Computer Vision

CS308

Feng Zheng

SUSTech CS Vision Intelligence and Perception

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

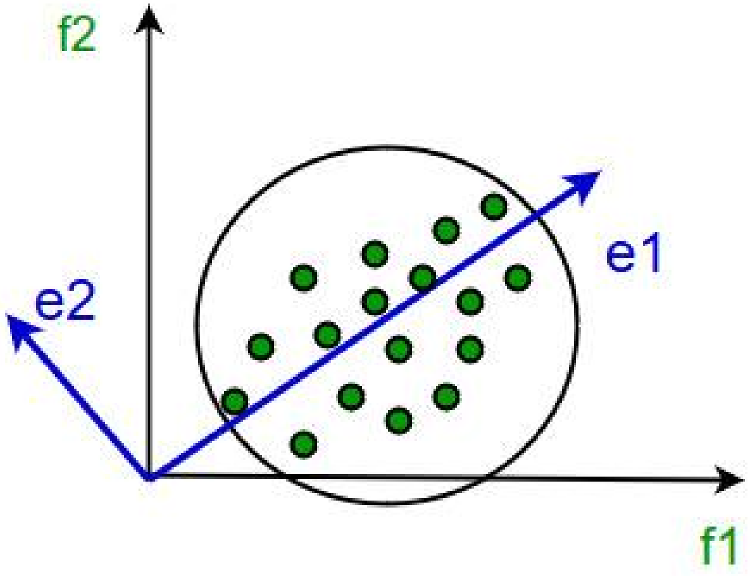
Content

- Brief Review
- Pattern recognition - classification
  - The perception
  - Support vector machine

Brief Review

Review

- Principal component analysis (PCA)
  - Why?
  - How?
- Two matrices
  - Covariance matrix
  - Similarity matrix



(b): $\alpha_i = X{v_i}^T = \sqrt{\lambda_i}\mu_i$; **(c):** $n\lambda_i = \tau_i$;

(d): $v_i x^T = \sum_{j=1}^{n} \alpha_i(j) x_j x^T$ **(x is a new sample)**

Machine Learning Problems

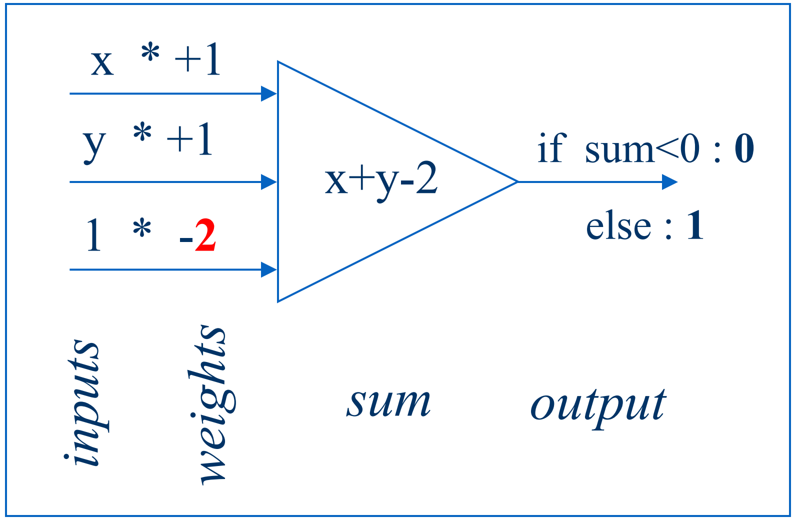
- Taxonomy

	Supervised Learning	Unsupervised Learning
Discrete	classification or categorization	clustering
Continuous	regression	dimensionality reduction

The Perception

Prehistory

- Seminal work
  - W.S. McCulloch & W. Pitts (1943). "A logical calculus of the ideas immanent in nervous activity", *Bulletin of Mathematical Biophysics*, 5, 115-137
  - Point out that simple artificial "neurons" could be made to perform basic logical operations such as AND, OR and NOT.



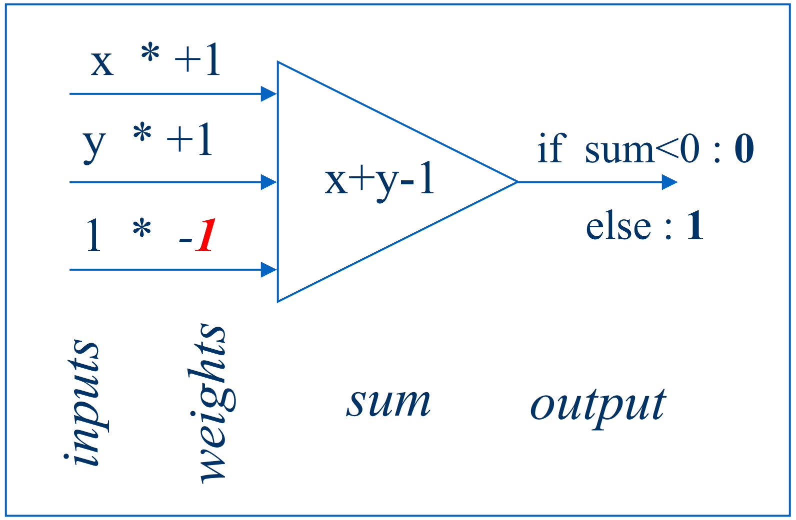
Truth Table for Logical AND

x	y	x & y
0	0	0
0	1	0
1	0	0
1	1	1

inputs *output*

Nervous Systems as Logical Circuits

- Groups of these "neuronal" logic gates could carry out *any* computation, even though each neuron was very limited.
  - Could computers built from these simple units reproduce the computational power of biological brains?
  - Were biological neurons performing logical operations?



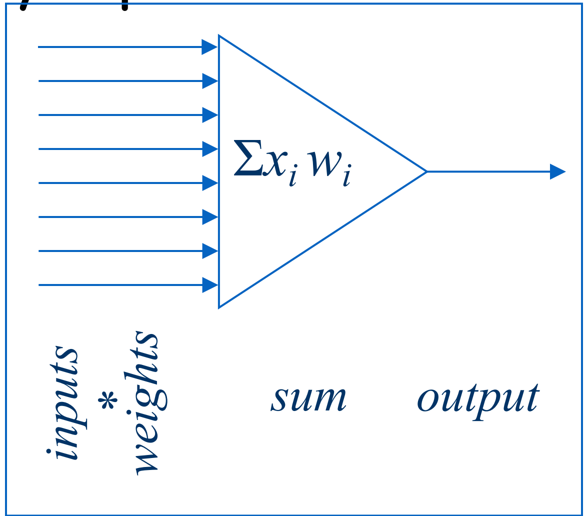
Truth Table for Logical OR

x	y	x | y
0	0	0
0	1	1
1	0	1
1	1	1

inputs   *output*

The Perceptron

- Pioneer work
  - Frank Rosenblatt (1962). Principles of Neurodynamics, Spartan, New York, NY.
  - Subsequent progress was inspired by the invention of learning rules inspired by ideas from neuroscience…
  - Rosenblatt's Perceptron could automatically learn to categorise or classify input vectors into types.



It obeyed the following rule:

If the sum of the weighted inputs exceeds a threshold, output 1, else output -1.

$$1 \text{ if } \Sigma\, input_i * weight_i > threshold$$

$$-1 \text{ if } \Sigma\, input_i * weight_i < threshold$$

Linear Neurons

- The neuron has a real-valued output which is a weighted sum of its inputs

Weight vector

$$\hat{y} = \sum_i w_i x_i = \mathbf{w}^T \mathbf{x}$$

Neuron's estimate of the desired output

Input vector

- The aim of learning is to minimize the discrepancy between the desired output and the actual output
  - ➢ How do we measure the discrepancies? Loss
  - ➢ Do we update the weights after every training case? Optimization
  - ➢ Why don't we solve it analytically? Noisy

A Motivating Example (Analytically)

- Each day you get lunch at the cafeteria.
  - Your diet consists of fish, chips, and beer
  - You get several portions of each
- The cashier only tells you the total price of the meal
  - After several days, you should be able to figure out the price of each portion
- Each meal price gives a linear constraint on the prices of the portions:

$$price = x_{fish} w_{fish} + x_{chips} w_{chips} + x_{beer} w_{beer}$$

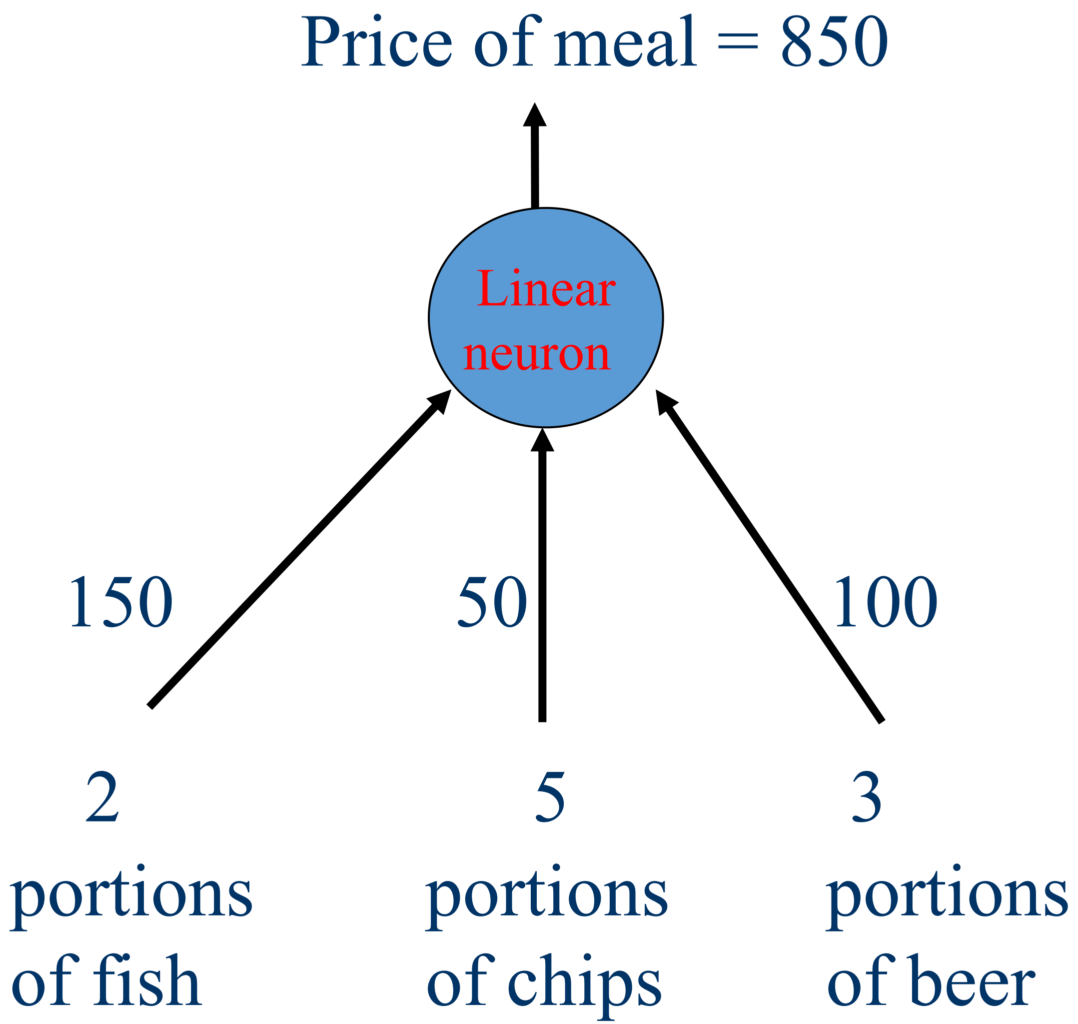
Two Ways to Solve the Equations

- 1. The obvious approach is just to solve a set of simultaneous linear equations, one per meal
- 2. But we want a method that could be implemented in a neural network
  - The prices of the portions are like the weights in of a linear neuron

$$\mathbf{w} = (w_{fish}, w_{chips}, w_{beer})$$

  - We will start with **guesses** for the weights and then adjust the guesses to give a better fit to the prices given by the cashier

The Cashier's Brain



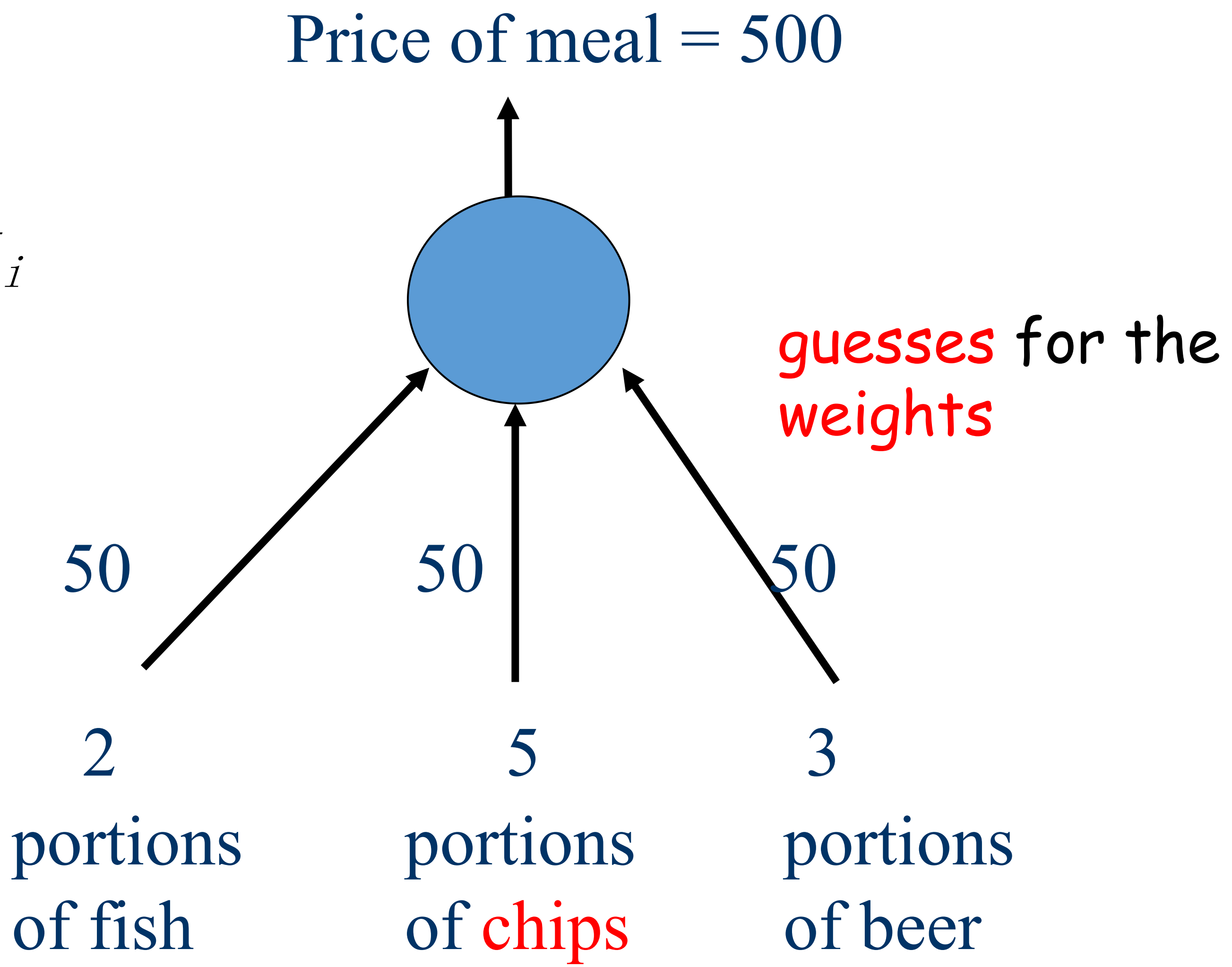
A model of the cashier's brain with arbitrary initial weights

- Residual error = 350
- The learning rule is:

$$\Delta w_i = \varepsilon \; x_i \,(y - \hat{y}) = 10 x_i$$

- With a learning rate ε of 1/35, the weight changes are +20, +50, +30
- This gives new weights of
- 70, 100, 80 ← 50, 50, 50
- Notice that the weight for chips got worse!

150, 50, 100



Behavior of the iterative learning procedure

- Do the updates to the weights always make them get closer to their correct values? No!
- Does the online version of the learning procedure eventually get the right answer? Yes, if the learning rate gradually decreases in the appropriate way.
- How quickly do the weights converge to their correct values? It can be very slow if two input dimensions are highly correlated (e.g. ketchup and chips).
- Can the iterative procedure be generalized to much more complicated, multi-layer, non-linear nets? YES!

Deriving the delta rule

- Define the error as the squared residuals summed over all training cases:

$$E = \tfrac{1}{2}\sum_n (y_n - \hat{y}_n)^2$$

- Now differentiate to get error derivatives for weights

$$\frac{\partial E}{\partial w_i} = \tfrac{1}{2}\sum_n \frac{\partial \hat{y}_n}{\partial w_i}\frac{\partial E_n}{\partial \hat{y}_n}$$

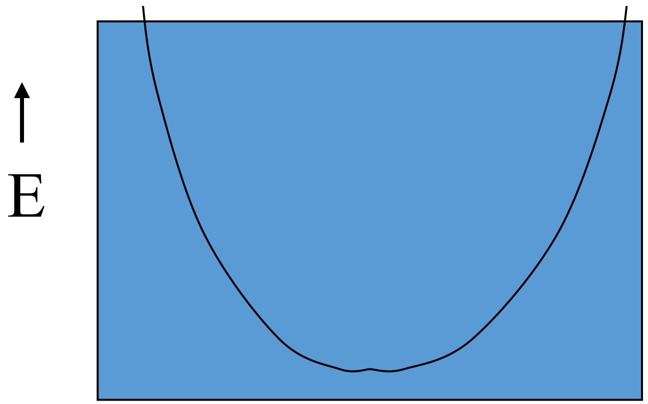
$$= -\sum_n x_{i,n}\,(y_n - \hat{y}_n)$$

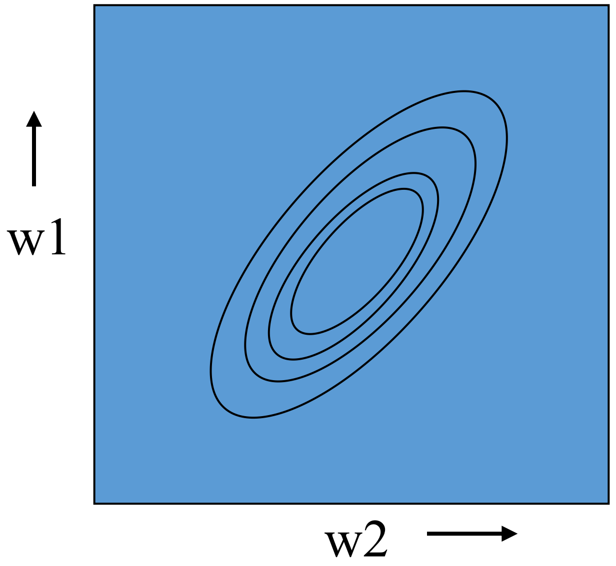
- The batch delta rule changes the weights in proportion to their error derivatives summed over all training cases

$$\Delta w_i = -\varepsilon \frac{\partial E}{\partial w_i}$$

The Error Surface

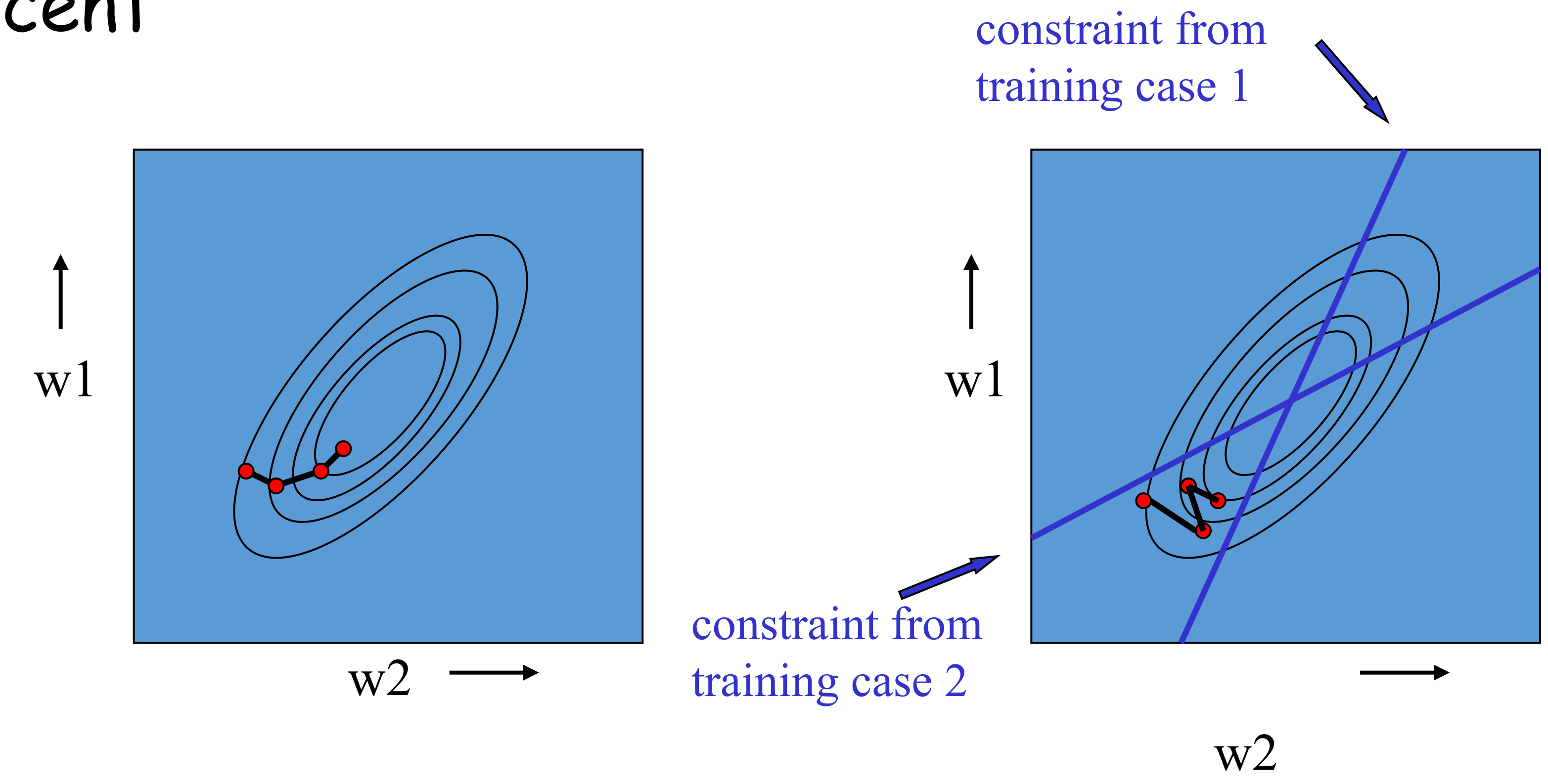
- The error surface lies in a space with a horizontal axis for each weight and one vertical axis for the error.
  - ➢ For a linear neuron, it is a quadratic bowl.
  - ➢ Vertical cross-sections are parabolas.
  - ➢ Horizontal cross-sections are ellipses.





Online versus batch learning

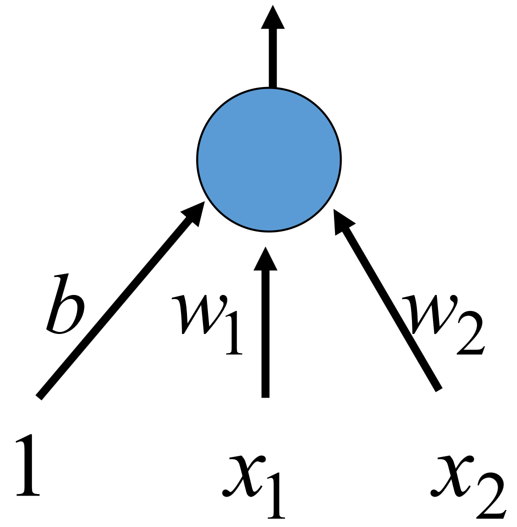
- Batch learning does steepest descent on the error surface
- Online learning zig-zags around the direction of steepest descent



Adding biases

- A linear neuron is a more flexible model if we include a bias
- We can avoid having to figure out a separate learning rule for the bias by using a trick:
    - A bias is exactly equivalent to a weight on an extra input line that always has an activity of 1

$$\hat{y} = b + \sum_i x_i w_i$$



Preprocessing the Input Vectors

- Instead of trying to predict the answer directly from the raw inputs we could start by extracting a layer of "features"
  - Sensible if we already know that certain combinations of input values would be useful
  - The features are equivalent to a layer of hand-coded non-linear neurons.

- So far as the learning algorithm is concerned, the hand-coded features are the input

Is Preprocessing Cheating?

- It seems like cheating if the aim to show how powerful learning is. The really hard bit is done by the preprocessing.
- Its not cheating if we learn the non-linear preprocessing
  - This makes learning much more difficult and much more interesting
- Its not cheating if we use a very big set of non-linear features that is task-independent
  - Support Vector Machines make it possible to use a huge number of features without much computation or data.

Statistical and ANN Terminology

- A perceptron model with a linear transfer function is equivalent to a possibly multiple or multivariate linear regression model [Weisberg 1985; Myers 1986].
- A perceptron model with a logistic transfer function is a logistic regression model [Hosmer and Lemeshow 1989].
- A perceptron model with a threshold transfer function is a linear discriminant function [Hand 1981; McLachlan 1992; Weiss and Kulikowski 1991]. An ADALINE is a linear two-group discriminant.

Transfer functions

- Determines the output from a summation of the weighted inputs of a neuron.

$$O_j = f_j\left(\sum_i w_{ij} x_i\right)$$

- Maps any real numbers into a domain normally bounded by 0 to 1 or -1 to 1, i.e. squashing functions. Most common functions are sigmoid functions:

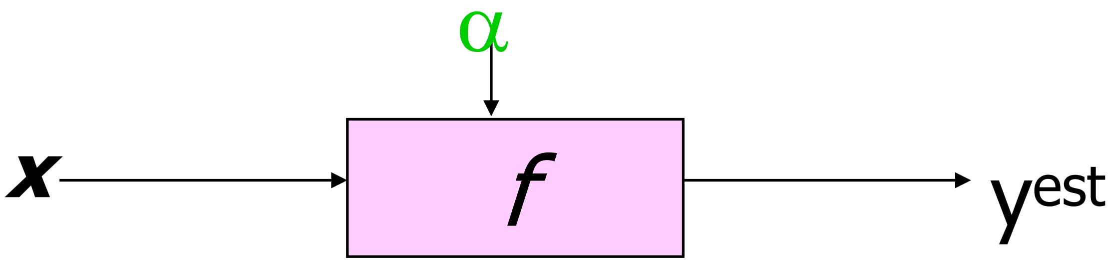
logistic: $$f(x) = \frac{1}{1+e^{-x}}$$

hyperbolic tangent: $$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Support Vector Machine

Linear Classifiers

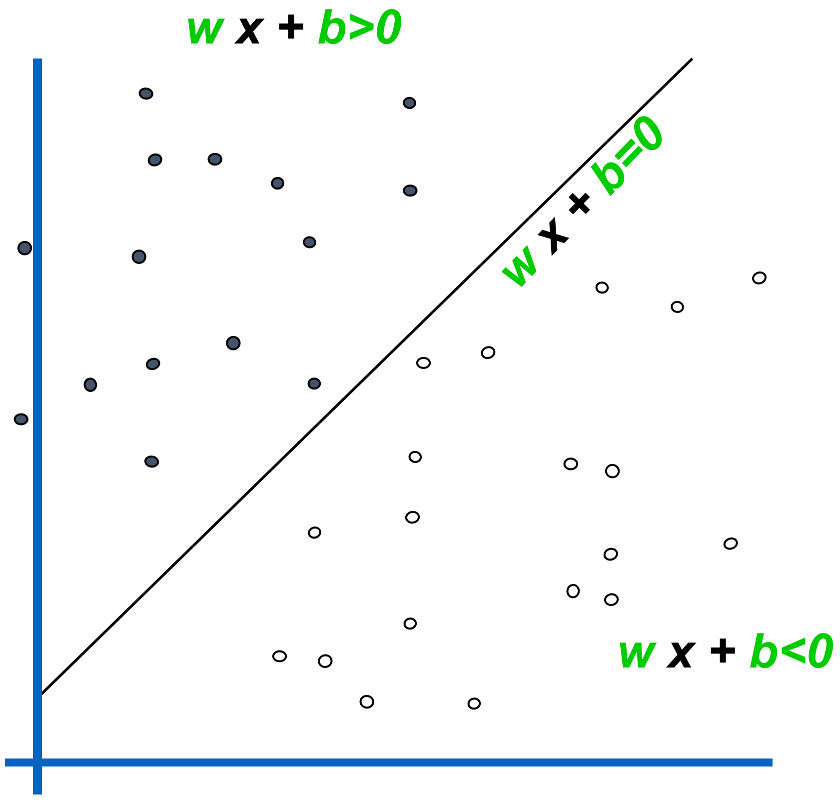
- How would you build the classifier?



$$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}\ \mathbf{x} + b)$$

denotes +1

denotes -1



Linear Classifiers

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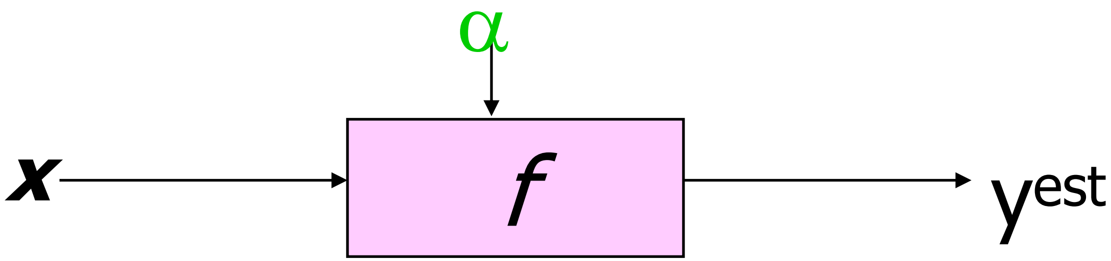

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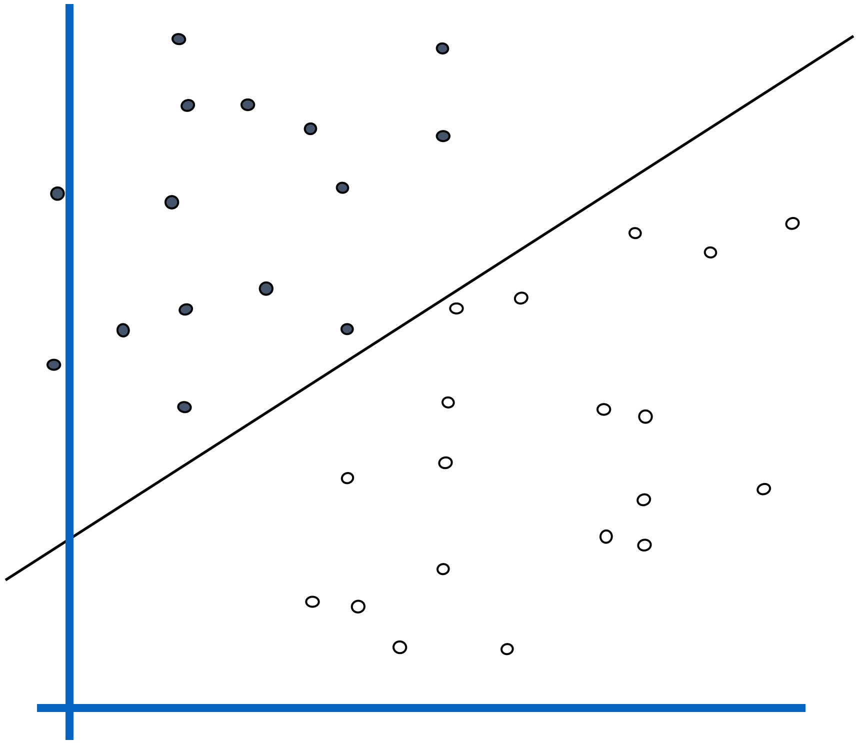

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Linear Classifiers

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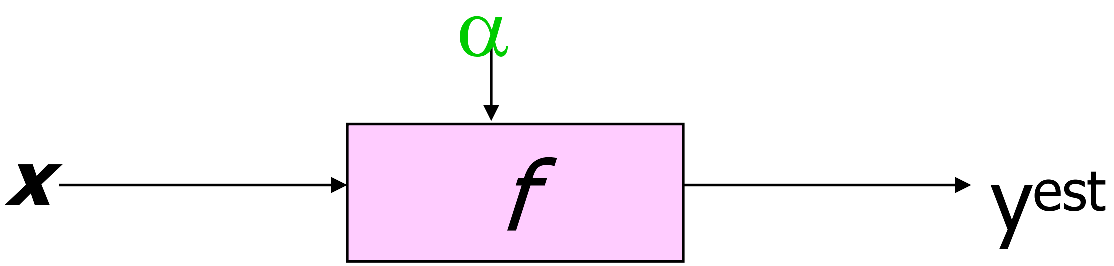
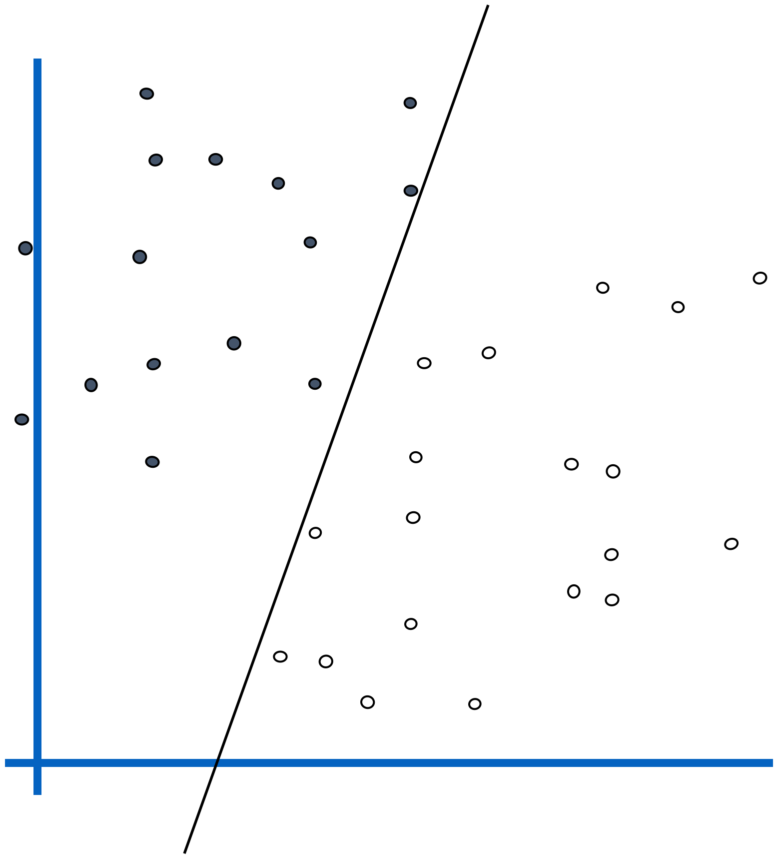


$$f(\mathbf{x},\mathbf{w},b) = \mathrm{sign}(\mathbf{w}\,\mathbf{x} + b)$$

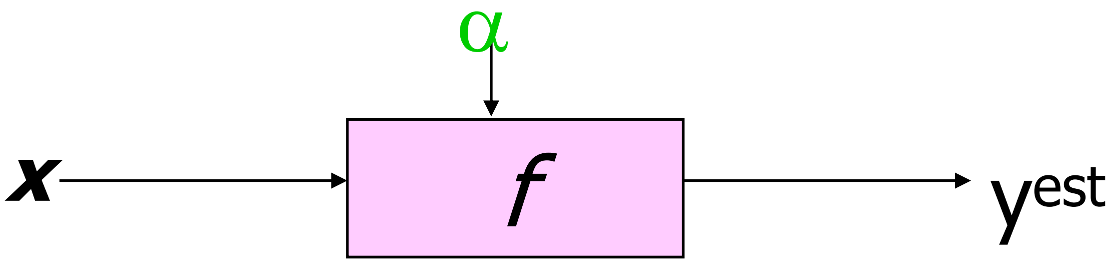


denotes +1

denotes -1

Linear Classifiers

- Any of these would be fine, but which is best?



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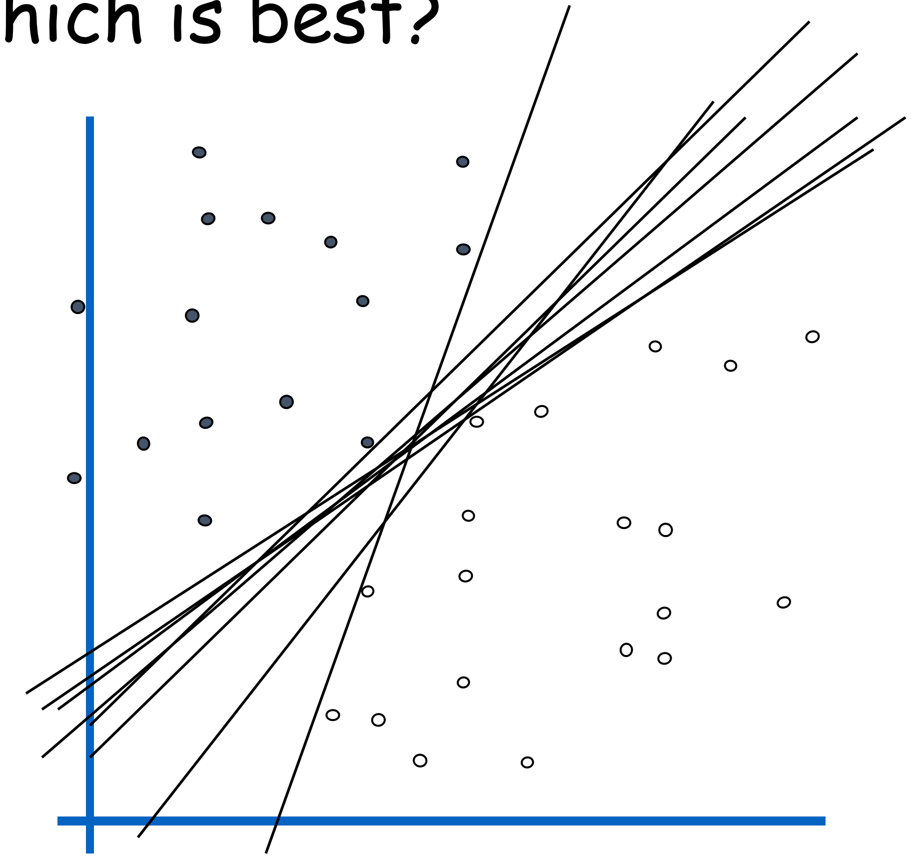
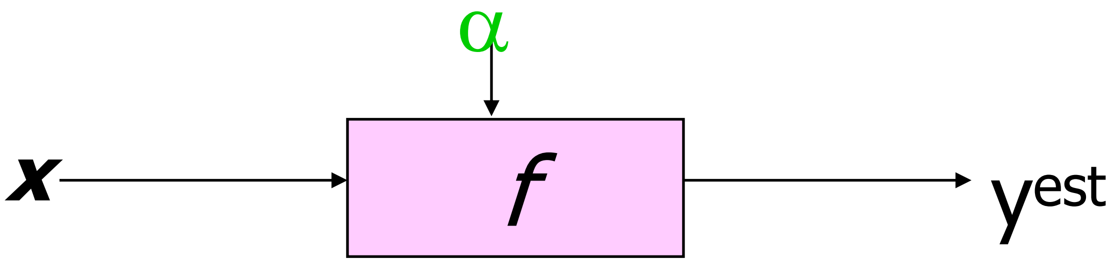


denotes +1

denotes -1

Linear Classifiers

- How would you classify this data?

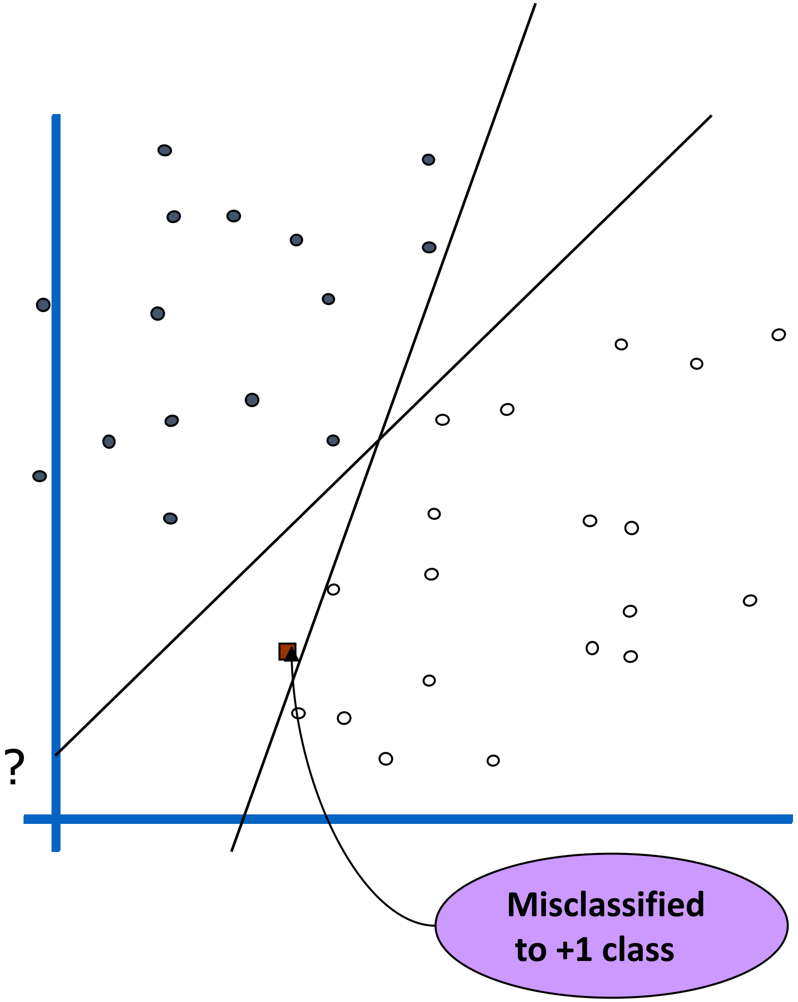


$$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}\,\mathbf{x} + b)$$

denotes +1

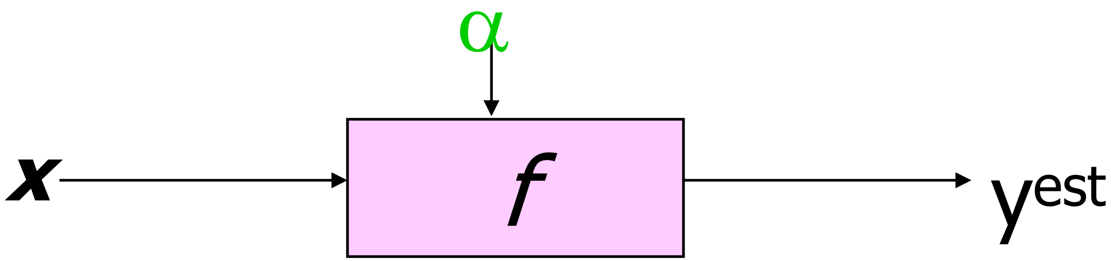
denotes -1

How would you classify this **new** data?



Classifier Margin

- How would you classify this data?

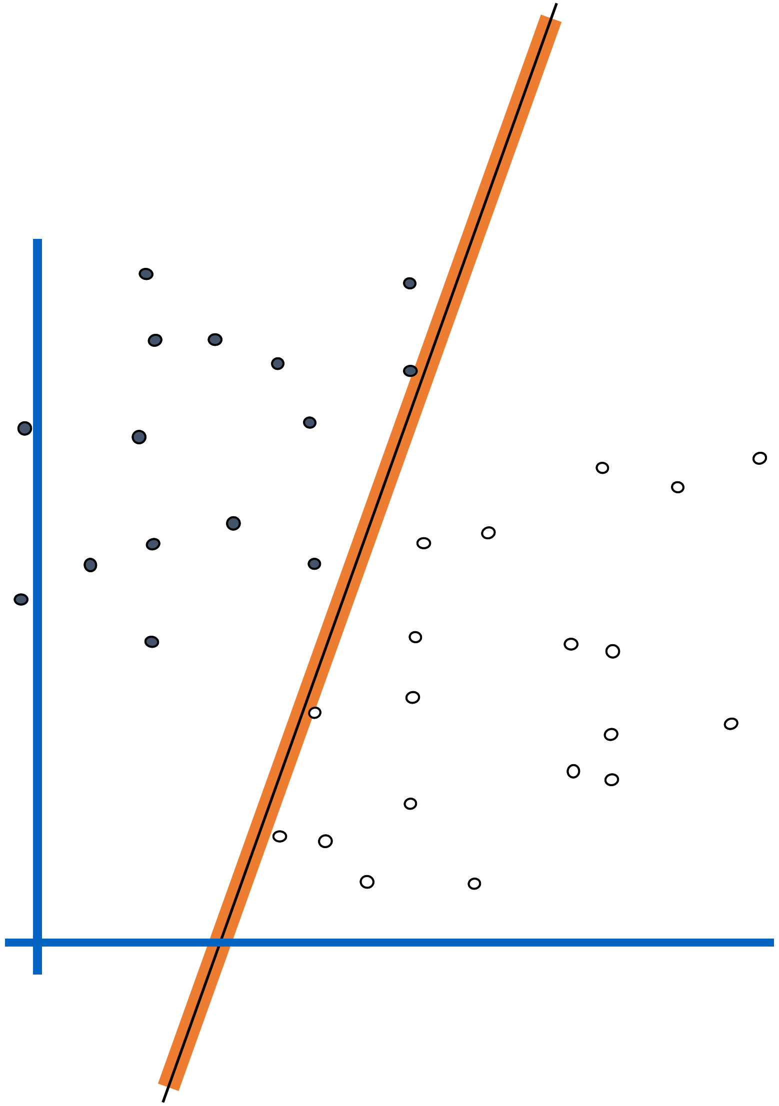


$$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}\ \mathbf{x} + b)$$

denotes +1

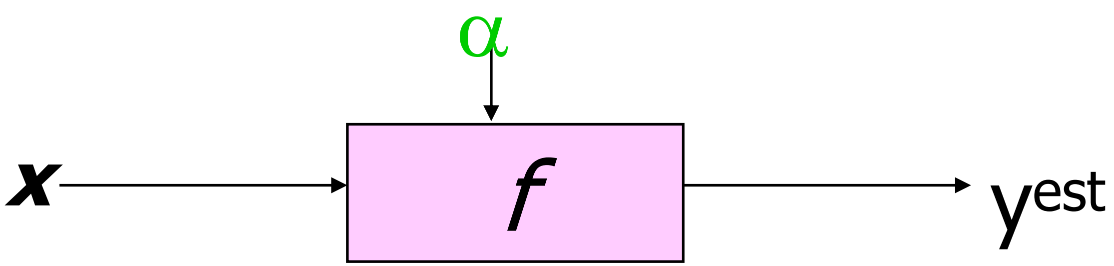
denotes -1

Define the margin of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.



Maximum Margin

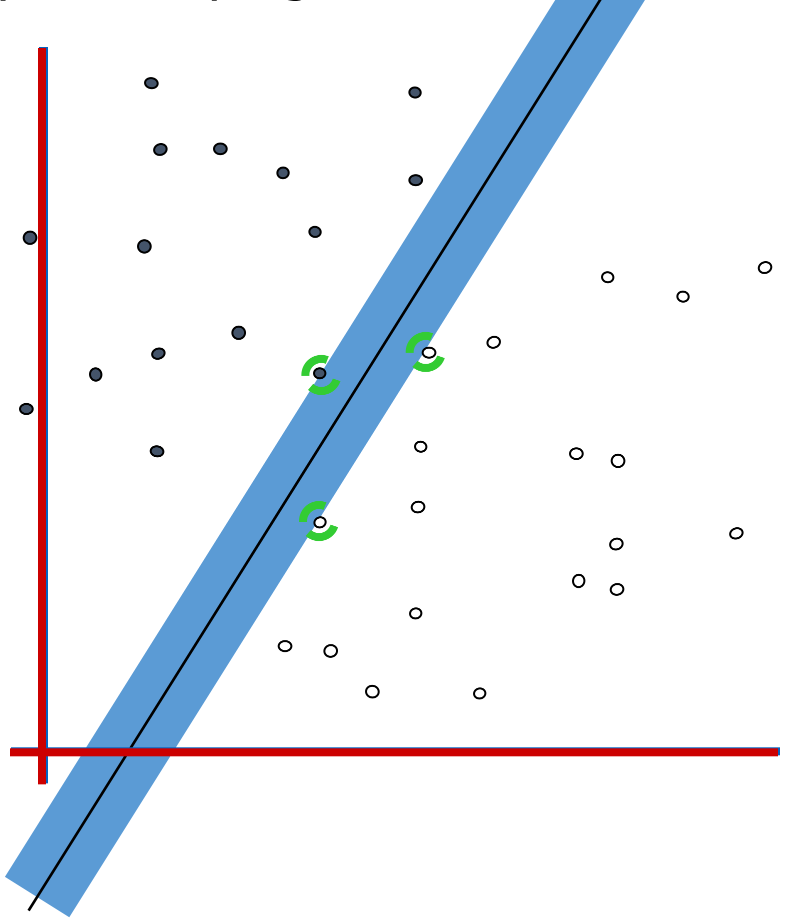
- This is the simplest kind of SVM



$$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}\,\mathbf{x} + b)$$

1. Maximizing the margin is good according to intuition and PAC theory
2. Implies that only support vectors are important; other training examples are ignorable.
3. Empirically it works very very well.

Support Vectors are those datapoints that the margin pushes up against

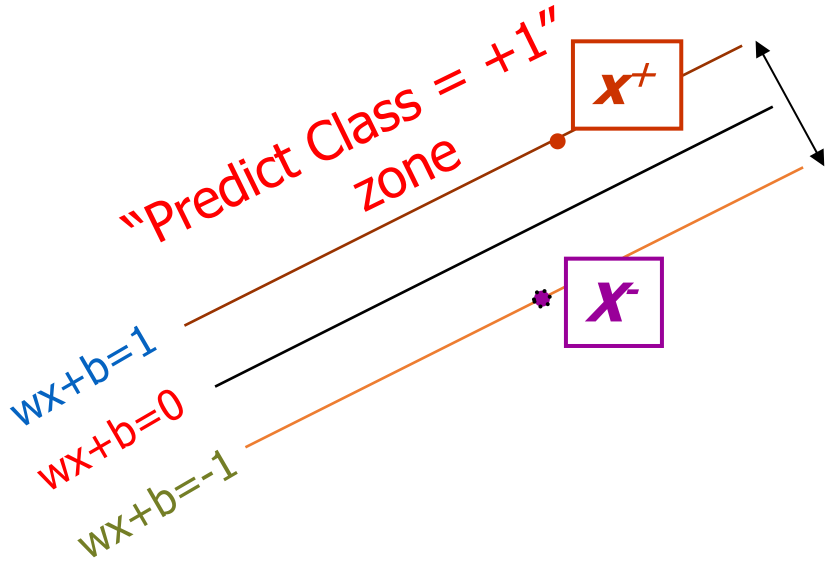


Linear SVM Mathematically

What we know:

- $\mathbf{w} \cdot \mathbf{x}^+ + b = +1$
- $\mathbf{w} \cdot \mathbf{x}^- + b = -1$
- $\mathbf{w} \cdot (\mathbf{x}^+ - \mathbf{x}^-) = 2$

$$M = \frac{(x^+ - x^-) \cdot w}{|w|} = \frac{2}{|w|}$$



M=Margin Width

Linear SVM Mathematically

- Goal:

1) Correctly classify all training data

$$wx_i + b \geq 1 \quad \text{if } y_i = +1$$

$$wx_i + b \leq -1 \quad \text{if } y_i = -1$$

$$y_i(wx_i + b) \geq 1 \quad \text{for all } i$$

2) Maximize the margin $M = \frac{2}{|w|}$

3) Same to minimize $\frac{1}{2} w^t w$

- We can formulate a Quadratic Optimization Problem and solve for w and b

Minimize $\Phi(w) = \frac{1}{2} w^t w$

Subject to $y_i(wx_i + b) \geq 1 \quad \forall i$

Solving the Optimization Problem

- Solving
  - Need to optimize a *quadratic* function subject to *linear* constraints
  - Quadratic optimization problems are a well-known class of mathematical programming problems, and many (rather intricate) algorithms exist for solving them
  - The solution involves constructing a *dual problem* where a ***Lagrange*** *multiplier* α_i is associated with every constraint in the primary problem:

Find $\mathbf{w}$ and b such that
$\mathbf{\Phi}(\mathbf{w}) = \frac{1}{2}\, \mathbf{w}^T\mathbf{w}$ is minimized;
and for all $\{(\mathbf{x_i}, y_i)\}$: $y_i\,(\mathbf{w}^T\mathbf{x_i} + b) \geq 1$

Find $\alpha_1 \ldots \alpha_N$ such that
$\mathbf{Q}(\boldsymbol{\alpha}) = \Sigma\alpha_i - \frac{1}{2}\Sigma\Sigma\alpha_i\alpha_j y_i y_j \mathbf{x_i}^T\mathbf{x_j}$ is maximized and
(1) $\Sigma\alpha_i y_i = 0$
(2) $\alpha_i \geq 0$ for all α_i

The Optimization Problem Solution

- The solution has the form:

$$\boxed{\mathbf{w} = \Sigma\alpha_i y_i \mathbf{x_i} \qquad b = y_k - \mathbf{w}^{\mathbf{T}}\mathbf{x_k} \text{ for any } \mathbf{x_k} \text{ such that } \alpha_k \neq 0}$$

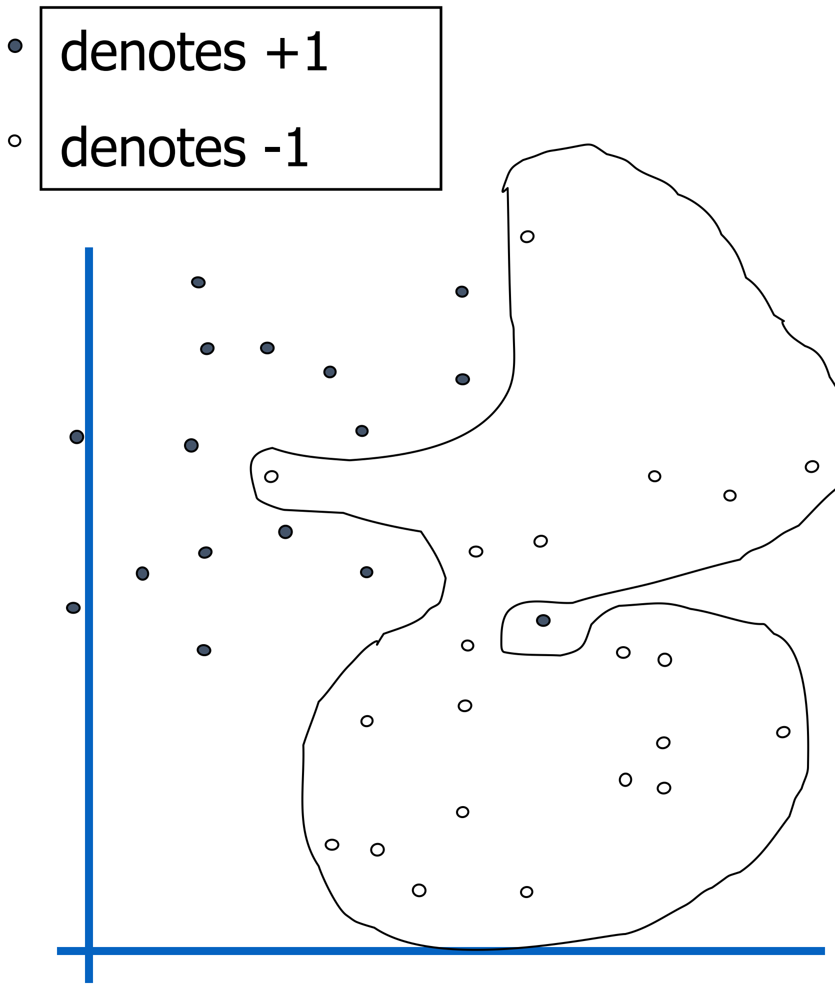
- Each non-zero α_i indicates that corresponding $\mathbf{x_i}$ is a support vector
- Then the classifying function will have the form:

$$\boxed{f(\mathbf{x}) = \Sigma\alpha_i y_i \mathbf{x_i}^{\mathbf{T}}\mathbf{x} + b}$$

- Notice that it relies on an *inner product* between the test point $\mathbf{x}$ and the support vectors $\mathbf{x_i}$ – we will return to this later
- Also keep in mind that solving the optimization problem involved computing the inner products $\mathbf{x_i}^{\mathbf{T}}\mathbf{x_j}$ between all pairs of training points

Dataset with noise

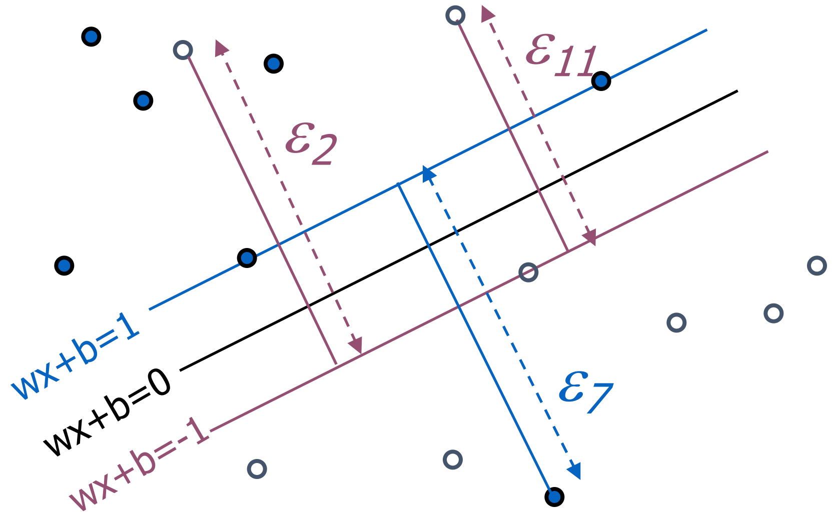
- Hard Margin: So far we require all data points be classified correctly
  - ➢ No training error
- What if the training set is noisy?
  - ➢ Solution 1: use very powerful kernels



Soft Margin Classification

- *Slack variables* ξ_i can be added to allow misclassification of difficult or noisy examples.
- What should our quadratic optimization criterion be?

$$\frac{1}{2}\mathbf{w}.\mathbf{w} + C\sum_{k=1}^{R}\varepsilon_k$$



Hard Margin v.s. Soft Margin

- The old formulation:

Find $\mathbf{w}$ and b such that

$\mathbf{\Phi}(\mathbf{w}) = ½\, \mathbf{w}^T\mathbf{w}$ is minimized and for all $\{(\mathbf{x_i}, y_i)\}$

$y_i\,(\mathbf{w}^T\mathbf{x_i} + b) \geq 1$

- The new formulation incorporating slack variables:

Find $\mathbf{w}$ and b such that

$\mathbf{\Phi}(\mathbf{w}) = ½\, \mathbf{w}^T\mathbf{w} + C\Sigma\xi_i$ is minimized and for all $\{(\mathbf{x_i}, y_i)\}$

$y_i\,(\mathbf{w}^T\mathbf{x_i} + b) \geq 1 - \xi_i$ and $\xi_i \geq 0$ for all i

- Parameter C can be viewed as a way to control **overfitting**.

Linear SVMs: Overview

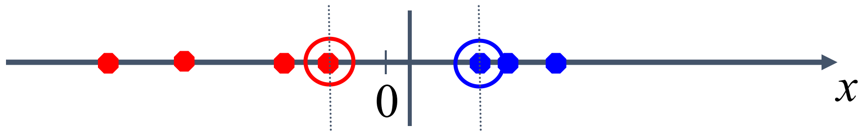
- The classifier is a *separating hyperplane.*
- Most "important" training points are support vectors; they define the hyperplane.
- QP can identify which training points x_i are support vectors with non-zero Lagrangian multipliers α_i.
- Both in the dual formulation of the problem and in the solution training points appear only inside dot products:

Find $\alpha_1 \ldots \alpha_N$ such that
$Q(\alpha) = \Sigma\alpha_i - \frac{1}{2}\Sigma\Sigma\alpha_i\alpha_j y_i y_j x_i^T x_j$ is maximized and
(1) $\Sigma\alpha_i y_i = 0$
(2) $0 \le \alpha_i \le C$ for all α_i

$$f(x) = \Sigma\alpha_i y_i x_i^T x + b$$

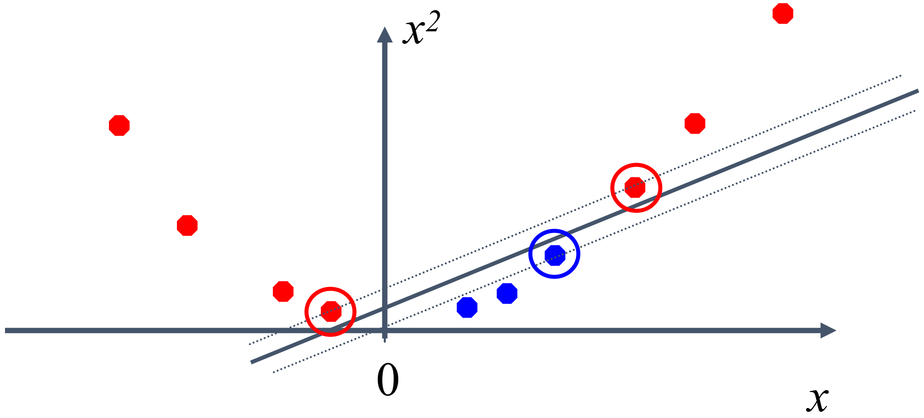
Non-linear SVMs

- Datasets that are linearly separable with some noise work out great:



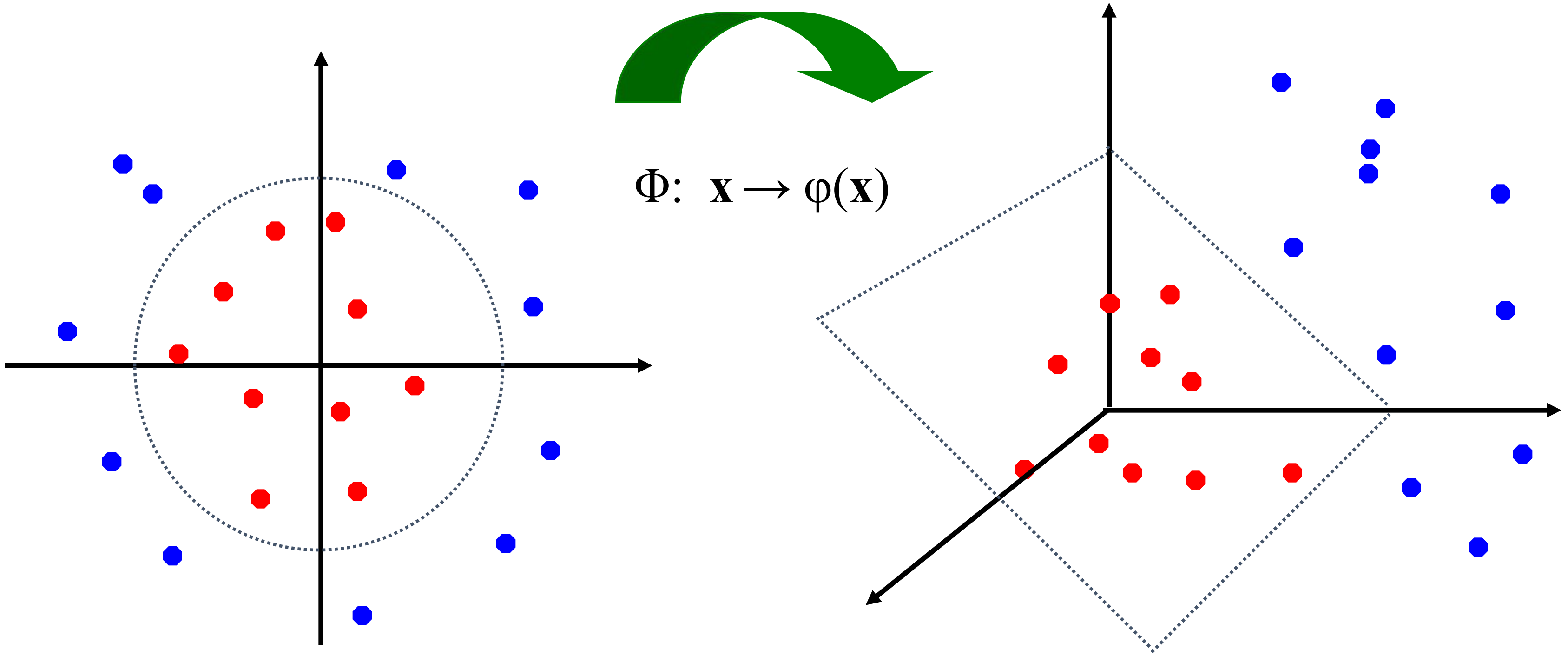
- But what are we going to do if the dataset is just too hard?

- How about mapping data to a higher-dimensional space?



Non-linear SVMs: Feature spaces

- General idea: the original input space can always be mapped to some higher-dimensional feature space where the training set is separable:



The "Kernel Trick"

- The linear classifier relies on dot product between vectors

$$K(\mathbf{x}_i,\mathbf{x}_j)=\mathbf{x}_i^T\mathbf{x}_j$$

- If every data point is mapped into high-dimensional space via some transformation

$$\Phi:\ \mathbf{x} \rightarrow \varphi(\mathbf{x}),$$

- The dot product becomes:

$$K(\mathbf{x}_i,\mathbf{x}_j)= \varphi(\mathbf{x}_i)^T\varphi(\mathbf{x}_j)$$

- A *kernel function* is some function that corresponds to an inner product in some expanded feature space.

The "Kernel Trick"

- Example:
  - 2-dimensional vectors

$$\mathbf{x}=[x_1\ \ x_2];\ \ \text{let } K(\mathbf{x}_i, \mathbf{x}_j)=(1+\mathbf{x}_i^T\mathbf{x}_j)^2,$$

  - Need to show that $K(\mathbf{x}_i, \mathbf{x}_j)=\varphi(\mathbf{x}_i)^T\varphi(\mathbf{x}_j)$:

$$K(\mathbf{x}_i, \mathbf{x}_j)=(1+\mathbf{x}_i^T\mathbf{x}_j)^2,$$

$$= 1+x_{i1}^2x_{j1}^2+2\,x_{i1}x_{j1}x_{i2}x_{j2}+x_{i2}^2x_{j2}^2+2x_{i1}x_{j1}+2x_{i2}x_{j2}$$

$$= [1\ \ x_{i1}^2\ \ \sqrt{2}\,x_{i1}x_{i2}\ \ x_{i2}^2\ \ \sqrt{2}x_{i1}\ \ \sqrt{2}x_{i2}]^T\,[1\ \ x_{j1}^2\ \ \sqrt{2}\,x_{j1}x_{j2}\ \ x_{j2}^2\ \ \sqrt{2}x_{j1}\ \ \sqrt{2}x_{j2}]$$

$$= \varphi(\mathbf{x}_i)^T\varphi(\mathbf{x}_j),$$

$$\text{where } \varphi(\mathbf{x}) = [1\ \ x_1^2\ \ \sqrt{2}\,x_1x_2\ \ x_2^2\ \ \sqrt{2}x_1\ \ \sqrt{2}x_2]$$

What Functions are Kernels?

- For some functions $K(x_i,x_j)$ checking that

  $K(x_i,x_j)= \varphi(x_i)^T\varphi(x_j)$ can be cumbersome.
- Mercer's theorem:
  - *Every semi-positive definite symmetric function is a kernel*
- Semi-positive definite symmetric functions correspond to a semi-positive definite symmetric Gram matrix:

K=

$K(\mathbf{x}_1,\mathbf{x}_1)$	$K(\mathbf{x}_1,\mathbf{x}_2)$	$K(\mathbf{x}_1,\mathbf{x}_3)$	…	$K(\mathbf{x}_1,\mathbf{x}_N)$
$K(\mathbf{x}_2,\mathbf{x}_1)$	$K(\mathbf{x}_2,\mathbf{x}_2)$	$K(\mathbf{x}_2,\mathbf{x}_3)$		$K(\mathbf{x}_2,\mathbf{x}_N)$
…	…	…	…	…
$K(\mathbf{x}_N,\mathbf{x}_1)$	$K(\mathbf{x}_N,\mathbf{x}_2)$	$K(\mathbf{x}_N,\mathbf{x}_3)$	…	$K(\mathbf{x}_N,\mathbf{x}_N)$

Examples of Kernel Functions

- Linear: $K(\mathbf{x_i},\mathbf{x_j}) = \mathbf{x_i}^T\mathbf{x_j}$

- Polynomial of power p: $K(\mathbf{x_i},\mathbf{x_j}) = (1 + \mathbf{x_i}^T\mathbf{x_j})^p$

- Gaussian (radial-basis function network):

$$K(\mathbf{x}_i, \mathbf{x}_j) = \exp\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma^2}\right)$$

- Sigmoid: $K(\mathbf{x_i},\mathbf{x_j}) = \tanh(\beta_0 \mathbf{x_i}^T\mathbf{x_j} + \beta_1)$

Non-linear SVMs Mathematically

- Dual problem formulation:

Find $\alpha_1 \ldots \alpha_N$ such that
$Q(\alpha) = \Sigma\alpha_i - \frac{1}{2}\Sigma\Sigma\alpha_i\alpha_j y_i y_j K(\mathrm{x_i}, \mathrm{x_j})$ is maximized and
(1) $\Sigma\alpha_i y_i = 0$
(2) $\alpha_i \geq 0$ for all α_i

- The solution is:

$$f(\mathrm{x}) = \Sigma\alpha_i y_i K(\mathrm{x_i}, \mathrm{x}) + b$$

- Optimization techniques for finding α_i's remain the same!

Nonlinear SVM - Overview

- SVM locates a separating hyperplane in the feature space and classify points in that space
- It does not need to represent the space explicitly, simply by defining a kernel function
- The kernel function plays the role of the dot product in the feature space.

Properties of SVM

- Flexibility in choosing a similarity function
- Sparseness of solution when dealing with large data sets
  - Only support vectors are used to specify the separating hyperplane
- Ability to handle large feature spaces
  - Complexity does not depend on the dimensionality of the feature space
- Overfitting can be controlled by soft margin approach
- Nice math property: a simple convex optimization problem which is guaranteed to converge to a single global solution
- Feature selection

Weakness of SVM

- It is sensitive to noise
  - ➢ A relatively small number of mislabeled examples can dramatically decrease the performance
- It only considers two classes
  - ➢ How to do multi-class classification with SVM?
  - ➢ Answer:

1) With m output, learn m SVM's

- ✓ SVM 1 learns "Output==1" vs "Output != 1"
- ✓ SVM 2 learns "Output==2" vs "Output != 2"
- ✓ SVM m learns "Output==m" vs "Output != m"

2) To predict the output for a new input, just predict with each SVM and find out which one puts the prediction the furthest into the positive region

Some Issues

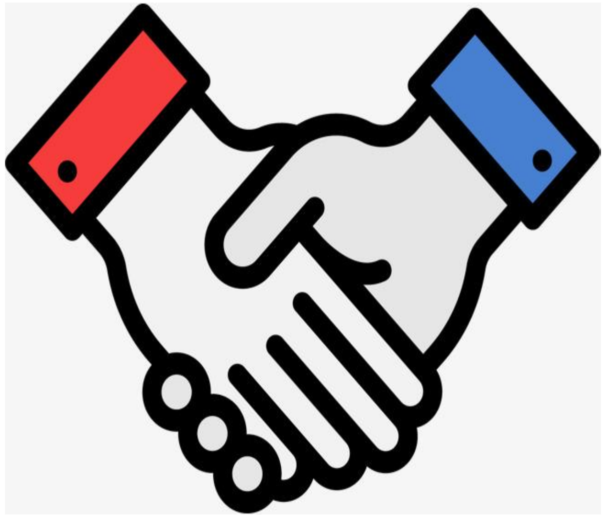
- Choice of kernel
  - Gaussian or polynomial kernel is default
  - If ineffective, more elaborate kernels are needed
  - Domain experts can give assistance in formulating appropriate similarity measures
- Choice of kernel parameters
  - e.g. σ in Gaussian kernel
  - σ is the distance between closest points with different classifications
  - In the absence of reliable criteria, applications rely on the use of a validation set or cross-validation to set such parameters.
- Optimization criterion – Hard margin v.s. Soft margin
  - A lengthy series of experiments in which various parameters are tested

Conclusions

Conclusion

- The perception
  - Online learning
  - One layer
  - Multiple classifiers

- Support vector machine
  - Maximum margin
  - Support vector
  - Kernel trick



Thanks

zhengf@sustc.edu.cn