1. **What derivative is approximated in the above formula?**

$$f(x-2h) = f(x) + (-2h) * f'(x) + \frac{(-2h)^2}{2!} * f''(x) + \frac{(-2h)^3}{3!} * f'''(x) + O(h^4)$$

$$f(x-h) = f(x) + (-h) * f'(x) + \frac{(-h)^2}{2!} * f''(x) + \frac{(-h)^3}{3!} * f'''(x) + O(h^4)$$

$$g(x,h) = \frac{f(x-2h) - 2f(x-h) + f(x)}{h^2} = f''(x) - hf'''(x) - O(h^2)$$

2. **What is the leading order truncation error ?**

$$truncation\ error = h|f'''(x)|$$

3. **What is the estimated round-off error as a function of $\mathbf{h}$? (Assume condition number is roughly 1).**

$$\mathrm{g(x,h)} = \frac{f(x-2h) - 2f(x-h) + f(x)}{h^2} = \frac{\delta^2 f}{\delta x^2}$$

Rounded number:

$$fl\left(\frac{\delta^2 f}{\delta x^2}\right) = \frac{\hat{f}\left(\hat{x} - 2\hat{h}\right) - 2f\left(\hat{x} - \hat{h}\right) + f(\hat{x})}{\hat{h}^2}$$

Let:

$$\hat{f}_1 = \hat{f}\left(\hat{x} - 2\hat{h}\right)(1 + \epsilon_1)$$

$$\hat{f}_2 = \hat{f}\left(\hat{x} - \hat{h}\right)(1 + \epsilon_2)$$

$$\hat{f}_3 = \hat{f}(\hat{x})(1 + \epsilon_3)$$

Thus,

$$fl\left(\frac{\delta^2 f}{\delta x^2}\right) \approx \frac{f_1 - 2f_2 + f_3}{h^2} + \frac{f_1\epsilon_1 - 2f_2\epsilon_2 + f_3\epsilon_3}{h^2}$$

$$\approx \frac{f_1 - 2f_2 + f_3}{h^2} + \frac{\epsilon_1 - 2\epsilon_2 + \epsilon_3}{h^2} f(x)$$

Last term is bounded by,

$$|\frac{\epsilon_1 - 2\epsilon_2 + \epsilon_3}{h^2} f(x)| \leq \frac{4\epsilon_{math}}{h^2}|f(x)|$$

Therefore,

$$round\ off\ error = \frac{4\epsilon_{math}}{h^2}|f(x)|$$

4. For what h is the minimal total error attained?

$$total\ error = F(x) = h|f'''(x)| + \frac{4\epsilon_{math}}{h^2}|f(x)|$$

$$F(x)' = |f'''(x)| - \frac{8\epsilon_{math}|f(x)|}{h^3} = 0$$

$$h = 2 * \sqrt[3]{\frac{\epsilon_{math}|f(x)|}{|f'''(x)|}}$$

5. What is the estimated minimal total error (for IEEE 64-bit arithmetic) ?

Plug $h = 2 * \sqrt[3]{\frac{\epsilon_{math}|f(x)|}{|f'''(x)|}}$ into $F(x) = h|f'''(x)| + \frac{4\epsilon_{math}}{h^2}|f(x)|$

$$minimal\ total\ error\ = 3 * (|f'''(x)|)^{2/3} * \sqrt[3]{\epsilon_{math}|f(x)|}$$