GEOPHYSICAL FLUID DYNAMICS

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§ Documentation §

Commonly used Formulas

各种坐标系中的矢量算子

- Cartesian coordinate system(x, y, z):

$$\textbf{Gradient } \nabla\psi = \frac{\partial\psi}{\partial x}\boldsymbol{i} + \frac{\partial\psi}{\partial y}\boldsymbol{j} + \frac{\partial\psi}{\partial z}\boldsymbol{k} \quad \textbf{Divergence } \nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\textbf{Rotation } \nabla\times\vec{V} = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\boldsymbol{i} + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\boldsymbol{j} + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\boldsymbol{k}$$

- Cylindrical coordinate system(r, θ, z):

$$\textbf{Gradient } \nabla\psi = \frac{\partial\psi}{\partial r}\boldsymbol{i} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\boldsymbol{j} + \frac{\partial\psi}{\partial z}\boldsymbol{z} \quad \textbf{Divergence } \nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial u}{\partial\theta}$$

$$\textbf{Rotation } \vec{k}\cdot(\nabla\times\vec{V}) = \frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial\theta} \quad \nabla_h^2\psi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

- Spherical coordinate system(λ, φ, r):

$$\textbf{Gradient } \nabla\psi = \frac{\boldsymbol{i}}{r\cos\varphi}\frac{\partial\psi}{\partial\lambda} + \frac{\boldsymbol{j}}{r}\frac{\partial\psi}{\partial\varphi} + \boldsymbol{k}\frac{\partial\psi}{\partial r} \quad \textbf{Divergence } \nabla\cdot\vec{V} = \frac{1}{r\cos\varphi}[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\varphi}(v\cos\varphi)]$$

$$\textbf{Rotation } \vec{k}\cdot(\nabla\times\vec{V}) = \frac{1}{r\cos\varphi}[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(u\cos\varphi)] \quad \nabla_h^2\psi = \frac{1}{r\cos^2\varphi}[\frac{\partial^2\psi}{\partial\lambda^2} + \cos\varphi\frac{\partial}{\partial\varphi}(cos\varphi\frac{\partial psi}{\partial\varphi})]$$

积分定理

(1) Divergence theorem

$$\int_V \nabla\cdot\vec{A}dV = \int_{\sum}\vec{A}\cdot\vec{n}d\sigma$$

式中 dV 为体积元,V 为体积,$d\sigma$ 为面积元,$\sum$ 为包围体积 V 的曲面,$\vec{n}$ 为曲面 $\sum$ 的外法线方向上的单位矢量。

$$\int_S \nabla\cdot\vec{F}dS = \oint_C \vec{F}\cdot\vec{n}dl$$

式中 $\vec{F}$ 为二维矢量,dS 为平面上面积元,S 为平面面积,C 为包围 S 的曲线,dl 为沿着 C 的线元,$\vec{n}$ 为曲线 C 的外法线方向上的单位矢量。

(2) Stokes' theorem

$$\int_V \nabla\times\vec{A}dV = \int_{\sum}\vec{n}\times\vec{A}d\sigma$$

$$\int_S \vec{k}\cdot\nabla\times\vec{F}dS = \oint_C \vec{F}\cdot\vec{\tau}dl$$

式中 $\vec{k}$ 是平面 S 的法线方向上的单位矢量,$\vec{\tau}$ 是曲线 C 的切线方向上的单位矢量,其他符号同上。

$$\int_V \nabla a dV = \oint_{\sum} a\vec{n}d\sigma$$

Section I 基础知识

1. 基本方程

- 地转 $(\Omega = 7.3\times 10^{-5} ms^{-1})$　薄层 $(\frac{H}{L} \ll 1)$　层化 $(\rho \neq constant)$
- 基本控制方程：

动量方程 $\rho(\frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V}) = -\nabla p - \rho\nabla\Phi + \vec{F}\ \left(\vec{F} = \mu\nabla^2\vec{V} + \frac{\mu}{3}\nabla(\nabla\cdot\vec{V})\right)$

连续方程 $\frac{d\rho}{dt} + \rho\nabla\cdot\vec{V} = 0$

$$\text{状态方程}\begin{cases}\text{海洋}\begin{cases}\rho = \rho_0[1-\alpha_T(T-T_0)+\alpha_S(S-S_0)] \\ \frac{dT}{dt} = K_T\nabla^2 T + \frac{Q_H}{\rho C_v} \\ \frac{dS}{dt} = K_S\nabla^2 S\end{cases} \\ \text{大气}\begin{cases}\rho = \frac{p}{RT} \\ \frac{dT}{dt} = \kappa\nabla^2 T - \frac{Q}{C_p}\end{cases}\end{cases}$$

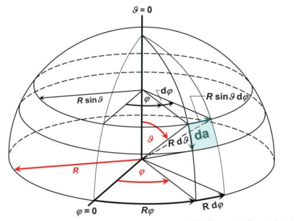
α_T, α_S 为热，盐膨胀系数
K_T, K_S 为热，盐扩散系数
Q_H 为热源或汇，C_v 为比热
R 为干空气气体常数，κ 为热扩散率，C_p 定压比热，Q 为加热率

- 无粘流体：

基本方程 $\frac{d\vec{V}}{dt} = -\frac{V_p}{\rho} - 2\vec{\Omega}\times\vec{v} + \vec{g}$

球坐标 (λ, θ, r)

$$\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uv\tan\theta}{r} + \frac{uw}{r}\right)\vec{i} + \left(\frac{dv}{dt} - \frac{u^2\tan\theta}{r} + \frac{vw}{r}\right)\vec{j} + \left(\frac{dw}{dt} - \frac{u^2+v^2}{r}\right)\vec{k}$$



$$\text{地转角速度}\begin{cases}\vec{\Omega} = \Omega\cos\theta\,\vec{j} + \Omega\sin\theta\,\vec{k} \\ 2\vec{\Omega}\times\vec{v} = (2\Omega w\cos\theta - 2\Omega v\sin\theta)\,\vec{i} + 2\Omega u\sin\theta\,\vec{j} - 2\Omega u\cos\theta\,\vec{k}\end{cases}$$

$$\text{分量形式为}\begin{cases}\frac{du}{dt} - (2\Omega + \frac{u}{r\cos\theta})(v\sin\theta - w\cos\theta) = -\frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial\lambda} \\ \frac{dv}{dt} + \frac{wv}{r} + (2\Omega + \frac{u}{r\cos\theta})u\sin\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \\ \frac{dw}{dt} - \frac{u^2+v^2}{r} - 2\Omega u\cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g\end{cases}$$

- 动量方程简化为：$\begin{cases}\frac{du}{dt} - fv - \frac{uv}{a}\tan\theta = -\frac{1}{\rho a\cos\theta}\frac{\partial p}{\partial\lambda} \\ \frac{dv}{dt} + fu + \frac{u2}{a}\tan\theta = -\frac{1}{\rho a}\frac{\partial p}{\partial\theta} \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\end{cases}$

其中：$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial z}$
$f = 2\Omega\sin\theta$(科氏参数)

- 连续方程为：$\frac{d\rho}{dt} + \rho\left[\frac{1}{a\cos\theta}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}(v\cos\theta) + \frac{\partial w}{\partial z}\right] = 0$

2. 基本原理

- f 平面近似

这是对地转参数采用的一种近似，在中纬度地区，可以将 f 作为常数处理。

纬度为 $\theta_0(\delta x, \delta y, \delta z) \approx (a\cos\theta_0\delta\lambda, a\delta\theta_0, \delta z)$　$f_0 = 2\Omega\sin\theta_0 = constant$

$$\text{简化动力方程}\begin{cases}\frac{\partial u}{\partial t} + (\vec{V}\cdot\nabla)u - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + (\vec{V}\cdot\nabla)v + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} = -\rho g\end{cases}$$

$$\frac{d\vec{V}}{dt} + \vec{f_0}\times\vec{V} = -\frac{1}{\rho}\nabla_z p$$
$$\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

- β 平面近似

  流体运动的尺度足够大（或者在赤道附近），要考虑地球曲率的影响；为了即使用直角坐标系，又把 f 随纬度的变化考虑在内，这种近似为 β 平面近似:

$$f = f_0 + \beta y, \beta = \frac{\partial f}{\partial y} = \frac{\partial(2\Omega\sin\theta)}{a\partial\theta} = \frac{2\Omega\cos\theta_0}{a}$$

- 静力近似 $\frac{\partial p}{\partial z} = -\rho g$

  假定 p(x,y,z)=$\overline{p}$(z)+p'(x,y,z,t), $|p'|\ll\overline{p}$ ρ(x,y,z)=$\overline{\rho}$(z)+ρ'(x,y,z,t), $|\rho'|\ll\overline{\rho}$

$$\frac{\partial(\overline{p}+p')}{\partial z} = -(\overline{\rho}+\rho')g$$
$$\frac{\partial p'}{\partial z} = -\rho' g$$

  在静力近似中，可以使用扰动量来代替平均量，将精确度提高两个量级。

  静力近似不适用的情况：

$$\frac{dw}{dt} - 2\Omega u\cos theta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$
$$= -\frac{1}{\overline{\rho}(1+\frac{\rho'}{\overline{\rho}})}\frac{\partial(\overline{p}+p')}{\partial z} - g \approx -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\overline{\rho}}g$$

  由水平动量方程可得: $p' \sim 2\Omega\rho UL \Rightarrow \frac{2\Omega u\cos\theta}{\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}} \sim \frac{H}{L} \ll 1$

  为了满足静力近似，即 $\frac{dw}{dt} \ll \frac{\rho'}{\overline{\rho}}g : T \sim \frac{L}{U} \Rightarrow W \ll \frac{L\rho'}{U\overline{\rho}}g$

- **Boussinesq** 近似

  在密度变化不大的水体中，除了浮力项中的密度外，其他项的密度可以当作常数。假定

$$\rho = \rho_0 + \partial\rho(x,y,z,t)$$
$$= \rho_0 + \tilde{\rho}(z) + \rho'(x,y,z,t)$$
$$= \tilde{\rho}(z) + \rho'(x,y,z,t)$$

  $\rho_0 = constant$ $|\partial\rho|,|\partial\rho|,|\rho'|\ll\rho_0$ $|\partial p|\ll\tilde{p}$

  海洋: $\rho_0 \cong 1000kg/m^3$ $\tilde{\rho} \cong 10kg/m^3$ $\rho' \cong 0.1kg/m^3$

  大气: $\rho_0 \cong 0.5kg/m^3$ $\tilde{\rho} \cong 0.5kg/m^3$ $\rho' \cong 0.005kg/m^3$

  $p = \overline{p} + \partial p(x,y,z,t)$

  带入动量方程可得：

$$(\rho_0 + \partial\rho)\left(\frac{d\overrightarrow{V}}{dt} + 2\Omega\times\overrightarrow{V}\right) = -\nabla\partial p - \frac{\partial\overline{p}}{\partial z}\overrightarrow{k} - g(\rho_0+\partial\rho)\overrightarrow{k}$$

  带入静力近似 $\frac{\partial\overline{p}}{\partial z}$=-$\rho_0 g$，方程可写为: $(\rho_0+\partial\rho)\left(\frac{d\overrightarrow{V}}{dt} + 2\Omega\times\overrightarrow{V}\right) = -\nabla\partial p - g\partial\rho\overrightarrow{k}$

  由 $|\partial\rho|\ll\rho_0$,动量方程进而可写为：

$$\frac{d\overrightarrow{V}}{dt} + 2\Omega\times\overrightarrow{V} = -\frac{\nabla\partial p}{\rho_0} - \frac{g\partial\rho}{\rho_0}\overrightarrow{k}\,(-\frac{g\partial p}{\rho_0}\text{为浮力})$$

  连续方程: $\frac{d\rho}{dt} + \rho\nabla\cdot\overrightarrow{V} \Rightarrow \frac{d\partial\rho}{dt} + (\rho_0+\partial\rho)\nabla\cdot\overrightarrow{V} = 0$

  Q: 因为 $\frac{d}{dt} \sim \nabla\cdot\overrightarrow{V}$，因此又可写成 $\nabla\cdot\overrightarrow{V} = 0$

- **Brunt-V**$\ddot{a}$**is**$\ddot{a}$**l**$\ddot{a}$ 频率

  不可压缩流体（海洋）: $\rho = \rho(z)$

流体微团 A 由 z 升到 $z+dz$，$\delta\rho$ 为流体微团 A 和它周围的密度差：

$$\delta\rho = rho_A - \rho_B = -\frac{\partial\rho}{\partial z}dz$$

流体微团所受到的恢复力：$F = -g\delta\rho = g\frac{\partial\rho}{\partial z}dz$，由牛顿第二定律可得：

$$\rho\frac{\partial^2(dz)}{\partial t^2} - g\frac{\partial\rho}{\partial z} = 0 \implies \frac{\partial^2(dz)}{\partial t^2} + N^2 dz = 0 \quad \left(N^2 = -\frac{g}{\rho}\frac{\partial\rho}{\partial z}\right)$$

– 当 $N^2 > 0$ 时，方程的通解为：$dz(t) = A\cos Nt + B\sin Nt$

$$When\ \ t=0,\ \ dz(t)=0 \implies A=0 \implies dz(t) = B\sin Nt = z_0 \sin Nt$$

该解表明流体微团在平衡高度上下震荡，z_0 时震荡的振幅，其频率即为 Brunt-Väisälä 频率或称为浮力频率：$N = \left[-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\right]^{1/2}$

– 当 $N^2 < 0$ 时，方程的通解为：$dz(t) = Ae^{i\sqrt{N^2}t} + Be^{-i\sqrt{N^2}t}$

$$When\ \ t=0,\ \ dz(t)=0 \implies A+B=0 \implies dz(t) = B\left(e^{-i\sqrt{N^2}t} - e^{i\sqrt{N^2}t}\right)$$

因为 $i\sqrt{N^2}$ 和 $-i\sqrt{N^2}$ 是实数，该解表明流体微团离平衡高度越来越远。

可压缩流体（大气）：$\rho = \rho(z)$，$p = p(z)$

流体微团 A 由 z 升到 $z+dz$，它的密度可写为：$\rho_A + \Delta\rho_A = \rho_A(z) + \frac{\partial\rho}{\partial p}\frac{dp}{dz}dz$

流体微团 B 的密度为：$\rho_B = \rho_A(z) + \frac{\partial\rho}{\partial z}dz$

$$\delta\rho = \rho_A + \Delta\rho_A - \rho_B = \frac{\partial\rho}{\partial p}\frac{dp}{dz}dz - \frac{\partial\rho}{\partial z}dz$$

$$\implies -\frac{g\delta\rho}{\rho} = -\frac{g}{\rho}\left(\frac{\partial\rho}{\partial p}\frac{dp}{dz} - \frac{\partial\rho}{\partial z}\right)dz = -N^2 dz,\ \ N^2 = \frac{g}{\rho}\left(\frac{\partial\rho}{\partial p}\frac{dp}{dz} - \frac{\partial\rho}{\partial z}\right)$$

理想大气中

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{C_p}},\ \ \rho = \frac{p}{RT} \implies \rho = \frac{p_0}{R\cdot\theta}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}},\ \text{其中}\ \gamma = \frac{C_p}{C_v}, C_v = C_p - R$$

$$N^2 = \frac{g}{\rho}\left(\frac{\partial\rho}{\partial p}\frac{dp}{dz} - \frac{\partial\rho}{\partial z}\right) = g\left(\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial\rho}{\partial z}\right) = \frac{g}{\theta}\left(\frac{\partial\theta}{\partial z}\right)$$

Prove:

Here we prove: $\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial\rho}{\partial z} = \frac{1}{\theta}\frac{\partial\theta}{\partial z}$

Set $\frac{\partial F}{\partial z} = F'$ and $F = f_1 f_2$, then we have: $F' = f_1' f_2 + f_2' f_1$ and $\frac{F'}{F} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$.

For the formula $\rho = \frac{p_0}{R\cdot\theta}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}$, set $F = \rho$, $f_1 = \frac{1}{\theta}$, $f2 = \frac{p_0}{R}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}$

$$\frac{1}{rho}\frac{\partial\rho}{\partial z} = \frac{\rho'}{\rho} = \theta\cdot\left(-\frac{1}{\theta^2}\right)\theta' + \left[\frac{p_0}{R}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}\right]^{-1}\cdot\left[\frac{p_0}{R}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}\right]'$$

$$= -\frac{\theta'}{\theta} + \left[\frac{p_0}{R}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}\right]^{-1}\cdot\frac{1}{\gamma}\frac{p_0}{R}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}-1}\cdot\frac{1}{p_0}p'$$

$$= -\frac{\theta'}{\theta} + \frac{1}{\gamma P}p' = -\frac{1}{\theta}\frac{\partial\theta}{\partial z} + \frac{1}{\gamma P}\frac{\partial p}{\partial z}$$

- 地转平衡 (**Geostrophic Balance**)

  Rossby Number: $R_0 \equiv \frac{U^2}{L}/fU \equiv \frac{H}{fL}$，旋转时间尺度 $(\frac{1}{f})$/平流时间尺度 $(\frac{L}{U})$，相对涡度 $(\frac{U}{L})$/行星涡度 (f)

  $R_0 \ll 1$，得到地转平衡方程: $\overrightarrow{f} \times \overrightarrow{V} = -\frac{1}{\rho}\nabla_z p$, $\overrightarrow{f} = 2\Omega\sin\theta\,\overrightarrow{k}$

  直角坐标系: $fu_g = -\frac{1}{\rho}\frac{\partial p}{\partial y}$ $fv_g = \frac{1}{\rho}\frac{\partial p}{\partial x}$

  球坐标系: $fu_g = -\frac{1}{\rho a}\frac{\partial p}{\partial \theta}$ $fv_g = \frac{1}{a\rho\cos\theta}\frac{\partial p}{\partial \lambda}$

  $(u_g,\ v_g)$ 为地转速度

Section II 无粘浅水理论

Section III 行星边界层

Section IV 层结流体中的准地转运动