GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6254 Spring 2023

Problem Set #1

Assigned: 12 Jan

Due Date: 18 Jan

Please complete each problem before looking at the solutions.
Your skills gained from completing this homework will be tested with an in-class, closed-book quiz.

Please contact the TAs for clarification on the instructions in the homework assignments.

Bins and Marbles

Problem 1 We have 2 opaque bags, each containing 2 balls. One bag has 2 black balls and the other has a black ball and a white ball. You pick a bag at random and then pick one of the balls in that bag at random. When you look at the ball, it is black. You now pick the second ball from that same bag. What is the probability that this ball is also black?

(a) 1/4

(b) 1/3

(c) 1/2

(d) 2/3

(e) 3/4

Consider a sample of 10 marbles drawn from a bin containing red and green marbles. The probability that any marble we draw is red is $\mu = 0.55$ (independently, with replacement). We address the probability of getting no red marbles ($v = 0$) in the following problems.

Problem 2 We draw only one such sample. Compute the probability that $v = 0$. The closest answer is ("closest answer" means: —your answer - given option— is closest to 0):

(a) $7.331x10^{-6}$

(b) $3.405x10^{-4}$

(c) 0.289

(d) 0.450

(e) 0.550

Problem 3 We draw 1,000 independent samples. Compute the probability that (at least) one of the samples has ($v = 0$) The closest answer is:

(a) $7.331x10^{-6}$

(b) $3.405x10^{-4}$

(c) 0.289

(d) 0.450

(e) 0.550

Feasibility of Learning Consider a Boolean target function over a 3-dimensional input space $\mathcal{X} = \{0, 1\}^3$ (instead of our $+/-1$ binary convention, we use 0,1 here since it is standard for Boolean functions). We are given a data set $\mathcal{D}$ of five examples represented in the table below, where $y_n = f(\mathbf{x}_n)$ for $n = 1, 2, 3, 4, 5$.

$\mathbf{x}_n$			y_n
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1

Note that in this simple Boolean case, we can enumerate the entire input space (since there are only $2^3 = 8$ distinct input vectors), and we can enumerate the set of all possible target functions (there are only $2^{2^3} = 256$ distinct Boolean function on 3 Boolean inputs). Let us look at the problem of learning f. Since f is unknown except inside $\mathcal{D}$, any function that agrees with $\mathcal{D}$ could conceivably be f. Since there are only 3 points in $\mathcal{X}$ outside $\mathcal{D}$, there are only $2^3 = 8$ such functions. The remaining points in $\mathcal{X}$ which are not in $\mathcal{D}$ are: 101, 110, and 111. We want to determine the hypothesis that agrees the most with the possible target functions. In order to quantify this, count how many of the 8 possible target functions agree with each hypothesis on all 3 points, how many agree on just 2 of the points, on just 1 point, and how many do not agree on any points. The final score for each hypothesis is computed as follows: Score = (# of target functions agreeing with hypothesis on all 3 points)x3 + (# of target functions agreeing with hypothesis on exactly 2 points)x2 + (# of target functions agreeing with hypothesis on exactly 1 point)x1 + (# of target functions agreeing with hypothesis on 0 points) x0.

Problem 4 Which hypothesis g agrees the most with the possible target functions in terms of the above score?

(a) g returns 1 for all three points.

(b) g returns 0 for all three points.

(c) g is the XOR function applied to $\mathbf{x}$, i.e., if the number of 1s in $\mathbf{x}$ is odd, g returns 1; if it is even, g returns 0.

(d) g returns the opposite of the XOR function: if the number of 1s is odd, it returns 0, otherwise returns 1.

(e) They are all equivalent (equal scores for g in [a] through [d]).

Hoeffding Inequality

Run a computer simulation for flipping 1,000 virtual fair coins. Flip each coin independently 10 times. Focus on 3 coins as follows: c_1 is the first coin flipped, c_{rand} is a coin chosen randomly from the 1,000, and c_{min} is the coin which had the minimum frequency of heads (pick the earlier one in case of a tie). Let v_1, v_{rand}, and v_{min} be the fraction of heads obtained for the 3 respective coins out of the 10 tosses. Run the experiment 100,000 times in order to get a full distribution of v_1, v_{rand}, and v_{min} (note that c_{rand} and c_{min} will change from run to run).

Problem 5 The average value of v_{min} is closest to:

(a) 0

(b) 0.01

(c) 0.1

(d) 0.5

(e) 0.67

Problem 6 Which coin(s) has a distribution of v that satisfies the (single-bin) Hoeffding Inequality?

(a) c_1 only

(b) c_{rand} only

(c) c_{min} only

(d) c_1 and c_{rand}

(e) c_{min} and c_{rand}