

# 复变函数与积分变换作业 (第6册)

## 第十一次作业

教学内容: 5.3 利用留数计算实积分 5.4 对数留数与辐角原理 7.1 Fourier 积分公式 7.2 Fourier 变换

1 计算下列积分:

$$(1) \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, 0 < b < a;$$

$$\text{解: 设 } z = e^{i\theta}, \cos\theta = \frac{z^2+1}{2z}$$

原式

$$= \oint_{|z|=1} \frac{1}{a+b\frac{z^2+1}{2z}} \frac{1}{iz} dz$$

$$= \frac{1}{i} \oint_{|z|=1} \frac{2}{bz^2+2az+b} dz = 2\pi i \frac{2}{i} \operatorname{Res}\left[\frac{1}{bz^2+2az+b}, \frac{-a+\sqrt{a^2-b^2}}{b}\right] = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$(2) \int_{-\infty}^{+\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx;$$

$$\text{令 } f(z) = \frac{z^2-z+2}{z^4+10z^2+9} \text{ 则 } f(z) \text{ 在上半平面有两个一级极点 } z_1=i, z_2=3i$$

$$\operatorname{Res}[f(z), z_1] = \frac{z^2-z+2}{(z^4+10z^2+9)'} \Big|_{z_1=i} = -\frac{1}{16}(1+i)$$

$$\operatorname{Res}[f(z), z_1] = \frac{z^2-z+2}{(z^4+10z^2+9)'} \Big|_{z=3i} = \frac{3-7i}{48}$$

$$\int_{-\infty}^{+\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx = 2\pi i \left[ -\frac{1}{16}(1+i) + \frac{3-7i}{48} \right] = \frac{5}{12}\pi$$

$$(3) \int_0^{+\infty} \frac{dx}{1+x^4}$$

解：令  $f(z) = \frac{1}{1+z^4}$   $f(z)$  在上半平面上有两个一级极点

$$z_1 = \frac{\sqrt{2}}{2}(1+i) \quad z_2 = \frac{\sqrt{2}}{2}(-1+i)$$

$$\operatorname{Res}[f(z), z_1] = \frac{1}{(1+z^4)'} \Big|_{z_1 = \frac{\sqrt{2}}{2}(1+i)} = -\frac{\sqrt{2}}{8}(1+i)$$

$$\operatorname{Res}[f(z), z_2] = \frac{1}{(1+z^4)'} \Big|_{z_1 = \frac{\sqrt{2}}{2}(-1+i)} = \frac{\sqrt{2}}{8}(1-i)$$

$$\int_0^{+\infty} \frac{dx}{1+x^4} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = 2\pi i \left[ -\frac{\sqrt{2}}{8}(1+i) + \frac{\sqrt{2}}{8}(1-i) \right] = \frac{\sqrt{2}}{4} \pi$$

$$(4) \int_0^{+\infty} \frac{x \sin ax}{x^2 + b^2} dx, (a > 0, b > 0);$$

$$\text{解：令 } f(z) = \frac{ze^{iaz}}{z^2 + b^2}$$

$$\int_{-\infty}^{+\infty} \frac{xe^{iax}}{x^2 + b^2} dx = 2\pi i [f(z), bi] = 2\pi i \frac{ze^{iaz}}{(z^2 + b^2)'} \Big|_{z=bi} = \pi i e^{-ab}$$

$$\text{所以 } \int_0^{+\infty} \frac{xe^{iax}}{x^2 + b^2} dx = \frac{1}{2} \pi e^{-ab}$$

$$(5) \int_{-\infty}^{+\infty} \frac{\cos x dx}{(x^2 + 4x + 5)^2}$$

所求的积分是积分  $I = \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2 + 4x + 5)^2} dx$  的实部

$$\text{而 } I = 2\pi i \operatorname{Res} \left[ \frac{e^{iz}}{(z^2 + 4z + 5)^2}, -2+i \right]$$

$$= 2\pi i \left[ \frac{e^{iz}}{(z+2+i)^2} \right]' \Big|_{z=-2-i}$$

$$= \pi e^{-1-2i}$$

所以  $\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2 + 4x + 5)^2} dx = \frac{\pi}{e} \cos 2$

2. 证明方程  $z^7 - z^3 + 12 = 0$  的根都在圆环域  $1 \leq |z| \leq 2$  内。

证明：当  $|z| < 2$  时，取  $f(z) = z^7$ ， $g(z) = 12 - z^3$ ，当  $|z| = 2$  时，

$$|g(z)| = |12 - z^3| \leq |12 + z^3| \leq 20 < |z^7| = |f(z)|$$

所以， $z^7 - z^3 + 12 = 0$  的根与  $z^7$  的根的个数相同，因此， $z^7 - z^3 + 12 = 0$  的根全部在  $|z| = 2$  内部

当  $|z| < 1$  时，取  $f(z) = 12$ ， $g(z) = z^7 - z^3$

$$\text{当 } |z| = 1 \text{ 时， } |f(z)| > |z^7| + |z^3| \geq |z^7 - z^3| = |g(z)|$$

故  $z^7 - z^3 + 12 = 0$  的根与 12 的根个数相同，即在  $|z| = 1$  内无根。

综上所述， $z^7 - z^3 + 12 = 0$  的根都在圆环域  $1 \leq |z| \leq 2$  内

3、求下列函数的 Fourier 积分变换

$$(1) f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \\ 0 & \text{其它} \end{cases}$$

$$\begin{aligned} \text{解： } \mathcal{F}[f(t)] &= \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \int_{-1}^0 -e^{-it} dt + \int_0^1 e^{-it} dt \\ &= \frac{1}{i\omega} \cdot e^{-i\omega t} \Big|_{-1}^0 - \frac{1}{i\omega} \cdot e^{-i\omega t} \Big|_0^1 = \frac{1}{i\omega} (1 - e^{i\omega} - e^{-i\omega} + 1) = \frac{-2i}{\omega} (1 - \cos \omega) \end{aligned}$$

$$(2) f(t) = \begin{cases} e^t & t \leq 0 \\ 0 & t > 0 \end{cases}$$

$$\begin{aligned} \text{解： } \mathcal{F}[f(t)] &= \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^0 e^t e^{-i\omega t} dt = \int_{-\infty}^0 e^{(1-i\omega)t} dt \\ &= \frac{1}{1-i\omega} \cdot e^{(1-i\omega)t} \Big|_{-\infty}^0 = \frac{1}{1-i\omega} \end{aligned}$$

4 求下列函数的 Fourier 变换，并证明所列的积分等式

$$(1) \quad f(t) = e^{-|t|} \cos t, \quad \text{证明} \int_0^{+\infty} \frac{\omega^2 + 2}{\omega^4 + 4} \cos \omega t d\omega = \frac{\pi}{2} e^{-|t|} \cos t$$

解:

$$\begin{aligned} F(\omega) &= \mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} e^{-|r|} \cos t e^{-i\omega t} dt = \int_{-\infty}^{+\infty} e^{-|r|} \frac{e^{it} + e^{-it}}{2} e^{-i\omega t} dt \\ &= \frac{1}{2} \left\{ \int_{-\infty}^0 e^{[1+i(1-\omega)]t} dt + \int_{-\infty}^0 e^{[1-i(1+\omega)]t} dt + \int_0^{+\infty} e^{[-1+i(1-\omega)]t} dt + \int_0^{+\infty} e^{[-1-i(1+\omega)]t} dt \right\} \\ &= \frac{1}{2} \left\{ \frac{e^{[1+i(1-\omega)]t} \Big|_{-\infty}^0}{1+i(1-\omega)} + \frac{e^{[1-i(1+\omega)]t} \Big|_{-\infty}^0}{1-i(1+\omega)} + \frac{e^{[-1+i(1-\omega)]t} \Big|_0^{+\infty}}{-1+i(1-\omega)} + \frac{e^{[-1-i(1+\omega)]t} \Big|_0^{+\infty}}{-1-i(1+\omega)} \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{1+i(1-\omega)} + \frac{1}{1-i(1+\omega)} + \frac{1}{-1+i(1-\omega)} + \frac{1}{-1-i(1+\omega)} \right\} = \frac{2\omega^2 + 4}{\omega^4 + 4} \end{aligned}$$

$f(t)$  的积分表达式为

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\omega^2 + 4}{\omega^4 + 4} e^{i\omega t} d\omega = \frac{1}{\pi} \int_0^{+\infty} \frac{2\omega^2 + 4}{\omega^4 + 4} \cos \omega t d\omega$$

$$\text{因此有} \int_0^{+\infty} \frac{2\omega^2 + 4}{\omega^4 + 4} \cos \omega t d\omega = \frac{\pi}{2} f(t) = \frac{\pi}{2} e^{-|t|} \cos t$$

$$(2) \quad f(t) = e^{-\beta|t|} (\beta > 0), \quad \text{证明} \int_0^{+\infty} \frac{\cos \omega t}{\beta^2 + \omega^2} d\omega = \frac{\pi}{2\beta} e^{-\beta|t|}$$

$$\text{解: } F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} e^{-\beta|r|} e^{-i\omega t} dt = 2 \int_0^{+\infty} e^{-\beta t} \cos \omega t dt = 2 \int_0^{+\infty} e^{-\beta t} \frac{e^{i\omega t} + e^{-i\omega t}}{2} dt$$

$$= \int_0^{+\infty} [e^{-(\beta-i\omega)t} + e^{-(\beta+i\omega)t}] dt = \frac{e^{-(\beta-i\omega)t} \Big|_0^{+\infty}}{-(\beta-i\omega)} + \frac{e^{-(\beta+i\omega)t} \Big|_0^{+\infty}}{-(\beta+i\omega)}$$

$$= \frac{1}{\beta-i\omega} + \frac{1}{\beta+i\omega} = \frac{2\beta}{\beta^2 + \omega^2}$$

$f(t)$  的积分表达式为

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\beta}{\beta^2 + \omega^2} e^{i\omega t} d\omega = \frac{2}{\pi} \int_0^{+\infty} \frac{2\beta}{\beta^2 + \omega^2} \cos \omega t d\omega$$

$$\text{即: } \int_0^{+\infty} \frac{\cos \omega t}{\beta^2 + \omega^2} d\omega = \frac{\pi}{2\beta} e^{-\beta|t|}$$

## 第十二次作业

**教学内容：** 7.3  $\delta$  函数及其 Fourier 变换； 7.4 Fourier 变换的性质

1. 填空

(1)  $f(t) = \frac{1}{2}[\delta(t+a) + \delta(t-a)]$  Fourier 变换为  $\cos \omega a$

(2) 函数  $F(\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$  的 Fourier 逆变换为  $\cos \omega_0 t$

(3)  $f(t) = \sin t \cos t$  Fourier 变换为  $\frac{\pi i}{2}[\delta(\omega + 2) - \delta(\omega - 2)]$

2. 若  $F(\omega) = \mathcal{F}[f(t)]$ , 证明

$$\mathcal{F}[f(t) \cos \omega_0 t] = \frac{1}{2}[F(\omega - \omega_0) + F(\omega + \omega_0)]$$

$$\mathcal{F}[f(t) \sin \omega_0 t] = \frac{1}{2i}[F(\omega - \omega_0) - F(\omega + \omega_0)]$$

$$\text{证: } \mathcal{F}[f(t) \cos \omega_0 t] = \int_{-\infty}^{+\infty} f(t) \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} e^{-i\omega t} dt$$

$$= \frac{1}{2} \left[ \int_{-\infty}^{+\infty} f(t) e^{i(\omega - \omega_0)t} dt + \int_{-\infty}^{+\infty} f(t) e^{i(\omega + \omega_0)t} dt \right]$$

$$= \frac{1}{2} [F(\omega - \omega_0) + F(\omega + \omega_0)]$$

$$\mathcal{F}[f(t) \sin \omega_0 t] = \int_{-\infty}^{+\infty} f(t) \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} e^{-i\omega t} dt$$

$$= \frac{1}{2i} \left[ \int_{-\infty}^{+\infty} f(t) e^{-i(\omega - \omega_0)t} dt - \int_{-\infty}^{+\infty} f(t) e^{-i(\omega + \omega_0)t} dt \right]$$

$$= \frac{1}{2i} [F(\omega - \omega_0) - F(\omega + \omega_0)]$$

3. 求下列函数的 Fourier 变换

$$(1) f(t) = e^{2it} \sin t$$

解：因为  $\mathcal{F}[\sin t] = i\pi[\delta(\omega + 1) - \delta(\omega - 1)]$ ，由位移性质得

$$\mathcal{F}[e^{2it} \sin t] = i\pi[\delta(\omega - 1) - \delta(\omega - 3)]$$

$$(2) f(t) = \sin^2 t$$

$$\begin{aligned} \text{解：} \mathcal{F}[\sin^2 t] &= \mathcal{F}\left[\frac{1}{2}(1 - \cos 2t)\right] = \frac{1}{2}\mathcal{F}[1] - \frac{1}{2}\mathcal{F}[\cos 2t] \\ &= \pi\delta(\omega) - \frac{\pi}{2}[\delta(\omega + 2) + \delta(\omega - 2)] \end{aligned}$$

$$(3) f(t) = e^{i\omega_0 t} u(t)$$

解：由像函数的位移性质及  $\mathcal{F}[u(t)] = \frac{1}{i\omega} + \pi\delta(\omega)$  得

$$\mathcal{F}[e^{i\omega_0 t} u(t)] = \frac{1}{i(\omega - \omega_0)} + \pi\delta(\omega - \omega_0)$$

$$(4) f(t) = e^{-\beta t} u(t) \cdot \cos \omega_0 t$$

$$\begin{aligned} \text{解：} \mathcal{F}(\omega) &= \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{+\infty} e^{-\beta t} u(t) \cos \omega_0 t e^{-i\omega t} dt = \int_0^{+\infty} e^{-\beta t} \frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{2} e^{-i\omega t} dt \\ &= \frac{1}{2} \int_0^{+\infty} (e^{-(\beta + i(\omega - \omega_0))t} + e^{-(\beta + i(\omega + \omega_0))t}) dt = \frac{1}{2} \left( \frac{1}{\beta + i(\omega - \omega_0)} + \frac{1}{\beta + i(\omega + \omega_0)} \right) \\ &= \frac{\beta + i\omega}{(\beta + i\omega)^2 + \omega_0^2} \end{aligned}$$

4 设  $\mathcal{F}[f(t)] = F(\omega)$ ， $a$  为非零常数，试证明

$$(1) \mathcal{F}[f(at - t_0)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-i\frac{\omega}{a}t_0}$$

$$(2) \mathcal{F}[f(t_0 - at)] = \frac{1}{|a|} F\left(-\frac{\omega}{a}\right) e^{-i\frac{\omega}{a}t_0}$$

证明：(1) 由定义有

$$\mathcal{F}[f(at-t_0)] = \int_{-\infty}^{+\infty} f(at-t_0)e^{-i\omega t} dt$$

$$(\text{令 } at-t_0 = u, \text{ 且 } a > 0) = \int_{-\infty}^{+\infty} f(u)e^{-i\omega \frac{u+t_0}{a}} \frac{1}{a} du$$

$$(\text{u 换为 t}) = \frac{1}{a} e^{-i\frac{\omega}{a}t_0} \int_{-\infty}^{+\infty} f(t)e^{-i\frac{\omega}{a}t} dt$$

$$= \frac{1}{a} F\left(\frac{\omega}{a}\right) e^{-i\frac{\omega}{a}t_0}$$

$$\text{当 } a < 0 \text{ 时, } \mathcal{F}[f(at-t_0)] = -\frac{1}{a} F\left(\frac{\omega}{a}\right) e^{-i\frac{\omega}{a}t_0}$$

$$\text{因此 } \mathcal{F}[f(at-t_0)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-i\frac{\omega}{a}t_0}$$

注：也可以由位移性质和相似性质加以证明。例如令  $g(t) = f(at)$  由位移性质得

$$\mathcal{F}[f(at-t_0)] = \mathcal{F}\left[f\left[a\left(t-\frac{t_0}{a}\right)\right]\right] = \mathcal{F}\left[g\left(t-\frac{t_0}{a}\right)\right] = \mathcal{F}\left[g(t)e^{-i\omega \frac{t_0}{a}}\right]$$

$$= \mathcal{F}\left[f(at)e^{-i\omega \frac{t_0}{a}}\right] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-i\frac{\omega}{a}t_0} \quad (\text{相似性质})$$

(2) 在结论 (1) 中取  $a, t_0$  分别为  $-a, -t_0$  即得。

注：此题也可由定义出发证明，或利用位移性质和相似性质证明。

**5** 已知  $F(\omega) = \mathcal{F}[f(t)]$ ，利用 Fourier 变换的性质求下列函数的 Fourier 变换

(1)  $tf(t)$

解：由像函数的微分性质，有  $\mathcal{F}[tf(t)] = -\frac{1}{i} F'(\omega)$

(2)  $(t-2)f(t)$

解：由线性性质及像函数的微分性质

$$\mathcal{F}[(t-2)f(t)] = \mathcal{F}[tf(t)] - 2\mathcal{F}[f(t)] = -\frac{1}{i} F'(\omega) - 2F(\omega)$$

(3)  $tf'(t)$

解：由微分性质

$\mathcal{F}[f'(t)] = i\omega F(\omega)$ ，再由像函数的微分性质，有

$$\mathcal{F}[tf'(t)] = -\frac{1}{i} \frac{d}{d\omega} [i\omega F(\omega)] = -F(\omega) - \omega F'(\omega).$$

(4)  $f(1-t)$

解：由相似性质  $\mathcal{F}[f(-t)] = F(-\omega)$

再由位移性质

$$\mathcal{F}[f(1-t)] = e^{-i\omega} F(-\omega)$$

6. 求函数  $f(t) = \sin(5t + \frac{\pi}{3})$  的 Fourier 变换.

$$\begin{aligned} \text{解：} \mathcal{F}(\omega) &= \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{+\infty} \sin(5t + \frac{\pi}{3}) e^{-i\omega t} dt \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \sin(5t + \sqrt{3} \cos 5t) e^{-i\omega t} dt \\ &= \frac{1}{2} i\pi [\delta(\omega + 5) - \delta(\omega - 5)] + \frac{\sqrt{3}}{2} \pi [\delta(\omega + 5) + \delta(\omega - 5)] \\ &= \frac{1}{2} \pi [(\sqrt{3} + i)\delta(\omega + 5) + (\sqrt{3} - i)\delta(\omega - 5)] \end{aligned}$$

$$\text{解二：由于 } f(t) = \sin(5t + \frac{\pi}{3}) = \frac{1}{2} \sin 5t + \frac{\sqrt{3}}{2} \cos 5t$$

$$\text{所以 } \mathcal{F}[f(t)] = \frac{i\pi}{2} [\delta(\omega + 5) - \delta(\omega - 5)] + \frac{\sqrt{3}}{2} \pi [\delta(\omega + 5) + \delta(\omega - 5)]$$