Supplementary Material
Weird scaling for 2-D avalanches: Curing the faceting, and scaling in the lower critical dimension

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I. SIMULATIONS

Experience simulating the RFIM on a square lattice has revealed a propensity for faceting in which the shape of the avalanche size distribution becomes dependent on properties of the lattice for small avalanche sizes. To mitigate this effect, we perform simulations on a periodic Voronoi lattice where, for each value of r, we consider 100 distinct lattices of size 1000x1000. Voronoi cells were chosen by generating random coordinates between 0 and 1 and constructing the cells with a 2D implementation of Voro++ [S1] provided by C. H. Rycroft. Examples of the avalanche behavior for different values of r are shown in Figure S1.

We note that much larger simulations have been done on the square lattice, including a thorough analysis of results from a $131,072^2$ lattice [S2, S3]. In analysis of in house simulations on a square lattice, however, we encountered long, unnaturally straight avalanche boundaries. We found these distortions strongly affected the shape of the size distribution for small disorders and served to effectively decreased the system size, a difficulty which became dramatically more pronounced as the disorder decreased. In addition to lattice dependent effects infecting the distributions for larger and larger avalanche sizes approaching the critical point, this effective reduction of system size encouraged the use of a Voronoi lattice.

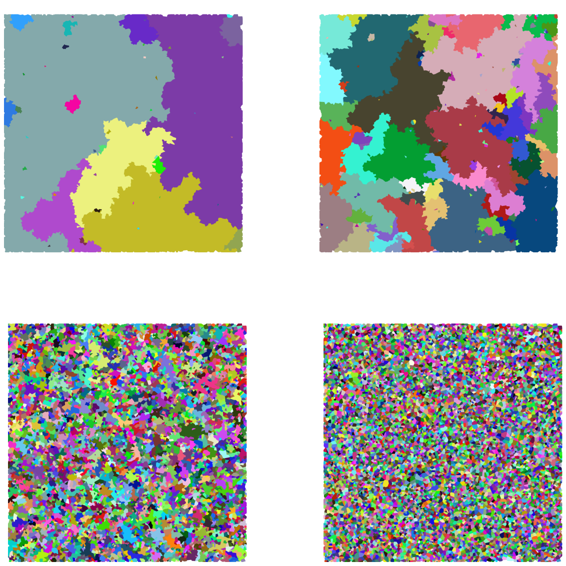

FIG. S1: $r = 0.5, 1.0, 5.0, 50.0$ from left to right, top to bottom

From the simulations we extract two quantities of interest: the area weighted avalanche size distribution $A(s|r)$ [S4] and the change in magnetization of the sample with respect to the field $\frac{dM}{dh}(h|r)$. Alternatively, we may write these as $A(s|w)$ and $\frac{dM}{dh}(h|w)$ where w is a function of r as defined in Appendix II.

II. NORMAL FORM OF THE RG DISORDER FLOW

In the equilibrium model, the flow equation is found to be $dw/d\ell = -(\epsilon/2)w + Aw^3 + h.o.t.$ where $\epsilon = D - 2$ and $w = r/J$ [S5]. For the NE-RFIM, however, r has the symmetry $r \leftrightarrow -r$ while J lacks this symmetry due to the external field. This implies $w \not\leftrightarrow -w$ and suggests that the RG flow for w in the NE-RFIM must include a squared order term.

In principle, there are an infinite series of terms. Assuming the lower critical dimension $D = 2$, we have $\epsilon = 0$, and may choose a scale for the disorder r_s such that the prefactor of the squared order term in the flow equation of w is equal to one. Taking $J = 1$, the choice we make for w is $w = (r - r_c)/r_s$ where r_c defines the critical disorder. The generic form for the flow equation of w is given by

$$\frac{dw}{d\ell} = w^2 + B_1 w^3 + B_2 w^4 + \ldots \tag{S1}$$

Using only analytic changes of variables, it is possible to remove all terms of $O(4)$ or higher without removing any universal behavior. To demonstrate, consider the change of variables $w = \tilde{w} + b_1\tilde{w}^2 + b_2\tilde{w}^3 + b_3\tilde{w}^4 + \ldots$ The resulting flow equation takes the form:

$$\frac{d\tilde{w}}{d\ell} = \tilde{w}^2 + B_1\tilde{w}^3 + (B_1 b_2 + b_2^2 + B_2 - b_3)\tilde{w}^4 + \ldots \tag{S2}$$

With an appropriate choice of b_2 in terms of b_3, the coefficient of $\tilde{w}^4$ may easily be set to zero. Likewise, all higher order terms may be systematically removed. Dropping the tildes and subscripts for clarity, the final form of the flow equation is given by

$$\frac{dw}{d\ell} = w^2 + Bw^3 \tag{S3}$$

III. SCALING COLLAPSE FORM FOR $\frac{dM}{dh}(h|w)$

The scaling form for the magnetization as a function of field in three dimensions would be

$$M_{\mathrm{3D}}(h|w) \sim w^{\beta}\mathcal{M}(h-h_c)/w^{\beta\delta}) \tag{S4}$$

yielding a 3D scaling form

$$\frac{dM_{\mathrm{3D}}}{dh}(h|w) = w^{\beta-\beta\delta}\frac{d\mathcal{M}_{\mathrm{3D}}}{dh}((h-h_c)/w^{\beta\delta}) \tag{S5}$$

In two dimensions, $w^{\beta\delta}$ is simply replaced by $\eta(w)$ from Eq. 4.

But what of the term w^{β}? It is quite typical for critical exponents to take saturating values in the lower critical dimension. We know that $\beta\delta \to \infty$ as $d \to 2$, but that does not tell us how β varies with dimension. Numerical simulations in higher dimensions [S6] show β decreasing from its mean-field value $\beta_{\mathrm{MF}} = 1/2$ in $d = 6$ down to $\beta_{\mathrm{3D}} = 0.035 \pm 0.0280$ in three dimensions. It is natural to expect that $\beta = 0$ in two dimensions, and that the universal scaling function $\mathcal{M}$ varies from -1 to 1 as the field increases (saturating the behavior). This implies that

$$\frac{dM}{dh}(h|w) = \eta(w)^{-1}\frac{d\mathcal{M}}{dh}(v) \tag{S6}$$

where the invariant scaling combination

$$v = (h - h_{\max})/\eta(w). \tag{S7}$$

It is traditional to scale with $h-h_c$, but since $h_{\max}-h_c \propto \eta(w)$, scaling to $h_{\max}$ is equivalent. The form chosen for the universal scaling function $\frac{d\mathcal{M}}{dh}(h|w)$ is given in Appendix VI.

IV. CORRELATION LENGTH

The correlation length may be calculated directly from the flow equation of the disorder. For the transcritical form

$$\frac{dw}{d\ell} = w^2 + Bw^3 \tag{S8}$$

we have

$$\int_{\ell_0}^{\ell^*} d\ell = \int_{w_0}^{w^*}\frac{dw}{w^2+Bw^3} \tag{S9}$$

where (w_0, ℓ_0) denotes an initial point and (w^*, ℓ^*) a fixed point of the RG, a constant. Performing the integration and letting $(w_0, \ell_0) \to (w, \ell)$ we obtain

$$\ell \sim B\log\left(B + \frac{1}{w}\right) - \frac{1}{w} \tag{S10}$$

We have

$$\xi \sim \exp(-\ell) \tag{S11}$$

hence

$$\xi \sim \left(\frac{1}{w} + B\right)^{-B}\exp\left(\frac{1}{w}\right) \tag{S12}$$

V. INVARIANT SCALING COMBINATIONS

A. Power Law Form

As our invariant parameter combinations are unorthodox, we provide here a thorough derivation and a comparison to the usual power law ‘homogeneous’ variables seen at the usual hyperbolic fixed points. The invariant scaling combinations corresponding to traditional power law scaling may be simply derived from the flow equations in 3 and higher dimensions. We have

$$\begin{aligned}\frac{dw}{d\ell} &= \frac{1}{\nu}w\\ \frac{ds}{d\ell} &= -\frac{1}{\sigma\nu}s\\ \frac{dh}{d\ell} &= \frac{\beta\delta}{\nu}h\end{aligned} \tag{S13}$$

Taking $(dw/d\ell)/(ds/d\ell)$ and integrating gives

$$\int_{w_0}^{w^*}\frac{dw}{(1/\nu)w} = \int_{s_0}^{s^*}\frac{ds}{(-1/\sigma\nu)s} \tag{S14}$$

Performing the integral and working through the algebra

$$\begin{aligned}&\log\ w^* - \log\ w_0 = -\sigma(\log\ s^* - \log\ s_0)\\ &\Rightarrow \sigma\log(s_0) + \log\ w_0 = \sigma\log\ s^* + \log\ w^*\\ &\Rightarrow s_0^{\sigma}w_0 = \text{constant}\end{aligned} \tag{S15}$$

where (w^*, s^*) corresponds to the fixed point of the RG and is hence a constant. The invariant scaling combination in this instance is thus

$$s^{\sigma}w \tag{S16}$$

which agrees with the results in 3 and higher dimensions [S6]. Similarly for h we have

$$\int_{w_0}^{w^*}\frac{dw}{(1/\nu)w} = \int_{h_0}^{h^*}\frac{dh}{(\beta\delta/\nu)h} \tag{S17}$$

Performing the integral and working through the algebra

$$\begin{aligned}
&\beta\delta(\log\ w^* - \log\ w_0) = \log\ h^* - \log\ h_0 \\
&\Rightarrow \log\ h_0 - \beta\delta \log\ w_0 = \log\ h^* - \beta\delta \log\ w^* \\
&\Rightarrow h_0 w_0^{-\beta\delta} = \text{constant}
\end{aligned} \tag{S18}$$

The invariant scaling combination is hence

$$h/w^{\beta\delta} \tag{S19}$$

which again agrees with the literature [S6].

B. Transcritical Form

The flow equations using the transcritical form for the disorder are as follows

$$\begin{aligned}
\frac{dw}{d\ell} &= w^2 + Bw^3 \\
\frac{ds}{d\ell} &= -d_f s - Csw \\
\frac{dh}{d\ell} &= \lambda_h h + Fhw
\end{aligned} \tag{S20}$$

As before, we take the integral of $dw/d\ell$ over $ds/d\ell$ and obtain

$$\int_{s_0}^{s^*} (1/s)\ ds = \int_{w_0}^{w^*} \frac{-d_f - Cw}{w^2 + Bw^3}\ dw \tag{S21}$$

Solving for s_0 we have

$$s_0 = \left(B + \frac{1}{w_0}\right)^{-Bd_f + C} \exp\left(\frac{d_f}{w_0}\right) f(w^*, s^*) \tag{S22}$$

where $f(w^*, s^*)$ denotes a function of w^* and s^* and is therefore constant. The invariant scaling combination in this case is then

$$\frac{s}{\Sigma_{\text{th}}(w)} \tag{S23}$$

where

$$\Sigma_{\text{th}}(w) = \left(B + \frac{1}{w}\right)^{-Bd_f + C} \exp\left(\frac{d_f}{w}\right) \tag{S24}$$

Likewise for h we obtain an invariant scaling combination

$$\frac{h}{\eta_{\text{th}}(w)} \tag{S25}$$

where

$$\eta_{\text{th}}(w) = \left(B + \frac{1}{w}\right)^{B\lambda_h - F} \exp\left(-\frac{\lambda_h}{w}\right) \tag{S26}$$

C. Alternative Transcritical Form

Applying our methods to the 2D equilibrium NE-RFIM, we find that the fixed point is given by a pitchfork bifurcation corresponding to

$$\frac{dw}{d\ell} = w^3 - Dw^5 \tag{S27}$$

In this instance, however, the behavior of the correlation length suggests an alternative choice for the normal form

$$\frac{dw}{d\ell} = \frac{w^3}{1 + Dw^2} \tag{S28}$$

as discussed in [S7]. This form, while retaining the pitchfork behavior, produces a well behaved correlation function that is also able to capture higher order corrections to scaling which we expect to become important further from the critical point. We may apply the same procedure in the non-equilibrium case, although the function for the correlation length here appears well behaved. This yields an alternative form for the transcritical bifurcation given by

$$\begin{aligned}
\frac{dw}{d\ell} &= \frac{w^2}{1 - Bw} \\
\frac{ds}{d\ell} &= -d_f s - Csw \\
\frac{dh}{d\ell} &= \lambda_h h + Fhw
\end{aligned} \tag{S29}$$

As before, to determine $\Sigma(w)$, we take the integral of $dw/d\ell$ over $ds/d\ell$ and obtain

$$\int_{s_0}^{s^*} (1/s)\ ds = \int_{w_0}^{w^*} \frac{-d_f - Cw}{w^2/(1 - Bw)}\ dw \tag{S30}$$

Solving for s_0 we have

$$s_0 = w_0^{Bd_f - C} \exp\left(\frac{d_f}{w_0} + BCw_0\right) f(w^*, s^*) \tag{S31}$$

where $f(w^*, s^*)$ denotes a function of w^* and s^* and is therefore constant. The invariant scaling combination in this case is then

$$\frac{s}{\Sigma_{\text{alt}}(w)} \tag{S32}$$

where

$$\Sigma_{\text{alt}}(w) = w^{Bd_f - C} \exp\left(\frac{d_f}{w} + BCw\right) \tag{S33}$$

Likewise for h we obtain an invariant scaling combination

$$\frac{h}{\eta_{\text{alt}}(w)} \tag{S34}$$

where

$$\eta_{\text{alt}}(w) = w^{-B\lambda_h + F} \exp\left(-\frac{\lambda_h}{w} - BFw\right) \tag{S35}$$

D. Pitchfork Form

The flow equations using a pitchfork form for the disorder are as follows

$$\begin{aligned}\frac{dw}{d\ell} &= w^3 + Bw^5 \\ \frac{ds}{d\ell} &= -d_f s - Csw \\ \frac{dh}{d\ell} &= \lambda_h h + Fhw\end{aligned} \tag{S36}$$

As before, we take the integral of $dw/d\ell$ over $ds/d\ell$ and obtain

$$\int_{s_0}^{s^*} (1/s)\ ds = \int_{w_0}^{w^*} \frac{-d_f - Cw}{w^3 + Bw^5}\ dw \tag{S37}$$

Solving for s_0 we have

$$\begin{aligned}s_0 \sim & w_0^{Bd_f}(1+Bw_0^2)^{-\frac{Bd_f}{2}} \\ & \times \exp\left(\frac{d_f}{2w_0^2} + \frac{C}{w_0} + \sqrt{B}C\arctan(\sqrt{B}w_0)\right)\end{aligned} \tag{S38}$$

The invariant scaling combination in this case is then

$$\frac{s}{\Sigma_{\rm pf}(w)} \tag{S39}$$

where

$$\begin{aligned}\Sigma_{\rm pf}(w) =& w^{Bd_f}(1+Bw^2)^{-\frac{Bd_f}{2}} \\ & \times \exp\left(\frac{d_f}{2w^2} + \frac{C}{w} + \sqrt{B}C\arctan(\sqrt{B}w)\right)\end{aligned} \tag{S40}$$

Likewise for h we obtain an invariant scaling combination

$$\frac{h}{\eta_{\rm pf}(w)} \tag{S41}$$

where

$$\begin{aligned}\eta_{\rm pf}(w) =& w^{-B\lambda_h}(1+Bw^2)^{\frac{B\lambda_h}{2}} \\ & \times \exp\left(-\frac{\lambda_h}{2w^2} - \frac{F}{w} - \sqrt{B}F\arctan(\sqrt{B}w)\right)\end{aligned} \tag{S42}$$

VI. UNIVERSAL SCALING FUNCTION FORMS

In order to perform our fits, we choose functional forms for the universal scaling functions. For the area weighted avalanche size distribution, we choose

$$\mathcal{A}(v_s) = \frac{1}{\mathcal{A}_N} v_s^a \exp(v_s^b) \tag{S43}$$

where the leading power law v_s^x has been absorbed into v_s^a here and $\mathcal{A}_N$ is the normalization factor $\mathcal{A}_N = \left[\Gamma\left(\frac{a}{b}\right)\gamma\left(\frac{a}{b}, \Sigma(w)^{-2b}\right)\right]/b$ where γ denotes the regularized upper incomplete gamma function.

For $d\mathcal{M}/dh$ we choose

$$\frac{d\mathcal{M}}{dh}(v_h) = \frac{1}{\frac{d\mathcal{M}}{dh}_N}\exp\left[\left(\frac{-v_h^2}{a+bv_h+cv_h^2}\right)^{d/2}\right] \tag{S44}$$

where $\frac{d\mathcal{M}}{dh}_N$ is a normalization factor computed as a sum of $\frac{dM}{dh}$ over the data range.

VII. FORM COMPARISON

In the lower critical dimension, we expect the fixed point to be governed by a transcritical bifurcation. Assuming compact avalanches, this yields directly

$$\Sigma_{\rm th}(w) = \Sigma_s\left(B + \frac{1}{w}\right)^{-Bd_f}\exp\left(\frac{d_f}{w}\right) \tag{S45}$$

where Σ_s is an unknown scale factor. We may compare this functional form for Σ with that derived assuming power law scaling and one assuming a pitchfork bifurcation as appears in the equilibrium model. For the power law case we have the invariant scaling combination

$$s^\sigma w \tag{S46}$$

which is equilvalent to

$$s/w^{-1/\sigma} \tag{S47}$$

such that $\Sigma_{\rm pl}(w)$ for the power law case would be given by:

$$\Sigma_{\rm pl}(w) = \Sigma'_s w^{-1/\sigma} \tag{S48}$$

where Σ'_s is an unknown scale factor determined by fitting to a power law form. For the pitchfork case we have from Appendix V D

$$\begin{aligned}\Sigma_{\rm pf}(w) =& \Sigma''_s w^{Bd_f}(1+Bw^2)^{-\frac{Bd_f}{2}} \\ & \times \exp\left(\frac{d_f}{2w^2} + \frac{C}{w} + \sqrt{B}C\arctan(\sqrt{B}w)\right)\end{aligned} \tag{S49}$$

where Σ''_s is the unknown scale factor. We have that $w = (r - r_c)/r_s$ for each of the functional forms considered. The comparison of the fits are shown in Figure 3.

VIII. PARAMETER VALUES

The parameter values corresponding to resonable fits are highly variable. For example, we may restrict $\lambda_h = 1$ corresponding to the Harris criteria for this model [S6]

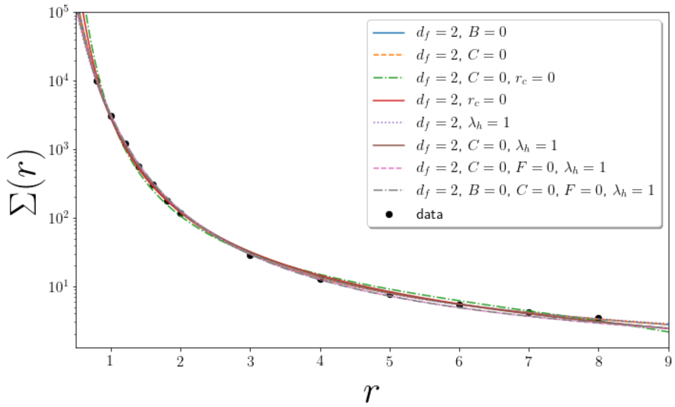

FIG. S2: Fit comparisons $\Sigma_{\rm th}(w)$, transcritical form

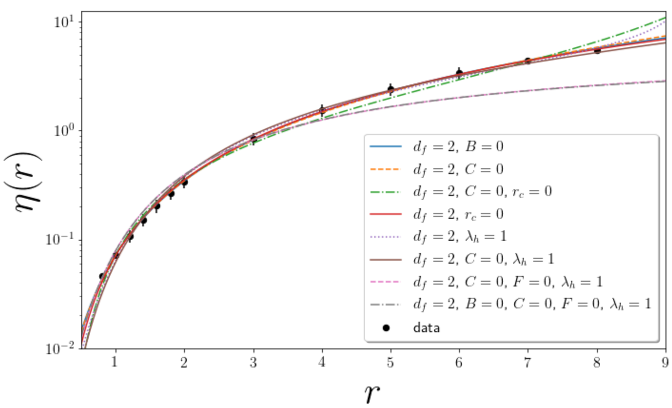

FIG. S3: Fit comparisons $\eta_{\rm th}(w)$, transcritical form

and still obtain an acceptable fit. A wide range of fits with various restrictions are shown in Figures S2, S3, S4, and S5. Corresponding best fit parameter values are shown in Tables S1 and S2. As anticipated, the alternative form for the transcritical bifurcation is able to better capture the behavior far from the critical point.

[S1] C. H. Rycroft, Chaos **19**, 041111 (2009).
[S2] D. Spasojević, S. Janićević, and M. Knežević, Phys. Rev. Lett. **106**, 175701 (2011).
[S3] D. Spasojević, S. Janićević, and M. Knežević, Phys. Rev. E **84**, 051119 (2011).
[S4] Y. J. Chen, S. Papanikolaou, J. Sethna, S. Zapperi, and G. Durin, Phys. Rev. E **84**, 061103 (2011).
[S5] A. J. Bray and M. A. Moore, J. Phys. C: Solid State Physics **18**, L927 (1985).
[S6] O. Perković, K. Dahmen, and J. P. Sethna, arXiv:cond-mat/9609072 (1996).
[S7] A. Raju, C. B. Clement, L. X. Hayden, J. P. Kent-Dobias, D. B. Liarte, D. Z. Rocklin, and J. P. Sethna, Phys. Rev. X **9**, 021014 (2019).

r_s	5.11 ± 0.54	5.49 ± 0.18	2.91 ± 0.27	2.04 ± 0.34	6.89 ± 0.63	4.93 ± 0.15	6.78 ± 0.20	7.40 ± 0.12
r_c	-0.42 ± 0.11	-0.46 ± 0.06	**0**	**0**	-0.65 ± 0.10	4.93 ± 0.15	-0.64 ± 0.06	-0.70 ± 0.03
Σ_s	1.24 ± 0.66	1.11 ± 0.05	5.27 ± 1.96	14.40 ± 8.84	0.68 ± 0.22	1.64 ± 0.04	0.64 ± 0.04	0.54 ± 0.005
η_s	3.16 ± 0.11	3.11 ± 0.44	1.02 ± 0.52	0.50 ± 0.41	6.26 ± 0.25	4.33 ± 0.42	5.55 ± 0.52	6.08 ± 0.82
d_f	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
λ_h	0.44 ± 0.04	0.52 ± 0.07	0.61 ± 0.05	0.24 ± 0.08	**1**	**1**	**1**	**1**
B	**0**	-0.15 ± 0.01	-0.27 ± 0.03	0.039 ± 0.007	-0.69 ± 0.21	-0.25 ± 0.03	-0.09 ± 0.01	**0**
C	0.46 ± 0.10	**0**	**0**	1.76 ± 0.28	-1.38 ± 0.52	**0**	**0**	**0**
F	1.72 ± 0.08	1.33 ± 0.12	0.73 ± 0.02	2.02 ± 0.13	-0.37 ± 0.35	0.45 ± 0.06	**0**	**0**

TABLE S1: Table of the best fit values corresponding to Figures S2 and S3. Values in bold correspond to values fixed in the fit.

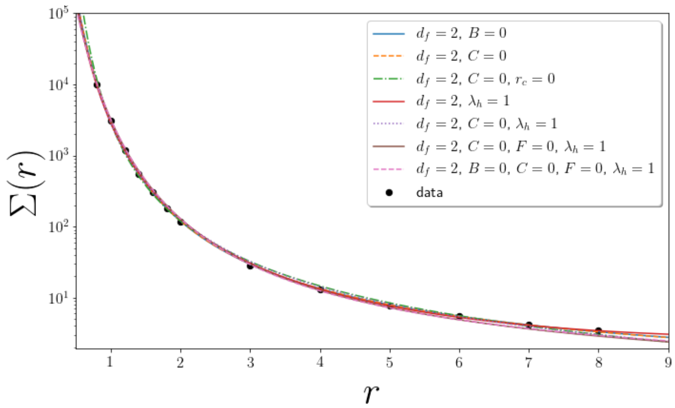

FIG. S4: Fit comparisons $\Sigma_{\rm alt}(w)$, alternative transcritical form

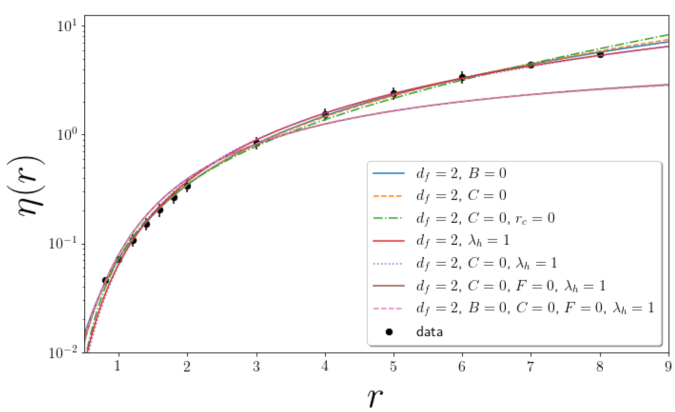

FIG. S5: Fit comparisons $\eta_{\rm alt}(w)$, alternative transcritical form

r_s	5.10 ± 0.54	5.05 ± 0.36	2.12 ± 0.33	1.81 ± 0.08	3.62 ± 0.18	6.57 ± 0.39	7.40 ± 0.12
r_c	-0.42 ± 0.11	-0.42 ± 0.09	**0**	-0.15 ± 0.07	-0.29 ± 0.09	-0.62 ± 0.10	-0.70 ± 0.03
Σ_s	1.24 ± 0.66	-1.27 ± 0.38	13.16 ± 5.72	21.44 ± 1.31	3.19 ± 0.45	-0.67 ± 0.32	0.54 ± 0.005
η_s	3.16 ± 0.11	2.49 ± 0.20	0.56 ± 0.40	0.69 ± 0.04	2.48 ± 0.09	5.42 ± 0.17	6.08 ± 0.82
d_f	**2**	**2**	**2**	**2**	**2**	**2**	**2**
λ_h	0.44 ± 0.04	0.54 ± 0.08	0.70 ± 0.05	**1**	**1**	**1**	**1**
B	**0**	-0.24 ± 0.01	-0.76 ± 0.14	-1.70 ± 0.16	-0.56 ± 0.05	-0.13 ± 0.01	**0**
C	0.46 ± 0.10	**0**	**0**	-0.31 ± 0.04	**0**	**0**	**0**
F	1.72 ± 0.08	1.22 ± 0.16	0.45 ± 0.04	-0.031 ± 0.038	0.35 ± 0.03	**0**	**0**

TABLE S2: Table of the best fit values corresponding to Figures S4 and S5. Values in bold correspond to values fixed in the fit.