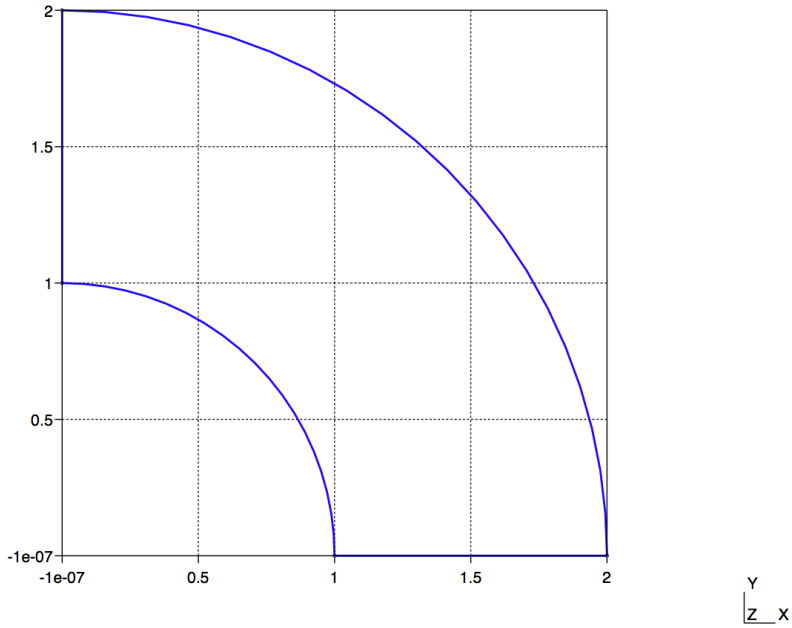
Poisson's Equation on an Annular Section

10 points



Consider the 2D Poisson's equation

$$-\nabla \cdot \nabla u = f, \quad \vec{x} \in \Omega,$$

where Ω is a quarter of an annulus domain, as shown above.
Your task is to solve this problem using FEM with quadratic triangle elements.
1. Assuming forcing $f = 1$, and the exact solution $\tilde{u}$ to be only a function of radius $r = \sqrt{x^2 + y^2}$, not of the angle θ. In that case, the PDE becomes

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{u}(r)}{dr}\right) = 1.$$

Homogeneous Dirichlet boundary conditions are on the two arcs, that is $\tilde{u}(r=1) = 0$, and $\tilde{u}(r=2) = 0$. Find the exact solution $\tilde{u}(r)$.
2. Numerically solve the 2D problem with the same forcing $f(x, y) = 1$ and Dirichlet conditions given as in part 1, on all three of the meshes provided. Assume homogeneous Neumann conditions on the two straight sides $(x \in [1, 2], y = 0)$, and $(x = 0, y \in [1, 2])$. Show a plot of your solution on the mesh with the finest resolution.
3. Find the relative pointwise infinity norm errors in the solution for all three meshes, and find the convergence rate of your solver.
Consider, what is a good measurement for Δx when you find your convergence rate?
Does your convergence rate depend on Gaussian quadrature order? Why or why not?
Instructions:
A detailed example of solving Poisson's equation on a square domain using FEM with linear triangles is provided here (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/fesimple_detail.ipynb), along with two text files containing the mesh data for a square domain: vertices (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/mesh.v), and elements (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/mesh.e). It is recommended you work through this example and reuse ideas from it for this homework.
Three sets of quadratic triangle mesh with increasing resolution of the same annular section domain are provided here, mesh 1 (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/aq2.msh), mesh 2 (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/aq3.msh), and mesh 3 (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/aq4.msh).
A template solution notebook (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/hw4_tmpl.ipynb) is provided which reads in the mesh, and contains a couple of examples for plotting the mesh and a function on the mesh points. An additional file gmsh.py (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/gmsh.py) is needed for reading in the mesh file.
Gauss quadrature will be needed. This code (/course/cs555-s18/file-version/14f970fef9bc565937a4bb9e3a27ef7b997d17c4/handouts/trigauss.py) generates quadrature points and weights inside a reference triangle.

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You may do either of the following:

*   create the PDF using software (meaning, LaTeX!). (preferred)

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    If you decide to use your phone for scanning, make sure to use an app
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Submit your response to each problem in this homework as a separate PDF. Code
will sometimes be uploaded and tested (do so when prompted), but your response
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