MA202 Complex Analysis Midterm-24Spring

Problem 1. 15 pts

Description

Let f:$\mathbb{C} \to \mathbb{C}$ be a complex valued function which is written as

$$f(x, y) = u(x, y) + iv(x, y)$$

where u, v are continuously differentiable.

(a) Let $g(z) = \overline{f(\bar{z})}$, show that g is holomorphic if and only if f is holomorphic.
(b) Suppose $u(x, y) = xy - x + y$, find all possible v such that f is holomorphic.

Answer

(a)

$$g(z) = \overline{f(\bar{z})} = \overline{f(x - iy)} = u(x, -y) - iv(x, -y)$$

Suppose $u'(x, y) = u(x, -y)$, $v'(x, y) = -v(x, -y)$.

If f is holomorphic, by **Cauchy-Riemann** equations we know:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

thus:

$$\frac{\partial u'}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v'}{\partial y}$$

$$\frac{\partial u'}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial v'}{\partial x}$$

which means g follows **Cauchy-Riemann** equations, thus g is holomorphic.

Converse is the same proof.

(b)
By **Cauchy-Riemann** equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y - 1$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x - 1$$

so we have:

$$v(x, y) = y^2 - y + h(x)$$

$$v(x, y) = -x^2 - x + h(y)$$

In conclusion, v will be $v(x, y) = y^2 - y - x^2 - x + C$, C is a constant.

Problem 2. 10 pts

Description

Let $f(z)$ be a continuous function defined in the unit disk $D_1 = \{z \mid |z| < 1\}$.
Assuming $f(z)^5$ and $f(z)^7$ are holomorphic, show that $f(z)$ itself is holomorphic.

Answer

Notice that:

$$f(z) = \frac{(f(z)^5)^3}{(f(z)^7)^2}$$

then by **Proposition 2.2** in Chapter 1 and we are done.

Problem 3. 15 pts

Description

Consider the meromorphic function $f(z) = \frac{1}{e^{z^2}+1}$.

(a) Find all the poles of $f(z)$.
(b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series expression of $f(z)$ centered at 0, find the radius of convergence and explain your answer.

Answer

(a) The poles occur where $e^{z^2} + 1 = 0$, thus $e^{z^2} = -1$, then $z^2 = i\pi + 2k\pi i (k \in \mathbb{Z})$.

Thus:

$$z = \pm\sqrt{i\pi(1+2k)} = \pm\sqrt{\pi(1+2k)} \cdot e^{\frac{i\pi}{4}} = \pm\sqrt{\pi(1+2k)} \cdot \frac{1+i}{\sqrt{2}}, k \in \mathbb{Z}$$

(b) By (a), the nearest pole to $z = 0$ is:

$$z = \pm\sqrt{\pi} \cdot \frac{1+i}{\sqrt{2}} = \pm\sqrt{\frac{\pi}{2}}(1+i)$$

and the distance is $d = |z| = \sqrt{\pi}$.

So $f(z)$ is holomorphic in the disk of radius $R < \sqrt{\pi}$, thus the radius of convergence is $R = \sqrt{\pi}$.

Problem 4. 15 pts

Description

Consider the strip $S = \{z \mid 0 < \Im(z) < 1\}$. Let $f : S \to \mathbb{C}$ be a holomorphic function on S such that it extends continuously to the closure $\bar{S}$ and has real values on the boundary.

(a) Show that there is an entire function F whose restriction to S is f. (Hints: use Schwarz reflection principle).
(b) Assuming $f : \bar{S} \to \mathbb{C}$ is bounded, show that f is a constant function.

Answer

(a) Since f is real-valued on the boundaries, we can extend f across both $\Im(z) = 0$ and $\Im(z) = 1$.

By **Schwarz reflection principle**, define:

$$F(z) = \overline{f(\bar{z})}, \Im(z) < 0$$

then F is holomorphic on $\{z | -1 < \Im(z) < 1\}$;

define:

$$F(z) = \overline{f(2i - \bar{z})}, \Im(z) > 1$$

then F is holomorphic on $\{z | 0 < \Im(z) < 2\}$.

Repeat these reflections, we can extend F to the whole $\mathbb{C}$ which means F is **entire**.

(b) Since f is bounded on $\bar{S}$, and reflections preserve boundedness, thus F is bounded on $\mathbb{C}$.

By **Liouville's theorem**, F is constant, thus f is constant.

Problem 5. 15 pts

Description

For each of the following functions $f(z)$, determine the type of singularity, and compute the residue at the point z if it is a pole.

(a)

$$f(z) = \frac{e^z - e^{-z}}{z^3(e^z + e^{-z})}, at\ z = -\frac{\pi i}{2}.$$

(b)

$$f(z) = \sin(\frac{1}{z}), at\ z = 0.$$

Answer

(a) When $z = -\frac{\pi i}{2}$, $e^z + e^{-z} = 0$ but $z^3 \neq 0$ and $e^z - e^{-z} \neq 0$, thus $z = -\frac{\pi i}{2}$ is a **pole**.

For $\frac{d}{dz}(e^z + e^{-z}) = e^z - e^{-z} \neq 0$ when $z = -\frac{\pi i}{2}$, thus $z = -\frac{\pi i}{2}$ is a **single pole**.

The residue is:

$$res_{-\frac{\pi i}{2}} f = \lim_{z \to -\frac{\pi i}{2}} (z + \frac{\pi i}{2}) \frac{\sinh z}{z^3 \cosh z}$$

Using **L'Hopital's rule**:

$$res_{-\frac{\pi i}{2}} f = \frac{\sinh(-\frac{\pi i}{2})}{(-\frac{\pi i}{2})^3 \cdot \sinh(-\frac{\pi i}{2})} = \frac{1}{(-\frac{\pi i}{2})^3} = \frac{8}{\pi^3 i}$$

(b) The Taylor series of $sin(\frac{1}{z})$ is:

$$sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} - \dots$$

So the negative powers of z is infinite, thus $z = 0$ is an **essential singularity**.

Problem 6. 10 pts

Description

Calculate the following integral:

$$I = \int_0^{2\pi} \frac{1}{\sqrt{2} - \cos\theta} d\theta$$

Answer

Suppose $z = e^{i\theta}$, then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+z^{-1}}{2}$, and the contour becomes an unit disk. Thus:

$$I = \oint_{|z|=1} \frac{1}{\sqrt{2} - \frac{z+z^{-1}}{2}} \cdot \frac{dz}{iz}$$

$$I = \frac{2}{i} \oint_{|z|=1} \frac{1}{-z^2 + 2\sqrt{2}z - 1} dz$$

The poles occur where $-z^2 + 2\sqrt{2}z - 1 = 0$, thus $z_{1,2} = \sqrt{2} \pm 1$. Only $z_2 = \sqrt{2} - 1$ is inside the unit disk since $|z_2| < 1$.

So the residue at $z = z_2$ is:

$$res_{z_2} f = \lim_{z \to z_2} (z - z_2) \cdot \frac{1}{-(z - z_1)(z - z_2)} = \frac{1}{z_1 - z_2} = \frac{1}{2}$$

So I is:

$$I = \frac{2}{i} \cdot 2\pi i \cdot res_{z_2} f = 2\pi$$

Answer:

$$\int_0^{2\pi} \frac{1}{\sqrt{2} - \cos\theta} d\theta = 2\pi$$

Problem 7. 20 pts

Description

Consider a function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{\sin z}{z}$$

(a) Show that f has a removable singularity at 0 and find the value that f can be extended continuously to 0.
(b) Show that $f(z) \neq 0$ for $|z| \leq 3$.
(c) Show that for any holomorphic function g defined in a neighborhood of the closed

unit disk $\{z \mid |z| \leq 1\}$, we have the integral formula

$$g(z) = \frac{1}{2\pi i} \int_{C_1} \frac{g(w)}{\sin(w-z)} dw$$

for $|z| < 1$. Here C_1 is the unit circle oriented in counterclockwise direction.

Answer

(a) We know $\frac{\sin z}{z}$ has the limit 1 in the neighorhood of $z = 0$, and using Taylor series expansion:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \ldots$$

So this series is defined and analytic at $z = 0$, thus f has a removable singularity at 0. Define $f(0) = \lim_{z \to 0} \frac{\sin z}{z} = 1$, then f is continuous at 0.

(b) The zeros occur where $\sin z = 0$ and $z \neq 0$, thus $z = k\pi \ (k \in \mathbb{Z} \setminus \{0\})$. The closest zero to 0 is at $z = \pi$, and in this case $|z| > 3$, thus when $|z| \leq 3$, $f(z) \neq 0$.

(c) For $|z| < 1$ and g is holomorphic in $|z| \leq 1$, $\frac{g(w)}{\sin(w-z)}$ has a simple pole at $w = z$ (simple zero of $\sin(w-z)$).

So the residue is:

$$res_z \frac{g(w)}{\sin(w-z)} = \lim_{w \to z} \frac{g(w)}{\frac{\sin(w-z)}{w-z}} = g(z)$$

Thus:

$$\int_{C_1} \frac{g(w)}{\sin(w-z)} dw = 2\pi i \cdot g(z)$$

Thus the formula holds:

$$g(z) = \frac{1}{2\pi i} \int_{C_1} \frac{g(w)}{\sin(w-z)} dw, \ |z| < 1$$