Variational Formulation of the Stokes Problem [1]

- Recall for the Poisson equation we have equivalence between orthogonality and error minimization: *Find* $u \in X_0^N \subset \mathcal{H}_0^1$ *s.t.*

$$a(v,u) \;=\; (v,f)\;\forall\; v \,\in X_0^N \quad \Longleftrightarrow \quad a(v,u) \;=\; a(v,\tilde{u})\;\forall\; v \,\in X_0^N.$$

- The equivalent variational (minimization) formulation is,

$$\begin{aligned} u &= \operatorname*{argmin}_{w\in X_0^N}\; a(\tilde{u}-w,\tilde{u}-w) \;=\; \operatorname*{argmin}_{w\in X_0^N}\; a(e,e), \quad e := \tilde{u}-w \\ &= \operatorname*{argmin}_{w\in X_0^N}\; \tilde{I}(w), \qquad \tilde{I}(w) := a(\tilde{u}-w,\tilde{u}-w). \end{aligned}$$

- Note that the quadratic functional $\tilde{I}$ is

$$\tilde{I}(w) \;=\; a(w,w) \;-\; 2a(w,\tilde{u}) \;+\; a(\tilde{u},\tilde{u}).$$

- Minimizing $\tilde{I}(w)$ is equivalent to finding the w that minimizes

$$\begin{aligned} \mathcal{I}(w) &= a(w,w) \;-\; 2a(w,\tilde{u}) \\ &= a(w,w) \;-\; 2(w,f). \end{aligned}$$

- For $w = \tilde{u}$, $\tilde{I} \;=\; 0$, whereas $\mathcal{I} \;=\; -a(\tilde{u},\tilde{u}) \;<\; 0$.
- The *standard variational formulation* is based on $\mathcal{I}$,

$$\boxed{u \;=\; \operatorname*{argmin}_{w\in X_0^N}\; \mathcal{I}(w) \;=\; \operatorname*{argmin}_{w\in X_0^N}\; a(w,w) \;-\; 2(w,f).}$$

- This form doesn't expressly depend on $\tilde{u}$.
- So we have, even in the continuous case,

$$-\nabla\tilde{u} \;=\; f, \quad \tilde{u} = g \text{ on } \partial\Omega \quad \Longleftrightarrow \quad \tilde{u} \;=\; \operatorname*{argmin}_{w\in\mathcal{H}_b^1}\; \mathcal{I}(w).$$

- **$\tilde{u}$ is the solution of a minimization problem.**

[1] Following notes from A.T. Patera...

- Consider now,

$$\tilde{\mathbf{u}} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} \tilde{u}_x \\ \tilde{u}_y \end{pmatrix} \qquad \tilde{\mathbf{v}} = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} \tilde{v}_x \\ \tilde{v}_y \end{pmatrix}$$

$$a(\mathbf{v},\mathbf{u}) := \sum_{i=1}^{d} a(v_i,u_i) = \sum_{i=1}^{d} \int_{\Omega} \nabla v_i \cdot \nabla u_i \, dV$$

$$(\mathbf{v},\mathbf{u}) := \sum_{i=1}^{d} (v_i,u_i) = \sum_{i=1}^{d} \int_{\Omega} v_i u_i \, dV.$$

- We can write a system of equations

$$-\nabla^2 \tilde{\mathbf{u}} = \mathbf{f}, \quad \begin{cases} \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega_D \\ \nabla \mathbf{u}_i \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial\Omega_N \\ (\textit{Can also mix } \partial\Omega_{D_i} \textit{ and } \partial\Omega_{N_i}.) \end{cases}$$

  in equivalent variational form

$$\tilde{\mathbf{u}} = \underset{\mathbf{w}\in\mathcal{H}_b^1}{\operatorname{argmin}} \; a(\mathbf{w},\mathbf{w}) - 2(\mathbf{w},\mathbf{f}).$$

- Our ***discrete*** (i.e. finite-dimensional) variational statement is

$$\boxed{\mathbf{u} = \underset{\mathbf{w}\in X_b^N}{\operatorname{argmin}} \; a(\mathbf{w},\mathbf{w}) - 2(\mathbf{w},\mathbf{f}),}$$

  which gives rise to the familiar system

$$\begin{bmatrix} A & \\ & A \end{bmatrix} \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} = \begin{pmatrix} RQ^T B_L \underline{f}_{L,1} \\ RQ^T B_L \underline{f}_{L,2} \end{pmatrix} = \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \end{pmatrix} \tag{1}$$

  or, $\mathbf{A}\underline{\mathbf{u}} = \underline{\mathbf{b}}$.

- The system matrix $\mathbf{A}$ is SPD and amounts to two independent Poisson solves.

- In this case, the equations are decoupled, but they needn't be (e.g., when the viscosity is variable there is a coupling through the stress tensor, $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$).

- **Stokes** – Incompressibility constraint:

$$
\begin{aligned}
-\frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad + \quad \frac{\partial \tilde{p}}{\partial x_i} \quad &= \; f_i \text{ in } \Omega, \quad i = 1, \ldots, d \\
-\frac{\partial u_j}{\partial x_j} \qquad\qquad\qquad &= \; 0, \quad u_i \; = \; 0 \text{ on } \partial\Omega.
\end{aligned}
$$

- Vector form:

$$
\begin{aligned}
-\nabla^2 \tilde{\mathbf{u}} \; - \; \nabla \tilde{p} \; &= \; \mathbf{f} \quad \text{in } \Omega, \\
-\nabla \cdot \tilde{\mathbf{u}} \qquad\quad &= \; 0, \quad \tilde{\mathbf{u}} \; = \; 0 \text{ on } \partial\Omega.
\end{aligned}
$$

- Equivalent (*constrained*) minimization problem:

$$
\begin{aligned}
\tilde{\mathbf{u}} \quad &= \quad \underset{\mathbf{w} \in \mathcal{H}_0^1, \nabla \cdot \mathbf{w} = 0}{\operatorname{argmin}} \quad a(\mathbf{w}, \mathbf{w}) \; - \; 2(\mathbf{w}, \mathbf{f}) \\
&= \quad \underset{\mathbf{w} \in \mathcal{H}_0^1, \nabla \cdot \mathbf{w} = 0}{\operatorname{argmin}} \quad \mathcal{I}(\mathbf{w}).
\end{aligned}
$$

Constrained Minimization Example in $\mathbb{R}^n$

- Consider the SPD matrix A and quadratic functional

$$\mathcal{I}(\underline{w}) \quad := \quad \frac{1}{2}\underline{w}^T A \underline{w} \;-\; \underline{w}^T \underline{f}$$

  and linear constraint

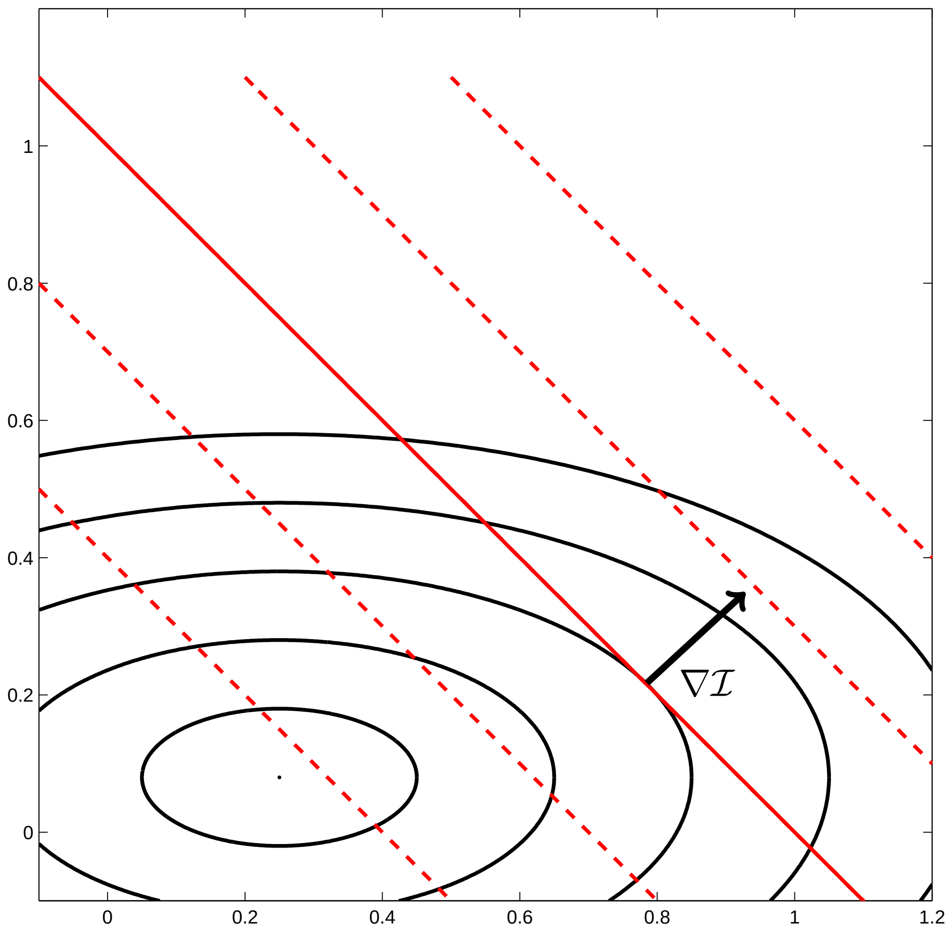
$$\underline{d}(\underline{w}) \quad := \quad D\underline{w} - \underline{g} \;=\; 0.$$

- Note that

$$\begin{aligned} \frac{\partial \mathcal{I}}{\partial w_i} \quad &= \quad (A\underline{w})_i \;-\; f_i, \\ \frac{\partial^2 \mathcal{I}}{\partial w_i \partial w_j} \quad &= \quad A_{ij}. \end{aligned}$$

- Perturbation $\underline{v}$:

$$\begin{aligned} \mathcal{I}(\underline{w}+\underline{v}) \quad &= \quad \mathcal{I}(\underline{w}) \;+\; \frac{\partial I}{\partial w_i} v_i \;+\; \frac{1}{2}\,\underline{w}^T A \underline{w} \\ \underline{d}(\underline{w}+\underline{v}) \quad &= \quad D(\underline{w}+\underline{v}) \;-\; \underline{g} \;=\; D\underline{w} \;-\; \underline{g} \;+\; D\underline{v} \\ &= \quad \underline{d}(\underline{w}) \;+\; D\underline{v} \\ &= \quad \underline{d}(\underline{w}) \;+\; \frac{\partial d_k}{\partial w_j} v_j. \end{aligned}$$



- Note that, at the solution point, $\nabla\mathcal{I} = c\nabla d$.
- That is, the isosurface of $\mathcal{I}$ is tangent to the constraint space.

- Introduce the Lagrangian and the Lagrange multiplier, $\underline{q}$:

$$L(\underline{w}, \underline{q}) \quad := \quad \mathcal{I}(\underline{w}) \; - \; \underline{q}^T \underline{d}(\underline{w})$$

- Perturbations:

$$\begin{aligned} L(\underline{w}+\underline{v}, \underline{q}+\underline{r}) \quad &= \quad L(\underline{w}, \underline{q}) \\ &+ \quad \underline{v}^T (A\underline{w} \; - \; \underline{f}) \; - \; \underline{q}^T D\underline{v} \; - \; \underline{r}^T (D\underline{w} - \underline{g}) \\ &+ \quad \underbrace{\frac{1}{2}\underline{v}^T A \underline{v}}_{>0\,\forall\,\underline{v}\neq 0} \; - \quad \underbrace{\underline{r}^T D\underline{v}}_{=0\,\forall\,\underline{v}\,\text{s.t.}\,D\underline{v}=0} \quad . \end{aligned}$$

- At solution point $(\underline{u}, \underline{p})$, require first variation to vanish (recall plot):

$$\begin{aligned} \delta L \quad &= \quad \underline{v}^T \left(A\underline{w} \; - \; D^T\underline{q} \; - \; \underline{f}\right) \; - \; \underline{r}^T \left(D\underline{w} \; - \; \underline{g}\right) \\ &= \quad 0 \,\forall\, \underline{v} \in \mathbb{R}^n, \; \underline{r} \in \mathbb{R}^m. \end{aligned}$$

$$\begin{array}{lll} \underline{r} = 0, \; \underline{v}\,\text{arbitrary:} & A\underline{u} - D^T\underline{p} & = \underline{f} \\ \underline{v} = 0, \; \underline{r}\,\text{arbitrary:} & -D\underline{u} & = -\underline{g}. \end{array}$$

- Note that $\forall\, \underline{v}$ s.t. $D\underline{v} \; = \; 0$,

$$\mathcal{I}(\underline{u} \, + \, \underline{v}) \quad \geq \quad \mathcal{I}(\underline{u}),$$

  which implies that we have minimized $\mathcal{I}$ in the constraint space.

- The system for our constrained minimization example is thus

$$\begin{bmatrix} A & -D^T \\ -D & 0 \end{bmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} \quad = \quad \begin{pmatrix} \underline{f} \\ -\underline{g} \end{pmatrix}.$$

Example

$$\underline{d}(\underline{w}) \;=\; w_1 + w_2 - 1 = [1\;1]\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - 1$$

$$\mathcal{I}(\underline{w}) \;=\; \frac{1}{2}w_1^2 + w_2^2 + w_1 + w_2$$

$$\nabla \mathcal{I} = \begin{pmatrix} w_1 + 1 \\ 2w_2 + 1 \end{pmatrix}$$

- Governing system:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ p \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

- Solution: $p = \frac{5}{3}$, $u_1 = p - 1 = \frac{2}{3}$, $u_2 = \frac{1}{2}(p-1) = \frac{1}{3}$.

Saddle Problem Example

- Constrained minimization problems are often referred to as *saddle problems.*
- The stationary point is *not* a local minima.
- Along the constraint, it is a minimum.
- Orthogonal to the constraint one has a descent direction.
- Consider the minimization problem: *Find* x^* *that minimizes* $\mathcal{I}(x) = \frac{1}{2}x^2$ *subject to the constraint* $d(x) = x - 1 = 0$*.*

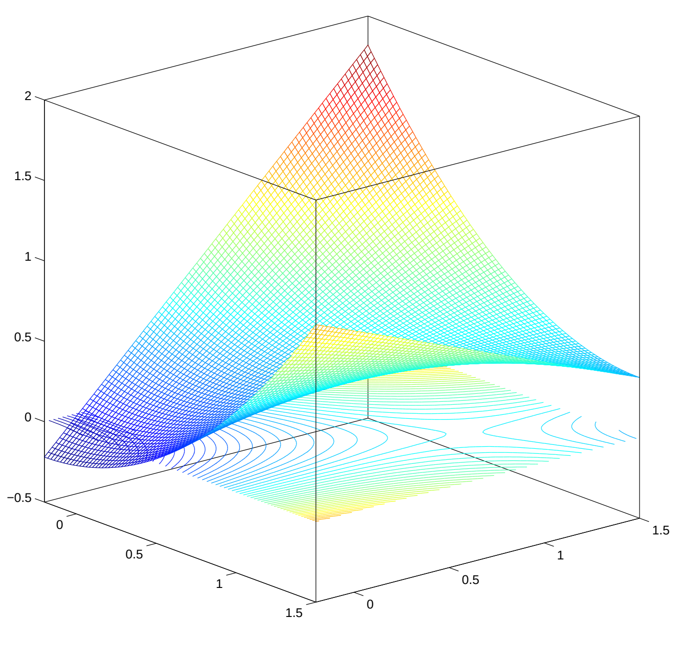
$$x^* \quad = \quad \underset{x \in \mathbb{R};\, \mathrm{s.t.}\, x=1}{\operatorname{argmin}} \quad \mathcal{I}(x).$$

- Clearly, $\mathcal{I}_{\min} := \mathcal{I}(x^*) = \frac{1}{2}$.
- With Lagrange multiplier y, the Lagrangian for this case is

$$\begin{aligned} L(x,y) &= \overbrace{\frac{1}{2}x^2}^{\mathcal{I}(x)} - y \cdot d(x) \\ &= \frac{1}{2}x^2 - y(x-1). \end{aligned}$$

At the solution point (x^*, y^*), we require stationarity,

$$\left.\begin{array}{rcl} \frac{\partial L}{\partial x} &=& x - y = 0 \\ \frac{\partial L}{\partial y} &=& 1 - x = 0 \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{l} x^* = y^* = 1 \\ L(x^*, y^*) = \mathcal{I}_{\min} = \frac{1}{2}. \end{array}$$



Stokes Problem

- Returning to our PDE, we see that the discrete form of the constrained minimization problem will be

$$\left[\begin{array}{ccc} A & & -D_x^T \\ & A & -D_y^T \\ -D_x & -D_y & \end{array}\right] \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ -\underline{g} \end{pmatrix}.$$

- Or, in vector form,

$$\left[\begin{array}{cc} \mathbf{A} & -\mathbf{D}^T \\ -\mathbf{D} & 0 \end{array}\right] \begin{pmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{f}} \\ -\underline{g} \end{pmatrix}. \tag{2}$$

- $\mathbf{A}$ is symmetric positive definite (SPD) and thus invertible.
- For large (e.g., $n \gg 10^4$) systems, particularly in three space dimensions, an effective solution strategy is the *Uzawa algorithm*, in which we carry out formal block-Gaussian elimination.
- The first equation remains unchanged, while the second is modified to generate a zero in the [2,1] position:

$$\left[\begin{array}{cc} \mathbf{A} & -\mathbf{D}^T \\ 0 & S \end{array}\right] \begin{pmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{f}} \\ \underline{g} - \mathbf{D}\mathbf{A}^{-1}\underline{\mathbf{f}} \end{pmatrix}. \tag{3}$$

- Here, S is the *Schur complement*,

$$S = D\mathbf{A}^{-1}D^T.$$

- S is clearly symmetric.
- S will be positive definite if

$$\beta^2 := \lambda_{\min}(S) = \min_{\underline{q} \in \mathbb{R}^m;\, \underline{q} \neq 0} \frac{\underline{q}^T S \underline{q}}{\underline{q}^T \underline{q}} > 0. \tag{4}$$

- If $\beta = 0$, it implies that there are spurious pressure modes $\hat{\underline{p}} \neq 0$ such that $\mathbf{D}^T \hat{\underline{p}} = 0$ and $S\underline{p} = \tilde{\underline{g}}$ is therefore not solvable.
- Can show that β is the *inf-sup* constant,

$$\beta = \inf_{\underline{q}} \sup_{\underline{\mathbf{v}}} \frac{\underline{\mathbf{v}}^T \mathbf{D}^T \underline{q}}{(\underline{\mathbf{v}}^T \mathbf{A} \underline{v})^{\frac{1}{2}} (\underline{q}^T \underline{q})^{\frac{1}{2}}}. \tag{5}$$

- Having $\beta > 0$ implies that the discretization is *div-stable*.

- We look at these issues momentarily, but return now to the Uzawa system (3) under the assumption that $\beta > 0$.

- Note that $\mathbf{A}$ and $\mathbf{D}$ are typically large and sparse, whereas $\mathbf{A}^{-1}$ is *full* (or, at least, it's diagonal blocks are full).

- The idea behind the Uzawa algorithm is to use conjugate gradient (CG) iteration to solve $S\underline{p} = \underline{\tilde{g}}$, with S being SPD.

- Each CG iteration requires a matrix-vector product of the form $\underline{w} = S\underline{s}$, where $\underline{s}$ is a search direction in the CG scheme.

- These mat-vecs are effected as follows:

$$\begin{aligned}
\underline{\mathbf{v}} &= \mathbf{D}^T \underline{s} \\
\text{Solve } \mathbf{A}\underline{\mathbf{w}} &= \underline{\mathbf{v}} \\
\underline{w} &= \mathbf{D}\underline{\mathbf{w}},
\end{aligned}$$

  which requires $d = 2$ or 3 Poisson solves in A per CG iteration.

- The Poisson solves in A can be solved with multigrid or a combination of multigrid plus CG or GMRES.

- Fortunately, S tends to be well-conditioned, so relatively few outer CG iterations (in S) are required. (This is *not* true for the *unsteady* Stokes, or Navier-Stokes, problem.)

- Heuristically,

$$S \approx \mathbf{D}\left[\mathbf{D}^T\mathbf{B}\mathbf{D}\right]^{-1}\mathbf{D} \;\sim \mathbf{B}^{-1},$$

  and can thus be effectively preconditioned by the block-diagonal mass matrix $\mathbf{B}$.

- Assuming div-stability (i.e., β bounded away from zero independent of mesh resolution), the mass-matrix preconditioned Uzawa scheme generally leads to a condition number that is bounded independent of the mesh resolution.

  - The number of iterations does not depend on the mesh resolution.
  - The number of iterations *does* depend on the domain shape and boundary conditions.

- The condition that $\beta > 0$, independent of the mesh parameter, is known as the *inf-sup* or Ladyzenskaja-Babuska-Brezzi (LBB) condition.

- Note the importance of $\beta > 0$ for uniqueness of $\underline{p}$.

  - Suppose we have two solutions $\underline{p}_1$ and $\underline{p}_2$ satisfying

$$\begin{array}{rcl} S\underline{p}_1 & = & \underline{g} - \mathbf{D}\mathbf{A}^{-1}\underline{\mathbf{f}} \\ S\underline{p}_2 & = & \underline{g} - \mathbf{D}\mathbf{A}^{-1}\underline{\mathbf{f}} \\ \hline S(\underline{p}_1 - \underline{p}_2) & = & 0 \end{array}$$

  - Since $\beta > 0$ implies that S is SPD, we know that

$$S(\underline{p}_1 - \underline{p}_2) = 0 \quad \Longrightarrow \quad (\underline{p}_1 - \underline{p}_2) = 0,$$

    and the pressure is thus unique.

  - As noted earlier, the LBB condition is also important for solvability and conditioning of the discrete Stokes system.

Demonstration of ***inf-sup*** Constant

$$
\begin{aligned}
\beta^2 &:= \min_{\underline{q} \in \mathbb{R}^m} \frac{\underline{q}^T S \underline{q}}{\underline{q}^T \underline{q}} \\
&= \min_{||\underline{q}||=1} \underline{q}^T S \underline{q} \\
&= \min_{||\underline{q}||=1} \underline{\mathbf{w}}^T \mathbf{A}^{-1} \underline{\mathbf{w}}, \;\; \underline{\mathbf{w}} := \mathbf{D}^T \underline{q}
\end{aligned}
$$

Some Unstable/Stable Examples

- Revisit our minimization problem in $\mathbb{R}^2$.

- $P_1 - P_0$ (continuous velocity, discontinuous pressure)

- $Q_1 - Q_0$ (continuous velocity, discontinuous pressure)

- $P_2 - P_1$ (continuous velocity, continuous pressure) – Taylor-Hood.

  Key results for Taylor-Hood:

$$
\begin{aligned}
||\tilde{\mathbf{u}} - \mathbf{u}||_{H^1} &\sim O(h^2) \\
||\tilde{\mathbf{u}} - \mathbf{u}||_{L^2} &\sim O(h^3) \\
||\tilde{p} - p||_{L^2} &\sim O(h^2).
\end{aligned}
$$