Chapter 4: Linear Programming and Simplex Method

Chao Wang

SUSTech

Outline

Linear Programming Problems

Canonical Form and Standard Form

The general form of an LP problem is

$$\begin{array}{lll} \text{Max/Min} & z = c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & (\leq, =, \geq) b_1 \\ & \vdots & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & (\leq, =, \geq) b_m \end{array} \tag{1}$$

两种特别形式

<1>. <2>. standard form

- Canonical form:

$$\begin{array}{lll} \text{Maximize} & z = c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq b_1 \\ & \vdots & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq b_m \\ \text{where} & x_i \geq 0, i = 1, 2, \ldots, n. & \end{array} \tag{2}$$

not must. 存在可行解.

If all $b_i \geq 0$, then the form is called a **feasible canonical form**.

Canonical form and standard form

(2).

- Standard form:

区分CF.

$$
\begin{array}{rll}
\text{Maximize} & c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\
& \vdots & \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = b_m \\
\text{and} & x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 &
\end{array} \tag{3}
$$

★ must

where $b_i \geq 0$. 就算<0可转化为≥0

①
- Min program, i.e.

$$\text{Min} \qquad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n.$$

This is equivalent to Min $\underset{}{\overset{\text{添加}-}{\rightleftharpoons}}$ Max.

但不要忘记原问题条件.

$$\text{Max} \qquad z' = -z = (-c_1)x_1 + (-c_2)x_2 + \ldots + (-c_n)x_n = -z.$$

②
- $(\geq)$ type constraint, i.e.

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i.$$

This is equivalent to

$$\sum_{j=1}^{n}(-a_{ij})x_j \leq -b_i.$$

Reduction to canonical form and standard form

$a^T x \geq b$

$\Leftrightarrow -a^T x \leq -b \Leftrightarrow a^T x - S = b$

$S \geq 0$

③ - $(=)$ type constraint, i.e.

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

This is equivalent to

取并. $\Leftrightarrow a_{ij} x_j \geq b_i$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{and} \quad \sum_{j=1}^{n} (-a_{ij}) x_j \leq (-b_i).$$

④ - Free variables, i.e. x_j is unrestricted in sign. Let

$$x_j = x_j^+ - x_j^-, \qquad |x_j| = x_j^+ + x_j^-$$

where $x_j^+ \geq 0$ and $x_j^- \geq 0$. We substitute $x_j^+ - x_j^-$ for x_j everywhere in the LP. The problem then has $(n+1)$ non-negative variables $x_1, x_2, \ldots, x_j^+, x_j^-, \ldots, x_n$.

①. cf. max $-2x_1 - 4x_2 \; (x_2^+ - x_2^-)$

subject to $-x_1 - x_2 \le -3$

$3x_1 + 2x_2 \le 14$

$-3x_1 - 2x_2 \le -14$

$x_1 \ge 0 \quad x_2^+ \ge 0 \quad x_2^- \ge 0$

②. Standard form

max $-2x_1 - 4(x_2^+ - x_2^-)$

s.t. $+(-x_1 - x_2^+ + x_2^- + x_3) = +3$

$3x_1 + 2(x_2^+ - x_2^-) = 14$

$x_1 \ge 0, \; x_2^+ \ge 0, \; x_2^- \ge 0$

Example

The problem

$x_1, x_2 \ge 0.$ 但未限制

$$
\begin{array}{lrcrcl}
\text{Minimize} & 2x_1 & + & 4x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \geq & 3 \\
 & 3x_1 & + & 2x_2 & = & 14 \\
 & & & x_1 & \geq & 0
\end{array} \tag{4}
$$

We introduce a slack variable x_3 such that $x_1 + x_2 - x_3 = 3$.
Let $x_2 = x_2^+ - x_2^-$, then the problem is equivalent to the standard form problem

$$
\begin{array}{lrcrcrcrcl}
\text{Maximize} & -2x_1 & - & 4x_2^+ & + & 4x_2^- & & & & \\
\text{subject to} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 & = & 3 \\
 & 3x_1 & + & 2x_2^+ & - & 2x_2^- & & & = & 14 \\
 & \multicolumn{9}{c}{x_1, x_2^+, x_2^-, x_3 \geq 0}
\end{array} \tag{5}
$$

Linearization of special nonlinear models

Problems Involving Absolute Values

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{n} c_i |x_i| \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \end{aligned} \tag{6}$$

1 Approach 1: let $x_i = x_i^+ - x_i^-$

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{n} c_i (x_i^+ + x_i^-) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x}^+ - \mathbf{A}\mathbf{x}^- \geq \mathbf{b} \\ & \mathbf{x}^+, \mathbf{x}^- \geq \mathbf{0} \end{aligned} \tag{7}$$

where $\mathbf{x}^+ = (x_1^+, \ldots, x_n^+)^T$ and $\mathbf{x}^- = (x_1^-, \ldots, x_n^-)^T$.

Linearization of special nonlinear models

Problems Involving Absolute Values

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{n} c_i |x_i| \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \end{aligned} \tag{8}$$

2 Approach 2: let $|x_i|$ be the smallest number z_i that satisfies $x_i \leq z_i$ and $-x_i \leq z_i$

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{n} c_i z_i \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & x_i \leq z_i, \qquad i = 1, \ldots, n, \\ & -x_i \leq z_i, \quad i = 1, \ldots, n. \end{aligned} \tag{9}$$

Linearization of special nonlinear models

① Min $2x_1^+ + 2x_1^- + x_2$
s.t. $x_1^+ - x_1^- + x_2 \geq 4$.
$x_1^+, x_1^-, x_2 \geq 0$.

② Min $2z_1 + x_2$
s.t. $x_1 + x_2 \geq 4$.
$x_1 \leq z_1$
$-x_1 \leq z_1$

Example

Consider the problem

$$\begin{array}{rl} \text{Minimize} & 2|x_1| + x_2 \\ \text{subject to} & x_1 + x_2 \geq 4. \end{array} \tag{10}$$

Our first reformulation yields

$$\begin{array}{rrcl} \underset{z, x_2}{\text{Minimize}} & 2z_1 + x_2 & & \\ \text{subject to} & x_1 + x_2 & \geq & 4 \\ & x_1 & \leq & z_1 \\ & -x_1 & \leq & z_1, \end{array} \tag{11}$$

while the second yields

$$\begin{array}{rccccccl} \text{Minimize} & 2x_1^+ & + & 2x_1^- & + & x_2 & & \\ \text{subject to} & x_1^+ & - & x_1^- & + & x_2 & \geq & 4 \\ & & & & & x_1^+ & \geq & 0 \\ & & & & & x_1^- & \geq & 0 \end{array} \tag{12}$$

Linearization of special nonlinear models

Min-Max Problem

取函数组中最大函数的最小值.

$$\text{Minimize: Maximum}\{\mathbf{c}^T\mathbf{x}, \mathbf{d}^T\mathbf{x}, \ldots\} \tag{13}$$

First we introduce a new decision variable w. We add new constraints, one for every sub-function in the objective function:

$$\begin{aligned} w &\geq \mathbf{c}^T\mathbf{x} \\ w &\geq \mathbf{d}^T\mathbf{x} \\ &\vdots \end{aligned} \tag{14}$$

Then the problem can be reformulated as

$$\begin{aligned} \text{Minimize} \quad & w \\ \text{subject to} \quad & w \geq \mathbf{c}^T\mathbf{x} \\ & w \geq \mathbf{d}^T\mathbf{x} \\ & \vdots \end{aligned} \tag{15}$$

Max-Min Problem

$$\begin{array}{c}\text{Maximize: Minimum}\{\mathbf{c}^T\mathbf{x}, \mathbf{d}^T\mathbf{x}, \ldots\} \\ \Downarrow \\ \begin{array}{rc} \text{Maximize} & w \\ \text{subject to} & w \le \mathbf{c}^T\mathbf{x} \\ & w \le \mathbf{d}^T\mathbf{x} \\ & \vdots \end{array}\end{array} \tag{16}$$

Multiplication

$\log xyz = \log x + \log y + \log z.$

$$
\begin{array}{lrl}
\text{Maximize} & xyz & \\
\text{subject to} & xy & \leq 1 \\
 & yz & \leq 1 \\
 & xz & \leq 2 \\
 & x, y, z & \geq 0
\end{array}
\tag{17}
$$

→ log定义域.

Take the $\log_2()$ of the objective function. Maximizing the $\log_2$ of something is equivalent to maximizing it owing to the monotonicity of $\log_2$. Thus our non-linear optimization is transformed:

→ 当作整体才可作为线性.

$$
\begin{array}{lrl}
\text{Maximize} & \overset{\hat{x}}{\boxed{\log_2 x}} + \overset{\hat{y}}{\boxed{\log_2 y}} + \overset{\hat{z}}{\boxed{\log_2 z}} & \\
\text{subject to} & \log_2 x + \log_2 y & \leq 0 \\
 & \log_2 y + \log_2 z & \leq 0 \\
 & \log_2 x + \log_2 z & \leq 1 \\
 & (\log_2 x, \log_2 y, \log_2 z \geq -\infty) &
\end{array}
\tag{18}
$$

→ 注意复合函数增减性.

Minimize z.
s.t. $x_1 + x_2 \geq 15$
$x_2 \leq 7$
$x_1 - x_2 \leq z$
$x_2 - x_1 \leq z$.

Example

$$\begin{aligned} \text{Minimize} \quad & \max\{x_1 - x_2, x_2 - x_1\} \\ \text{subject to} \quad & x_1 + x_2 \geq 15 \\ & x_2 \leq 7 \end{aligned} \tag{19}$$

We can equivalently reformulated as

$$\begin{aligned} \text{Minimize} \quad & w \\ \text{subject to} \quad & w \geq x_1 - x_2 \\ & w \geq x_2 - x_1 \\ & x_1 + x_2 \geq 15 \\ & x_2 \leq 7 \end{aligned} \tag{20}$$

Graphical Analysis

Graphical representation and solution

Solve

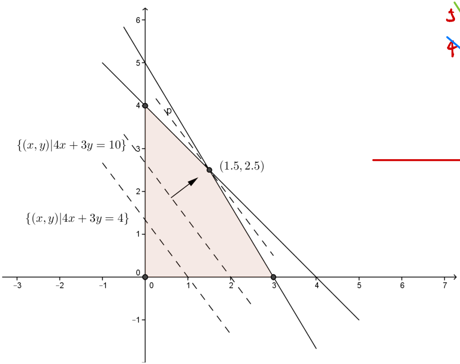
$$\begin{array}{ll} \text{Max} & z = 4x + 3y \\ \text{Subject to} & x + y \leq 4 \\ & 5x + 3y \leq 15 \\ & x \geq 0, \quad y \geq 0 \end{array} \tag{21}$$

转化为与y轴相交点

两点确定一线.

① $x+y=4$.

② $5x+3y=15$



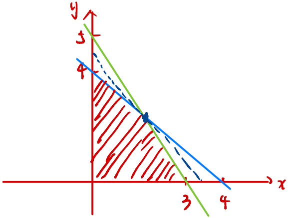


Figure: An LP with a bounded nonempty feasible region

Graphical representation and solution

Unbounded solution: Consider an LP:

$$
\begin{array}{ll}
\text{Max} & z = 2x + 5y \\
\text{Subject to} & -3x + 2y \le 6 \\
 & x + 2y \ge 2 \\
 & x \ge 0, \quad y \ge 0
\end{array}
\tag{22}
$$

$5y = -2x + z.$

$y = -\frac{2}{5}x + \frac{z}{5}.$

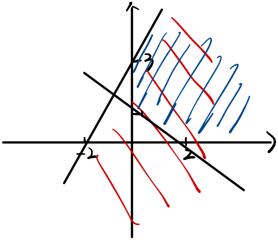
$y = 3 \quad x = -2.$

$-3x + 2y = 6$

$x + 2y = 2.$

$2y = -x + 2$

$x = 2 \; y = 1.$



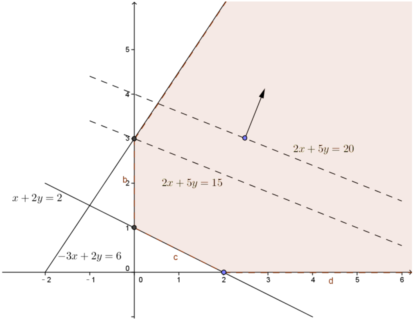


Figure: An LP with unbounded feasible region

Graphical representation and solution

Unbounded feasible region with a finite optimal solution: Consider an LP:

$$\begin{array}{cl} \text{Max} & z = -4x + 2y \\ \text{Subject to} & -3x + 2y \leq 6 \\ & x + 2y \geq 2 \\ & x \geq 0, \quad y \geq 0 \end{array} \tag{23}$$

unbounded region

也可存在 bounded solution.

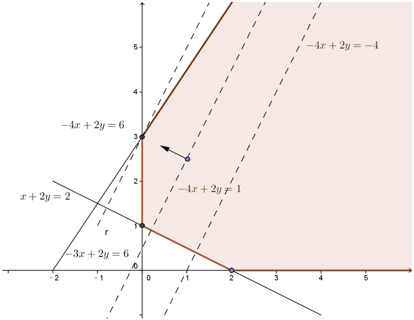


Figure: An LP with unbounded feasible region and finite optimal solution.

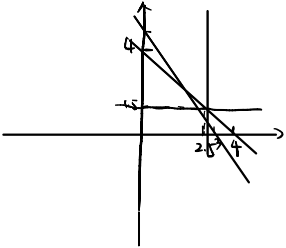
Graphical representation and solution

LP is infeasible: Consider an LP:

$$
\begin{array}{ll}
\text{Max} & z = 3x + 2y \\
\text{Subject to} & x + y \le 4 \\
& 5x + 3y \le 15 \\
& x \ge 2.5, \quad y \ge 1.5
\end{array}
\tag{24}
$$

$2y = -3x + z.$

$y = -\frac{3}{2}x + \frac{z}{2}.$



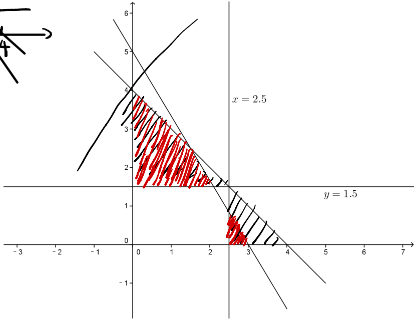


Figure: An LP with no solution.

Graphical representation and solution

Infinitely many optimal solutions: Consider an LP:

$$\begin{array}{rl} \text{Max} & z = 15x + 9y \\ \text{Subject to} & x + y \leq 4 \\ & 5x + 3y \leq 15 \\ & x \geq 0, \quad y \geq 0 \end{array} \tag{25}$$

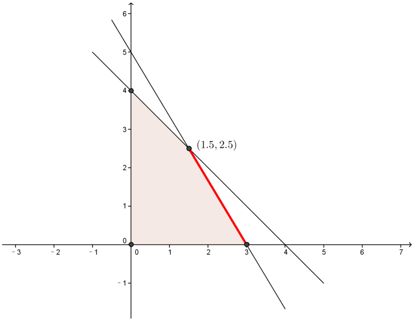


Figure: An LP with infinite number of optimal solutions.

Graphical representation and solution

To summarize the insights obtained, we have the following possibilities:

1. There exists a unique optimal solution.
2. There exists multiple optimal solutions.
3. The optimal cost is $\pm\infty$, and no feasible solution is optimal.
4. The feasible region is empty.

Summary

- Canonical form and standard form:
  - definitions
  - how to reduce to these forms
- Linearzation of special nonlinear models
  - problems involving absolute values
  - min-max problem
  - ...
- Graphical representation and solution

Fundamental Theorem of LP

Basic solution

Now consider adding the non-negativity constraints, i.e.

Standard form. $m\times n$ $m<n$ $\operatorname{rank} A = m$.

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}. \end{cases} \tag{26}$$

$A = \boxed{B \mid N}$

B invertible.

Definition

A vector $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{A}\mathbf{x} = \mathbf{b}$ is said to be a feasible solution for the constraints.

- A feasible solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ that is also a basic solution is said to be a **basic feasible solution (BFS)**; $(x_B, 0)$ $x_B \geq 0$
- if this solution is non-degenerate then it is called a **non-degenerate basic feasible solution (NBFS)**, $(x_B, 0)$ $x_B > 0$
- otherwise it is a **degenerate basic feasible solution (DBFS)**.

$(x_B, 0)$ $\exists i,$ $(x_B)_i = 0$ $x_B \geq 0$.

We shall be mostly concerned with non-degenerate basic feasible solution. Hence frequently we shall write only BFS for NBFS; and specify degeneracy as exception.

Basic solution

Corresponding to a linear program in standard form

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{27}$$

Definition

- A feasible solution to the constraints that achieves the maximum value of the objective function subject to those constraints is said to be an **optimal feasible solution**. 最优
- If this solution is basic, it is an **optimal basic feasible solution**

Theorem (Fundamental Theorem of Linear Programming)

Given a linear program in standard form (27), where $\mathbf{A}$ *is an* $m \times n$ *matrix of rank* m*,*

(a) *If there is a feasible solution, there is a basic feasible solution;*

(b) *If there is an optimal feasible solution, there is an optimal basic feasible solution.*

(optimal) feasible solution → (optimal) basic feasible solution.

Fundamental theorem of linear programming

max $c^T x$
s.t. $Ax=b$
$x \geq 0$ } ⇒ feasible

Proof:
(a) Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is a feasible solution. Then, in terms of the columns of $\mathbf{A}$, this solution satisfies:

$Ax = b$

$[\;][\;] = b$

列向量.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \quad (28)$$

Assume that exactly p of the variables x_i are greater than zero, and for convenience, that they are the first p variables. Thus 其他 = 0

排序.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b} \quad (29)$$

rank $A = m$

Case 1: Assume $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ are linearly independent. Then $p \leq m$.
If $p = m$, the solution is basic and the proof is complete.
If $p < m$, then, since $\mathbf{A}$ has rank m, $m - p$ vectors can be found from the remaining $n - p$ vectors so that the resulting set of m vectors is linearly independent.

$\{a_{p+1} \sim a_n\}$ selects $n-p$ vectors

$x_B = [x_1, \ldots, x_p, 0, 0 \ldots]$

Fundamental theorem of linear programming

Case 2: Assume $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ are linearly dependent. Then there are constants $y_1, y_2, \ldots, y_p$, at least one of which can be assumed to be positive, such that

$$y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \cdots + y_p\mathbf{a}_p = \mathbf{0}. \tag{30}$$

至少一个
at least $y_i > 0$.
(若所有 $y_i < 0$ ⇒ 添加 -)

Multiplying this equation by a scalar ϵ and subtract it from (29), we obtain

$$(x_1 - \epsilon y_1)\mathbf{a}_1 + (x_2 - \epsilon y_2)\mathbf{a}_2 + \cdots + (x_p - \epsilon y_p)\mathbf{a}_p = \mathbf{b} \tag{31}$$

This equation holds for every ϵ, and for each ϵ the components $x_j - \epsilon y_j$ correspond to a solution of the linear equalities -although they may violate $x_i - \epsilon y_i \geq 0$. Denoting $\mathbf{y} = (y_1, y_2, \ldots, y_p, 0, 0, ..., 0)$, we see that for any ϵ, $\mathbf{x} - \epsilon\mathbf{y}$ is a solution to the equalities. Since we assume at least one y_i is positive, at least one component will decrease as ϵ is increased. We increase ϵ to the first point where one or more components become zero. Specifically, we set

$\exists\, x_i - \varepsilon y_i = 0$. $x_i / y_i \geq \varepsilon$.

$$\epsilon = \min\{x_i/y_i : y_i > 0\}. \tag{32}$$

$y_i < 0$ 时 $x_i - \varepsilon y_i \geq 0$ 必成立.

挪动位置.(降了一维).

For this value of ϵ the solution is feasible and has at most $p - 1$ positive variables.

Repeating this process if necessary, we can eliminate positive variables until we have a feasible solution with corresponding columns that are linearly independent. At that point Case 1 applies.

降维至到所有都线性无关.

worst case: 只剩1个.

Fundamental theorem of linear programming

max $C^T x$

we Just need to show the BFS is optimial.

(b) Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an optimal feasible solution and, as in the proof of (a) above, suppose there are exactly p positive variables $x_1, x_2, \ldots, x_p$.
Case 1, corresponding to linear independence, is exactly the same as before.
Case 2 also goes exactly the same as before, but it must be shown that for any ϵ is optimal. To show this, note that the value of the solution $\boxed{\mathbf{x} - \epsilon\mathbf{y}}$ is

当前的$\hat{x}$. $C^T y \neq 0$.

$$\mathbf{c}^T\mathbf{x} - \epsilon\mathbf{c}^T\mathbf{y} \tag{33}$$

$C^T x$ max.

For ϵ sufficiently small in magnitude, $\mathbf{x} - \epsilon\mathbf{y}$ is a feasible solution for positive or negative values of ϵ. Thus we conclude that $\mathbf{c}^T\mathbf{y} = 0$. For, if $\mathbf{c}^T\mathbf{y} \neq 0$, an ϵ of small magnitude and proper sign could be determined so as to render (33) smaller than $\mathbf{c}^T\mathbf{x}$ while maintaining feasibility. This would violate the assumption of optimality of $\mathbf{x}$ and hence we must have $\mathbf{c}^T\mathbf{y} = 0$. Having established that the new feasible solution with fewer positive components is also optimal, the remainder of the proof may be completed exactly as in part (a). 对y添加了约束条件. 使得前后之值一定.
(must)

What does this theorem tell us about?

This theorem reduces the task of solving a linear program to that of searching over basic feasible solutions. For a problem having n variables and m constraints the number of basic solutions is at most

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

n个纵向量中选m个无关 构成 B.
(rank = m)

Simplex Method

Simplex method by example

Consider the following LP. Convert to equalities by adding slack variables

$$
\begin{aligned}
\max 3x_1 + 2x_2 & \\
x_1 + x_2 &\leq 80 \\
2x_1 + x_2 &\leq 100 \\
x_1, x_2 &\geq 0, \overset{x_1}{\cancel{x_4}} \leq 40
\end{aligned}
$$

$$
\begin{array}{llcccccccccccl}
\text{Max} & 3x_1 + 2x_2 \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & & & & & = & 80 \\
 & 2x_1 & + & x_2 & & & + & x_4 & & & = & 100 \\
 & x_1 & & & & & & & + & x_5 & = & 40 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

- Starting feasible solution:
  Set variables x_1, x_2 to zero and set slack variables to the values on the right-hand side. $\rightarrow$ yields a feasible solution:
  $x_1 = x_2 = 0, x_3 = 80, x_4 = 100, x_5 = 40$

Simplex method by example

$$
\begin{array}{rlrr}
x_3 & = 80 & -x_1 & -x_2 \\
x_4 & = 100 & -2x_1 & -x_2 \\
x_5 & = 40 & -x_1 & \\
\hline
z & = 0 & +3x_1 & +2x_2 \\
\hline
\end{array}
$$

- x_1, x_2 are non-basic variables
- x_3, x_4, x_5 are basic variables
- $\{x_3, x_4, x_5\}$ is a basis
- $z = 0$
- How to improve the solution: try to increase x_1 from 0 in hopes of improving the value of z e.g.:
  - try $x_1 = 20, x_2 = 0$ and substitute into the dictionary to obtain the values of x_3, x_4, x_5 and $z \longrightarrow x_3 = 60, x_4 = 60, x_5 = 20$ with value $z = 60 \rightarrow$ feasible
  - try again $x_1 = 40, x_2 = 0 \longrightarrow x_3 = 40, x_4 = 20, x_5 = 0$ with value $z = 120 \rightarrow$ feasible now
  - try $x_1 = 50, x_2 = 0 \longrightarrow x_3 = 30, x_4 = 0, x_5 = -10 \rightarrow$ not feasible

Simplex method by example

How much we can increase x_1 before a (dependent) variable becomes negative?

$$
\begin{array}{rlrrr}
x_3 & = 80 & -t & -x_2 & \geq 0 \\
x_4 & = 100 & -2t & -x_2 & \geq 0 \\
x_5 & = 40 & -t & & \geq 0 \\
\hline
z & = 0 & +3t & +2x_2 & \\
\hline
\end{array}
\Rightarrow
\begin{array}{rl}
t & \leq 80 \\
t & \leq 50 \\
t & \leq 40
\end{array}
\Rightarrow t \leq 40.
$$

- x_1 is entering variable
- x_5 is departing variable
- the current solution is $x = (40, 0, 40, 20, 0)$ with $z = 120$.

Simplex method by example

- The above analysis can be streamlined into the following simple "ratio" test.

$$
\begin{array}{rlrrr}
x_3 & = 80 & x_1 & -x_2 & \frac{80}{1} = 80 \\
x_4 & = 100 & -2x_1 & -x_2 & \frac{100}{\cancel{50}\,2} = 50 \\
x_5 & = 40 & -x_1 & & \frac{40}{1} = 40 \\
\hline
z & = 0 & +3x_1 & +2x_2 & \\
\hline
\end{array}
\Rightarrow \boxed{\text{minimum radio achieved with } x_5}
$$

- Express x_1 from the equation for

$$x_5 : x_5 = 40 - x_1 \rightarrow x_1 = 40 - x_5$$

$$
\begin{array}{rlrr}
x_1 & = & (40 - x_5) & \\
x_3 & = 80 & -(40 - x_5) & -x_2 \\
x_4 & = 100 & -2(40 - x_5) & -x_2 \\
\hline
z & = 0 & +3(40 - x_5) & +2x_2 \\
\hline
\end{array}
\Rightarrow
\begin{array}{rlrr}
x_1 & = 40 & & -x_5 \\
x_3 & = 40 & -x_2 & +x_5 \\
x_4 & = 20 & -x_2 & +2x_5 \\
\hline
z & = 120 & +2x_2 & -3x_5 \\
\hline
\end{array}
$$

Simplex method by example

- we repeat: we increase $x_2 \to$ entering variable, ratio test:
  - x_1 does not contain $x_2 \to$ no constraint
  - $x_2 : \frac{40}{1} = 40$
  - $x_4 : \frac{20}{1} = 20$

  minimum achieved for $x_4 \to$ departing variable

$$
\begin{array}{llccc}
x_1 & = 40 & & -x_5 \\
x_2 & = & (20 - x_4 + 2x_5) & \\
x_3 & = 40 & -(20 - x_4 + 2x_5) & +x_5 \\
\hline
z & = 120 & +2(20 - x_4 + 2x_5) & -3x_5 \\
\hline
\end{array}
\Rightarrow
\begin{array}{llcc}
x_1 & = 40 & & -x_5 \\
x_2 & = 20 & -x_4 & +2x_5 \\
x_3 & = 20 & +x_4 & -x_5 \\
\hline
z & = 160 & -2x_2 & +x_5 \\
\hline
\end{array}
$$

- Similarly we have

$$
\begin{array}{llcc}
x_1 & = 40 & & -(20 + x_4 - x_3) \\
x_2 & = 20 & -x_4 & +2(20 + x_4 - x_3) \\
x_5 & = & & (20 + x_4 - x_3) \\
\hline
z & = 160 & -2x_4 & +(20 + x_4 - x_3) \\
\hline
\end{array}
\Rightarrow
\begin{array}{llcc}
x_1 & = 20 & +x_3 & -x_4 \\
x_2 & = 60 & -2x_3 & +x_4 \\
x_5 & = 20 & -x_3 & +x_4 \\
\hline
z & = 180 & -x_3 & -x_4 \\
\hline
\end{array}
$$

- no more improvement possible $\to$ optimal solution

Simplex algorithm

- **Preparation:** find a starting feasible solution/dictionary
  1. Convert to the canonical form (constraints are equalities) by adding slack variables $x_{n+1}, \ldots, x_{n+m}$
  2. Construct a starting dictionary - express slack variables and objective function z
  3. If the resulting dictionary is feasible, then we are done with preparation If not, try to find a feasible dictionary using the Phase I. method (next lecture). Simplex step (maximization LP): try to improve the solution

Simplex algorithm

- **Simplex step:** $\exists i\ (x_N)_i \uparrow$? $z \uparrow$? $\rightarrow$ not optimal

1. (Optimality test): If no variable appears with a positive coefficient in the equation for $z \rightarrow$ STOP, current solution is optimal
   - set non-basic variables to zero
   - read off the values of the basic variables and the objective function
2. Else pick a variable x_i having positive coefficient in the equation for z
   $x_i \equiv$ entering variable
3. Ratio test: in the dictionary, find an equation for a variable x_j in which
   - x_i appears with a negative coefficient $-a$
   - the ratio $\frac{b}{a}$ is smallest possible
4. $x_j \equiv$ departing variable $\rightarrow$ construct a new dictionary by pivoting:
   - express x_i from the equation for x_j,
   - add this as a new equation,
   - remove the equation for x_j,
   - substitute x_i to all other equations (including the one for z)
5. Repeat from 1 .

Simplex in matrix form

- Standard LP:

$$\begin{aligned} \text{Max} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{Subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$[B \;\; N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

$$\Rightarrow \quad Bx_B = b - Nx_N$$

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

- Partitioning matrix: $\mathbf{A} = [\mathbf{B}, \mathbf{N}], \mathbf{x} = [\mathbf{x}_B, \mathbf{x}_N]$

$x_N = 0$

- Objective function:

$$\begin{aligned} z &= \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N \\ &= \mathbf{c}_B^T(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N) + \mathbf{c}_N^T\mathbf{x}_N \\ &= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} - (\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T)\mathbf{x}_N \\ &= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} - \sum_{j\in J}\underline{(\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{a}_j - c_j)}x_j, \end{aligned}$$

$\geq 0 \rightarrow$ optimal

where J is the index set of the nonbasic variables. N中的列向量.

- Solution

$$\begin{aligned} x_B &= \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \\ &= \mathbf{B}^{-1}\mathbf{b} - \sum_{j\in J}(\mathbf{B}^{-1}\mathbf{a}_j)x_j, \end{aligned}$$

Simplex in matrix form

- $z = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in J}(\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{a}_j - c_j)x_j$
- reduced cost: $z_j - c_j$, where $z_j = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{a}_j$.
- if the coefficient $-(z_j - c_j) > 0$, then increasing x_j will increase z.

Optimality conditions (Max problem)

The basic feasible solution $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$ will be optimal if

$$-(z_j - c_j) = -(\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{a}_j - c_j) \leq 0, \quad \text{for all } j \in J$$

or, equivalently, if

$$z_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{a}_j - c_j \geq 0, \quad \text{for all } j \in J$$

Simplex in matrix form

- Determining the Entering Variable x_k with a reduced cost $z_k - c_k < 0$
- Determining the Departing Variable x_r:
  - x_N is changed as $x_N = (0, \dots, 0, x_k, 0, \dots, 0)^T$
  - x_B becomes

$$
\begin{aligned}
x_B &= \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in J} (\mathbf{B}^{-1}\mathbf{a}_j) x_j \\
&= \mathbf{B}^{-1}\mathbf{b} - x_k(\mathbf{B}^{-1}\mathbf{a}_k) \\
&= \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} - x_k \begin{pmatrix} y_{1,k} \\ y_{2,k} \\ \vdots \\ y_{m,k} \end{pmatrix} \geq \mathbf{0}
\end{aligned}
$$

第k列.

where $\mathbf{B}^{-1}\mathbf{b} = \boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_m)^T$ and $\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$

  - z becomes $\mathbf{c}_B^T \boldsymbol{\beta} - x_k(z_k - c_k)$.
- minimum ratio: $x_k \leq \min \left\{ \frac{\beta_i}{y_{i,k}} \mid y_{i,k} > 0 \right\} = \frac{\beta_r}{y_{r,k}}$

Simplex in tabular

$$\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} \tag{34}$$

$$z + (\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T)\mathbf{x}_N = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} \tag{35}$$

	z	$\mathbf{x}_N$	$\mathbf{x}_B$	RHS	
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$	(rows 1- m)
z	1	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T$	$\mathbf{0}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$	(row 0)

	x_1	x_2		x_n	RHS
$x_{B,1}$	$y_{1,1}$	$y_{1,2}$	$\cdots$	$y_{1,n}$	β_1
$x_{B,2}$	$y_{2,1}$	$y_{2,2}$	$\cdots$	$y_{2,n}$	β_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$x_{B,m}$	$y_{m,1}$	$y_{m,2}$	$\cdots$	$y_{m,n}$	β_m
z	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	$\mathbf{c}_B^T\boldsymbol{\beta}$

Pivot operation

A new canonical representation is derived by solving for z and the new set of basic variables in terms of the new set of nonbasic variable. This process is called **pivot** and results in a new canonical representation. This process can simply be demonstrated using tableau. $y_{r,k}$ is called the *pivot element*, row r is called the *pivot row*, column k is called the pivot column. Use elementary row operations on the old tableau so that the column associated with x_k in the new tableau consists of all zero elements except for a 1 at the pivot position $y_{r,k}$.

	k	j
r	$y^*_{r,k}$	$y_{r,j}$
i	$y_{i,k}$	$y_{i,j}$

becomes

	k	j
r	1	$y_{r,j}/y_{r,k}$
i	0	$y_{i,j} - y_{i,k}y_{r,j}/y_{r,k}$

Example

Consider the LP problem:

$$
\begin{array}{ll}
\text{Max} & z = 3x_1 + x_2 + 3x_3 \\
\text{Subject to} & 2x_1 + x_2 + x_3 \leq 2 \\
 & x_1 + 2x_2 + 3x_3 \leq 5 \\
 & 2x_1 + 2x_2 + x_3 \leq 6 \\
 & x_1, x_2, x_3 \geq 0
\end{array}
$$

$z_i - c_i$
$\downarrow$
$c_B^T B^{-1} a_i$

By adding slack variables x_4, x_5 and x_6, we have the following initial tableau.
Tableau 1:

pivot $B^{-1}N$

x_4, x_5, x_6 一定构成一组基变量 x_B.

	x_1	x_2	x_3	x_4	x_5	I x_6	**b**	Ratio
x_4	2	1^*	1	1	0	0	2	$\frac{2}{1} = 2^*$ ✓
x_5	1	2	3	0	1	0	5	$\frac{5}{2} = 2.5$
x_6	2	2	1	0	0	1	6	$\frac{6}{2} = 3$
z	-3	-1	-3	0	0	0	0	

Initial tableau, current BFS is $\mathbf{x} = [0, 0, 0, 2, 5, 6]^T$ and $z = 0$.

Example

We choose x_2 as the entering variable to illustrate that any nonbasic variable with negative coefficient can be chosen as entering variable. The smallest ratio is given by x_4 row. Thus x_4 is the leaving variable.

Tableau 2:

as entering variable. 只有大于0的才需要作 Ratio; 目的是变号.

		x_1	x_2	x_3	x_4	x_5	x_6	**b**	Ratio
①	x_2	2	1	1	1	0	0	2	$\frac{2}{1} = 2$
②→2① → ②	x_5	-3	0	1^*	-2	1	0	1	$\frac{1}{1} = 1^*$
③ - 2① → ③	x_6	-2	0	-1	-2	0	1	2	$-$
④ + ① → ④	z	-1	0	-2	1	0	0	2	$z_i - c_i$

Current BFS is $\mathbf{x} = [0, 2, 0, 0, 1, 2]^T$ and $z = 2$.

Example

Tableau 3:

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}$	Ratio
x_2	5^*	1	0	3	-1	0	1	$\frac{1}{5}$
x_3	-3	0	1	-2	1	0	1	—
x_6	-5	0	0	-4	1	1	3	—
z	-7	0	0	-3	2	0	4	

Current BFS is $\mathbf{x} = [0, 1, 1, 0, 0, 3]^T$ and $z = 4$.

Example

$$\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$$
$$z + (\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T)\mathbf{x}_N = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$$

	z	$\mathbf{x}_N$	$\mathbf{x}_B$	RHS	
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$	(rows 1- m)
z	1	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T$	$\mathbf{0}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$	(row 0)

Tableau 4:

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}$	
x_1	1	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$	–
x_3	0	$\frac{3}{5}$	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{8}{5}$	–
x_6	0	1	0	-1	0	1	4	–
z	0	$\frac{7}{5}$	0	$\frac{6}{5}$	$\frac{3}{5}$	0	$\frac{27}{5}$	→ 定

直到都为正数.

$z + a + b + c = \frac{27}{5}$

$z \leq \frac{27}{5}$.

Optimal tableau, optimal BFS is $\mathbf{x} = [1/5, 0, 8/5, 0, 0, 4]^T$ and $z^* = 27/5$. We note that the extreme point sequence that the simplex method passes through are $\{x_4, x_5, x_6\} \to \{x_2, x_5, x_6\} \to \{x_2, x_3, x_6\} \to \{x_1, x_3, x_6\}$.

Big-M Method

Big-M method

Consider the following LP.

$$\begin{array}{ll} \text{Max} & z = x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 4 \\ & x_1 + 2x_2 = 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Putting into standard form by adding the surplus variable x_3, Tableau 0:

	x_1	x_2	x_3	**b**
	2	1	-1	4
	1	2	0	6
z	-1	-1	0	0

	x_1	x_2	x_3	b
x_2	$\frac{1}{2}$	1	0	3
x_3	$-\frac{3}{2}$	0	1	-1
z	$-\frac{1}{2}$	0	0	3

How to select starting basic variables?

Big-M method

If x_2 and x_3 are chosen as starting basic variables, then we have Tableau 1′:

	x_1	x_2	x_3	$\mathbf{b}$
x_2	$\frac{1}{2}$	1	0	3
x_3	$-\frac{3}{2}$	0	1	-1
z	$-\frac{1}{2}$	0	0	3

This illustrates that the starting basic solution may sometimes be infeasible.

Big-M method

- A standard form of LP:

$$\begin{aligned} \text{Max} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

- Add suitable number of artificial variables $x_{a1}, x_{a2}, \cdots, x_{am}$ to it to get a starting identity matrix. The corresponding prices for the artificial variables are $-M$ for maximization problem, where M is sufficiently large.

BFS $\begin{pmatrix} x \\ x_a \end{pmatrix}$

x_a is 0 $\Rightarrow$ x is FS origin.

$$\begin{aligned} \text{Max} \quad & z = \mathbf{c}^T\mathbf{x} - M \cdot \mathbf{1}^T\mathbf{x}_a \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{I}_m\mathbf{x}_a = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

where $\mathbf{x}_a = (x_{a1}, x_{a2}, \cdots, x_{am})^T$ and $\mathbf{1}$ is the vector of all ones. We observe that $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_a = \mathbf{b}$ is a feasible starting BFS. Moreover, any solution to $\mathbf{A}\mathbf{x} + \mathbf{I}_m\mathbf{x}_a = \mathbf{b}$ which is also a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ must have $\mathbf{x}_a = \mathbf{0}$. Thus, we have to drive $\mathbf{x}_a = \mathbf{0}$ if possible.

Big-M method

$$
\begin{array}{ll}
\text{Max} & z = x_1 + x_2 \\
\text{subject to} & 2x_1 + x_2 \geq 4 \\
& x_1 + 2x_2 = 6 \\
& x_1, x_2 \geq 0
\end{array}
$$

形式转化需要.　　　　　　　　单位阵的补充.

Introducing surplus variable x_3 and artificial variables x_4 and x_5 yields,

人工变量.

$$
\left\{
\begin{array}{rcrcrcrcrcl}
2x_1 & + & x_2 & -x_3 & + & x_4 & & & = & 4 \\
x_1 & + & 2x_2 & & & & + & x_5 & = & 6 \\
z - x_1 & - & x_2 & & + & Mx_4 & + & Mx_5 & = & 0
\end{array}
\right.
$$

可以直接作为单位阵的一部分.

Big-M method

Now the columns corresponding to x_4 and x_5 form an identity matrix. In tableau form, we have

目的: 人工变量为0时才能保持问题性质不变

引入 M 无限大 (or 无限小)

max z

min z.

为了保持Max (Min) 人工变量 = 0.

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	2	1	-1	1	0	4
x_5	1	2	0	0	1	6
z	-1	-1	0	M	M	0

Notice that in the x_0 row, the reduced cost coefficients that correspond to the basic variables x_4 and x_5 are not zero.

Step 1: 消除 M.

		x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$	
①	x_4	2^*	1	-1	1	0	4	$\frac{4}{2}$
②	x_5	1	2	0	0	1	6	$\frac{6}{1}$
③	z	$-(1+3M)$	$-(1+3M)$	M	0	0	$-10M$	

③ - M① - M②

Big-M method

We note that once an artificial variable becomes non-basic, it can be dropped from consideration in subsequent calculations.

	x_1	x_2	x_3	x_5	$\mathbf{b}$
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_5	0	$\frac{3}{2}^*$	$\frac{1}{2}$	1	4
z	0	$-\frac{1+3M}{2}$	$-\frac{1+M}{2}$	0	$2-4M$

After we eliminate all the artificial variables we have

	x_1	x_2	x_3	$\mathbf{b}$
x_1	1	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_2	0	1	$\frac{1}{3}^*$	$\frac{8}{3}$
z	0	0	$-\frac{1}{3}$	$\frac{10}{3}$

Big-M method

After one iteration, we get the final optimal tableau.

	x_1	x_2	x_3	$\mathbf{b}$
x_1	1	2	0	6
x_3	0	3	1	8
z	0	1	6	6

Thus the optimal solution is $\mathbf{x}^* = (6, 0, 8)^T$ with $z^* = 6$.

Exercise

Using big-M method to solve the following LPP:

$$\begin{aligned}
\text{min} \quad & z = x_1 + x_2 - 3x_3 \\
\text{subject to} \quad & x_1 - 2x_2 + x_3 \leq 11 \\
& 2x_1 + x_2 - 4x_3 \geq 3 \\
& x_1 - 2x_3 = 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

$$
\begin{aligned}
\text{min} \quad & z = x_1 + x_2 - 3x_3 \\
\text{subject to} \quad & x_1 - 2x_2 + x_3 \leq 11 \\
& 2x_1 + x_2 - 4x_3 \geq 3 \\
& x_1 - 2x_3 = 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

max $z = -x_1 - x_2 + 3x_3$

$x_1 - 2x_2 + x_3 + x_4 = 11$

$-2x_1 - x_2 + 4x_3 + x_5 = -3$

$x_1 - 2x_3 = 1$

$x_1, x_2, x_3 \geq 0.$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	-2	3	1	0	0	11
x_5	-2	-1	4	0	1	0	-3
x_6	1	0	-2	0	0	1	1
z	1	1	-3	M	M	M	0

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	-2	3	1	0	0	11
x_5	-2	-1	4	0	1	0	-3
x_6	1	0	-2	0	0	1	1
z	1	1	-3	M	M	M	0

$-3-3M-4M+2M$

	x_1	x_2	x_3	x_4	x_5	x_6		
x_4	1	-2	3	1	0	0	11	$\frac{11}{3}$
x_5	-2	-1	4	0	1	0	-3	$-\frac{3}{4}$ ✓.
x_6	1	0	-2	0	0	1	1.	
z	1	$1+3M$	$-3-5M^*$	0	0	0	$-9M$	

Two-phase Method

Two-phase method

- The big-M method is sensitive to round-off error when being implemented on computers. The two-phase method is used to circumvent this difficulty.
- Two-phase method
  1. PHASE I: (Search for a Starting BFS)

     先不管原问题 最优即 $x_a=0$ ↓ 返回了原问题.

$$\begin{array}{lr} \text{Min} & z = \mathbf{1}^T \mathbf{x}_a \\ \text{subject to} & \mathbf{A}\mathbf{x} + \mathbf{I}_m \mathbf{x}_a = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

     若 $x_a > 0$ 则无可行解.

     Since $\mathbf{b} \geq \mathbf{0}$, the initial BFS satisfies $\mathbf{x}_a \geq \mathbf{0}$. Notice that $z^* \geq 0$ and the possible minimum value of z^* is zero.
  2. PHASE II: (Conclude with an Optimal BFS)

Two-phase method

At the end of Phase I, i.e. when the optimality condition is satisfied or $z^* = 0$, we have one of the following two possibilities:

1. $z^* > 0$, in this case, no feasible solution exists for our original problem.
2. $z^* = 0$, in this case, all artificial variables equal zero, i.e. we have found a BFS to the original problem.

Two-phase method

Consider the following LP.

$$\begin{array}{lrcrcrcrcrcl}
\text{Min} & & & & z = -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\
\text{subject to} & -x_1 & + & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & 7 \\
& -x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & + & x_5 & = & 6 \\
& -x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & x_5 & = & 4 \\
\end{array}$$

$$x_1 \text{ free}, x_2, x_3, x_4, x_5 \geq 0.$$

Since x_1 is free, it can be eliminated by solving for x_1 in terms of the other variables from the first equation and substituting everywhere else. This can be done nicely using our pivot operation on the following simplex tableau:

x_1	x_2	x_3	x_4	x_5	**b**
-1^*	1	2	1	2	7
-1	2	3	1	1	6
-1	1	1	2	1	4
-2	4	7	1	5	0

Initial tableau

Two-phase method

不需要变换 x_1，消除 x_1 的影响

We select any non-zero element in the first column as our pivot element - this will eliminate x_1 from all other rows:

x_1	x_2	x_3	x_4	x_5	**b**	
-1^*	1	2	1	2	7	$\leftarrow (*)$
0	-1	-1	0	1	1	
0	0	1	-1	1	3	
0	2	3	-1	1	-14	

Equivalent Problem

Saving the first row $(*)$ for future reference only, we carry on only the sub-tableau with the first row *and* the first column deleted. There is no obvious basic feasible solution, so we use the two-phase method:

Two-phase method

首先要消除y_1, y_2：将其从x_B变为x_N → $\underbrace{y_1, y_2}_{=0.}$

After making $\mathbf{b} \geq \mathbf{0}$, we introduce artificial variables $y_1 \geq 0$ and $y_2 \geq 0$ to give the artificial problem:

x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{b}$
-1	-1	0	1	1	0	1
0	1	-1	1	0	1	3
0	0	0	0	1	1	0

Initial Tableau for Phase I

$$\mathbf{c}_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Transforming the last row to give a tableau in canonical form, we get

	x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{b}$
y_1	-1	-1	0	1^*	1	0	1
y_2	0	1	-1	1	0	1	3
z	1	0	1	-2	0	0	-4

First tableau for Phase I

We carry out the pivot operations with the indicated pivot elements:

	x_2	x_3	x_4	x_5	y_1	y_2	**b**
x_5	-1	-1	0	1	1	0	1
y_2	1^*	2	-1	0	-1	1	2
z	-1	-2	1	0	2	0	-2

Second tableau for Phase I

	x_2	x_3	x_4	x_5	y_1	y_2	**b**
x_5	0	1	-1	1	0	1	3
x_2	1	2	-1	0	-1	1	2
z	0	0	0	0	1	1	0

Final tableau for Phase I

Two-phase method

At the end of Phase I, we go back to the equivalent reduced problem (i.e. discarding the artificial variables y_1, y_2):

	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_5	0	1	-1	1	3
x_2	1	2	-1	0	2
z	2	3	-1	1	-14

Initial Tableau for Phase II

This is transform into

	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_5	0	1	-1	1	3
x_2	1	2^*	-1	0	2
z	0	-2	2	0	-21

Initial Tableau for Phase II

Two-phase method

Pivoting as shown gives

	x_2	x_3	x_4	x_5	**b**
x_5	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1	2
x_3	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1
z	1	0	1	0	-19

Final Tableau for Phase II

The solution $x_3 = 1$, $x_5 = 2$ can be inserted in the expression $(*)$ for x_1 giving

$$x_1 = -7 + 2(1) + 2(2) = -1.$$

Thus the final solution is $x^* = [-1, 0, 1, 0, 2]^T$ with $z^* = 19$.

Exercise

Use two-phase method to solve the following LPPs

(a)

$$\begin{aligned} \text{Max} \quad & x_1 - x_2 + x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 2x_3 \leq 4 \\ & 2x_1 - 3x_2 + x_3 \leq -5 \\ & -x_1 + x_2 - 2x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \text{Min} \quad & x_0 = 4x_1 + 3x_3 \\ \text{subject to} \quad & 3x_1 + 6x_2 + 3x_3 - 4x_4 = 12 \\ & 2x_1 + x_3 = 4 \\ & 3x_1 - 6x_2 + 4x_4 = 0 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

and answer the following question: Is the optimal solution unique? If it is not unique, find all optimal solutions.

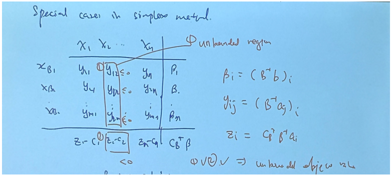
Special Cases in Simplex Method

Special cases in simplex method

two-phase method: $z_0 > 0$ ($\exists x \in x_a \neq 0$)

big-M method: $x_a \neq 0$

- no feasible solution
- unbounded solutions
- infinite number of optimal solution
- degeneracy and cycling



No feasible solution

Consider the following linear programming problem.

$$
\begin{array}{llcccc}
\text{Max} & & x_0 = 2x_1 + x_2 & & & \\
\text{subject to} & -x_1 & + & x_2 & \geq & 2 \\
& x_1 & + & x_2 & \leq & 1 \\
& & x_1, x_2 \geq 0 & & &
\end{array}
$$

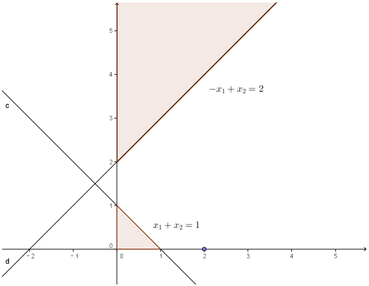


Figure: No Feasible Region

No feasible solution

In terms of the methods of artificial variable techniques, the solution at optimality could include one or more artificial variables at a positive level (i.e. as a non-zero basic variable). In such a case the corresponding constraint is violated and the artificial variable cannot be driven out of the basis. The feasible region is thus empty.

Example

Big-M method:
Using a surplus variable x_3, an artificial variable x_4 and a slack variable x_5, the augmented system is:

$$\begin{array}{rcccccccl} -x_1 & + & x_2 & -x_3 & + & x_4 & & = & 2 \\ x_1 & + & x_2 & & & & +x_5 & = & 1 \\ x_0 - 2x_1 & - & x_2 & & + & Mx_4 & & = & 0 \end{array}$$

We will show the artificial variable is not zero in an optimal solution.

No feasible solution

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	-1	1	-1	1	0	2
x_5	1	1	0	0	1	1
x_0	-2	-1	0	M	0	0

After elimination of the M in the x_4 column, we have the initial tableau:

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	-1	1	-1	1	0	2
x_5	1	1^*	0	0	1	1
x_0	$-2+M$	$-1-M$	M	0	0	$-2M$

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	-2	0	-1	1	-1	1
x_2	1	1	0	0	1	1
x_0	$-1+2M$	0	M	0	$1+M$	$1-M$

No feasible solution

Since M is a very large number, $-1 + 2M$ is positive. Hence all entries in the x_0 row are nonnegative. Thus we have reached an optimal point. However, we see that the artificial variable $x_4 = 1$, which is not zero. That means that the solution just found is not a solution to our original problem. Indeed the $\mathbf{x}$ that satisfies $\mathbf{Ax} + \mathbf{Ix}_a = \mathbf{b}$ with $\mathbf{x}_a \neq \mathbf{0}$ is not a solution to $\mathbf{Ax} = \mathbf{b}$.

Unbounded solutions

Theorem

Consider an LPP in feasible canonical form. If in the simplex tableau, there exists a nonbasic variable x_j such that $y_{ij} \leq 0$ for all $i = 1, 2, \cdots, m$, i.e. all entries in the x_j column are non positive, then the feasible region is unbounded. If moreover that $z_j - c_j < 0$, then there exists a feasible solution with at most $m + 1$ variables nonzero and the corresponding value of the objective function can be set arbitrarily large.

Proof

Let $\mathbf{x}_B$ be the current BFS with $\mathbf{B}\mathbf{x}_B = \mathbf{b}$. Let the columns of $\mathbf{B}$ be denoted by $\mathbf{b}_i$. Then we have

$$\mathbf{B}\mathbf{x}_B = \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i = \mathbf{b}.$$

$m \times m$ $\quad$ $m \times 1$

Let $\mathbf{a}_j$ be the column of $\mathbf{A}$ that corresponds to the variable x_j. We have

$$\mathbf{a}_j = \mathbf{B}\mathbf{y}_j = \sum_{i=1}^{m} y_{ij}\mathbf{b}_i.$$

$B^{-1}[\ ]$

Hence for all $\theta > 0$, we have

方法：构造了一组新的可行解.

$$\begin{aligned}
\mathbf{b} &= \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i - \theta\mathbf{a}_j + \theta\mathbf{a}_j \\
&= \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i - \theta\sum_{i=1}^{m} y_{ij}\mathbf{b}_i + \theta\mathbf{a}_j \\
&= \sum_{i=1}^{m} \underbrace{(x_{B_i} - \theta y_{ij})}_{\text{基变量}}\mathbf{b}_i + \underbrace{\theta}_{\text{非基变量}}\mathbf{a}_j.
\end{aligned}$$

当θ↑ $\quad$ $y_{ij} < 0$.

Proof

Thus we obtain a new nonbasic solution of $m+1$ nonzero variables. This solution is feasible as $x_{B_i} - \theta y_{ij} \geq 0$, for all i.
Moreover, the value of x_j, which is equal to θ, can be set arbitrarily large, indicating that the feasible region is unbounded in the x_j direction.
If moreover that $c_j > z_j$, then the value of the objective function can be set arbitrarily large since

新的z值.

$$
\begin{aligned}
\hat{z} &= \sum_{i=1}^{m}(x_{B_i} - \theta y_{ij})c_{B_i} + \theta c_j \\
&= \sum_{i=1}^{m} x_{B_i} c_{B_i} - \theta \sum_{i=1}^{m} y_{ij} c_{B_i} + \theta c_j \\
&= \mathbf{c}_B \mathbf{x}_B - \theta \mathbf{c}_B^T \mathbf{y}_j + \theta c_j \\
&= z - \theta z_j + \theta c_j \\
&= z + \theta(c_j - z_j).
\end{aligned}
$$

Example

This is an example where the feasible region and the optimal value of the objective function are unbounded. Consider the LPP

$$
\begin{array}{rrcrcl}
\text{Max} & & & x_0 = 2x_1 + x_2 & & \\
\text{subject to} & x_1 & - & x_2 & \leq & 10 \\
 & 2x_1 & - & x_2 & \leq & 40 \\
 & & & x_1, x_2 \geq 0 & &
\end{array}
$$

The initial tableau is

	x_1	x_2	x_3	x_4	**b**
x_3	1	−1	1	0	10
x_4	2	−1	0	1	40
x_0	−2	−1	0	0	0

No positive ratio exists in ~~x_3~~ x_2 column. Hence ~~x_3~~ x_2 can be increased without bound while maintaining feasibility.

Example

Consider the LPP

$$\begin{array}{rrcrcr} \text{Max} & & x_0 = 6x_1 - 2x_2 & & & \\ \text{subject to} & 2x_1 & - & x_2 & \le & 2 \\ & x_1 & & & \le & 4 \\ & & x_1, x_2 \ge 0 & & & \end{array}$$

The computation goes as follows

	x_1	x_2	x_3	x_4	**b**
x_3	2^*	-1	1	0	2
x_4	1	0	0	1	4
x_0	-6	2	0	0	0

$$\downarrow$$

Example

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	1
x_4	0	$\frac{1}{2}^*$	$-\frac{1}{2}$	1	3
x_0	0	-1	3	0	6

$$\downarrow$$

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_1	1	0	0	1	4
x_2	0	1	-1	2	6
x_0	0	0	2	2	12

Optimal tableau

Infinite number of optimal solutions

- Zero reduced cost coefficients for non-basic variables at optimality indicate alternative optimal solutions, since if we pivot in those columns, x_0 value remains the same after a change of basis for a different BFS.
- The set of alternative optimal solutions is given by the convex combination of optimal BFS solutions.
- Suppose $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^p$ are extreme point optimal solutions, then $\mathbf{x} = \sum_{k=1}^{p} \lambda_k \mathbf{x}^k$, where $0 \leq \lambda_k \leq 1$ and $\sum_{k=1}^{p} \lambda_k = 1$ is also an optimal solution. In fact, if $\mathbf{c}^T \mathbf{x}^k = z_0$ for $1 \leq k \leq p$, then

$$\mathbf{c}^T \mathbf{x} = \sum_{k=1}^{p} \lambda_k \mathbf{c}^T \mathbf{x}^k = \sum_{k=1}^{p} \lambda_k z_0 = z_0.$$

存在 x_N 在 z 中的一行为 0.

Infinite number of optimal solutions

Consider

$$\begin{array}{rcccl} \text{Max} & & x_0 = 4x_1 + 14x_2 & & \\ \text{subject to} & 2x_1 & + & 7x_2 & \leq & 21 \\ & 7x_1 & + & 2x_2 & \leq & 21 \\ & & x_1, x_2 \geq 0 & & \end{array}$$

	x_1	x_2	x_3	x_4	**b**
x_3	2	7^*	1	0	21
x_4	7	2	0	1	21
x_0	-4	-14	0	0	0

$$\downarrow$$

	x_1	x_2	x_3	x_4	**b**
x_2	$\frac{2}{7}$	1	$\frac{1}{7}$	0	3
x_4	$\frac{45}{7}^*$	0	$-\frac{2}{7}$	1	15
x_0	$\boxed{0}$	0	2	0	42

(0, 3, 0, 15).

$\downharpoonleft\upharpoonright$ 结果不会发生改变.

	x_1	x_2	x_3	x_4	**b**
x_2	0	1	$\frac{7}{45}$	$-\frac{2}{45}$	$\frac{7}{3}$
x_1	1	0	$-\frac{2}{45}$	$\frac{7}{45}^*$	$\frac{7}{3}$
x_0	0	0	2	$\boxed{0}$	42

只需 $x_3=0$

存在无穷多解.

Degeneracy and cycling

存在 $x_B = 0$.

- Degenerate basic solutions are basic solutions with one or more basic variables at zero level.
- Degeneracy occurs when one or more of the constraints are redundant.

Consider the following LLP.

$$\begin{array}{rl} \text{Max} & x_0 = 2x_1 + x_2 \\ \text{Subject to} & 4x_1 + 3x_2 \leq 12 \\ & 4x_1 + x_2 \leq 8 \\ & 4x_1 - x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	4	3	1	0	0	12
x_4	4^{**}	1	0	1	0	8
x_5	4^{*}	-1	0	0	1	8
x_0	-2	-1	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	4	1	0	-1	4
x_4	0	2^*	0	1	-1	0
x_1	1	$-1/4$	0	0	$1/4$	2
x_0	0	$-3/2$	0	0	$1/2$	4

Degenerate Vertex $\{x_4 = 0 \text{ and basic}\}$

$\downarrow$

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	0	1	-2	1^*	4
x_2	0	1	0	$1/2$	$-1/2$	0
x_1	1	0	0	$1/8$	$1/8$	2
x_0	0	0	0	$3/4$	$-1/4$	4

Degenerate Vertex $\{x_2 = 0 \text{ and basic}\}$

$\downarrow$

	x_1	x_2	x_3	x_4	x_5	**b**
x_5	0	0	1	-2	1	4
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_0	0	0	$1/4$	$1/4$	0	5

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	2^{**}	1	-1	0	4
x_1	1	$1/4$	0	$1/4$	0	2
x_5	0	-2	0	-1	1	0
x_0	0	$-1/2$	0	$1/2$	0	4

Degenerate Vertex $\{x_5 = 0 \text{ and basic}\}$

$\downarrow$

	x_1	x_2	x_3	x_4	x_5	**b**
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_5	0	0	1	-2	1	4
x_0	0	0	$1/4$	$1/4$	0	5

多个可行主元

- degeneracy guarantees the existence of more than one feasible pivot element, i.e. *tie-ratios* exist. For example, in the first tableau, the ratios for variables x_4 and x_5 are both equal to 2. 多比值相等.
- When an LP is degenerate, i.e. its feasible region possesses degenerate vertices, **cycling** may occur as follows:
  - Suppose the current basis is $\mathbf{B}$ and such that this basis $\mathbf{B}$ yields a degenerate BFS.
  - moving from a degenerate vertex (BFS) to another degenerate vertex does not affect (i.e. increase or decrease) the objective function value
  - It is then possible for the simplex procedure to start from the current (degenerate) basis $\mathbf{B}$, and after some p iterations, to return to $\mathbf{B}$ with no change in the objective function value as long as all vertices in-between are degenerate.
  - a further p iterations will again bring us back to this same basis $\mathbf{B}$.

Example

$$\begin{array}{rl} \text{Max} & x_0 = 20x_1 + \frac{1}{2}x_2 - 6x_3 + \frac{3}{4}x_4 \\ \text{Subject to} & x_1 \le 2 \\ & 8x_1 - x_2 + 9x_3 + \frac{1}{4}x_4 \le 11 \\ & 12x_1 - \frac{1}{2}x_2 + 3x_3 + \frac{1}{2}x_4 \le 24 \\ & x_2 \le 1 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

Tableau 0:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_5	1^*	0	0	0	1	0	0	0	2
x_6	8	-1	9	$1/4$	0	1	0	0	16
x_7	12	$-1/2$	3	$1/2$	0	0	1	0	24
x_8	0	1	0	0	0	0	0	1	1
x_0	-20	$-1/2$	6	$-3/4$	0	0	0	0	0

$\frac{2}{1} = 2$

$\frac{16}{8} = 2$

$\frac{24}{12} = 2.$

Example

Tableau 1:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\mathbf{b}$
x_1	1	0	0	0	1	0	0	0	2
x_6	0	-1	9	$1/4^*$	-8	1	0	0	0
x_7	0	$-1/2$	3	$1/2$	-12	0	1	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$-1/2$	6	$-3/4$	20	0	0	0	40

Tableau 2:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\mathbf{b}$
x_1	1	0	0	0	1	0	0	0	2
x_4	0	-4	36	1	-32	4	0	0	0
x_7	0	$3/2$	-15	0	4^*	-2	1	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$-7/2$	33	0	-4	3	0	0	40

Example

Tableau 3:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	$-3/8$	$15/4$	0	0	$1/2$	$-1/4$	0	2
x_4	0	8^*	-84	1	0	-12	8	0	0
x_5	0	$3/8$	$-15/4$	0	1	$-1/2$	$1/4$	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	-2	18	0	0	1	1	0	40

Tableau 4:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	$-3/16$	$3/64$	0	$-1/16$	$1/8$	0	2
x_2	0	1	$-21/2$	$1/8$	0	$-3/2$	1	0	0
x_5	0	0	$3/16^*$	$-3/64$	1	$1/16$	$-1/8$	0	0
x_8	0	0	$21/2$	$-1/8$	0	$3/2$	-1	1	1
x_0	0	0	-3	$1/4$	0	-2	3	0	40

Example

Tableau 5:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_2	0	1	0	$-5/2$	56	2^*	-6	0	0
x_3	0	0	1	$-1/4$	$16/3$	$1/3$	$-2/3$	0	0
x_8	0	0	0	$5/2$	-56	-2	6	1	1
x_0	0	0	0	$-1/2$	16	-1	1	0	40

Tableau 6:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_6	0	$1/2$	0	$-5/4$	28	1	-3	0	0
x_3	0	$-1/6$	1	$1/6$	-4	0	$1/3^*$	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$1/2$	0	$-7/4$	44	0	-2	0	40

Example

Tableau 7:

与表格1相同了,

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_6	0	-1	9	$1/4^*$	-8	1	0	0	0
x_7	0	$-1/2$	3	$1/2$	-12	0	1	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$-1/2$	6	$-3/4$	20	0	0	0	40

degenerate

不会再发生改变.

Note that Tableau 1 is the same as Tableau 7. Thus starting from the basis $\mathbf{B} = \{x_1, x_6, x_7, x_8\}$, we have moved to $\{x_1, x_4, x_7, x_8\}$, to $\{x_1, x_4, x_5, x_8\}$, to $\{x_1, x_2, x_5, x_8\}$, to $\{x_1, x_2, x_3, x_8\}$ to $\{x_1, x_6, x_3, x_8\}$ and finally back to $\{x_1, x_6, x_7, x_8\}$ in six iterations, or a cycle of period $p = 6$. To break the cycle, bring in x_4 and remove x_7. Then the next iteration yields the optimal solution $\mathbf{x}^* = [2, 1, 0, 1, 0, 3/4, 0, 0]^T$ with $x_0^* = 41.25$.

Bland's rule

- To get out of cycling, one way is to try a different pivot element.
- Bland's rule for selecting entering and leaving basic variables
  1. The variables are ordered $\boxed{\text{arbitrarily}}$ from x_1 to x_n without loss of generality.
  2. Among all the nonbasic variables with negative coefficients in the objective function row, the one with the smallest index is chosen to enter.
  3. The leaving basic variables is the variable with the smallest index of all the variables who tie in the usual minimum ratio test.
- Proof are provided in the BB system. Please read it if you are interested.