7. (1) Once continuously differentiable?

Yes. Since it's a piecewise quadratic function and there're n data points. There are $n-1$ $a_i x^2 + b_i x_i + c_i = f(x)$. Thus, there're $3n-3$ unknowns. Interpolating the given data gives $2n-2$ equation, because each of the $n-1$ quadratic need to match the two data points at either end of its interval. Once continuously differentiable gives $n-2$ equations, because the derivatives of the quadratic on either side need to match of each of $n-2$ interior data points.

There're $2n-2 + n-2 = 3n-4$ equations. $3n-3 > 3n-4$. Therefore, there are solutions when once continuously differentiable.

(2) Twice continuously differentiable?

No. Beside $3n-4$ equations mentioned above, there're additional $n-2$ equations imposed by second continuous derivative

There're $3n-4 + n-2 = 4n-6$ equations.

Unknowns $\geq$ Equations $\Rightarrow$ $3n-3 > 4n-6$
$$n \leq 3$$

$n=3$ is the maximum value.