CH.6 Laplace Transform

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1 Where does the Laplace transform come from?

How to solve this mistery that where the Laplace transform come from? The starting point is thinking about power series. The power series looks like

$$\mathcal{F}(x) = \sum_{n=0}^{+\infty} f_n x^n = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

This series is the summation of x's with increasing positive integer powers. The coefficients here are assigned numbers at $n = 0, 1, 2, \cdots$. If we write it using a computer notation.

$$\mathcal{F}(x) = \sum_{n=0}^{+\infty} f[n] x^n = f[0] + f[1]x + f[2]x^2 + f[3]x^3 + \cdots$$

Next, we would like to investigate the relation of $\mathcal{F}(x)$ and the coefficients. Specifically, for different coefficients $\overrightarrow{f[n]} = [f[0], f[1], f[2], \cdots]$, how to calculate $\mathcal{F}(x)$?

For example, if $\overrightarrow{f[n]}$ is a unit vector, (e.g. $\overrightarrow{f[n]} = [1, 1, 1, \cdots]$)

$$\mathcal{F}(x) = \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$$

except for this is the wrong answer! It is not true for every value of x. It is only true for the series to converge. This answer is true, if and only if its suport set is restricted to $|x| < 1$. As such

$$\mathcal{F}(x) = \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1$$

If $f[n] = \frac{1}{n!}$ results in $\overrightarrow{f[n]} = \left[1, 1, \frac{1}{2!}, \frac{1}{3}!, \cdots\right]$, the power series looks like

$$\mathcal{F}(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n = e^x$$

This identity here is valid for any arbitrary value x

I am not satisfied with this discrete representation. I want more! I want a continuous analogy. The discrete variable $n = 0, 1, 2, \cdots$ are replaced by a continuous variable t. And, in the continuous case, summation is approximated by integral.

$$\mathcal{F}(x) = \int_0^{+\infty} f(t)x^t dt$$

In general, we do not not like the variable x appearing at the base ,making the integral very difficult to compute. How to fix this?

$$\mathcal{F}(x) = \int_0^{+\infty} f(t)x^t dt = \int_0^{+\infty} f(t) \left(e^{\ln x}\right)^t dt$$

One problem solved, but here comes another two. Given any value x, can I calculate the integral? For any $|x| \geq 1$, since t goes to infinity, the integral is quite unlikely to converge. Then, we have to restrict that $|x| < 1$.

Another problem is if $-1 < x < 0$, since t can value from 0 to 1, a negative number with fractional power may give you a complex number, which is very undesired. For example

$$-1^{\frac{1}{2}} = \sqrt{-1} = j$$

Hence, we have to restrict that $0 < x < 1$. Look at the blue expression again. If $x \in (0,\ 1)$, we have $\ln x \in (-\infty,\ 0)$. The integral is taken over all the positive numbers, but the power is negative??? How confusing! Then, we apply variable replacement

$$s = -\ln x \in (0,\ +\infty)$$

As such, everything is all set. Then, what happened to the integral?

$$\mathcal{F}(s) = \int_0^{+\infty} f(t)e^{-st} dt$$

This is the unilateral Laplace transform of $f(t)$.

2 Calculate Laplace Transform

Laplace transform is defined over the whole complex plane (e.g. the variable in Laplace transform $s = \sigma + j\omega$ is a complex number). Fourier transform is a special case, when $\sigma = 0$ if and only if the ROC of Laplace transform contains the imagine axis. In your text book, the author stated '*Laplace transform of a signal may exist when the Fourier transform of the same signal does not*'. This can be explained by the ROC of Laplace transform does not include the imagine axis.

As such, we can calculate Laplace transform following similar steps in carrying out the Fourier transform. In addition, we need to discuss the region of convergence of corresponding Laplace transform.

Example.1 Calculate the Laplace transform of the following signals

(a). $x(t) = \cos(\omega_0 t)u(t)$

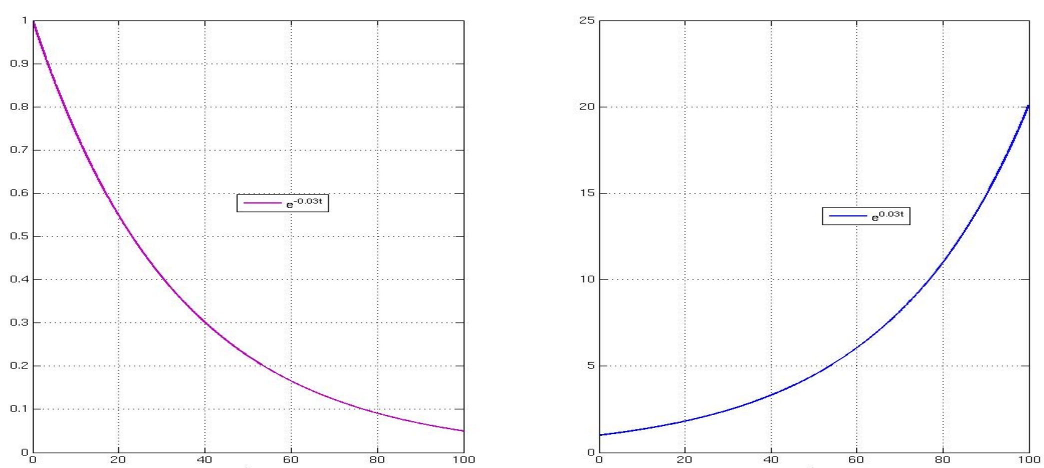
(b). $x(t) = \int_{-\infty}^{t} e^{-2\tau} u(\tau) d\tau$
(c). $x(t) = -e^{at} u(-t+b)$, where a is a positive real constant and b is a real constant
Solution
(a). By definition, the Laplace transform of $x(t)$ is equivalent to solve the following integral

$$
\begin{aligned}
\mathscr{L}[\cos(\omega_0 t)u(t)] &= \int_{-\infty}^{\infty} \cos(\omega_0 t) u(t) e^{-st} dt = \int_{0}^{\infty} \cos(\omega_0 t) e^{-st} dt \\
&= \frac{1}{2} \int_{0}^{\infty} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-st} dt \qquad s = \sigma + j\omega \\
&= \frac{1}{2} \left[\frac{1}{j\omega_0 - \sigma - j\omega} e^{j(\omega_0 - \omega)t} e^{-\sigma t} \Big|_{0}^{+\infty} + \frac{1}{-j\omega_0 - \sigma - j\omega} e^{j(-\omega_0 - \omega)t} e^{-\sigma t} \Big|_{0}^{+\infty} \right]
\end{aligned}
$$

I will use the first part as an illustration to discuss the ROC when you compute Laplace transform by definition.

$$
\frac{1}{j\omega_0 - \sigma - j\omega} e^{j(\omega_0 - \omega)t} e^{-\sigma t} \Big|_{0}^{+\infty} = \frac{1}{j\omega_0 - \sigma - j\omega} e^{j(\omega_0 - \omega)+\infty} e^{-\sigma+\infty} - 1
$$

$e^{j(\omega_0-\omega)t}$ is a complex number with magnitude one. As t grows bigger, this complex number runs along unit ring anti-clock-wise. I do not care the exact value of it when t approaches infinity. I just know, the limit, $e^{j(\omega_0-\omega)\infty}$, is still a complex number on this unit ring. And it is not infinity.
$e^{-\sigma t}$ is another story. The sketch or the limit at $t = \infty$ depends on σ as illustrated in the figure bellow If $\sigma < 0$, $-\sigma > 0$. As t goes to infinity, the limit is also infinity, which means



the integral does not converge. The limit exists only when $\sigma > 0$, which is equivalent to $\mathrm{Re}[s] > 0$. Correspondingly, the limit $e^{-\sigma\infty}$ is 0 when $\mathrm{Re}[s] > 0$. Restrict the real part of s, the integral can be carried out as

$$\begin{aligned}\mathscr{L}[\cos(\omega_0 t)u(t)] &= \frac{1}{2}\left[\frac{-1}{j\omega_0 - \sigma - j\omega} + \frac{-1}{-j\omega_0 - \sigma - j\omega}\right] \\ &= \frac{1}{2}\left[\frac{1}{j\omega_0 + s} - \frac{1}{j\omega_0 - s}\right] \\ &= \frac{s}{s^2 + \omega_0^2}, \quad \mathrm{Re}[s] > 0\end{aligned}$$

(b). $x(t) = \int_{-\infty}^{t} f_1(\tau)d\tau$, where $f_1(t) = e^{-2t} f_2(t)$ and $f_2(t) = u(t)$.
Apply Time domain integration, we have

$$\mathscr{L}_x(s) = \frac{1}{s}\mathscr{L}_{f_1}(s), \quad R_x = R_{f_1} \cap \mathrm{Re}[s] > 0$$

Apply Laplace domain shifting, we obtain

$$\mathscr{L}_{f_1}(s) = \mathscr{L}_{f_2}(s+2), \quad R_{f_1} = R_{f_2} - 2$$

Laplace transform of $f_2(t)$ is specified as

$$\mathscr{L}_{f_2}(s) = \mathscr{L}[u(t)] = \frac{1}{s}, \quad R_{f_2} = \mathrm{Re}[s] > 0$$

Now, we are ready to characterize the Laplace transform of $x(t)$

$$\mathscr{L}_x(s) = \frac{1}{s}\mathscr{L}_{f_1}(s) = \frac{1}{s}\mathscr{L}_{f_2}(s+2) = \frac{1}{s(s+2)}$$

The ROC can be determined by

$$R_x = R_{f_1} \cap \mathrm{Re}[s] > 0 = [R_{f_2} - 2] \cap \mathrm{Re}[s] > 0 = \mathrm{Re}[s] > -2 \cap \mathrm{Re}[s] > 0 = \mathrm{Re}[s] > 0$$

The final answer is

$$\mathscr{L}_x(s) = \frac{1}{s(s+2)}, \quad \mathrm{Re}[s] > 0$$

(c). $x(t) = -e^{at} f_1(t)$, where $f_1(t) = f_2(-t)$, $f_2(t) = f_3(t+b)$ and $f_3(t) = u(t)$
Apply Laplace domain shifting and linearity, we have

$$\mathscr{L}_x = -\mathscr{L}_{f_1}(s-a), \quad R_x = R_{f_1} + a$$

Apply Time domain scaling, we have

$$\mathscr{L}_{f_1}(s) = \mathscr{L}_{f_2}(-s), \quad R_{f_1} = -R_{f_2}$$

Apply Time domain shifting, we have

$$\mathscr{L}_{f_2}(s) = e^{bs}\mathscr{L}_{f_3}(s), \quad R_{f_2} = R_{f_3}$$

The Laplace transform of $f_3(t)$ can be calculated as

$$\mathscr{L}_{f_3}(s) = \frac{1}{s}, \quad R_{f_3} = \text{Re}[s] > 0$$

By substituting each equation to the previous one, we obtain

$$\mathscr{L}_{f_2}(s) = e^{bs}\frac{1}{s}, \quad \mathscr{L}_{f_1} = -e^{-bs}\frac{1}{s}, \quad \mathscr{L}_x = e^{-b(s-a)}\frac{1}{s-a}$$

The ROC of $x(t)$ is determined as

$$R_{f_1} = \text{Re}[s] < 0, \text{as such}, R_x = \text{Re}[s] < a$$

The final result is

$$\mathscr{L}_x(s) = e^{-b(s-a)}\frac{1}{s-a}, \quad \text{Re}[s] < a$$

As I stated, calculation Laplace transform is similar to calculating Fourier transform, except that we need to determine the ROC of corresponding Laplace transform.

3 Inverse Laplace Transform

Calculate inverse Laplace transform by the definition is torturing as the contour integration is not easily to solve. However, with partial fraction expansion, we can construct a lot of Laplace transform pairs.
Example.2 Find all possible inverse Laplace transform of

$$H(s) = \frac{7s-1}{s^2-1}$$

Solution With partial fractional expansion

$$H(s) = \frac{7s-1}{s^2-1} = \frac{7s-1}{(s-1)(s+1)} = \frac{A_1}{s-1} + \frac{A_2}{s+1}$$

where the coefficients are

$$A_1 = (s-1)H(s)\Big|_{s-1=0} = \frac{7s-1}{s+1}\Big|_{s=1} = 3$$
$$A_2 = (s+1)H(s)\Big|_{s+1=0} = \frac{7s-1}{s-1}\Big|_{s=-1} = 4$$

We obtain this expansion as

$$H(s) = \frac{3}{s-1} + \frac{4}{s+1}$$

We can easily figure out that $H(s)$ has two poles $s = 1$ and $s = -1$. The possible ROC should be $\mathrm{Re}[s] < -1$, $\mathrm{Re}[s] > 1$ or $-1 < \mathrm{Re}[s] < 1$. From the Laplace transform table, we note that

$$\mathscr{L}^{-1}\left(\frac{1}{s+a}\right) = \begin{cases} e^{-at}u(t), & \mathrm{Re}[s] > -a \\ -e^{-at}u(-t), & \mathrm{Re}[s] < -a \end{cases} \tag{1}$$

For $\mathrm{Re}[s] < -1$, We note that

$$\begin{aligned} h(t) &= 3\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) + 4\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = 3[-e^{t}u(-t)] + 4[-e^{-t}u(-t)] \\ &= [-3e^{t} - 4e^{-t}]u(-t), \quad \mathrm{Re}[s] < -1 \end{aligned} \tag{2}$$

I will solve this by baby-steps. Hopefully, you can figure out how to proceed for the rest of the two cases.

$$\begin{aligned} &\mathscr{L}^{-1}\left(\frac{1}{s-1}\right), \text{ Compare with Eq. (1), } a = -1, -a = 1; \\ &\text{The ROC is } \mathrm{Re}[s] < -1, \text{ of course } \mathrm{Re}[s] < 1 \rightarrow \mathrm{Re}[s] < -a \\ &\text{Then, } \mathscr{L}^{-1}\left(\frac{1}{s-1}\right) = -e^{t}u(-t) \quad \mathrm{Re}[s] < -1 \end{aligned}$$

Accordingly,

$$\begin{aligned} &\mathscr{L}^{-1}\left(\frac{1}{s+1}\right), \text{ Compare with Eq. (1), } a = 1, -a = -1; \\ &\text{The ROC is } \mathrm{Re}[s] < -1 \rightarrow \mathrm{Re}[s] < -a \\ &\text{Then, } \mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = -e^{-t}u(-t) \quad \mathrm{Re}[s] < -1 \end{aligned}$$

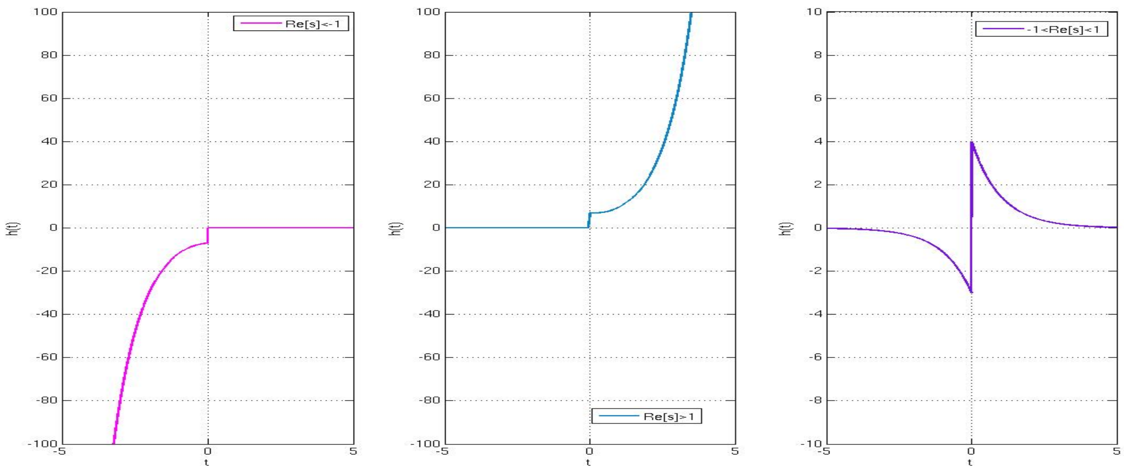
Substitute these two inverse Laplace transform back to the expression of $h(t)$, we got the result in Eq. (2). For $\mathrm{Re}[s] > 1$, We note that

$$\begin{aligned} h(t) &= 3\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) + 4\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = 3[e^{t}u(t)] + 4[e^{-t}u(t)] \\ &= [3e^{t} + 4e^{-t}]u(t), \quad \mathrm{Re}[s] > 1 \end{aligned}$$

For $-1 < \mathrm{Re}[s] < 1$

$$\begin{aligned} h(t) &= 3\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) + 4\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = 3[-e^{t}u(-t)] + 4[e^{-t}u(t)] \\ &= -3e^{t}u(-t) + 4e^{-t}u(t), \quad -1 < \mathrm{Re}[s] < 1 \end{aligned}$$

This example tells us, ROC means a lot!!!The mathematical expression combined with its ROC defines a Laplace transform pair.



4 LTI system analysis

I will skip the differential equation part. We can solve it following exactly identical steps using Fourier analysis.

4.1 bilateral Laplace transform for all zeros initial condition

In your coming study, courses like ELEC 360, control theory, focuses on characterizing system input-output relation with Laplace transform. Let us see how to proceed
Example.3 Consider the LTI system with input $x(t)$, output $y(t)$, and system function $H(s)$, as shown in the figure below. Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are causal LTI systems with system functions $H_1(s)$ and $H_2(s)$, respectively, given by

$$H_1(s) = \frac{1}{s-1}, \quad H_2(s) = A$$

(a). Find an expression for $H(s)$ in terms of $H_1(s)$ and $H_2(s)$.
(b). Determine for what values of A the system is BIBO stable.
SolutionFrom the diagram, we have the following equations

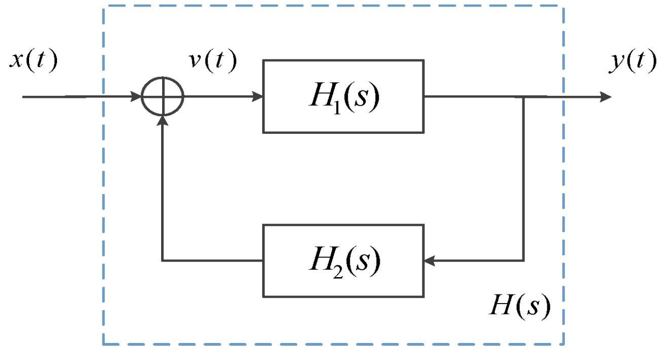
$$y(t) = v(t) * h_1(t), \quad v(t) = x(t) + y(t) * h_2(t)$$

Take Laplace transform on both sides

$$\mathscr{L}_y(s) = \mathscr{L}_v(s)\mathscr{L}_{h_1}(s), \quad \mathscr{L}_v(s) = \mathscr{L}_x(s) + \mathscr{L}_y(s)\mathscr{L}_{h_2}(s)$$

Replace $\mathscr{L}_v(s)$ by $\mathscr{L}_x(s) + \mathscr{L}_y(s)\mathscr{L}_{h_2}(s)$, we obtain

$$\begin{aligned}
\mathscr{L}_y(s) &= \Big[\mathscr{L}_x(s) + \mathscr{L}_y(s)\mathscr{L}_{h_2}(s)\Big]\mathscr{L}_{h_1}(s) \\
&= \mathscr{L}_x(s)\mathscr{L}_{h_1}(s) + \mathscr{L}_y(s)\mathscr{L}_{h_2}(s)\mathscr{L}_{h_1}(s) \\
\Big[1 - \mathscr{L}_{h_2}(s)\mathscr{L}_{h_1}(s)\Big]\mathscr{L}_y(s) &= \mathscr{L}_x(s)\mathscr{L}_{h_1}(s)
\end{aligned}$$



We obtain the system function as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 - H_1(s)H_2(s)} = \frac{\frac{1}{s-1}}{1 - \frac{A}{s-1}} = \frac{1}{s - (A+1)}$$

Since the two sub-system $\mathscr{H}_1$ and $\mathscr{H}_2$ are causal. we have the ROC as $\mathrm{Re}[s] > A + 1$.
(b). System is stable if and only if the ROC includes imagine axis. To guarantee $\mathrm{Re}[s] > A+1$ include the imagine axis, $A + 1 < 0$. We have $A < -1$

4.2 unilateral Laplace transform for non-zero initial conditions

Example.4 Suppose that we have a incrementally-linear TI system with input $x(t)$ and output $y(t)$ characterized by the differential equation

$$y''(t) + 7y'(t) + 12y(t) = x(t)$$

If $y(0^-) = -1$, $y'(0^-) = 0$ and $x(t) = u(t)$, find $y(t)$
Solution Take unilateral Laplace transform on both sides

$$\mathscr{U}\mathscr{L}[y''(t)] + 7\mathscr{U}\mathscr{L}[y'(t)] + 12\mathscr{U}\mathscr{L}[y(t)] = \mathscr{U}\mathscr{L}[x(t)]$$

From the unilateral Laplace transform table, we have

$$\begin{aligned}
\mathscr{U}\mathscr{L}[y''(t)] &= s^2Y(s) - sy(0^-) - y'(0^-) \\
\mathscr{U}\mathscr{L}[y'(t)] &= sY(s) - y(0^-) \\
\mathscr{U}\mathscr{L}[y(t)] &= Y(s) \\
\mathscr{U}\mathscr{L}[x(t)] &= \mathscr{U}\mathscr{L}[u(t)] = \frac{1}{s}
\end{aligned}$$

I have to stress, you can check out everything in your text book except fot $\mathscr{U}\mathscr{L}[y''(t)]$. Denote $z(t) = y'(t)$, the unilateral Laplace transform of $z(t)$ is

$$Z[s] = \mathscr{U}\mathscr{L}[y'(t)] = sY(s) - y(0^-)$$

Apply the time domain differentiation, we have

$$\begin{aligned}
\mathscr{U}\mathscr{L}[y''(t)] &= \mathscr{U}\mathscr{L}[z'(t)] \\
&= sZ(s) - z(0^-) \\
&= s[sY(s) - y(0^-)] - y'(0^-) \\
&= s^2Y(s) - sy(0^-) - y'(0^-)
\end{aligned}$$

Substituting all the unilateral Laplace transform in differential equation and after several carrying out manipulations, the Laplace transform of output $y(t)$ is given by

$$Y(s) = \frac{-s^2 - 7s + 1}{s(s+3)(s+4)}$$

The partial fractional expansion of $Y(s)$ is

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+3} + \frac{A_3}{s+4}$$

Solve the coefficients by

$$\begin{aligned}
A_1 &= sY(s)\Big|_{s=0} = \frac{-s^2 - 7s + 1}{(s+3)(s+4)}\Big|_{s=0} = \frac{1}{12} \\
A_2 &= (s+3)Y(s)\Big|_{s=-3} = \frac{-s^2 - 7s + 1}{s(s+4)}\Big|_{s=-3} = -\frac{13}{3} \\
A_3 &= (s+4)Y(s)\Big|_{s=-4} = \frac{-s^2 - 7s + 1}{s(s+3)}\Big|_{s=-4} = \frac{13}{4}
\end{aligned}$$

The Laplace transform can be calculated as

$$Y(s) = \frac{1}{12}\left(\frac{1}{s}\right) - \frac{13}{3}\left(\frac{1}{s+3}\right) + \frac{13}{4}\left(\frac{1}{s+4}\right)$$

$y(t)$ can be calculated by taking inverse unilateral Laplace transform

$$\begin{aligned}
y(t) &= \frac{1}{12}\mathscr{U}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{13}{3}\mathscr{U}\mathscr{L}^{-1}\left[\frac{1}{s+3}\right] + \frac{13}{4}\mathscr{U}\mathscr{L}^{-1}\left[\frac{1}{s+4}\right] \\
&= \frac{1}{12}u(t) - \frac{13}{3}e^{-3t}u(t) + \frac{13}{4}e^{-4t}u(t)
\end{aligned}$$