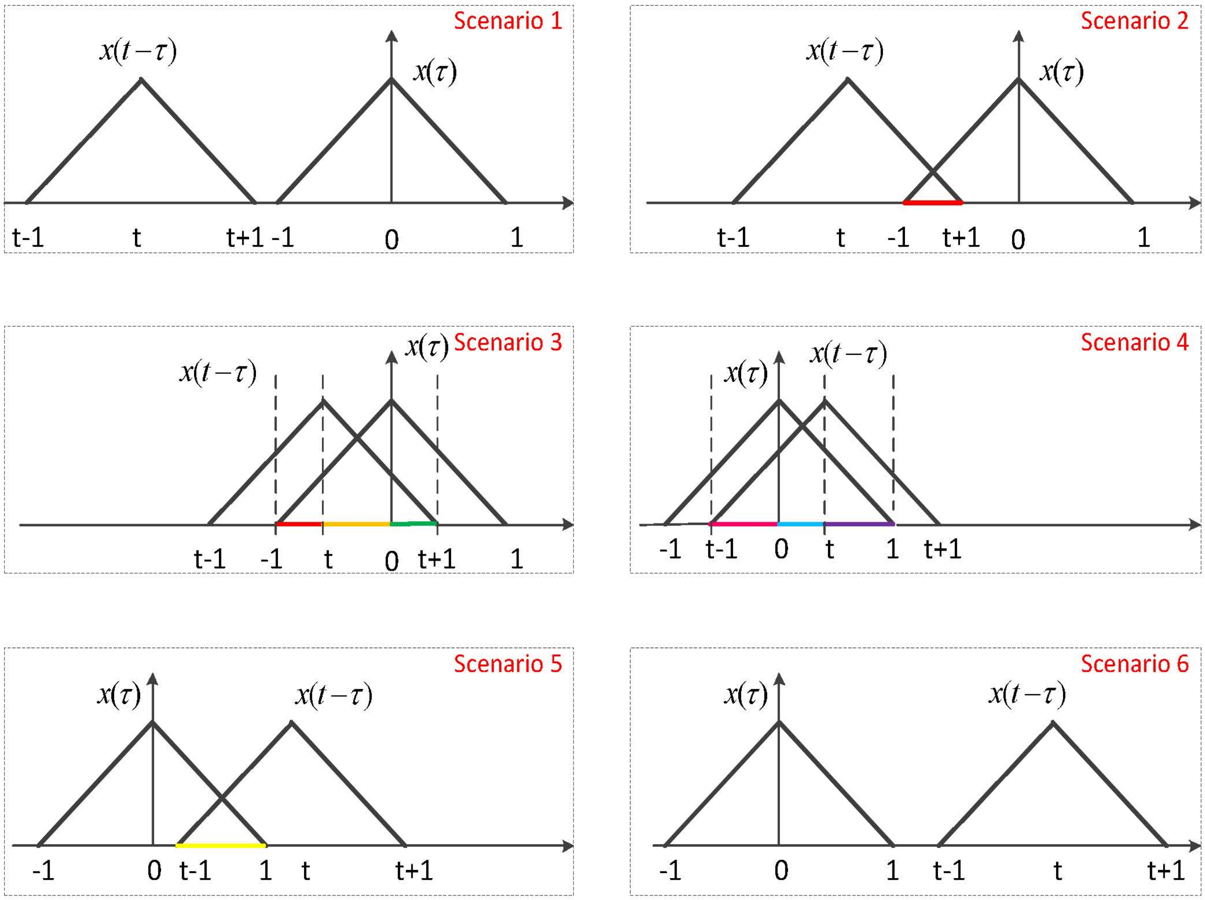
Note that the integration is taken with respect to $\tau$. Hence, $t$ is supposed to be a constant when you carry out this integration.
$x(t)$ and $x(\tau)$ have exactly same mathematical expression. $x(t-\tau)$ is obtained by firstly shifting $x(\tau)$ by $t$, then, flipping the whole thing w.r.t. the vertical axis. Or simply replace $t$ by $t-\tau$ in both expressions and support set. Hence,

$$x(t-\tau) = \begin{cases} 1+\tau-t, & t-1 \leq \tau < t; \\ 1-\tau+t, & t \leq \tau \leq t+1. \end{cases}$$

The sketches of $x(\tau)$ and $x(t-\tau)$ are shown



Increasing $t$, equivalent to slowly dragging $x(t-\tau)$ to right. We can classify the following six cases, for each of which the expressions of $x(\tau)$ and $x(t-\tau)$ over the overlapping region are different.




Then, we calculate the convolution integral for each scenario.
*Scenario 1* This scenario occurs only when $t+1<-1$. Obviously, $x(\tau)$ and $x(t-\tau)$ have no overlapped region. We obtain here

$$y(t)=x(t)*x(t)=\int_{-\infty}^{\infty}x(\tau)x(t-\tau)d\tau=0, \quad t<-2$$

*Scenario 2* This scenario occurs only when $-1<t+1<0$. Obviously, over the overlapped region, $x(\tau)$ has a positive slope and $x(t-\tau)$ have negative slope, which yields

$$\begin{aligned}y(t)=x(t)*x(t)&=\int_{-\infty}^{\infty}x(\tau)x(t-\tau)d\tau\\&=\int_{-1}^{t+1}(1-\tau+t)(1+\tau)d\tau\\&=\frac{1}{6}(t+2)^3, \quad -2\le t<-1\end{aligned}$$

*Scenario 3* We keep sliding this reversed and shifted triangle for different value of $t$ until the mathematical expression over the over lapped region changes. But do not slide too far. Scenario 3 occurs when $0 < t+1 < 1$. We arrive at

$$\begin{aligned} y(t) = x(t) * x(t) &= \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau \\ &= \int_{-1}^{t} (1+\tau-t)(1+\tau)d\tau + \int_{t}^{0} (1-\tau+t)(1+\tau)d\tau + \int_{0}^{t+1} (1-\tau)(1-\tau+t)d\tau \\ &= -\frac{1}{2}t^3 - t^2 + \frac{2}{3}, \quad -1 \le t < 0 \end{aligned}$$

*Scenario 4* When the front edge of $x(t-\tau)$ passes through 1

$$\begin{aligned} y(t) = x(t) * x(t) &= \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau \\ &= \int_{t-1}^{0} (1+\tau-t)(1+\tau)d\tau + \int_{0}^{t} (1+\tau-t)(1-\tau)d\tau + \int_{t}^{1} (1-\tau)(1-\tau+t)d\tau \\ &= \frac{1}{2}t^3 - t^2 + \frac{2}{3}, \quad 0 \le t < 1 \end{aligned}$$

*Scenario 5* We are almost there...

$$\begin{aligned} y(t) = x(t) * x(t) &= \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau \\ &= \int_{t-1}^{1} (1+\tau-t)(1-\tau)d\tau \\ &= -\frac{1}{6}(t-2)^3, \quad 1 \le t < 2 \end{aligned}$$

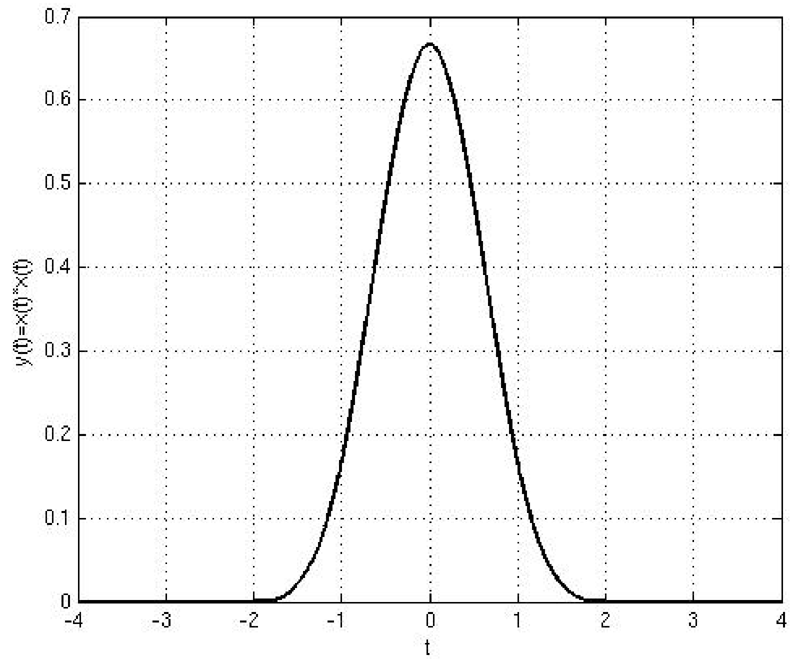
*Scenario 6* Finally, when the two functions have no overlapped region again, the convolution equals zero again.

$$y(t) = x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau = 0, \quad 2 < t$$

Finally, the convolution can be expressed in a piece wise function

$$y(t) = x(t) * x(t) = \begin{cases} 0, \quad t < -2 \\ \frac{1}{6}(t+2)^3, \quad -2 \le t < -1 \\ -\frac{1}{2}t^3 - t^2 + \frac{2}{3}, \quad -1 \le t < 0 \\ \frac{1}{2}t^3 - t^2 + \frac{2}{3}, \quad 0 \le t < 1 \\ -\frac{1}{6}(t-2)^3, \quad 1 \le t < 2 \\ 0, \quad 2 < t \end{cases}$$

The following figure illustrates the result of our analytical results.

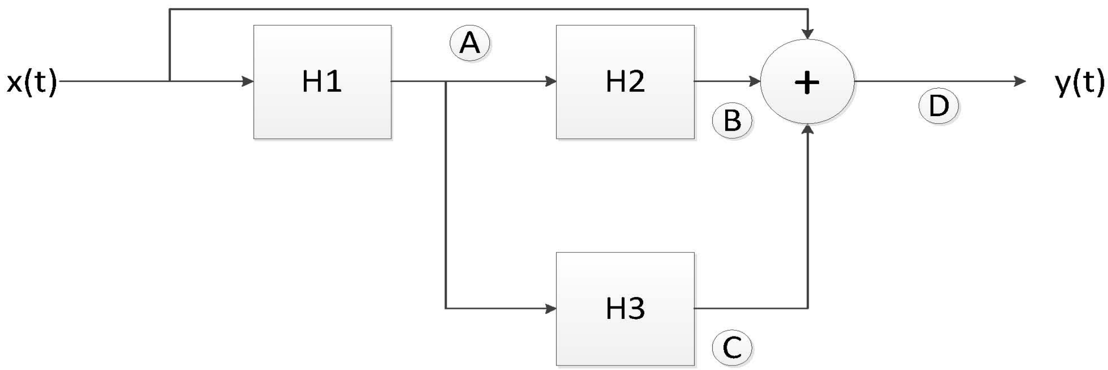



### 2.3 Visualizing convolution

What kind of effect does LTI system perform onto the input signal. Some people prefer so-called flipping and dragging procedure. However, I would like to say LTI system make the input smooth and expanded. Convolution averages $x(t)$ over $h(t)$.

## 3 Now we are ready

We KNOW a LTI system is fully identified by its impulse response. We also KNOW, given arbitrary input signal $x(t)$, the corresponding output can be calculated by convolution. Now, we are ready to investigate the practical LTI systems.
**Example** Consider the system with input $x(t)$ and output $y(t)$ as shown in the figure below. Suppose that the systems $H1$, $H2$, and $H3$ are LTI systems with impulse responses $h_1(t)$, $h_2(t)$, and $h_3(t)$, respectively
(a) Find the impulse response h(t) of the overall system in terms of $h_1(t)$, $h_2(t)$, and $h_3(t)$.
(b) Determine the impulse response h(t) in the specific case that $h_1(t) = \delta(t+1)$, $h_2(t) = \delta(t)$, and $h_3(t) = \delta(t)$.
**Solution**
(a).Signal at point A: $a(t) = x(t) * h_1(t)$
Signal at point B: $b(t) = a(t) * h_2(t) = x(t) * h_1(t) * h_2(t)$
Signal at point C: $c(t) = a(t) * h_3(t) = x(t) * h_1(t) * h_3(t)$
Signal at point D: $d(t) = x(t) + x(t) * h_1(t) * h_2(t) + x(t) * h_1(t) * h_3(t)$. Applying $x(t) =$




$x(t) * \delta(t)$, we obtain

$$y(t) = x(t) * [\delta(t) + h_1(t) * h_2(t) + h_1(t) * h_3(t)]$$

The overall impulse response of the system is $h(t) = \delta(t) + h_1(t) * h_2(t) + h_1(t) * h_3(t)$
(b).

$$\begin{aligned} h(t) &= h_1(t) + h_1(t) * h_2(t) + h_1(t) * h_3(t) \\ &= \delta(t+1) * \delta(t) + \delta(t+1) * \delta(t) + \delta(t) \\ &= \delta(t+1) + \delta(t+1) + \delta(t) \\ &= 2\delta(t+1) + \delta(t) \end{aligned}$$

# 4 Property of LTI system

| | |
|---|---|
| LTI system is memoryless | iff its impulse response $h(t) = 0$ for all $t \neq 0$ |
| LTI system is BIBO stable | iff its impulse response $h(t)$ is absolutely integrable i.e. $\int_{-\infty}^{\infty} \lvert h(t) \rvert dt \leq \infty$ |
| LTI system is invertible | iff it simpulse response $h(t)$ and the inverse $h^{inv}(t)$ satisfy $h(t) * h^{inv}(t) = \delta(t)$ |