Stokes Equations

Stokes equations are given by

$$\begin{aligned} \nabla p - \nabla \cdot \nabla \mathbf{u} &= \mathbf{f} \qquad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \qquad \text{in } \Omega. \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{u}_d \end{aligned}$$

with inhomogeneous Dirichlet boundary conditions imposed on velocity $\mathbf{u}$ on $\partial\Omega$. There are no essential boundary conditions on pressure p (in other words, homogeneous Neumann for pressure). Forcing $\mathbf{f}$ will be taken as $\mathbf{0}$.

Your task

Your will solve Stokes equations problems in two parts. In part 1 you will work on the Wannier-Stokes problem. In part 2, you will choose one problem out of the three options provided.

You are encouraged to explore with your solver, and come up with new additional problems in various geometry.

You will submit one writeup pdf containing all of your project problems on the first page of the flow on Relate, and your codes (either in py or ipynb) on the next page 2.

Problem part 1.

Due to Wannier's work on lubrication, an exact solution to Stokes flow is derived. This describes a steady 2D flow in a semi-infinite plane. The horizontal wall at the bottom, coinciding with the horizontal axis, moves with constant velocity U in its own plane. A fixed cylinder of radius R is located with its center at a distance d above the wall. Exact solution is given by [2] and shown in appendix. This is already implemented for you in the template file.

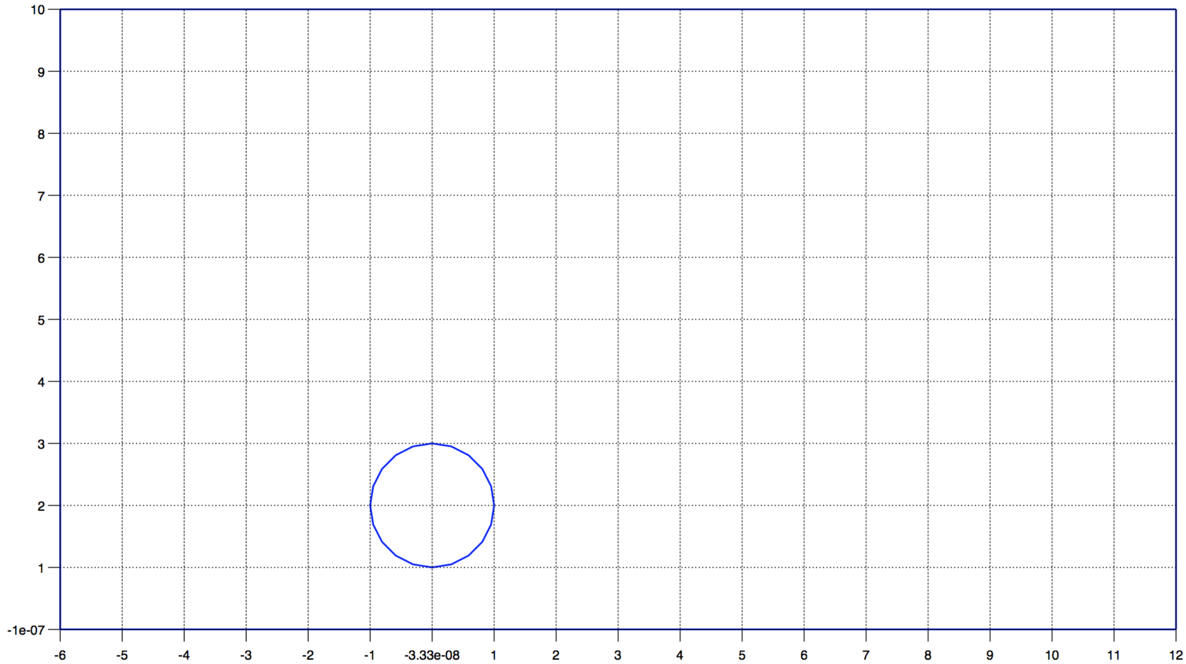


Figure 1: Geometry for Wannier-Stokes flow

Use the exact solution for your inhomogeneous Dirichlet boundary conditions.

Solve this problem by building a Stokes equations solver using Taylor-Hood $(\mathcal{P}_2, \mathcal{P}_1)$ element FEM. More instructions on FEM are in a later section. Report the following:

- For the most refined mesh, plot your solution fields(u, v and p).
- For a set of meshes with increasing resolution, compute the $\mathcal{L}^2$ norm of errors for each velocity component, and pressure:
$$\|u - \tilde{u}\|_{\mathcal{L}^2}, \|v - \tilde{v}\|_{\mathcal{L}^2}, \|p - \tilde{p}\|_{\mathcal{L}^2}$$
, and find convergence rates based on this norm.
- Find and plot pressure profiles on two cross-sections, $x \in [-6, 12]\,, y = 0.5$ and $x \in [-6, 12]\,, y = 1$. Use both your numerical solution and the exact pressure solution. You should have four lines in one plot. *Hint:* `scipy.interpolate.LinearNDInterpolator` is a good starting place.

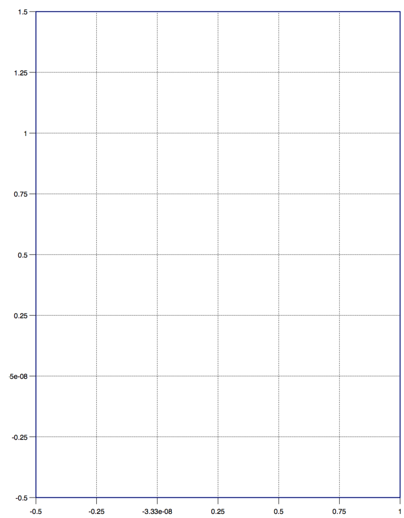
Problem part 2.

Choose one out of the following three problems

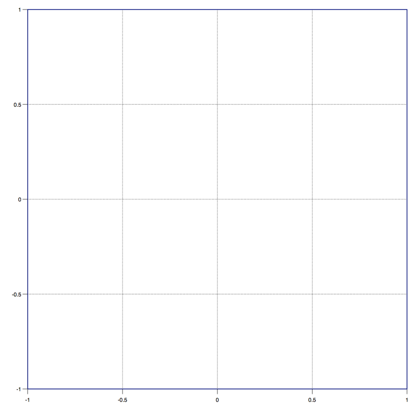
a Stokes extension of Kovasznay flow.

b Lid-driven cavity problem.

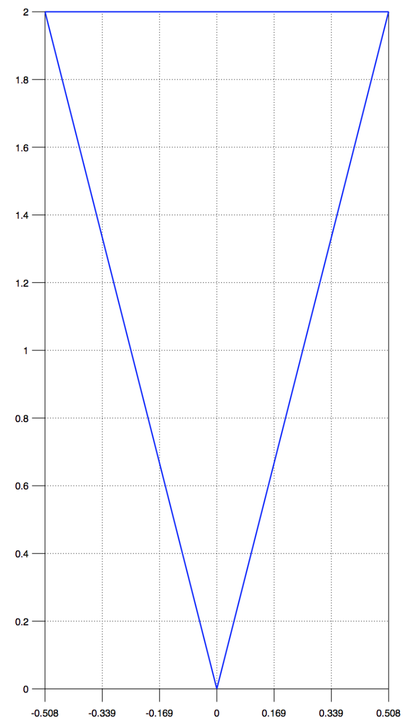
c Flow in a wedge.



(a) Kovasznay flow



(b) Lid-driven cavity



(c) Flow in a wedge

Figure 2: Geometry for optional problems

2a. Kovasznay flow problem

Kovasznay flow is an analytical steady-state solution to the Navier-Stokes equations that is similar to the 2D flow field behind an array of cylinders (see [1]).

$$
\begin{aligned}
u &= 1 - e^{\lambda x} \cos 2\pi y, \\
v &= \frac{\lambda}{2\pi} e^{\lambda x} \sin 2\pi y, \\
\lambda &= -2\pi.
\end{aligned}
$$

Taking to the viscous limit, one obtains a solution to the Stokes problem. Domain of the problem is chosen to be a rectangle, $x \in [-0.5, 1]$, $y \in [-0.5, 1.5]$.

Solve this problem using your Stokes equations solver and report the following:

- For the most refined mesh, plot your solution fields(u, v and p).
- Compare with the exact solution and find the $\mathcal{L}^2$ norm of error in velocities, and the rates of convergence.
- Describe what you see in the pressure solution field. Is this what you expected? If yes, explain why. If not, what did you expect?

2b. Lid-driven cavity

This is a classic problem in testing Stokes flow solvers. The domain is a square, $[-1, 1]^2$; the velocity boundary conditions are homogeneous on the three fixed lower walls, while the top lid is sliding at a unit horizontal component.

Solve this problem using your Stokes equations solver and report the following:

- For the most refined mesh that you ran, plot your solution fields(u, v and p).
- Tabulate your pressure values at the top corners (top left and top right) for all three meshes. What is the trend of pressure values there as you refine the mesh?

2c. Wedge flow

This 2D problem studies flow in a wedge-shaped domain. See [3]. As the top lid moves at unit velocity, a series of Moffatt eddies is generated. All the mesh files provided have an aperture angle of 28.5°. Solve this problem using your Stokes equations solver and report the following:

- For the most refined mesh, plot your solution fields(u, v and p).
- Tabulate your pressure values at the top corners (top left and top right) for all three meshes. What is the trend of pressure values there as you refine the mesh?

Weak form

First of all separate inhomogeneous Dirichlet boundary conditions by $\mathbf{u} = \mathbf{u}_b + \mathbf{u}_0$, $\mathbf{v} = \mathbf{v}_b + \mathbf{v}_0$, where

$$\begin{aligned} \nabla p - \nabla \cdot \nabla \mathbf{u}_0 &= \mathbf{f} + \nabla \cdot \nabla \mathbf{u}_b = \mathbf{f}' \qquad && \text{in } \Omega, \\ \nabla \cdot \mathbf{u}_0 &= -\nabla \cdot \mathbf{u}_b = g && \text{in } \Omega, \\ \mathbf{u}_0\big|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

This way, the inhomogeneous boundary conditions only exist in the right-hand side of the governing equation.

Next we solve the new system. For spatial discretization, you will use $(\mathcal{P}_2, \mathcal{P}_1)$ element (also known as Taylor-Hood element). That is, velocity $u \in \mathcal{P}_2$ is piecewise quadratic(same space for v), and pressure $p \in \mathcal{P}_1$ is piecewise linear. In other words, velocities are defined on all 6 nodes on a quadratic triangle, but pressure is only defined on the 3 vertices of the triangle.

Taking into account that we are now only working with functions $\mathbf{u}_0$ that satisfy the homogeneous Dirichlet boundary conditions, the final space for $\mathbf{u}_0$ is $\mathbb{V} = (\mathcal{P}_2)^2 \cap \mathcal{H}_0^1(\Omega)$, and for pressure $\mathbb{P} = \mathcal{P}_1 \cap \mathcal{L}^2(\Omega)$.

Weak form is obtained by taking test functions $\mathbf{v} \in \mathbb{V}, q \in \mathbb{P}$, multiplying the governing equations and integrated over the entire domain Ω. After integration by parts, the weak form reads: find $\mathbf{u}$ in $\mathbb{V}$, p in $\mathbb{P}$ such that

$$\begin{aligned} \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{u}_0 \,\mathrm{d}V - \int_\Omega p \nabla \cdot \mathbf{v} \,\mathrm{d}V &= \int_\Omega \mathbf{f}' \cdot \mathbf{v} \,\mathrm{d}V, \qquad && \forall \mathbf{v} \in \mathbb{V}, \\ -\int_\Omega q \nabla \cdot \mathbf{u}_0 \,\mathrm{d}V &= \int_\Omega q \nabla \cdot \mathbf{u}_b \,\mathrm{d}V, && \forall q \in \mathbb{P}. \end{aligned}$$

Note that $\mathbf{f}' = \mathbf{f} + \nabla \cdot \nabla \mathbf{u}_b$.

There are in fact three equations. After global-assembly the complete system is,

$$\begin{bmatrix} \bar{A}_u & 0 & -\bar{D}_x^T \\ 0 & \bar{A}_v & -\bar{D}_y^T \\ -\bar{D}_x & -\bar{D}_y & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} \bar{f}'_u \\ \bar{f}'_v \\ \bar{g}'_p \end{bmatrix} = \begin{bmatrix} \bar{f}_u - \bar{A}_u \bar{u}_b \\ \bar{f}_v - \bar{A}_v \bar{v}_b \\ \bar{D}_x \bar{u}_b + \bar{D}_y \bar{v}_b \end{bmatrix}.$$

Note that since pressure only has three points per element, it will have less degrees of freedom than the velocities. So $\bar{D}_x$ and $\bar{D}_y$ are rectangular, and are "short-and-fat" (more columns than rows).

Denote restriction matrix by R that masks off boundary points, then we have that the actual degrees of freedom for velocities, denoted by u and v, satisfy $u = R\bar{u}_0, \bar{u}_0 = R^T u$ (same applies to v and v_0). Pressure p does not have any boundary condition, so the unknowns $p = \bar{p}$. Modified systems reads

$$\begin{aligned} R\bar{A}_u R^T u - R\bar{D}_x^T p &= R\bar{f}_u', \\ R\bar{A}_v R^T v - R\bar{D}_y^T p &= R\bar{f}_v', \\ -\left(\bar{D}_x R^T u + \bar{D}_y R^T v\right) &= \bar{g}_p'. \end{aligned}$$

Take $A_u = R\bar{A}_u R^T$, and $A_v = R\bar{A}_v R^T$. Now we are ready to solve this system. Treat the first two equations first,

$$\begin{aligned} u &= A_u^{-1}\left(R\bar{f}_u' + R\bar{D}_x^T p\right), \\ v &= A_v^{-1}\left(R\bar{f}_v' + R\bar{D}_y^T p\right). \end{aligned}$$

Substitute that into the equation for p,

$$-\left(\bar{D}_x R^T A_u^{-1}\left(R\bar{f}_u' + R\bar{D}_x^T p\right) + \bar{D}_y R^T A_v^{-1}\left(R\bar{f}_v' + R\bar{D}_y^T p\right)\right) = \bar{g}_p',$$

then,

$$\begin{aligned} -\left(\bar{D}_x R^T A_u^{-1} R\bar{D}_x^T + \bar{D}_y R^T A_v^{-1} R\bar{D}_y^T\right) p &= \bar{g}_p' + \bar{D}_x R^T A_u^{-1} R\bar{f}_u' + \bar{D}_y R^T A_v^{-1} R\bar{f}_v'. \\ &= \bar{f}_p. \end{aligned}$$

Uzawa operator is defined to be $S := -\left(\bar{D}_x R^T A_u^{-1} R\bar{D}_x^T + \bar{D}_y R^T A_v^{-1} R\bar{D}_y^T\right)$. Solve $Sp = f_p$ to obtain pressure solution, then substitute back for velocity components. Lastly, the final solution reads

$$\begin{aligned} &p \text{ is a solution to } Sp = f_p, \\ &\bar{u} = R^T\left(A_u^{-1}\left(R\bar{f}_u' + R\bar{D}_x^T p\right)\right) + \bar{u}_b, \\ &\bar{v} = R^T\left(A_v^{-1}\left(R\bar{f}_v' + R\bar{D}_y^T p\right)\right) + \bar{v}_b. \end{aligned}$$

That completes the solver.

Coding

Much of your last homework is preparing you for this project, so reuse your code from there.

Appendix

Wannier-Stokes flow exact solution

$$\begin{aligned} u &= U - F\ln\left(\frac{K_1}{K_2}\right) - \frac{2(A+Fy)}{K_1}\left\{(s+y) + \frac{K_1}{K_2}(s-y)\right\} \\ &\quad - \frac{B}{K_1}\left\{(s+2y) - \frac{2y(s+y)^2}{K_1}\right\} - \frac{C}{K_2}\left\{(s-2y) + \frac{2y(s-y)^2}{K_2}\right\}, \\ v &= \frac{2x}{K_1K_2}(A+Fy)(K_2-K_1) - \frac{2Bxy}{K_1^2}(s+y) - \frac{2Cxy}{K_2^2}(s-y), \\ p &= -\frac{4Bx(s+y)}{K_1^2} - \frac{4Cx(s-y)}{K_2^2} - \frac{16Fsxy}{K_1K_2}. \end{aligned}$$

where

$$K_1 = x^2 + (s+y)^2,\ K_2 = x^2 + (s-y)^2,\ \Gamma = \frac{d+s}{d-s},\ s = \sqrt{d^2 - R^2}$$

$$A = -\frac{Ud}{\ln \Gamma},\ B = \frac{2(d+s)U}{\ln \Gamma},\ C = \frac{2(d-s)U}{\ln \Gamma},\ F = \frac{U}{\ln \Gamma}.$$

For the mesh provided, the following constants are assumed $d = 2, R = 1$ and $U = 4$. This is implemented in the template notebook file.

References

[1] L. I. G. Kovasznay. "Laminar flow behind a two-dimensional grid". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 44.1 (1948), pp. 58–62.

[2] M. K. Maslanik, R. L. Sani, and P. M. Gresho. "An isoparametric finite-element stokes flow test problem". In: *International Journal for Numerical Methods in Biomedical Engineering* 6.6 (1990), pp. 429–436.

[3] H. K. Moffatt. "Viscous and resistive eddies near a sharp corner". In: *Journal of Fluid Mechanics* 18.1 (1964), pp. 1–18.