1. (1) $f(x,y) = x^2 - 4xy + y^2$

$\nabla f = \begin{bmatrix} 2x - 4y \\ 2y - 4x \end{bmatrix} = 0 \Rightarrow$ critical point : $[0,0]^T$

$Hf = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$, $(2-\lambda)(2-\lambda) - 16 = 0$ $\Rightarrow \lambda = -2, 6$

Thus, it's a saddle point and there is no global minimum or maximum.

(2) $f(x,y) = x^4 - 4xy + y^4$

$\nabla f = \begin{bmatrix} 4x^3 - 4y \\ 4y^3 - 4x \end{bmatrix} = 0 \Rightarrow$ critical point: $\begin{cases} [0,0]^T \\ [1,1]^T \\ [-1,-1]^T \end{cases}$

$Hf = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$

At $[0,0]^T$, $\lambda^2 - 16 = 0 \Rightarrow \lambda = \pm 4 \Rightarrow$ saddle point

At $[1,1]^T$, $(12-\lambda)^2 - 16 = 0 \Rightarrow \lambda = 8, 16 \Rightarrow$ local minimum

At $[-1,-1]^T$, $(12-\lambda)^2 - 16 = 0 \Rightarrow \lambda = 8, 16 \Rightarrow$ local minimum.

Since $f(x,y)$ is coercive, it has global minimum on $[1,1]^T$, $[-1,-1]^T$

(3) $f(x,y) = 2x^3 - 3x^2 - 6xy(x-y-1)$

$\nabla f = \begin{bmatrix} 6x^2 - 6x - 6y(2x-y-1) \\ -6x(x-2y-1) \end{bmatrix} = 0 \Rightarrow$ critical points : $\begin{cases} [0,0]^T \\ [0,-1]^T \\ [1,0]^T \\ [-1,-1]^T \end{cases}$

$Hf = \begin{bmatrix} 12x - 12y - 6 & -12x + 12y + 6 \\ -12x + 12y + 6 & 12x \end{bmatrix}$

At $[0,0]^T$, $(6-\lambda)(-\lambda)-36=0 \Rightarrow \lambda = -3 \pm 3\sqrt{5} \Rightarrow$ saddle point

At $[0,-1]^T$, $(6-\lambda)(-\lambda)-36=0 \Rightarrow \lambda = 3 \pm 3\sqrt{5} \Rightarrow$ saddle point

At $[1,0]^T$, $(6-\lambda)(12-\lambda)+36=0 \Rightarrow \lambda = 9 \pm 3\sqrt{5} \Rightarrow$ local minimum

At $[-1,-1]^T$, $(6-\lambda)(-12-\lambda)-36=0 \Rightarrow x = -9 \pm 3\sqrt{5} \Rightarrow$ local maximum

Since $\lim\limits_{x,y \to +\infty} f(x,y) = +\infty$, $\lim\limits_{x,y \to -\infty} f(x,y) = -\infty$, It's not corcercive.

Therefore, there is no global maximum or minimum.

(4) $f(x,y) = (x-y)^4 + x^2 - y^2 - 2x + 2y + 1$

$$\nabla f = \begin{bmatrix} 4(x-y)^3 + 2x - 2 \\ -4(x-y)^3 - 2y + 2 \end{bmatrix} \Rightarrow \text{critical point: } [1,1]^T$$

$$Hf = \begin{bmatrix} 12(x-y)^2 + 2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 - 2 \end{bmatrix},$$

At $[1,1]^T$, $(2-\lambda)(-2-\lambda)=0$.

$\Rightarrow \lambda = 2$ or -2.

It's a saddle point and there is no global maximum or minimum.