MA201a ODE-A Midterm-24Spring

Problem 1. 20 pts

Description

(a) Find the general solution of $y' = 4y + x^2 e^{4x}$.

(b) Solve the initial value problem

$$y' = \frac{1+3x^2}{3y^2-6y}\ ,\ y(0) = 1$$

and determine the interval in which the solution is valid.

Answer

(a) The integral factor:

$$\mu(x) = e^{\int -4dx} = e^{-4x}$$

So:

$$e^{-4x}y' - 4e^{-4x}y = x^2$$

$$(e^{-4x}y)' = x^2$$

$$e^{-4x}y = \frac{x^3}{3} + C$$

$$y = \frac{x^3 e^{4x}}{3} + Ce^{4x}$$

(b)

$$(3y^2 - 6y)dy = (1 + 3x^2)dx$$

Integrate both sides:

$$y^3 - 3y^2 = x + x^3 + C$$

For $y(0) = 1$, then $C = -2$.

So the solution is:

$$y^3 - 3y^2 = x + x^3 - 2$$

The solution is valid where f and $\frac{\partial f}{\partial y}$ are continuous,

$$f = \frac{1+3x^2}{3y^2-6y} \ , \ \frac{\partial f}{\partial y} = \frac{6x(3y^2-6y)-(6y-6)(1+3x^2)}{(3y^2-6y)^2}$$

So $3y^2 - 6y \neq 0$, which means $y \neq 0$ and $y \neq 2$, then $x \neq -1$ and $x \neq 1$.

So the interval in which the solution is valid is $-1 < x < 1$.

Problem 2. 30 pts

Description

Solve the following problems.

(a) $y^{(6)} + y = 0$

(b) $y' = \frac{x-y-1}{x+y-2}$

(c) $y'' - 3y' - 4y = 2e^{-t}$

Answer

(a) The characteristic equation:

$$r^6 + 1 = 0$$

Solutions:

$$r = e^{i\frac{\pi+2k\pi}{6}} \ , k = 0, 1, 2, 3, 4, 5.$$

which can be written as:

$$r_1 = \frac{\sqrt{3}}{2} + \frac{i}{2} \ , \ r_2 = i \ , \ r_3 = -\frac{\sqrt{3}}{2} + \frac{i}{2} \ , \ r_4 = -\frac{\sqrt{3}}{2} - \frac{i}{2} \ , \ r_5 = -i \ , \ r_6 = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

The solution is:

$$y(t) = C_1 e^{\frac{\sqrt{3}}{2}x} \cos\frac{x}{2} + C_2 e^{\frac{\sqrt{3}}{2}x} \sin\frac{x}{2} + C_3 e^{-\frac{\sqrt{3}}{2}x} \cos\frac{x}{2} + C_4 e^{-\frac{\sqrt{3}}{2}x} \sin\frac{x}{2} + C_5 \cos x + C_6 \sin x$$

(b)

$$(x-y-1) - (x+y-2)y' = 0$$

So suppose $M(x,y) = x - y - 1$, $N(x,y) = -(x+y-2)$, then:

$$M_y = -1 = N_x$$

which means that is an exact equation.

There exists a function $\psi(x, y)$, such that:

$$\begin{cases} \psi_x = M = x - y - 1 \\ \psi_y = N = -(x + y - 2) \end{cases}$$

Thus:

$$\psi(x, y) = \frac{x^2}{2} - xy - x + h_1(y) = -\frac{y^2}{2} - xy + 2y + h_2(x)$$

We can take:

$$\psi(x, y) = \frac{x^2}{2} - \frac{y^2}{2} - xy - x + 2y$$

So the solution is:

$$\frac{y^2}{2} + (x - 2)y - \frac{x^2}{2} + x = C$$

$$y = 2 - x \pm \sqrt{2x^2 - 6x + C}$$

(c) First solve $y'' - 3y' - 4y = 0$.

The characteristic equation:

$$r^2 - 3r - 4 = 0$$

Solution:

$$r_1 = 4 \ , \ r_2 = -1$$

So the complementary solution:

$$y(t) = C_1 e^{4t} + C_2 e^{-t}$$

Suppose $Y(t) = Ate^{-t}$, then it holds:

$$Y'' - 3Y' - 4Y = 2e^{-t}$$

$$A(t - 2)e^{-t} - 3A(1 - t)e^{-t} - 4Ate^{-t} = 2e^{-t}$$

$$(A + 3A - 4A)te^{-t} + (-2A - 3A)e^{-t} = 2e^{-t}$$

thus $A = -\frac{2}{5}$, so $Y(t) = -\frac{2}{5}te^{-t}$.

The general solution is:

$$y(t) = C_1 e^{4t} + C_2 e^{-t} - \frac{2}{5}te^{-t}$$

Problem 3. 20 pts

Description

Provide $y = t$ is one of the solution for

$$t^2y'' - 2ty' + 2y = 0\ ,\ t > 0$$

Then find the general solution for

$$t^2y'' - 2ty' + 2y = 4t^2\ ,\ t > 0$$

Answer

If $y = t$, then $y' = 1$ and $y'' = 0$, and $t^2y'' - 2ty' + 2y = 0$ holds.

Suppose $y_1(t) = t$, and $y_2(t) = v(t)y_1(t)$, then

$$y_2'(t) = v(t) + tv'(t)\ ,\ y_2''(t) = tv''(t) + 2v'(t)$$

$$t^3v''(t) + 2t^2v'(t) - 2tv(t) - 2t^2v'(t) + 2tv(t) = 0$$

$$t^3v''(t) = 0$$

So $v(t) = C_1t + C_2$. Take $C_1 = 1$, $C_2 = 0$, then $v(t) = t$, which means $y_2(t) = t^2$.

Thus, the complementary solution is $y(t) = C_3t + C_4t^2$.

Suppose $Y(t) = tu_1(t) + t^2u_2(t)$, then

$$Y'(t) = u_1(t) + tu_1'(t) + 2tu_2(t) + t^2u_2'(t)$$

Let $tu_1'(t) + t^2u_2'(t) = 0$, then

$$Y'(t) = u_1(t) + 2tu_2(t)$$

$$Y''(t) = u_1'(t) + 2u_2(t) + 2tu_2'(t)$$

Thus:

$$t^2u_1'(t) + 2t^2u_2(t) + 2t^3u_2'(t) - 2tu_1(t) - 4t^2u_2(t) + 2tu_1(t) + 2t^2u_2(t) = 4t^2$$

$$t^2u_1'(t) + 2t^3u_2'(t) = 4t^2$$

The solution is:

$$u_1'(t) = -4\ ,\ u_2'(t) = \frac{4}{t}$$

$$u_1(t) = -4t\ ,\ u_2(t) = 4\ln t$$

So the particular solution is:

$$Y(t) = -4t^2 + 4t^2 \ln t$$

The general solution is:

$$y(t) = C_1 t + C_2 t^2 + 4t^2 \ln t$$

Problem 4. 20 pts

Description

Consider the differential equation $2ydx + (x+y)dy = 0$.

(a) Without finding it, prove that the equation has an integrating factor that is a function of y.

(b) Find the integrating factor and use it to solve (implicity) the differential equation.

Answer

(a) We know $M(x,y) = 2y$, $N(x,y) = x + y$, and suppose $\mu(x,y)$ is the integrating factor. Then:

$$(\mu M)_y = (\mu N)_x$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

$$2y\mu_y - (x+y)\mu_x + \mu = 0$$

Let $\mu_x = 0$, then:

$$2y\mu_y + \mu = 0$$

$$\mu(y) = \frac{C}{\sqrt{y}}$$

So this kind of integrating factor exists.

(b) Take $C = 1$, thus $\mu(y) = \frac{1}{\sqrt{y}}$. Suppose $M'(x,y) = \mu(y)M(x,y) = 2\sqrt{y}$, $N'(x,y) = \mu(y)N(x,y) = \frac{x}{\sqrt{y}} + \sqrt{y}$, then:

$$M'_y = \frac{1}{\sqrt{y}} = N'_x$$

Thus there exist a function $\psi(x,y)$, such that:

$$\begin{cases} \psi_x = M' = 2\sqrt{y} \\ \psi_y = N' = \frac{x}{\sqrt{y}} + \sqrt{y} \end{cases}$$

$$\psi(x,y) = 2x\sqrt{y} + h_1(y) = \frac{2}{3}y^{\frac{3}{2}} + 2x\sqrt{y} + h_2(x)$$

Thus we can take:

$$\psi(x,y)=\frac{2}{3}y^{\frac{3}{2}}+2x\sqrt{y}$$

So the implicity solution is:

$$\frac{2}{3}y^{\frac{3}{2}}+2x\sqrt{y}=C$$

Problem 5. 10 pts

Description

Solve the initial value problem

$$y'=y^{\frac{1}{4}}\ ,\ y(0)=0$$

Answer

When $y\neq 0$,

$$y^{-\frac{1}{4}}dy=dt$$
$$\frac{4}{3}y^{\frac{3}{4}}=t+C$$
$$y=(\frac{3}{4}t+C)^{\frac{4}{3}}$$

For $y(0)=0$, thus $C=0$, then:

$$y=(\frac{3}{4}t)^{\frac{4}{3}}$$

When $y=0$, the equation holds.

Actually, we can combine two cases and the solution becomes:

$$y(t)=\begin{cases}0\ , & t\leq a\\ (\frac{3}{4}(t-a))^{\frac{4}{3}}\ , & t>a\end{cases}$$

where $a\geq 0$.