MATH 235 Linear Algebra 2

University of Waterloo - Spring 2024

24th June 2024

Contents

1 Abstract Vector Spaces

1.1 The Definition of a Vector Space

Lecture 1

Before we formally define a vector space, let's introduce some examples of vector spaces.

Example (Vector Spaces).

1. $V = \mathbb{R}^2$. We have $\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ and $c\vec{x} = c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$.
2. $V = \mathcal{P}_n(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R}\}$ = polynomiasl of degree $\leq n$ with coefficients in $\mathbb{R}$.

   We can add: $(a_0 + \cdots + a_n x^n) + (b_0 + \cdots + b_n x^n) = (a_0 + b_0) + \cdots + (a_n + b_n)x^n$.

   We can scale: $c(a_0 + a_1 + \cdots + a_n x^n) = (ca_0) + (ca_1)x + \cdots + (ca_n)x^n$.
3. $V = M_{m\times n}(\mathbb{R}) = \left\{ \begin{bmatrix} \\ & a_{ij} & \\ \\ \end{bmatrix} : a_{ij} \in \mathbb{F} \right\}$.

   Add: $\begin{bmatrix} a_{ij} \end{bmatrix} + \begin{bmatrix} b_{ij} \end{bmatrix} = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}$ AND Scale: $c\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} ca_{ij} \end{bmatrix}$.

   Oberservation: $\mathbb{R}^4$ vs. $\mathcal{P}_3(\mathbb{R})$ vs. $M_{2\times 2}(\mathbb{R})$. These are essentially the same. This is an example of an isomorphism (identical form). In general, $\mathbb{R}^{mn} \cong \mathcal{P}_{mn-1}(\mathbb{R}) \cong M_{m\times n}(\mathbb{R})$.
4. $V = \{\text{functions } f : \mathbb{R} \to \mathbb{R}\}, f, g \in V$. We can add them to get $(f+g)(x) + f(x) + g(x)$ and we can scale by $c \in \mathbb{R}$ to get $(cf)(x) = c(f(x))$.
5. $C = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}$. If $f, g \in V$, then $f + g \in V$. If $c \in \mathbb{R}, f \in V$, then $cf \in V$. This is a subspace of V.
6. $D = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is infinitely differentiable}\}$ is also a subspace of V.

Note. The vector spaces in example 5 and example 6 are **isomorphic**. But example 4 and 6 & example 5 and 6 are both not.

Definition (Vector Space). A vector space over $\mathbb{F}$ is a set V together with $+ : V \times V \to V$ (vector addition) so that $\forall \vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$ and an operation $\cdot : \mathbb{F} \times V \to V$ (scalar multiplication) so that $\forall c \in \mathbb{F}$ and $\vec{x} \in V$, $c \cdot \vec{x} \in V$.

These operations must satisfy the following **vector space axioms**:

1. $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$.
2. $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$.
3. $\forall \vec{x}, \vec{y} \in V$ and $c \in \mathbb{F}, c \cdot (\vec{x} + \vec{y}) = c \cdot \vec{x} + c \cdot \vec{y}$.
4. $\forall \vec{x} \in V$ and $c, d \in \mathbb{F}, (c + d) \cdot \vec{x} = c \cdot \vec{x} + d \cdot \vec{x}$.
5. $\forall \vec{x} \in V$ and $c, d \in \mathbb{F}, (cd) \cdot \vec{x} = c \cdot (d \cdot \vec{x})$.
6. $\forall \vec{x} \in V, 1 \cdot \vec{x} = \vec{x}$.
7. $\exists \vec{0} \in V$ s.t. $\forall \vec{x} \in V, \vec{0} + \vec{x} = \vec{x} + \vec{0} = \vec{x}$.
8. $\forall \vec{x} \in V, \exists (-\vec{x}) \in V$ s.t. $\vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = \vec{0}$.

Remark.

1. There is a unique $\vec{0}$ as in axiom 7.
2. $\forall \vec{x}$, there is a unique $-\vec{x}$ as in axiom 8.

Lecture 2

Example. $V = \mathbb{F}^n, \mathcal{P}_n(\mathbb{F}), M_{m\times n}(\mathbb{F})$ with usual $+$ and $\cdot$ are vector spaces.

In $\mathbb{F}^n$, $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$; in $\mathcal{P}_n(\mathbb{F})$, $\vec{0} = 0 + 0x + \cdots + 0x^n$; and in $M_{m\times n}(\mathbb{F})$, $\vec{0} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$.

Example. The following is NOT a vector space!

$V = \mathbb{R}^2$, define $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_2 \\ x_2 + y_1 \end{bmatrix}$, and $c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_2 \\ cx_1 \end{bmatrix}$. Notice that multiple axioms fail, for example, in this case $1 \cdot \vec{x} \neq \vec{x}$.

Example. $V = \mathbb{R}^2$, define $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{bmatrix}$, and $c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 - c + 1 \\ cx_2 - c + 1 \end{bmatrix}$.

Claim: V is a vector space.

Check the multiplicative identity: $1 \cdot \vec{x} = \begin{bmatrix} x_1 - 1 + 1 \\ x_2 - 1 + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}$. ✓

Find the zero vector: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \iff \begin{bmatrix} x_1 + z_1 - 1 \\ x_2 + z_2 - 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Therefore, $\vec{0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ✓

Find the additive inverse: let $\vec{x} + \vec{y} = \vec{0}$.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \iff \begin{bmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore. $-\vec{x} = \vec{y} = \begin{bmatrix} 2 - x_1 \\ 2 - x_2 \end{bmatrix}$. ✓

Proposition. Let V be a vector space over $\mathbb{F}$. Then

(a) $\vec{0} \in V$ is *unique*.

(b) Let $\vec{x} \in V$. The additive inverse of $\vec{x}$ is *uniquely determined* by $\vec{x}$.

(c) $\forall \vec{x} \in V, 0 \cdot \vec{x} = \vec{0}$.

(d) $\forall \vec{x} \in V, (-1) \cdot \vec{x} = -\vec{x}$.

Remark. We can use part (c) and part (d) to find the zero vector and the additive inverse, respectively. To find $\vec{0}$, scale by 0. To find $-\vec{x}$, scale by -1.

Example (**Above example continued**).

To find $\vec{0}$: $\quad 0 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0x_1 - 0 + 1 \\ 0x_2 - 0 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find $-\vec{x}$: $\quad (-1) \cdot \vec{x} = \begin{bmatrix} -1x_1 - (-1) + 1 \\ -1x_2 - (-1) + 1 \end{bmatrix} = \begin{bmatrix} 2 - x_1 \\ 2 - x_2 \end{bmatrix}$.

Proof.

For part (a), Suppose $\vec{0}_1$ and $\vec{0}_2$ satisfy $\vec{x} + \vec{0}_1 = \vec{x}$ and $\vec{x} + \vec{0}_2 = \vec{x}, \forall \vec{x} \in V$. Plug in $\vec{x} = \vec{0}_2$, then $\vec{0}_2 + \vec{0}_1 = \vec{0}_2$. Similarly, plug in $\vec{x} = \vec{0}_1$ is a zero vector, we have $\vec{0}_1 + \vec{0}_2 = \vec{0}_1$. This implies that $\vec{0}_1 = \vec{0}_2$.

For part (b), let $\vec{x}, \vec{y} \in V$ be such that $\vec{x} + \vec{y} = \vec{y} + \vec{x} = \vec{0}$. By a vector space axiom, $\exists(-\vec{x}) \in V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$.

$$\begin{aligned} &\implies \vec{x} + (-\vec{x}) = \vec{x} + \vec{y} \\ &\implies \vec{y} + (\vec{x} + (-\vec{x})) = \vec{y} + (\vec{x} + \vec{y}) \\ &\implies (\vec{y} + \vec{x}) + (-\vec{x}) = (\vec{y} + \vec{x}) + \vec{y} \\ &\implies \vec{0} + (-\vec{x}) = \vec{0} + \vec{y}. \end{aligned}$$

Hence, $-\vec{x} = \vec{y}$.

For part (c), we have $0 \cdot \vec{x} = (0+0) \cdot \vec{x} = 0 \cdot \vec{x} + 0 \cdot \vec{x}$. Then, add the additive inverse $-(0 \cdot \vec{x})$ to both sides:

$$\begin{aligned} -(0 \cdot \vec{x}) + 0 \cdot \vec{x} &= -(0 \cdot \vec{x}) + 0 \cdot \vec{x} + 0 \cdot \\ \implies \vec{0} &= \vec{0} + 0 \cdot \vec{x} \\ \implies \vec{0} &= 0 \cdot \vec{x}. \end{aligned}$$

For part (d), add $\vec{x}$ to the LHS, we have

$$\vec{x} + (-1) \cdot \vec{x} = 1 \cdot \vec{x} + (-1) \cdot \vec{x} = (1 + (-1)) \cdot \vec{x} = 0 \cdot \vec{x}.$$

Using part (c), we can conclude that $\vec{x} + (-1) \cdot \vec{x} = \vec{x} + (-\vec{x}) = \vec{0}$. □

Lecture 3

1.2 Subspaces

Definition (**Subspace**). Let V be a vector space over $\mathbb{F}$ and $W \subseteq V$ a subset. We call W a **subspace** of V if W, using the same addition and scalar multiplication from V, is itself a vector space over $\mathbb{F}$.

Theorem (**The Subspace Test**).
Let V be a vector space over $\mathbb{F}$ and let W be a subset of V. Then W is a subspace of V if and only if the following three conditions hold.

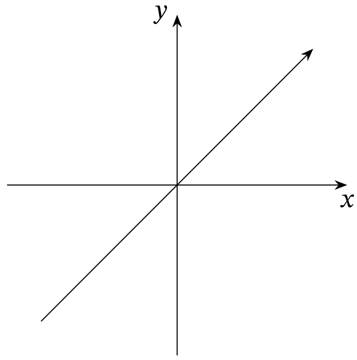
(a) W is non-empty OR $\vec{0}_V \in W$.

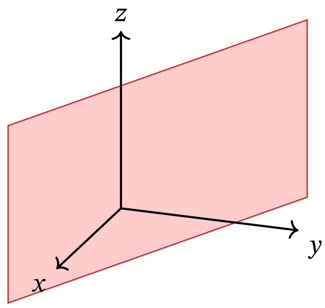
(b) $\forall \vec{w}_1, \vec{w}_2 \in W$, we have $\vec{w}_1 + \vec{w}_2 \in W$. (closed under addition)

(c) $\forall c \in \mathbb{F}$ and $\forall \vec{w} \in W$, we have $c\vec{w} \in W$. (closed under scalar multiplication)

Remark.

- When seeing ”subspace”, we should visualize a line through the origin for $\mathbb{R}^2$ and possibly a plane through the origin for $\mathbb{R}^3$:





- Let V be a vector space. Suppose that W is a subspace of V, then they have the same zero vector (i.e. $\vec{0}_V = \vec{0}_W$).

Example. $W = \{p(x) \in \mathcal{P}_n(\mathbb{F}) : p(2) = p(3)\}$.

This is a subspace. Using the subspace test:

1. $\vec{0} = 0 + 0x + \cdots + 0x^n$.
2. Let $p(x), q(x) \in W$. We have $p(2) + q(2) = p(3) + q(3)$, therefore closed under $+$, $p + q \in W$.
3. Let $p(x) \in W, c \in \mathbb{F}$. We have $(cp)(x) = c(p(x))$.
   $\implies (cp)(2) = c(p(2)) = c(p(3)) = (cp)(3)$. Thus, closed under $\cdot$, $cp \in W$.

Example. $W = \{A \in M_{2\times 2}(\mathbb{F}) : \operatorname{tr}(A) = 0\}$.

This is a subspace.

1. $\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$, so $\vec{0} \in W$.
2. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$, therefore closed under $+$.
3. $\operatorname{tr}(cA) = c \operatorname{tr}(A)$, therefore closed under $\cdot$.

Example (Not a subspace). $W = \left\{A \in M_{n\times n}(\mathbb{F}) : A^2 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right\}$.

Let $A = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix} \in W$. Notice that W is not closed under $+$ since $A + B \notin W$.
Thus not a subspace.

Example. $W = \{p(x) \in \mathcal{P}_2(\mathbb{R}) : p'(1) = 0\}$.

The intuition used in this example is important. Let $p(x) = a + bx + cx^2$. Then $p'(x) = b + 2cx$.

$\implies p'(1) = b + 2c = 0$ (essentially solving for nullspace). Thus W is a subspace.

(Since c depends on b, so the degree of freedom reduced by 1, it's 2 now. The dimension is 2, so W is a subspace of $\mathcal{P}_2(\mathbb{R})$).

Example (Not a subspace). $W = \{p(x) \in \mathcal{P}_2(\mathbb{R}) : p'(1) = 4\}$.

Similar to the previous example. Here we have $p'(1) = b + 2c = 4$. Not a nullspace and hence not a subspace.

Corollary. Let V be a vector space over $\mathbb{F}$ and suppose that U is a subspace of V. Let $\vec{0}_V$ and $\vec{0}_U$ denote the zero vectors in V and U, respectively. Then $\vec{0}_U = \vec{0}_V$. In particular, the zero vector in V is in U: $\vec{0}_V \in U$.

Lecture 4

Example. $V = M_{2\times 2}(\mathbb{R})$, $W = \left\{A \in V : A^T = -A\right\}$. Check: W is a subspace of V (matrices satisfy $A^T = -A$ are skew symmetric).

Solution:

$$\begin{aligned} A \in W \iff A^T = -A &\iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &\iff a = -a,\ c = -b,\ b = -c,\ d = -d \\ &\iff a = 0,\ d = 0,\ b = -c \\ &\iff A = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $W = \left\{\begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} : c \in \mathbb{R}\right\} = \left\{c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : c \in \mathbb{R}\right\}$.

Definition (Span, Linear Combination). Let $S = \{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq V$. The span of S is

$$\mathrm{Span}\,(S) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k : a_i \in \mathbb{F}\}.$$

A vector of the form $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$ is called a **linear combination** of the vectors $\vec{v}_1, \dots, \vec{v}_k$. By convention, we define Span $\varnothing = \{\vec{0}\}$.

Proposition. Let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a subset of a vector space V. Then $W = \operatorname{Span}(S)$ is a subspace of V.

Proof. Apply the Subspace Test.

1. $\vec{0} = 0\vec{v}_1 + \cdots + 0k \in W$. ✓
2. Let $\vec{w}_1, \vec{w}_2 \in W = \operatorname{Span}(S)$. Then $\vec{w}_1 = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$ and $\vec{w}_2 = b_1\vec{v}_1 + \cdots + b_k\vec{v}_k$. $\implies \vec{w}_1 + \vec{w}_2 = (a_1 + b_1)\vec{v}_1 + \cdots + (a_k + b_k)\vec{v}_k \in W$. ✓
3. Similar to above. ✓

Thus, W is a subspace of V. □

Example. $V = \mathcal{P}_2(\mathbb{R})$, $W = \{p(x) \in V : p'(1) = p(-1)\}$. Let's show that $W = \operatorname{Span}(?)$.

$$\begin{aligned} p(x) = a + bx + cx^2 \in W &\iff p'(1) = p(-1) \quad (p'(x) = b + 2cx) \\ &\iff b + 2c = a - b + c \\ &\iff a = 2b + c \\ &\iff p(x) = (2b + c) + bx + cx^2 = b(2 + x) + c(1 + x^2). \end{aligned}$$

So, we obtain $W = \operatorname{Span}\{2 + x, 1 + x^2\}$.

1.3 Bases and Dimension

Definition (Spanning Set, Spans).
A set of vectors $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ in a vector space V is a **spanning set** for V if $\operatorname{Span}(S) = V$. We also say that S **spans** V.

Example. $V = M_{2\times 2}(\mathbb{F})$, $W = \{A \in V : \operatorname{tr}(A) = 0\}$. Find a finite spanning set for W.

Solution: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W \iff a + d = 0 \iff d = -a.$

$\therefore A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Hence, $W = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$.

Example. Find a spanning set for $W = \left\{A \in M_{2\times 2}(\mathbb{F}) : A\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}A\right\}$. Guess the number of vectors in the spanning set before doing it!

Example. In $\mathcal{P}_1(\mathbb{R})$,

$$\begin{aligned} W = \text{Span}\{1, 1+x\} &= \{a + b(1+x) : a, b \in \mathbb{F}\} \\ &= \{(a+b) + bx : a, b \in \mathbb{F}\} \\ &= \{c + dx : c, d \in \mathbb{F}\}. \end{aligned}$$

In the last equality, the direction $\subseteq$ is obvious.
For $\supseteq$: Given $c + dx$, find (if possible) a & b s.t. $c + dx = (a+b) + bx \iff b = d$.
So $c = a + b \implies a = c - d$.
$\therefore (a+b) + bx = c + dx = (c - d + d) + dx$. Thus $\mathcal{P}_1(\mathbb{F}) = \text{Span}\{1, 1+x\} = \text{Span}\{1, x\}$.

Lecture 5

Definition (Linearly Independent, Linearly Dependent).
A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in a vector space V is **linearly independent** if the only solution to the equation

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$$

is $t_1 = \cdots = t_k = 0$. The set is **linearly dependent** otherwise.
By convention, the empty set $\varnothing$ is linearly independent.

Remark. We can think of a linearly independent set to be a set of vectors that ”point in different directions” so that no vectors in the set is ”redundant”.

Proposition. A subset $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ of vector space V is **linearly dependent** $\iff$ at least one vector in S is a linear combination of other vectors in S.

Proof. For the forward direction ($\Rightarrow$): assume that S ia linearly dependent. Then, there is a solution to $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ with some $c_i \neq 0$.

$$\implies \sum_{j\neq i} c_j\vec{v}_j = (-c_i\vec{v}_i)$$
$$\implies \vec{v}_i = \frac{\sum_{j\neq i} c_j\vec{v}_j}{-c_i}$$

which is a linear combination of the other vectors in S.

For the backward direction ($\Leftarrow$): assume that one vector in S, say $\vec{v}_i$ is a linear combination of other vectors. Then, we have $\vec{v}_i = \sum_{j\neq i} d_j\vec{v}_j$.

$$\implies d_1\vec{v}_1 + \cdots + (-1)\vec{v}_i + \cdots + d_k\vec{v}_k = \vec{0}.$$

We found a solution to $d_1\vec{v}_1 + \cdots + d_k\vec{v}_k = \vec{0}$ with some $d_i \neq 0$. Thus S is linearly dependent. $\square$

Example. $V = \mathcal{P}_2(\mathbb{R})$, $S = \{1-x, 1+x, 1-x^2, 1+x^2\}$. Is S linearly independent?
Solution: Consider the equation $c_1(1-x)+c_2(1+x)+c_3(1-x^2)+c_4(1+x^2) = \vec{0} = 0+0x+0x^2$. Equate the coefficients of 1, x, x^2:

$$\begin{cases} c_1 - c_2 + c_3 + c_4 &= 0 \\ -c_1 + c_2 &= 0 \\ -c_3 + c_4 &= 0 \end{cases} \iff A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{aligned} S \text{ is linearly independent} &\iff \text{The only solution is the trivial solution.} \\ &\iff \text{nullity}(A) = 0 \\ &\iff \text{rank}(A) = 4 \end{aligned}$$

However, rank $(A) \leq 3$, since A is 3×4. So S is linearly dependent.

Alternate solution: we can solve the system to get $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

Example. $V = \mathcal{P}_2(\mathbb{R})$, $S = \{1+x, 1+x^2\}$. Does S span V?

Solution: To show $V = \operatorname{Span}(S)$, we must show that $V \subseteq \operatorname{Span}(S)$ and $V \supseteq \operatorname{Span}(S)$.

Notice that $V \supseteq \operatorname{Span}(S)$ is obvious. For $V \subseteq \operatorname{Span}(S)$, we must show that every $a + bx + cx^2 \in \operatorname{Span}(S)$.

$\therefore$ Solve $a + bx + cx^2 = c_1(1+x) + c_2(1+x^2)$ for c_1, c_2.

Equate the coefficients of $1, x, x^2$:

$$\begin{cases} a &= c_1 + c_2 \\ b &= c_1 \\ c &= c_2 \end{cases} \iff A = \left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{array}\right]$$

$$\begin{aligned} S \text{ spans } V &\iff \text{The above system has a solution } \forall a, b, c \\ &\iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \operatorname{Col}(A) \ \forall a, b, c \\ &\iff \operatorname{Col}(A) = \mathbb{R}^3 \\ &\iff \dim(\operatorname{Col}(A)) = 3 \\ &\iff \operatorname{rank}(A) = 3. \end{aligned}$$

However, $\operatorname{rank}(A) \leq 2$ since A is 3×2. So S cannot span V.

Definition (Basis). Let V be a vector space. A **basis** for V is a set $B \subseteq V$ that

(1) is linearly independent and

(2) spans V.

Remark. A basis is essentially the smallest set that can build your vector space.

Example (Bases).

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$. The standard basis for $\mathbb{F}^n$.
2. $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$. The standard basis for $\mathcal{P}_n(\mathbb{F})$.

3. $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. The standard basis for $M_{2\times 2}(\mathbb{F})$.

4. $\mathcal{B} = \{E_{11}, E_{12}, \ldots, E_{ij}, \ldots, E_{mn}\}$, where $E_{ij} = \begin{cases} 1, & \text{in } (i,j) \text{ entry} \\ 0, & \text{else} \end{cases}$ is an $m \times n$ matrix. The standard basis for $M_{m\times n}(\mathbb{F})$.

5. $\varnothing$ is a basis for the zero vector space $\{\vec{0}\}$.

Remark. **(WARNING!!!)** A general vector space does **NOT** have a standard basis.

Lecture 6

Example (**More bases**).

1. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

2. $\{1 + 2x, 3 + 3x\}$ is a basis for $\mathcal{P}_1(\mathbb{R})$ (exercise).

3. Let $W = \{A \in M_{2\times 2}(\mathbb{F}) : \operatorname{tr}(A) = 0\}$ in the previous lecture, we found a spanning set: $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. We check that $\mathcal{B}$ is linearly independent:
   Consider $c_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This implies that $\begin{bmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, $c_1 = c_2 = c_3 = 0$, so $\mathcal{B}$ is linearly independent and hence a basis for W.

Theorem. Every vector space has a basis. We will prove a special case.

Remark. This result is DIFFICULT to prove.

Example. Consider $V = C([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous on } [0,1]\}$, the vector space of real-valued continuous functions on $[0,1]$. The basis for V exists but we cannot write it down.

Lemma. Let V be a vector space over $\mathbb{F}$ and suppose that $V = \operatorname{Span}(\{\vec{v}_1, \dots, \vec{v}_n\})$. If $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a linearly Independent set in V, then $k \leq n$.

Definition (Finite-dimensional, Infinite-dimensional). A vector space V is **finite-dimensional** if it has a finite spanning set (basis). Otherwise, we say the vector space is **infinite-dimensional**, we write $\dim(V) = \infty$.

Example.

1. $\mathbb{F}^n$, $M_{m\times n}(\mathbb{F})$, $\mathcal{P}_n(\mathbb{F})$ and W from the previous example are finite-dimensional.
2. $C([0,1])$ is infinite-dimensional (exercise).

Theorem. Every finite-dimensional vector space has a basis.

Proof. V has a finite spanning set $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ if S is linearly independent - Done!
Suppose S is linearly dependent. Then some vector, say $\vec{v}_k$, is a linear combination of the other vectors. Delete $\vec{v}_k$ from S to create $S' = \{\vec{v}_1, \dots, \vec{v}_{k-1}\}$. Notice that $\operatorname{Span}(S') = \operatorname{Span}(S) = V$. Repeat the process until you find a basis. □

Remark.

1. This proof gives an algorithm for finding a basis:
   - start with a spanning set
   - remove linearly dependent vectors one at a time until you get a linearly independent subset.
2. Convention: $\varnothing$ is lienarly independent, and is a basis for the zero vector space $\{\vec{0}\}$.

Definition (Dimension). If V is finite-dimensional, we define $\dim(V)$ to be the size of any finite basis of V.

Note. This definition is broken: we need to know that any two bases for V have the same size.

Example.

1. $\dim(\mathbb{F}^n) = n$, because its standard basis has size n.
2. $\dim(M_{m\times n}(\mathbb{F})) = mn$.
3. $\dim(\mathcal{P}_n(\mathbb{F})) = n + 1$.
4. $\dim(\{\vec{0}\}) = 0$.
5. $\dim(C([0,1])) = \infty$.

Proposition. Let V be an n-dimensional vector space over $\mathbb{F}$. Assume that V has a basis $\mathcal{B}$ of size n. Let $S \subseteq V$ of size k. Then

(a) If $k > n$, then S is linearly dependent.

(b) If $k < n$, then S cannot span V.

(c) If $k = n$, then S is linearly independent $\iff$ $\mathrm{Span}(S) = V$.

Corollary. Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\mathcal{C} = \{\vec{u}_1, \dots, \vec{u}_k\}$ are both bases of a vector space V. Then $k = n$ (or $|\mathcal{B}| = |\mathcal{C}|$).

Remark. If $\mathcal{B}$ is a finite-dimensional, then $|\mathcal{B}| =$ the size of $\mathcal{B}$.

Proof. Use the previous proposition with $S = \mathcal{C}$. Since $\mathcal{C}$ is lienarly independent, then $|\mathcal{C}| \le |\mathcal{B}|$ and since $\mathcal{C}$ spans V and $\mathcal{B}$ is linearly independent, then $|\mathcal{C}| \ge |\mathcal{B}|$. Thus, $|\mathcal{C}| = |\mathcal{B}|$. $\square$

Note. This result fixes the "broken" part of our deifinition for $\dim(V)$.

Lecture 7

Why is the following true (from a proposition in the last lecture)?

$\Longrightarrow$ Suppose $\dim(V) = n$ and $S \subseteq V$ has size k. If $k = n$, then S is linearly independent $\iff$ S spans V.

Example. $V = \mathcal{P}_2(\mathbb{R})$, $S = \left\{1 + x, 1 - x^2, 2x + x^2\right\}$ with $\dim(V) = 3$.

Claim. S is a basis for V.

Proof. First, show that S is linearly independent.

Consider $c_1(1+x)+c_2(1-x^2)+c_3(2x+x^2) = \vec{0}(=0+0x+0x^2)$. Equate the coefficients:

$$\begin{cases} c_1 + c_2 & = 0 \\ c_1 + 2c_3 & = 0 \\ -c_2 + c_3 & = c_0 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right].$$

$$\begin{aligned} \therefore S \text{ is linearly independent } &\iff c_1 = c_2 = c_3 = 0 \\ &\iff \text{The only solution to the system is the trivial solution} \\ &\iff \operatorname{nullity}(A) = 0. \end{aligned}$$

We have nullity$(A) = 0$ after row reduction. Thus, S is linearly independent.

Next, we show that Span $(S) = V$. We need to check if every vector $\vec{v} \in V$ is a linear combination of the vectors in S. Solve: $c_1(1+x)+c_2(1-x^2)+c_3(2x+x^2) = a+bx+cx^2$. Equate the coefficients we obtain:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 2 & b \\ 0 & -1 & 1 & c \end{array}\right].$$

$$\begin{aligned} \therefore \operatorname{Span}(S) = V &\iff \text{The system has a solution } \forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \\ &\iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \operatorname{Col}(A) \ \forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \\ &\iff \operatorname{Col}(A) = \mathbb{R}^3 \\ &\iff \dim(\operatorname{Col}(A)) = \dim(\mathbb{R}^3) \\ &\iff \operatorname{rank}(A) = 3 \end{aligned}$$

Observation: In this example:

- S is linearly independent $\iff$ nullity $(A) = 0$.
- S spans $V \iff \operatorname{rank}(A) = \underbrace{3}_{\dim(V)}$.

But because A is $\underbrace{3}_{\dim(V)} \times \underbrace{3}_{|S|}$, then $\operatorname{rank}(A) + \operatorname{nullity}(A) = \#$ of columns $= 3$. Thus,

$$\operatorname{rank}(A) = 3 \iff \operatorname{nullity}(A) = 0.$$

Therefore, S is linearly independent $\iff$ S spans V. $\square$

Note. A is 3×3, the first '3' implies $\dim(V)$ as there there are 3 rows, and each row represents an equation. The second '3' implies $|S|$ as there are 3 columns, and each column represents coefficients of a corresponding variable.

Remark (Exercise). We ended up with the same square matrix, the proof for showing both linearly independence and span used the same argument. Prove the general proposition instead of using an example.

Theorem. Let V be a finite-dimensional vector space over $\mathbb{F}$ and let U be a subspace of V. Then, $\dim(U) \leq \dim(V)$ with equality $\iff$ $U = V$.

Proof. Let $\mathcal{B}$ be a basis for U. Since $U \subseteq V$, then $\mathcal{B} \subseteq V$. Since $\mathcal{B}$ is linearly independent, then $\underbrace{|\mathcal{B}|}_{\dim(U)} \leq \dim(V)$.

Suppose now $\dim(U) = \dim(V)$ (so $|B| = \dim(V)$). Since $\mathcal{B}$ is a basis for U, then $\operatorname{Span}(B) = U$. Since $\mathcal{B} \subseteq V$ and since $\mathcal{B}$ is linearly independent, we have $\mathcal{B}$ spans V. Thus, $V = \operatorname{Span}(B) = U$. Conversely, if $U = V$, then it follows that $\dim(U) = \dim(V)$. $\square$

Lemma. Let V be a vector space, let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a subset of V, and let $U = \operatorname{Span}(S)$. Then every vector in U can be expressed in a unique way as a linear combination of the vectors in S $\iff$ if S is linearly independent.

Coordinates

In $\mathbb{R}^3$, we have standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and every $\vec{v} \in \mathbb{R}^3$ is of the form

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3.$$

In general, if V is an n-dimensional vector space with basis $\mathcal{B} = \left\{\vec{b}_1, \dots, \vec{b}_n\right\}$, then every $\vec{v} \in V$ can be written as

$$\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n.$$

Fact (Unique Representation Theorem): The a_i are uniquely determined by $\vec{v}$. They are called the $\mathcal{B}$-coefficients of $\vec{v}$. We introduce coordinate vector $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

Example. $V = \mathcal{P}_2(\mathbb{R})$, $\mathcal{B} = \left\{1, x, x^2\right\}$. Let $p(x) = a + bx + cx^2$ with $\mathcal{B}$-coefficients a, b, c. Then $[p(x)]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Lecture 8

Theorem (Unique Representation Theorem).
Let V be a vector space and let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of V. Then $\forall \vec{v} \in V$, $\exists$ unique scalars $x_1, \dots, x_n \in \mathbb{F}$ s.t.

$$\vec{v} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n.$$

Definition (Ordered Basis). Let V be a finite-dimensional vector space over $\mathbb{F}$. An **ordered basis for** V is a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ for V together with a fixed ordering.

Definition (Coordinate Vector, $\mathcal{B}$-coordinate).
Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an ordered basis for a vector space V. If $\vec{x} \in V$ is written as

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

then the **coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$**, or the $\mathcal{B}$**-coordinates of $\vec{x}$** is

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Example (**Ordered Bases**). In $\mathbb{R}^2$, consider $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. As sets, $\mathcal{B}_1 = \mathcal{B}_2$, but as orderbases, $\mathcal{B}_1 \neq \mathcal{B}_2$. So if we have $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. On the other hand if $[\vec{x}]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\vec{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Example. Let $V = \{A \in M_{2\times 2}(\mathbb{R}) : \operatorname{tr}(A) = 0\}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \in V$. Then, $[A]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

Example (**Problem 1**). Let $V = \mathcal{P}_2(\mathbb{F})$ and $\{\mathcal{B} = -1 + 2x + 2x^2, 2 + x^2, -3 + x\}$. Show that $\mathcal{B}$ is a basis for V and determine $[1 + x]_{\mathcal{B}}$ and $\left[1 + x + x^2\right]_{\mathcal{B}}$.

Solution: Since $\dim(V) = 3$ and $|\mathcal{B}| = 3$, it suffices to check $\operatorname{Span}(\mathcal{B}) = V$ or $\mathcal{B}$ is linearly independent. In this case, we will show that $\operatorname{Span}(\mathcal{B}) = V$. Take any $p(x) \in V$, say $p(x) = a + bx + cx^2$ and find $c_1, c_2, c_3 \in \mathbb{F}$ s.t. $p(x) = c_1(-1 + 2x + 2x^2) + c_2(2 + x^2) + c_3(-3 + x)$.
Equate the coefficients, we have

$$\begin{cases} -c_1 + 2c_2 - c_3 &= a \\ 2c_1 + c_3 &= b \\ 2c_1 + c_2 &= c \end{cases} \iff \left[\begin{array}{ccc|c} -1 & 2 & -3 & a \\ 2 & 0 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right] \iff \left[\begin{array}{ccc|c} 1 & 0 & 0 & a + 3b - 2c \\ 0 & 1 & 0 & -2a - 6b + 5c \\ 0 & 0 & 1 & -2a - 5b + 4c \end{array}\right]$$

So $[1 + x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 + 3 - 0 \\ -2 - 6 + 0 \\ -2 - 5 + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -7 \end{bmatrix}$ and $\left[1 + x + x^2\right]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}$.

Example (**Problem 2**). Let $V = \mathbb{F}^3$ and $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$. Show that $\mathcal{B}$ is a basis for V and determine $[\vec{v}]_{\mathcal{B}}$ and $[\vec{u}]_{\mathcal{B}}$, where $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Remark. This is the same as Problem 1.

Abstract Vector Space vs. $\mathbb{F}^n$

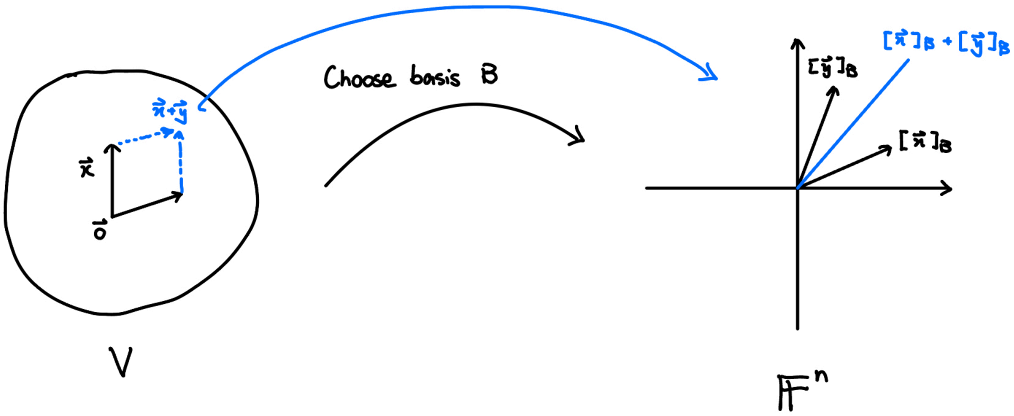


Figure 1: Translation between Abstract Vector Space and $\mathbb{F}^n$.

Theorem (Linearity of Taking Coordinates).

Let V be a finite-dimensional vector space over $\mathbb{F}$ with ordered basis $\mathcal{B}$. Then

$$[\vec{x}+\vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}} \quad \text{and} \quad [c\vec{x}]_{\mathcal{B}} = c[\vec{x}]_{\mathcal{B}}$$

$\forall \vec{x}, \vec{y} \in V$ and all $c \in \mathbb{F}$.

Proof. Exercise: prove addition.

We will prove scalar multiplication. Suppose $\mathcal{B} = \left\{\vec{b}_1, \ldots, \vec{b}_n\right\}$. Suppose $\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$.

Then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. So $c\,[\vec{x}]_{\mathcal{B}} = c\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$.

Now, $c\vec{x} = c(a_1\vec{b}_1 + \cdots + a_n\vec{b}_n) = (ca_1)\vec{b}_1 + \cdots + (ca_n)\vec{b}_n$. Thus,

$$[c\vec{x}]_{\mathcal{B}} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix} = c\,[\vec{x}]_{\mathcal{B}}\,.$$

□

2 Linear Transformations

2.1 Linear Transformations Between Abstract Vectors

Definition (Linear Transformation, Linear Map, Linearity).
If V and W are vector spaces over $\mathbb{F}$, a function $L : V \to W$ is called a **linear transformation** (or **linear map**) if it satisfies the **linearity** properties:

1. $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$, and
2. $L(c\vec{x}) = cL(\vec{x})$

$\forall \vec{x}, \vec{y} \in V, c \in \mathbb{F}$.

Remark. We can combine the two properties and check if $L(c\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}), \forall \vec{x}, \vec{y} \in V$ and $c \in \mathbb{F}$.

Example. $[\ \ ] : V \to \mathbb{F}^n$ is a linear map.

Question: Does L respect zero vector & additive inverse?

Lecture 9

Well, I came late...

Proposition. Let $L : V \to W$ be a linear map, and let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Then

$$L(\vec{0}_V) = \vec{0}_W$$

Observation: The definition of "linear map" is essentially "a function that respects vector space structure".

Let $L : V \to W$ be a linear map. L respects $+$ and $\cdot$. What else does it respect?

(a) Does a linear map respect $\vec{0}$?
Yes! $L(\vec{0}_V) = L(0 \cdot \vec{0}_V) = 0 \cdot L(\vec{0}_V) = \vec{0}_W$.

(b) Does a linear map respect additive inverse?
Yes! $L(-\vec{x}) = L((-1)\vec{x}) = (-1)L(\vec{x}) = -L(\vec{x})$.

(c) What else? Does L respect spans? Does L respect linear independence?

Example (Basic Examples & Linear Maps).

1. The zero map $L : V \to W, L(\vec{v}) = \vec{0}_W, \forall \vec{v}$ (Notation: can also write $\vec{v} \mapsto \vec{0}$).

2. The identity map $L : V \to W$ or $\vec{v} \mapsto \vec{v}, L(\vec{v}) = \vec{v}, \forall \vec{v}$.

Remark. Given a function $L : V \to W$, we will often write $\vec{v} \mapsto \vec{w}$ to mean that L sends $\vec{v}$ to $\vec{w}$.

Example (Fundamental Examples).

1. Let $\mathcal{B}$ be an ordered basis for V. The coordinate map $[\ \ ]_{\mathcal{B}} : V \to \mathbb{F}^n$ or $\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$ is a linear map.

   Special case: $V = \mathcal{P}_3(\mathbb{R}), \mathcal{B} = \{1, x, x^2, x^3\}$, then the $\mathcal{B}$-coordinate map is $[\ \ ]_{\mathcal{B}} : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$
   $$\left(a + bx + cx^2 + dx^3 \mapsto \left[a + bx + cx^2 + dx^3\right]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right).$$

2. Let $A \in M_{m\times n}(\mathbb{F})$. Define $L_A : \mathbb{F}^n \to \mathbb{F}^m, \vec{x} \mapsto A\vec{x}$ by $L_A(\vec{x}) = A\vec{x}$. This is linear.

   Special case: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \in M_{2\times 3}(\mathbb{F})$ and $L_A : \mathbb{F}^3 \to \mathbb{F}^2, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b + c \\ 2 + 3b + 4c \end{bmatrix}$.

Example (More Examples).

1. The differentiation map $D : \mathcal{P}_n(\mathbb{F}) \to \mathcal{P}_{n-1}(\mathbb{F}), p(x) \mapsto p'(x)$ is linear. $D(cp(x) + q(x)) = (cp(x) + q(x))' = cp'(x) + q'(x) = cD(p(x)) + D(q(x))$.

2. The integration map $I : \mathcal{P}_n(\mathbb{F}) \to \mathcal{P}_{n+1}(\mathbb{F})$ is linear.

3. The evaluation map, fix $\alpha \in \mathbb{F}$ in general is a linear map. Define $ev_\alpha : \mathcal{P}_n(\mathbb{F}) \to \mathbb{F}$.

   Check: $\mathrm{ev}_\alpha(cp(x) + q(x)) = cp(\alpha) + q(\alpha) = c\,\mathrm{ev}_\alpha(p(x)) + \mathrm{ev}_\alpha(q(x))$.

4. Transpose is linear. $L : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$.

   Check:: $L(cA + B) = (cA + B)^T = (cA)^T + B^T = c(A^T) + B^T = cL(A) + L(B)$.

5. Trace is linear. $\mathrm{tr} : M_{m\times n}(\mathbb{F}) \to \mathbb{F}, A \mapsto \mathrm{tr}(A)$.

Example (Non-Linear Examples).

1. Determinant is not linear except for $n = 1$. $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$.

   If $n = 1$, this is linear (identity map $\mathbb{F} \to \mathbb{F}$).

   If $n > 1$, $\det(A + B) \neq \det(A) + \det(B)$, and $\det(cA) = c^n \det(A) \neq c \det(A)$, in general.

2. $L : M_{2\times 3}(\mathbb{F}) \to \mathcal{P}_2(\mathbb{F})$, $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \mapsto (a+b)+(c+d)x+x^2$ is NOT linear as it does not respect to $\vec{0}$, i.e. $L(\vec{0}) \neq \vec{0}$ since $L\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0 + 0x + x^2 \neq 0 + 0x + 0x^2$.

Problem: $L : \mathcal{P}_2(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ or $p(x) \mapsto \begin{bmatrix} p(1) & p'(1) \\ p''(1) & p'''(1) \end{bmatrix}$. Is this linear?

Yes! (intuition: differentiation and evaluation are both liner) We will come back to this next lecture.

2.2 Rank and Nullity

* The two most important features of a linear map are:

1. what it destroys
2. what it creates.

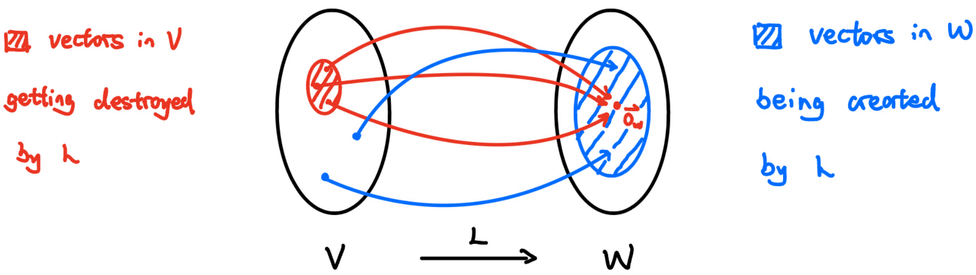


Figure 2: Two features of linear mapping.

Definition (Kernel, Nullspace, Range).

Let $L : V \to W$ be a linear map. The **kernel** (or **nullspace**) of L is

$$\mathrm{Ker}\,(L) = \{\vec{x} \in V : L(\vec{x}) = \vec{0}\}.$$

The **range** (or **image**) of L is

$$\text{Range}(L) = \{L(\vec{x}) \in W : \vec{x} \in V\}.$$

Remark. Kernel → vectors "destroyed" by L & Range → vectors "created" by L.
Referring to Figure 2 above, the red block represents $\text{Ker}(L)$, whereas the blue block represents $\text{Range}(L)$.

Note. $\text{Ker}(L) \subseteq V$ (domain) and $\text{Range}(L) \subseteq W$ (codomain).

Lecture 10

Theorem. Let V and W be vector spaces over $\mathbb{F}$, and let $L : V \to W$ be a linear map. Then

(a) $\text{Ker}(L)$ is a subspace of V, and

(b) $\text{Range}(L)$ is a subspace of W.

Proof. By the Subspace Test - Exercise. □

Definition (Rank, Nullity). Let V and W be vector spcaces over $\mathbb{F}$. The **rank** of a linear map $L : V \to W$ is the dimension of the range of L. The **nullity** of L is the dimension of the kernel (nullspace) of L. That is,

$$\text{rank}(L) = \dim(\text{Range}(L)) \quad \text{and} \quad \text{nullity}(L) = \dim(\text{Ker}(L)).$$

Note. These are VERY important numerical invariants of L.

Example.

1. Let $Z : V \to W$ be the zero map ($Z(\vec{v}) = \vec{0}\ \forall \vec{v} \in V$). Then $\text{Ker}(Z) = V$ and $\text{Range}(Z) = \{\vec{0}\}$.

2. Idendity map $\text{id} : V \to V$ ($\text{id}(\vec{v}) = \vec{v}$). Then $\text{Ker}(\text{id}) = \{\vec{0}\}$ and $\text{Range}(\text{id}) = V$.

3. $\mathcal{B}$-coordinates map. Suppose $\mathcal{B}$ is an ordered basis for V and $\dim(V) = n$, $[\ \ ]_{\mathcal{B}} : V \to \mathbb{F}^n$ or $\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$. Then, $\text{Ker}([\ \ ]_{\mathcal{B}}) = \{\vec{0}_V\}$.

   Proof. Let $\vec{v} \in \text{Ker}([\ \ ]_{\mathcal{B}}) \iff [\vec{v}]_{\mathcal{B}} = \vec{0}_{\mathbb{F}^n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \iff \vec{v} = 0\vec{b}_1 + 0\vec{b}_2 + \cdots + 0\vec{b}_n$

$\iff \vec{v} = \vec{0}_V$. $\square$

And $\text{Range}([\ \ ]_{\mathcal{B}}) = \mathbb{F}^n$.

Proof. $\text{Range}([\ \ ]_{\mathcal{B}}) \subseteq \mathbb{F}^n$ by definition. So just need to prove $\supseteq$. Take $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$. Consider

$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n \in V$. Then $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, so $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \text{Range}([\ \ ]_{\mathcal{B}})$. $\square$

In addition, $\text{nullity}([\ \ ]_{\mathcal{B}}) = 0$ and $\text{rank}([\ \ ]_{\mathcal{B}}) = \dim(\mathbb{F}^n) = n$.

4. ***Matrix mapping. Let $A \in M_{m\times n}(\mathbb{F})$. Let $L_A : \mathbb{F}^n \to \mathbb{F}^m$ or $\vec{x} \mapsto A\vec{x}$. Then $\text{Ker}(L_A) = \left\{\vec{x} \in \mathbb{F}^n : L_A(\vec{x}) = \vec{0}\right\} = \left\{\vec{x} \in \mathbb{F}^n : A\vec{x} = \vec{0}\right\} = \text{Null}(A)$.

   And $\text{Range}(L_A) = \{\vec{w} \in \mathbb{F}^m : \vec{w} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{F}^n\} = \cdots = \text{Col}(A)$. (IMPORTANT: must be able to show this...)

   Also, $\text{nullity}(L_A) = \dim(\text{Null}(A)) = \text{nullity}(A)$ and $\text{rank}(L_A) = \dim(\text{Col}(A)) = \text{rank}(A)$.

5. Differentiation. $D : \mathcal{P}_n(\mathbb{F}) \to \mathcal{P}_{n-1}(\mathbb{F})$ or $p(x) \mapsto p'(x)$.
   Then $\text{Ker}(D) = \left\{a_0 1 + 0x + 0x^2 + \cdots + 0x^n : a_0 \in \mathbb{F}\right\} = \text{Span}\{1\}$.

   Proof.

$$\begin{aligned} p(x) = a_0 + a_1x + \cdots + a_nx^n \in \text{Ker}(D) &\iff p'(x) = \vec{0} \\ &\iff a_1 2a_2x + \cdots + na_nx^n = 0 + 0x + \cdots + 0x^n \\ &\iff a_1 = a_2 = \cdots = a_n = 0 \\ &\iff p(x) = a_1 1 \end{aligned}$$

$\square$

   And $\text{Range}(D) = \mathcal{P}_{n-1}(\mathbb{F})$.

   Proof. Given $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} = D(a_0x + \frac{1}{2}a_1x^2 + \cdots + \frac{a_{n-1}}{n}x^n)$. So every $p(x) \in \mathcal{P}_{n-1}(\mathbb{F})$ is D(something) and is in $\text{Range}(D)$.

Also, $\text{nullity}(D) = \dim(\text{Ker}(D)) = \dim(\text{Span}\{1\}) = 1$ and $\text{rank}(D) = \dim(\text{Range}(D)) = \dim(\mathcal{P}_{n-1}(\mathbb{F})) = n$. $\square$

Lecture 11

Warm-up: Let $L : \mathbb{R}^3 \to \mathcal{P}_1(\mathbb{R})$ be defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a+b+c)x$. Find $\text{rank}(L)$ and $\text{nullity}(L)$.

Solution: Since $\text{rank}(L) = \dim(\text{Range}(L))$, we can claim the following by inspection (intuition).

Claim: $\text{Range}(L) = \text{Span}\{x\}$.

Proof. For ($\Rightarrow$),

$$\begin{aligned} p(x) \in \text{Range}(L) &\iff p(x) = L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) \text{ for some } \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3. \\ &\iff p(x) = (a+b+c)x \\ &\iff p(x) \in \text{Span}\{x\}. \end{aligned}$$

For ($\Leftarrow$), take $\alpha x \in \text{Span}\{x\}$ and note that $\alpha x = L\left(\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}\right)$, so $\alpha x \in \text{Range}(L)$.

Therefore, we have $\text{rank}(L) = \dim(\text{Range}(L)) = \dim(\text{Span}\{x\}) = 1$. $\square$

Next, for $\text{nullity}(L)$, we know that $\text{nullity}(L) = \dim(\text{Ker}(L))$. Then

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{Ker}(L) &\iff L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = 0 + 0x \\ &\iff (a+b+c)x = 0 + 0x \\ &\iff a+b+c = 0 \\ &\iff a = -b-c \\ &\iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b-c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Therefore, $\text{Ker}(L) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ and the two vectors are clearly linearly independent. Thus, $\text{nullity}(L) = \dim(\text{Ker}(L)) = 2$.

Remark. This warm-up example leads to the following important theorem.

Theorem (Rank-Nullity Theorem).
Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional and $\dim(V) = n$. Let $L : V \to W$ be a linear map. Then $\text{rank}(L) + \text{nullity}(L) = n$.

Remark. This is also refered to the Fundamental Theorem of Linear Algebra.

Important Fact from Linear Algebra 1: $\dim((A)) = \text{nullity}(A)$ and $\dim(\text{Col}(A)) = \text{rank}(A)$.

Example.

1. Consider the $\mathcal{B}$-coordinate map $[\ \ ]_{\mathcal{B}} : V \to \mathbb{F}^n$, where V is n-dimensional. By definition, $\text{Range}([\ \ ]_{\mathcal{B}}) = \mathbb{F}^n$, so $\text{rank}([\ \ ]_{\mathcal{B}}) = n \implies \text{nullity}([\ \ ]_{\mathcal{B}}) = 0$.

2. Consider $\text{tr} : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$. Let $W = \text{Ker}(\text{tr}) = \{A \in M_{n\times n}(\mathbb{F}) : \text{tr}(A) = 0\}$. What is $\dim(W)$?
   Solution: $\dim(W) = \textit{nully}(\text{tr}) = \dim(M_{n\times n}(\mathbb{F})) - \text{rank}(\text{tr}) = n^2 - 1$.
   Note that $\text{rank}(\text{tr}) = 1$ because $\text{rank}(\text{tr}) = \dim(\text{Range}(\text{tr}))$ and $\text{Range}(\text{tr})$ is a subspace of $\mathbb{F}$. But $\mathbb{F}$ only has two subspaces, $\{\vec{0}\}$ and itself. So $\text{Range}(\text{tr}) = \mathbb{F}$.

3. (Exercise): $L : V \to \mathbb{F}$ is a non-zero map. Prove that $\text{nullity}(L) = \dim(V) - 1$.

Definition (Injective, Surjective).
The linear map $L : V \to W$ is

(a) **injective** (or **one-to-one**) if $\text{Ker}(L) = \{\vec{0}\}$ ($\iff$ $\text{nullity}(L) = 0$)

(b) **surjective** (or **onto**) if $\text{Range}(L) = W$ ($\iff$ $\text{rank}(L) = \dim(W)$).

Example. $\text{tr} : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$.
tr is surjective since $\text{rank}(\text{tr}) = 1 = \dim(\mathbb{F})$. But tr is not injective since $\text{nullity}(\text{tr}) = n^2 - 1 \neq 0$ when $n \neq 1$.

Lecture 12

Example.

$L : M_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a + (b+c)x + (a-2d)x^2$. Is L injective? Surjective?

Solution 1: (Direct from definition)

For Ker(L):

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Ker}(L) &\iff a + (b+c)x + (a-2d)x^2 = \vec{0} \\ &\iff a = 0, b + c = 0, a - 2d = 0 \\ &\quad \vdots \\ &\iff a = d = 0, c = -b \\ &\iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

Therefore, $\mathrm{Ker}(L) = \mathrm{Span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. Thus L is not injective since $\mathrm{Ker}(L) = \{\vec{0}\}$.

For Range(L):

$$\begin{aligned} p(x) \in \mathrm{Range}(L) &\iff p(x) = a + (b+c)x + (a-2d)x^2 \\ &\iff p(x) = a(1+x^2) + bx + cx + d(-2x^2) \\ &\iff p(x) \in \mathrm{Span}\{1 + x^2, x, x, -2x^2\} \end{aligned}$$

Therefore $\{1 + x^2, x, -2x^2\}$ is linearly independent and hence a basis for Range(L). So $\mathrm{rank}(L) = \dim(\mathrm{Range}(L)) = 3 = \dim(\mathcal{P}_2(\mathbb{R}))$, L is surjective.

Solution 2: (Try Rank-Nullity Theorem)

We have, $\dim(M_{2\times 2}(\mathbb{R})) = 4 = \mathrm{rank}(L) + \mathrm{nullity}(L)$. Once we have rank **OR** nullity, we get the other one! This cuts work in half.

Since $\dim(\mathcal{P}_2(\mathbb{R})) = 3$, then $\mathrm{rank}(L) \leq \dim(\mathcal{P}_2(\mathbb{R})) = 3 \implies \mathrm{nullity}(L) \neq 1$. Thus, L is not injective.

Note. Don't forget that we still need to show some work to check surjective if using solution 2.

Theorem. Let $L : V \to W$ be a linear map between finite-dimensional vector spaces.

(a) If $\dim(V) > \dim(W)$, then L cannot be injective.

(b) If $\dim(V) < \dim(W)$, then L cannot be surjective.

(c) If $\dim(V) = \dim(W)$, then L is injective $\iff$ if L is surjective.

Proof. We will prove (a).

By Rank-Nullity Theorem, we have $\dim(V) = \text{nullity}(L) + \text{rank}(L)$. And $\text{rank}(L) = \dim(\text{Range}(L)) \leq \dim(W) < \dim(V)$. If $\text{nullity}(L) = 0$, then $\dim(V) = \text{rank}(L) < \dim(V)$, a contradiction! $\square$

Example (Above Example Continued)**.**
$L : M_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a + (b+c)x + (a-2d)x^2$.

This looks a lot like $T : \mathbb{R}^4 \to \mathbb{R}^3$, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mapsto \begin{bmatrix} a \\ b+c \\ a-2d \end{bmatrix}$.

Remark (**Refresher**)*.* Every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by a matrix $T(\vec{x}) = A\vec{x}$ for some $A \in M_{m\times n}(\mathbb{R})$. For our example: $T : \mathbb{R}^4 \to \mathbb{R}^3$, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mapsto \begin{bmatrix} a \\ b+c \\ a-2d \end{bmatrix}$. We want $A \in M_{3\times 4}(\mathbb{R})$ so that

$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b+c \\ a-2d \end{bmatrix}$. We discover: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$.

Summary: If $T : \mathbb{R}^n \to \mathbb{R}^m$, let $A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & | & | \end{bmatrix}$, then $T(\vec{x}) = A\vec{x}$.

Note. A knows everything about T!!!

Lecture 13

3 Diagonalizability

3.1 Eigenvectors and Diagonalization

3.2 Diagonalization

3.3 Applications of Diagonalization

4 Inner Product Spaces

4.1 Inner Products

Definition (Inner Product, Conjugate Symmetry, Linearity in First Argument, Positive-Definite).
Let V be a vector space over $\mathbb{F}$. An **inner product** on V is a function

$$\langle \, , \, \rangle : V \times V \to \mathbb{F}$$

s.t. $\forall \vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha \in \mathbb{F}$,

1. $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$.
2. $\langle \alpha \vec{v}, \vec{w} \rangle = \alpha \langle \vec{v}, \vec{w} \rangle$.
3. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$.
4.

4.2 Orthogonality and Norm

4.3 Orthonormal Bases

4.4 Projections

4.5 The Gram-Schmidt Orthogonalization Procedure

4.6 Projection onto a Subspace and Orthogonal Complements

4.7 Application: Method of Least Squares