PMATH 333 Introduction to Real Analysis

University of Waterloo - Winter 2025

16th April 2025

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1 Real Numbers

Lecture 1, 2025/01/06

Definition 1.1 (Notation). We use the following notation:

- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$.
- $\mathbb{N} = \{1, 2, 3, \dots\}$.
- $\mathbb{Q} = \left\{ \frac{x}{y} : x, y \in \mathbb{Z}, y \neq 0 \right\}$.
- $\mathbb{R}$ is the set of real numbers.

Proposition 1.1 (Axioms of Addition and Multiplication).
There exists a set $\mathbb{R}$ with binary operations $+$ and $\cdot$ ($+, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$), where we write $x + y$ and $x \cdot y$, the sum and product of $x, y \in \mathbb{R}$, respectively. They satisfy the following:

A1 **Commutativity of** $+$: $x + y = y + x$.

A2 **Associatively of** $\cdot$: $(x + y) + z = x + (y + z)$.

A3 **Existence of additive identity**: there exists a real number $0 \in \mathbb{R}$ such that $x + 0 = x$ for all $x \in \mathbb{R}$.

A4 **Existence of additive inverses**: for all $x \in \mathbb{R}$, $\exists -x \in \mathbb{R}$ such that $x + (-x) = 0$.

M1 **Commutativity of** $\cdot$: $x \cdot y = y \cdot x$.

M2 **Associativity of** $\cdot$: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

M3 **Existence of multiplicative identity**: there exists $1 \in \mathbb{R}$ with $1 \neq 0$ such that $1 \cdot x = x$ for all $x \in \mathbb{R}$.

M4 **Existence of multiplicative inversese**: $\forall x \neq 0$, $\exists x^{-1} \in \mathbb{R}$ such that $x \cdot x^{-1} = 1$.

D **Distributivity**: $x \cdot (y + z) = x \cdot y + x \cdot z$.

Note. These axioms do not quite axiomatize $\mathbb{R}$ (still need “order” and “completeness”).

Question: Which axioms do $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ satisfy?

Proposition 1.2 (Axioms of Order). There is a relation $<$ on $\mathbb{R}$ (i.e. a subset of $\mathbb{R} \times \mathbb{R}$ where we write $x < y$ if (x, y) is the set) such that

O1 **Trichotomy**: given $x, y \in \mathbb{R}$, exactly one of the following holds:

$$x < y, \quad x = y, \quad y < x.$$

O2 **Transitivity**: for $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

O3 **Invariance of** $<$: for $x, y \in \mathbb{R}$, if $x < y$, then $x + z < y + z$ for all $z \in \mathbb{R}$.

O4 ???: for $x, y \in \mathbb{R}$, if $x < y$ and $z \in \mathbb{R}$ with $0 < z$, then $x \cdot z < y \cdot z$.

Question: What about $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$?

Note. We still cannot distinguish $\mathbb{R}$ form $\mathbb{Q}$.

Proposition 1.3 (**Axiom of Completeness**).
Every non–empty subset $A \subseteq \mathbb{R}$ which is bounded above (i.e. $\exists M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$) has a least upper bound (i.e. M is an upper bound for A and if wherever N is an upper bound for A, $M \leq N$).

Example. Think of $\mathbb{R}$ as you know it. $(0, 1)$ has upper bound 6000. Its least upper bound is 1 (requires proof).

Completeness actually distinguishes $\mathbb{R}$ form $\mathbb{Q}$.

Example. Say $\pi = 3.14159\ldots$. Then $A = \{3, 3.1, 3.14, 3.141, \ldots\}$ has a least upper bound π, which is <u>NOT</u> rational.

Remark (**Replacement**). If $a = b$ in any sets, then any expression involving a (e.g. $a \cdot a$) can be replaced with the same expression with b in place of a.

Example (**What we can do with these axioms?**).

(1) $0 \in \mathbb{R}$ is unique (as an additive identity).

(2) $1 \in \mathbb{R}$ is unique (as a multiplicative identity).

(3) $-x$ is unique (as an additive inverse).

(4) x^{-1} is unique (as a multiplicative inverse).

(5) Completeness of $\mathbb{R}$ allows us to define square roots: $\forall x > 0, \exists! \sqrt{x}$ s.t. $(\sqrt{x}) \cdot (\sqrt{x}) = x$.
Consider $S = \{t \in \mathbb{R} : x < t^2\}$ and work with lower bounds!

(6) $0 \cdot x = 0$ and $x \cdot 0 = 0$ $\forall x \in \mathbb{R}$.

(7) $xy = 0 \implies x = 0$ or $y = 0$.

Proof of (6) and (7).

(6)

$$
\begin{aligned}
0 = 0 + 0 \quad &(\text{by A3}) \\
\implies x \cdot 0 = x \cdot (0+0) \quad &\text{by replacement} \\
\implies x \cdot 0 = x \cdot 0 + x \cdot 0 \quad &\text{by D} \\
\implies x \cdot 0 + (-(x \cdot 0)) = (x \cdot 0 + x \cdot 0) + (-(x \cdot 0)) \quad &\text{by replacement} \\
\implies x \cdot 0 + (-(x \cdot 0)) = x \cdot 0 + (x \cdot 0 + (-(x \cdot 0))) \quad &\text{by A2} \\
\implies 0 = x \cdot 0 + (x \cdot 0 + (-(x \cdot 0))) \quad &\text{by A4} \\
\implies 0 = x \cdot 0 + 0 \quad &\text{by A4} \\
\implies 0 \cdot x = 0 \quad &\text{by M1.}
\end{aligned}
$$

(7) Suppose $x \cdot y = 0$. There are two cases: $x = 0$ or $x \neq 0$. If $x = 0$, we are done. If $x \neq 0$, then we have

$$
\begin{aligned}
xy = 0 \quad & \\
\implies x^{-1} \cdot (xy) = x^{-1} \cdot 0 \quad &\text{by replacement} \\
\implies x^{-1} \cdot (xy) = 0 \quad &\text{by (6)} \\
\implies (x^{-1} \cdot x) \cdot y = 0 \quad &\text{by M2} \\
\implies 1 \cdot y = 0 \quad &\text{by M4} \\
\implies y = 0 \quad &\text{by M3.}
\end{aligned}
$$

□

2 Topology of $\mathbb{R}^n$

Definition 2.1 (Notation). Let $n \in \mathbb{N}$. We define

1. $\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}\ \forall i \in \{1, 2, \dots, n\}\}$.
2. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, the **dot/inner product** is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

3. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n, \alpha \in \mathbb{R}$,
   - Vector Addition: $x + y = (x_1 + y_1, \dots, x_n + y_n)$.
   - Scalar Multiplication: $\alpha x = (\alpha x_1, \dots, \alpha x_n)$.

Lemma 2.1. For $x, y, z \in \mathbb{R}^n, \alpha, \beta \in \mathbb{R}$, the following hold:

(1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.

(2) $\langle x, y \rangle = \langle y, x \rangle$.

(3) $\langle x, x \rangle \geq 0$ with equality if and only if $x = \vec{0}$.

Proof. Exercise. □

Definition 2.2 (Euclidean Norm). The **Euclidean norm** on $\mathbb{R}^n$ is given by

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + \cdots + x_n^2}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Note. In $\mathbb{R}^2$, this is the distance from the origin to the point (x_1, x_2). If $n = 1$, this is just the absolute value: $\|x\| = |x| = \sqrt{x^2}$, where $|x|$ is the <u>maximum</u> of x and $-x$.

Lemma 2.2. Let $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then

(1) $\|\alpha x\| = |\alpha| \|x\|$ (homogeneity).

(2) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = \vec{0}$ (positive definiteness).

Proof. Exercise. □

Note. The sum of two sides of a triangle will be at least the length of the third side.

Proposition 2.3 (Cauchy–Schwarz Inequality).
For all $x, y \in \mathbb{R}^n$, we have

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

and equality holds $\iff$ $x = \vec{0}$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.

Proof. For $x, y \in \mathbb{R}^n$, then

$$\begin{aligned} 2\|x\|^2\|y\|^2 - 2|\langle x, y\rangle|^2 &= 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 y_j^2 - 2\left(\sum_{i=1}^{n} x_i y_i\right)^2 \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 y_j^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j x_j y_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_j^2) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i y_j - x_j y_i)^2 \geq 0 \end{aligned}$$

Thus, $2\|x\|^2\|y\|^2 - 2|\langle x, y\rangle|^2 \geq 0$. Then,

$$\implies \|x\|^2\|y\|^2 \geq |\langle x, y\rangle|^2 \implies \|x\|\|y\| \geq |\langle x, y\rangle|.$$

For equality, suppose $|\langle x, y\rangle| = \|x\|\|y\|$. If $x = y = \vec{0}$, this is true. If either x or y is not $\vec{0}$. WLOG, say $x \neq \vec{0}$. So, $x_i \neq 0$ for some $1 \leq i \leq n$. WLOG, say $x_1 \neq 0$. But now, since $\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i y_j - x_j y_i)^2 = 0$. Then

$$\begin{aligned} &\implies x_i y_j - x_j y_i = 0 \quad \forall 1 \leq i, j \leq n \\ &\implies x_1 y_j - x_j y_1 = 0 \\ &\implies y_j = \frac{y_1}{x_1} x_j \implies y = \frac{y_1}{x_1} x. \end{aligned}$$

$\square$

Proposition 2.4 (**Triangle Inequality**).
For $x, y \in \mathbb{R}^n$, we have
$$\|x+y\| \le \|x\| + \|y\|.$$

Proof. We have

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&\le \langle x,x\rangle + |\langle x,y\rangle| + |\langle x,y\rangle| + \langle y,y\rangle \\
&\le \langle x,x\rangle + \|x\|\|y\| + \|x\|\|y\| + \langle y,y\rangle \quad \text{by Cauchy–Schwarz} \\
&= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

Thus, we have $\|x+y\| \le \|x\| + \|y\|$ by taking square roots. □

Proposition 2.5 (**Triangle Inequality 2**).
For $x, y \in \mathbb{R}^n$, we have
$$\|x+y\| = \|x\| + \|y\| \iff x = \vec{0} \text{ or } y = cx \text{ for some } c \ge 0.$$

Proof. By Proposition 2.4, we have

$$\begin{aligned}
\|x+y\| = \|x\| + \|y\| &\iff 2\langle x,y\rangle = 2\|x\|\|y\| \\
&\iff \langle x,y\rangle = \|x\|\|y\|
\end{aligned}$$

By Cauchy–Schwarz, $x = \vec{0}$ or $y = cx$ for some $c \ge 0$. Note that if $x \ne \vec{0}$, plugging in $y = cx$ gives

$$\|x\|\|y\| = \langle x,y\rangle = \langle x,cx\rangle = c\langle x,x\rangle = c\|x\|^2 \implies c = \frac{\|y\|}{\|x\|} \ge 0.$$

□

Remark. See A1 for “Reverse Triangle Inequality”.

Note. There are other notions of distance.

Definition 2.3 (**Norm**). A function $\rho : \mathbb{R}^n \to [0, \infty)$ is called a **norm** if

(1) $\rho(x) \geq 0$ and $\rho(x) = 0 \iff x = 0$ (positive definiteness).

(2) $\rho(\alpha x) = |\alpha|\rho(x)$ for all $\alpha \in \mathbb{R}$ (homogeneity).

(3) $\rho(x + y) \leq \rho(x) + \rho(y)$ (triangle inequality).

Example. We proved that $\|\cdot\|$, the Euclidean norm, is a norm.

Note. There are more examples on A1: $\|\cdot\|_1, \|\cdot\|_\infty$. Also, $\|\cdot\|_p$-norms ($\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$) for $p \in (1, \infty)$.

Lecture 3, 2025/01/10

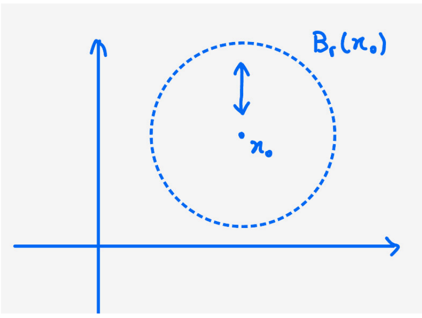
Definition 2.4 (**Open/Closed Balls**).

Let $x_0 \in \mathbb{R}^n$, $r > 0$. We say

(1) $B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$ is the **open ball** of radius r around x_0.

(2) $\overline{B_r(x_0)} = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$ is the **closed ball** of radius r around x_0.

Remark (**Why "balls"?**).



For example, in $\mathbb{R}^3$, $\overline{B_1(\vec{0})}$ gives you the closed sphere.

In $\mathbb{R}$, we have $B_\delta(a) = (a - \delta, a + \delta)$. **Question**: Is $[0, 1)$ a ball?

Definition 2.5 (**Open/Closed Sets**).

(1) A subset $A \subseteq \mathbb{R}^n$ is **open** if $\forall x \in A, \exists r > 0$ s.t. $B_r(x) \subseteq A$.

(2) A subset $A \subseteq \mathbb{R}^n$ is **closed** if $A^c := \mathbb{R}^n \setminus A$ is open.

Note. A subset is open if every point in it has an open ball around it which is contained in the set.

Example.

(1) $\varnothing$ and $\mathbb{R}^n$ are both open and closed.

(2) $B_r(x)$ is open.

Proof. Pick $y \in B_r(x)$. We want to find $s > 0$ s.t. $B_s(y) \subseteq B_r(x)$. Let $s = r - \|y - x\|$. Then, for $z \in B_s(y)$, we have

$$\|z - x\| = \|z - y + y - x\| \leq \|z - y\| + \|y - x\| < s + \|y - x\| = r.$$

Thus, by definition, $z \in B_r(x)$. Therefore, $B_s(y) \subseteq B_r(x)$. □

Is $B_r(x)$ closed? We will look at $\mathbb{R}^n \setminus B_r(x)$, choose y such that $\|y - x\| = r$. Then, $\forall s > 0$, $B_s(y) \nsubseteq \mathbb{R}^n \setminus B_r(x)$ (it contains points in and out of $\mathbb{R}^n \setminus B_r(x)$). Thus, $B_r(x)$ is not closed.

(3) $\overline{B_r(x)}$ is closed, but not open.

(4) $E = \{(x_1, \dots, x_n) : x_1 + \cdots + x_n < 1\}$ is open.

(5) $E = [0, 1)$ is neither open (no open ball around 0) nor closed ($\mathbb{R} \setminus [0, 1) = (-\infty, 0) \cup [1, \infty)$ is not open, as there is no open ball at 1).

Remark.

(1) Not open $\not\Longrightarrow$ closed (e.g. $[0, 1) \subseteq \mathbb{R}$).

(2) Closed $\not\Longrightarrow$ not open (e.g. $\mathbb{R}^n$ is closed).

What about unions of open balls? Their union is open!

Proposition 2.6.

(1) An arbitrary union of open sets in $\mathbb{R}^n$ is open.

(2) A finite intersection of open sets in $\mathbb{R}^n$ is open.

Proof.

(1) Let $(U_i)_{i \in I}$ be a collection of open subsets of $\mathbb{R}^n$. Let $U = \bigcup_{i \in I} U_i$ and $y_0 \in U$. Then, $\exists i_0 \in I$ s.t. $y_0 \in U_{i_0}$, which is open. Thus, $\exists r > 0$ s.t. $B_r(y_0) \subseteq U_{i_0} \subseteq U$. Therefore, U is open, since y_0 was arbitrary.

(2) Let $U_1, \dots, U_k \subseteq \mathbb{R}^n$ be open sets. Let $U = \bigcap_{i=1}^k U_i$ and $x \in U$ be arbitrary. Then, $x \in U \subseteq U_i$

for all $i \in \{1, \dots, k\}$. Since each U_i is open, so $\exists r_i > 0$ s.t. $B_{r_i}(x) \subseteq U_i$. Let $r = \min\{r_1, \dots, r_k\}$. Then, $r > 0$ and $B_r(x) \subseteq B_{r_i}(x) \subseteq U_i \ \forall i \in \{1, \dots, k\} \implies B_r(x) \subseteq U$. Thus, U is open.

□

Note. For $t \leq s$, $B_t(x) \subseteq B_s(x)$.

Corollary 2.7.

(1) An arbitrary intersection of closed sets in $\mathbb{R}^n$ are closed.

(2) A finite union of closed sets in $\mathbb{R}^n$ are closed.

Proof.

(1) Let $(E_i)_{i \in I}$ be a collection of closed subsets of $\mathbb{R}^n$. Let $E = \bigcap_{i \in I} E_i$. By De Morgan's Law and the previous proposition (1), we have

$$\left(\bigcap_{i \in I} E_i\right)^c = \bigcup_{i \in I} E_i^c.$$

Since each E_i^c is open, we have $\left(\bigcap_{i \in I} E_i\right)^c$ is open, so $E = \bigcap_{i \in I} E_i$ is closed.

(2) Let $E_1, \dots, E_k \subseteq \mathbb{R}^n$ be closed sets. Let $E = \bigcup_{i=1}^k E_i$. By De Morgan's Law and the previous proposition (2), we have

$$\left(\bigcup_{i=1}^k E_i\right)^c = \bigcap_{i=1}^k E_i^c.$$

Since each E_i^c is open, we have $\left(\bigcup_{i=1}^k E_i\right)^c$ is open, so $E = \bigcup_{i=1}^k E_i$ is closed.

□

Exercise: Prove (2) of Proposition 2.6.

Remark. We need the finite hypothesis. For example,

(1) Look at $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then, $\bigcap_{n=1}^\infty U_n = \{0\}$, which is <u>not</u> open.

Note. $\{x\} \subseteq \mathbb{R}^n$ is closed but not open $\forall x \in \mathbb{R}^n$.

(2) Take any set $S \subseteq \mathbb{R}^n$, say $S = \bigcup_{s \in S} \{s\}$ is an infinite union of closed sets. Take $S = B_r(x)$, which is <u>not</u> closed.

Example (**More Examples of Open/Closed Sets**).

(1) Open intervals: for $-\infty \leq a < b \leq \infty$, (a, b) is open.

(2) Closed intervals: if $a < b$, $a, b \in \mathbb{R}$, then the closed interval $[a, b]$ is closed, since $[a, b]^c = (-\infty, a) \cup (b, \infty)$ is open.

Example (**More on Above Remark**).

We know that <u>finite</u> intersections of open sets are open. Finiteness is necessary, for example

(1) For $n \geq 1$, $\left(0, 1 + \frac{1}{n}\right)$ is open. However, $\bigcap_{n \geq 1} \left(0, 1 + \frac{1}{n}\right) = (0, 1]$ is not open, since eventually for any $c > 1$, $1 + \dfrac{1}{n} < c$ and for any $r > 0$, $B_r(1) = (1 - r, 1 + r) \not\subseteq (0, 1]$.

Note. A **topology** is a pair $(X, (U_i)_{i \in I})$ consisting of a set X (e.g. $\mathbb{R}^n$) and some notion of open set satisfying (1) and (2) of Proposition 2.6

Exercise: Every open set in $\mathbb{R}^n$ can be written as a union of open balls (i.e. open balls of some radius $r > 0$ around some point $x \in \mathbb{R}^n$).

Note. In A1 Q5, we consider other norms on $\mathbb{R}^n$, namely $\|\cdot\|_1, \|\cdot\|_\infty$. The definition of an open set in $\mathbb{R}^n$ could equivalently use $\|\cdot\|_1$ or $\|\cdot\|_\infty$.

Exercise: Prove above note using A1 Q5 (every norm on $\mathbb{R}^n$ gives the same open sets).

3 Interior and Closure

Definition 3.1 (Interior, Closuere, Boundary). Let $A \subseteq \mathbb{R}^n$.

(1) The **interior** of A is $A^\circ = \{x \in A : \exists r > 0 \text{ s.t. } B_r(x) \subseteq A\}$.

(2) The **closure** of A is $\overline{A} = \{x \in \mathbb{R}^n : \forall r > 0, B_r(x) \cap A \neq \varnothing\}$.

(3) The **boundary** of A is $\partial A = \overline{A} \setminus A^\circ$.

Note. We will see that

(1) A° is the biggest open set in A.

(2) $\overline{A}$ is the smallest closed set containing A.

Intuition:

(1) $x \in A^\circ$ if it cannot be "approached" using points outside of A (i.e. A^c).

(2) $x \in \overline{A}$ if x can be "approached" using points in A.

Proposition 3.1 (Properties of Interior).

Let $A \subseteq \mathbb{R}^n$.

(1) $A^\circ = \bigcup\{U \subseteq A : U \text{ is open }\}$.

(2) A° is open.

(3) A is open $\iff A = A^\circ$.

Proof.

(1) Suppose $U \subseteq A$ is open. By the definition of openness, for every $x \in U$, there exists $r > 0$ s.t. $B_r(x) \subseteq U \subseteq A$. Hence, $U \subseteq A^\circ$, so $\bigcup\{U \subseteq A : U \text{ is open }\} \subseteq A^\circ$.
Conversely, for $x \in A^\circ$. Then, there exists $r > 0$ s.t. $B_r(x) \subseteq A$. Since $B_r(x)$ is open, $x \in B_r(x) \subseteq A$, so $x \in \bigcup\{U \subseteq A : U \text{ is open }\}$. Hence, $A^\circ \subseteq \bigcup\{U \subseteq A : U \text{ is open }\}$.
Therefore, $A^\circ = \bigcup\{U \subseteq A : U \text{ is open }\}$.

(2) By (1), A° is a union of open sets, hence by Proposition 2.6, A° is open.

(3) If $A = A^\circ$, then by (2), A° is open, hence A is open. Conversely, if A is open, then by (1), $A \subseteq \bigcup\{U \subseteq A : U \text{ is open }\} = A^\circ \subseteq A$, so $A = A^\circ$.

□

Lecture 5, 2025/01/15

Note. $x \in \overline{A} \iff \forall$ open sets $U \subseteq A^c$, $x \notin U$. Otherwise, if $x \in U$, then $\exists r > 0$ s.t. $B_r(x) \subseteq U \subseteq A^c$, meaning $B_r(x) \cap A = \varnothing$.

Proposition 3.2 (**Properties of Closure**).
Let $A \subseteq \mathbb{R}^n$.

(1) $\overline{A} = \bigcap\{C \subseteq \mathbb{R}^n : C \supseteq A \text{ and } C \text{ is closed}\}$.
(2) $\overline{A}$ is closed.
(3) $A = \overline{A} \iff A$ is closed.

Proof.

(1) If $C \subseteq \mathbb{R}^n$ satisfies $C \supseteq A$ and is closed, then C^c is open and $C^c \subseteq A^c$. From the note above, $\overline{A} \cap C^c = \varnothing$, so $\overline{A} \subseteq C$. Hence $\overline{A} \subseteq \bigcap\{C \subseteq \mathbb{R}^n : C \supseteq A \text{ and } C \text{ is closed}\}$.
Conversely, suppose $x \in \bigcap\{C \subseteq \mathbb{R}^n : C \supseteq A \text{ and } C \text{ is closed}\}$. Then, for every open set $U \subseteq A^c$, $x \notin U$. By the note above, it implies that $x \in \overline{A}$. Hence, $\bigcap\{C \subseteq \mathbb{R}^n : C \supseteq A \text{ and } C \text{ is closed}\} \subseteq \overline{A}$. Therefore, $\overline{A} = \bigcap\{C \subseteq \mathbb{R}^n : C \supseteq A \text{ and } C \text{ is closed}\}$.

(2) $\overline{A}$ is closed because intersections of closed sets are closed by Corollary 2.7.

(3) If $A = \overline{A}$, then by (2), A is closed. Conversely, if A is closed, then the intersection in (1) is A, so $A = \overline{A}$.

□

Exercise:

(1) $A^\circ = \left(\overline{(A^c)}\right)^c$.
(2) $\overline{A} = ((A^c)^\circ)^c$.

Example (**Examples of Interior, Closure, Boundary**).

(1) For $x \in \mathbb{R}^n$, the set $\{x\}$ (singleton set) is closed. So $\overline{\{x\}} = \{x\}$ by Proposition 3.2. On the other hand, $\{x\}^\circ = \varnothing$ since $\forall r > 0$, $B_r(x) \nsubseteq \{x\}$. Finally, $\partial\{x\} = \overline{\{x\}} \setminus \{x\}^\circ = \{x\}$.
(2) For $a, b \in \mathbb{R}$ with $a < b$, the "half-open" interval $(a, b]$ is neither open nor closed (not closed because $\forall r > 0$, $B_r(a) = (a - r, a + r)$ has non-empty intersection with $(a, b]$, so $a \in \overline{(a, b]}$, but $a \notin (a, b]$. So $(a, b] \neq \overline{(a, b]}$, so $(a, b]$ is not closed by Proposition 3.2).

The closure is $\overline{(a,b]} = [a,b]$, for any $x \in [a,b]$, $\forall r > 0$, $B_r(x) \cap (a,b] \neq \varnothing$ but for any $y \notin [a,b]$, $\exists r > 0$ s.t. $B_r(y) \cap (a,b] = \varnothing$. For example, if $y > b$, then take $r = \frac{|y-b|}{2}$.

The interior is $(a,b]^\circ = (a,b)$ since for any $x \in (a,b)$, $\exists r > 0$ s.t. $B_r(x) \subseteq (a,b)$, but this is not true for $x = b$.

Hence, the boundary $\partial(a,b] = \overline{(a,b]} \setminus (a,b]^\circ = [a,b] \setminus (a,b) = \{a,b\}$.

(3) $\mathbb{Z} \subseteq \mathbb{R}$ is closed because $\mathbb{Z}^c = \bigcup_{n\in\mathbb{Z}}(n, n+1)$ is open (union of open sets). Hence by Proposition 3.2, $\overline{\mathbb{Z}} = \mathbb{Z}$. On the other hand, for every $r > 0$, $n \in \mathbb{Z}$, $B_r(n)$ contains non-integers, so in particular, $B_r(n) \nsubseteq \mathbb{Z}$. Hence, $\mathbb{Z}^\circ = \varnothing$. Therefore, $\partial\mathbb{Z} = \overline{\mathbb{Z}} \setminus \mathbb{Z}^\circ = \mathbb{Z}$.

(4) $\mathbb{Q}$ (the rational numbers) is neither open nor closed.

For $q \in \mathbb{Q}$, $r > 0$, $B_r(q)$ contains irrational numbers, so $B_r(q) \nsubseteq \mathbb{Q}$. Hence, $\mathbb{Q}^\circ = \varnothing$. So by Proposition 3.1, $\mathbb{Q}$ is not open. The same idea shows that $\mathbb{Q}^c$ (irrational numbers) is not open, so $\mathbb{Q}$ is not closed.

For every irrational $x \in \mathbb{Q}^c$, $\forall r > 0$, $B_r(x) \cap \mathbb{Q} \neq \varnothing$. So $x \in \overline{\mathbb{Q}}$. Hence, $\overline{\mathbb{Q}} = \mathbb{R}$.

Hence, the boundary $\partial\mathbb{Q} = \overline{\mathbb{Q}} \setminus \mathbb{Q}^\circ = \mathbb{R} \setminus \varnothing = \mathbb{R}$.

Note.

- The boundary is the set of points that are in between the interior and the exterior.
- We used the completeness axiom of $\mathbb{R}$ for the argument in example (4) (we will cover this later).

Exercise: For $x \in \mathbb{R}^n$, $r > 0$, we know $B_r(x)$ is open. Show: $\overline{B_r(x)} = \{y \in \mathbb{R}^n : \|x - y\|_2 \leq r\}$. So $\partial B_r(x) = \{y \in \mathbb{R}^n : \|x - y\|_2 = r\}$.

4 Completeness of $\mathbb{R}$

Definition 4.1 (**Upper Bound, Bounded Above, Least Upper Bound**).

Let $A \subseteq \mathbb{R}$.

(1) $c \in \mathbb{R}$ is an **upper bound** for A if $a \leq c$ for all $a \in A$.

(2) A is **bounded from above** if there exists an upper bound (at least one) for A.

(3) $c \in \mathbb{R}$ is a **least upper bound** for A if

 (a) c is an upper bound for A.

 (b) $c \leq d$ if d is an upper bound for A.

Lemma 4.1. A set $A \subseteq \mathbb{R}^n$ has at most one least upper bound.

Proof. Let $c, d \in \mathbb{R}$ be least upper bounds for A. By definition, $c \leq d$ and $d \leq c$, so $c = d$. □

Note. This says we can talk about the least upper bound for a set when it exists.

Remark (**Notation**). If $A \subseteq \mathbb{R}$ has a least upper bound, we will denote it by $\sup A$.

Note. sup stands for **supremum**, which means the same thing as least upper bound.

Example.

(1) For a finite set $A = \{a_1, \dots, a_n\} \subseteq \mathbb{R}$, $\sup A = \max A$. To see this, note that $\max A$ is an upper bound for A, also, $\max A = a_i$ for some i. So if $c \in \mathbb{R}$ is an upper bound, then $a_j \leq c$ for all j, so $\max A \leq c$.

(2) $\sup[0, 1) = 1$. Note that $\max[0, 1)$ does not exist.

First, 1 is an upper bound for $[0, 1)$. If $c \in \mathbb{R}$ is an upper bound for $[0, 1)$, then $c \geq d$ whenever $d < 1$. This forces $c \geq 1$ (if $c < 1$, then $1 - \frac{|1-c|}{2} < 1$ and $1 - \frac{|1-c|}{2} \in [0, 1)$, so c could not be an upper bound). Hence, $\sup[0, 1) = 1$.

Note. For $A \subseteq \mathbb{R}$, $\sup A$ is not necessarily in A (unlike max).

Proposition 4.2. Let $A \subseteq \mathbb{R}$ and suppose $\sup A$ exists. Then,

(1) $\forall \epsilon > 0, \exists x \in A$ s.t. $\sup A - \epsilon < x \leq \sup A$.

(2) Hence, $\sup A \in \overline{A}$.

Proof.

(1) Suppose for the sake of contradiction that it fails. Then $\exists \epsilon > 0$ s.t. $\forall x \in A, x \leq \sup A - \epsilon$. This means that $\sup A - \epsilon$ is an upper bound for A contradicting the definition of $\sup A$. Hence, the result holds.

(2) Recall $\overline{A} = \{x \in \mathbb{R} : \forall r > 0, B_r(x) \cap A \neq \varnothing\}$. From (1),

$$\begin{aligned} \forall \epsilon > 0,\ \sup A - \epsilon < x \text{ for some } x \in A &\iff \sup A - x < \epsilon \text{ for some } x \in A \\ &\iff x \in B_\epsilon(\sup A) \text{ for some } x \in A \\ &\iff B_\epsilon(\sup A) \cap A \neq \varnothing. \end{aligned}$$

Since this holds for all $\epsilon > 0$, $\sup A \in \overline{A}$.

$\square$

Question: When does $A \subseteq \mathbb{R}$ have a least upper bound?

Proposition 4.3 (Axiom of Completeness).
If $A \subseteq \mathbb{R}$ is non-empty and bounded from above, then $\sup A$ exists.

Proposition 4.4 (Archimedean Property of $\mathbb{R}$).
For $a, b \in \mathbb{R}$, with $a, b > 0$. Then, $\exists n \in \mathbb{N}$ s.t. $na > b$.

Proof. Suppose such n does not exist. Then for every $n \in \mathbb{N}$, $n \leq \frac{b}{a}$. Hence, $\frac{b}{a}$ is an upper bound for $\mathbb{N}$. Therefore, $\mathbb{N}$ is bounded above, so $\sup \mathbb{N}$ exists by the axiom of completeness. By Proposition 4.2, $\sup \mathbb{N} \in \overline{\mathbb{N}} = \mathbb{N}$, because $\mathbb{N}$ is closed. So there is a largest natural number, which is false. $\square$

Theorem 4.5 (**Density of $\mathbb{Q}$ in $\mathbb{R}$**).

For $a, b \in \mathbb{R}$ with $a < b$, $\exists q \in \mathbb{Q}$ s.t. $a < q < b$. Equivalently, $(a, b) \cap \mathbb{Q} \neq \varnothing$.

Remark (**Terminology**).

If $A \subseteq B \subseteq \mathbb{R}^n$ satisfies $\overline{A} = B$, then A is dense in B. Thus, above theorem says $\overline{\mathbb{Q}} = \mathbb{R}$.

Proof. We can assume that $0 < a$, since otherwise, we can choose $N \in \mathbb{N}$ s.t. $a + N > 0$. Then if $q \in \mathbb{Q}$ satisfies $a + N < q < b + N$, then $a < q - N < b$ and $q - N \in \mathbb{Q}$.

By the Archimedean property of $\mathbb{R}$,

$$\begin{aligned} \exists n \in \mathbb{N} \text{ s.t. } n > \frac{1}{b-a} &\implies nb - na > 1 \\ &\implies 1 + na < nb. \end{aligned}$$

Let $m = \lfloor na \rfloor + 1$, where $\lfloor x \rfloor$ is the largest integer $\leq x$. Then,

$$m - 1 \leq na < m \tag{1}$$

$$\implies m \leq na + 1 < nb \tag{2}$$

Now, putting (1) and (2) together gives $na < m < nb$. Thus, we have $a < \frac{m}{n} < b$, and $\frac{m}{n} \in \mathbb{Q}$. $\square$

Corollary 4.6. Every open interval in $\mathbb{R}$ contains a rational number.

Corollary 4.7. For $n \geq 1$, $\overline{\mathbb{Q}^n} = \mathbb{R}^n$. In particular, $\overline{\mathbb{Q}} = \mathbb{R}$.

Remark. This says that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Proof. We first prove the case when $n = 1$. Fix $c \in \mathbb{R}$, we want to show that $c \in \overline{\mathbb{Q}}$. For $r > 0$, $B_r(c) = (c - r, c + r)$, and by Theorem 4.5, $B_r(c) \cap \mathbb{Q} \neq \varnothing$. Since r was arbitrary, $c \in \overline{\mathbb{Q}}$. Hence, since c was arbitrary, $\overline{\mathbb{Q}} = \mathbb{R}$.

Now, let $n \geq 1$. Fix $v = (v_1, \dots, v_n) \in \mathbb{R}^n$. Choose $r > 0$. For each $1 \leq i \leq n$, choose $q_i \in \mathbb{Q}$ using

Theorem 4.5 such that

$$|q_i - v_i| < \frac{r}{\sqrt{n}} \quad (\iff q_i \in B_{\frac{r}{\sqrt{n}}}(v_i)).$$

Note. We often don't know how to choose the RHS of the above inequality. We choose it after we have worked out the problem... See below.

Let $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$. Then,

$$\begin{aligned}\|q - v\|_2 &= \left((q_1 - v_1)^2 + \cdots + (q_n - v_n)^2\right)^{1/2} \\ &< \Bigg(\underbrace{\frac{r^2}{n} + \cdots + \frac{r^2}{n}}_{n \text{ times}}\Bigg)^{1/2} = r.\end{aligned}$$

Hence, $q \in B_r(v)$. Hence, $B_r(v) \cap \mathbb{Q}^n \neq \varnothing$. Hence, since v was arbitrary, $\overline{\mathbb{Q}^n} = \mathbb{R}^n$. □

Proposition 4.8. For $x > 0$, there exists a unique $y > 0$ such that $y^2 = x$.

Remark (**Terminology**). We write $y = \sqrt{x} = x^{1/2}$.

Proof. Let $S = \{y \in \mathbb{R} : y > 0 \text{ and } y^2 < x\}$. Then, S is bounded from above because we can choose $n \in \mathbb{N}$ s.t. $n^2 \geq x$. Then for $y \in S$, $y \leq n$ because otherwise, if $y > n$, then $y^2 > n^2 \geq x$, contradicting that $y \in S$. Also, $S \neq \varnothing$ since $0 \in S$. Hence, by the completeness axiom, $s = \sup S$ exists. We will prove that $s^2 = x$.

First, $s^2 \geq x$. To see this, suppose otherwise (for the sake of contradiction), then $s^2 < x$. Then, $\exists \epsilon > 0$ s.t. $(s + \epsilon)^2 = s^2 + 2s\epsilon + \epsilon^2 < x$. Then, $s + \epsilon \in S$, contradicting that s is the supremum of S. Hence, $s^2 \geq x$.

Now, suppose (again for a contradiction) that $s^2 > x$. Then, $\exists \epsilon > 0$ s.t. $(s-\epsilon)^2 = s^2 - 2s\epsilon + \epsilon^2 > x$. This would imply that $s - \epsilon$ is an upper bound for S because if $y \in S$ and $y > s - \epsilon$, then $y^2 > (s - \epsilon)^2 > x$, contradicting $y^2 < x$. But $s = \sup S$ and $s - \epsilon < s$, so this is a contradiction. Hence, $s^2 = x$.

To show uniqueness, suppose $y^2 = x = z^2$. Then, $0 = y^2 - z^2 = (y - z)(y + z)$. Since $y, z > 0$, $y - z = 0 \implies y = z$. □

Exercise: Generalize the above proof to show that for $x > 0$ and $n \in \mathbb{N}$, there exists a unique $y > 0$ such that $y^n = x$.

Remark (**Terminology**). We write $y = \sqrt[n]{x} = x^{1/n}$ for the n-th root of x.

Definition 4.2 (**Lower Bound, Bounded Below, Greatest Lower Bound**).

Let $A \subseteq \mathbb{R}$.

(1) $c \in \mathbb{R}$ is a **lower bound** for A if $c \leq a$ for all $a \in A$.

(2) A is **bounded below** if it has a lower bound.

(3) $c \in \mathbb{R}$ is a **greatest lower bound** for A if

  (a) c is a lower bound for A.

  (b) $c \geq d$ if d is a lower bound for A.

Lemma 4.9. For $A \subseteq \mathbb{R}$, let $-A = \{-a : a \in A\}$. Then,

(1) $c \in \mathbb{R}$ is a lower bound for $A \iff -c$ is an upper bound for $-A$.

(2) $c \in \mathbb{R}$ is a greatest lower bound for $A \iff -c$ is a least upper bound for $-A$.

Proof.

(1) We have $c \in \mathbb{R}$ is a lower bound for A if

$$\begin{aligned} c \leq a \quad \forall a \in A &\iff -a \leq -c \quad \forall a \in A \\ &\iff b \leq -c \quad \forall b \in -A \end{aligned}$$

i.e. $-c$ is an upper bound for $-A$.

(2) We have that $c \in \mathbb{R}$ is a greatest lower bound for A if c is a lower bound for A and $c \geq d$ for all lower bounds d of A. This is equivalent to $-c$ being an upper bound for $-A$ and $-d \geq -c$ for all upper bounds $-d$ of $-A$. This is equivalent to $-c$ being an upper bound for $-A$ and $b \geq -c$ for all upper bounds b for $-A$. That is, $-c$ is a least upper bound for $-A$.

□

Corollary 4.10. Let $A \subseteq \mathbb{R}$.

(1) A has at most one greatest lower bound.

(2) If A is non-empty and bounded below, then it has a greatest lower bound.

Proof. Exercise. □

Remark (**Notation**). If $A \subseteq \mathbb{R}$ has a greatest lower bound, we denote it by $\inf A$ (infimum).

5 Sequences and Limits

5.1 Convergence of Sequences and Cauchy Sequences

Definition 5.1 (Convergence, Divergence).

Let (x_n) be a sequence in $\mathbb{R}^d$.

(1) (x_n) **converges** to $x \in \mathbb{R}^d$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|x_n - x\| < \epsilon \; \forall n \geq N$.

We write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$ or even $x_n \to x$.

(2) (x_n) **diverges** if it does not converge to any $x \in \mathbb{R}^d$.

Note. $x_n \to x$ in $\mathbb{R}^d \iff \|x_n - x\| \to 0$ in $\mathbb{R}$ because $\|x_n - x\| \to 0$ in $\mathbb{R}$ means $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|\|x_n - x\| - 0| = \|x_n - x\| < \epsilon \; \forall n \geq N$.

Lemma 5.1. A sequence (x_n) in $\mathbb{R}^d$ converges to $x \in \mathbb{R}^d \iff \forall$ open sets $U \ni x, \exists N \in \mathbb{N}$ s.t. $x_n \in U \; \forall n \geq N$.

Proof. We will prove both directions.

($\Rightarrow$): Suppose $x_n \to x$. Let $U \ni x$ be an open set. We must show that $\exists N \in \mathbb{N}$ s.t. $x_n \in U \; \forall n \geq N$. By the definition of an open set, $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U$. By the definition of convergence, $\exists N \in \mathbb{N}$ s.t. $\|x_n - x\| < \epsilon \; \forall n \geq N$. Thus, $x_n \in B_\epsilon(x) \subseteq U \; \forall n \geq N$.

($\Leftarrow$): Conversely, suppose that for every open set $U \ni x, \exists N \in \mathbb{N}$ s.t. $x_n \in U \; \forall n \geq N$. We must show that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|x_n - x\| < \epsilon \; \forall n \geq N$. For $\epsilon > 0$, by assumption, since $B_\epsilon(x)$ is open, $\exists N \in \mathbb{N}$ s.t. $x_n \in B_\epsilon(x) \; \forall n \geq N$. Thus, $\|x_n - x\| < \epsilon \; \forall n \geq N$. Since $\epsilon > 0$ was arbitrary, we conclude that $x_n \to x$. $\square$

Proposition 5.2 (Uniqueness of Limits for Sequences).

A sequence (x_n) in $\mathbb{R}^d$ has at most one limit.

Proof. Let $x, y \in \mathbb{R}^d$ be limits of (x_n). So $x_n \to x$ and $x_n \to y$. Then, $\forall \epsilon > 0, \exists M, N$ s.t. $\|x_n - x\| < \epsilon$ $\forall n \geq M$ and $\|x_n - y\| < \epsilon \; \forall n \geq N$. Let $K = \max\{M, N\}$. Then, $\|x_k - x\| < \epsilon$ and $\|x_k - y\| < \epsilon \; \forall k \geq K$. Now, by the triangle inequality,

$$\|x - y\| = \|x - x_k + x_k - y\| \leq \|x - x_k\| + \|x_k - y\| < 2\epsilon.$$

Since this holds for every $\epsilon > 0$, this foces $\|x - y\| = 0$, i.e. $x = y$. $\square$

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Remark (**Terminology**). We say (x_n) converges if it converges to some $x \in \mathbb{R}^d$. We say (x_n) in $\mathbb{R}^d$ does not converge to $y \in \mathbb{R}^d$ if $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N}$ s.t. $\|x_n - y\| \geq \epsilon$ for some $n \geq N$.

Example. Let $x_n = (-1)^{n+1}$, $n \in \mathbb{N}$. Then, $x_1 = 1, x_2 = -1, x_3 = 1, \ldots$. This sequence does not converge to any $y \in \mathbb{R}$.
To see this, fix $y \in \mathbb{R}$. Then, either $|y - 1| \geq 1$ or $|y - (-1)| \geq 1$. So, for any $N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|y - x_n| \geq 1$ (because x_n alternates between 1 and -1).

Question: Can we test whether (x_n) in $\mathbb{R}^d$ converges without checking the definition against $x \in \mathbb{R}^d$?
$\implies$ We seek an "intrinsic" characterization of convergence.

Definition 5.2 (**Bounded, Cauchy Sequence**).
Let (x_n) be a sequence in $\mathbb{R}^d$.

(1) (x_n) is **bounded** if $\exists R > 0$ s.t. $\|x_n\| \leq R \ \forall n \in \mathbb{N}$. Equivalently, $x_n \in \overline{B_R(0)} \ \forall n \in \mathbb{N}$.

(2) (x_n) is **Cauchy** (a Caushy sequence) if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|x_n - x_m\| < \epsilon, \forall n, m \geq N$.

Note. We will eventually see that (x_n) converges $\iff$ (x_n) is Cauchy and bounded.

Proposition 5.3. If (x_n) is a convergent sequence, then it is bounded and Cauchy.

Proof. Choose $x \in \mathbb{R}^d$ s.t. $\lim x_n = x$. For boundedness, choose $\epsilon = 1$. Then by definition, $\exists N \in \mathbb{N}$ s.t. $\|x_n - x\| < 1 \ \forall n \geq N$. By the reverse triangle inequality (A1),

$$\|x_n\| - \|x\| \leq \|x_n - x\| < 1 \quad \forall n \geq N$$

This gives

$$\|x_n\| \leq \|x\| + 1 \quad \forall n \geq N.$$

Choose $R = \max(1 + \|x\|, \|x_1\|, \ldots, \|x_{N-1}\|)$. Then $\|x_n\| \leq R \ \forall n \in \mathbb{N}$, so (x_n) is bounded.

For Cauchy-ness, choose $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\|x_n - x\| < \frac{\epsilon}{2} \ \forall n \geq N$. Then, $\forall n, m \geq N$,

$$\|x_m - x_n\| \leq \|x_m - x\| + \|x_n - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, (x_n) is Cauchy. $\square$

Note (**Idea**). A sequence (x_n) in $\mathbb{R}^d$ converges $\iff$ each coordinate converges.

Proposition 5.4. Let (x_n) be a sequence in $\mathbb{R}^d$. Write $x_n = (x_{n1}, \dots, x_{nd})$ for each $n \in \mathbb{N}$. Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Then, $x_n \to x \iff x_{ni} \to x_i \ \forall 1 \le i \le d$.

Proof. First suppose $x_{ni} \to x_i \ \forall 1 \le i \le d$. Then, $\forall \epsilon > 0$, $\exists N_i \in \mathbb{N}$ s.t. $|x_{ni} - x_i| < \frac{\epsilon}{d}$ if $n \ge N_i$. Take $N = \max(N_1, \dots, N_d)$. Then,

$$|x_{ni} - x_i| < \frac{\epsilon}{d} \quad \text{if } n \ge N \quad \forall i = 1, \dots, d.$$

Then by A1 Q5,

$$\begin{aligned} \|x_n - x\|_2 \le \|x_n - x\|_1 &= \sum_{i=1}^{d} |x_{ni} - x_i| \\ &< \sum_{i=1}^{d} \frac{\epsilon}{d} = \epsilon \quad \text{if } n \ge N. \end{aligned}$$

Hence, $x_n \to x$.

Conversely, suppose $x_n \to x$. Then, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\|x_n - x\|_2 < \epsilon \ \forall n \ge N$. By A1 Q5, for $1 \le i \le d$,

$$\begin{aligned} |x_{ni} - x_i| &\le \max(|x_{n1} - x_1|, \dots, |x_{nd} - x_d|) \\ &= \|x_n - x\|_\infty \\ &\le \|x_n - x\|_2 < \epsilon \quad \text{if } n \ge N. \end{aligned}$$

Hence, $x_{ni} \to x_i$ for all $1 \le i \le d$. $\square$

Theorem 5.5. Let $A \subseteq \mathbb{R}^d$. For $x \in \mathbb{R}^d$,

$$x \in \overline{A} \iff \exists \text{ a sequence } (a_n) \text{ in } A \text{ such that } a_n \to x.$$

Proof. First suppose $a_n \to x$ for some (a_n) in A. We must show $x \in \overline{A}$, where $\overline{A} = \{x \in \mathbb{R}^d : \forall r > 0, B_r(x) \cap A \neq \emptyset\}$. For $r > 0$, by the definition of convergence, $\exists N \in \mathbb{N}$ s.t. $\|a_n - x\| < r \ \forall n \ge N$. In particular, $\|a_N - x\| < r$, i.e. $a_N \in B_r(x)$, so $B_r(x) \cap A \neq \emptyset$. Since $r > 0$ was arbitrary, $x \in \overline{A}$.

Conversely, suppose $x \in \overline{A}$, we must build (a_n) in A s.t. $a_n \to x$. For each $n \in \mathbb{N}$, $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$. Choose $a_n \in B_{\frac{1}{n}}(x) \cap A$. Then,

$$\|a_n - x\| < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

This gives us a sequence (a_n) in A. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. For $n \geq \mathbb{N}$,

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

so by construction of (a_n), $\|a_n - x\| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Hence, $a_n \to x$. $\square$

Example. $A = (0, 1]$. Then, $\overline{A} = [0, 1]$. But $0 \notin A$. However, $0 = \lim \frac{1}{n}$ and $\frac{1}{n} \in A$.

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Proposition 5.6. Let (x_n) and (y_n) be sequences in $\mathbb{R}^d$ s.t. $x_n \to x$ and $y_n \to y$ for some $x, y \in \mathbb{R}^d$. Then,

(1) $x_n + y_n \to x + y$.

(2) $cx_n \to cx$ for any $c \in \mathbb{R}$.

(3) For $d = 1$, $x_n \cdot y_n \to x \cdot y$. Hence if $d \geq 1$, $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Note. (1) and (2) show that limits are linear.

Proof.

(1) & (2) Exercises.

(3) Suppose $d = 1$. For $\epsilon > 0$, choose $N \in \mathbb{N}$ s.t. $|x_n - x| < \min\left(\frac{\epsilon}{2(|y|+1)}, 1\right)$ and $|y_n - y| < \min\left(\frac{\epsilon}{2(|x|+1)}, 1\right)$ $\forall n \geq N$. Then, by A1, we have $|x_n y_n - xy| < \epsilon$.

If $d > 1$, we can use the fact that a sequence in $\mathbb{R}^d$ converges $\iff$ the coordinates converge, plus the case $d = 1$. Details are left as exercises.

$\square$

Example.

(1) For a sequence (x_n) in $\mathbb{R}$, the sequence $(|x_n|)$ satisfies $|x_n| \to 0 \iff x_n \to 0$.

Proof. First, suppose $x_n \to 0$. Then, given $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $|x_n - 0| = |x_n| < \epsilon$, for $n \geq N$. Then, $||x_n| - 0| = |x_n| < \epsilon$ for $n \geq N$. Hence, $|x_n| \to 0$.

Conversely, if $|x_n| \to 0$, then given $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $||x_n| - 0| = |x_n| < \epsilon$ for $n \geq N$. Hence, $|x_n - 0| = |x_n| < \epsilon$ for $n \geq N$. Thus, $x_n \to 0$. □

(2) A constant sequence $x_n = v \in \mathbb{R}^d$ for all $n \in \mathbb{N}$ where v is fixed, converges to v.

Proof. For $\epsilon > 0$, we can take $N = 1$. For $n \geq N$, $\|x_n - v\| = \|v - v\| = 0 < \epsilon$. Hence, $x_n \to v$. □

(3) For $k \in \mathbb{N}$ fixed, let $x_n = \frac{1}{n^k}$, $n \in \mathbb{N}$. Then, $x_n \to 0$.

Proof. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. For $n \in \mathbb{N}$, $n^k \geq n \implies \frac{1}{n^k} \leq \frac{1}{n}$. Thus for $n \geq N$, $\frac{1}{n} \leq \frac{1}{N}$. So, $|x_n - 0| = |x_n| = \frac{1}{n^k} \leq \frac{1}{n} < \frac{1}{N} < \epsilon$. Hence, $x_n \to 0$. □

(4) Let $x_n = \frac{1}{n^2+3n+1}$, $n \in \mathbb{N}$. Show that $x_n \to 0$.

Proof. Note that $n^2 + 3n + 1 \geq n^2$, so $\frac{1}{n^2+3n+1} \leq \frac{1}{n^2}$. From here, we can use (3). □

(5) Let $x_n = \frac{1}{n^{27}-3n+1}$, $n \in \mathbb{N}$.

Proof. For n sufficiently large, say $n \geq N \in \mathbb{N}$, we have $n^{27} - 3n + 1 \geq n$, so $\frac{1}{n^{27}-3n+1} \leq \frac{1}{n}$. Then for $n \geq N$, $|x_n - 0| = |x_n| < \epsilon$ if $\frac{1}{N} < \epsilon$. Hence, $x_n \to 0$. □

(6) Unbounded sequences do not converge.

Proof. Let (x_n) be a sequence s.t. $\nexists R > 0$ s.t. $x_n \in \overline{B_R(0)}$, $\forall n \in \mathbb{N}$. Then (x_n) does not converge. This follows from the result that convergent sequences are bounded and Cauchy by Proposition 5.3. □

(7) Fix $r \in \mathbb{R}$. Let $x_n = r^n$, $n \in \mathbb{N}$. Talk about its convergence.

Proof. To consider the convergence of (x_n), we must consider several cases:

(a) If $r = 1$, $x_n = 1\ \forall n \in \mathbb{N}$, so $x_n \to 1$.

(b) If $r = -1$, $x_n = (-1)^n\ \forall n \in \mathbb{N}$, is alternating between $1, -1$, so does not converge by a previous example.

(c) If $|r| > 1$, write $|r| = 1 + c$ for $c > 0$. We will show that $|r^n| = |r|^n$ is unbounded. Note that $|r|^n = (1 + c)^n = \sum_{k=0}^{n} \binom{n}{k} c^k > \binom{n}{1} c = nc$. And nc is unbounded because of Archimedean property of $\mathbb{R}$. Hence, $(|r|^n)$ is unbounded, so (x_n) does not converge.

(d) If $|r| < 1$, $x_n \to 0$. It suffices to show that $|r|^n \to 0$ by (1). Write $|r| = \frac{1}{1+c}$ for

$c > 0$. As in (c), $(1+c)^n > cn$. So $|r|^n = \frac{1}{(1+c)^n} < \frac{1}{nc} = \frac{1}{c} \cdot \frac{1}{n} \to 0$. Hence, given $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $\frac{1}{nc} < \epsilon$ for $n \geq N$. Then, $|r|^n < \frac{1}{nc} < \epsilon$ for $n \geq N$. Hence, $|r|^n \to 0$.

□

Exercises: Fill in the details to above proofs.

5.2 Monotone Sequences and Subsequences

Definition 5.3 (Increasing/Decreasing, Monotone Sequences).
Let (x_n) be a sequence in $\mathbb{R}$. We say that (x_n) is **increasing** (strictly increasing) if $x_n \leq x_{n+1}$ ($x_n < x_{n+1}$) for all $n \in \mathbb{N}$. We say that (x_n) is **decreasing** (strictly decreasing) in the obvious way. We say that (x_n) is **monotone** if it is increasing or decreasing.

Recall: consider $r \in \mathbb{R}$, let $c_n = 1 + r + \cdots + r^n$, $n \in \mathbb{N}$. Note that

$$\begin{aligned} rc_n &= r + r^2 + \cdots + r^{n+1} \\ c_n - rc_n &= 1 - r^{n+1} \\ c_n(1-r) &= 1 - r^{n+1} \\ c_n &= \frac{1-r^{n+1}}{1-r}. \end{aligned}$$

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Definition 5.4 (Bounded from Above/Below, Bounded).
A sequence (x_n) in $\mathbb{R}$ is **bounded from above** if the set $\{x_n : n \in \mathbb{N}\}$ is bounded from above. Similarly for **bounded from below** We say that (x_n) is **bounded** if it is bounded from above and below.

Example.

(1) $x_n = (-1)^n$, $n \in \mathbb{N}$, is bounded but not monotone.
(2) $x_n = n$, $n \in \mathbb{N}$, is monotone (strictly increasing) but not bounded.
(3) For $r \in (0,1)$, $x_n = r^n$, $n \in \mathbb{N}$, is both monotone (strictly decreasing) and bounded.

Theorem 5.7 (**Monotone Convergence Theorem for Sequences in $\mathbb{R}$**). Let (x_n) be a bounded monotone sequence in $\mathbb{R}$. Then, (x_n) converges if

(1) (x_n) is increasing and $x_n \to \sup\{x_n : n \in \mathbb{N}\}$ (bounded from above).

(2) (x_n) is decreasing and $x_n \to \inf\{x_n : n \in \mathbb{N}\}$ (bounded from below).

Proof. Suppose that (x_n) is increasing. By boundedness, $\{x_n : n \in \mathbb{N}\}$ is bounded from above, so $s := \sup\{x_n : n \in \mathbb{N}\}$ exists. By the ϵ-characterization of the sup, given $\epsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $s - \epsilon < x_n \leq s$. But by the fact that (x_n) is increasing, $s - \epsilon < x_n \leq x_{n+1} \leq \cdots \leq s$. This says for $m \geq n$, $|x_m - s| < \epsilon$. Hence, $x_n \to s$. The proof in the decreasing case is similar. $\square$

Exercise: Prove above for the decreasing case.

Exercise: Prove all results for inf that have been proven for sup (prepare for tests).

Example. Let $x_n = \sum_{k=0}^{n} \frac{1}{2^k}$, $n \geq 0$. Then from last lecture,

$$x_n = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n} \leq 2.$$

So (x_n) is bounded, it is also clearly strictly increasing. Hence, by the monotone convergence theorem, $x_n \to \sup\{x_n : n \in \mathbb{N} \cup \{0\}\}$.

Definition 5.5 (**Subsequence**). Let (x_n) be a sequence in $\mathbb{R}^d$. A **subsequence** of (x_n) is a sequence obtained by deleting elements in (x_n) without changing the order. We often write $(x_{n_k})_{k\in\mathbb{N}}$ or simply (x_{n_k}) to mean that we have taken the elements $x_{n_1}, x_{n_2}, \ldots$ from (x_n), where $n_1 < n_2 < \cdots$.

Example.

(1) Let $(x_n) = (-1)^{n+1}$, $n \in \mathbb{N}$. The sequence $(x_{2n})_{n\in\mathbb{N}}$ is a subsequence. Note that $n_k = 2k$ with the notation from the definition.

$$(x_n) = (1, -1, 1, -1, \ldots)$$
$$(x_{2n}) = (-1, -1, \ldots)$$

(2) Let $(x_n) = n$, $n \in \mathbb{N}$. Then, $(1, 3, 5, \dots)$ is a subsequence. It is $(x_{2n-1})_{n\in\mathbb{N}}$. However, $(1, 2, 3, 3, 5, 6, \dots)$ is not a subsequence.

(3) Let (x_n) be a sequence in $\mathbb{R}^d$ s.t. $x_n \to 0$. Then, $\exists$ a subsequence (x_{n_k}) s.t. $\left\|x_{n_k}\right\| < \frac{1}{k}$, $\forall k \in \mathbb{N}$.

Proof. By the definition of $x_n \to 0$, setting $\epsilon = \frac{1}{k}$, there is $N_k \in \mathbb{N}$ s.t. $\|x_n\| < \frac{1}{k}$, $\forall n \geq N_k > N_{k-1}$. Take $n_k = N_k$ for all $k \in \mathbb{N}$. Then, $\left\|x_{n_k}\right\| < \frac{1}{k}$ for all $k \in \mathbb{N}$. Also, $\underbrace{n_1}_{N_1} < \underbrace{n_2}_{N_2} < \cdots$. Thus, if $m \geq k$, then $\left\|x_{n_m}\right\| < \frac{1}{k}$ because $n_m \geq N_k$. □

Lemma 5.8. Let (x_n) be a convergent sequence in $\mathbb{R}^d$, say $x_n \to x \in \mathbb{R}^d$. Then, for every subsequence $(x_{n_k})_{k\in\mathbb{N}}$, $x_{n_k} \to x$ as $k \to \infty$.

Proof. Choose $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\|x_n - x\| < \epsilon$ for all $n \geq N$. Choose $K \in \mathbb{N}$ s.t. $n_K \geq N$. Then, for $k \geq K$, $n_k \geq n_K \geq N$, so $\left\|x_{n_k} - x\right\| < \epsilon$. Hence, $x_{n_k} \to x$. □

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5.3 Bolzano-Weierstrass Theorem

Theorem 5.9 (Bolzano-Weierstrass Theorem).
Let (x_n) be a sequence in $\mathbb{R}^d$. If (x_n) is bounded, then (x_n) has a convergent subsequence.

Proof. We need an important lemma to prove this theorem.

Lemma 5.10. Every sequence in $\mathbb{R}$ has a monotone subsequence.

Note. Not assuming boundedness, so we cannot claim convergence (e.g. $x_n = n$).

Proof of Lemma 5.10. Let (x_n) be a sequence in $\mathbb{R}$. Say $n \in \mathbb{N}$ is a <u>peak</u> for (x_n) if $x_n \geq x_m$ for all $m \geq n$. We need to look at two cases.

Case 1: There are infinitely many peaks.

Then, we can take peaks $n_1 < n_2 < \cdots$. By the definition of a peak, $x_{n_k} \geq x_{n_\ell}$ for all $\ell \geq k$. Equivalently, (x_{n_k}) is decreasing, so monotone.

Case 2: There are finitely many peaks (maybe even none).

Let $N \in \mathbb{N}$ be the largest (furthest) peak if it exists, otherwise, let $N = 0$. Let $n_1 = N + 1$. Since n_1 is not a peak, there is $n_2 > n_1$ s.t. $x_{n_1} < x_{n_2}$. Since $n_2 > n_1 > N$, n_2 is also not a peak, so there is $n_3 > n_2$ s.t. $x_{n_2} < x_{n_3}$. We can continue this process to get a subsequence (x_{n_k}) s.t. $x_{n_1} < x_{n_2} < x_{n_3} < \cdots$. Hence, (x_{n_k}) is increasing, so monotone. $\square$

Now, to prove the theorem, first assume $d = 1$. Let (x_n) be a sequence in $\mathbb{R}$. By the lemma, there is a monotone subsequence (x_{n_k}). Since (x_n) is bounded, (x_{n_k}) is also bounded. By the monotone convergence theorem, (x_{n_k}) converges. Hence, (x_n) has a convergent subsequence.

Now, suppose that $d > 1$. First, apply the case when $d = 1$ to find a subsequence of (x_n) where the first coordinate converges. Now, pass to a further subsequence such that the second coordinate converges. By the lemma about subsequences of convergent sequences, the first coordinate still converges. Repeating this process d times, we obtain a subsequence (x_{n_k}) of (x_n) where each coordinate converges. Hence, (x_n) has a convergent subsequence. $\square$

Definition 5.6 (Compactness). A subset of $\mathbb{R}^n$ is **compact** if it is closed and bounded.

Note. The above is a definition/theorem. We will talk about compactness later.

Remark. The Bolzano-Weierstrass theorem is where we first introduce compactness. A restatement of the BW theorem is that every sequence in a compact set has a convergent subsequence.

Application of Bolzano-Weierstrass Theorem to Norms

Definition 5.7 (Equivalence of Norms).
Two norms on $\mathbb{R}^d$, $\|\cdot\|_a$ and $\|\cdot\|_b$, are **equivalent** if there are constants $C, D > 0$ s.t.

$$C\|x\|_a \le \|x\|_b \le D\|x\|_a \quad \forall x \in \mathbb{R}^d.$$

Note. Equivalence of norms is an equivalence relation (see A2).

Exercise: Prove the note above.

Theorem 5.11. If $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms on $\mathbb{R}^d$, then they have the same open (and closed) sets and the same convergent sequences.

Proof. Exactly the same as for $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ in A1. $\square$

Proposition 5.12. Let $\|\cdot\|_a$ be a norm on $\mathbb{R}^d$.

(1) There is $D > 0$ s.t. $\|x\|_a \le D\|x\|_\infty$ for all $x \in \mathbb{R}^d$.

(2) There is $C > 0$ s.t. $\|x\|_\infty \le C\|x\|_a$ for all $x \in \mathbb{R}^d$.

Hence, $\frac{1}{D}\|x\|_a \le \|x\|_\infty \le C\|x\|_a$ for all $x \in \mathbb{R}^d$. Thus, $\|\cdot\|_a$ is equivalent to $\|\cdot\|_\infty$.

Corollary 5.13. All norms on $\mathbb{R}^d$ are equivalent.

Proof. For any norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^d$, they are both equivalent to $\|\cdot\|_\infty$ by Proposition 5.12. By the transitivity, $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent. $\square$

Proof of Proposition 5.12.

(1) Let $e_1, \dots, e_d$ be any basis (including standard basis) of $\mathbb{R}^d$. Then for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $x = x_1e_1 + \cdots + x_de_d$. Then,

$$\begin{aligned}\|x\|_a = \|x_1e_1 + \cdots + x_de_d\|_a \le |x_1|\|e_1\|_a + \cdots + |x_d|\|e_d\|_a &\le \|x\|_\infty\|e_1\|_a + \cdots + \|x\|_\infty\|e_d\|_a \\ &= (\|e_1\|_a + \cdots + \|e_d\|_a)\|x\|_\infty.\end{aligned}$$

Thus, we can take $D = \|e_1\|_a + \cdots + \|e_d\|_a$.

(2) Note that $\|x\|_\infty \le C\|x\|_a$ is equivalent to (by dividing by $\|x\|_\infty$) $C\|x\|_a \ge 1$ for all $x \in \mathbb{R}^d$ with $\|x\|_\infty = 1$. Suppose that towards a contradiction that this fails. Then for all $n \in \mathbb{N}$, there is $x_n \in \mathbb{R}^d$ s.t. $\|x_n\|_\infty = 1$ but $n\|x_n\|_a < 1$. Equivalently, $\|x_n\|_a < \frac{1}{n}$. Thus, $x_n \to 0$ in the $\|\cdot\|_a$ norm. By the Bolzano-Weierstrass theorem, since (x_n) is bounded with respect to $\|\cdot\|_\infty$, there is a subsequence (x_{n_k}) convergent with respect to $\|\cdot\|_\infty$. Note that $\{x \in \mathbb{R}^d : \|x\|_\infty = 1\}$ is the unit ball, which is closed with respect to $\|\cdot\|_\infty$. Hence, $x_{n_k} \to x$ with $\|x\|_\infty = 1$. Now, $\left\|x_{n_k} - x\right\|_a \le D\left\|x_{n_k} - x\right\|_\infty$ by (1). So $x_{n_k} \to x \ne 0$ with respect to $\|\cdot\|_a$ and also that $\left\|x_{n_k}\right\|_a \to 0$. This is a contradiction.

$\square$

Proposition 5.14. $A \subseteq \mathbb{R}^d$ is closed and bounded $\iff$ every sequence of points in A has a subsequence which converges to a point in A.

Proof.
($\Rightarrow$): Suppose $A \subseteq \mathbb{R}^d$ is closed and bounded. Let (a_n) be an arbitrary sequence in A. Since A is bounded, and so by the Bolzano-Weierstrass theorem, $\exists$ a subsequence (a_{n_k}) which converges, say $\lim_{k\to\infty} a_{n_k} = a$, since A is closed, it contains its limit points, so $a \in A$.
($\Leftarrow$): Suppose that every sequence in A has a subsequence which converges and has a limit in A. We first show that A is bounded. Assume towards a contradiction that A is not bounded, then we can find a sequence (a_n) in A such that $\|a_n\| \geq n$ for all $n \in \mathbb{N}$. Then, any subsequence of (a_{n_k}) is also unbounded. But convergence sequences are bounded. Hence, (a_n) does not have a subsequence which converges. This is a contradiction. Hence, A is bounded.

To prove that A is closed, we prove that $A = \overline{A}$. Note that $A \subseteq \overline{A}$ is true. Let $a \in \overline{A}$, then there is a sequence (a_n) in A such that $\lim_{n\to\infty} a_n = a$. By assumption, there is a subsequence (a_{n_k}) such that $\lim_{k\to\infty} a_{n_k} = b \in A$. Since any subsequence of a convergent sequence converges to the same limit, we have $a = b \in A$. Hence, $\overline{A} \subseteq A$. Thus, $A = \overline{A}$, so A is closed. □

Example.
(1) $[0, 1]$ is closed and bounded. So every sequence in $[0, 1]$ has a subsequence which converges to a point in $[0, 1]$.
(2) $\mathbb{R}^d$ is closed but not bounded.
(3) $(0, 1]$ is not closed. Consider the sequence $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then $x_n \to 0$, but $0 \notin (0, 1]$.

Note. In $\mathbb{R}^d$, closed + bounded is equivalent to "compact". We will see this later.

Other Applications of Bolzano-Weierstrass Theorem

Recall that a sequence (x_n) in $\mathbb{R}^d$ is **Cauchy** if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\|x_n - x_m\| < \epsilon \ \forall n, m \geq N$.

Lemma 5.15. A Cauchy sequence in $\mathbb{R}^d$ is bounded.

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}^d$. Let $\epsilon = 2$. By Cauchy-ness, $\exists N \in \mathbb{N}$ s.t. $m, n \geq N$, then

$\|x_n - x_m\| < 2$. Take $L = \max(\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 2)$. If $1 \le n \le N$, then $\|x_n\| \le L$, clearly. If $n \ge N$, then

$$\begin{aligned}\|x_n\| &= \|x_n - x_N + x_N\| \\ &\le \|x_n - x_N\| + \|x_N\| \quad \text{by Triangle inequality} \\ &< \|x_N\| + 2 \le L.\end{aligned}$$

$\square$

Proposition 5.16. Cauchy sequences in $\mathbb{R}^d$ converge.

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}^d$. By Lemma 5.15, (x_n) is bounded. By the Bolzano-Weierstrass theorem, there is a subsequence (x_{n_k}) which converges, say $x_{n_k} \to x \in \mathbb{R}^d$.

Claim. $\lim_{n\to\infty} x_n = x$.

Proof of Claim. Let $\epsilon > 0$. Since $x_{n_k} \to x$, $\exists N_0 \in \mathbb{N}$ s.t. $\forall k \ge N_0$, $\left\|x_{n_k} - x\right\| < \frac{\epsilon}{2}$. Since (x_n) is Cauchy, $\exists N_1 \in \mathbb{N}$ s.t. $\forall m, n \ge N_1$, we have $\|x_n - x_m\| < \frac{\epsilon}{2}$. Now, if $k \ge N := \max(N_0, N_1)$, then

$$\begin{aligned}\|x_n - x\| = \left\|x_n - x_{n_k} + x_{n_k} - x\right\| &\le \underbrace{\left\|x_n - x_{n_k}\right\|}_{n, n_k \ge N_1} + \underbrace{\left\|x_{n_k} - x\right\|}_{k \ge N_0} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus, $x_n \to x$. $\square$

This completes the proof. $\square$

Remark. Obviously, by last lecture, the choice of norm does not matter.

Example. Note that $\mathbb{Q} \subseteq \mathbb{R}$ does not have this property, i.e. there are Cauchy sequences in $\mathbb{Q}$ which do not converge in $\mathbb{Q}$. For example, consider $a_1 = 1.4, a_2 = 1.41, a_3 = 1.414, a_4 = 1.4142, \dots$. This is a Cauchy sequence in $\mathbb{Q}$, it converges to $\sqrt{2} \notin \mathbb{Q}$.

Proof. If $N \in \mathbb{N}$ and $n, m \ge N$, a_n, a_m agree on the first N decimal places. Hence, $|a_n - a_m| < 10^{-N}$. But $\lim_{n\to\infty} a_n = \sqrt{2} \notin \mathbb{Q}$ (ex). Also, (a_n) in $\mathbb{Q}$ is bounded, $\sup\{a_n : n \in \mathbb{N}\} = \sqrt{2}$. $\square$

Note.

(1) We started with the completeness as an axiom (i.e. every non-empty set bounded above in $\mathbb{R}$ has a least upper bound in $\mathbb{R}$).

(2) Every bounded monotone sequence in $\mathbb{R}$ converges (i.e. MCT)

(3) The Bolzano-Weierstrass theorem.

(4) Cauchy $\implies$ convergence.

These are actually all equivalent (i.e. if we assume one, we can prove the others).

I do not think we formally defined boundedness for sets in $\mathbb{R}^d$, but here it is.

Definition 5.8 (**Bounded Set in $\mathbb{R}^d$**).
Let $A \subseteq \mathbb{R}^d$. We say that A is **bounded** if $\exists R > 0$ s.t. $A \subseteq B_R(0)$.

6 Countability

Lecture 14, 2025/02/05

Definition 6.1 (**Same Cardinality**).

If A, B are sets, we say A and B have the same cardinality, i.e. $|A| = |B|$, if there exists a bijection $f : A \to B$.

Example. Finite sets have the same cardinality $\iff$ they have the same number of elements. For example, $|\{1, 2, 3\}| = |\{4, 5, 6\}|$. For finite sets, we write $|A| = n$ if A has n elements.

What about infinite sets? For example, do $\mathbb{N}$ and $\mathbb{R}$ have the same cardinality? What about $\mathbb{Q}$ and $\mathbb{Z}$?

Remark. We write $|A| \leq |B|$ if there exists an injection $f : A \to B$.

Exercise: If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Definition 6.2 (**Countable, Uncountable**).

Let A be a set. We say:

(1) A is **countable** (or finite) if there is a bijection $f : A \to \mathbb{N}$ (i.e. $|A| = |\mathbb{N}|$).

(2) A is **uncountable** if A is infinite and not countable.

Example. $\mathbb{N}$ is countable. We will prove that infinite subsets of $\mathbb{N}$ are countable.

Lemma 6.1 (**Well-Ordering Principle**).

Every non-empty subset of $\mathbb{N}$ has a minimum element.

Note. Well-ordering $\iff$ Induction. Also, we assume induction in PMATH 333.

Proof. Say $\varnothing \neq X \subseteq \mathbb{N}$ does not have a minimum ellement. So $1 \notin X$ (base case). Moreover, if $1, \dots, n \notin X$, then $n + 1 \notin X$. Thus, X cannot have a minimum element. Therefore, $n \notin X$ for all $n \in \mathbb{N}$, i.e. $X = \varnothing$. This is a contradiction. $\square$

Proposition 6.2. Any infinite subset of $\mathbb{N}$ is countable.

Proof. Let $A \subseteq \mathbb{N}$ be infinite (so $A \neq \varnothing$). By the Well-Ordering Principle, $\exists$ a smallest element $a_1 \in A$. Then, $A \backslash \{a_1\} \neq \varnothing$ is infinite. By the Well-Ordering Principle again, $\exists$ a smallest element $a_2 \in A \backslash \{a_1\}$. Continue this process to get a sequence $a_1, a_2, \ldots$ in A and $a_1 < a_2 < \cdots$. Moreover, every element of A is a_k for some $k \in \mathbb{N}$ because for each $a \in A$, $\{n \in A : n \leq a\}$ is finite and if $|\{n \in A : n \leq a\}| = k$, then $a = a_k$. Definte $f : A \to \mathbb{N}$ by $f(a_k) = k$. This is a bijection. Thus, A is countable. $\square$

Exercise: Verify that f is a bijection.

Corollary 6.3. Let A and B be sets with B countable.

(1) If $\exists$ an injection $f : A \to B$, then A is countable.

(2) If $\exists$ a surjection $g : B \to A$, then A is countable.

Proof.

(1) Since B is countable, there is a bijection, say $\phi : B \to \mathbb{N}$ is a bijection. Then, $\phi \circ f : A \to \mathbb{N}$ is an injection (composition of injection is an injection). Then, $|A| = |\phi \circ f(A)|$. By Proposition 6.2, since $\phi \circ f(A)$ is an infinite subset of $\mathbb{N}$, so $|\phi \circ f(A)| = |\mathbb{N}|$. Thus, A is countable.

(2) We use ϕ as above. Then, $g \circ \phi^{-1} : \mathbb{N} \to A$ is surjective. For each $a \in A$, pick $n_a \in \mathbb{N}$ with $g \circ \phi^{-1}(n_a) = a$ (note that $n_a \neq n_b$ for $a \neq b$). Define $h : A \to \mathbb{N}$ by $h(a) = n_a$. This is a bijection. Thus, A is countable.

$\square$

Exercise: Verify that h is a bijection.

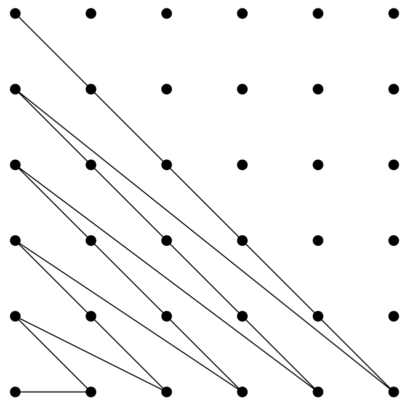
Example. $\mathbb{Z}$ is countable. Note that $\mathbb{Z} = \mathbb{N} \sqcup \mathbb{Z}_{\leq 0}$. We can define $f : \mathbb{Z} \to \mathbb{N}$ by

$$f(k) = \begin{cases} 2(k-1)+1 & \text{if } k \in \mathbb{N} \\ 2(-k)+2 & \text{if } k \in \mathbb{Z}_{\leq 0}. \end{cases}$$

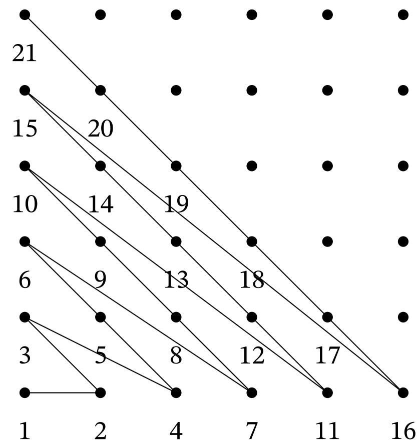
Exercise: Verify that f is a bijection.

Proposition 6.4. $\mathbb{N} \times \mathbb{N}$ is countable.

Proof. We draw the following grid and look at its diagonals.



We can label the dots.



For $n \in \mathbb{N}$, let $D_n = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i + j = n + 1\}$. Note that D_n is finite and $|D_n| = n$. Moreover, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Define a bijection $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ as follows. For $m \in \mathbb{N}$, pick $n \in \mathbb{N}$ s.t.

$$\frac{n(n-1)}{2} \leq m < \frac{n(n+1)}{2}.$$

Let $k = m - \frac{n(n-1)}{2}$ and let $\phi(m) = (k, n + 1 - k)$. For example, $m = 1 \implies n = 1 \implies k = 1 \implies \phi(1) = (1, 1)$. And, if $m = 2$, then $\frac{n(n-1)}{2} \leq 2 < \frac{n(n+1)}{2}$ implies $n = 2$ and $k = 1$. Thus, $\phi(2) = (1, 2)$. $\square$

Corollary 6.5. If A and B are sets, then, A and B are countable $\implies A \times B$ is countable.

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Corollary 6.6. A countable union of countable sets is countable.

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a collection of a countable sets. Let $B_1 = A_1$ and $B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ for $n \geq 2$. Note that $A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$. Let us enumerate each B_n, say, $B_n = \{b_1^{(n)}, b_2^{(n)}, \dots\}$. Define $f : A = \bigcup_{n\in\mathbb{N}} B_n \to \mathbb{N} \times \mathbb{N}$ by $f(b_i^{(n)}) = (i, n)$. By construction, $b_i^{(n)} \neq b_j^{(m)}$ for $i \neq j$ or $n \neq m$. So $(i, n) = (j, m) \iff b_i^{(n)} = b_j^{(m)}$. Thus, f is injective. Thus, $|A| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. Thus, A is countable. □

So far, we know $|\mathbb{N}| = |\mathbb{Z}|$, what about $\mathbb{Q}$?

Proposition 6.7. $\mathbb{Q}$ is countable.

Note. Think about it, every rational can be written uniquely as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b > 0$ and $\gcd(a, b) = 1$.

Proof. We will inject $\mathbb{Q}$ into $\mathbb{Z} \times \mathbb{N}$, which we know is countable. Referring to the note above, every $r \in \mathbb{Q}$ can be writen as $r = \frac{a}{b}$ for $a \in \mathbb{Z}$, $b \in \mathbb{N}$ and $\gcd(a, b) = 1$. This is unique. Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f(n) = (a, b)$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, $\gcd(a, b) = 1$ satisfying $r = \frac{a}{b}$. This is an injection by uniqueness of this decomposition. Thus, $\mathbb{Q}$ is countable. □

Example.

(1) $\mathbb{Q} \times \mathbb{Q}$ is countable.
(2) $\mathbb{Q} \cap [0, 1]$ is countable.
(3) $\mathbb{R}$ is uncountable! We will use Cantor's Diagonal Argument to prove this.

Every real number has a decimal expansion. Let us restrict ourselves to $[0, 1)$. For every $x \in [0, 1)$, there exists a unique $x_n \in \{0, 1, \dots, 9\}$ such that

$$\sum_{i=1}^{n} \frac{a_i}{10^i} \leq x \leq \sum_{i=1}^{n} \frac{a_i}{10^i} + \frac{1}{10^{n+1}}.$$

We write $x = 0.x_1x_2x_3\cdots$. The only thing to be cautious about is if there are infinitely many 9s, e.g. $0.999\cdots = 1$. But it is unique in $[0,1)$.

Proposition 6.8. $\mathbb{R}$ is uncountable.

Proof. It suffices that $\mathbb{R}$ has an uncountable subset. Let us look at $[0,1)$. Suppose for a contradiction that $[0,1)$ is countable. Then, $[0,1) = \{x_1, x_2, ...\}$. Write $x_1 = 0.x_{11}x_{12}x_{13}\cdots$, $x_2 = 0.x_{21}x_{22}x_{23}\cdots$, and so on. Define the numbers that are not on the list:

$$a_k = \begin{cases} 7 & \text{if } x_{kk} \in \{0,1,2,3,4\} \\ 2 & \text{if } x_{kk} \in \{5,6,7,8,9\} \end{cases}$$

There are no issues with non-uniqueness of decimal expansion since $9 = 0.a_1a_2a_3\cdots$ has only 2s and 7s. Clearly $a \neq x_k$ for any k, because otherwise, $x_{kk} = a_k$ for some k, but $|a_k - x_{kk}| \geq 2$. This is a contradiction, so $[0,1)$ is uncountable. Thus, $\mathbb{R}$ is uncountable. □

Note. The argument is called Cantor's Diagonal Argument. It can be applied more generally.

Theorem 6.9 (**Cantor's Theorem**).
If X is any set, then $|P(X)| \neq |X|$, where $P(X)$ is the power set of X.

Proof. Exercise. Use diagonalization, so there are lots of "infinities". □

Question: How big is $\mathbb{R}$ relative to $\mathbb{N}$? We know $\mathbb{R}$ is bigger but do we have $|\mathbb{R}| = |P(\mathbb{N})|$?

Answer: It turns out that is is impossible to say with our standard axioms of set theory (ZFC). Look up continuum hypothesis.

7 Compactness

Lecture 16, 2025/02/10

Recall: Bolzano-Weierstrass Theorem says every bounded sequence in $\mathbb{R}^d$ has a convergent subsequence.

Combine the Bolzano-Weierstrass Theorem and Theorem 5.5, we have the following.

Corollary 7.1. If $A \subseteq \mathbb{R}^d$ is closed and bounded, then every sequence in A has a convergent subsequence converging to a point in A.

Proof. If (x_n) is a sequence in A, since A is bounded, (x_n) is bounded. By Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}), say $x_{n_k} \to x \in \mathbb{R}^d$. By Theorem 5.5, $x \in \overline{A} = A$. $\square$

Motivation: We want to isolate the properties that make this corollary true.

Definition 7.1 (**Open Cover, Open Subcover, Compactness**).
Let $A \subseteq \mathbb{R}^d$.

(1) An **open cover** of A is a family of <u>open sets</u> $(U_i)_{i\in I}$ such that $A \subseteq \bigcup_{i\in I} U_i$.

(2) Given an open cover $(U_i)_{i\in I}$ of A, an **open subcover** of A is a subfamily $(U_j)_{j\in J}$ for $J \subseteq I$ such that $A \subseteq \bigcup_{j\in J} U_j$.

(3) A is **compact** if, whenever $(U_i)_{i\in I}$ is an open cover of A, $\exists$ a <u>finite</u> subcover of A.

Note. We use family and set interchangeably.

Remark. A is **compact** means if $(U_i)_{i\in I}$ are open sets with $A \subseteq \bigcup_{i\in I} U_i$, then $\exists U_{i_1}, \dots, U_{i_n}$ such that $A \subseteq \bigcup_{k=1}^{n} U_{i_k}$.

Example.

(1) A finite set $A = \{a_1, \dots, a_n\} \subseteq \mathbb{R}^d$ is compact.

Proof. To see this, let $(U_i)_{i\in I}$ be an open cover of A, so $A \subseteq \bigcup_{i\in I} U_i$. For each $1 \le j \le n$, choose U_{ij} with $a_j \in U_{ij}$. Then, $(U_{i1}, \dots, U_{in})$ is an open subcover of A. Hence, A is compact. $\square$

(2) $\mathbb{R}^d$ is not compact.

Proof. The family of open balls $(B_n(0))_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}^d$ with $0 \in \mathbb{R}^d$. However, there is no finite subcover of $\mathbb{R}^d$. To see this, suppose towards a contradiction that $(B_{n_1}(0), \dots, B_{n_k}(0))$ is a finite subcover of $\mathbb{R}^d$. Take $M = \max(n_1, \dots, n_k)$. Then, $\mathbb{R}^d \subseteq \bigcup_{i=1}^k B_{n_i}(0) \subseteq B_M(0)$, which is a contradiction since $\mathbb{R}^d$ is unbounded. □

(3) $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is not compact.

Proof. Let $U_m = \left(\frac{1}{m}, 2\right)$, $m \in \mathbb{N}$. Then, $A \subseteq \bigcup_{m\in\mathbb{N}} U_m$, so $(U_m)_{m\in\mathbb{N}}$ is an open cover of A. However, it has no finite subcover because for any $m \in \mathbb{N}$, $\frac{1}{m} \in A$ but $\frac{1}{m} \notin U_m$.

Exercise: Fill in the rest of this argument. □

Goal: Prove the Heine-Borel Theorem which says that a subset of $\mathbb{R}^d$ is compact $\iff$ it is closed and bounded.

Lemma 7.2. Compact subsets of $\mathbb{R}^d$ are bounded.

Proof. Let $A \subseteq \mathbb{R}^d$ be compact. We want to show that $\exists R > 0$ s.t. $A \subseteq B_R(0)$. The family of open sets $(B_n(0))_{n\in\mathbb{N}}$ is an open cover for $\mathbb{R}^d$, so it is also an open cover for A. By the compactness of A, $\exists n_1, \dots, n_k \in \mathbb{N}$ s.t. $A \subseteq \bigcup_{i=1}^k B_{n_i}(0)$. Let $N = \max(n_1, \dots, n_k)$. Then, $A \subseteq \bigcup_{i=1}^k B_{n_i}(0) \subseteq B_N(0)$. Thus, A is bounded. □

Note (**Information on Test 1**).

(1) Study the two assignments...

(2) State definitions (e.g. Cauchy sequences).

(3) For example, in the proof of the Monotone Convergence Theorem, ... This can be a question.

(4) Mostly from Chapter 5, about 2-3 questions.

(5) There will be a very similar question to the proof of the Squeeze Theorem: if $x_n \le y_n \le z_n$ and $x_n \to a$, $z_n \to a$, then $y_n \to a$.

Lemma 7.3. If $C \subseteq \mathbb{R}^d$ is compact, then it is closed.

Proof. Let $C \subseteq \mathbb{R}^d$ be compact. We will prove that $C = \overline{C}$. Equivalently, we will prove that if (x_n) is a sequence in C with $x_n \to x \in \mathbb{R}^d$, then $x \in C$. Suppose (x_n) as above with $x_n \to x \in \mathbb{R}^d \setminus C$. For each $y \in C$, choose an open ball B_y around y such that there is some open ball around x that does not intesect B_y (exercise). The family $(B_y)_{y\in C}$ is an open cover of C. By the compactness of C, there is a finite subcover, say $B_{y_1}, \dots, B_{y_n}$. For each of $y_1, \dots, y_n$, by the construction of B_{y_i}, there is an open ball $U_{y_i} \ni x$ and $B_{y_i} \cap U_{y_i} = \varnothing$. The set $U := \bigcap_{i=1}^n U_{y_i}$ is open and $x \in U$. But $C \subseteq \bigcup_{i=1}^n B_{y_i}$ and $\left(\bigcup_{i=1}^n B_{y_i}\right) \cap U = \varnothing$, so $C \cap U = \varnothing$. However, since $x_n \to x$ and $x_n \in C$ for all n, $x \in \overline{C} = \{z \in \mathbb{R}^d : V \cap C \neq \varnothing \ \ \forall \text{ open } V \ni z\}$. Since $C \cap U = \varnothing$, this is a contradiction. Therefore, whenever (x_n) is a sequence in C with $x_n \to x \in \mathbb{R}^d$, $x \in C$. By Proposition 3.2, $C = \overline{C}$, so C is closed. □

Lemma 7.4. If $D \subseteq \mathbb{R}^d$ is compact and $C \subseteq \mathbb{R}^d$ is closed with $C \subseteq D$, then C is also compact (i.e. Compactness is inherited by closed subsets).

Proof. Let $(U_i)_{i\in I}$ be an open cover of C. We must show that it admits a finite subcover of C. Since C is closed, $U := \mathbb{R}^d \setminus C$ is open, and $(U_i)_{i\in I} \cup \{U\}$ is an open cover of D. By the compactness of D, there is a finite subcover, say $U_{i_1}, \dots, U_{i_n}, U$ (note that adding U does not change being a finite subcover of D). Since $D \subseteq \left(\bigcup_{k=1}^n U_{i_k}\right) \cup U$ and $U \cap C = \varnothing$, $C \subseteq \bigcup_{k=1}^n U_{i_k}$. Hence, $U_{i_1}, \dots, U_{i_n}$ is a finite subcover of C. □

Recall: $\mathbb{R}^d = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d \text{ times}}$. More generally, the Cartesian product of d sets $A_1, \dots, A_d$ is

$$A_1 \times \cdots \times A_d = \{(a_1, \dots, a_d) : a_i \in A_i\}.$$

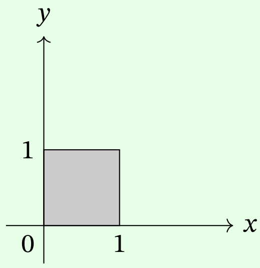
Definition 7.2 (Closed d-box).
A **closed d-box** in $\mathbb{R}^d$ is a set of the form

$$B = [a_1, b_1] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d$$

for closed intervals $[a_i, b_i] \subseteq \mathbb{R}$.

Note. $B = \{(x_1, \dots, x_d) : a_i \le x_i \le b_i, \forall i = 1, \dots, d\}$.

Example. For $d = 2$ and $B = [0,1] \times [0,1]$, we have the following.



Lemma 7.5. d-boxes $[a_1, b_1] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d$ are closed and bounded.

Exercise: Prove the lemma.

Theorem 7.6 (**Heine-Borel Theorem**).
A set $C \subseteq \mathbb{R}^d$ is compact $\iff$ it is closed and bounded.

Note (**Strategy for proof**). We know that compact $\implies$ closed. For the other direction, C is closed and bounded, C is closed subset of some closed d-cube $C \subseteq D$. By Lemma 7.4, it suffices to prove that D is compact.

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Proof of Heine-Borel Theorem.
($\Rightarrow$): This follows from Lemma 7.3 and Lemma 7.2.

($\Leftarrow$): Let $C \subseteq \mathbb{R}^d$ be closed and bounded. Since C is bounded, there is a d-box $D =: [-r, r]^d \supseteq C$ (D is a d-cube with radius (side length) $2r$). By Lemma 7.4, it suffices to prove that D is compact. Let $D_1 = D$. Suppose for the sake of contradiction that D_1 is not compact. Then, there is an infinite open cover $(U_i)_{i \in I}$ of D_1 with no finite subcover of D_1. Divide D_1 into 2^d closed sub-d-cubes with radius r.

Note. E.g. if $d = 2$, then $D_1 = [-r, 0] \times [-r, 0] \cup [-r, 0] \times [0, r] \cup [0, r] \times [-r, 0] \cup [0, r] \times [0, r]$.

Since no finite subset of $(U_i)_{i \in I}$ covers D_1, one of these sub-d-cubes also cannot be covered with any finite subset of $(U_i)_{i \in I}$. Call this sub-d-cube D_2. Proceeding in this way, we obtain a nested sequence of d-cubes $D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$ such that each D_n has radius $\frac{r}{2^{n-2}}$ and D_n connot be covered by finitely many $(U_i)_{i\in I}$. For each $n \geq 1$, choose $x_n \in D_n$. Then, for $n, m \geq N$,

$$\|x_n - x_m\| \leq \frac{r}{2^{N-2}}.$$

Hence, (x_n) is Cauchy. Hence, $\exists x \in \mathbb{R}^d$ s.t. $x_n \to x$. But x_n is eventually in each D_m (i.e. for n sufficiently large, in this case, $n > m$, $x_n \in D_m$). Furthermore, each D_m is closed. Hence, $x \in D_m$ for all $m \geq 1$. Since $(U_i)_{i\in I}$ covers D_1 and $x \in D_1$, $x \in U_{i_0}$ for some i_0. Since U_{i_0} is open, $\exists r > 0$ s.t. $B_r(x) \subseteq U_{i_0}$. Then for sufficiently large n, $D_n \subseteq B_r(x) \subseteq U_{i_0}$. In particularly, U_{i_0} is a finite open cover of D_n. This contradicts the fact that D_n has no finite open cover consisting of U_i's. Hence, $D_1 = D$ is compact. $\square$

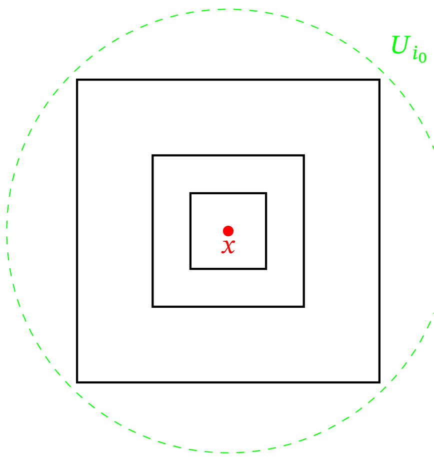

Figure 1: Final contradiction: limit point covered by finite subcover.

8 Functions

We will consider functions from $\mathbb{R}^m \to \mathbb{R}^n$.

8.1 Limits of Functions

Definition 8.1 (Limit of a Function).

Let $A \subseteq \mathbb{R}^m$ be a subset, $f : A \to \mathbb{R}^n$ be a function.

(1) For $a \in A$ and $y \in \mathbb{R}^n$, we write

$$\lim_{x \to a} f(x) = y \quad \text{in } A$$

if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\|f(x) - y\|_2 < \epsilon$ whenever $\|x - a\|_2 < \delta$ for all $x \in A$.

We also write $f(x) \to y$ as $x \to a$.

(2) We say that $f(x)$ has a limit in A as $x \to a$ if $\exists y \in \mathbb{R}^d$ s.t. $f(x) \to y$ as $x \to a$ in A.

Note. If $A = \mathbb{R}^m$, we often omit the phrase "in A".

We can use sequences to understand limits of functions.

Proposition 8.1. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$ be a function. The following are equivalent

(1) $f(x) \to y$ as $x \to a$ in A.

(2) For every sequence (x_n) in A with $x_n \to a$, the sequence $(f(x_n))$ converges to y.

Proof.

(1) $\implies$ (2): Suppose $f(x) \to y$ as $x \to a$ in A. Then, $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\|f(x) - y\|_2 < \epsilon$ if $\|x - a\|_2 < \delta$ for all $x \in A$. Let (x_n) be a sequence in A with $x_n \to a$. Then, $\forall \epsilon' > 0, \exists N \in \mathbb{N}$ s.t. $\|x_n - a\|_2 < \epsilon'$ for all $n \geq N$. We want to show that given $\epsilon > 0, \exists M \in \mathbb{N}$ s.t. $m \geq M$, then $\|f(x_m) - y\|_2 < \epsilon$. Choose $\delta > 0$ s.t. $\|f(x) - y\|_2 < \epsilon$ if $\|x - a\|_2 < \delta$ for all $x \in A$. Then, choose $M \in \mathbb{N}$ s.t. $\|x_m - a\|_2 < \delta$ for all $m \geq M$. Then, for $m \geq M$, we have $\|f(x_m) - y\|_2 < \epsilon$. Hence, $(f(x_n))$ converges to y.

(2) $\implies$ (1): We will prove the contrapositive. Suppose $f(x) \not\to y$ as $x \to a$ in A. Then, $\exists \epsilon > 0$ s.t. $\forall \delta > 0, \exists x \in A$ s.t. $\|x - a\|_2 < \delta$ but $\|f(x) - y\|_2 \geq \epsilon$.

Note. The negation of $p \implies q$ is $p \wedge \neg q$.

Then for each $n \in \mathbb{N}$, $\exists x_n \in A$ s.t. $\|x_n - a\|_2 < \frac{1}{n}$ but $\|f(x_n) - y\|_2 \geq \epsilon$. Then, the sequence (x_n) converges to a, but $f(x_n) \not\to y$ because $\|f(x_n) - y\|_2 \geq \epsilon$ for all $n \in \mathbb{N}$. Hence, (2) fails. $\square$

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Corollary 8.2 (**Uniqueness of Limits for Functions**).
Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Then, f has at most one limit as $x \to a$.

Note. That is, if $f(x) \to y_1$ and $f(x) \to y_2$ as $x \to a$, then $y_1 = y_2$.

Proof. Suppose that $f(x) \to y_1$ and $f(x) \to y_2$ as $x \to a$. Choose $x_n \in A$ s.t. $x_n \to a$. Then, by Proposition 8.1, $f(x_n)$ converges to y_1 and y_2. By Proposition 5.2, limits of sequences are unique, hence $y_1 = y_2$. $\square$

Corollary 8.3. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Write $f(x) = (f_1(x), \dots, f_n(x))$, $x \in A$, where $f_i : A \to \mathbb{R}$ for all $1 \leq i \leq n$. Then,

$$f(x) \to y \text{ as } x \to a \text{ in } A \iff f_i(x) \to y_i \text{ for all } 1 \leq i \leq n \text{ as } x \to a \text{ in } A.$$

Proof. Exercise using Proposition 8.1. $\square$

Example.

(1) Consider the Euclidean norm $\|\cdot\|_2 : \mathbb{R}^m \to \mathbb{R}$ (as a function on $\mathbb{R}^m$).

Claim. $\|x\|_2 \to \|a\|_2$ as $x \to a$.

Proof. We must show that given $\epsilon > 0$, $\exists \delta > 0$ s.t.

$$\|x - a\|_2 < \delta \implies \Big\| \|x\|_2 - \|a\|_2 \Big\|_2 < \epsilon.$$

Note. The above is equivalent to saying $\Big\| \|x\|_2 - \|a\|_2 \Big\| < \epsilon$ whenever $\|x - a\|_2 < \delta$. Also, $\Big\| \|x\|_2 - \|a\|_2 \Big\| = |\|x\|_2 - \|a\|_2|$.

By the reverse triangle inequality, we have

$$\left|\|x\|_2 - \|a\|_2\right| \leq \|x - a\|_2.$$

Therefore if we choose $\delta = \epsilon$, and $\|x - a\|_2 < \delta$, then

$$\left|\|x\|_2 - \|a\|_2\right| < \delta = \epsilon$$

from above. □

(2) For $A \subseteq \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$, the choice of A does matter. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then, $f(x)$ does not have a limit as $x \to a$ for any $a \in \mathbb{R}$. To see this, note that for $\epsilon = \frac{1}{2}$, then for any $\delta > 0$, there are irrational $x \in \mathbb{R}$ s.t. $|x - a| < \delta$, and rational $x \in \mathbb{R}$ s.t. $|x - a| < \delta$. Thus, $|f(x) - y| \not< \frac{1}{2}$ for any $y \in \mathbb{R}$ if $|x - a| < \delta$. Now, let $A = \mathbb{Q}$. Let $g = f\big|_A$ i.e. g is the restriction of f to A. So, g is the constant function 1 on A, clearly $g(x) \to 1$ as $x \to a$ in A for any $a \in A$.

8.2 Continuity of Functions

Definition 8.2 (Continuity).

Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$.

(1) For $a \in A$, we say f is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a) \quad \text{in } A.$$

(2) We say f is **continuous** if f is continuous at every $a \in A$.

Remark. This says that f is continuous at $a \in A$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \|x - a\|_2 < \delta \implies \|f(x) - f(a)\|_2 < \epsilon \quad \forall x \in A.$$

Proposition 8.4. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Then f is continuous at $a \in A \iff$ for every sequence (x_n) in A with $x_n \to a$, we have $f(x_n) \to f(a)$.

Note. This is a direct consequence of Proposition 8.1 by taking $y = f(a)$. The sequence $(f(x_n))$ converges to $f(a)$.

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Proposition 8.5. Let $A \subseteq \mathbb{R}^m$ and let $f, g : A \to \mathbb{R}^n$ be continuous at $a \in A$. Then, the functions

$$f + g,\ cf : A \to \mathbb{R}^n$$

are continuous at a. Furthermore,

$$f \cdot g : A \to \mathbb{R}^n$$

is also continuous at a (where $f \cdot g$ is the dot product of f and g).

Note. For $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$ with $f_i,\ g_i : A \to \mathbb{R}$, let $c \in \mathbb{R}$, then

$$\begin{aligned}(f+g)(x) &= ((f_1+g_1)(x), \dots, (f_n+g_n)(x)),\\ cf(x) &= (cf_1(x), \dots, cf_n(x)),\\ (f \cdot g)(x) &= (f_1 \cdot g_i)(x) + \dots + (f_n \cdot g_n)(x).\end{aligned}$$

If $n = 1$, the dot product is the ordinary multiplication.

Proof. The continuity of $f+g$ and cf follows from the characterization of limits of functions in terms of sequences, plus the fact that limits of sequences respect addition and scalar multiplication, i.e.

- $\lim(x_n + y_n) = \lim x_n + \lim y_n$, if these limits exist.
- Similarly, $\lim(cx_n) = c \lim x_n$.

Similarly for $f \cdot g$. □

Corollary 8.6. Let V_a be the set of all functions

$$V_a = \{f : A \to \mathbb{R}^n : f \text{ is continuous at } a \in A\}.$$

Then, V_a is a vector space. In particular,

$$V = \{f : A \to \mathbb{R}^n : f \text{ is continuous at all } a \in A\}$$

is also a vector space.

Proof. By Proposition 8.5, we know that V_a is a vector space since it is closed under addition and scalar multiplication. Next, V is a vector space because

$$V = \bigcap_{a \in A} V_a$$

and intersections of vector spaces are vector spaces. □

Proposition 8.7. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Write $f = (f_1, \dots, f_n)$, where $f_i : A \to \mathbb{R}$ for all $1 \leq i \leq n$. Then,

$$f \text{ is continuous at } a \in A \iff f_i \text{ is continuous at } a \text{ for all } 1 \leq i \leq n.$$

Proof. Apply the sequential characterization of limits. □

Example.

(1) Write $\mathbb{R}[x]$ for the vector space of all polynomials in x. View $p \in \mathbb{R}[x]$ as a function $p : \mathbb{R} \to \mathbb{R}$. Then, p is continuous on all of $\mathbb{R}$.

Proof. First consider $p(x) = x$. This is continuous: fix $a \in \mathbb{R}$, choose $\epsilon > 0$, we must find $\delta > 0$ s.t.

$$|x - a| < \delta \implies |x - a| = |p(x) - p(a)| < \epsilon.$$

Taking $\delta = \epsilon$ suffices. Hence, $p(x) = x$ is continuous. □

(2) Scalars (constant) polynomials are also continuous.

Now, starting with $p(x) = x, 1$, which are continuous from above, and taking sums, products and scalar multiples (in any order), we can obtain any polynomial in $\mathbb{R}[x]$. Since these operations preserve continuity by Proposition 8.5, we conclude that all polynomials are continuous.

(3) The Euclidean norm on $\mathbb{R}^n$, $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ is continuous (on all of $\mathbb{R}^n$).

Proof. We show that $\lim\limits_{x \to a} \|x\|_2 = \|a\|_2$. □

(4) More generally, we can use the equivalence of norms on $\mathbb{R}^n$ to show that every norm on $\mathbb{R}^n$ is continuous.

(5) For $K_1, \dots, K_m \in \mathbb{N}$, the function $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x) = x_1^{K_1} \cdots x_m^{K_m}$ for $x = (x_1, \dots, x_m)$, is continuous on $\mathbb{R}^m$.

Proof. To see this, first define

$$r, s : \mathbb{R}^m \to \mathbb{R}^m \quad \text{by} \quad r(x) = (x_1^{K_1}, \dots, x_m^{K_m}) \quad \text{and} \quad s(x) = (1, \dots, 1).$$

Then, r is continuous because each coordinate function is a polynomial in one variable and therefore continuous by (1) and Proposition 8.5. Also, s is continuous because it is a constant function. Then, $f = r \cdot s$ is continuous, being a dot product of continuous functions. □

(6) Let $\mathbb{R}[x_1, \dots, x_m]$ denote the vector space of polynomials over $\mathbb{R}$ in variables $x_1, \dots, x_m$. E.g. $p(x_1, \dots, x_m) = x_1^2 x_2^2 + \cdots + 17 x_{m-1} x_m^7$. Then, $p : \mathbb{R}^m \to \mathbb{R}$ is continuous for all $p \in \mathbb{R}[x_1, \dots, x_m]$.

Proof. This is an easy exercise using Proposition 8.5. □

(7) For $n \in \mathbb{N}$, the n-th root function $f : [0, \infty) \to \mathbb{R}$ by $f(x) = x^{1/n}$ is continuous at every $x \in [0, \infty)$.

Proof. We will have to use the δ-ϵ definition of continuity for this one.

Case 1: $x = 0$. We must show that for $\epsilon > 0$, there is $\delta > 0$ s.t.

$$|x - 0| < \delta \implies \left|x^{1/n} - 0^{1/n}\right| < \epsilon.$$

Equivalently,

$$x < \delta \implies x^{1/n} < \epsilon.$$

But $x^{1/n} \leq \epsilon \iff x \leq \epsilon^n$. Hence, taking $\delta = \epsilon^n$ suffices.

Case 2: $x > 0$. Again, given $\epsilon > 0$, we must show that there is $\delta > 0$ s.t.

$$|x - a| < \delta \implies \left|x^{1/n} - a^{1/n}\right| < \epsilon.$$

Let $r = x^{1/n}$, $s = a^{1/n}$. Then,

$$\begin{aligned} |x-a| = |r^n - s^n| &= |r-s||r^{n-1} + r^{n-2}s + \cdots + rs^{n-2} + s^{n-1}| \\ &= \left|x^{1/n} - a^{1/n}\right||r^{n-1} + r^{n-2}s + \cdots + rs^{n-2} + s^{n-1}| \\ \implies \left|x^{1/n} - a^{1/n}\right| &= \frac{|x-a|}{|r^{n-1} + r^{n-2}s + \cdots + rs^{n-2} + s^{n-1}|} \\ &= \frac{|x-a|}{r^{n-1} + r^{n-2}s + \cdots + rs^{n-2} + s^{n-1}} \\ &\leq \frac{|x-a|}{s^{n-1}} = \frac{|x-a|}{a^{\frac{n-1}{n}}}. \end{aligned}$$

Therefore, taking $\delta = \epsilon a^{\frac{n-1}{n}}$, then

$$\left|x^{1/n} - a^{1/n}\right| \leq \frac{|x-a|}{a^{\frac{n-1}{n}}} < \frac{\epsilon a^{\frac{n-1}{n}}}{a^{\frac{n-1}{n}}} = \epsilon.$$

Hence, f is continuous at a. □

Exercise: Prove (4) and (6).

Lecture 21, 2025/03/05

Quotients

Proposition 8.8. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}$. Suppose $f(a) \neq 0$ and f is continuous at $a \in A$. Then, $\exists r > 0$ s.t. $f(x) \neq 0$ $\forall x \in B_r(a) \cap A$ and $\frac{1}{f} : B_r(a) \cap A \to \mathbb{R}$ is continuous at a.

Proof. By assumption, $c := |f(a)| \neq 0$. By the continuity of f at a, for $\epsilon = \frac{c}{2}$, $\exists \delta > 0$ s.t.

$$\|x-a\|_2 < \delta \implies |f(x) - f(a)| < \frac{c}{2}.$$

By the reverse triangle inequality, we have

$$\begin{aligned} |f(a)| - |f(x)| \leq |f(x) - f(a)| &< \frac{c}{2} \\ \implies |f(x)| > |f(a)| - \frac{c}{2} = c - \frac{c}{2} &= \frac{c}{2} > 0. \end{aligned} \tag{$\star$}$$

Equivalently, $|f(x)| > \frac{c}{2} > 0\ \forall x \in B_\delta(a)$. Now, we want to show that $\frac{1}{f} : B_\delta(a) \cap A \to \mathbb{R}$ is continuous at a. Given $\delta' > 0$ s.t.

$$\|x-a\|_2 < \delta' \implies \left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| < \epsilon' \quad \forall x \in B_\delta(a) \cap A.$$

Write $\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| = \left|\frac{f(a)-f(x)}{f(x)f(a)}\right|$. By $(\star)$, we have

$$|f(x)| > \frac{c}{2} \implies \frac{1}{|f(x)|} < \frac{2}{c}. \tag{1}$$

By the continuity of f (again), we can choose $\delta' > 0$ s.t.

$$\|x-a\|_2 < \delta' \implies |f(x)-f(a)| < \frac{\epsilon' c^2}{2}. \tag{2}$$

Now, for $\|x-a\|_2 < \delta'$, we have (by (1) and (2)),

$$\left|\frac{f(x)-f(a)}{f(x)f(a)}\right| < \frac{\epsilon' c^2/2}{(c/2)c} = \epsilon'.$$

The conclusion is that for $x \in B_\delta(a) \cap A$ with $\|x-a\|_2 < \delta$, we have

$$\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| < \epsilon'.$$

Hence, $\frac{1}{f}$ is continuous at a. □

Example. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then, f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and discontinuous at $(0,0)$.

Proof. To see this, first note that the function x^2+y^2 is continuous at every point and nonzero on $\mathbb{R}^2 \setminus \{(0,0)\}$. By Proposition 8.8, $\frac{1}{x^2+y^2}$ is continuous at every point except $(0,0)$ (why?). Hence, so is $\frac{xy}{x^2+y^2}$ continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. It remains to show that f is discontinuous at

$(0,0)$. Observe that if $y = x \neq 0$, then

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}.$$

Let $z_n = \left(\frac{1}{n}, \frac{1}{n}\right)$. Then, $z_n \to (0,0)$. But, $f(z_n) = \frac{1}{2} \to \frac{1}{2} \neq f(0,0) = 0$. By our sequential characterization of continuity, f is discontinuous at $(0,0)$. $\square$

8.3 Continuity and Topology

Definition 8.3 (**Relative Open/Closed Sets**).
Let $A \subseteq \mathbb{R}^m$. For $E \subseteq A$,

(1) E is **open relative to** A (or **relatively open** in A) if $\forall x \in E, \exists r > 0$ s.t. $B_r(x) \cap A \subseteq E$.

(2) E is **closed relative to** A (or **relatively closed** in A) if $A \setminus E$ is open relative to A.

Proposition 8.9. Let $E \subseteq A \subseteq \mathbb{R}^m$.

(1) E is open relative to $A \iff \exists$ open $G \subseteq \mathbb{R}^m$ s.t. $E = G \cap A$.

(2) E is closed relative to $A \iff \exists$ closed $F \subseteq \mathbb{R}^m$ s.t. $E = F \cap A$.

Example. Let $A = \mathbb{Z} \subseteq \mathbb{R}$ and $E = \{0\}$. Then, E is both relatively open and relatively closed in A (**relatively clopen**).

Proof. To see relative openness, let $G = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then, G is open and $E = G \cap A$. So relative openness follows from Proposition 8.9. To see relative closedness, let $F = E$, then F is closed in $\mathbb{R}$ and $E = F \cap A$. So again by Proposition 8.9, E is relatively closed in A. $\square$

Note. In $\mathbb{R}$, the only clopen sets are $\varnothing$ and $\mathbb{R}$ (exercise).

Lecture 22, 2025/03/07

Idea: Think of A as a "new" topological space with its own open/closed subsets "inherited" from $\mathbb{R}^m$.

Note. There is a convention in analysis that we use G for open sets and F for closed sets.

Proof of Proposition 8.9. Assuming (1), then (2) follows easily: the following statement

$$A \setminus E \text{ is relatively open in } A \iff \exists \text{ open } G \subseteq \mathbb{R}^m \text{ such that } A \setminus E = G \cap A$$

is equivalent to

$$E = G^c \cap A = (\mathbb{R}^m \setminus G) \cap A$$

and $\mathbb{R}^m \setminus G$ is closed if G is open. Now, we prove (1).

($\Rightarrow$): Suppose $E \subseteq A$ is relatively open in A. Then, $\forall x \in E$, $\exists r > 0$ s.t. $B_r(x) \cap A \subseteq E$. Let G be the union of all such $B_r(x)$'s. Then, $G \subseteq \mathbb{R}^m$ is open (in $\mathbb{R}^m$). Then,

$$E \subseteq G \cap A \subseteq E.$$

Hence, $E = G \cap A$.

($\Leftarrow$): Suppose $E = G \cap A$ for open $G \subseteq \mathbb{R}^m$. For $x \in E \subseteq G$, since G is open, there is $r > 0$ s.t. $B_r(x) \subseteq G$. Then, $B_r(x) \cap A \subseteq G \cap A = E$. Hence, E is relatively open in A. $\square$

Definition 8.4 (**Neighborhood in $\mathbb{R}^m$, Neighborhood Relative to A**).

1. For $x \in \mathbb{R}^m$, a **neighborhood** of x is an open set $U \subseteq \mathbb{R}^m$ s.t. $x \in U$.
2. For $A \subseteq \mathbb{R}^m$ and $x \in A$, a **neighborhood of x relative to A** is an open subset relative to A containing x, i.e. $V \subseteq A$ s.t. $x \in V$.

Note. $V \subseteq A$ is a neighborhood of $x \in A$ relative to $A \iff \exists$ a neighborhood $U \subseteq \mathbb{R}^m$ of x s.t. $V = U \cap A$. This follows immediately from (1) of Proposition 8.9.

Definition 8.5 (**Preimage, Image**).

Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$.

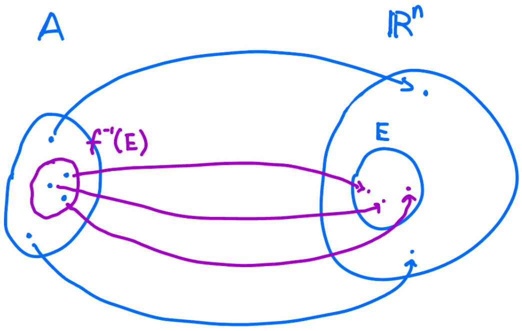
(1) For $E \subseteq \mathbb{R}^n$, the **preimage** of E (with respect to f) is

$$f^{-1}(E) = \{x \in A : f(x) \in E\}.$$

(2) For $B \subseteq A$, we will write the **image** of B as

$$f(B) = \{f(x) : x \in B\}.$$

Note. $f^{-1}(E)$ is not the same as the inverse of f (f may not be invertible). However, if f^{-1} (the inverse of f) exists, then (1) and (2) are compatible. Below is a plot showing the preimage of f.



Exercise: Let $A \subseteq \mathbb{R}^m$ and let $f; A \to \mathbb{R}^n$. For $(B_i)_{i\in I}$, subsets of $\mathbb{R}^n$. Show that

(1) $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$.

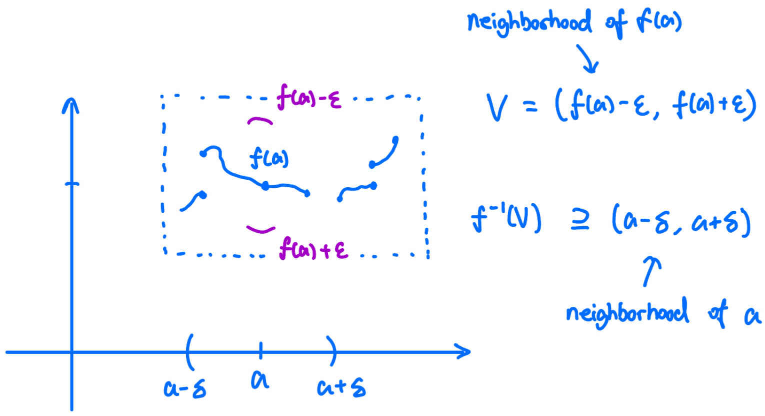
(2) $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$.

This will probably be on A4.

Continuity (revisited)

Proposition 8.10. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. For $a \in A$, f is continuous at $a \iff$ for every neighborhood $V \subseteq \mathbb{R}^n$ of $f(a)$, the preimage $f^{-1}(V) \subseteq A$ contains a neighborhood of a relative to A.

Note. The condition means $\exists U \subseteq A$ is relatively open in A with $a \in U$ s.t. $f(U) \subseteq V$. This is equivalent to $U \subseteq f^{-1}(V)$. We can draw the following plot to help visualize.



Proof.

($\Rightarrow$): Suppose f is continuous at a. Then for $\epsilon > 0$, $\exists \delta > 0$ s.t.

$$\|x - a\|_2 < \delta \implies \|f(x) - f(a)\|_2 < \epsilon \quad \forall x \in A. \tag{$\star$}$$

Let $V \subseteq \mathbb{R}^n$ be a neighborhood of $f(a)$. Since V is open, there is $\epsilon > 0$ s.t. $B_\epsilon(f(a)) \subseteq V$. By continuity, there is $\delta > 0$ s.t. ($\star$) is satisfied. In other words,

$$\|x - a\|_2 < \delta \implies \|f(x) - f(a)\|_2 < \epsilon \quad \forall x \in A$$

is equivalent to

$$x \in B_\delta(a) \cap A \implies f(x) \in B_\epsilon(f(a)).$$

So, setting $U = B_\delta(a) \cap A$, we have U is a relative neighborhood of a and $f(U) \subseteq V$.

($\Leftarrow$): Suppose that $\forall$ neighborhoods $V \subseteq \mathbb{R}^n$ of $f(a)$, there is a neighborhood relative to A of a s.t. $U \subseteq f^{-1}(V)$ ($f(U) \subseteq V$). Given $\epsilon > 0$, take $V = B_\epsilon(f(a))$. By assumption, there is a relative neighborhood in A of a, say $U \subseteq A$ s.t. $U \subseteq f^{-1}(V)$. By the definition of relatively open subsets of A, there is $\delta > 0$ s.t. $B_\delta(a) \cap A \subseteq U$. This translates to

$$\|x - a\|_2 < \delta \implies \|f(x) - f(a)\|_2 < \epsilon \quad \forall x \in A.$$

Hence, f is continuous at a. $\square$

Note. $f : \underbrace{X}_{=A} \to \underbrace{Y}_{=\mathbb{R}^n}$ is continuous if $\forall$ open $V \subseteq Y$ with $f(x) \in V$, $\exists$ open $U \subseteq X$ with $x \in U$ s.t. $U \subseteq f^{-1}(V)$.

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Corollary 8.11. A function $f : A \to \mathbb{R}^n$ is continuous $\iff$ $\forall$ open sets $V \subseteq \mathbb{R}^n$, the preimage $f^{-1}(V) \subseteq A$ is open relative to A.

Proof.

($\Rightarrow$): Suppose f is continuous (at every point). Let $V \subseteq \mathbb{R}^n$ be open. For $a \in A$ with $f(a) \in V$, V is a neighborhood of $f(a)$. By Propostion 8.10, $f^{-1}(V)$ will contain a neighborhood of a relative to A.

Note that $a \in f^{-1}(V) \iff f(a) \in V$. Hence, $f^{-1}(V)$ contains a neighborhood relative to A of every point in $f^{-1}(V)$. Therefore, $f^{-1}(V)$ is open relative to A.

($\Leftarrow$): The converse is clear because if $f^{-1}(V)$ is open relative to A for every open $V \subseteq \mathbb{R}^n$, then $f^{-1}(V)$ contains a neighborhood relative to A of every point in $f^{-1}(V)$. In particular, for $a \in f^{-1}(V)$, and $V \subseteq \mathbb{R}^n$, there is a neighborhood of a relative to A. By Proposition 8.10, f is continuous at a. $\square$

Question: Why look at preimages?

Answer: We are forced to.

Example. It is not always true for f continuous that $f(U)$ is open when $U \subseteq A$ is open relative to A. Consider $f(x) = 0$. This is continuous, but $f((0,1)) = \{0\}$ is not open despite $(0,1)$ being open.

8.4 Continuity and Compactness

Proposition 8.12. Let $A \subseteq \mathbb{R}^m$ be non-empty and let $f : A \to \mathbb{R}^n$ be continuous. If A is compact, then $f(A)$ is compact.

Note. The image of a compact set under a continuous function is compact. In addition, if A is empty, then $f(A) = \varnothing$ is also compact.

Proof. Let A be compact and f be continuous. Let (V_i) be an open cover of $f(A)$. We must show that there is a finite subcover. For each i, by the continuity of f, $f^{-1}(V_i)$ is relatively open in A by Corollary 8.11. By our characterization of relatively open sets, there is an open set $U_i \subseteq \mathbb{R}^n$ s.t. $f^{-1}(V_i) = A \cap U_i$. The open sets (U_i) are an open cover of A because

$$A = f^{-1}(f(A)) \subseteq f^{-1}\left(\bigcup_i V_i\right) \overset{\text{A4 Q1}}{=} \bigcup_i f^{-1}(V_i) = \bigcup_i (A \cap U_i).$$

By the compactness of A, there is $U_{i_1}, \dots, U_{i_n}$ s.t. $A \subseteq U_{i_1} \cup \dots \cup U_{i_n}$. Hence,

$$\begin{aligned} A = A \cap A &\subseteq \left(U_{i_1} \cap A\right) \cup \dots \cup \left(U_{i_n} \cap A\right) \\ &= f^{-1}(V_{i_1}) \cup \dots \cup f^{-1}(V_{i_n}) \\ &\overset{\text{A4 Q1}}{=} f^{-1}(V_{i_1} \cup \dots \cup V_{i_n}). \end{aligned}$$

Therefore, $f(A) \subseteq V_{i_1} \cup \dots \cup V_{i_n}$. So $f(A)$ has a finite cover from (V_i). Hence, $f(A)$ is compact. $\square$

Corollary 8.13. Let $\varnothing \neq A \subseteq \mathbb{R}^m$ be compact. Let $f : A \to \mathbb{R}$ be continuous. Then,

(1) f is bounded on A, i.e. $\exists r > 0$ s.t. $|f(x)| \leq r \ \forall x \in A$.

(2) $\sup\{f(x) : x \in A\}$ and $\inf\{f(x) : x \in A\}$ exist and attained, i.e. $\exists a, b \in A$ s.t. $f(a) = \sup\{f(x) : x \in A\}$ and $f(b) = \inf\{f(x) : x \in A\}$.

Proof. By Proposition 8.12, $f(A)$ is compact. Hence by Heine-Borel, $f(A)$ is closed and bounded.

(1) Boundedness of $f(A)$ means there is $r > 0$ s.t. $f(A) \subseteq \overline{B_r(0)}$. This is equivalent to $|f(x)| \leq r$ $\forall x \in A$.

(2) Since $f(A)$ is bounded, $\sup f(A)$ and $\inf f(A)$ exist. By the closedness of $f(A)$, $\sup f(A)$ and $\inf f(A)$ belong to $f(A)$. Hence, there is $c, d \in f(A)$ s.t. $c = \inf f(A)$ and $d = \sup f(A)$, where $c = f(a)$ and $d = f(b)$ for some $a, b \in A$. □

Example (The compactness condition is necessary in Corollary 8.13).
Consider $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Then, f is unbounded, and so does not attain a maximum despite being continuous. The problem is that $(0, 1)$ is not closed, so not compact.

Lecture 24, 2025/03/12

8.5 Continuity and Connectedness

Definition 8.6 (Separated Sets).
Let $A, B \subseteq \mathbb{R}^d$. We say that A and B are **separated** if $\overline{A} \cap B = \varnothing = A \cap \overline{B}$.

Definition 8.7 (Connected Set).
A set $C \subseteq \mathbb{R}^d$ is **connected** if there does not exist separated sets $A, B \subseteq \mathbb{R}^d$ s.t. $A \cup B = C$.

Example. A set $C \subseteq \mathbb{R}$ is connected $\iff$ whenever $x, z \in C$ with $x < z$, then $(x, z) \subseteq C$. This means that connected sets in $\mathbb{R}$ are intervals.

Theorem 8.14. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. If A is connected and f is continuous, then $f(A)$ is connected.

Sketch Proof.
Suppose for the sake of contradiction that $f(A) = X \cup Y$ for separated sets $X, Y \subseteq \mathbb{R}^n$. Then by A4 Q1, we have

$$A = f^{-1}(f(A)) = f^{-1}(X) \cup f^{-1}(Y)$$

and the fact that X and Y are separated implies that $f^{-1}(X)$ and $f^{-1}(Y)$ are separated (exercise). This contradicts the connectedness of $f(A)$. Hence, $f(A)$ is connected. □

Corollary 8.15 (**Intermediate Value Theorem**).
Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}$. If f is continuous and A is connected, then whenever $f(a) \leq y \leq f(b)$ for $a, b \in A$ and $y \in \mathbb{R}$, then $\exists c \in A$ s.t. $f(c) = y$.

Proof. By Theorem 8.14, $f(A)$ is connected. By our characterization of connected sets in $\mathbb{R}$, for $a, b \in A$ s.t. $f(a) \leq y \leq f(b)$, $y \in f(A)$. Hence, $\exists c \in A$ s.t. $f(c) = y$. □

8.6 Uniform Continuity

Motivation: We are interested in taking limits of continuous functions.

Definition 8.8 (**Uniform Continuity**).
Let $A \subseteq \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. We say that f is **uniformly continuous** on A if

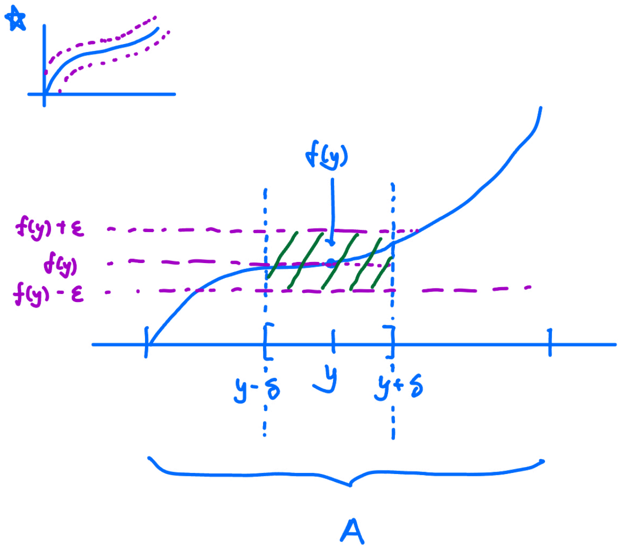
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. whenever } \|x - y\|_2 < \delta \implies \|f(x) - f(y)\|_2 < \epsilon \quad \forall x, y \in A.$$

Remark. Recall that f is continuous on A if $\forall \epsilon > 0$ and $a \in A$, $\exists \delta > 0$ s.t.

$$\|x - a\|_2 < \delta \implies \|f(x) - f(a)\|_2 < \epsilon \quad \forall x \in A.$$

Note that δ depends on a here. In the definition of uniform continuity, δ does not depend on a.

Below is a plot showing the uniform continuity of a function.



Note. Uniform continuity $\implies$ (ordinary) continuity.

Example. Let $f(x) = x^2$.

Claim. This function is uniformly continuous on $[-r, r]$ for any $r > 0$, but not on $\mathbb{R}$.

Proof. For uniform continuity on $[-r, r]$, fix $r > 0$. Choose $\epsilon > 0$. We want to find $\delta > 0$ s.t. $|x^2 - y^2| < \epsilon$ if $|x - y| < \delta$ $\forall x, y \in [-r, r]$. Note that

$$|x^2 - y^2| = |x - y||x + y|.$$

Choose $\delta = \frac{\epsilon}{2r}$. If $|x - y| < \delta = \frac{\epsilon}{2r}$, then

$$\begin{aligned} |x^2 - y^2| &= |x - y||x + y| \\ &\leq |x - y|(|x| + |y|) \\ &< \frac{\epsilon}{2r}(2r) = \epsilon. \end{aligned}$$

Hence, f is uniformly continuous on $[-r, r]$. However, f is not uniformly continuous on $\mathbb{R}$, i.e. (negation) $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in \mathbb{R}$ s.t. $|x - y|$ but $|x^2 - y^2| \geq \epsilon$. Choose $\epsilon = 1$. For $\delta > 0$, choose $x = \frac{1}{\delta}$, $y = \frac{1}{\delta} + \frac{\delta}{2}$. Then, $|x - y| = \frac{\delta}{2} \leq \delta$. But,

$$\begin{aligned}|x^2 - y^2| &= |x-y||x+y| \\ &= \frac{\delta}{2}\left(\frac{1}{\delta} + \frac{1}{\delta} + \frac{\delta}{2}\right) \\ &= 1 + \frac{\delta^2}{4} \geq 1.\end{aligned}$$

Hence, f is not uniformly continuous on $\mathbb{R}$. □

Exercise: Show that the Euclidean norm is uniformly continuous on $\mathbb{R}^n$.

Theorem 8.16. Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. If A is compact and f is continuous, then f is uniformly continuous.

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Proof. Since f is continuous, given $y \in A$ and $\epsilon > 0$, $\exists \delta_y > 0$ s.t. ,

$$\|x - y\| < \delta_y \implies \|f(x) - f(y)\| < \frac{\epsilon}{2} \quad \forall x \in A.$$

Note that the family of open balls $\left(B_{\frac{\delta_y}{2}}(y)\right)_{y \in A}$ is an open cover of A. By the compactness of A, there is $B_{\frac{\delta_{y_1}}{2}}, \dots, B_{\frac{\delta_{y_n}}{2}}$ that covers A. Let $\delta = \min\left(\frac{\delta_{y_1}}{2}, \dots, \frac{\delta_{y_n}}{2}\right)$. For $x, y \in A$, suppose that $\|x - y\|_2 < \delta$. Since we have an open cover of A, there is some i s.t. $y \in B_{\frac{\delta_{y_i}}{2}}(y_i)$. Then,

$$\begin{aligned}\|x - y_i\| = \|(x-y) + (y - y_i)\| &< \|x - y\| + \|y - y_i\| \\ &< \frac{\delta_{y_i}}{2} + \frac{\delta_{y_i}}{2} = \delta_{y_i}.\end{aligned}$$

Hence, $\|f(x) - f(y_i)\| < \frac{\epsilon}{2}$. Also, $\|f(y) - f(y_i)\| < \frac{\epsilon}{2}$. So

$$\begin{aligned}\|f(x) - f(y)\| &= \|(f(x) - f(y_i)) + (f(y_i) - f(y))\| \\ &\leq \|f(x) - f(y_i)\| + \|f(y_i) - f(y)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, f is uniformly continuous. □

Remark. This gives us a plentiful supply of uniformly continuous functions.

8.7 Uniform Convergence

We want to consider limits of sequences of functions.

Definition 8.9 (**Pointwise Convergence**).
Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. A sequence of functions $(f_k)_{k\in\mathbb{N}}$ with $f_k : A \to \mathbb{R}^n$ **converges pointwise** to f if

$$\forall x \in A, \lim_{k\to\infty} f_k(x) = f(x).$$

Note. For each $x \in A$, $(f_k(x))$ is an ordinary sequence in $\mathbb{R}^n$ and we are asking for convergence (in the usual sence) for each x.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$. Let $f_k : \mathbb{R} \to \mathbb{R}$ given by $f_k(x) = x + \frac{x^2}{2^k}$. Then for each $x \in \mathbb{R}$,

$$|f_k(x) - f(x)| = \left|\frac{x^2}{2^k}\right| = \frac{|x|^2}{2^k} \to 0 \quad \text{as } k \to \infty.$$

So $(f_k(x))$ converges to $f(x)$. Hence, $f_k \to f$ pointwise as $k \to \infty$.

Example. Let $f_k : [-1,1] \to \mathbb{R}$ given by

$$f_k(x) = \begin{cases} -1 & \text{if } x \le -\frac{1}{2^k}, \\ 2^k x & \text{if } x \in \left(-\frac{1}{2^k}, \frac{1}{2^k}\right), \\ 1 & \text{if } x \ge \frac{1}{2^k}. \end{cases}$$

Let $f : [-1,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Now, for $x \in [-1,1]$,

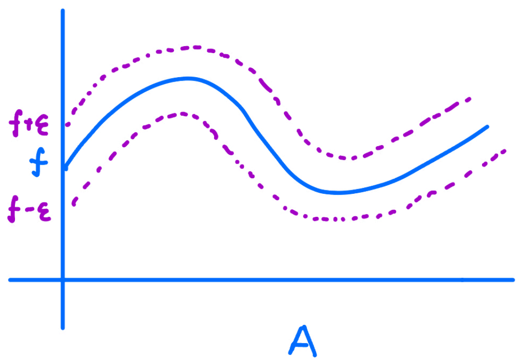
$$|f_k(x) - f(x)| \to 0 \quad \text{as } k \to \infty.$$

Therefore, $f_k \to f$ pointwise as $k \to \infty$, each f_k is continuous (so uniformly continuous by Theorem 8.16). But, f is not continuous. This tells us that if we want to perserve (uniform) continuity, pointwise convergence is not strong enough.

Definition 8.10 (**Uniform Convergence**).
Let $A \subseteq \mathbb{R}^m$ and let $f_k : A \to \mathbb{R}^n$, $k \in \mathbb{N}$. We say that (f_k) **converges uniformly** to a function $f : A \to \mathbb{R}^n$ if

$$\forall \epsilon > 0, \exists K \in \mathbb{N} \text{ s.t. } \|f_k(x) - f(x)\|_2 < \epsilon \quad \forall x \in A, \forall k \geq K.$$

Note. Uniform convergence asks that for sufficiently large k, the graph of f_k to be contained between the dotted lines as shown below.



Example. Recall the previous example. We saw that $f_k \to f$ pointwise. However, this convergence is not uniform because $f_k\left(\frac{1}{2^{k+1}}\right) = 2^k\left(\frac{1}{2^{k+1}}\right) = \frac{1}{2}$. So $\left|f_k\left(\frac{1}{2^{k+1}}\right) - f\left(\frac{1}{2^{k+1}}\right)\right| = \frac{1}{2}$. So for $0 < \epsilon < \frac{1}{2}$, and any $K \in \mathbb{N}$, take $k \geq K$, then $\left|f_k\left(\frac{1}{2^{k+1}}\right) - f\left(\frac{1}{2^{k+1}}\right)\right| = \frac{1}{2} > \epsilon$. This negates the definition of uniform convergence.

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Example. Let $f_k : [-1, 1] \to \mathbb{R}$ and $f : [-1, 1] \to \mathbb{R}$ be given by

$$f_k(x) = x + \frac{x^2}{2^k} \quad \text{and} \quad f(x) = x.$$

Then, $f_k \to f$ uniformly.

Proof. Choose $\epsilon > 0$. We have

$$|f_k(x) - f(x)| = \left|\frac{x^2}{2^k}\right| \le \frac{1}{2^k} \quad \forall x \in [-1,1].$$

Choose $K \in \mathbb{N}$ s.t. $\frac{1}{2^K} < \epsilon$. Then, for $k \ge K$, $\frac{1}{2^k} \le \frac{1}{2^K}$, and from above,

$$|f_k(x) - f(x)| \le \frac{1}{2^k} < \epsilon \quad \forall x \in [-1,1].$$

Since $\epsilon > 0$ was arbitrary, $f_k \to f$ uniformly. □

Theorem 8.17. Let $A \subseteq \mathbb{R}^m$ and let $f_k : A \to \mathbb{R}^n$ and $f : A \to \mathbb{R}^n$. Suppose that $f_k \to f$ uniformly and each f_k is uniformly continuous. Then, f is uniformly continuous.

Proof. Choose $\epsilon > 0$. Since $f_k \to f$ uniformly, then $\exists K \in \mathbb{N}$ s.t. $k \ge K \implies \|f_k(x) - f(x)\| < \frac{\epsilon}{3}$. Since f_k is uniformly continuous, $\exists \delta > 0$ s.t. $\|x - y\| < \delta \implies \|f_k(x) - f_k(y)\| < \frac{\epsilon}{3}$, $\forall x, y \in A$. We are now ready to prove that f is uniformly continuous. For $x, y \in A$ with $\|x - y\| < \delta$, we have

$$\begin{aligned}\|f(x) - f(y)\| &= \|(f(x) - f_k(x)) + (f_k(x) - f_k(y)) + (f_k(y) - f(y))\| \\ &\le \|f(x) - f_k(x)\| + \|f_k(x) - f_k(y)\| + \|f_k(y) - f(y)\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Hence, f is uniformly continuous. □

Example. We previously had an example of uniformly continuous functions $f_k : [-1,1] \to \mathbb{R}$ and $f : [-1,1] \to \mathbb{R}$ s.t. $f_k \to f$ pointwise but not uniformly. We proved by hand that the convergence was not uniform.This follows easily now, from the fact that f is not continuous.

Corollary 8.18. Let $U \subseteq \mathbb{R}^m$ be open and let $f_k : U \to \mathbb{R}^n$ be continuous and $f : U \to \mathbb{R}^n$. If $f_k \to f$ uniformly, then f is continuous.

Note. We are not assuming compactness of U or uniform continuity of the f_k's. E.g. we could take $U = \mathbb{R}^m$. The conclusion is slightly weaker: continuity, but not uniform continuity.

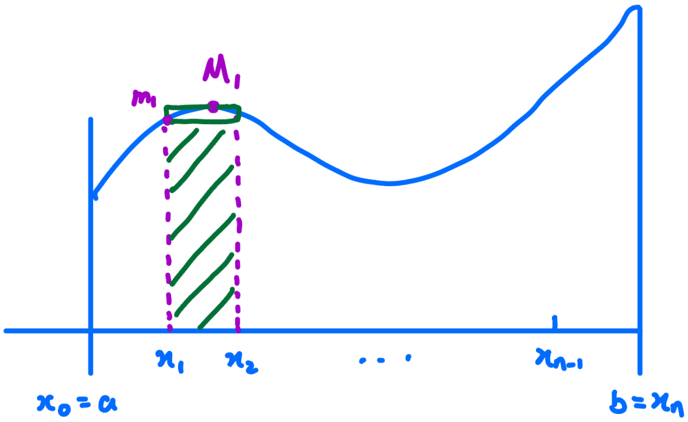
Proof. Choose $a \in U$. We must show that f is continuous at a. By openness of U, $\exists r > 0$ s.t. $B_{2r}(a) \subseteq U$. Then, $\overline{B_r(a)} \subseteq U$. Note that $\overline{B_r(a)}$ is compact (i.e. it is closed and bounded). By Theorem 8.16, each f_k is uniformly continuous on $\overline{B_r(a)}$. Also by our assumption that $f_k \to f$ uniformly on $\overline{B_r(a)}$. By Theorem 8.17, f is uniformly continuous on $\overline{B_r(a)}$. In particular, f is continuous at a. $\square$

9 Integration

9.1 Integration

Recall: for $f : [a, b] \to \mathbb{R}$. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Then, P "splits" $[a, b]$ into n intervals: $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$. For $0 \le i \le n-1$, let

$$m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\}$$
$$M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\}.$$



The area of the rectangle determined by $m_i, [x_i, x_{i+1}]$ is $m_i(x_{i+1} - x_i)$, and the area of the rectangle determined by $M_i, [x_i, x_{i+1}]$ is $M_i(x_{i+1} - x_i)$.

The lower sum corresponding to P is

$$L(f, P) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i).$$

The upper sum corresponding to P is

$$U(f, P) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i).$$

The idea is that if f is "reasonable", then we expect that

$$L(f, P) \approx \int_a^b f(x)\,dx \approx U(f, P).$$

To make this precise, we take larger and larger partitions. If $P \subseteq Q$, for a partition Q of $[a,b]$, we can prove that

$$\underbrace{L(f,P) \leq L(f,Q)}_{\text{monotone}} \leq \underbrace{U(f,Q) \leq U(f,P)}_{\text{monotone}}.$$

By monotonicity and MCT,

$$L(f) := \lim L(f,P)$$
$$U(f) := \lim U(f,P).$$

exist. We define f to be integrable if $L(f) = U(f)$, and we write

$$L(f) = U(f) = \int_a^b f(x)\,dx.$$

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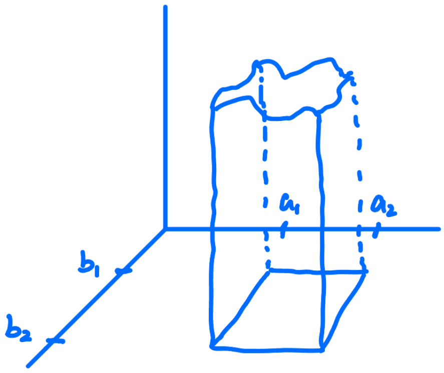
We consider integration of functions $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$. The integral $\int f$ should give us the "volume" under the graph of f. We will restrict to the case when A is a rectangle in $\mathbb{R}^2$: $A = [a_1, a_2] \times [b_1, b_2]$. The case of $\mathbb{R}^m$ with $m > 2$ is very similar.



Definition 9.1 (**Partitions**).
Let $A = [a_1, a_2] \times [b_1, b_2] \subseteq \mathbb{R}$ be a rectangle. A **partition** of A is a pair $P = (P_1, P_2)$, where

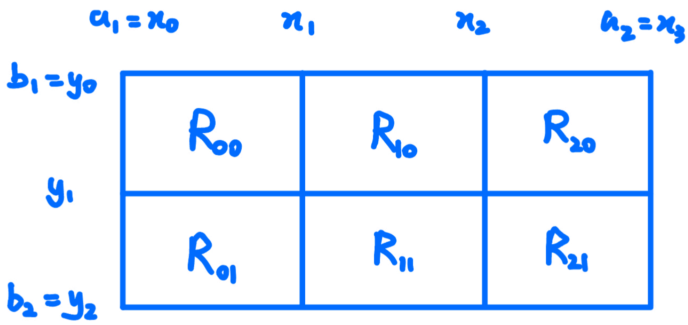
$$P_1 = \{a_1 = x_0 < x_1 < \cdots < x_m = a_2\}$$
$$P_2 = \{b_1 = y_0 < y_1 < \cdots < y_n = b_2\} \quad \text{for } m, n \in \mathbb{N}.$$

This divides A into $m \times n$ rectangles. For $0 \le i \le m-1$, $0 \le j \le n-1$, we get a rectangle

$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}].$$

Let $|R_{ij}|$ denote the area of R_{ij}, so $|R_{ij}| = (x_{i+1} - x_i) \cdot (y_{j+1} - y_j)$.

Note. An example of a partition looks like the following.



For partitions $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$, say that Q refines P (gives us more rectangle) if

$$P_1 \subseteq Q_1, \quad P_2 \subseteq Q_2.$$

This implies that the rectangle determined by Q subdivides the rectangles determined by P. We will use the notation $P \preceq Q$ to denote that Q refines P (i.e. $Q_1 \supseteq P_1$ and $Q_2 \supseteq P_2$). And $\preceq$ is the partial order for partition.

For arbitrary partitions P, Q of A, it may not be true that either $P \preceq Q$ or $Q \preceq P$. But there is always a partition $R = (R_1, R_2)$ of A such that

$$P \preceq R \quad \text{and} \quad Q \preceq R.$$

We can take $R = (R_1, R_2)$ defined by $R_1 = P_1 \cup Q_1$ and $R_2 = P_2 \cup Q_2$.

Definition 9.2 (**Upper and Lower Sums**).
Let $A = [a_1, a_2] \times [b_1, b_2] \subseteq \mathbb{R}^2$. Let $f : A \to \mathbb{R}$ be bounded. Let P be a partition of A

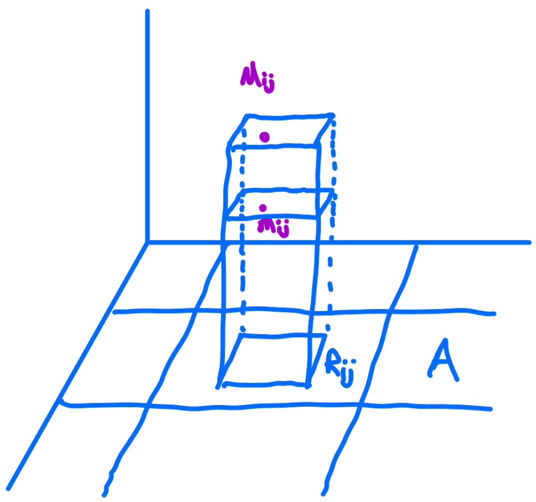
determining the rectangles $(R_{ij})_{\substack{0\le i\le m-1\\ 0\le j\le n-1}}$. Let

$$m_{ij} = \inf\{f(x) : x \in R_{ij}\}$$
$$M_{ij} = \sup\{f(x) : x \in R_{ij}\}.$$

The **upper and lower sums** of f with respect to P are

$$L(f,P) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} m_{ij}|R_{ij}|$$
$$U(f,P) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} M_{ij}|R_{ij}|.$$

Note. Clearly, $L(f,P) \le U(f,P)$. Here is a plot:



Lemma 9.1. Let $A \subseteq \mathbb{R}^2$ be a rectangle and let $f : A \to \mathbb{R}$ be bounded. Let P, Q be partitions of A with $P \preceq Q$ ($Q_1 \supseteq P_1, Q_2 \supseteq P_2$). Then,

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Proof. Suppose that

$$P_1 = \{x_0 < \cdots < x_m\}, \quad P_2 = \{y_0 < \cdots < y_n\}$$
$$Q_1 = \{x_0 < \cdots < x_k < c < x_{k+1} < \cdots < x_m\}, \quad Q_2 = P_2.$$

Let

$$\alpha_j = \inf\{f(x) : x \in [x_k, c] \times [y_j, y_{j+1}]\}$$
$$\beta_j = \inf\{f(x) : x \in [c, x_{k+1}] \times [y_j, y_{j+1}]\}.$$

Let $\gamma := \inf\{f(x) : x \in [x_k, x_{k+1}] \times [y_j, y_{j+1}]\}$. We can observe that

$$\alpha_j \geq \gamma \quad \text{and} \quad \beta_j \geq \gamma.$$

Now, we have

$$\begin{aligned} L(f,Q) - L(f,P) &= \sum_{j=0}^{n-1}(c - x_k)(y_{j+1} - y_j)\alpha_j + \sum_{j=0}^{n-1}(x_{k+1} - c)(y_{j+1} - y_j)\beta_j - \sum_{j=0}^{n-1}(x_{k+1} - x_k)(y_{j+1} - y_j)\gamma \\ &\geq \sum_{j=0}^{n-1} [(c - x_k) + (x_{k+1} - c) - (x_{k+1} - x_k)]\,\gamma = 0. \end{aligned}$$

Hence, $L(f,P) \leq L(f,Q)$. We know that $L(f,Q) \leq U(f,Q)$, and a similar argument gives us $U(f,Q) \leq U(f,P)$. The case of an extra point in Q_2 instead of Q_1 is similar. In the general case, we can iterate this argument. $\square$

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Definition 9.3 (**Integrals (Darboux), Integrable**).
Let $A \subseteq \mathbb{R}^2$ be a rectangle and let $f : A \to \mathbb{R}$ be bounded. The lower and upper integrals of f are

$$L(f) := \sup_P L(f,P)$$
$$U(f) := \inf_P U(f,P)$$

where P is taken to be a partition of A. We say f is **integrable** if $L(f) = U(f)$. In this case, we write

$$\int_A f \quad \text{or} \quad \int_A f(x)\,dx \quad \text{or} \quad \int_A f(x,y)\,dx\,dy$$

for $L(f) = U(f)$. Note that x, (x,y) are "dummy" variables.

Note. Lemma 9.1 implies that $L(f) \le U(f)$.

Lemma 9.2 (**Linearity of Integration**).
Let $A \subseteq \mathbb{R}^2$ be a rectangle and let $f, g : A \to \mathbb{R}$ be integrable. Then, for $c \in \mathbb{R}$, $f + cg$ is integrable and

$$\int_A (f + cg) = \int_A f + c\int_A g.$$

Proof Sketch. This follows from the linearity of summation and A5. □

Lemma 9.3 (**Integrability Criterion**).
Let $A \subseteq \mathbb{R}^2$ be a rectangle and let $f : A \to \mathbb{R}$ be bounded. Then,

$$f \text{ is integrable} \iff \forall \epsilon > 0, \exists \text{ partition } P \text{ of } A \text{ s.t. } U(f,P) - L(f,P) < \epsilon.$$

Proof. Note that $U(f) - L(f) \le U(f,P) - L(f,P)$ for any partition P of A by Lemma 9.1. □

Theorem 9.4. Let $A = [a_1, a_2] \times [b_1, b_2] \subseteq \mathbb{R}^2$ and let $f : A \to \mathbb{R}$. Suppose that f is continuous. Then, f is integrable.

Proof. We use the integrability criterion. For this, fix $\epsilon > 0$, we must find a partition P of A s.t. $U(f,P) - L(f,P) < \epsilon$. Since A is compact by Lemma 7.5 and Heine-Borel, and f is continuous, f is actually uniformly continuous by Theorem 8.16. By uniform continuity of f, there is $\delta > 0$ s.t. $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \frac{\epsilon}{2|A|}$, for all $x, y \in A$, where $|A| = (a_2 - a_1)(b_2 - b_1)$. Choose a partition $P = (P_1, P_2)$, say

$$P_1 = \{a_1 = x_0 < x_1 < \cdots < x_m = a_2\}$$
$$P_2 = \{b_1 = y_0 < y_1 < \cdots < y_n = b_2\}$$

s.t. for each rectangle R_{ij}, the maximum distance between any point $x, y \in R_{ij}$ is less than δ. For $x, y \in R_{ij}$, $\|x - y\| < \delta$, so

$$\|f(x) - f(y)\| < \frac{\epsilon}{2|A|}.$$

So, $\underbrace{\sup_{x \in R_{ij}} f(x)}_{M_{ij}} - \underbrace{\inf_{x \in R_{ij}} f(x)}_{m_{ij}} < \frac{\epsilon}{2|A|} < \frac{\epsilon}{|A|}$. Now,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{ij}|R_{ij}| - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij}|R_{ij}| \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (M_{ij} - m_{ij})|R_{ij}| \\ &< \frac{\epsilon}{|A|} \sum_{\substack{0 \le i \le m-1 \\ 0 \le j \le n-1}} |R_{ij}| = \frac{\epsilon}{|A|}|A| = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows from the integrability criterion that f is integrable. □

Lecture 29, 2025/03/26

9.2 Repeated Integrals

We would like to replace integrals in $\mathbb{R}^2$ with two integrals in $\mathbb{R}$.

Let $A = [a_1, a_2] \times [b_1, b_2] \subseteq \mathbb{R}^2$. Let $f : A \to \mathbb{R}$ be continuous. For $y \in [b_1, b_2]$, define

$$f_y : [a_1, a_2] \to \mathbb{R} \quad \text{by} \quad f_y(x) = f(x, y).$$

Then, f_y is continuous, so we can define

$$\mathbb{R} \ni F_y = \int_{a_1}^{a_2} f_y(x)\, dx = \int_{a_1}^{a_2} f(x, y)\, dx.$$

Exercise: The map $[b_1, b_2] \to \mathbb{R}$ by $y \mapsto F_y$ is continuous.

By the exercise, $y \to F_y$ is continuous, we can compute

$$I_{xy} = \int_{b_1}^{b_2} F_y \, dy = \int_{b_1}^{b_2} \underbrace{\left(\int_{a_1}^{a_2} f(x, y) \, dx \right)}_{y \text{ constant}} dy.$$

We can also do this in the other order to get

$$I_{yx} = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} f(x, y) \, dy \right) dx.$$

Theorem 9.5 (**Fubini's Theorem**).

Let $A \subseteq \mathbb{R}^2$ be a rectangle, i.e. $A = [a_1, a_2] \times [b_1, b_2]$. Let $f : A \to \mathbb{R}$ be continuous. Then,

$$\int_A f = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} f(x, y) \, dy \right) dx = \int_{b_1}^{b_2} \left(\int_{a_1}^{a_2} f(x, y) \, dx \right) dy.$$

Proof Sketch. Expand lower and upper sums in $\mathbb{R}$ (twice) and then in $\mathbb{R}^2$ with respect to sufficiently refined partitions, then show they get arbitrarily close. $\square$

Note. Take PMATH 450!

Example. Let $A = [0, 2] \times [0, 1]$. Define $f : A \to \mathbb{R}$ by $f(x, y) = xe^y$. By Fubini's Theorem,

$$\int_A f = \int_0^1 \left(\int_0^2 xe^y \, dx \right) dy = \int_0^1 \left[\frac{1}{2} x^2 e^y \right]_0^2 dy = \int_0^1 2e^y \, dy = [2e^y]_0^1 = 2(e - 1).$$

Exercise: Integrate with respect to y, then x.

9.3 Non-Rectangular Domains

What about $C \subseteq \mathbb{R}^2$ where C is not a rectangle but we want to compute $\int_C f$ for $f : C \to \mathbb{R}$?

Definition 9.4 (**Characteristic Function**).

For $C \subseteq \mathbb{R}^2$, the characteristic function of C is $\chi_C : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases}$$

Definition 9.5 (**Jordan Measurable**).

A subset $C \subseteq \mathbb{R}^2$ is **Jordan measurable** if there is a rectangle $A \subseteq \mathbb{R}^2$ with $C \subseteq A$ such that the characteristic function χ_C is integrable on A, i.e. $\displaystyle\int_A \chi_C$ exists.

Exericse: If $\displaystyle\int_A \chi_C$ exists for some rectangle $A \subseteq \mathbb{R}^2$ with $C \subseteq A$, then $\displaystyle\int_B \chi_C$ exists for any rectangles $B \subseteq \mathbb{R}^2$ with $C \subseteq B$.

Many sets are Jordan measurable. But many are not, here is an example.

Example. Let $C = \{(x, y) \in \mathbb{R}^2 : x, y \in [-1, 1] \text{ and } x, y \in \mathbb{Q}\}$ is not Jordan measurable.

Definition 9.6 (**Integral of** f **over** C).

For $C \subseteq \mathbb{R}^2$ Jordan measurable, $A \subseteq \mathbb{R}^2$ a rectangle with $C \subseteq A$, and a continuous function $f : A \to \mathbb{R}$, the integral $\displaystyle\int_C f$ is defined to be $\displaystyle\int_A \chi_C f$.

10 Spaces of Continuous Functions

Lecture 30, 2025/03/28

10.1 Spaces of Continuous Functions

Definition 10.1 (Spaces of Continuous Functions).
Let $X \subseteq \mathbb{R}^d$ be compact. Let $C(X) := C(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$. This is the **space of continuous functions** on X.

Note. This is a vector space with respect to

$$(f + cg)(x) = f(x) + cg(x) \quad \text{for } f, g \in C(X),\ x \in X,\ c \in \mathbb{R}.$$

Observe from previous results that $f + cg \in C(X)$. 0 is the zero function, i.e. $0(x) = 0$ for all $x \in X$.

Example. Let $X = [0, 1]$. Then

$$X = \pi = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\} = \overline{B_1(0)} \setminus B_1(0) = \partial B_1(0).$$

Recall: Each $f \in C(X)$ is actually uniformly continuous because X is compact. Also, since $f(X)$ is compact, f attains its maximum and minimum on X, i.e. $\exists a, b \in X$ s.t.

$$f(a) = \inf_{x \in X} f(x) \quad \text{and} \quad f(b) = \sup_{x \in X} f(x).$$

Exercise: Check above (a good exam question).

Definition 10.2 (Uniform Norm on $C(X)$).
For $X \subseteq \mathbb{R}^d$ compact, we define a **norm** on $C(X)$ by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|, \quad f \in C(X).$$

We call this the **uniform norm** on $C(X)$.

Recall: Let V be a vector space. A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ s.t.

(1) Positive Definiteness: $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0 \iff v = 0$.

(2) Positive Homogeneity: $\|cv\| = |c|\|v\|$ for all $v \in V$ and $c \in \mathbb{R}$.

(3) Triangle Inequality: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

Fact: The uniform norm on $C(X)$ is a norm

Proof.

(1) It is clear that $\|f\|_\infty = \sup_{x \in X} |f(x)| \geq 0$ for all $f \in C(X)$. If $f = 0$, then $\|f\|_\infty = 0$. If $\|f\|_\infty = 0$, then $f(x) = 0$ for all $x \in X$. Thus, $f = 0$.

(2) For $f \in C(X)$, $c \in \mathbb{R}$,

$$\begin{aligned} \|cf\|_\infty &= \sup_{x \in X} |cf(x)| \\ &= \sup_{x \in X} |c||f(x)| \\ &= |c| \sup_{x \in X} |f(x)| = |c|\|f\|_\infty. \end{aligned}$$

(3) For $f, g \in C(X)$,

$$\begin{aligned} \|f + g\|_\infty &= \sup_{x \in X} |f(x) + g(x)| \\ &\leq \sup_{x \in X} (|f(x)| + |g(x)|) \quad \text{Triangle Inequality in } \mathbb{R} \\ &\leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| \\ &= \|f\|_\infty + \|g\|_\infty. \end{aligned}$$

□

Remark. If $|X| = \infty$, then $\dim(C(X)) = \infty$. If $|X| = d \leq \infty$, then $C(X) \cong \mathbb{R}^d$.

Exercise: Prove the remark above.

Definition 10.3 (Normed Vector Space).
A vector space V along with a norm $\|\cdot\|$ on V is called a **normed vector space**.

Example.

(1) $\mathbb{R}^d$ with one of the norms we have considered, e.g. $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, is normed.

(2) $C(X)$ for $X \subseteq \mathbb{R}^d$ compact with the uniform norm $\|\cdot\|_\infty$ is normed.

Note. Most concepts we have considered for $\mathbb{R}^d$ generalize the normed vector spaces. For example, we have the following.

Definition 10.4 (Convergence in Normed Vector Space).
Let $(V, \|\cdot\|)$ be a normed vector space. For a sequence (v_n) in V,we say that v_n **converges** to $v \in V$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \implies \|v_n - v\| < \epsilon.$$

We write $v_n \to v$.

Proposition 10.1. Let $X \subseteq \mathbb{R}^d$ be comapct. A sequence of functions (f_n) in $C(X)$ converges to f in $C(X)$ with respect to $\|\cdot\|_\infty \iff f_n \to f$ uniformly.

Proof.
($\Rightarrow$): Note that $f_n \to f$ with respect to $\|\cdot\|_\infty$ means that

$$\begin{aligned}\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \implies \|f_n - f\|_\infty < \epsilon &\iff \sup_{x\in X} |f_n(x) - f(x)| < \epsilon \\ &\iff |f_n(x) - f(x)| < \epsilon \quad \forall x \in X.\end{aligned}$$

Therefore, $f_n \to f$ uniformly.

($\Leftarrow$): Converse is very similar (move the implications). Exercise. □

Other results generalize to normed vector spaces too.

Proposition 10.2. If (f_n) in $C(X)$ converges to f in $C(X)$, then (f_n) is Cauchy, i.e.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m, n \geq N \implies \|f_n - f_m\|_\infty < \epsilon.$$

Proof. Same as the proof for $\mathbb{R}^d$. □

Recall: In $\mathbb{R}^d$, (x_n) is Cauchy $\iff$ (x_n) converges.

Definition 10.5 (**Completeness of Normed Vector Space**).

A normed vector space $(V, \|\cdot\|)$ is **complete** if every Cauchy sequence in V converges in V.

Note. Completeness means there are sufficiently many points.

Example.

(1) $\mathbb{R}^d$ is complete by our results on Cauchy sequences.

(2) Consider $\mathbb{Q}$ as a vector space over $\mathbb{Q}$. We can turn it into a normed vector space with the usual absolute value $|\cdot|$. However, $\mathbb{Q}$ is not complete because we have Cauchy sequences in $\mathbb{Q}$ that converges (in $\mathbb{R}$) to points in $\mathbb{R} \setminus \mathbb{Q}$.

Fact: If $(V, \|\cdot\|)$ is incomplte, we can always complete it, meaning that there is a complete extension $(\overline{V}, \|\cdot\|)$ obtained by consider equivalence classes of Cauchy sequences in V as in A1.

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Proposition 10.3. $(C(X), \|\cdot\|_\infty)$ is complete.

Proof. Let (f_n) be Cauchy in $C(X)$. We must show that $\exists f \in C(X)$ s.t. $f_n \to f$ with respect to $\|\cdot\|_\infty$. Define $f : X \to \mathbb{R}$ in the following way: for $x \in X$, $|f_m(x) - f_n(x)| \le \|f_m - f_n\|_\infty$, so the sequence $(f_n(x))$ is Cauchy in $\mathbb{R}$. Since it is a sequence in $\mathbb{R}$, it converges to a point $f(x) \in \mathbb{R}$. This defined f. We must show that $f \in C(X)$ and $f_n \to f$ with respect to $\|\cdot\|_\infty$. We know that f_n is uniformly continuous because X is compact and f_n is continuous (by Cauchyness), and we know that uniformly convergent sequences of uniformly continuous functions converge to uniformly continuous functions. So it suffices to prove $f_n \to f$ uniformly. Choose $\epsilon > 0$. We want to prove that $\exists N \in \mathbb{N}$ s.t. if $n \ge N$, then

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in X.$$

Since (f_n) is Cauchy with respect to $\|\cdot\|_\infty$, there is $N \in \mathbb{N}$ s.t. if $m, n \ge N$, then

$$\|f_m - f_n\|_\infty < \frac{\epsilon}{2}.$$

Equivalently,

$$\sup_{x \in X} |f_m(x) - f_n(x)| < \frac{\epsilon}{2} \quad \text{or} \quad |f_m(x) - f_n(x)| < \frac{\epsilon}{2} \quad \forall x \in X.$$

Note that f is the pointwise limit of f_n. Taking $n \to \infty$ gives

$$|f_m(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon \quad \forall x \in X.$$

This proves that (f_n) converges uniformly to f, giving the result. □

Note. The last inequality has $\leq$ instead of $<$. Consider an example: $1 - \frac{1}{n} < 1$ for all $n \in \mathbb{N}$. But $1 - \frac{1}{n} \to 1$ as $n \to \infty$.

Theorem 10.4 (Weierstrass).

Let $X = [a, b] \subseteq \mathbb{R}$ be a closed (hence compact) interval. Note that for a polynomial $p(x) = c_0 + c_1 x + \cdots c_n x^n$, $p(x) \in C(X)$. Let $\mathcal{P}$ denote the set of all polynomials in $C(X)$. Then, $\mathcal{P}$ is dense in $C(X)$ with respect to $\|\cdot\|_\infty$.

Note. This says that for every continuous function $f \in C(X)$ and every $\epsilon > 0$, there is a polynomial $p(x)$ s.t.

$$\sup_{x \in X} |f(x) - p(x)| < \epsilon.$$

Note. We can use a method called Lagrange interpolation to find a polynomial $p(x)$.

10.2 Application of the Completeness of $C(X)$

Definition 10.6 (Convergence of Series in $C(X)$).

Let $X \subseteq \mathbb{R}^m$ be compact. Let (f_n) be a sequennce in $C(X)$. We say the series $\sum_{n \in \mathbb{N}} f_n$ converges to $f \in C(X)$ if the sequence of partial sums (s_n) in $C(X)$, $s_n = \sum_{k=1}^n f_k$ converges to f (with respect to $\|\cdot\|_\infty$).

Example. We are interested in power series:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{for } (c_n) \text{ in } \mathbb{R}.$$

We want to know if things like $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ are true and continuous.

Recall: For (c_n) in $\mathbb{R}$, the series $\sum_n c_n$ is **absolutely convergent** if the series $\sum_n |c_n|$ converges. That is, absolute convergence $\implies$ convergence.

Theorem 10.5 (Weierstrass M-Test).

Let $X \subseteq \mathbb{R}^d$ be compact. Let (f_n) be a sequence in X. Suppose that there is a sequence (M_n) in $\mathbb{R}$ with $M_n \geq 0$ s.t.

(1) $\sum M_n < \infty$ (converges).

(2) $\|f_n\|_\infty \leq M_n$ for all $n \in \mathbb{N}$.

Then, $\sum_n f_n$ converges.

Example. Applying to constant sequence (c_n) in $\mathbb{R}$ with $M_n = |c_n|$ recovers the result that absolute convergence $\implies$ convergence in $\mathbb{R}$.

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Proof of Theorem 10.5. Since $\sum M_n$ converges, the partial sums (t_n),

$$t_n = \sum_{k=1}^{n} M_k$$

converge, hence are a Cauchy sequence. Then, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|t_m - t_n| < \epsilon \quad \forall m, n \geq N.$$

Suppose that $m < n$, then

$$\begin{aligned} |t_m - t_n| &= |M_{m+1} + \cdots + M_n| \\ &= M_{m+1} + \cdots + M_n < \epsilon. \end{aligned}$$

For $n \in \mathbb{N}$, let

$$s_n = \sum_{k=1}^{n} f_k.$$

We must show that (s_n) converges, which is equivalent to showing (s_n) is Cauchy. We estimate, for $n > m \geq N$,

$$\begin{aligned} \|s_m - s_n\|_\infty &= \|f_{m+1} + \cdots + f_n\|_\infty \\ &\leq \|f_{m+1}\|_\infty + \cdots + \|f_n\|_\infty \\ &\leq M_{m+1} + \cdots + M_n \\ &< \epsilon. \end{aligned}$$

Hence (s_n) is Cauchy, as required. □

Note. We just used the completeness of $C(X)$ (i.e. Cauchy sequences in $C(X)$ converge in $C(X)$).

Corollary 10.6. Let (a_n) be a sequence in $\mathbb{R}$, $r > 0$ s.t.

$$\sum_n |a_n| r^n < \infty$$

Then the power series

$$\sum_n a_n x^n$$

converges uniformly (with respect to $\|\cdot\|_\infty$) to a continuous function on $[-r, r]$.

Proof. Let $X = [-r, r]$. This is compact, so we can apply Theorem 10.5. Let $f_n = a_n x^n$, $M_n = |a_n| r^n$. Then,

$$\begin{aligned} \|f_n\|_\infty &= \|a_n x^n\|_\infty \\ &= |a_n| \|x^n\|_\infty \\ &\leq |a_n| r^n \leq M_n. \end{aligned}$$

By Theorem 10.5, $\sum_n f_n$ converges uniformly to a continuous function in X. □

Example. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges uniformly to a continuous function on every interval $[-r, r]$

for $r > 0$ because $\sum_{n=0}^{\infty} \frac{1}{n!} r^n < \infty \; \forall r > 0$.

Note. The easiest way to show that it converges may be to use the ratio test.

END OF PMATH 333!

A Review

A.1 Topology

Open and closed sets

- $U \subseteq \mathbb{R}^d$ is open if $\forall x \in U, \exists r > 0$ s.t. $B_r(x) \subseteq U$.
- $C \subseteq \mathbb{R}^d$ is closed if $C^c := \mathbb{R}^d \setminus C$ is open.

Exercise: Prove the following equivalence definitions of the interior and closure of a set.

Interior and closure of a set

- $\text{Int}(X) = X^\circ =$ biggest open set contained in X.
- $\text{Cl}(X) = \overline{X} =$ smallest closed set containing $X = \{x \in \mathbb{R}^d : \forall r > 0, B_r(x) \cap X \neq \varnothing\}$.

Properties of Open and Closed Sets

Example.
- Unions of open sets are open.
- Intersection of finitely many open sets are open.

When studying, always try to drop assumptions and find counterexamples. For example, if finiteness is dropped in the second statement above, consider $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is closed.

Relatively open and closed Sets

- $A \subseteq \mathbb{R}^d$, $U \subseteq A$ is relatively open if $\forall x \in U, \exists r > 0$ s.t. $B_r(x) \cap A \subseteq U$.
- Similarly for relatively closed.

Example. The definitions for relatively open and closed sets can be counter intuitive. For example, if $A \subseteq \mathbb{R}$ and $X \subseteq A$ is both relatively open and closed (with respect to A), then $X = A$ or $X = \varnothing$. This is not true for open and closed sets in $\mathbb{R}^d$. For example, $A = \mathbb{N}$, then $\{1\}$ is both relatively open and closed in A.

A.2 Convergence of Sequences, inf, sup, Completeness of $\mathbb{R}$

Convergence

(x_n) converges in $\mathbb{R}^d$ if $\forall \epsilon > 0$, $\exists N$ s.t. $\|x_n - x\| < \epsilon \ \forall n \geq N$ for some x.

Remark. $\|x_n - x\| \leq \epsilon$ is equivalent to $x_n \in B_\epsilon(x)$.

Note. To prove a sequence not converges, we negate the statement.

Negation: $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N}$ s.t. $\|x_n - x\| \geq \epsilon$ for some $n \geq N$.

This is "extrinsic": need to refer to x.

"Intrinsic": Cauchy sequences (x_n) is Cauchy if $\forall \epsilon > 0$, $\exists N$ s.t. $\|x_n - x_m\| < \epsilon$ for all $n, m \geq N$.

Theorem A.1. (x_n) converges in $\mathbb{R}^d$ $\iff$ if is Cauchy.

- Lower Bound, Upper Bounds of $A \subseteq \mathbb{R}$.
- $\sup A$ = least upper bound of A.
- $\inf A$ = greatest lower bound of A.

We do not have above without the Axiom of Completeness: if $A \subseteq \mathbb{R}$ is bounded, then $\sup A$ exists.

- ϵ-characterization of sup (and inf): $c = \sup A \iff$
  - c is an upper bound for A.
  - $\forall \epsilon > 0$, $\exists a \in A$ such that $c - \epsilon < a \leq c$.

If A is closed, then $\sup A$, $\inf A$, if they exist, belong to A. In general, they belong to $\overline{A}$ if they exist.

Understand the followng.

Proposition A.2. For $x > 0$, $\exists! y > 0$ s.t. $y^2 = x$ (so $y = \sqrt{x}$).

A.3 Compactness (most important)

$C \subseteq \mathbb{R}$ is **compact** if for any open cover $\{U_i\}$ of C ($C \subseteq \bigcup_i U_i$), there is a finite open subcover $\{U_{i_1}, \dots, U_{i_n}\}$ of C ($C \subseteq \bigcup_{j=1}^{n} U_{i_j}$).

Proposition A.3. $B \subseteq C$ and C is compact, B is closed $\implies$ B is compact.

Note. Make sure to know the definitions well! For example, need to show there is an open cover before showing there is a finite open subcover.

Subsequences

Consider $(x_n)_{n\in\mathbb{N}}$ a sequence.

$$\underset{\uparrow}{x_1},\ x_2,\ \underset{\uparrow}{x_3},\ x_4,\ x_5,\ \underset{\uparrow}{x_6},\ \dots$$

We have $n_1 = 1, n_2 = 3, n_3 = 6, \dots$.

Monoton Convergence Theorem

Theorem A.4. If (x_n) is bounded and monotone, then it converges.

Proof Sketch.
Assume $x_1 \le x_2 \le \dots$. Then the proof is about understanding the least upper bound of the sequence (we applied the ϵ-characterization of sup). $\square$

Note. This is the theorem that unlocks compactness for us and is where subsequences come in.

Lemma A.5. We can extract a monotone subsequence from any sequence.

Corollary A.6. If (x_n) is bounded, then MCT $\implies$ that subsequence converges.

Theorem A.7 (Bolzano-Weierstrass).
Every bounded sequence has a convergent subsequence.

Corollary A.8. If $A \subseteq \mathbb{R}^d$ is closed and bounded, then every sequence in A has a convergent subsequence converging to something in A.

This corollary leads to the Heine-Borel theorem.

Theorem A.9 (Heine-Borel).
$C \subseteq \mathbb{R}^d$ is compact $\iff$ it is closed and bounded.

A.4 Functions

Continuity of Functions (most important)

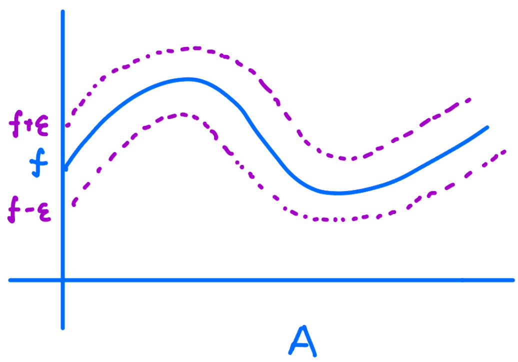
$A \subseteq \mathbb{R}^m$, $f : A \to \mathbb{R}^n$ is continuous at $a \in A$ means

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } \|x - a\| < \delta \implies \|f(x) - f(a)\| < \epsilon \ \ \forall x \in A.$$

Sequential Characterization

f is continuous at $a \iff \forall$ sequences (x_n) in A with $x_n \to a$, $f(x_n) \to f(a)$.

Uniform Continuity



This means

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ (independent of points) s.t. } \|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon \ \ \forall x, y \in A.$$

Remark (**Motivation for Uniform Continuity**).
Later, combined with uniform convergence, we have a way to ensure limits of (uniformly) continuous functions are continuous.

Example. Quadratic polynomials are not uniformly continuous on $\mathbb{R}$ (check!).

Theorem A.10. If $A \subseteq \mathbb{R}^d$ is compact, then any continuous function $f : A \to \mathbb{R}^n$ is uniformly continuous.

Topological Characterization of Continuity

$f : A \to \mathbb{R}^n$ is continuous at $a \in A \iff \forall$ neighborhood V of $f(a)$ (open set containing $f(a)$), $\exists$ neighborhood U of a relative to A (open set relative to A containing a) such that $U \subseteq f^{-1}(V)$.

A.5 Integration

We just finished Integration... Also important!