homework4

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1. The Poission's equation $-\nabla \cdot \nabla u = f$, written in the polar coordinates becomes

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{u}}{dr}\right) = 1$$

The solution is easy to get by integrating twice, and I found the general solution is

$$\tilde{u} = -\frac{1}{4}r^2 + A\ln r + B$$

By boundary condition $\tilde{u}(r=1) = 0$ and $\tilde{u}(r=2) = 0$, we have

$$\tilde{u} = -\frac{1}{4}r^2 + \frac{3}{4}\frac{\ln r}{\ln 2} + \frac{1}{4}$$

2. Solving the equation with finite element method on a mesh made of triangle elements. The idea is based on WRT:

$$\int -v\nabla \cdot \nabla u dA = \int vf dA \quad \Rightarrow \quad \int \nabla v \nabla u dA = \int vf dA$$

if we break the integration region into a lot of element regions, we have

$$\int_{\Omega} \nabla v \nabla u dA = \sum \int_{\Omega} vf dA \quad \Rightarrow \quad \sum_e \int_{\Omega^e} \nabla v \nabla u dA = \sum_e \int_{\Omega^e} vf dA$$

Transform each triangle into standard triangle shape with new coordinates y^1, y^2 with the original coordinate as x^1, x^2. Then we have

$$\int_{\Omega^e} \nabla v \nabla u dA = \int_{\hat{\Omega}^e} \nabla v \nabla u \det(J) da \quad \text{with} \quad J = \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^1} \end{bmatrix}$$

Then expand u and v in terms of the basis function ϕ_i defined on the standard domain and change of the derivative from repsect to x's to y's. Then we have

$$\int_{\Omega^e} \nabla v \nabla u dA = \int_{\hat{\Omega}^e} v^i \frac{\partial \phi_i}{\partial y^m}\frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial x^k}\frac{\partial \phi_j}{\partial y^n} u^j \det(J) d^2y$$

where Einstein summation convension is used. In this problem, we are mapping six points in standard domain $\hat{\Omega}$ to Ω. Thus, we are using second order polynomial to map y^i to x^i. Thus, we can assume

$$x^1 = A_1 + B_1y^1 + C_1y^2 + D_1y^1y^2 + E_1(y^1)^2 + F_1(y^2)^2$$
$$x^2 = A_2 + B_2y^1 + C_2y^2 + D_2y^1y^2 + E_2(y^1)^2 + F_2(y^2)^2$$

we need to map six pairs of y to six pairs of x. Then we have the linear system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0.5 & 0 & 0 & 0.25 & 0 \\ 1 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \\ C_1 & C_2 \\ D_1 & D_2 \\ E_1 & E_2 \\ F_1 & F_2 \end{bmatrix} = \begin{bmatrix} x^1_{(1)} & x^2_{(1)} \\ x^1_{(2)} & x^2_{(2)} \\ x^1_{(3)} & x^2_{(3)} \\ x^1_{(4)} & x^2_{(4)} \\ x^1_{(5)} & x^2_{(5)} \\ x^1_{(6)} & x^2_{(6)} \end{bmatrix}$$

Obviously, we can determine the coefficients for each element. As the left matrix is fixed we do not need to take its inverse for each element, we just need the inverse once and apply it to each element. Therefore, we can calculate the matrix element of $\partial x/\partial y$. However, we need its inverse in the integration. Thus, we just need an inverse of 2×2 matrix. Obviously, $\det(J)$ has order 2 and each element in the matrix $\partial x/\partial y$ has order 1. Then the matrix $\partial y/\partial x$ has order -1. Also, with the known formula of ϕ, we can calulate its derivatives and the derivatives have order 1. We can write the integration in the way

$$\int_{\Omega^e} \nabla v \nabla u dA = v^i \left(\int_{\hat{\Omega}^e} \frac{\partial \phi_i}{\partial y^m} \frac{\partial y^m}{\partial x^k} \frac{\partial y^n}{\partial x^k} \frac{\partial \phi_j}{\partial y^n} \det(J) d^2 y \right) u^j$$

The matrix element inside the bracket is an integration for a order 2 polynomial , and thus we just need an order 2 Guassian quadrature on the triangle domain.

Next, consider $\int v f dA$, we convert it to the standard triangle domain, we find the integration is $v^i \int \phi_i f \det(J) d^2 y$ and therefore, the integrand is a order 4 polynomial. Thus, the right hand side or the "b" matrix needs an order 4 Gaussian quadrature on the triangle domain.

Then, we assmebly the A matrices on each element togethor. It looks hard but is actually simple when we notice that the the assmeble procedure is just to add node coefficients in different elements when this node belongs to more than one elements.

The Neumann boundary condition is weakly applied automatically in the above procedure as proved in lectures. The Dirichlet boundary condition can be applied by changing the A coefficients corresponding to the boundary node to $A_{ij} = \delta_{ij}$ when u_i is a boundary point and change $b_i = 0$.

With the constructed matrices, I solved the system and the plot is given below

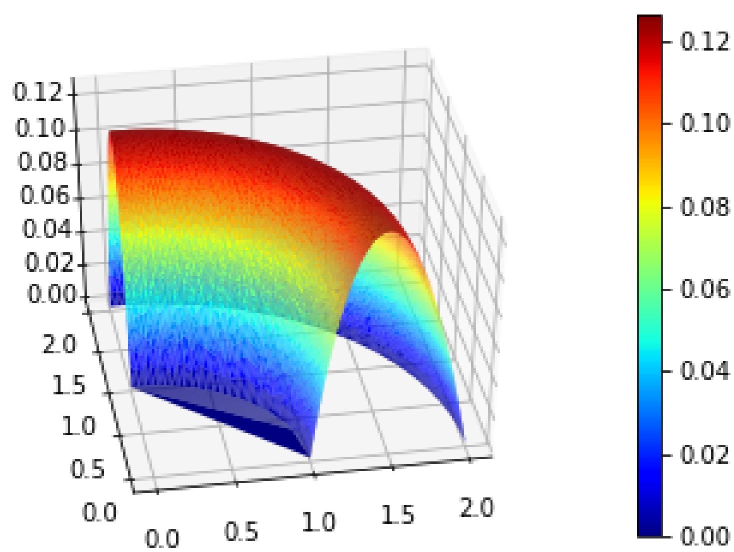

Figure 1: The solution solved numerically on the finest mesh of the given meshes.

we can see from solution that the variaiton only depends on the radial coordinates and on the Neumann boundary, the perpendicular direction of the slope of the variation is zero. Also, the values are zero on the Dirichlet boundary. Also, the infinity norm of the relative error is 0.0006239257994707043.

3 Using all three given meshes, I can calculate the infinity norm of relative error in all three cases. Also, If I define $\Delta x = \sqrt{1/n_e}$, I can have the following plot.

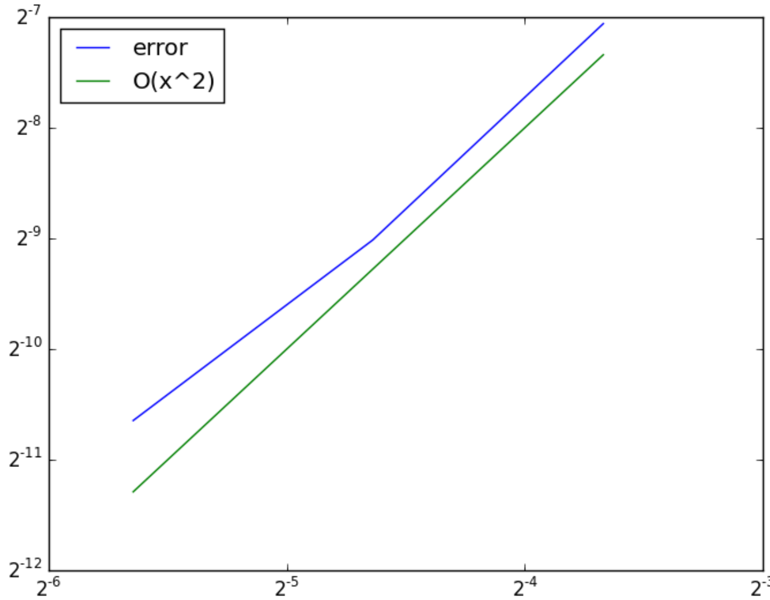

Figure 2: The infinity norm of relative error in all three cases and the a plot for $e \sim (\Delta x)^2$

Thus, I found the convergence rate is second order.

I increased the Gauss quadrature order in the integrations, the error does not change much although there is a slight increase from 0.0006239257994707043 to 0.0006243102802560895 with relative change of 0.06% when the order is increased to 8. However, if I change the order of Gaussian quadrature from 2 to 1 for the A matrix elements, the error blows up to 10^{19}.

Thus, if the Guassian quadrature order is above 2, the convergence rate should not change since the integration is already exact when the order is 2 and increase the quadrature order would not affect anything. When the quadrature order is below 2, the system is unstable and thus the convergence rate is not well defined.