CH.5 Continuous-Time Fourier Transform

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1 THEOREM

1.1 Transfer from periodic phenomena to aperiodic phenomena

Basically, Fourier series has two disadvantages in modeling practical phenomena. Firstly, not all the phenomena are periodic. Secondly, in practise, all phenomena are time constrained. In other words, it has to start somewhere and end somewhere else. Nothing lasts forever. I love you forever is a lie! However, in term of the building block, sine and cosine last forever. How to extend periodicity to aperiodicity? A very naive move is suppose that we have a very big period T to include the whole function and let T go to infinity, see Fig. 1. Does this approach work?

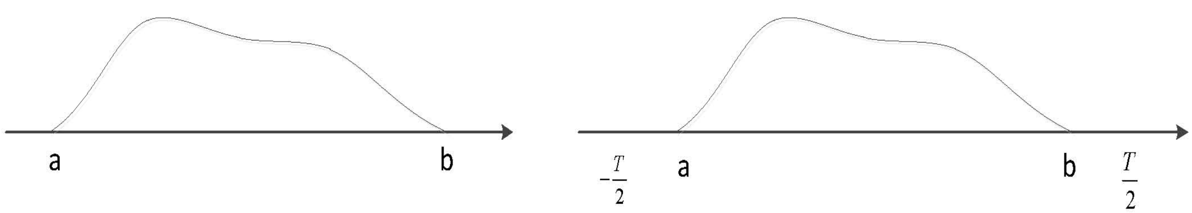


Figure 1: For aperiodic time-limited signal, suppose that we have a very big period.

Suppose that it works and see what the consequences are? Mathematically the coefficient is given by

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\frac{2\pi k}{T}t}dt = \frac{1}{T}\int_a^b f(t)e^{-j\frac{2\pi k}{T}t}dt \tag{1}$$

The magnitude of C_k can be calculated as

$$\begin{aligned} |C_k| &= \frac{1}{T}\left|\int_a^b f(t)e^{-j\frac{2\pi k}{T}t}dt\right| \le \frac{1}{T}\int_a^b \left|f(t)\right|\left|e^{-j\frac{2\pi k}{T}t}\right|dt \\ |C_k| &\le \frac{1}{T}\int_a^b |f(t)|dt \end{aligned} \tag{2}$$

Here $\int_a^b |f(t)|dt$ is a definite integration. The exact value of it does not matter. We know the result has to be a constant. Denote $M = \int_a^b |f(t)|dt$. As such,

$$\begin{aligned} |C_k| &\leq \frac{1}{T}M \\ \lim_{T\to\infty} |C_k| &\leq \lim_{T\to\infty} \frac{1}{T}M = 0 \end{aligned} \tag{3}$$

All the coefficients die!!! It seems that when T go to infinity, all the coefficients go to zero. We get nothing! How to save it? From Eq. (1), we can see the bad guy is $\frac{1}{T}$ in the front. It murders the coefficient since T goes to infinity, $\frac{1}{T}$ goes to zero. Then we need to figure out how to get rid of this painful $\frac{1}{T}$. Even though k is the index representing frequency, $\frac{k}{T}$ is something really matters.

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi\left(\frac{k}{T}\right)t}dt = \frac{1}{T}\mathscr{F}\left[f\left(\frac{k}{T}\right)\right] \tag{4}$$

$\mathscr{F}\left[f\left(\frac{k}{T}\right)\right]$ is just one notation representing the integration in Eq. (4). As such, the kth coefficient is written by the product of $\frac{1}{T}$ and $\mathscr{F}\left[f\left(\frac{k}{T}\right)\right]$. Applying Fourier synthesis formula, we have

$$\begin{aligned} f(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi\left(\frac{k}{T}\right)t} &= \sum_{k=-\infty}^{\infty} \mathscr{F}\left[f\left(\frac{k}{T}\right)\right] e^{j2\pi\left(\frac{k}{T}\right)t}\frac{1}{T} \\ &= \int_{-\infty}^{\infty} \mathscr{F}\left[f(s)\right] e^{j2\pi st}ds \end{aligned} \tag{5}$$

where the last equality is carried out by making T go to infinity. You can trace the derivation in Eq. (5) by analogy with Fig. 2.

Now, we can write the formulas of so-called Fourier transform and inverse Fourier transform.

$$\mathscr{F}\left[f(s)\right] = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}dt \qquad f(t) = \int_{-\infty}^{\infty} \mathscr{F}\left[f(s)\right] e^{j2\pi st}ds \tag{6}$$

To be consistent with the representation in your text book. I will use the following formulas in my slide as well. Eq. (6) and Eq. (7) are identical. I personally prefer Eq. (6) because I do not have to care about the $\frac{1}{2\pi}$ in the front. But since this is a new topic for you guys, I should try me best not to confuse you.

$$\mathscr{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}(\omega)e^{j\omega t}d\omega \tag{7}$$

The constant coefficient comes from the fact that $\omega = 2\pi s$ resulting in $d\omega = 2\pi ds$. As such, $ds = \frac{1}{2\pi}d\omega$. **Example 1** Using the Fourier transform analysis equation, find the Fourier transform of each of the following signals:

(a) $f(t) = 1$

(b) $f(t) = \cos\omega_0 t$

(c) $f(t) = \mathrm{sgn}(t)$;

(d) $f(t) = 3[u(t) - u(t-2)]$;

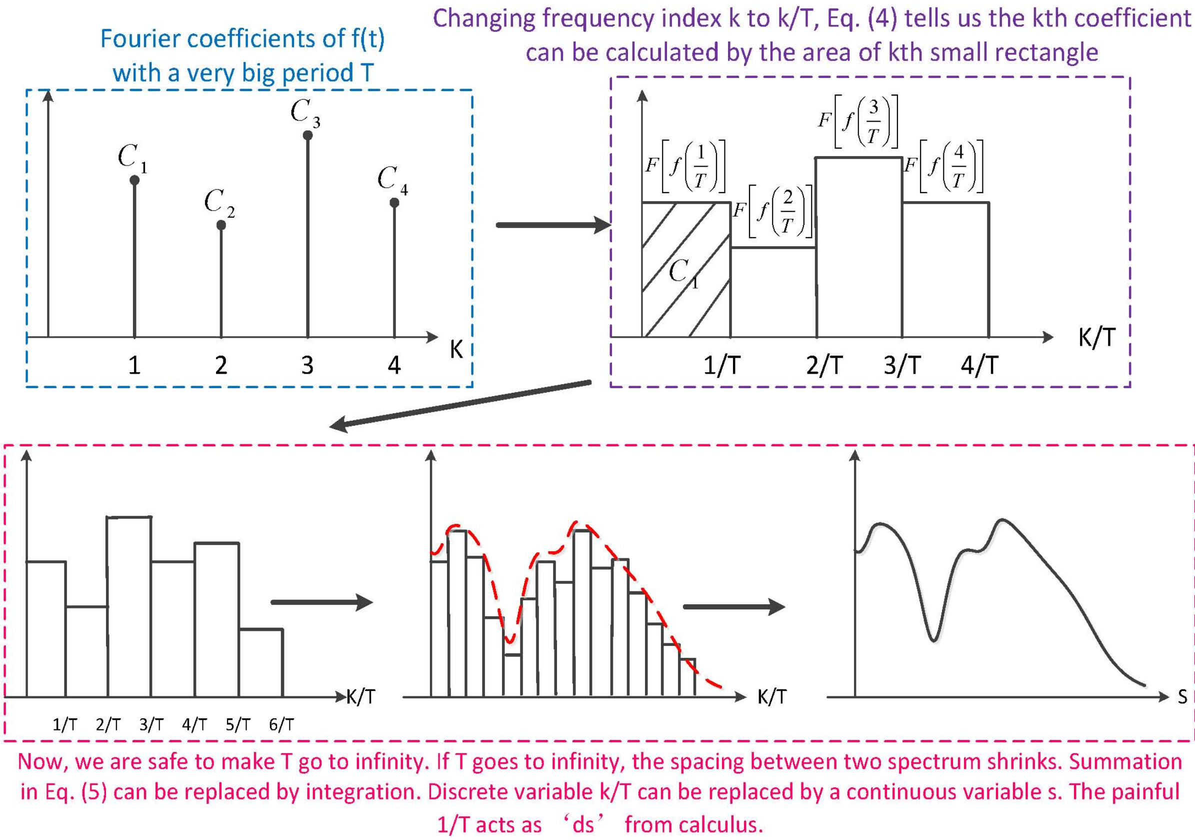


Figure 2: Not a formal mathematical derivation of inverse Fourier transform. But this is the most straight forward way I can show you where the formulas come from.

(e)$x(t) = e^{-|t|}$.

Solution

(a)

$$\mathscr{F}[1] = \int_{-\infty}^{\infty} e^{-j\omega t} dt \tag{8}$$

Generally, this integration can not be solved by Newton-Leibniz integration. However, with theory of distributions, δ-function and shifted δ-function can be expressed as

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t} dt; \qquad \delta(\omega - \omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt \tag{9}$$

Therefore

$$\mathscr{F}[1] = 2\pi\delta(\omega); \qquad \mathscr{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0) \tag{10}$$

(b)

$$\begin{aligned}
\mathscr{F}[\cos\omega_0 t] = \int_{-\infty}^{\infty} \cos\omega_0 t e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t} dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} + e^{-j(\omega+\omega_0)t} dt = \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t} d\right. \\
&= \frac{1}{2}[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)] \\
&= \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]
\end{aligned} \tag{11}$$

(c)$f(t) = \text{sgn}(t)$ is actually defined as a piece wise function. $\text{sgn}(t) = 1$ for any $t \geq 0$ and $\text{sgn}(t) = -1$ for any $t \leq 0$. This function is not integrable. Hence, you can not expect to calculate the Fourier transform by definition. Watch me solving this problem. It is magic. we can redefine $\text{sgn}(t)$

$$\text{sgn}(t) = u(t) - u(-t) = \lim_{a\to 0}[e^{-at}u(t) - e^{at}u(-t)] \tag{12}$$

Thus

$$\begin{aligned}
\mathscr{F}[\text{sgn}(t)] &= \mathscr{F}\left[\lim_{a\to 0}[e^{-at}u(t) - e^{at}u(-t)]\right] = \lim_{a\to 0}\left\{\mathscr{F}\left[e^{-at}u(t)\right] - \mathscr{F}\left[e^{at}u(-t)\right]\right\} \\
&= \lim_{a\to 0}\left(\int_{0}^{\infty} e^{-at}e^{-j\omega t} - \int_{-\infty}^{0} e^{at}e^{-j\omega t}\right) = \lim_{a\to 0}\left(\frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty} - \frac{1}{a-j\omega}e^{(a-j\omega)t}\Big|_{-\infty}^{0}\right) \\
&= \lim_{a\to 0}\left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right) = \lim_{a\to 0}\frac{-2j\omega}{a^2+\omega^2} = \frac{2}{j\omega}
\end{aligned} \tag{13}$$

This can be an explanation of "suppose the Fourier transform of $\text{sgn}(t)$ is $\frac{2}{j\omega}$" in Example 5.6 in your text book. $u(t)$ is also non-integrable, which means you can not solve the Fourier transform by definition. That is why we have to represent $u(t) = \frac{1+\text{sgn}(t)}{2}$. The result of Example 5.6 is pretty useful.

$$\mathscr{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega} \tag{14}$$

(d)

$$\begin{aligned}
\mathscr{F}\left[3[u(t) - u(t-2)]\right] &= \int_{-\infty}^{\infty} 3[u(t) - u(t-2)]e^{-j\omega t} dt = 3\int_{0}^{2} e^{-j\omega t} dt = 3\left[-\frac{1}{j\omega}e^{-j\omega t}\right]\Bigg|_{0}^{2} \\
&= -\frac{3}{j\omega}\left(e^{-j2\omega} - 1\right) = \frac{j3}{\omega}e^{-j\omega}\left(e^{-j\omega} - e^{j\omega}\right) = \frac{j3}{\omega}e^{-j\omega}(-2j\sin(\omega)) \\
&= 6e^{-j\omega}\frac{\sin(\omega)}{\omega} = 6e^{-j\omega}\text{sinc}(\omega)
\end{aligned} \tag{15}$$

(e)

$$
\begin{aligned}
\mathscr{F}\left[e^{-|t|}\right] &= \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{(1-j\omega)t}dt + \int_{0}^{\infty} e^{(-1-j\omega)t}dt = \frac{1}{1-j\omega}e^{(1-j\omega)t}\Big|_{-\infty}^{0} - \frac{1}{1+j\omega}e^{(-1-j\omega)t}\Big|_{0}^{\infty} \\
&= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}
\end{aligned}
\tag{16}
$$

1.2 Convergence of Fourier transform

The convergence can be investigated by analogy with convergence of Fourier series. In chapter 4, we discussed the convergence of the infinite sum. Here, we focus on the convergence of infinite integration. **Example. 5.4** was in the previous Mid-term two.

1.3 Property of Fourier transform

The property of Fourier transform is summarized in the table below. You can simply remem-

Table 1: Fourier Transform Properties

Property	Time Domain	Frequency Domain
Linearity	$af_1(t) + bf_2(t)$	$a\mathscr{F}_1(\omega) + b\mathscr{F}_2(\omega)$
Time-Domain Shifting	$f(t-t_0)$	$e^{-j\omega t_0}\mathscr{F}(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}f(t)$	$\mathscr{F}(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}\mathscr{F}\left(\frac{\omega}{a}\right)$
Conjugation	$f^*(t)$	$\mathscr{F}^*(-\omega)$
Duality	$\mathscr{F}(t)$	$2\pi f(-\omega)$
Time-Domain Convolution	$f_1(t) * f_2(2)$	$\mathscr{F}_1(\omega)\mathscr{F}_2(\omega)$
Frequency-Domain Convolution	$f_1(t)f_2(2)$	$\frac{1}{2\pi}\mathscr{F}_1(\omega) * \mathscr{F}_2(\omega)$
Time-Domain Differentiation	$\frac{df(t)}{dt} / \frac{d^k f(t)}{dt^k}$	$j\omega\mathscr{F}(\omega)/(j\omega)^k\mathscr{F}(\omega)$
Frequency-Domain Differentiation	$tf(t)/t^k f(t)$	$j\frac{d\mathscr{F}(\omega)}{d\omega}/j^k\frac{d^k\mathscr{F}(\omega)}{d\omega^k}$
Time-Domain Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}\mathscr{F}(\omega) + \mathscr{F}(0)\delta(\omega)$
ParsevalâĂŹs Relation	$\int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert\mathscr{F}(\omega)\rvert^2 d\omega$	

ber all of the properties. Now, we have two approaches to calculate the Fourier transform of function $f(t)$. The first one is calculating by definition which is given by Eq. (7). Or, you can apply the properties to constructing the Fourier transform.

Example 2 Use Fourier transform table and properties of the Fourier transform to find the Fourier transform of each of the signals below.

(a). $f(t) = \cos t \times u(t)$;
(b). $f(t) = 6[u(t) - u(t-3)]$;
(c). $f(t) = \frac{1}{t}$;
(d). $f(t) = (t+1)\mathrm{rect}(2t)$;

(e). $f(t) = e^{-j3t}\sin(5t-2)$;

Solution

(a). $f(t) = f_1(t)f_2(t)$, where $f_1(t) = \cos t$ and $f_2(t) = u(t)$. The Fourier transform of $f_1(t)$ and $f_2(t)$ are given by

$$\mathscr{F}_1(\omega) = \pi[\delta(\omega-1)+\delta(\omega+1)]; \qquad \mathscr{F}_2(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

Applying Frequency-domain convolution, the Fourier transform of $f(t)$ can be calculated as

$$\begin{aligned}
\mathscr{F}[f(t)] &= \frac{1}{2\pi}\mathscr{F}_1(\omega) * \mathscr{F}_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{F}_1(\sigma)\mathscr{F}_2(\omega-\sigma)d\sigma \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\Big[\delta(\sigma-1)+\delta(\sigma+1)\Big]\left[\pi\delta(\omega-\sigma)+\frac{1}{j(\omega-\sigma)}\right]d\sigma \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\left[\pi\delta(\sigma-1)\delta(\omega-\sigma)+\delta(\sigma-1)\frac{1}{j(\omega-\sigma)}+\pi\delta(\sigma+1)\delta(\omega-\sigma)+\delta(\sigma+1)\frac{1}{j(\omega-\sigma)}\right]d\sigma \\
&= \frac{1}{2}\left[\pi\delta(\omega-1)+\frac{1}{j(\omega-1)}+\pi\delta(\omega+1)+\frac{1}{j(\omega+1)}\right] \\
&= \frac{\pi}{2}\Big[\delta(\omega-1)+\delta(\omega+1)\Big]-\frac{j\omega}{\omega^2-1}
\end{aligned}$$

(b). $f(t) = 6[f_1(t)-f_2(t)]$, where $f_1(t) = u(t)$ and $f_2(t) = u(t-3)$. The Fourier transform of $f_1(t)$ is $\mathscr{F}_1(\omega) = \pi\delta(\omega)+\frac{1}{j\omega}$. Applying Time-domain shifting, the Fourier transform of $f_2(t)$ is given by

$$\mathscr{F}_2(\omega) = \mathscr{F}[f_1(t-3)] = e^{-j3\omega}\mathscr{F}_1(\omega) = e^{-j3\omega}\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$$

Applying Linearity, the Fourier transform of $f(t)$ can be calculated as

$$\begin{aligned}
\mathscr{F}[f(t)] &= 6\Big[\mathscr{F}_1(\omega)-\mathscr{F}_2(\omega)\Big] = 6\left[\pi\delta(\omega)+\frac{1}{j\omega}-e^{-j3\omega}\pi\delta(\omega)-\frac{e^{-j3\omega}}{j\omega}\right] = 6\left[\pi\delta(\omega)+\frac{1}{j\omega}-\pi\delta(\omega)-\frac{e^{-j3\omega}}{j\omega}\right] \\
&= \frac{6}{j\omega}\left(1-e^{-j3\omega}\right) = \frac{6}{j\omega}e^{-j\frac{3\omega}{2}}\left(e^{j\frac{3\omega}{2}}-e^{-j\frac{3\omega}{2}}\right) = \frac{6}{j\omega}e^{-j\frac{3\omega}{2}}\left[2j\sin\left(\frac{3\omega}{2}\right)\right] = \frac{12}{\omega}e^{-j\frac{3\omega}{2}}\sin\left(\frac{3\omega}{2}\right) \\
&= \frac{3\omega}{2}\frac{12}{\omega}e^{-j\frac{3\omega}{2}}\frac{\sin\left(\frac{3\omega}{2}\right)}{\frac{3\omega}{2}} = 18e^{-j\frac{3\omega}{2}}\text{sinc}\left(\frac{3\omega}{2}\right)
\end{aligned}$$

(c). By definition, the integration is damn hard to solve! However, we can use some trick. We know

$$\mathscr{F}[\text{sgn}(t)] = \frac{2}{j\omega}$$

We can use the duality property of the Fourier transform to deduce

$$\mathscr{F}\left[\frac{2}{jt}\right] = 2\pi\text{sgn}(-\omega) = -2\pi\text{sgn}(\omega)$$

If we denote $\hat{f}(t) = \frac{2}{jt}$, then, $\frac{1}{t} = \hat{f}(\frac{j}{2}t)$. With Time-domain scaling property, we have

$$\mathscr{F}\left[\frac{1}{t}\right] = \mathscr{F}\left[\hat{f}\left(\frac{j}{2}t\right)\right] = \frac{j}{2}\mathscr{F}\left[\hat{f}(t)\right] = \frac{j}{2}\mathscr{F}\left[\frac{2}{jt}\right] = \frac{j}{2}[-2\pi\mathrm{sgn}(\omega)] = -j\pi\mathrm{sgn}(\omega)$$

(d). $f(t) = tf_1(t) + f_1(t)$, where $f_1(t) = \mathrm{rect}(2t)$. From Fourier transform table, we notice

$$\mathscr{F}\left[\mathrm{rect}\left(\frac{t}{T}\right)\right] = T\mathrm{sinc}\left(\frac{\omega T}{2}\right)$$

Let $T = \frac{1}{2}$, We have

$$\mathscr{F}[f_1(t)] = \mathscr{F}_1(\omega) = \frac{1}{2}\mathrm{sinc}\left(\frac{\omega}{4}\right)$$

Then, applying Frequency-domain differentiation property, we arrive at

$$\mathscr{F}[tf_1(t)] = j\frac{d}{d\omega}\mathscr{F}_1(\omega) = \frac{j}{2}\frac{d}{d\omega}\left[\frac{\sin\left(\frac{\omega}{4}\right)}{\frac{\omega}{4}}\right] = \frac{j}{2\omega}\cos\left(\frac{\omega}{4}\right) - \frac{2j}{\omega^2}\sin\left(\frac{\omega}{4}\right)$$

Applying the Linearity, we have

$$\begin{aligned}\mathscr{F}(\omega) &= \mathscr{F}[f_1(t)] + \mathscr{F}[tf_1(t)] \\ &= \frac{1}{2}\mathrm{sinc}\left(\frac{\omega}{4}\right) + \frac{j}{2\omega}\cos\left(\frac{\omega}{4}\right) - \frac{2j}{\omega^2}\sin\left(\frac{\omega}{4}\right) \\ &= \frac{1}{2}\left(1 - \frac{j}{\omega}\right)\mathrm{sinc}\left(\frac{\omega}{4}\right) + \frac{j}{2\omega}\cos\left(\frac{\omega}{4}\right)\end{aligned}$$

(e). $f(t) = e^{-j3t}f_1(t)$, where $f_1(t) = \sin(5t-2)$. Applying Time-domain shifting and scaling property.

$$\mathscr{F}[\sin(t)] = \frac{\pi}{j}\Big[\delta(\omega-1) - \delta(\omega+1)\Big]$$

$$\mathscr{F}[\sin(t-2)] = \frac{\pi e^{-j2\omega}}{j}\Big[\delta(\omega-1) - \delta(\omega+1)\Big]$$

$$\mathscr{F}_1(\omega) = \mathscr{F}[\sin(5t-2)] = \frac{\pi e^{-\frac{j2\omega}{5}}}{5j}\left[\delta\left(\frac{\omega}{5}-1\right) - \delta\left(\frac{\omega}{5}+1\right)\right]$$

Applying Frequency-domain shifting, we obtain

$$\begin{aligned}\mathscr{F}(\omega) = \mathscr{F}_1(\omega+3)\mathscr{F}[\sin(5t-2)] &= \frac{\pi}{5j}e^{-\frac{j2(\omega+3)}{5}}\left[\delta\left(\frac{\omega+3}{5}-1\right) - \delta\left(\frac{\omega+3}{5}+1\right)\right] \\ &= -\frac{j\pi}{5}e^{-j2}\delta\left(\frac{\omega-2}{5}\right) + \frac{j\pi}{5}e^{j2}\delta\left(\frac{\omega+8}{5}\right)\end{aligned}$$

The last equality follows equivalent property of δ-function.
Example 3 Use Fourier transform of $x(t)$ to represent Fourier transform of $y(t)$.
(a). $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;

(b). $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(c). $f(t) = \frac{d}{dt}[x(t) * x(t)]$;
Solution
(a). $y(t) = f_1(2t)$ where $f_1(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$$\mathscr{F}_1(\omega) = \mathscr{F}[f_1(t)] = \frac{1}{j\omega}\mathscr{X}(\omega) + \pi\mathscr{X}(0)\delta(\omega)$$

$$\mathscr{Y}(\omega) = \mathscr{F}[f_1(2t)] = \frac{1}{2}\mathscr{F}_1\Big(\frac{\omega}{2}\Big) = \frac{1}{2}\left[\frac{1}{j(\omega/2)}\mathscr{X}\Big(\frac{\omega}{2}\Big) + \pi\mathscr{X}(0)\delta\Big(\frac{\omega}{2}\Big)\right] = \frac{1}{j\omega}\mathscr{X}\Big(\frac{\omega}{2}\Big) + \frac{\pi}{2}\mathscr{X}(0)\delta\Big(\frac{\omega}{2}\Big)$$

(b). $y(t) = \int_{-\infty}^{t} f_1(\tau)d\tau$, where $f_1(t) = x^2(t)$. We have

$$\mathscr{Y}(\omega) = \frac{1}{j\omega}\mathscr{F}_1(\omega) + \pi\mathscr{F}_1(0)\delta(\omega), \text{ where } \mathscr{F}_1(\omega) = \frac{1}{2\pi}\mathscr{X}(\omega) * \mathscr{X}(\omega)$$

$$\begin{aligned}\text{As such } \mathscr{Y}(\omega) &= \frac{1}{j\omega}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{X}(\sigma)\mathscr{X}(\omega-\sigma)d\sigma\right] + \pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{X}(\sigma)\mathscr{X}(-\sigma)d\sigma\right]\delta(\omega)\\ &= \frac{1}{j2\pi\omega}\int_{-\infty}^{\infty}\mathscr{X}(\sigma)\mathscr{X}(\omega-\sigma)d\sigma + \frac{1}{2}\delta(\omega)\int_{-\infty}^{\infty}\mathscr{X}(\sigma)\mathscr{X}(-\sigma)d\sigma\end{aligned}$$

(c). $y(t) = \frac{d}{dt}f_1(t)$, where $f_1(t) = x(t) * x(t)$. Then,

$$\begin{aligned}\mathscr{Y}(\omega) &= j\omega\mathscr{F}_1(\omega), \text{where } \mathscr{F}_1(\omega) = \mathscr{X}^2(\omega)\\ \text{As such } \mathscr{Y}(\omega) &= j\omega\mathscr{X}^2(\omega)\end{aligned}$$

1.4 Fourier transform of periodic function

Suppose that $x(t)$ is a periodic function with period T and $x_T(t)$ is the truncated version of $x(t)$ with time duration T. Fourier transform of periodic function $x(t)$ can be calculated by

$$\mathscr{F}[x(t)] = \sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\mathscr{F}_{x_T}\Big(\frac{2\pi k}{T}\Big)\delta\Big(\omega - \frac{2\pi k}{T}\Big) \tag{17}$$

where $\mathscr{F}_{x_T}(\cdot)$ is the Fourier transform of $x_T(t)$
Example 4 Find the Fourier transform of each of the periodic signals shown below.
Solution The period of $x(t)$ is $T = 3$. Next we need to define the truncated version $x_T(t)$.

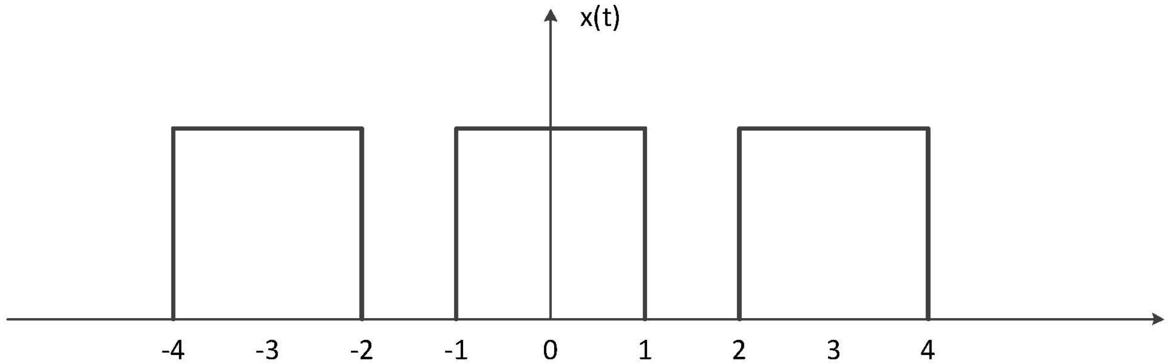
$$x_T(t) = \text{rect}\Big(\frac{t}{2}\Big)$$

Then, the Fourier transform of $x_T(t)$ can be calculated as

$$\mathscr{F}_{x_T}(\omega) = 2\text{sinc}(\omega)$$

From all these, the Fourier transform of $x(t)$ is given by

$$\mathscr{F}[x(t)] = \sum_{k=-\infty}^{\infty}\frac{2\pi}{3}2\text{sinc}\Big(\frac{2\pi k}{3}\Big)\delta\Big(\omega - \frac{2\pi k}{3}\Big)$$



1.5 Spectrum

$\mathscr{F}(\omega)$ is the Fourier transform of $f(t)$. $|\mathscr{F}(\omega)|$ is so-called magnitude spectrum and $\arg\Big[\mathscr{F}(\omega)\Big]$ is phase spectrum.

Example 5 Compute the frequency spectrum of the signal specified below. Also find and plot the corresponding magnitude and phase spectra.

(a). $f(t) = e^{-at}u(t)$, where a is a positive real constant.

Solution The Fourier transform of $f(t)$ can be calculated as

$$\begin{aligned}
\mathscr{F}(\omega) = \mathscr{F}[f(t)] &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{1}{a+j\omega}\int_{0}^{\infty} e^{-(a+j\omega)t}d[-(a+j\omega)t] \\
&= -\frac{1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]\Bigg|_{0}^{\infty} = -\frac{1}{a+j\omega}(0-1) \\
&= \frac{1}{a+j\omega}
\end{aligned}$$

Then we need to further simplify $\mathscr{F}(\omega)$ to isolate the real part and imagine part.

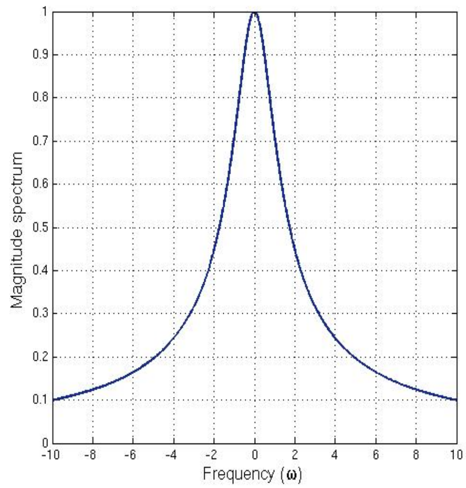
$$\mathscr{F}(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{(a+j\omega)(a-j\omega)} = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - \frac{\omega}{a^2+\omega^2}j$$

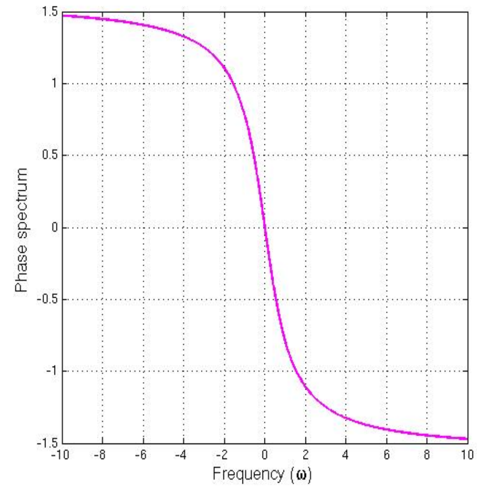
The magnitude spectrum can be calculated as

$$|\mathscr{F}(\omega)| = \sqrt{\left(\frac{a}{a^2+\omega^2}\right)^2 + \left(-\frac{\omega}{a^2+\omega^2}\right)^2} = \sqrt{\frac{1}{a^2+\omega^2}}$$

The phase spectrum is given by

$$\arg\Big[\mathscr{F}(\omega)\Big] = \arg\left[\frac{1}{a+j\omega}\right] = \arg 1 - \arg(a+j\omega) = -\arg(a+j\omega) = -\arctan\left(\frac{\omega}{a}\right)$$





2 APPLICATION

2.1 Fourier transform and LTI system

The input-output relation of LTI system is characterized by continuous-time convolution given by

$$y(t) = x(t) * h(t)$$

where $h(t)$ is the impulse response of the system. From Fourier transform, if we take Fourier transform on both sides, anything can be easier. The painful convolution can be transformed into multiplication by applying Time domain convolution property. In other words, the input-output relation can also be characterized by the multiplication defined below

$$\mathscr{Y}(\omega) = \mathscr{X}(\omega)\mathscr{H}(\omega)$$

where $\mathscr{Y}(\omega)$, $\mathscr{X}(\omega)$ and $\mathscr{H}(\omega)$ are the Fourier transform of output, input and impulse response, respectively. And you can see $\mathscr{H}(\cdot)$ is a function of frequency. Therefore, it is named frequency response. In general, $\mathscr{H}(\omega)$ is a complex function.

Example 6 Suppose that we have the LTI systems defined by the differential/integral equations given below, where $x(t)$ and $y(t)$ denote the system input and output, respectively. Find the frequency response of each of these systems.

(a). $\frac{d}{dt}y(t) + 2y(t) + \int_{-\infty}^{t} 3y(\tau)d\tau + 5\frac{d}{dt}x(t) - x(t) = 0$

Solution Taking derivative on both sides, the problem is simplified since $\frac{d}{dt}\Big(\int_{-\infty}^{t} 3y(\tau)d\tau\Big) = 3y(t)$. I want to talk more on this calculus property. In general,

$$\frac{d}{dt}\Big(\int_{-\infty}^{f(t)} g(\tau)d\tau\Big) = g[f(t)]\left(\frac{d}{dt}f(t)\right)$$

For our problem, if we take derivative on both sides

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 3y(t) + 5\frac{d^2}{dt^2}x(t) - \frac{d}{dt}x(t) = 0$$

Then, take Fourier transform on both sides

$$(j\omega)^2\mathscr{Y}(\omega) + 2(j\omega)\mathscr{Y}(\omega) + 3\mathscr{Y}(\omega) + 5(j\omega)^2\mathscr{X}(\omega) - (j\omega)\mathscr{X}(\omega) = 0$$

Isolate unknows, we have

$$\Big[-\omega^2 + 2j\omega + 3\Big]\mathscr{Y}(\omega) = \Big[5\omega^2 + j\omega\Big]\mathscr{X}(\omega)$$

By cross-multiplication, the frequency response is

$$\mathscr{H}(\omega) = \frac{\mathscr{Y}(\omega)}{\mathscr{X}(\omega)} = \frac{5\omega^2 + j\omega}{-\omega^2 + 2j\omega + 3}$$

You should also know how to solve the differential equation from the frequency response.

2.2 Filtering

Low pass filter allows frequency component less than threshold ω_L pass through (i.e. times 1). Frequency greater than ω_L will be eliminated (i.e. multiply by 0). Therefore, the frequency response of Low-pass filter has to have the form

$$\mathscr{H}_L(\omega) = \text{rect}\Big(\frac{\omega}{2\omega_L}\Big) = \begin{cases} 1, & \text{for } |\omega| \leq \omega_L \\ 0, & \text{otherwise} \end{cases}$$

Accordingly, the impulse response of a Low-pass filter is solved by calculating the inverse Fourier transform.

$$h_L(t) = \frac{\omega_L}{\pi}\text{sinc}(\omega_L t)$$

High pass filter is defined on the contrary of Low pass filter. As such, the frequency response and impulse response of a High pass filter are

$$\mathscr{H}_H(\omega) = 1 - \text{rect}\Big(\frac{\omega}{2\omega_H}\Big) = \begin{cases} 1, & \text{for } |\omega| \geq \omega_H \\ 0, & \text{otherwise} \end{cases} \qquad h_H(t) = \delta(t) - \frac{\omega_H}{\pi}\text{sinc}(\omega_H t)$$

Band pass filter allows frequency component within a particular frequency band to pass. Hence,

$$\mathscr{H}_B(\omega) = \text{rect}\Big(\frac{\omega - \omega_a}{2\omega_b}\Big) + \text{rect}\Big(\frac{\omega + \omega_a}{2\omega_b}\Big) = \begin{cases} 1, & \text{for } \omega_a - \omega_b \leq |\omega| \leq \omega_a + \omega_b \\ 0, & \text{otherwise} \end{cases}$$

$$h_B(t) = \frac{2\omega_b}{\pi}\Big[\text{sinc}\Big(\omega_b t\Big)\Big]\cos(\omega_a t)$$

ω_a is central frequency of the filter and ω_b is the cut-off frequency.

Example 7 Suppose that we have a LTI system with input $x(t)$ and output $y(t)$, and impulse response $h(t)$, where

$$h(t) = \delta(t) - 300\text{sinc}(300\pi t)$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t)$

$$x(t) = \frac{1}{2} + \frac{3}{4}\cos(200\pi t) + \frac{1}{2}\cos(400\pi t) - \frac{1}{4}\cos(600\pi t).$$

Solution Let us check what effect the system performs onto the input from a frequency perspective.

$$\mathscr{F}[h(t)] = \mathscr{F}[\delta(t)] - \mathscr{F}\left[300\text{sinc}(300\pi t)\right] \tag{18}$$

$$= 1 - \text{rect}\left(\frac{\omega}{600\pi}\right) = \begin{cases} 1 & \text{for } |\omega| \geq 300\pi \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Obviously, the system is a high-pass filter with cut-off frequency 300π. Next, let us compute the Fourier transform of $x(t)$.

$$\mathscr{F}[x(t)] = \pi\delta(\omega) + \frac{3}{4}[\delta(\omega-200\pi)+\delta(\omega+200\pi)] + \frac{1}{2}[\delta(\omega-400\pi)+\delta(\omega+400\pi)] - \frac{1}{4}[\delta(\omega-600\pi)+\delta(\omega+600\pi)]$$

Then

$$\begin{aligned}\mathscr{Y}(\omega) &= \mathscr{X}(\omega)\mathscr{H}(\omega) \\ &= \frac{1}{2}[\delta(\omega-400\pi)+\delta(\omega+400\pi)] - \frac{1}{4}[\delta(\omega-600\pi)+\delta(\omega+600\pi)]\end{aligned}$$

The output of this high-pass filter to this input $x(t)$ is given by

$$y(t) = \mathscr{F}^{-1}\Big[\mathscr{Y}(\omega)\Big] = \frac{1}{2}\cos(400\pi t) - \frac{1}{4}\cos(600\pi t) \tag{20}$$

2.3 Modulation

Voice signal, for example, typically have information in the range of 0 to 3.4 kHz. It is impractical to transmit such low-frequency signal mainly because low-frequency results in large wave-length. Antenna size is proportional to the wave-length of transmitted signal. We need to change the frequency range associated with a signal before transmission. This is so-called Modulation. Basically, we apply convolution property of δ-function or equivalently, frequency domain shifting property of Fourier transform.

Example 8 Consider the system shown below in with input $x(t)$ and output $z(t)$, where

$$\mathscr{G}(\omega) = \begin{cases} 2 \text{ for } |\omega| \leq 100\pi \\ 0 \text{ otherwise} \end{cases} \tag{21}$$

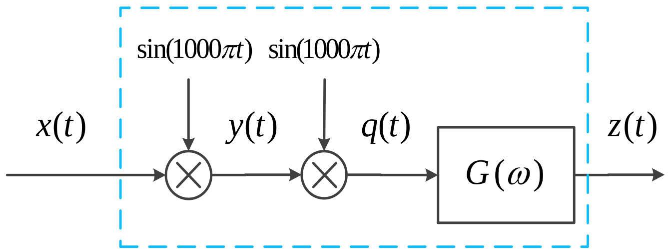
Let $\mathscr{X}(\omega)$, $\mathscr{Z}(\omega)$, $\mathscr{Y}(\omega)$, and $\mathscr{Q}(\omega)$ denote the Fourier transform of $x(t)$, $z(t)$, $y(t)$ and $q(t)$, respectively. For $\mathscr{X}(\omega)$ given below, sketch $\mathscr{Z}(\omega)$, $\mathscr{Y}(\omega)$, and $\mathscr{Q}(\omega)$.

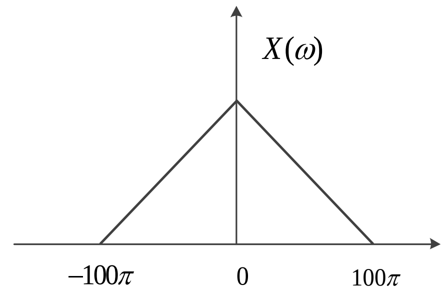
Solution From the system blocking diagram, we have

$$y(t) = x(t)\sin(1000\pi t)$$

Take Fourier transform on both sides

$$\mathscr{Y}(\omega) = \mathscr{F}[x(t)\sin(1000\pi t)]$$





Time domain multiplication results in frequency domain convolution.

$$\begin{aligned}\mathscr{Y}(\omega) &= \frac{1}{2\pi}\mathscr{F}[x(t)] * \mathscr{F}[\sin(1000\pi t)] = \frac{1}{2\pi}\mathscr{X}(\omega) * \frac{\pi}{j}[\delta(\omega - 1000\pi) - \delta(\omega + 1000\pi)] \\ &= \frac{1}{2j}\mathscr{X}(\omega) * [\delta(\omega - 1000\pi) - \delta(\omega + 1000\pi)] \\ &= \frac{1}{2j}[\mathscr{X}(\omega) * \delta(\omega - 1000\pi) - \mathscr{X}(\omega) * \delta(\omega + 1000\pi)]\end{aligned}$$

Applying convolution property of δ function, we arrive at

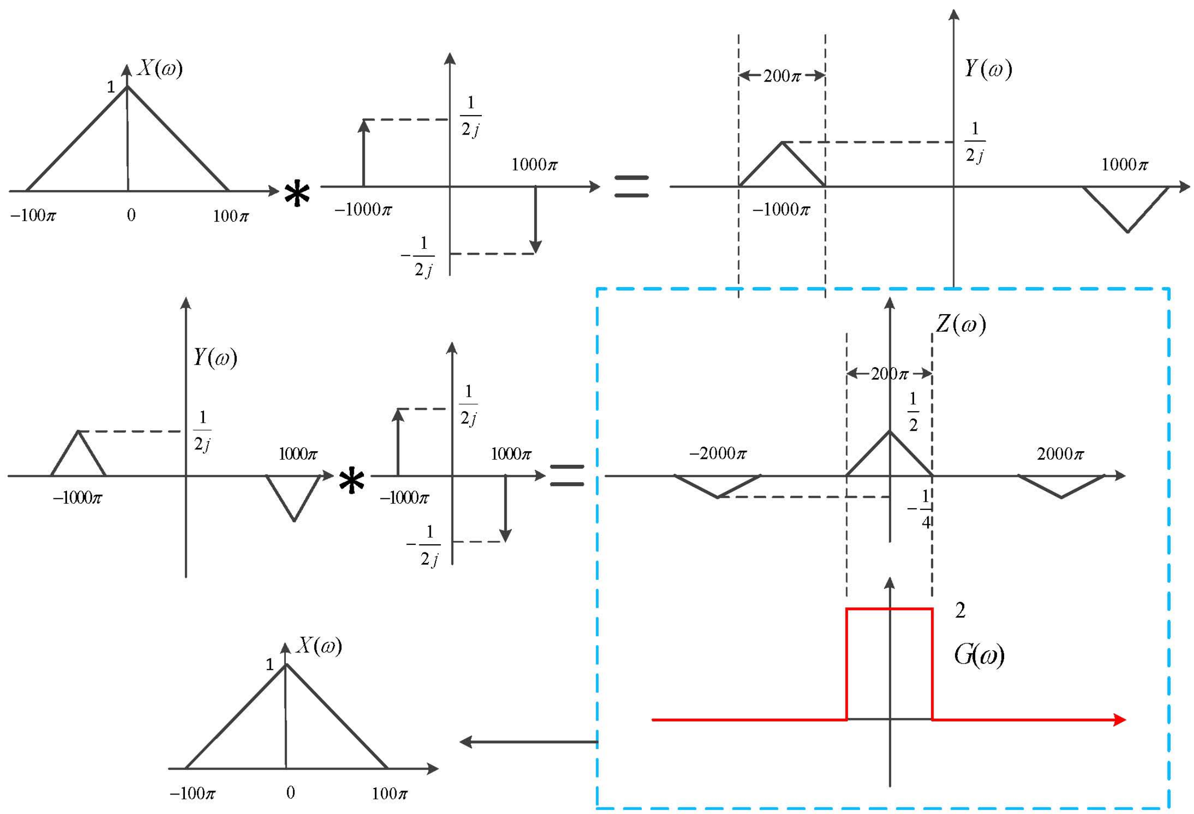
$$\mathscr{Y}(\omega) = \frac{1}{2j}[\mathscr{X}(\omega - 1000\pi) - \mathscr{X}(\omega + 1000\pi)]$$

Similarly, $q(t) = y(t)\sin(1000\pi t)$, We can immediately write

$$\begin{aligned}\mathscr{Q}(\omega) &= \frac{1}{2j}[\mathscr{Y}(\omega) * \delta(\omega - 1000\pi) - \mathscr{Y}(\omega) * \delta(\omega + 1000\pi)] \\ &= \frac{1}{2j}\Bigg\{\frac{1}{2j}[\mathscr{X}(\omega - 1000\pi) - \mathscr{X}(\omega + 1000\pi)] * \delta(\omega - 1000\pi) \\ &\qquad - \frac{1}{2j}[\mathscr{X}(\omega - 1000\pi) - \mathscr{X}(\omega + 1000\pi)] * \delta(\omega + 1000\pi)\Bigg\} \\ &= -\frac{1}{4}\Bigg[\mathscr{X}(\omega - 1000\pi) * \delta(\omega - 1000\pi) - \mathscr{X}(\omega + 1000\pi) * \delta(\omega - 1000\pi) \\ &\qquad - \mathscr{X}(\omega - 1000\pi) * \delta(\omega + 1000\pi) + \mathscr{X}(\omega + 1000\pi) * \delta(\omega + 1000\pi) \\ &= -\frac{1}{4}\Bigg[\mathscr{X}(\omega - 2000\pi) - 2\mathscr{X}(\omega) + \mathscr{X}(\omega + 1000\pi)\Bigg] \\ &= \frac{1}{2}\mathscr{X}(\omega) - \frac{1}{4}\mathscr{X}(\omega - 2000\pi) - \frac{1}{4}\mathscr{X}(\omega + 2000\pi)\end{aligned}$$

Finally,

$$\mathscr{Z}(\omega) = \mathscr{Q}(\omega)\mathscr{G}(\omega) = 2\left(\frac{1}{2}\mathscr{X}(\omega)\right) = \mathscr{X}(\omega)$$



After so many operations, you got your $x(t)$ back. Ah...what can I say....

2.4 Sampling and Interpolation

Before we move forward, I need to remind you the sampling property and shifting property of $\delta(t)$.

$$f(t)\delta(t-a) = f(a)\delta(t-a), \quad f(t) * \delta(t-a) = f(t-a) \tag{22}$$

Sampling is to discretizing the signal $f(t)$ to facilitate computer manipulation. For a computer, anything continuous is not accepted since continuous signal requires infinitely many memory space.

Suppose that you have a continuous function also known as analogue signal, you just sample it with sampling period T_s. Mathematically, we have

$$\begin{aligned} f[k] &= f(t)\sum_{k=-\infty}^{\infty}\delta(t-kT_s) = \sum_{k=-\infty}^{\infty} f(t)\delta(t-kT_s) \quad \text{applying sampling property} \\ &= \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t-kT_s) \end{aligned} \tag{23}$$

To characterize the frequency domain behaviour I need to apply a new notation Ш. A

Ш-function is the impulse train with spacing one. $Ш_{T_s}$ is the impulse train with spacing T_s.

$$Ш = \sum_{k=-\infty}^{\infty} \delta(t-k), \quad Ш_{T_s} = \sum_{k=-\infty}^{\infty} \delta(t-kT_s) \tag{24}$$

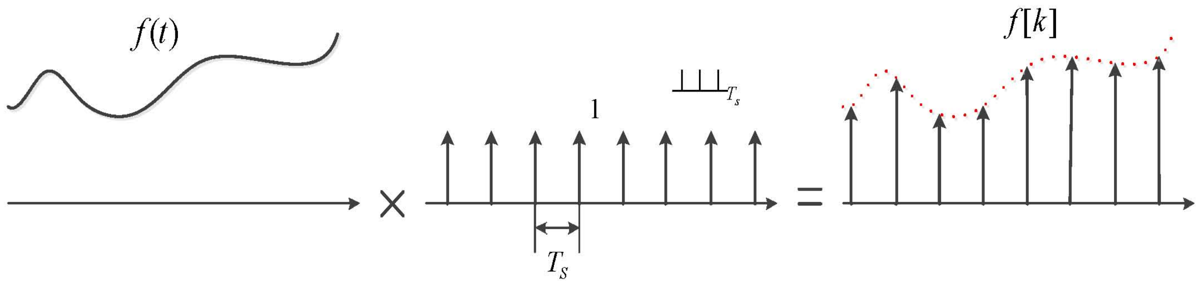
The Fourier transform of Ш and $Ш_{T_s}$ are given by

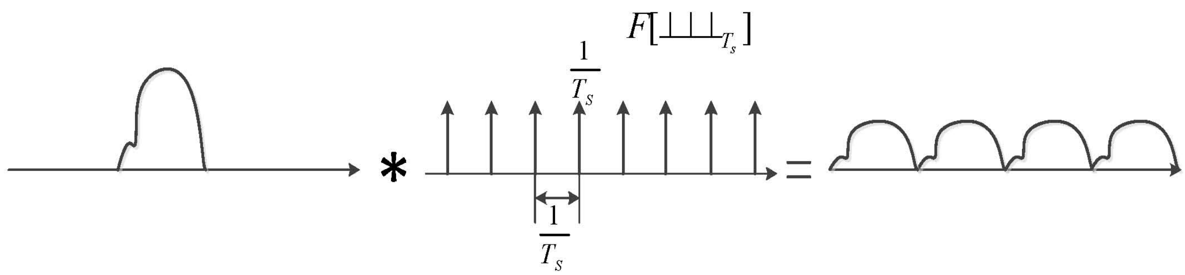
$$\mathscr{F}[Ш] = Ш = \sum_{k=-\infty}^{\infty} \delta(\omega-k), \quad \mathscr{F}[Ш_{T_s}] = \frac{1}{T_s} Ш_{\frac{1}{T_s}} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{k}{T_s}\right) \tag{25}$$

Note that Ш is a function of time, whereas, Fourier transform of Ш is a function of frequency ω. As you can see, the Fourier transform of Ш is itself. And $\mathscr{F}[Ш_{T_s}]$ is a modified Ш-function with magnitude $\frac{1}{T_s}$ and spacing $\frac{1}{T_s}$. Now, we are ready to investigate the spectrum of sampled signal $f[k]$.

$$\begin{aligned}
\mathscr{F}\left[f[k]\right] &= \mathscr{F}\left[f(t) \times Ш_{T_s}\right] = \frac{1}{2\pi}\mathscr{F}\left[f(t)\right] * \mathscr{F}[Ш_{T_s}] = \frac{1}{2\pi}\mathscr{F}(\omega) * \frac{1}{T_s} Ш_{\frac{1}{T_s}} \\
&= \frac{1}{2\pi}\left[\mathscr{F}(\omega) * \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{k}{T_s}\right)\right] = \frac{1}{2\pi}\left[\sum_{k=-\infty}^{\infty} \mathscr{F}(\omega) * \delta\left(\omega - \frac{k}{T_s}\right)\right] \text{ applying shifting property} \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi T_s}\mathscr{F}\left(\omega - \frac{k}{T_s}\right) = \sum_{k=-\infty}^{\infty} \frac{\omega_s}{2\pi}\mathscr{F}\left(\omega - k\omega_s\right)
\end{aligned} \tag{26}$$

The figure below characterize the effect of sampling in both time domain and frequency domain.

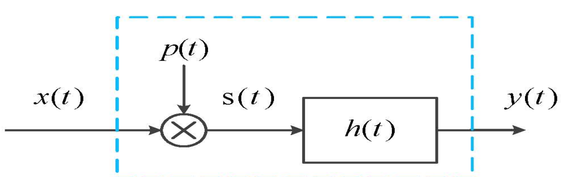


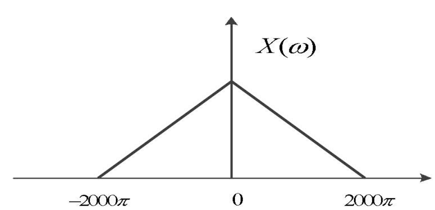


Example 9 Consider the system shown below in Figure A with input $x(t)$ and output $y(t)$. Let $\mathscr{X}(\omega)$, $\mathscr{P}(\omega)$, $\mathscr{S}(\omega)$, $\mathscr{H}(\omega)$, and $\mathscr{Y}(\omega)$ denote the Fourier transforms of $x(t)$, $p(t)$, $s(t)$, $h(t)$, and $y(t)$, respectively. Suppose that

$$p(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - \frac{k}{1000}\right), \quad \mathscr{H}(\omega) = \frac{1}{1000}\text{rect}\left(\frac{\omega}{2000\pi}\right) \tag{27}$$

(a) Derive an expression for $\mathscr{S}(\omega)$ in terms of $\mathscr{X}(\omega)$. Derive an expression for $\mathscr{Y}(\omega)$ in terms of $\mathscr{S}(\omega)$ and $\mathscr{H}(\omega)$.
(b) Suppose that $\mathscr{X}(\omega)$ is as shown in Figure C. Using the results of part (a), plot $\mathscr{S}(\omega)$ and $\mathscr{Y}(\omega)$. Indicate the relationship (if any) between the input $x(t)$ and output $y(t)$ of the system.





Solution

$$\begin{aligned}
\mathscr{F}[s(t)] &= \mathscr{F}[x(t)p(t)] = \frac{1}{2\pi}\mathscr{X}(s) * \mathscr{P}(s) = \frac{1}{2\pi}\mathscr{X}(s) * \sum_{k=-\infty}^{\infty} 1000\delta(s - 1000k) \\
&= \frac{1000}{2\pi}\sum_{k=-\infty}^{\infty} \mathscr{X}(s) * \delta(s - 1000k) = \frac{1000}{2\pi}\sum_{k=-\infty}^{\infty} \mathscr{X}(s - 1000k) \\
&= 1000\sum_{k=-\infty}^{\infty} \mathscr{X}(\omega - 2000\pi k)
\end{aligned} \tag{28}$$

$h(t)$ represents a low-pass filter. If $\mathscr{Y}(\omega) \neq \mathscr{X}(\omega)$, Nyquist sampling theory is not satisfied. Sampling distorts the signal. I will leave the rest to you.