DIGITAL LOGIC

Chapter 3 part1:
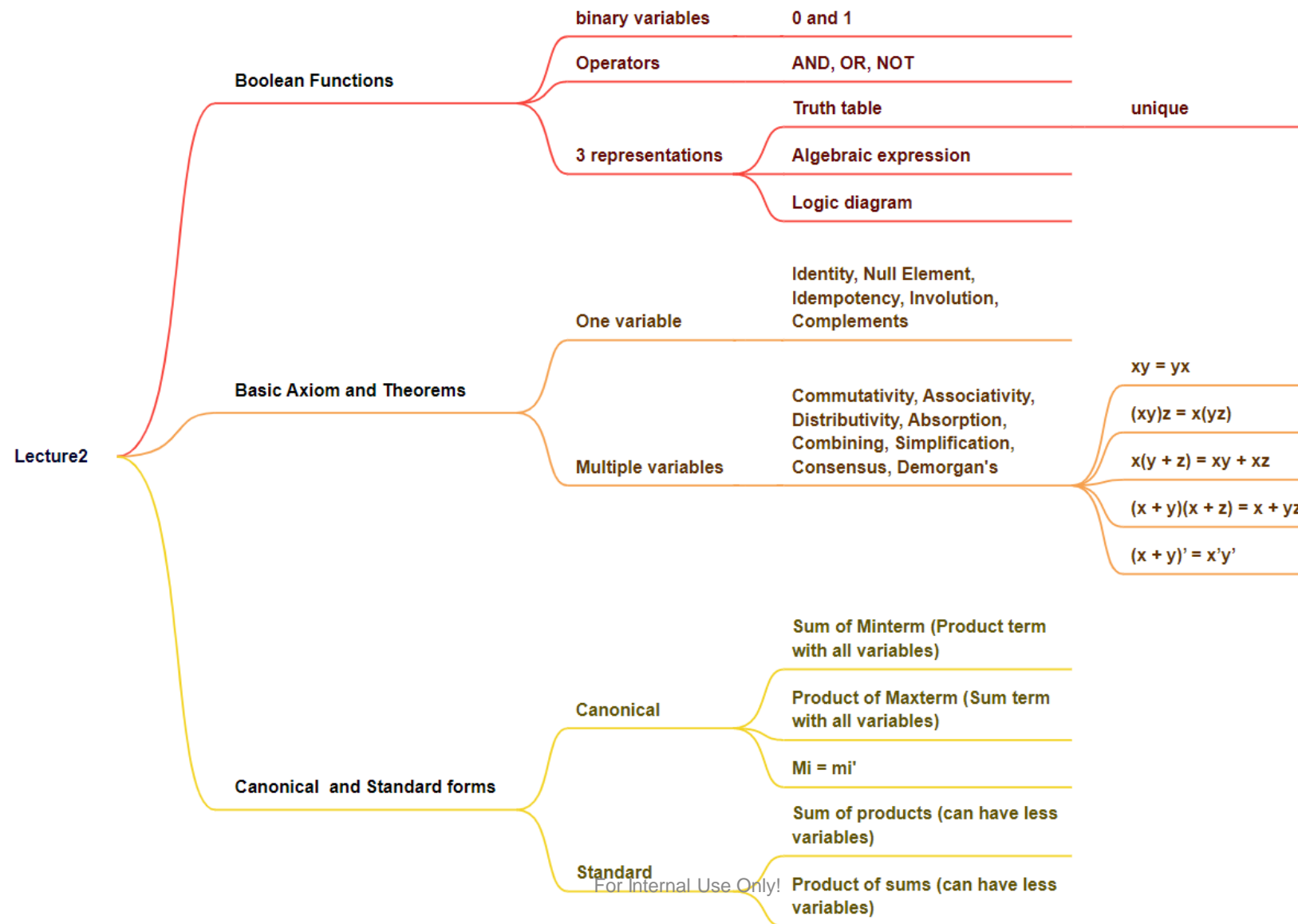## Gate-Level Minimization

2024 Fall

Today's Agenda

- Recap

- Context
  - Gate level minimization using the Map Method
  - Product of sums simplification
  - Don't Care Conditions

- Reading: Textbook, Chapter 3.1-3.5

Recap

Outline

- **Map Method Simplification**

- Product of sums simplification

- Don't Care Conditions

Boolean function simplification

- A function's truth-table representation is unique, while its algebraic expression is not unique.
- Complexity of digital circuit (gate count) ∝ complexity of algebraic expression (literal count)
  - F=x'y'z+x'yz+xy' (3 AND term, 8 literals)
  - F=x'z+xy' (2 AND terms, 4 literals)
- The simplest algebraic expression is one that has minimum number of terms with the smallest possible number of literals in each term
- Methods for gate-level minimization:
  - **Algebraic method**(逻辑代数): Boolean algebra (Last lecture)
  - **Karnaugh map**(卡诺图): the map method (This lecture)

Karnaugh Map (K-map)

- An array of squares each representing one minterm to be minimized
- Each K-map defines a unique Boolean function
  - A Boolean function can be represented by a truth table, a Boolean expression, or a map
- K-map is a visual diagram of all possible ways a function may be expressed
- Used for manual minimization of Boolean functions

Merging Minterms

- In function F, m1 and m3 in the truth table differ only in one position

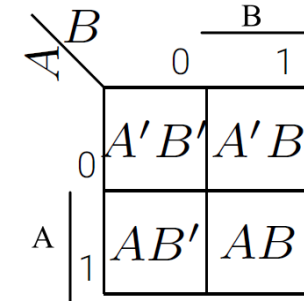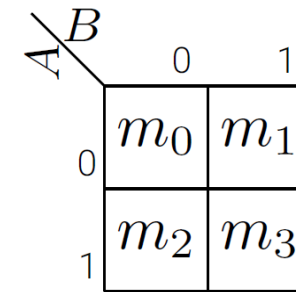 001

  ➡️   0?1

  011

  - ? matches either 0 or 1

- The minterms in a function can be merged to form a simpler product term
  - 001 → x'y'z
  - 011 → x'yz
  - 0?1 → x'y'z+x'yz = x'z(y'+y) = x'z

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Two-Variable Map

- A two-variable map
  - is A truth table in square diagram
  - 4 minterms: A'B', A'B, AB', AB
  - row 0 stands for A'; row 1 stands for A
  - column 0 stands for B'; column 1 stands for B

	A	B	F
m_0	0	0	0
m_1	0	1	1
m_2	1	0	1
m_3	1	1	1
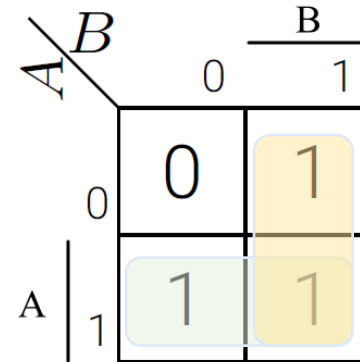
It's ok for groups to overlap, if that makes them larger

$$m_1+m_2+m_3 = A'B+AB' +AB = A+B$$

Three-variable Map

- Minterms are arranged in the Gray-code sequence
- Any 2 (horizontally or vertically) adjacent squares differ by exactly 1 variable, which is complemented in one square and uncomplemented in the other.
- Any 2 minterms in adjacent squares that are ORed together will cause a removal of the different variable (adjacent applies not only the middle squares but also the boundary squares)

Three-variable Map

- Example (adjacent squares)
- $m_5 + m_7 = AB'C + ABC = AC(B+B') = AC$
- $m_0 + m_2 = A'B'C' + A'BC' = A'C'(B+B') = A'C'$
- $m_4 + m_6 = AB'C' + ABC' = AC'(B'+B) = AC'$
- $m_1 + m_3 + m_5 + m_7$

  $= A'B'C + A'BC + AB'C + ABC = A'C(B+B') + AC(B+B')$

  $= A'C + AC = C$
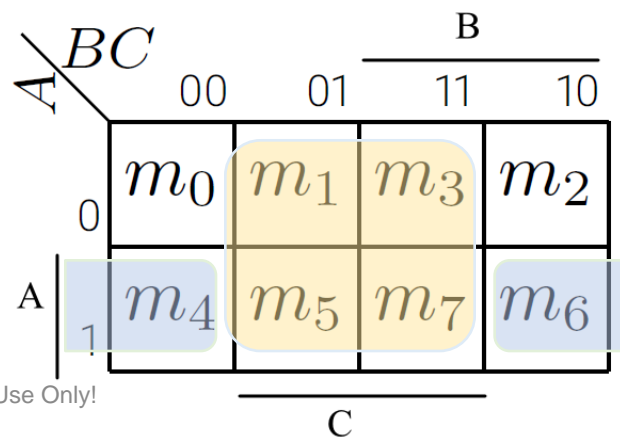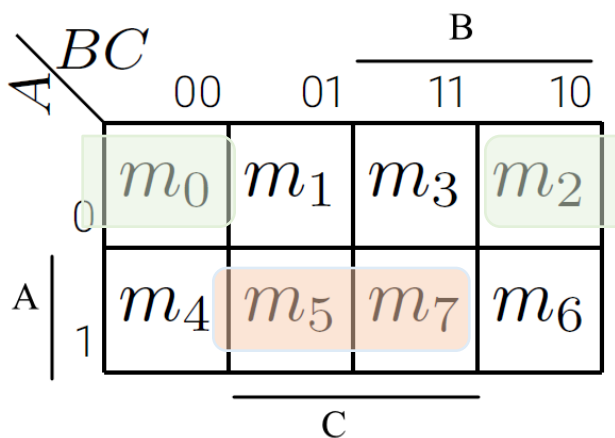
Example

- Simplify the following Boolean functions.

F = A'BC+A'BC'+AB'C'+AB'C
= A'B + AB'

F = A'BC+AB'C'+ABC'+ABC
= BC + AC'

Green circle: A'BC+A'BC' = A'B
Red circle: AB'C'+AB'C = AB'

Green circle: A'BC + ABC = BC
Red circle: AB'C' + ABC' = AC'

Example

- Simplify the following Boolean functions.

$F = \sum(1, 2, 3, 5, 7) = C + A'B$

$F= \sum(0, 2, 4, 5, 6) = C' + AB'$

It's ok for groups to overlap,
if that makes them larger

Exercise

- Simplify the following Boolean function.
  - F = A'C + A'B + AB'C + BC
    = ?

Exercise

- Simplify the following Boolean function.
  - F = A'C + A'B + AB'C + BC = ?
- Solution:
  - Express it in sum of minterms.
  - Find the minimal sum of products expression.
  - F = A'C + A'B + AB'C + BC
    = A'C(B+B') + A'B(C+C') + AB'C + (A+A')BC
    = A'BC + A'B'C + ~~A'BC~~ + A'BC' + AB'C + ABC + ~~A'BC~~
    = ∑(1, 2, 3, 5, 7) = C + A'B

Four-Variable Map

- The map
  - 16 minterms
  - Combinations of 2, 4, 8, and 16 adjacent squares

Implicants

- Implicant: any product term that implies the function
  - A product term that makes a function to be true

	minterm	implicant	
m_1	√	√	1-minterm
m_2	√	X	0-minterm
x'z	X	√	1 product term

- Prime implicant (PI) (质蕴含)
  - A 1-product term obtained by combining the maximum possible number of adjacent squares in the map.
- Essential prime implicant (EPI) (基本质蕴含)
  - If a minterm in a square is covered by only one prime implicant

x	y	z	F	
0	0	0	0	m_0
0	0	1	1	m_1
0	1	0	0	m_2
0	1	1	1	m_3
1	0	0	1	m_4
1	0	1	1	m_5
1	1	0	0	m_6
1	1	1	0	m_7

Tips for simplification

- Simplification Steps:
  - Determine all essential prime implicants.
  - Find other prime implicants that cover remaining minterms.
  - Logical sum all prime implicants.
- Tips:
  - Minimize the number of groups
  - Maximize the group size
  - It's ok for groups to overlap, if that makes them larger

Example

- Simplify the following Boolean functions
- $F = m_1 + m_5 + m_{10} + m_{11} + m_{12} + m_{13} + m_{15}$
  - start with EPIs
  - Green circle: $A'B'C'D + A'BC'D = A'C'D$
  - Purple circle: $ABC'D' + ABC'D = ABC'$
  - Red circle: …
  - …
  - $F = A'C'D + ABC' + ACD + AB'C$

- This reduced expression is not a unique one
  - If pairs are formed in different ways, the simplified expression will be different.
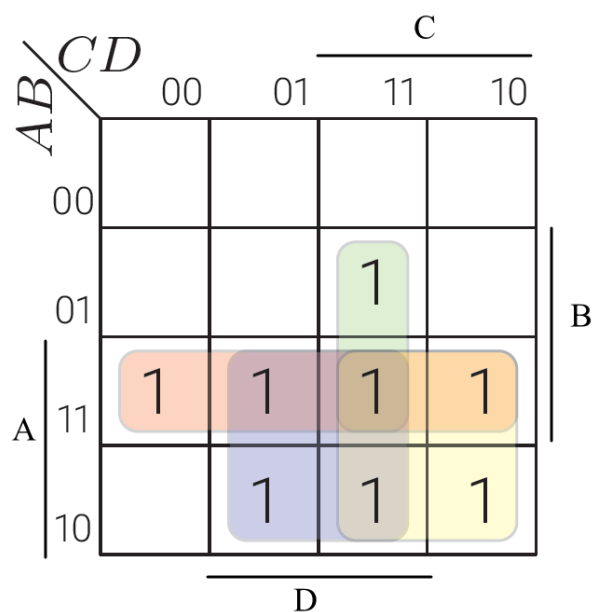 - $F = A'C'D + ABC' + ABD + AB'C$
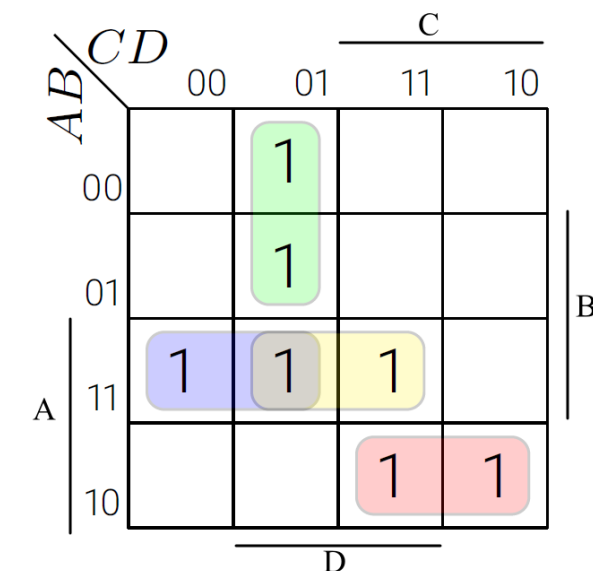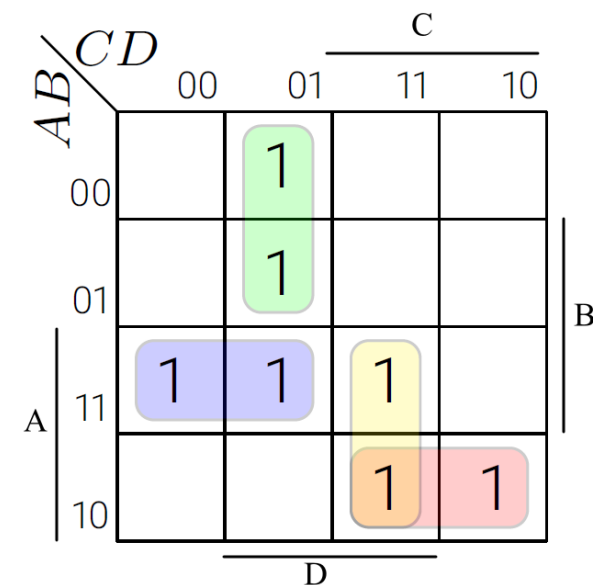
Example

- Simplify the following Boolean functions.

$F = \sum(7, 9, 10, 11, 12, 13, 14, 15)$
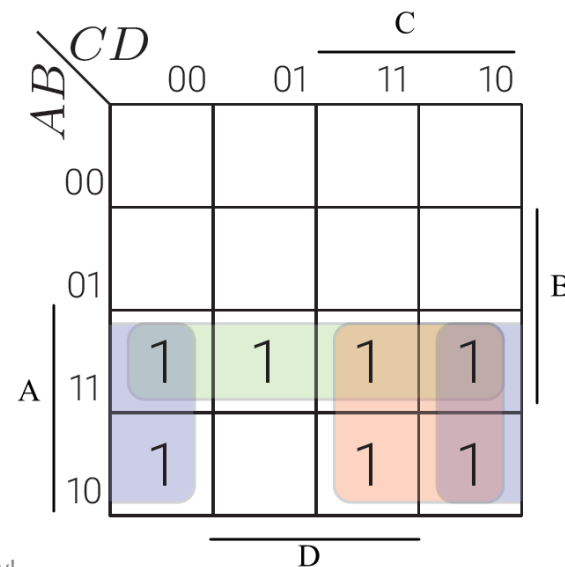$= AB + AC + AD + BCD$

$F(A,B,C,D) = ABCD + AB'C'D' + AB'C + AB$
$= ABCD + AB'C'D' + AB'C(D + D')$
$\quad + AB(C + C')(D + D')$
$= \ldots$
$= \sum(8, 10, 11, 12, 13, 14, 15)$
$= AB + AC + AD'$

Exercise

- Simplify the following Boolean functions.

F(A,B,C,D)
= ∑(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)
= ?

F(A,B,C,D)
= A′B′C′ + B′CD′ + A′BCD′ + AB′C′
= ?

Exercise

- Simplify the following Boolean functions.

F(A,B,C,D)
= $\sum$(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)
= C' + A'D' + BD'

F = A'B'C' + B'CD' + A'BCD' + AB'C'
= A'B'C'(D + D') + B'CD'(A + A') + A'BCD'+ AB'C'(D + D')
= A'B'C'D + A'B'C'D'+ AB'CD' + A'B'CD'+ A'BCD'+ AB'C'D + AB'C'D'
= $\sum$(0, 1, 2, 6, 8, 9, 10)
= B'C'+ B'D'+ A'CD'

K-map Summary

• Any 2k adjacent squares, k=0,1,...,n, in an n-variable map represent an area that gives a product term of n-k literals

K	# of adjacent squares	# of literals left in a term in an n-variable map		
		n=2	n=3	n=4
0	1	2	3	4
1	2	1	2	3
2	4	0	1	2
3	8		0	1
4	16			0

• Five-Variable Map
  • Map for more than four variables becomes complicated
  • Five-variable map: two four-variable map (one on the top of the other), contains 2^5 or 32 cells.

Outline

- Map Method Simplification
- **Product of sums simplification**
- Don't Care Conditions

Product of Sums Simplification

- Previous Examples are Sum of Product Simplification
  - E.g. F = AB + A'D + AB'C (Product of sum form)

- How to find Product of Sum simplification
  - E.g. F = (A+B)(B+C') (Sum of Product form)

- POS simplification Steps
  - Simplified F' in the form of sum of products
    - Group adjacent 0-minterms squares together
  - Apply DeMorgan's theorem F = (F')'
  - F': sum of products → F: product of sums

Example

- Simplify the Boolean function into product of sums form:
  - $F(A,B,C,D) = \sum(2, 3, 7, 10, 11, 15)$
- Solution
  - Step1: group the 0-minterms to find F complement
    $F' = \sum(0, 1, 4, 5, 6, 8, 9, 12, 13, 14)$
    $= C' + BD'$ (Group 0 minterms)
  - Step2: find the complement again to get original F
    $F = (F')' = (C' + BD')'$
    $= C(B'+D)$ (DeMorgan's)

- simplify F(A, B, C, D) = ∑(0, 1, 2, 5, 8, 9, 10) into
  - sum-of-products form
    - F = ?
  - product-of-sums form
    - F = ?

Exercise

- simplify F(A, B, C, D) = ∑(0, 1, 2, 5, 8, 9, 10) into
  - sum-of-products form
    - F = B'D' + B'C' + A'C'D (Group 1-minterms)
  - product-of-sums form
    - F' = AB+CD+BD'   (Group 0-minterms)
    - F = (A'+B')(C'+D')(B'+D) (DeMorgan's)

Outline

- Map Method Simplification
- Product of sums simplification
- **Don't Care Conditions**

Don't care conditions

- Incompletely specified functions
  - Functions that have unspecified outputs for some input combinations
  - E.g. output are unspecified for 1010 to 1111 in 4-bit BCD code
- Don't-care conditions
  - Unspecified minterms of a function, don't-cares, Xs
  - Can be used on a map to provide further simplifications of the Boolean expression
  - Each X can be assigned an arbitrary value, 0 or 1, to help simplification procedure
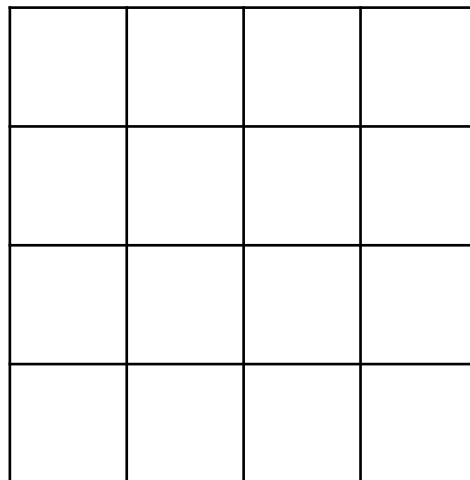
Example

- Simplify F(A, B, C, D) = ∑(1, 3, 7, 11, 15) with don't-care conditions d(A, B, C, D) = ∑(0, 2, 5).
  - F = A'B' + CD
  - or F = A'D + CD

  - Just make sure all 1 minterms are circled, thus both simplifications are acceptable

Exercise

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function F(A,B,C,D) = Σ(0, 2, 3, 6, 7) + d(8, 10, 11, 15)

Exercise

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function $F(A,B,C,D) = \Sigma(0, 2, 3, 6, 7) + d(8, 10, 11, 15)$
- $F = A'C + B'D'$