CH.4 Continuous-Time Fourier Series

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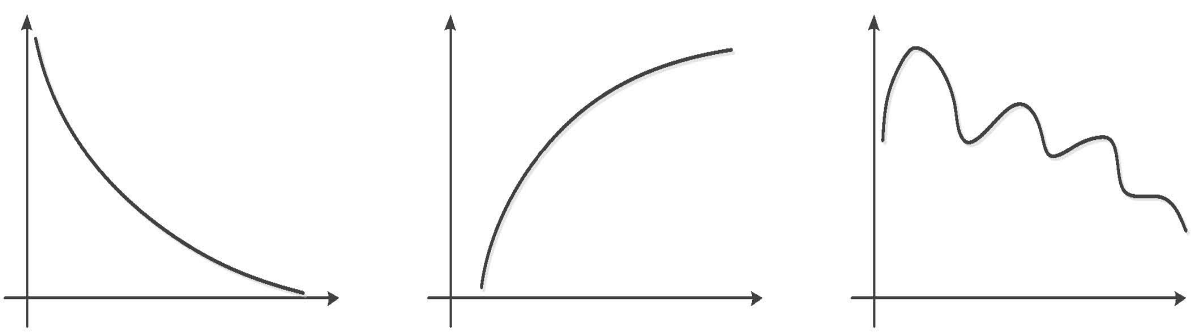
1 First step to Fourier analysis

1.1 My mathematical model is killing me!

The difference between mathematicians and engineers is mathematicians develop mathematical tools and construct mathematical models. Engineers use these tools and models to solve problems. Unfortunately, some of the models can be so difficult to work with



If your models are as simple as the first two, please say thank god. Otherwise, if it is as ugly as the last babe, just do not cry, it is not the end of the world. At least, we can try to fix this. Then, one naive idea flashed. Can we use the combination of the easy functions to represent the ugly ones?

1.1.1 what kind of combination?

Here, we focus on using linear combination of easy functions. By linear combination, it means summation and integral. Of course, non-linear combination works as well, but non-linear approach has a price to pay. It results in higher complexity and more computations which are not desired. Non-linear combination may give you a more accurate model though, the price is just to much. As engineer, we do not like perfectly good. We like sufficiently good. Empirically and experimentally, linear combination is just accurate enough! The problem we are interested in is changed into Can we use the linear combination of the easy functions to represent the ugly ones?

1.1.2 what kind of functions to start with?

We restrict ourselves in approximating periodic functions based on two reasons. Periodic phenomena can be very general in the real world. Mathematical approaches to model periodicity is possible because we have very simple functions exhibiting periodic behaviour, namely **sin** and **cos**. Then, the problem is changed again Can we use the linear combination of the easy periodic functions to represent the ugly periodic ones?. This is the question that chapter 4 tried to answer. Before Fourier started first derivation, he doubted himself too. Periodic phenomena can be super complicated and messy, but sine and cosine are, on the contrary, super simple.

2 Can we do it?

If the problem you are dealing with is like 'Can I do it?', please make a bold move. Suppose you can, then see what the consequences are. Let us assume that any periodic function, $f(t)$ with period 1, can be represented by the linear combination of sinusoid functions. And we know sinusoid function is characterized by three parameters, magnitude, frequency and phase. Linear combination of sinusoid functions means the summation of sinusoid functions with different magnitude, phase and frequency. Mathematically, It means

$$f(t) = \sum_{k=1}^{N} A_k \sin(2\pi kt + \theta_k) \tag{1}$$

We can loss the phase, so to speak. If you use the addition formula of sine and cosine which is the sine of the sum of two angles, we have

$$A_k \sin(2\pi kt + \theta_k) = A_k \sin(2\pi kt) \cos\theta_k + A_k \cos(2\pi kt) \sin\theta_k \tag{2}$$

let $a_k = A_k \cos\theta_k$ and $b_k = A_k \sin\theta_k$. a_k and b_k are related to A_k in the phase. There are a lot of information included in the phase. It is still there in the expression, but it is represented differently in the coefficient. As such, our idea can be rewritten as

$$f(t) = \sum_{k=1}^{N} \left[a_k \sin(2\pi kt) + b_k \cos(2\pi kt) \right] \tag{3}$$

A constant term is also allowed to shift the whole thing up/down

$$f(t) = C_0 + \sum_{k=1}^{N} \left[a_k \sin(2\pi kt) + b_k \cos(2\pi kt) \right]$$

This is maybe the last time sinusoid functions are represented by real sine and cosine. Kiss them goodbye. Applying the famous Euler's formula

$$\begin{aligned} f(t) &= C_0 + \sum_{k=1}^{N} \left[a_k \left(\frac{e^{j2\pi kt} - e^{-j2\pi kt}}{2j} \right) + b_k \frac{e^{j2\pi kt} + e^{-j2\pi kt}}{2} \right] \\ &= C_0 + \sum_{k=1}^{N} \left[a_k j \left(\frac{-e^{j2\pi kt} + e^{-j2\pi kt}}{2} \right) + b_k \left(\frac{e^{j2\pi kt} + e^{-j2\pi kt}}{2} \right) \right] \\ &= C_0 + \sum_{k=1}^{N} \left[\left(\frac{b_k}{2} - \frac{a_k}{2} j \right) e^{j2\pi kt} + \left(\frac{b_k}{2} + \frac{a_k}{2} j \right) e^{-j2\pi kt} \right] \\ &= \sum_{k=-N}^{N} C_k e^{j2\pi kt} \end{aligned} \tag{4}$$

The mysterious coefficient C_k is given by a piece-wise form.

$$C_k = \begin{cases} \frac{b_k}{2} - \frac{a_k}{2} j, & k > 0 \\ C_0, & k = 0 \\ \frac{b_k}{2} + \frac{a_k}{2} j, & k < 0. \end{cases} \tag{5}$$

Here we have an important sufficient necessary condition. If $f(t)$ is a real signal, the coefficients satisfy symmetric property, $C_k = C_{-k}^*$, whose inverse is also true.

Eq.(4) looks very much alike the expression in your text book Eq.(4.1). The differences are that, first of all, I apply a finite sum here. If finite sum works, I can avoid investigating the painful convergence of a infinite sum. I do not like putting myself in pain. No one likes. Secondly, I assume that the period of $f(t)$ is one. In your text book, $x(t)$ has a general period T. But it is pretty easy to generalize from period one to period T.

The LHS in Eq. (4) is the phenomena we are interested (The bad ass who is really difficult to work with. But we know it is a periodic function). The RHS is the linear combination of sinusoid functions. Hopefully, the equation is valid. $f(t)$ is given. And complex exponentials represent real sine and cosine waves. Everything is known except for the coefficients C_k's here. How to solve these coefficients is our next problem. Let us rewrite Eq. (4) in an expanded form.

$$f(t) = C_{-N} e^{-j2\pi Nt} + C_{-(N-1)} e^{-j2\pi(N-1)t} \cdots C_0 \cdots + C_{N-1} e^{j2\pi(N-1)t} + C_N e^{j2\pi Nt} \tag{6}$$

We have $2N + 1$ coefficients to solve. Suppose that we are solving the mth coefficient, $-N < m < N$.

$$\begin{aligned} C_m e^{j2\pi mt} &= f(t) - \sum_{k=-N,\ k\neq m}^{N} C_k e^{j2\pi kt} \\ C_m &= f(t) e^{-j2\pi mt} - \sum_{k=-N,\ k\neq m}^{N} C_k e^{j2\pi(k-m)t} \end{aligned} \tag{7}$$

Integral the whole thing over one period on both sides. Remember that $f(t)$ is periodic function with period one. As illustrated in tutorial 2, one period is enough to describe $f(t)$.

$$
\begin{aligned}
\int_0^1 C_m dt = C_m \int_0^1 dt = C_m = \int_0^1 & \left[f(t)e^{-j2\pi mt} - \sum_{k=-N,\ k\neq m}^{N} C_k e^{j2\pi(k-m)t} \right] dt \\
&= \int_0^1 f(t)e^{-j2\pi mt} dt - \int_0^1 \sum_{k=-N,\ k\neq m}^{N} C_k e^{j2\pi(k-m)t} dt \\
&= \int_0^1 f(t)e^{-j2\pi mt} dt - \sum_{k=-N,\ k\neq m}^{N} \int_0^1 C_k e^{j2\pi(k-m)t} dt
\end{aligned}
\tag{8}
$$

Let us take a closer look at the integration $\int_0^1 C_k e^{j2\pi(k-m)t} dt$.

$$
\begin{aligned}
\int_0^1 C_k e^{j2\pi(k-m)t} dt = C_k & \int_0^1 e^{j2\pi(k-m)t} dt \\
&= \frac{C_k}{j2\pi(k-m)} \int_0^1 e^{j2\pi(k-m)t} d\big[j2\pi(k-m)t\big] \\
&= \frac{C_k}{j2\pi(k-m)} e^{j2\pi(k-m)t} \Big|_0^1 \\
&= \frac{C_k}{j2\pi(k-m)} \left[e^{j2\pi(k-m)} - e^0 \right] = 0
\end{aligned}
\tag{9}
$$

Since $k \neq m$ and both k and m are integer, $k - m$ is a non-zero integer. The coefficient can be solved by

$$
C_m = \int_0^1 f(t)e^{-j2\pi mt} dt \tag{10}
$$

This expression is similar to Eq. (4.11) in your text book. The difference is that $x(t)$ in your text has period T.

This is the time to celebrate!!! Given any periodic function $f(t)$ with period **one**, it can be represented by the linear combination (Here, we are still talking about **finite sum**) of sinusoid functions. Mathematically, we have

$$
f(t) = \sum_{k=-N}^{N} C_k e^{j2\pi kt}, \quad C_k = \int_0^1 f(t)e^{-j2\pi kt} dt \tag{11}
$$

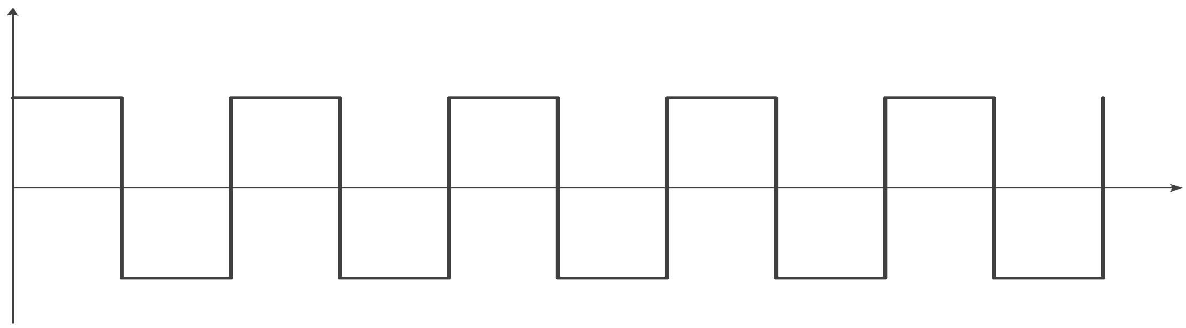
3 You can not always get what you want

It seems we have a super powerful mathematical model. We do not have to freak out if $f(t)$ is too complicated. As long as it is periodic, we can represent it by the finite sum of scaled sinusoid functions. I can not wait to try to implement this approach. Let us see some examples.

Example 1 $f(t)$ is periodic function with period one, whose expression over one period is given by

$$f(t) = \begin{cases} 1, & \text{for } 0 \leq t < \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} \leq t < 1 \end{cases} \tag{12}$$

Find out the Fourier series of $f(t)$
Solution The sketch of $f(t)$ is



The first step should always be calculating the 0th coefficient

$$C_0 = \int_0^1 f(t)dt = \int_0^{\frac{1}{2}} 1dt + \int_{\frac{1}{2}}^1 (-1)dt = t\Big|_0^{\frac{1}{2}} - t\Big|_{\frac{1}{2}}^1 = \left(\frac{1}{2} - 0\right) - \left(1 - \frac{1}{2}\right) = 0 \tag{13}$$

Other coefficients can be calculated by

$$\begin{aligned} C_k &= \int_0^1 f(t)e^{-j2\pi kt}dt = \int_0^{\frac{1}{2}} e^{-j2\pi kt}dt + \int_{\frac{1}{2}}^1 (-1)e^{-j2\pi kt}dt \\ &= \int_0^{\frac{1}{2}} e^{-j2\pi kt}dt - \int_{\frac{1}{2}}^1 e^{-j2\pi kt}dt = \frac{1}{-j2\pi k}\int_0^{\frac{1}{2}} e^{-j2\pi kt}d\Big(-j2\pi kt\Big) - \frac{1}{-j2\pi k}\int_{\frac{1}{2}}^1 e^{-j2\pi kt}d\Big(-j2\pi kt\Big) \\ &= \left(\frac{1}{-j2\pi k}e^{-j2\pi kt}\right)\Bigg|_0^{\frac{1}{2}} - \left(\frac{1}{-j2\pi k}e^{-j2\pi kt}\right)\Bigg|_{\frac{1}{2}}^1 = \left[\frac{j}{2\pi k}\Big(e^{-j\pi k} - 1\Big)\right] - \left[\frac{j}{2\pi k}\Big(e^{-j2\pi k} - e^{-j\pi k}\Big)\right] \\ &= \frac{j}{2\pi k}\Big(e^{-j\pi k} - 1 - e^{-j2\pi k} + e^{-j\pi k}\Big) = \frac{j}{\pi k}\Big(e^{-j\pi k} - 1\Big) \end{aligned} \tag{14}$$

Here, we need to further simplify the expression of the coefficient

$$C_k = \frac{j}{\pi k}\Big(e^{-j\pi k} - 1\Big) = \frac{j}{\pi k}\Big[(e^{-j\pi})^k - 1\Big] = \frac{j}{\pi k}\Big[(-1)^k - 1\Big] = \begin{cases} -\frac{2j}{\pi k} & \text{for k odd} \\ 0 & \text{for k even} \end{cases} \tag{15}$$

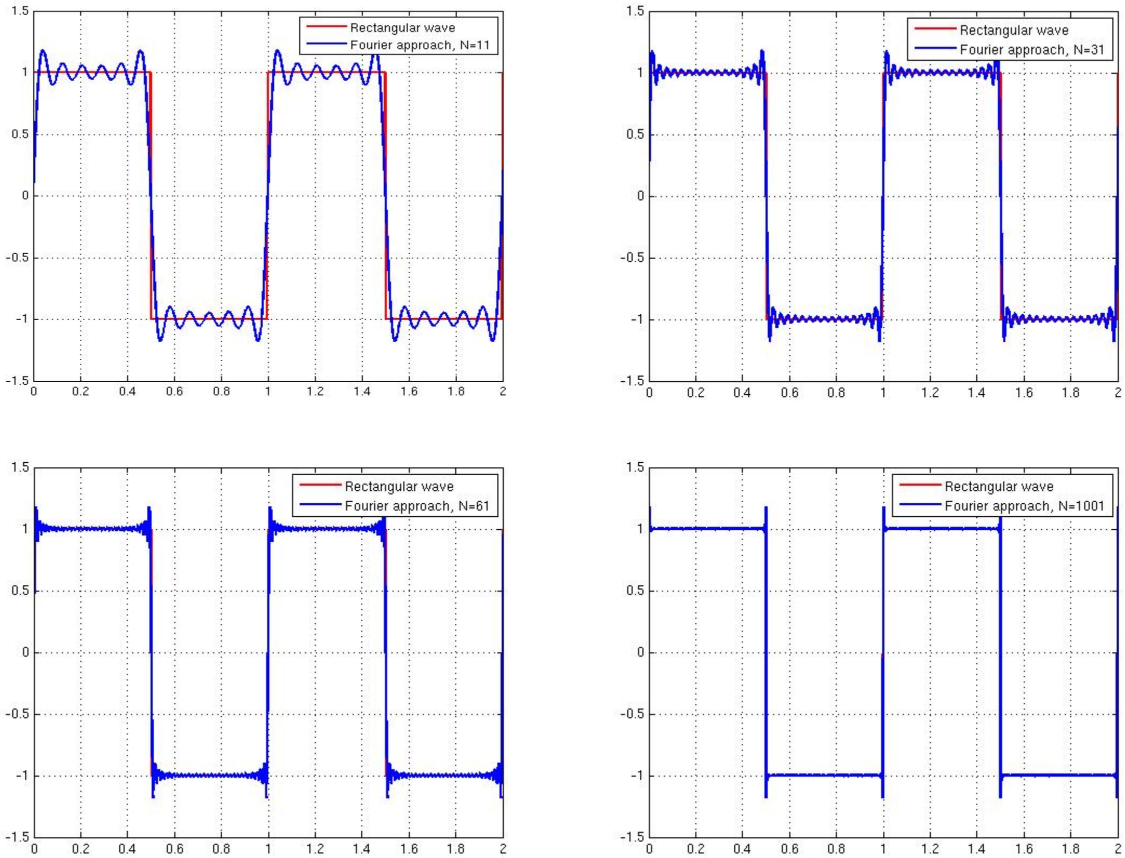


Figure 1: Illustration of the accuracy of Fourier approach approximating rectangular wave

Eventually, applying $k = 2n + 1$ and combining the negative power and positive power, the Fourier series of $f(t)$ is given by

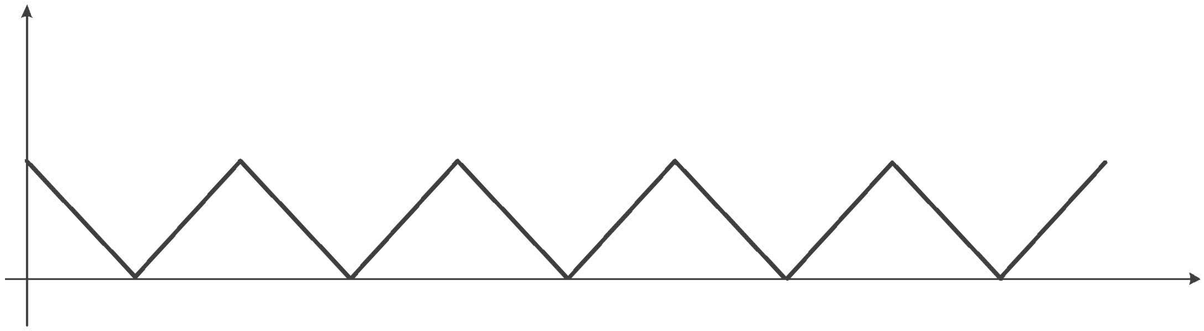
$$f(t) = \sum_{\text{k is odd}} -\frac{2j}{\pi k} e^{j2\pi kt} = \sum_{n=0}^{N} -\frac{2j}{\pi(2n+1)} 2j \sin(2\pi(2n+1)t) = \frac{4}{\pi} \sum_{n=0}^{N} \frac{1}{2n+1} \sin(2\pi(2n+1)t) \tag{16}$$

The behaviour of our model is strange (See Fig. 1). If Eq.(11) is valid, then the blue curve and red curve should perfectly match. As N gets bigger (i.e. harmonics with higher frequency are included in the summation), the accuracy gets better. You should know the common science in signal processing: Higher frequencies are required to produce a sharp edge.

Example 2 $f(t)$ is periodic function with period one, whose expression over one period is given by

$$f(t) = \begin{cases} \frac{1}{2} + t, & \text{for } -\frac{1}{2} \leq t < 0 \\ \frac{1}{2} - t, & \text{for } 0 \leq t < \frac{1}{2} \end{cases} \tag{17}$$

Find out the Fourier series of $f(t)$
Solution The sketch of $f(t)$ is



The first step should always be calculating the 0th coefficient

$$C_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)dt = \int_{-\frac{1}{2}}^{0} \frac{1}{2} + tdt + \int_{0}^{\frac{1}{2}} \frac{1}{2} - tdt = \frac{1}{4} \tag{18}$$

Other coefficients can be calculated by

$$\begin{aligned} C_k &= \int_0^1 f(t)e^{-j2\pi kt}dt = \int_{-\frac{1}{2}}^{0} \left(\frac{1}{2}+t\right) e^{-j2\pi kt}dt + \int_0^{\frac{1}{2}} \left(\frac{1}{2}-t\right) e^{-j2\pi kt}dt \\ &= \frac{1}{(2\pi k)^2}\left(1-e^{j\pi k}\right) - \frac{1}{j4\pi k} + \frac{1}{(2\pi k)^2}\left(1-e^{-j\pi k}\right) + \frac{1}{j4\pi k} \\ &= \frac{1}{(2\pi k)^2}\left(2 - e^{j\pi k} - e^{-j\pi k}\right) \end{aligned} \tag{19}$$

Here, we need to further simplify the expression of the coefficient

$$C_k = \frac{1}{(2\pi k)^2}\left(2 - e^{j\pi k} - e^{-j\pi k}\right) = \begin{cases} \dfrac{1}{\pi^2 k^2} & \text{for k odd} \\ 0 & \text{for k even} \end{cases} \tag{20}$$

The Fourier series is given by

$$f(t) = \frac{1}{4} + \sum_{\text{k is odd}} \frac{1}{\pi^2 k^2} e^{j2\pi kt} = \frac{1}{4} + \sum_{n=-N}^{N} \frac{1}{\pi^2(2n+1)^2} e^{j2\pi(2n+1)t} = \frac{1}{4} + \sum_{n=0}^{N} \frac{1}{\pi^2(2n+1)^2} \cos(2\pi(2n+1)t) \tag{21}$$

I just plot $f(t)$ and $\sum C_k e^{j2\pi kt}$ in Fig. 2. Somehow, they match well, but not perfectly.

4 Are we doomed?

Let me remind you what we have experienced. We have a great idea to represent any periodic functions by the summation of modified sinusoid functions. Good! We write the mathematical expression and simplify it. Very good! We successfully calculate all the coefficients.

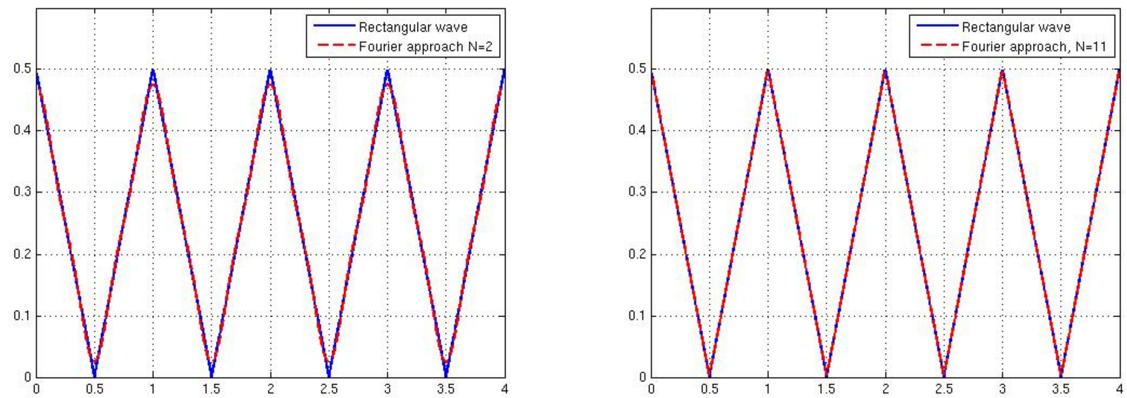


Figure 2: Illustration of the accuracy of Fourier approach approximating triangular wave

Super good! The linear combination does not match the function. NOT SO GOOD. But why? Trust me, any unreasonable phenomena has a very reasonable explanation.

4.1 Why does this approach fail?

Take a look at the formula again.

$$f(t) = \sum_{k=-N}^{N} C_k e^{j2\pi kt} \tag{22}$$

The RHS is the linear combination of real sine and cosine waves, which satisfy an **infinitely smooth** condition. Sinusoid functions are continuous, and continuous at any order of its derivative. Basically, the RHS is the finite sum of continuous functions, which is also continuous. However, sometime your $f(t)$ is not continuous, like the rectangular wave. Secondly, the RHS is the summation of differentiable functions, which is also differentiable. Unfortunately, $f(t)$ can be non-differentiable, like the triangular wave. I can go on delivering the bad news. I know I am annoying. Sorry. The RHS is infinitely differentiable, which means differentiate the RHS many times, as many as you want, the result is still continuous. This is so-called infinitely smooth.

In the real world, the periodic functions satisfying such condition are pretty rare. You can not expect to approximate a discontinuous function by a continuous one. Any lack of smoothness of your $f(t)$ at any order-derivative will ruin this approach. AS SUCH, finite sum will not work. In other words, discontinuity at any order of derivative of your $f(t)$ will force us to apply the infinite sum, which can be expressed as

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt}, \quad C_k = \int_0^1 f(t) e^{-j2\pi kt} dt \tag{23}$$

4.2 Infinite sum, what can I say about you?

If infinite summation applied, we have to deal with the painful problem named CONVERGENCY. What a tough world!!! Instead of deriving in detail, I will deliver the news directly. They converge, but from different perspective.

functions satisfying infinitely smooth condition: The Fourier series converge converges uniformly (i.e. pointwise convergence). This is the alternative representation of Theorem 4.1 in your text book.

$$\lim_{N\to\infty}\sum_{k=-N}^{N} C_k e^{j2\pi kt} = f(t) \tag{24}$$

functions satisfying square integrable condition: If $\int_0^1 |f(t)|^2 dt < \infty$, the Fourier series converges from a Mean square error perspective (i.e. MSE convergence). This is the alternative representation of Theorem 4.2 in your text book.

$$\lim_{N\to\infty}\int_0^1 \left| f(t) - \sum_{k=-N}^{N} C_k e^{j2\pi kt} \right|^2 dt = 0 \tag{25}$$

discontinuous functions satisfying Dirichlet condition: The Fourier series converges pointwise everywhere to $x(t)$, except at the points of discontinuity of $x(t)$. At each point $t = t_d$ of discontinuity of $x(t)$, the Fourier series converges to $\frac{1}{2}[x(t_d^-)+x(t_d^+)]$ where $x(t_d^-)$ and $x(t_d^+)$ denote the values of the signal on the left- and right-hand sides of the discontinuity, respectively.

This is good enough. Now we are safe to say We can use the infinite sum of sinusoid wave to represent periodic phenomena.

5 Fourier series: Signal decomposition

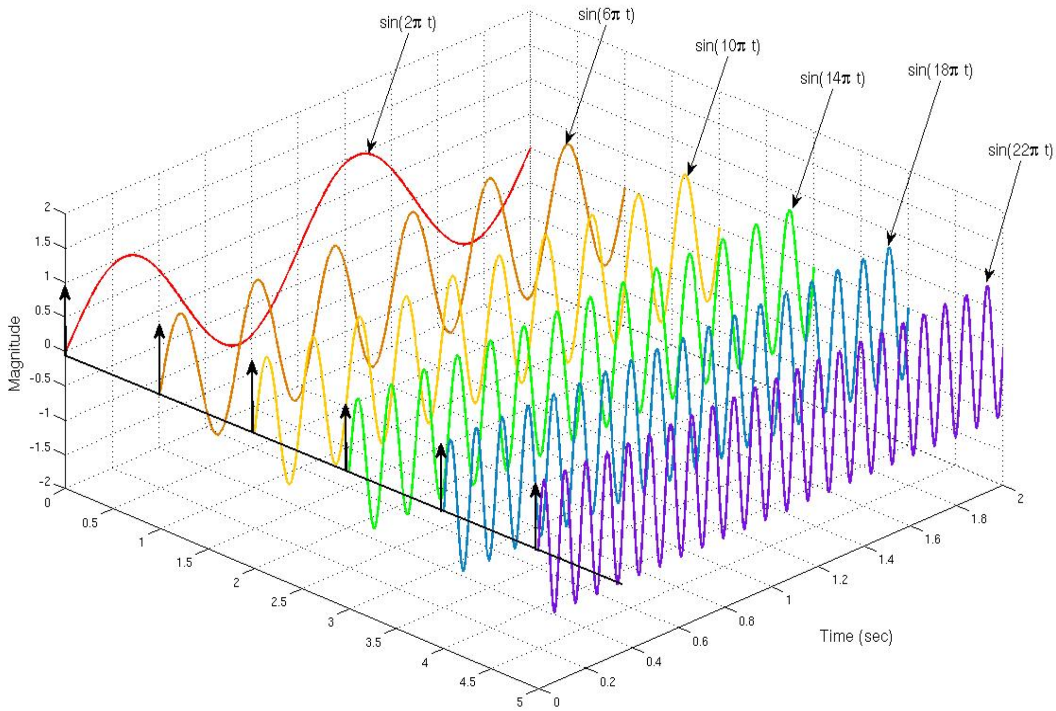
In the previous sections, we proved that the model works. With this Fourier series, what can we know about $f(t)$? That is so-called spectrum.

We have to recall Fourier analysis formula and Fourier synthesis formula.

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt}, \quad C_k = \int_0^1 f(t)e^{-j2\pi kt}dt \tag{26}$$

Here, $e^{j2\pi kt}$, $k = -\infty, \cdots, \infty$, k is integer, representing real sine and cosine functions, the magnitude of each of which is one. All the coefficients, including the information about $f(t)$, act as a scaler to modify the magnitude and phase of each sinusoid signal. In the following figure, I just plot the first five harmonics (when the period is one) in a 3-dimensional space. Scale each harmonic by the corresponding coefficient. Then project everything onto the time-magnitude plane, you get the time domain representation of $f(t)$. Similarly, project everything onto the frequency-magnitude plane, the frequency domain representation of $f(t)$ (i.e. Fourier series of $f(t)$) is obtained.

I have to remind you that Fig.(3) is just an illustration, which can help you understand the fact that Fourier analysis decomposes the signal. Since the coefficients are complex numbers, they can not be plotted.



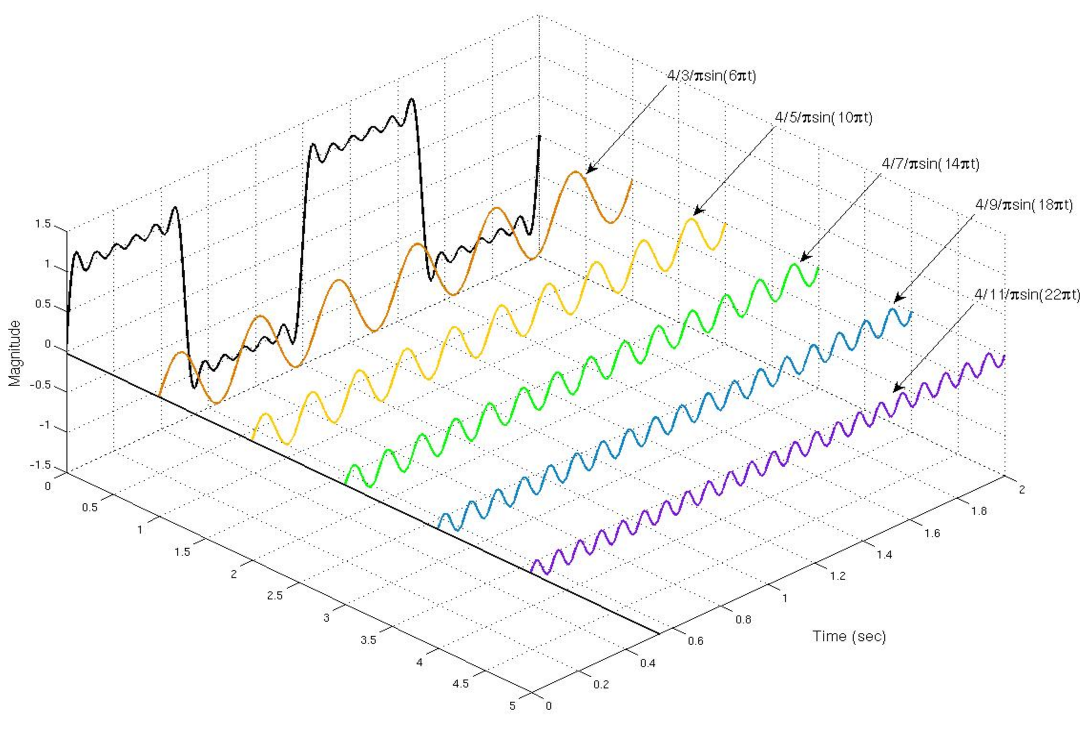


Figure 3: Illustration of Fourier decomposition

5.1 The most significant is the coefficient

Coefficients include all the information of $f(t)$. One way to investigate these useful information is so-called magnitude spectrum, which is the magnitude of each coefficient (i.e. $|C_k|$). Magnitude spectrum tells us the energy distribution in a signal $f(t)$ over different harmonics. Since the coefficients are complex numbers, the argument of C_k, $\arg(C_k)$, is also a function of frequency index k. The relation between argument of each coefficient and the frequency is called phase spectrum.

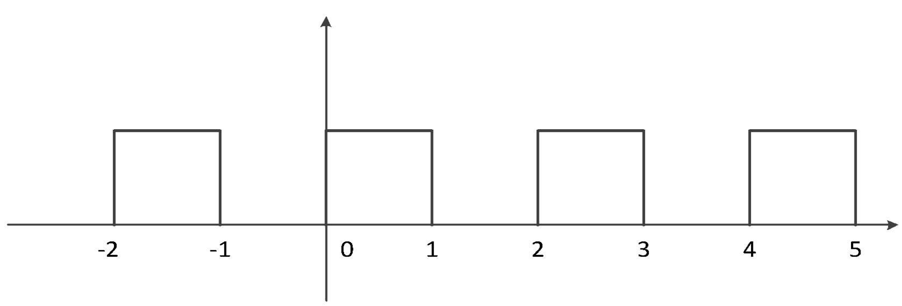
If $f(t)$ is periodic, then the spectrum of $f(t)$ is discrete. This is the first reciprocal relation between time domain and frequency domain.

5.1.1 spectrum

Magnitude spectrum tells you that how much strength $f(t)$ puts onto the kth harmonic. Calculating magnitude spectrum is actually calculating the absolute value of each coefficient. In term of our building blocks, sine and cosine functions, they all have magnitude 1. The coefficients act as scaler to make some of the harmonics stronger, some of them weaker.

Why do instruments sound different? More precisely, why do two instruments sound different even when they are playing the same note? It is because the note they produce is not a single sinusoid of a single frequency, not the A at 440 Hz, for example, but a sum (literally) of many sinusoids, each contributing a different amount. The complex wave that reaches your ear is the combination of many ingredients. Two instruments sound different because of the harmonics they produce and because of the strength of the harmonics. In all, two instruments sound different even when they play same note, because they generate different harmonics and they have different energy distribution among all the harmonics.

Example 3 Find the Fourier series coefficient sequence C_k of the periodic signal $f(t)$ shown in the figure below. Plot the frequency spectrum of this signal including the first five harmonics.



Solution: Here we need to generalize from period one to period T. When T is given, the Fourier coefficients are given by

$$C_k = \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi k}{T}t}dt \tag{27}$$

$f(t)$ is periodic with period $T = 2$. From the Fourier analysis formula, the 0th coefficient can be calculated by

$$C_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}\int_0^1 dt = \frac{1}{2} \tag{28}$$

Other coefficients are calculated as

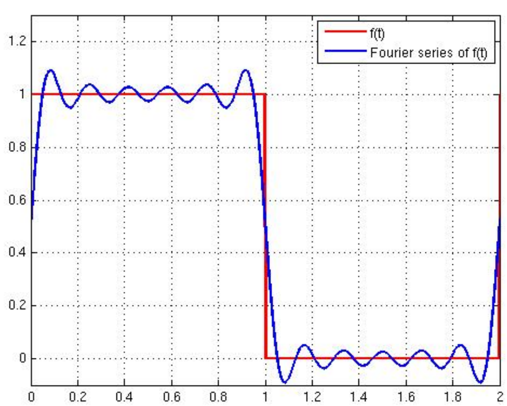
$$\begin{aligned} C_k &= \frac{1}{T}\int_T f(t)dt = \frac{1}{2}\int_0^2 f(t)e^{-j\frac{2\pi k}{T}t}dt = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \\ &= \frac{1}{-j2\pi k}\int_0^1 e^{-j\pi kt}d\left(-j\pi kt\right) \\ &= \frac{1}{-j2\pi k}\left(e^{-j\pi kt}\right)\Bigg|_0^1 = \frac{1}{-j2\pi k}\left(e^{-j\pi k}-1\right) \end{aligned} \tag{29}$$

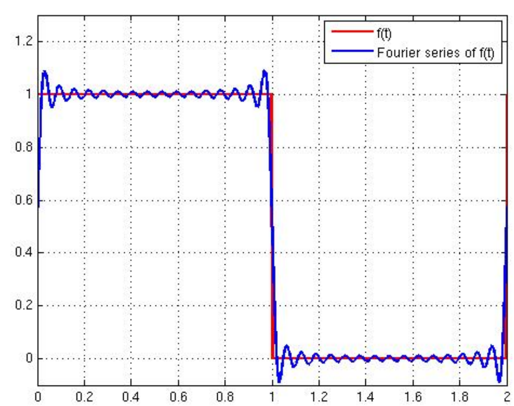
The coefficients can be further simplified into

$$C_k = \frac{1}{-j2\pi k}\left(e^{-j\pi k}-1\right) = \begin{cases} -\dfrac{j}{\pi k} & \text{for k odd} \\ 0 & \text{for k even} \end{cases} \tag{30}$$

The Fourier series of $f(t)$ is

$$\begin{aligned} f(t) &= \frac{1}{2} + \sum_{\text{k is odd}} -\frac{j}{\pi k}e^{j\pi kt} = \frac{1}{2} + \sum_{n=-\infty}^{\infty} -\frac{j}{\pi(2n+1)}e^{j\pi(2n+1)t} \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\pi(2n+1)t \end{aligned} \tag{31}$$

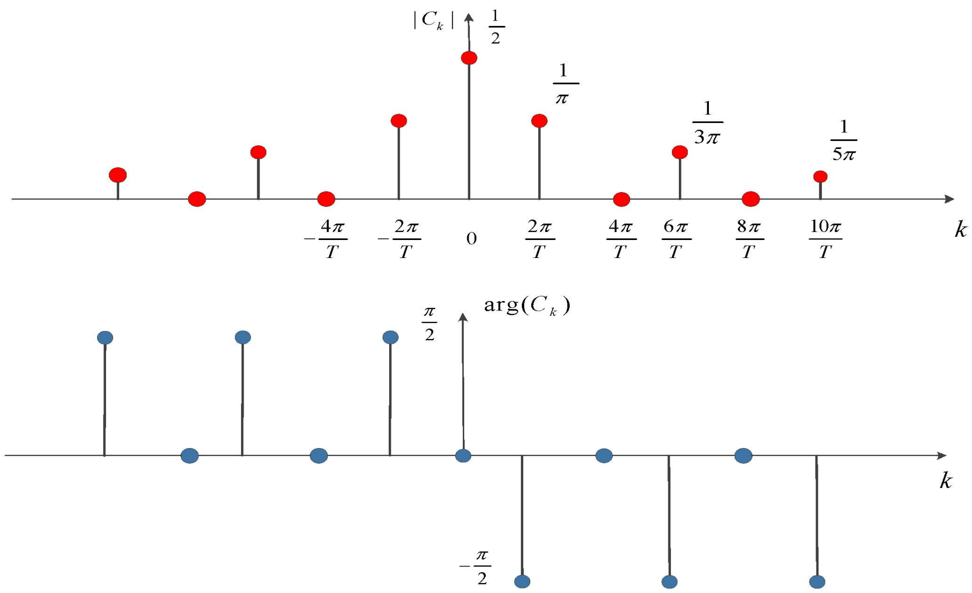




The magnitude and argument of each coefficient can be calculated as

$$\begin{aligned} |C_k| &= \left|-\frac{j}{\pi k}\right| = \frac{1}{\pi k} \quad \text{for k odd,} \quad C_0 = \frac{1}{2} \\ \arg(C_k) &= \arg\left(-\frac{1}{\pi k}j\right) = -\frac{\pi}{2} \quad \text{for k odd and } k>0 \end{aligned} \tag{32}$$

which yields the following spectrum



Compare the Fourier series of Eg. 1 and Eg. 3, can you figure out something interesting?

6 Application of Fourier series

6.1 Response to complex exponential of LTI system

We have known that input-output relation for a LTI system is characterized by its impulse response, $y(t) = x(t) * h(t)$. what if the input is a complex exponential? (i.e. $x(t) = e^{st}$, where s is a complex number)

$$\begin{aligned} y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= e^{st}H(s) \end{aligned} \tag{33}$$

$H(s)$ is namely the system function. In Chapter 6, you will see that this system function is actually the Laplace transform of impulse response of LTI system.

If the input is the linear combination of complex exponentials given by $x(t) = \sum_k \alpha_k e^{s_k t}$, the output can be calculated as

$$y(t) = \sum_k \alpha_k H(s_k) e^{s_k t} \tag{34}$$

Example 4 Suppose that we have the LTI system with the impulse response $h(t)$ given by

$$h(t) = \delta(t-1) \tag{35}$$

Find the system function $H(s)$. For input $x(t) = e^t\cos(\pi t)$, use the derived system function to determine the output $y(t)$.
Solution The system function can be obtained by

$$\begin{aligned} H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt &= \int_{-\infty}^{\infty} \delta(t-1)e^{-st}dt \\ &= \int_{-\infty}^{\infty}\left(e^{-st}\Big|_{t=1}\right)\delta(t-1)dt = e^{-s}\int_{-\infty}^{\infty}\delta(t-1)dt = e^{-s} \end{aligned} \tag{36}$$

The input $x(t)$ can be rewritten as

$$x(t) = \frac{1}{2}e^t\left(e^{j\pi t} + e^{-j\pi t}\right) = \frac{1}{2}e^{(1+j\pi)t} + \frac{1}{2}e^{(1-j\pi)t} \tag{37}$$

The corresponding output is given by

$$
\begin{aligned}
y(t) = \sum_k \alpha_k H(s_k) e^{s_k t} &= \frac{1}{2} H(1+j\pi) e^{(1+j\pi)t} + \frac{1}{2} H(1-j\pi) e^{(1-j\pi)t} \\
&= \frac{1}{2} e^{-1-j\pi} e^{(1+j\pi)t} + \frac{1}{2} e^{-1+j\pi} e^{(1-j\pi)t} = \frac{1}{2} e^{t-1+j\pi t-j\pi} + \frac{1}{2} e^{t-1-j\pi t+j\pi} \\
&= \frac{1}{2} e^{t-1} e^{(t-1)j\pi} + \frac{1}{2} e^{t-1} e^{-(t-1)j\pi} = \frac{1}{2} e^{t-1} \left[e^{(t-1)j\pi} + e^{-(t-1)j\pi} \right] \\
&= e^{t-1} \cos \pi (t-1)
\end{aligned}
\tag{38}
$$

6.2 Periodic signals pass through LTI system

With the help of Fourier series, any periodic signal $x(t)$ with period T can be characterized by the famous Fourier analysis formula and Fourier synthesis formula

$$
x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \qquad C_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \tag{39}
$$

where $\omega_0 = \frac{2\pi}{T}$ is the basic period. Obviously, periodic function is represented by the summation of complex exponentials. From Eq. (34), the output of periodic input can be calculated as

$$
y(t) = \sum_{k=-\infty}^{\infty} C_k H(jk\omega_0) e^{jk\omega_0 t} \tag{40}
$$

Example 5 Suppose that we have a LTI system with frequency response

$$
H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq 5 \\ 0 & \text{otherwise} \end{cases} \tag{41}
$$

Using frequency-domain methods, find the output y(t) of the system if the input x(t) is given by

$$
x(t) = 1 + 2\cos 2t + 2\cos 4t + \frac{1}{2}\cos 6t. \tag{42}
$$

Solution The input is the summation of cosine functions. $\cos 2\pi$ has period $T_1 = 2\pi/2 = \pi$. $\cos 4\pi$ has period $T_1 = 2\pi/4 = \pi/2$. $\cos 6\pi$ has period $T_1 = 2\pi/6 = \pi/3$. Recall the 'one-period-many-frequencies' property of sum of sinusoid waves, the summation can not start repeating until the slowest ingredient caught up. As such, $x(t)$ has period π. The basic frequency of $x(t)$ is $\omega_0 = 2$.

$$
\begin{aligned}
x(t) &= 1 + 2\cos 2t + 2\cos 4t + \frac{1}{2}\cos 6t \\
&= 1 + e^{j2t} + e^{-j2t} + e^{j4t} + e^{-j4t} + \frac{1}{4} e^{j6t} + \frac{1}{4} e^{-j6t} \\
&= \sum_{k=0,\ \pm 1,\ \pm 2,\ \pm 3} C_k e^{jk\omega_0 t}, \qquad \omega_0 = 2
\end{aligned}
\tag{43}
$$

The coefficients are given by $C_0 = 0,\ C_{\pm1} = 1,\ C_{\pm2} = 1,\ C_{\pm3} = \frac{1}{4}$. Next step, let us handle the frequency response

$$H(j\omega) = H(j\omega_0 k) = \begin{cases} 1 & \text{for } \ |\omega_0 k| \geq 5 \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

The output $y(t)$ has the form

$$y(t) = \sum_{k=0,\ \pm1,\ \pm2,\ \pm3}^{\infty} C_k H(jk\omega_0) e^{jk\omega_0 t} \tag{45}$$

$C_k H(jk\omega_0)$ can be calculated as

$$\begin{aligned} C_0 H(j(0 \times 2)) &= 0 \\ C_{\pm1} H(j(\pm1 \times 2)) &= 0 \\ C_{\pm2} H(j(\pm2 \times 2)) &= 0 \\ C_{\pm3} H(j(\pm3 \times 2)) &= \frac{1}{4} \end{aligned} \tag{46}$$

Accordingly,

$$y(t) = C_{-3} e^{-j6t} + C_3 e^{j6t} = \frac{1}{4}(e^{-j6t} + e^{j6t}) = \frac{1}{4} \times 2\cos(6t) = \frac{1}{2}\cos(6t) \tag{47}$$

As you can see, the system acts as a high-pass filter.

7 Summarize

The fundamental issue you have to know after finishing this chapter is summarized below

1. **Find the Fourier series of periodic function** $f(t)$: You need to identify all the coefficients. Firstly, you need to determine the basic frequency ω_0. Then, apply Fourier analysis formula to calculate C_0 and other coefficients C_k.

2. **Convergence of the series**: For infinitely smooth functions (or infinitely differentiable functions), the series converges pointwise and at the same rate. (See Eq. (24)). For square integrable functions (or finite power functions), the series converges from a Mean Square Error perspective. (See Eq. (25)). For functions satisfying Dirichlet condition, the series converges poinwise except for the discontinuity. At the discontinuity, it converges to a constant value given by $\frac{1}{2}[x(t_d^-) + x(t_d^+)]$.

3. **Property of the coefficients** Table 4.1 in your text book. Please do not just keep them in mind. You need to know how to apply it to solving problems.

4. **Spectrum** Magnitude spectrum, $|C_k|$, tells you the strength distribution of $f(t)$ over all the harmonics. If the relation of coefficients and harmonics is shown by the correspondence of argument of the coefficients and the frequency, you got the phase spectrum.

5. **The output of LTI system to a periodic input** All the results result from the output of LTI system to a complex exponential. One application of this is filtering. If you can understand Example 4.10 in your text book and the example in my slide, I think it is enough.