

# 华东理工大学

## 复变函数与积分变换作业本 (第4册)

### 第七次作业

教学内容: 4.1 复数项级数 4.2 幂级数

1. 判别下列复数项级数的收敛性, 若收敛, 求其极限, 其中  $n \rightarrow \infty$ .

$$(1) z_n = \frac{1+ni}{1+n};$$

$$\text{解: } z_n = \frac{1}{1+n} + \frac{n}{1+n}i$$

$$\lim_{n \rightarrow \infty} \frac{1}{1+n} = 0, \quad \lim_{n \rightarrow \infty} \frac{n}{1+n} = 1 \text{ 故 } z_n \text{ 收敛于 } i$$

$$(2) z_n = (-1)^n + \frac{i}{n+1};$$

解: 由于  $z_n$  的实部  $(-1)^n$  发散, 故  $z_n$  发散

$$(3) z_n = \left(1 + \frac{i}{2}\right)^{-n}.$$

$$\text{解: } z_n = \left(1 + \frac{i}{2}\right)^{-n} = \left(\frac{2}{\sqrt{5}} e^{-i\theta}\right)^n, \lim_{n \rightarrow \infty} \left(\frac{2}{\sqrt{5}} e^{-i\theta}\right)^n = 0, \text{ 故收敛, } \lim_{n \rightarrow \infty} z_n = 0$$

2. 判别下列级数的收敛情况:

$$(1) \sum_{n=1}^{\infty} \frac{i^n}{n};$$

解: 由  $i^n = \cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}$ ,  $\sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{2}}{n}$ ,  $\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n}$  为收敛的交错项实级数, 所以

$\sum_{n=1}^{\infty} \frac{i^n}{n}$  收敛, 但  $\left|\frac{i^n}{n}\right| = \frac{1}{n}$ , 故  $\sum_{n=1}^{\infty} \frac{i^n}{n}$  发散, 原级数条件收敛。

$$(2) \sum_{n=1}^{\infty} \frac{(6+5i)^n}{8^n};$$

解: 因  $\left|\frac{(6+5i)^n}{8^n}\right| = \left(\frac{\sqrt{61}}{8}\right)^n$ , 而  $\sum_{n=1}^{\infty} \left(\frac{\sqrt{61}}{8}\right)^n$  收敛, 故  $\sum_{n=1}^{\infty} \frac{(6+5i)^n}{8^n}$  绝对收敛。

$$(3) \sum_{n=1}^{\infty} \frac{\cos in}{2^n}.$$

$$\text{解: } \sum_{n=1}^{\infty} \frac{\cos in}{2^n} = \sum_{n=1}^{\infty} \frac{e^n + e^{-n}}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{e^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{e^{-n}}{2^{n+1}},$$

$$\text{因级数 } \sum_{n=1}^{\infty} \frac{e^n}{2^{n+1}} \text{ 发散, 故 } \sum_{n=1}^{\infty} \frac{\cos in}{2^n} \text{ 发散.}$$

3. 求下列幂级数的收敛半径:

$$(1) \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n;$$

$$\text{解: } R = \lim_{n \rightarrow \infty} \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!} = e$$

$$(2) \sum_{n=1}^{\infty} \frac{1}{(\ln in)^n} z^n;$$

$$\text{解: } R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}} = \lim_{n \rightarrow \infty} |\ln in| = \infty$$

$$(3) \sum_{n=1}^{\infty} (1+i)^n z^n;$$

$$\text{解: } R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}} = \lim_{n \rightarrow \infty} \frac{1}{|1+i|} = \frac{1}{\sqrt{2}}$$

$$(4) \sum_{n=0}^{\infty} \frac{z^n}{2^n + i3^n};$$

$$\text{解: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n + i3^n}{2^{n+1} + i3^{n+1}} \right| = \lim_{n \rightarrow \infty} \sqrt{\frac{2^{2n} + 3^{2n}}{2^{2n+2} + 3^{2n+2}}} = \frac{1}{3}, \text{ 收敛半径为 } 3;$$

$$(5) \sum_{n=1}^{\infty} \frac{n}{2^n} (z-i)^n.$$

$$\text{解: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \frac{1}{2}, \text{ 收敛半径为 } 2;$$

4. 把下列函数展开成  $z$  的幂级数, 并指出它的收敛半径:

$$(1) \frac{1}{(1+z^2)^2};$$

$$\begin{aligned}
 \text{解: } \frac{1}{(1+z^2)^2} &= \left( -\frac{1}{1+z^2} \right)' \cdot \frac{1}{2z} = \left[ -\sum_{n=0}^{\infty} (-1)^n z^{2n} \right]' \cdot \frac{1}{2z} \\
 &= \frac{1}{2z} \cdot \sum_{n=1}^{\infty} (-1)^{n+1} 2n \cdot z^{2n-1} \\
 &= \sum_{n=1}^{\infty} (-1)^{n+1} n z^{2n-2}
 \end{aligned}$$

$|z^2| < 1$ , 即收敛半径为 1;

(2)  $\sinh z$

$$\begin{aligned}
 \text{解: } \sinh z &= \frac{e^z - e^{-z}}{2} = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \right] = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1 + (-1)^{n+1}}{n!} z^n \\
 |z| &< +\infty;
 \end{aligned}$$

(3)  $\sin(1+z^2)$ ;

$$\begin{aligned}
 \text{解: } \sin(1+z^2) &= \sin 1 \cdot \cos z^2 + \cos 1 \cdot \sin z^2 \\
 &= \sin 1 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n}}{(2n)!} + \cos 1 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} \\
 |z| &< +\infty;
 \end{aligned}$$

## 第八次作业

教学内容: 4.3 解析函数的泰勒展开 4.4 洛朗级数

1. 求下列各函数在指定点处的 Taylor 展开式, 并指出它们的收敛半径:

(1)  $\frac{z-1}{z+1}, z_0 = 1$ ;

$$\begin{aligned}
 \text{解: } \frac{z-1}{z+1} &= 1 - \frac{2}{z+1} = 1 - \frac{2}{2+(z-1)} = 1 - \frac{1}{1+\frac{z-1}{2}} \\
 &= 1 - \sum_{n=0}^{\infty} (-1)^n \left( \frac{z-1}{2} \right)^n \\
 &= \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{z-1}{2} \right)^n
 \end{aligned}$$

$$\text{其中} \left| \frac{z-1}{2} \right| < 1, \text{ 即 } |z-1| < 2$$

$$(2) \frac{z}{(z+1)(z+2)}, z_0 = 2;$$

$$\begin{aligned} \text{解: } \frac{z}{(z+1)(z+2)} &= \frac{2}{z+2} - \frac{1}{z+1} \\ &= \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{4}} - \frac{1}{3} \cdot \frac{1}{1+\frac{z-2}{3}} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{4^n} - \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{3^n} \quad \left( \left| \frac{z-2}{4} \right| < 1, \left| \frac{z-2}{3} \right| < 1 \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{2^{2n+1}} - \frac{1}{3^{n+1}} \right) (z-2)^n \quad (|z-2| < 3) \end{aligned}$$

$$(3) \frac{1}{z^2}, z_0 = -1;$$

$$\begin{aligned} \text{解: } \frac{1}{z^2} &= \left( -\frac{1}{z} \right)' = \left( -\frac{1}{z+1-1} \right)' = \left[ \frac{1}{1-(z+1)} \right]' \\ &= \left[ \sum_{n=0}^{\infty} (z+1)^n \right]' = \sum_{n=1}^{\infty} n(z+1)^{n-1} \end{aligned}$$

$$\text{其中 } |z+1| < 1$$

$$(4) \frac{1}{4-3z}, z = 1+i;$$

$$\begin{aligned} \text{解: } \frac{1}{4-3z} &= \frac{1}{-3(z-1-i)+1-3i} = \frac{1}{1-3i} \cdot \frac{1}{1-\frac{3(z-1-i)}{1-3i}} \\ &= \frac{1}{1-3i} \cdot \sum_{n=0}^{\infty} \left[ \frac{3(z-1-i)}{1-3i} \right]^n \\ &= \sum_{n=0}^{\infty} \frac{3^n (z-1-i)^n}{(1-3i)^{n+1}} \\ \text{其中 } \left| \frac{3(z-1-i)}{1-3i} \right| &< 1, \text{ 即 } |z-(1+i)| < \frac{\sqrt{10}}{3} \end{aligned}$$

$$(5) \sin^2 z, z_0 = 0;$$

$$\text{解: } \sin^2 z = \frac{1}{2}(1 - \cos 2z) = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!}$$

其中  $|z| < +\infty$

$$(6) \cos z^2, z_0 = 0$$

$$\text{解: } \cos z^2 = 1 - \frac{1}{2!} z^4 + \frac{1}{4!} z^8 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{4n}; |z| < +\infty$$

2. 把下列各函数在指定的圆环域内展开成 Laurent 级数.

$$(1) \frac{1}{z(1-z)^2}, 0 < |z| < 1, 0 < |z-1| < 1$$

解: 在  $0 < |z| < 1$  内,

$$\frac{1}{z(1-z)^2} = \frac{1}{z} \cdot \left( \frac{1}{1-z} \right)' = \frac{1}{z} \cdot \left( \sum_{n=0}^{\infty} z^n \right)' = \sum_{n=1}^{\infty} n z^{n-2};$$

在  $0 < |z-1| < 1$  内,

$$\frac{1}{z(1-z)^2} = \frac{1}{(z-1)^2} \cdot \frac{1}{z-1+1} = \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n (z-1)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-2}$$

$$(2) \frac{1}{(z^2+1)(z-2)}, 1 < |z| < 2;$$

$$\text{解: } \frac{1}{(z^2+1)(z-2)} = \frac{-\frac{1}{5}z}{z^2+1} + \frac{-\frac{2}{5}}{z^2+1} + \frac{\frac{1}{5}}{z-2}$$

$$= -\frac{1}{5} z \frac{1}{z^2} \frac{1}{1+\frac{1}{z^2}} - \frac{2}{5} \frac{1}{z^2} \frac{1}{1+\frac{1}{z^2}} - \frac{1}{10} \frac{1}{1-\frac{z}{2}}$$

$$= -\frac{1}{5} \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n}} - \frac{2}{5} \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n}} - \frac{1}{10} \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

$$= -\frac{1}{5} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n+1}} - \frac{2}{5} \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n+2}} - \frac{1}{10} \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

(3)  $\frac{1}{z^2(z-i)}$ , 以  $i$  为中心的圆环;

解:  $\frac{1}{z^2(z-i)}$  有两个奇点,  $z_1=0$ ,  $z_2=i$ , 所以以  $z=i$  为中心的圆环域有:

$$0 < |z-i| < 1 \text{ 和 } 1 < |z-i| < +\infty,$$

在  $0 < |z-i| < 1$  内, 因  $\frac{1}{(1+z)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n z^{n-1}, |z| < 1$ ,

$$\text{故 } \frac{1}{z^2(z-i)} = \frac{1}{i^2(z-i)(1+\frac{z-i}{i})^2} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(z-i)^{n-2}}{i^{n+1}}$$

在  $1 < |z-i| < +\infty$  内展开, 得:

$$\frac{1}{z^2(z-i)} = \frac{1}{(z-i)^3} \cdot \frac{1}{\left(1+\frac{i}{z-i}\right)^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)i^n}{(z-i)^{n+3}}$$

3. 把下列各函数在指定圆环域内展成 Laurent 级数, 且计算其沿正向圆周  $|z|=6$  的积分值:

(1)  $\sin \frac{1}{1-z}$ ,  $z=1$  的去心邻域;

解: 由于  $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots \quad (|z| < +\infty)$

$$\text{所以 } \sin \frac{1}{1-z} = \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(2n+1)!} (z-1)^{-2n-1}$$

$$\text{于是 } \oint_{|z|=6} \sin \frac{1}{1-z} dz = - \oint_{|z|=6} \frac{1}{z-1} dz = -2\pi i;$$

(2)  $\frac{1}{z(z+1)^6}$ ,  $1 < |z+1| < \infty$ ;

$$\text{解: } \frac{1}{z(z+1)^6} = \frac{1}{(z+1)^6} \frac{1}{(z+1)-1} = \frac{1}{(z+1)^6} \frac{1}{1-\frac{1}{z+1}} = \sum_{n=0}^{+\infty} (z+1)^{-n-7}$$

$$\int_{|z|=6} \frac{1}{z(z+1)^6} dz = \sum_{n=0}^{+\infty} \oint_{|z|=6} \frac{1}{(z+1)^{n+7}} dz = 0$$

$$(3) \ln\left(\frac{z-i}{z+i}\right), 2 < |z+i| < \infty.$$

解:

$$\ln\left(\frac{z-i}{z+i}\right) = \ln\left(1 - \frac{2i}{z+i}\right) = -\frac{2i}{z+i} - \frac{1}{2}\left(\frac{2i}{z+i}\right)^2 - \frac{1}{3}\left(\frac{2i}{z+i}\right)^3 - \dots - \frac{1}{n}\left(\frac{2i}{z+i}\right)^n + \dots$$

$$\oint_{|z|=6} \ln\left(\frac{z-i}{z+i}\right) dz = -\int_{|z|=6} \frac{2i}{z+i} dz = 4\pi$$

$$4. \text{ 求函数 } e^{\frac{1-z^2}{z^2}} \sin \frac{1}{z^2} \text{ 在 } |z| > 0 \text{ 上的洛朗展开式。}$$

解:

$$e^{\frac{1-z^2}{z^2}} \sin \frac{1}{z^2} = e^{-1} e^{\frac{1}{z^2}} \sin \frac{1}{z^2}$$

$$= \frac{1}{2ei} (e^{\frac{1+i}{z^2}} - e^{\frac{1-i}{z^2}})$$

$$= \frac{1}{2ei} \sum_{n=0}^{\infty} \frac{(1+i)^n - (1-i)^n}{n! z^{2n}}$$

$$(1+i)^n = 2^{\frac{n}{2}} e^{n\frac{\pi}{4}i}$$

$$(1-i)^n = 2^{\frac{n}{2}} e^{-\frac{\pi}{4}i}$$

$$(1+i)^n - (1-i)^n = 2^{\frac{n}{2}} \cdot 2i \sin \frac{n\pi}{4}$$

$$\text{故 } e^{\frac{1-z^2}{z^2}} \sin \frac{1}{z^2} = \frac{1}{e} \sum_{n=0}^{\infty} (2^{\frac{n}{2}} \sin \frac{n\pi}{4}) / n! z^{2n}$$

$$5. \text{ 设 } \oint_{|\xi|=1} \frac{e^{\xi}}{(z\xi - \xi)^2} d\xi = \sum_{n=0}^{\infty} a_n z^{-n}, \text{ 求 } a_n?$$

$$\text{解: } \oint_{|\xi|=1} \frac{e^{\xi}}{(z-1)^2 \xi^2} d\xi = 2\pi i \frac{1}{(z-1)^2} = \frac{2\pi i}{z^2} \frac{1}{(1-\frac{1}{z})^2} = \sum_{n=1}^{\infty} \frac{2n\pi i}{z^{n+1}}$$

$$\text{因此 } a_0 = a_1 = 0, a_n = 2(n-1)\pi i, (n=2, 3, \dots)$$