

# 华东理工大学

## 复变函数与积分变换作业 (第7册)

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### 第十三次作业

教学内容: 7.5 Fourier 的卷积性质; 8.1 拉普拉斯变换的概念 8.2 拉普拉斯变换的性质。

1. 计算下列函数的卷积

$$f_1(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad f_2(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0 \end{cases}$$

解: 显然, 有  $f_1(t-\tau) = \begin{cases} 0 & t < \tau \\ 1 & t \geq \tau \end{cases}$

当  $t \leq 0$  时, 由于  $f_2(\tau)f_1(t-\tau) = 0$ , 所以  $f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_2(\tau)f_1(t-\tau)d\tau = 0$

当  $t > 0$  时  $f_1(t) * f_2(t) = \int_0^t f_2(\tau)f_1(t-\tau)d\tau = \int_0^t f_2(\tau)d\tau = \int_0^t e^{-a\tau}d\tau$   
$$= -\frac{1}{a} \cdot e^{-a\tau} \Big|_0^t = \frac{1}{a} \cdot (1 - e^{-at})$$

2. 已知  $f(t) = \cos \omega_0 t \cdot u(t)$ , 求  $\mathcal{F}[f(t)]$ .

解: 已知  $\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{i\omega}$  又

$$f(t) = \cos \omega_0 t \cdot u(t) = \frac{1}{2} [e^{i\omega_0 t} u(t) + e^{-i\omega_0 t} u(t)]$$

由位移性质有

$$\begin{aligned} \mathcal{F}[f(t)] &= \frac{1}{2} \left[ \pi\delta(\omega - \omega_0) + \frac{1}{i(\omega - \omega_0)} + \pi\delta(\omega + \omega_0) + \frac{1}{i(\omega + \omega_0)} \right] \\ &= \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] - \frac{\omega i}{\omega^2 - \omega_0^2} \end{aligned}$$

3. 填空

$$(1) \quad f(t) = \begin{cases} 2 & 0 \leq t < 2 \\ -3 & 2 \leq t < 4 \\ 0 & t \geq 4 \end{cases} \quad \text{的 Laplace 变换为 } F(s) = \frac{1}{s} (3e^{-4s} - 5e^{-2s} + 2)$$

$$(2) f(t) = e^{2t} + 5\delta(t) \text{ 的 Laplace 变换为 } F(s) = \frac{5s-9}{s-2}$$

$$(3) f(t) = \cos t \cdot \delta(t) - \sin t \cdot u(t) \text{ 的 Laplace 变换为 } F(s) = \frac{s^2}{s^2+1}$$

$$(4) f(t) = 1 - te^t \text{ 的 Laplace 变换为 } F(s) = \frac{1}{s} - \frac{1}{(s-1)^2}$$

$$(5) f(t) = t^3 - 2t + 1 \text{ 的 Laplace 变换为 } F(s) = \frac{1}{s^4}(s^3 - 2s^2 + 6)$$

$$(6) f(t) = e^{-2t} \cos 6t \text{ 的 Laplace 变换为 } F(s) = \frac{s+2}{(s+2)^2 + 36}$$

4. 求下列函数的 Laplace 变换.

$$(1) f(t) = (t-1)^2 e^t$$

$$\text{解: } \mathcal{L}[f(t)] = \mathcal{L}[(t-i)^2 e^t] = \mathcal{L}[(t^2 - 2t + 1)e^t]$$

$$= \frac{d^2}{ds^2} \mathcal{L}[e^t] + 2 \frac{d}{ds} \mathcal{L}[e^t] + \mathcal{L}[e^t] = \frac{s^2 - 4s + 5}{(s-1)^3}$$

$$(2) f(t) = t \cos 3t$$

$$\text{解: } \mathcal{L}[t \cos 3t] = -(\mathcal{L}[\cos 3t])'_s$$

$$= -\left(\frac{s}{s^2+9}\right)'$$

$$= \frac{s^2-9}{(s^2+9)^2}$$

$$(3) f(t) = t^n e^{at} \quad (n \text{ 为正整数})$$

解: 利用  $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$  及位移性质得

$$\mathcal{L}[f(t)] = \mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$$

$$(4) F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t^3 - 2t + 1]$$

$$= \mathcal{L}[t^3] - 2\mathcal{L}[t] + \mathcal{L}[1]$$

$$\begin{aligned}
&= \frac{3!}{s^4} - \frac{2}{s^2} + \frac{1}{s} \\
&= \frac{1}{s^4}(s^3 - 2s^2 + 6)
\end{aligned}$$

## 第十四次作业

教学内容：8.2 拉普拉斯的性质（续） 8.3 拉普拉斯逆变换

1. 求下列函数的 Laplace 变换

$$(1) f(t) = t \int_0^t e^{-3\tau} \sin 2\tau d\tau$$

**解：** 由积分性质  $\mathcal{L}[\int_0^t e^{-3\tau} \sin 2\tau d\tau] = \frac{1}{s} \mathcal{L}[e^{-3\tau} \sin 2\tau] = \frac{1}{s} \cdot \frac{2}{(s+3)^2 + 4}$

再由像函数的微分公式

$$\mathcal{L}[f(t)] = \mathcal{L}[t \int_0^t e^{-3\tau} \sin 2\tau d\tau] = -\frac{d}{ds} \left\{ \frac{2}{s[(s+3)^2 + 4]} \right\} = \frac{2(3s^2 + 12s + 13)}{s^2[(s+3)^2 + 4]^2}$$

$$(2) f(t) = \frac{\sin at}{t} \quad (a \text{ 为实数})$$

**解：** 利用像函数的积分性质

$$\begin{aligned}
F(s) &= \mathcal{L}\left[\frac{\sin at}{t}\right] = \int_s^\infty \mathcal{L}[\sin kt] ds = \int_s^\infty \frac{a}{s^2 + a^2} ds = \arctan \frac{s}{a} \Big|_s^\infty \\
&= \frac{\pi}{2} - \arctan \frac{s}{a} = \operatorname{arc cot} \frac{s}{a}.
\end{aligned}$$

$$(3) f(t) = \int_0^t te^{-3t} \sin 2tdt$$

**解：**  $\mathcal{L}[e^{-3t} \sin 2t] = \frac{2}{(s+3)^2 + 4}$

$$\begin{aligned}
\mathcal{L}[te^{-3t} \sin 2t] &= -\frac{d}{ds} \left[ \frac{2}{(s+3)^2 + 4} \right]' \\
&= \frac{4(s+3)}{[(s+3)^2 + 4]^2}
\end{aligned}$$

所以  $\mathcal{L}[f(t)] = \mathcal{L}\left[\int_0^t te^{-3t} \sin 2tdt\right]$

$$= \frac{1}{s} \cdot \frac{4(s+3)}{[(s+3)^2 + 4]^2}$$

$$(4) \quad f(t) = \int_0^t \frac{e^{-2t} \sin 3t}{t} dt$$

解

:

$$F(s) = \frac{1}{s} \mathcal{L}\left[\frac{e^{-2t} \sin 3t}{t}\right] = \frac{1}{s} \int_s^\infty \mathcal{L}[e^{-2t} \sin 3t] ds = \frac{1}{s} \int_s^\infty \frac{3}{(s+2)^2 + 9} ds = \frac{1}{s} \left(\frac{\pi}{2} - \arctan \frac{s+2}{3}\right)$$

2 计算下列积分

$$(1) \quad \int_0^{+\infty} \frac{\sin t}{t} e^{-t} dt$$

$$\text{解: } = \int_0^\infty \mathcal{L}[e^{-t} \sin t] ds = \int_0^\infty \frac{1}{(s+1)^2 + 1} ds = \arctan(s+1) \Big|_0^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

$$(2) \quad \int_0^{+\infty} \frac{1 - \cos t}{t} e^{-t} dt$$

$$\begin{aligned} \text{解: } \int_0^{+\infty} \frac{1 - \cos t}{t} e^{-t} dt &= \int_0^\infty \mathcal{L}[(1 - \cos t)e^{-t}] ds \\ &= \int_0^\infty \left(\frac{1}{s+1} - \frac{s+1}{(s+1)^2 + 1}\right) ds \\ &= \ln \frac{s+1}{\sqrt{(s+1)^2 + 1}} \Big|_0^\infty \\ &= \ln \sqrt{2} \end{aligned}$$

$$(3) \quad \int_0^{+\infty} t e^{-3t} \sin 2t dt$$

$$\text{解: 已知 } \mathcal{L}[\sin 2t] = \frac{2}{s^2 + 4}$$

$$\text{再由微分性质 } \mathcal{L}[t \sin 2t] = -\left(\frac{2}{s^2 + 4}\right)' = \frac{4s}{(s^2 + 4)^2}$$

$$\text{得 } \int_0^{+\infty} t e^{-3t} \sin 2t dt = \frac{4s}{(s^2 + 4)^2} \Big|_{s=3} = \frac{12}{169}$$

3. 求下列函数的 拉氏逆变换

$$(1) \quad F(s) = \frac{1}{s+1} - \frac{1}{s-1}, \quad (2) \quad F(s) = \frac{2s}{(s-1)^2}$$

$$(1) \quad \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = e^{-t} - e^t = -2\sinh t$$

$$\begin{aligned} (2) \quad \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{2s}{(s-1)^2}\right] = \mathcal{L}^{-1}\left[\frac{2}{s-1} + \frac{2}{(s-1)^2}\right] \\ &= 2e^{-t} - 2te^{-t} \end{aligned}$$

$$(3) \text{ 由于 } F(s) = \frac{1}{(s^2 + 2s + 2)^2} = \frac{1}{(s+1)^2 + 1} \cdot \frac{1}{(s+1)^2 + 1} = F(s) \cdot F(s)$$

由卷积性质

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{1}{[(s+1)^2 + 1]^2} \right] &= \mathcal{L}^{-1} [f(t) * f(t)] \\ &= \int_0^t e^{-\tau} \sin \tau \cdot e^{-(t-\tau)} \sin(t-\tau) d\tau = \frac{1}{2} e^t (\sin t - t \cos t) \end{aligned}$$

$$\begin{aligned} (4) \quad f(t) &= -\frac{1}{t} \mathcal{L}^{-1} [F'(s)] = -\frac{1}{t} \mathcal{L}^{-1} \left[ \left( \arctan \frac{a}{s} \right)' \right] \\ &= -\frac{1}{t} \mathcal{L}^{-1} \left[ -\frac{a}{s^2 + a^2} \right] \\ &= \frac{\sin at}{t} \end{aligned}$$

$$\begin{aligned} (5) \quad \mathcal{L}^{-1} [F(s)] &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} + \frac{e^{-2s}}{s^2} \right] \\ &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \right] + \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s^2} \right] \\ &= t + (t-2)u(t-2) \\ &= \begin{cases} 2(t-1) & t > 2 \\ t & 0 \leq t < 2 \end{cases} \end{aligned}$$

4 求下列拉氏卷积

$$(1) \quad t * t$$

$$\text{解: } t * t = \int_0^t \tau(t-\tau) d\tau = t \int_0^t \tau d\tau - \int_0^t \tau^2 d\tau = \frac{1}{2} t^3 - \frac{1}{3} t^3 = \frac{1}{6} t^3$$

$$(2) \quad \sin kt * \sin kt \quad (k \neq 0)$$

解:

$$\sin kt * \sin kt$$

$$= \int_0^t \sin kt \cdot \sin k(t-\tau) d\tau = \frac{1}{2} \int_0^t \cos(2kt - kt) d\tau - \frac{1}{2} \int_0^t \cos k\tau d\tau = \frac{1}{2k} \sin kt - \frac{1}{2} t \cos kt$$

$$5. \text{ 设 } \mathcal{L}[f(t)] = F(s), \text{ 利用卷积定理证明 } \mathcal{L} \left[ \int_0^t f(t) dt \right] = \mathcal{L} [f(t) * u(t)] = \frac{F(s)}{s}$$

$$\text{证: } \frac{F(s)}{s} = F(s) \cdot \frac{1}{s} = \mathcal{L} [f(t) * u(t)] = \int_0^t f(\tau) u(t-\tau) d\tau = \int_0^t f(t) dt$$

6. 求下列函数的逆变换

$$(1) F(s) = \frac{s}{(s-a)(s-b)}$$

$$\text{解法 1: } f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s}{(s-a)(s-b)}\right]$$

$$\begin{aligned} &= \operatorname{Res}\left[\frac{se^{st}}{(s-a)(s-b)}, a\right] + \operatorname{Res}\left[\frac{se^{st}}{(s-a)(s-b)}, b\right] \\ &= \frac{1}{a-b}(ae^{at} - be^{bt}) \end{aligned}$$

$$\text{解法 2: } f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s}{(s-a)(s-b)}\right]$$

$$\begin{aligned} &= \mathcal{L}^{-1}\left[\frac{1}{a-b}\left(\frac{a}{s-a} - \frac{b}{s-b}\right)\right] \\ &= \frac{1}{a-b}(a \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] - b \mathcal{L}^{-1}\left[\frac{1}{s-b}\right]) \\ &= \frac{1}{a-b}(ae^{at} - be^{bt}) \end{aligned}$$

$$(2) F(s) = \frac{s}{(s^2+1)(s^2+4)}$$

$$\text{解法 1: } f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{s}{s^2+1} - \frac{s}{s^2+4}\right)\right]$$

$$= \frac{1}{3}(\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]) = \frac{1}{3}(\cos t - \cos 2t)$$

解法 2:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$$

$$= \operatorname{Res}\left[\frac{se^{st}}{(s^2+1)(s^2+4)}, i\right] + \operatorname{Res}\left[\frac{se^{st}}{(s^2+1)(s^2+4)}, -i\right]$$

$$+ \operatorname{Res}\left[\frac{se^{st}}{(s^2+1)(s^2+4)}, 2i\right] + \operatorname{Res}\left[\frac{se^{st}}{(s^2+1)(s^2+4)}, -2i\right]$$

$$= \frac{ie^{it}}{2i(i^2+4)} + \frac{-ie^{-it}}{-2i(i^2+4)} + \frac{2ie^{2it}}{4i(4i^2+1)} + \frac{-2ie^{-2it}}{-4i(4i^2+1)}$$

$$= \frac{e^{it}}{6} + \frac{e^{-it}}{6} - \frac{e^{2it}}{6} - \frac{e^{-2it}}{6} = \frac{1}{3}(\cos t - \cos 2t)$$

$$(3) \quad F(s) = \frac{s+1}{9s^2+6s+5}$$

解:  $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s+1}{9s^2+6s+5}\right] = \mathcal{L}^{-1}\left[\frac{s+1}{9\left(s+\frac{1}{3}\right)^2+4}\right]$

$$= \frac{1}{9} \mathcal{L}^{-1}\left[\frac{s+\frac{1}{3}}{\left(s+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2} + \frac{\frac{2}{3}}{\left(s+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2}\right]$$

$$= \frac{1}{9} \left( \cos \frac{2}{3}t \cdot e^{-\frac{1}{3}t} + \sin \frac{2}{3}t \cdot e^{-\frac{1}{3}t} \right) = \frac{1}{9} \left( \cos \frac{2}{3}t + \sin \frac{2}{3}t \right) e^{-\frac{1}{3}t}$$

$$(4) \quad F(s) = \frac{2s+1}{s(s+1)(s+2)}$$

解:  $\frac{2s+1}{s(s+1)(s+2)} = \frac{1}{2s} + \frac{1}{s+1} - \frac{3}{2(s+2)}$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1, \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}, \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s(s+1)(s+2)}\right] = \frac{1}{2} + e^{-t} - \frac{3}{2}e^{-2t}$$

$$(5) \quad F(s) = \frac{2s^2+s+5}{s^3+6s^2+11s+6}$$

解:  $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{2s^2+s+5}{s^3+6s^2+11s+6}\right] = \mathcal{L}^{-1}\left[\frac{2s^2+s+5}{(s+1)(s+2)(s+3)}\right]$

$$= \operatorname{Re}s\left[\frac{(2s^2+s+5)e^{st}}{(s+1)(s+2)(s+3)}, -1\right] + \operatorname{Re}s\left[\frac{(2s^2+s+5)e^{st}}{(s+1)(s+2)(s+3)}, -2\right] +$$

$$\operatorname{Re}s\left[\frac{(2s^2+s+5)e^{st}}{(s+1)(s+2)(s+3)}, -3\right] =$$

$$\lim_{z \rightarrow -1} \frac{(2s^2+s+5)e^{st}}{(s+2)(s+3)} + \lim_{z \rightarrow -2} \frac{(2s^2+s+5)e^{st}}{(s+1)(s+3)} + \lim_{z \rightarrow -3} \frac{(2s^2+s+5)e^{st}}{(s+2)(s+1)}$$

$$= 3e^{-t} - 11e^{-2t} + 10e^{-3t}$$

