CS208 Theory Assignment 4

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Chapter 4 Exercise 2

Description

For each of the following two statements, decide whether it is true or false.
If it is true, give a short explanation. If it is false, give a counterexample.

(a) Suppose we are given an instance of the Minimum Spanning Tree Problem on a graph G, with edge costs that are all positive and distinct. Let T be a minimum spanning tree for this instance. Now suppose we replace each edge cost c_e by its square, c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

True or false? T must still be a minimum spanning tree for this new instance.

(b) Suppose we are given an instance of the Shortest $s-t$ Path Problem on a directed graph G. We assume that all edge costs are positive and distinct. Let P be a minimum-cost $s-t$ path for this instance. Now suppose we replace each edge cost c_e by its square, c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

True or false? P must still be a minimum-cost $s-t$ path for this new instance.

Analysis

(a) TURE. In Krustal's Algorithm, we consider edges in ascending order of weight but not the exact values of them. For each c_e is positive, square it do not change the relative order, so the orginal graph T still holds.

(b) FALSE. When we find the shortest path, we need to compare the total costs of different ways. Thus, it exists this case: suppose there are two ways from a to b: a to c to b and a to b directly, and the costs are $c_{ac} + c_{cb}$ and c_{ab}, with $c_{ac} + c_{cb} > c_{ab}$ but $c_{ac}^2 + c_{cb}^2 < c_{ab}^2$.

For example, let $c_{ac} = 3$, $c_{cb} = 4$, $c_{ab} = 6$, then $3 + 4 > 6$ but $3^2 + 4^2 < 6^2$. Thus we should choose a to b initialy, but choose a to c to b after changing c_e to e_e^2, which changes the graph P.

Chapter 4 Exercise 8

Description

Suppose you are given a connected graph G, with edge costs that are all distinct. Prove that G has a unique minimum spanning tree.

Analysis

Assume there are two different MST defined as T_1 and T_2, then there exist at least one edge, let's say e which has the smallest cost, only be contained in one of them. WLOG, suppose $e \in T_1$ and $e \notin T_2$.

Then add e to T_2, for T_2 is a MST, then $T_2 \cup \{e\}$ must contain a cycle C. For T_1 is a MST, it has no cycle, then at least one edge in C, let's say f which has the largest cost, do not be contained in T_1.

Then, consider the tree $T_2' = T_2 \cup \{e\} \setminus \{f\}$. For our assumptions above, c_e must less than c_f (otherwise for $c_e \neq c_f$, then $c_e > c_f$, T_1 will choose f not e), then $cost(T_2') < cost(T_2)$, which makes a contradiction.

So the MST of G is unique.

Chapter 4 Exercise 22

Description

Consider the Minimum Spanning Tree Problem on an undirected graph $G = (V, E)$, with a cost $c_e \geq 0$ on each edge, where the costs may not all be different. If the costs are not all distinct, there can in general be many distinct minimum-cost solutions. Suppose we are given a spanning tree $T \subseteq E$ with the guarantee that for every $e \in T$

, e belongs to *some* minimum-cost spanning tree in G. Can we conclude that T itself must be a minimum-cost spanning tree in G? Give a proof or a counterexample with explanation.

Analysis

FALSE. Easy to notice that if there exists a cycle $C = \{i, j, k\}$, and $c_{ij} = c_{ik} = c_{jk}$, then arbitrary two of the edges can be choosen to create MST. For T contains every edges that can be contained in MST, we have $\{i, j, k\} \in T$. Thus T contains a cycle which means T is not a tree.

Counterexample: $V = \{i, j, k\}$, $E = \{c_{ij}, c_{ik}, c_{jk}\}$, and $c_{ij} = c_{ik} = c_{jk} = 1$. Then all the MSTs are:

$$T_1 = \{i, j\} \; T_2 = \{i, k\} \; T_3 = \{j, k\}$$

and their costs are all 2. Thus $T = \{i, j, k\}$, which can not be a MST.