Fourier stability analysis, boundary layers - HW5:
Code attached here: https://github.com/lwyhasacat/HW5_Fourier.git

Exercise 1:

To compute the symbol of the right side of (2), we want to define the translation invariant operator M and compute $m(\theta)$.

For computing M, we want to compute the symbol $m(\theta)$ of the operator using a Fourier mode $V_j(\theta) = e^{i\theta j}$. We have the first order difference operator D_0, which is defined as:

$$(D_0 V)_j = \frac{1}{2\Delta x}(V_{j+1} - V_{j-1})$$

Applying D_0 to $V_j(\theta)$ gives us:

$$\begin{aligned}(D_0 V)_j &= \frac{1}{2\Delta x}(e^{i\theta(j+1)} - e^{i\theta(j-1)}) \\ &= \frac{1}{2\Delta x}(e^{i\theta}e^{i\theta j} - e^{-i\theta}e^{i\theta j}) \\ &= \frac{1}{2\Delta x}(e^{i\theta} - e^{-i\theta})e^{i\theta j} \\ &= \frac{i}{\Delta x}\sin(\theta)e^{i\theta j}\end{aligned}$$

We also have the second-order difference operator D_+D_-, which is defined as:

$$(D_+D_-V)_j = \frac{1}{\Delta x^2}(V_{j+1} - 2V_j + V_{j-1})$$

Applying D_+D_- to $V_j(\theta)$ gives us:

$$\begin{aligned}(D_+D_-V)_j &= \frac{1}{\Delta x^2}(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)}) \\ &= \frac{1}{\Delta x^2}(e^{i\theta} - 2 + e^{-i\theta})e^{i\theta j} \\ &= \frac{1}{\Delta x^2}(2\cos(\theta) - 2)e^{i\theta j}\end{aligned}$$

Then, for the operator M, we have:

$$\begin{aligned}MV_j(\theta) &= (-D_0 + \mu D_+D_-)V_j \\ &= \left(-\frac{i}{\Delta x}\sin(\theta) + \mu\frac{1}{\Delta x^2}(2\cos(\theta) - 2)\right)e^{i\theta j}\end{aligned}$$

Thus, the symbol $m(\theta)$ is:

$$m(\theta) = -\frac{i}{\Delta x}\sin(\theta) + \mu\frac{1}{\Delta x^2}(2\cos(\theta) - 2)$$

We want to show the semi-discrete scheme is von-Neumann stable. Since we have:

$$m(\theta) = -\frac{i}{\Delta x}\sin(\theta) + \mu\frac{1}{\Delta x^2}(2\cos(\theta) - 2)$$

and we know that the von-Neumann stability requires that:

$$|G(\theta)| = |1 + \Delta t \cdot m(\theta)| \leq 1$$

We can substitute $m(\theta)$ and get:

$$1 + \Delta t\left(-\frac{i}{\Delta x}\sin(\theta) + \mu\frac{1}{\Delta x^2}(2\cos(\theta) - 2)\right)$$

which means that for stability we would need:

$$\text{Real}\left(\mu\frac{\Delta t}{\Delta x^2}(2\cos(\theta) - 2)\right) \leq 0$$

Since $2\cos(\theta) - 2 \leq 0$ for all θ, the stability condition is met under the CFL condition:

$$\mu\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

We want to show that the scheme is second order accurate, which depends on the discretization of the differential operators. We first look at the central difference for the first derivative u_x at a grid point x_j:

$$(D_0 u)_j = \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x}$$

We can use the Taylor expansion at x_j and get:

$$u(x_{j+1}) = u(x_j) + u_x(x_j)\Delta x + \frac{u_{xx}(x_j)}{2}\Delta x^2 + O(\Delta x^3)$$

$$u(x_{j-1}) = u(x_j) - u_x(x_j)\Delta x + \frac{u_{xx}(x_j)}{2}\Delta x^2 - O(\Delta x^3)$$

Taking the difference and dividing by $2\Delta x$, we would still have the leading term $u_x(x_j)$, and the next significant term is $O(\Delta x^2)$. Thus, this is second order accuracy.

The central difference for the second derivative u_{xx} is:

$$(D_+ D_- u)_j = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{\Delta x^2}$$

If we use Taylor expansion again, we get:

$$u(x_{j+1}) = u(x_j) + u'(x_j)\Delta x + \frac{u''(x_j)}{2}\Delta x^2 + \frac{u'''(x_j)}{6}\Delta x^3 + O(\Delta x^4)$$

$$u(x_{j-1}) = u(x_j) - u'(x_j)\Delta x + \frac{u''(x_j)}{2}\Delta x^2 - \frac{u'''(x_j)}{6}\Delta x^3 + O(\Delta x^4)$$

Substituting these into the central difference formula:

$$(D_+ D_- u)_j = \frac{2u''(x_j)\Delta x^2 + O(\Delta x^4)}{\Delta x^2}$$

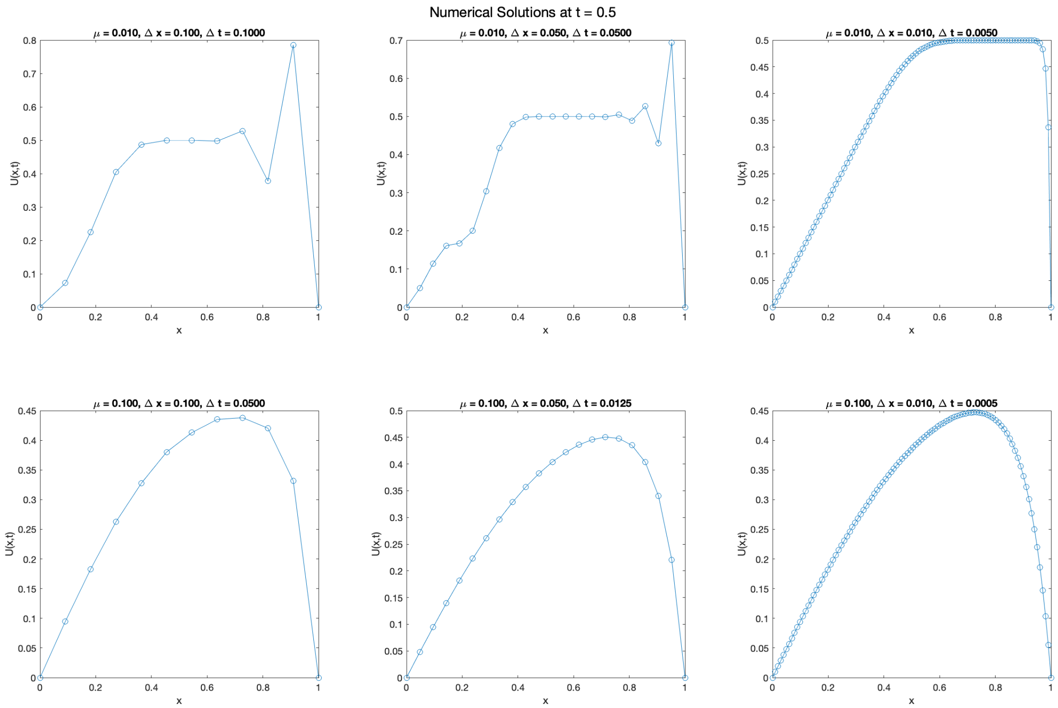
$$= u''(x_j) + O(\Delta x^2)$$

And again we can see that this is second order accuracy.

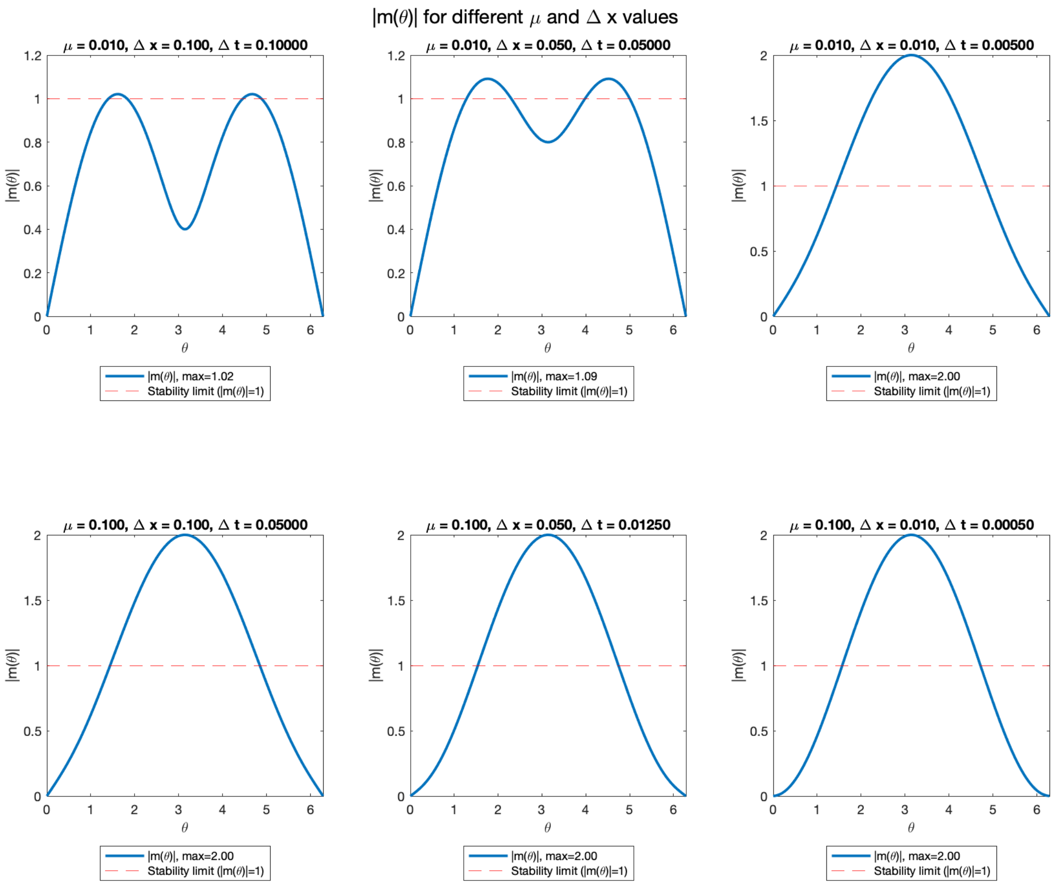
Because each discrete derivative gives us an error of $O(\Delta x^2)$, the total error at each grid point and time step is:

$$\Delta x \sum_{j=1}^{n} [u(x_j, t) - U_j(t)]^2 = O(\Delta x^4)$$

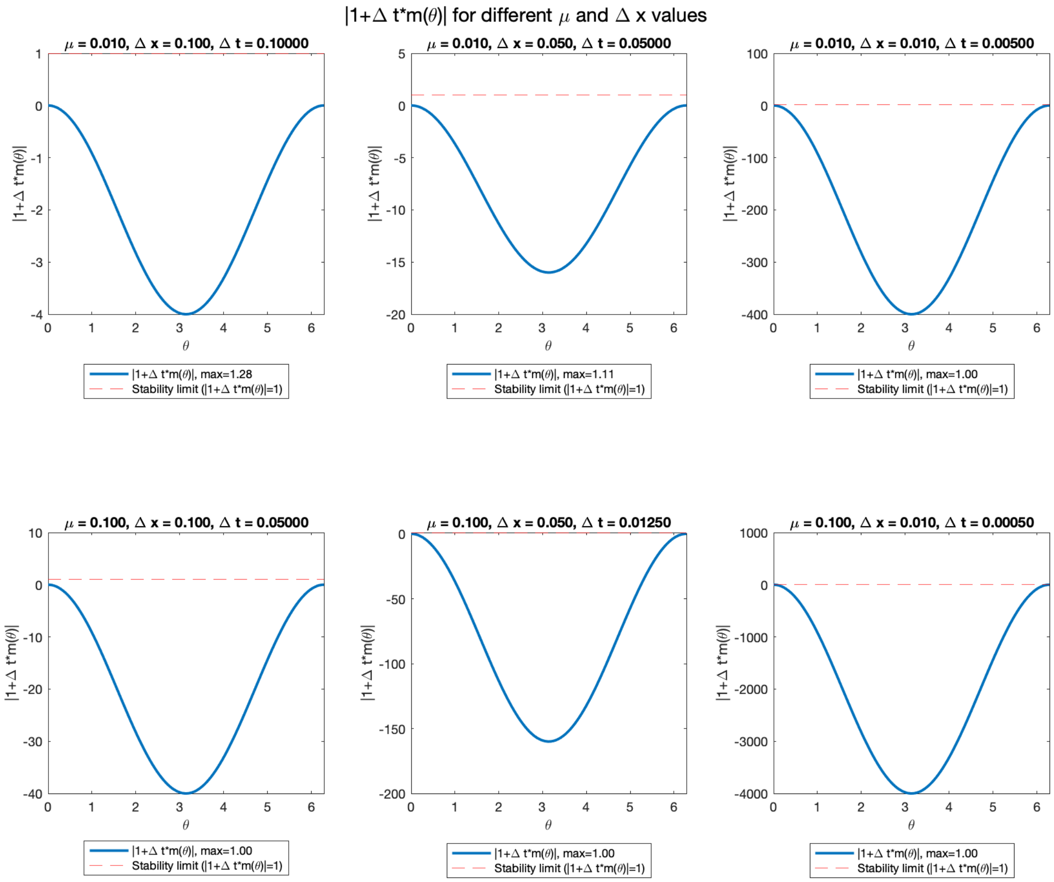
Exercise 2: Code see github page.
The numerical solutions using different μ, Δx, Δt are shown below.



My plot for $m(\theta)$ is shown below. The stability condition is set to be $max((m(\theta))) \leq 1$. Theoretically, none of the choices seem to be satisfying this condition. However, it may be too conservative and we will look at the amplification factor to better analyze the stability practically. In terms of whether it agrees with our formula of $m(\theta)$, we can see that the choices of μ, Δx, Δt can change the system stability.



To further analyze the error, we can also look at the amplification factor $G(\theta)$ given by $G(\theta) = 1 + \Delta t \cdot m(\theta)$ and the plot is shown below. The scheme is stable if $G(\theta) \leq 1$ for all θ, which is the same as $1 + \Delta t \cdot m(\theta) \leq 1$. We can see that our



Exercise 3: Movie see github page.
Initially, when μ small, the advection term dominates over the diffusion term, so the solution will primarily propagate in the direction of the advection term. Near the boundaries ($x = 0$ and $x = L$), there will be sharp gradients in u as the Dirichlet boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$ force the solution to be zero at the boundaries, while the source term $S(x) = 1$ wants to drive the solution upwards in the interior. Later, the solution will rise away from the boundary quickly because of the constant source term $S(x) = 1$. Near $x = 0$ and $x = L$, the sharp gradients or boundary layers form as the influence of the boundaries is countered by the source term and advection.

For the steady-state behavior, as $t \to \infty$, the solution reaches a steady state where the time derivative $\partial_t u = 0$. In this steady state, the solution profile in the interior (away from the boundaries) becomes relatively flat due to the dominance of the source term $S(x) = 1$. The boundary layers remain present at $x = 0$ and $x = L$.

Exercise 4:
The partly implicit scheme:

$$U^{(k+1)} = U^{(k)} + \Delta t \left[D_0 U^{(k)} + \mu D_+ D_- U^{(k+1)} + S_j \right]$$

We want to show that this scheme is first-order accurate when $\Delta t = \Delta x$ and $\mu = 1$. We know that the finite difference operators are defined as:

$$\begin{aligned} D_0 U_j^{(k)} &= \frac{U_{j+1}^{(k)} - U_{j-1}^{(k)}}{2\Delta x} \\ D_+ U_j^{(k)} &= \frac{U_{j+1}^{(k)} - U_j^{(k)}}{\Delta x} \\ D_- U_j^{(k)} &= \frac{U_j^{(k)} - U_{j-1}^{(k)}}{\Delta x} \end{aligned}$$

So the implicit diffusion term $\mu D_+ D_- U^{(k+1)}$ is:

$$\mu D_+ D_- U_j^{(k+1)} = \frac{U_{j+1}^{(k+1)} - 2U_j^{(k+1)} + U_{j-1}^{(k+1)}}{\Delta x^2}$$

Substituting these operators into the scheme, we have:

$$U_j^{(k+1)} = U_j^{(k)} + \Delta t \left[\frac{U_{j+1}^{(k)} - U_{j-1}^{(k)}}{2\Delta x} + \frac{U_{j+1}^{(k+1)} - 2U_j^{(k+1)} + U_{j-1}^{(k+1)}}{\Delta x^2} + S_j \right]$$

Rearranging gives us:

$$U_j^{(k+1)} - \frac{\Delta t}{\Delta x^2}(U_{j+1}^{(k+1)} - 2U_j^{(k+1)} + U_{j-1}^{(k+1)}) = U_j^{(k)} + \Delta t \left[\frac{U_{j+1}^{(k)} - U_{j-1}^{(k)}}{2\Delta x} + S_j \right]$$

For $\Delta t = \Delta x$:

$$U_j^{(k+1)} - \frac{1}{\Delta x}(U_{j+1}^{(k+1)} - 2U_j^{(k+1)} + U_{j-1}^{(k+1)}) = U_j^{(k)} + \Delta t \left[\frac{U_{j+1}^{(k)} - U_{j-1}^{(k)}}{2\Delta x} + S_j \right]$$

$$U_j^{(k+1)} - (U_{j+1}^{(k+1)} - 2U_j^{(k+1)} + U_{j-1}^{(k+1)}) = U_j^{(k)} + \Delta t \left[\frac{U_{j+1}^{(k)} - U_{j-1}^{(k)}}{2\Delta x} + S_j \right]$$

If we rewrite the equation in matrix form, we have:

$$\mathbf{A}\mathbf{U}^{(k+1)} = \mathbf{U}^{(k)} + \Delta t \left[\mathbf{D_0}\mathbf{U}^{(k)} + \mathbf{S} \right]$$

and $\mathbf{A}$ is a tridiagonal matrix representing the implicit part:

$$\mathbf{A} = \begin{bmatrix} 1+2\frac{\Delta t}{\Delta x^2} & -\frac{\Delta t}{\Delta x^2} & 0 & \cdots & 0 \\ -\frac{\Delta t}{\Delta x^2} & 1+2\frac{\Delta t}{\Delta x^2} & -\frac{\Delta t}{\Delta x^2} & \cdots & 0 \\ 0 & -\frac{\Delta t}{\Delta x^2} & 1+2\frac{\Delta t}{\Delta x^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\frac{\Delta t}{\Delta x^2} \end{bmatrix}$$

The residual should be of the order $O(\Delta t)$ or $O(\Delta x)$, indicating first-order accuracy. To determine the local truncation error, we expand the terms using Taylor series around the point (j, k).

$$U_{j+1}^{(k)} = U_j^{(k)} + \Delta x U_x + \frac{\Delta x^2}{2} U_{xx} + \frac{\Delta x^3}{6} U_{xxx} + O(\Delta x^4)$$
$$U_{j-1}^{(k)} = U_j^{(k)} - \Delta x U_x + \frac{\Delta x^2}{2} U_{xx} - \frac{\Delta x^3}{6} U_{xxx} + O(\Delta x^4)$$

Substituting these into the central difference term $D_0 U_j^{(k)}$:

$$\begin{aligned} D_0 U_j^{(k)} &= \frac{U_{j+1}^{(k)} - U_{j-1}^{(k)}}{2\Delta x} \\ &= \frac{\left(U_j^{(k)} + \Delta x U_x + \frac{\Delta x^2}{2} U_{xx} + \frac{\Delta x^3}{6} U_{xxx}\right) - \left(U_j^{(k)} - \Delta x U_x + \frac{\Delta x^2}{2} U_{xx} - \frac{\Delta x^3}{6} U_{xxx}\right)}{2\Delta x} \\ &= \frac{2\Delta x U_x + \frac{\Delta x^3}{3} U_{xxx}}{2\Delta x} \\ &= U_x + \frac{\Delta x^2}{6} U_{xxx} \end{aligned}$$

Then for the diffusion term:

$$U_{j+1}^{(k+1)} = U_j^{(k+1)} + \Delta x U_x^{(k+1)} + \frac{\Delta x^2}{2} U_{xx}^{(k+1)} + O(\Delta x^3)$$
$$U_{j-1}^{(k+1)} = U_j^{(k+1)} - \Delta x U_x^{(k+1)} + \frac{\Delta x^2}{2} U_{xx}^{(k+1)} + O(\Delta x^3)$$

Substituting these into the second derivative term $D_+D_-U_j^{(k+1)}$:

$$\begin{aligned}
D_+D_-U_j^{(k+1)} &= \frac{U_{j+1}^{(k+1)} - 2U_j^{(k+1)} + U_{j-1}^{(k+1)}}{\Delta x^2} \\
&= \frac{\left(U_j^{(k+1)} + \Delta x U_x^{(k+1)} + \frac{\Delta x^2}{2} U_{xx}^{(k+1)}\right) - 2U_j^{(k+1)} + \left(U_j^{(k+1)} - \Delta x U_x^{(k+1)} + \frac{\Delta x^2}{2} U_{xx}^{(k+1)}\right)}{\Delta x^2} \\
&= \frac{U_j^{(k+1)} + \Delta x U_x^{(k+1)} + \frac{\Delta x^2}{2} U_{xx}^{(k+1)} - 2U_j^{(k+1)} + U_j^{(k+1)} - \Delta x U_x^{(k+1)} + \frac{\Delta x^2}{2} U_{xx}^{(k+1)}}{\Delta x^2} \\
&= \frac{2\frac{\Delta x^2}{2} U_{xx}^{(k+1)}}{\Delta x^2} \\
&= U_{xx}^{(k+1)}
\end{aligned}$$

In the original scheme:

$$U_j^{(k+1)} = U_j^{(k)} + \Delta t \left(U_x + \frac{\Delta x^2}{6} U_{xxx} + U_{xx}^{(k+1)} + S_j \right)$$

Expanding $U_j^{(k+1)}$ using Taylor series at time $k+1$:

$$U_j^{(k+1)} = U_j^{(k)} + \Delta t U_t + \frac{\Delta t^2}{2} U_{tt} + O(\Delta t^3)$$

Since $\Delta t = \Delta x$ and assuming $\mu = 1$:

$$U_t + U_x + U_{xx} + S_j = 0$$

The residual is:

$$\text{Residual} = O(\Delta t^2) + O(\Delta x^3) = O(\Delta t)$$

Therefore the scheme is first-order accurate.

For Von Neumann stability analysis, we have:

$$D_0 U_j = \frac{i}{\Delta x}\sin(\theta)e^{i\theta j}, \quad D_+D_-U_j = \frac{2}{\Delta x^2}(\cos(\theta)-1)e^{i\theta j}$$

We assume an amplification factor $m(\theta)$ with $U_j^{(k)} = e^{i\theta j}$ such that the next time step $U_j^{(k+1)} = m(\theta)e^{i\theta j}$. Also, the source term should vary with the spatial domain, giving us $S_j e^{i\theta j}$. Substituting these into the scheme:

$$U_j^{(k+1)} = U_j^{(k)} + \Delta t\left[D_0 U_j^{(k)} + \mu D_+D_-U_j^{(k+1)} + S_j\right]$$

$$m(\theta)e^{i\theta j} = e^{i\theta j} + \Delta x\left[\frac{i}{\Delta x}\sin(\theta)e^{i\theta j} + \frac{2}{\Delta x^2}(\cos(\theta)-1)m(\theta)e^{i\theta j} + S_j e^{i\theta j}\right]$$
$$m(\theta) = 1 + \Delta x\left[\frac{i}{\Delta x}\sin(\theta) + \frac{2}{\Delta x^2}(\cos(\theta)-1)m(\theta) + S_j\right]$$

Solve for $m(\theta)$:

$$m(\theta) - \Delta x\frac{2}{\Delta x^2}(\cos(\theta)-1)m(\theta) = 1 + \Delta x\left[\frac{i}{\Delta x}\sin(\theta) + S_j\right]$$

$$m(\theta)\left[1 - \Delta x\frac{2}{\Delta x^2}(\cos(\theta)-1)\right] = 1 + \Delta x\left[\frac{i}{\Delta x}\sin(\theta) + S_j\right]$$

$$m(\theta) = \frac{1 + \Delta x\left[\frac{i}{\Delta x}\sin(\theta) + S_j\right]}{1 - \Delta x\frac{2}{\Delta x^2}(\cos(\theta)-1)}$$

Substituting $\Delta t = \Delta x$ and $\mu = 1$:

$$m(\theta) = \frac{1 + \Delta x\left[\frac{i}{\Delta x}\sin(\theta) + S_j\right]}{1 - \frac{2}{\Delta x}(\cos(\theta)-1)}$$
$$= \frac{1 + i\sin(\theta) + \Delta x S_j}{1 - 2(\cos(\theta)-1)}$$

Since $\Delta t = \Delta x$:

$$m(\theta) = \frac{1 + i\sin(\theta) + S_j\Delta x}{1 - 2(\cos(\theta)-1)}$$
$$= \frac{1 + i\sin(\theta) + S_j\Delta x}{3 - 2\cos(\theta)}$$

For stability, the magnitude of the amplification factor must satisfy $|m(\theta)| \leq 1$. We need to evaluate the magnitude of $m(\theta)$:

$$|m(\theta)| = \left|\frac{1 + i\sin(\theta) + S_j\Delta x}{3 - 2\cos(\theta)}\right|$$

Same as before, as long as we choose the appropriate Δx, Δt, and S_j, the scheme can be stable given appropriate conditions.

Exercise 5: Code see github page.

I tried to see if the computation is faster, but it doesn't seem so(if faster it's only faster by a little bit).

Exercise 6:
Using Runge-Kutta method, the advection equation is:

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

Using Crank-Nicolson method, the diffusion equation is:

$$\frac{\partial U}{\partial t} = \mu\frac{\partial^2 U}{\partial x^2}$$

For the advection part, use RK4:

$$U_j^{(k+1/2)} = U_j^{(k)}$$

We know that the RK4 scheme for the advection term $\frac{\partial U}{\partial x}$ is:

$$\begin{aligned}
k_1 &= -aD_0 U_j^{(k)} \\
k_2 &= -aD_0 \left(U_j^{(k)} + \frac{\Delta t}{2} k_1 \right) \\
k_3 &= -aD_0 \left(U_j^{(k)} + \frac{\Delta t}{2} k_2 \right) \\
k_4 &= -aD_0 \left(U_j^{(k)} + \Delta t k_3 \right) \\
U_j^{(k+1/2)} &= U_j^{(k)} + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

For the diffusive part, use Crank-Nicolson method:

$$\frac{U_j^{(k+1)} - U_j^{(k+1/2)}}{\Delta t/2} = \mu\frac{D_+D_-U_j^{(k+1)} + D_+D_-U_j^{(k+1/2)}}{2}$$

$$U_j^{(k+1)} = U_j^{(k+1/2)} + \frac{\Delta t\mu}{2}\left(D_+D_-U_j^{(k+1)} + D_+D_-U_j^{(k+1/2)} \right)$$

Written in matrix form:

$$\left(I - \frac{\Delta t\mu}{2} D_+D_- \right) U_j^{(k+1)} = \left(I + \frac{\Delta t\mu}{2} D_+D_- \right) U_j^{(k+1/2)}$$

The RK4 method is fourth-order accurate for the advection part, and the Crank-Nicolson method is second-order accurate for the diffusion part. Therefore, overall, using the combination of the two methods gives us second-order accuracy.

We can use Von Neumann stability analysis to check if the combined scheme is stable. As before, assume a Fourier mode $U_j^{(k)} = e^{i\theta j}$.

For the advection part, we already know that the amplification factor using RK4 is stable for appropriate Δt and Δx. For the diffusion part, we need to look at the eigenvalues of the operator.

The amplification factor G for the diffusion term is:

$$\left(1+\frac{\Delta t\mu}{2}\frac{2(\cos(\theta)-1)}{\Delta x^2}\right)G = 1-\frac{\Delta t\mu}{2}\frac{2(\cos(\theta)-1)}{\Delta x^2}$$

$$G = \frac{1-\frac{\Delta t\mu}{\Delta x^2}(\cos(\theta)-1)}{1+\frac{\Delta t\mu}{\Delta x^2}(\cos(\theta)-1)}$$

$$|G| = \left|\frac{1-\frac{\Delta t\mu}{\Delta x^2}(\cos(\theta)-1)}{1+\frac{\Delta t\mu}{\Delta x^2}(\cos(\theta)-1)}\right| \leq 1$$

Since $|G| \leq 1$, we know that the diffusion part is stable. Therefore, this split scheme is stable.