1. Proof:

Assume A is a lower triangular matrix

$\therefore A \in C^{m\times m}$

$$A = \begin{bmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

$$\therefore A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ & \bar{a}_{22} & & \\ & & \ddots & \\ 0 & & & \bar{a}_{mm} \end{bmatrix}$$

$$\therefore AA^* = \begin{bmatrix} a_{11}\bar{a}_{11} & a_{11}\bar{a}_{21} & \cdots & a_{11}\bar{a}_{m1} \\ a_{21}\bar{a}_{11} & a_{21}\bar{a}_{21}+a_{22}\bar{a}_{22} & \cdots & a_{21}\bar{a}_{m1}+a_{22}\cdot\bar{a}_{m2} \\ & & \ddots & \\ & & & \sum_{i=1}^{m} a_{mi}\,\overline{a_{mi}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

To satisfy above eqn $\therefore \forall i \neq j, a_{ij} = 0$

$\therefore$ Matrix A is diagonal

2.1 $A = A^H$

$A \cdot \vec{v} = \lambda \vec{v}$

$\Rightarrow \vec{v}^H A \vec{v} = \vec{v}^H \lambda \vec{v}$

$\therefore \vec{v}^H A \vec{v} = \lambda \vec{v}^H \vec{v}$ ①

Take hermitian on both sides

$(\vec{v}^H A \vec{v})^H = (\lambda \vec{v}^H \vec{v})^H$

$\vec{v}^H A^H \vec{v} = \lambda^H \vec{v}^H \vec{v}$ ②

$\because$ The left sides of ①, ② are equal

$\therefore \lambda \vec{v}^H \vec{v} = \lambda^H \vec{v}^H \vec{v}$

$\therefore (\lambda - \lambda^H) \vec{v}^H \vec{v} = 0$

$\therefore \lambda = \lambda^H$

All eigenvalues of A are real.

2.2 Two distinct eigenvalues of A are λ, μ

The eigenvectors are $\vec{v}, \vec{w}$ respectively.

$\therefore A\vec{v} = \lambda\vec{v}$

$A\vec{w} = \mu\vec{w}$

$\because A = A^*$

$\Rightarrow \vec{v}^* A\vec{w} = \vec{v}^* A^* \vec{w}$ ①

LHS of ① $= \vec{v}^*(A\vec{w}) = \vec{v}^*(\mu\vec{w}) = \mu\vec{v}^*\vec{w}$ ②

RHS of ① $= \vec{v}^* A^*\vec{w} = (A\vec{v})^*\vec{w} = (\lambda\vec{v})^*\vec{w} = \lambda(\vec{v}^*\vec{w})$ ③

$\Rightarrow \mu\vec{v}^*\vec{w} = \lambda\vec{v}^*\vec{w}$

$\Rightarrow (\lambda-\mu)\vec{v}^*\vec{w} = 0$

$\because \lambda, \mu$ are two distinct eigenvalues

$\therefore \lambda - \mu \neq 0$

$\therefore \vec{v}^*\vec{w} = 0$

$\therefore \vec{v}, \vec{w}$ are orthogonal to each other.

2.3 Let a_{ij}, b_{ij}, c_{ij} be the ijth elements of A, B, C

Assume $C = A + B$

$$a_{ij} = \overline{a_{ji}}$$

$$b_{ij} = \overline{b_{ji}}$$

$$\therefore c_{ij} = a_{ij} + b_{ij}$$

$$= \overline{a_{ji}} + \overline{b_{ji}}$$

$$= \overline{(a_{ji} + b_{ji})} = \overline{c_{ji}}$$

2.4 $$AA^{-1} = A^H A^{-1} = I$$

LHS, RHS are mulitipled with $(A^{-1})^H$

$$(A^{-1})^H A^H A^{-1} = (A^{-1})^H A A^{-1}$$

$$(A \cdot A^{-1})^H A^{-1} = (A^{-1})^H \cdot I$$

$$\Rightarrow A^{-1} = (A^{-1})^H$$

2.5 $$(AB)^H = \overline{(AB)^T} = \overline{B^T A^T} = B^H A^H$$

$$= BA$$

$(AB)^H = AB$ only if $BA = AB$

3.1 proof:

Matrix U is unitary, $\Rightarrow U^*U = UU^* = E$ ①

By schur decomposition, $U = ATA^*$,

A is unitary, and T is upper triangular matrix

From ①

$$(ATA^*)^* \cdot ATA^* = ATA^* \cdot (ATA^*)^*$$

$$AT^* \cdot \cancel{A^*A}TA^* = AT\cancel{A^* \cdot A}T^* \cdot A^*$$

$$\Rightarrow AT^* \cdot TA^* = ATT^* \cdot A^*, \quad T^*T = T^*T$$

$$\therefore T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ & t_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & t_{mm} \end{bmatrix} \quad T^* = \begin{bmatrix} \overline{t_{11}} & & & \\ \overline{t_{12}} & \overline{t_{22}} & & 0 \\ \vdots & & \ddots & \\ \overline{t_{1m}} & & & \overline{t_{mm}} \end{bmatrix}$$

$$TT^* = \begin{bmatrix} t_{11}\overline{t_{11}} & t_{12}\overline{t_{11}} & \cdots & t_{1m}\overline{t_{11}} \\ t_{11}\overline{t_{12}} & \cdots\cdots & & t_{1m}\overline{t_{12}} + t_{2m}\overline{t_{22}} \\ \vdots & & \ddots & \\ t_{11}\overline{t_{1m}} & \cdots & & \sum_{i=1}^{m} t_{im}\overline{t_{im}} \end{bmatrix} \quad TT^* = \begin{bmatrix} \sum_{i=1}^{m} t_{1m}\overline{t_{1m}} & \cdots & t_{1m}\overline{t_{mm}} \\ \vdots & & \\ t_{mm}\overline{t_{1m}} & \cdots & t_{mm}\overline{t_{mm}} \end{bmatrix}$$

$$\therefore \sum_{i=1}^{m} t_{i,m-1}\overline{t_{i,m-1}} = t_{m-1,m-1}\overline{t_{m-1,m-1}} + t_{m-1,m}\overline{t_{m-1,m}}$$

$$\sum_{i=1}^{m} t_{im}\overline{t_{im}} = t_{mm}\overline{t_{mm}} \Rightarrow t_{im} = 0, \ \forall i \neq m$$

Similarly: $t_{i,m-1} = 0$ for $\forall i \neq m-1$

$\therefore \forall i \neq j \quad t_{ij} = 0$

$\therefore$ Matrix T is diagonal.

$\therefore$ $U = ATA^*$ is diagonalizable.

32. proof.

$$UU^* = I$$

$$\Rightarrow U^{-1}UU^* = U^{-1}$$

$$\Rightarrow U^* = U^{-1}$$

33. $\langle Ux, Ux \rangle = (Ux)^* \cdot Ux$

$$= x^* \cdot U^* \cdot Ux$$

$$= x^* \cdot x = \langle x, x \rangle$$

3.4 Let λ be the eigenvalue of U with associated eigenvector $\vec{v} \neq 0$

$$\therefore \langle \vec{v}, \vec{v} \rangle = \langle U U^H \vec{v}, \vec{v} \rangle$$

$$= \langle U\vec{v}, U\vec{v} \rangle$$

$$= \langle \lambda \vec{v}, \lambda \vec{v} \rangle$$

$$= \lambda \cdot \bar{\lambda} \langle \vec{v}, \vec{v} \rangle$$

$$\Rightarrow (1 - \lambda \bar{\lambda}) \langle \vec{v}, \vec{v} \rangle = 0$$

$$\because \vec{v} \neq 0$$

$$\therefore \langle \vec{v}, \vec{v} \rangle \neq 0$$

$$\therefore 1 - \lambda \bar{\lambda} = 0$$

$$\Rightarrow \lambda \bar{\lambda} = 0$$

$$\Rightarrow |\lambda|^2 = 1$$

$$\Rightarrow |\lambda| = 1$$