没发答案，上传的是本人考时拍的，分数 3/3。

Computer Science

Quiz 2, Spring 2025

The quiz needs to be written in English. Quiz in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1. Use generating functions to determine the number of ways to insert tokens worth \$1, \$5, and \$10 into a vending machine to pay for an item that costs r dollars in both the cases

(a) when the order in which the tokens are inserted does not matter;

(b) when the order in which the tokens are inserted does matter.

You need to provide (i) the generating function and (ii) the term associated with the answer to the above question (e.g., "the coefficient of term x^{2}") for each case. You do NOT need to provide the derivation details.

(a) \$1 tokens: $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$

\$5 tokens: $1+x^5+x^{10}+\cdots=\frac{1}{1-x^5}$

\$10 tokens: $1+x^{10}+x^{20}+\cdots=\frac{1}{1-x^{10}}$

(i) ∴ the generating function is

$G(x)=\frac{1}{(1-x)(1-x^5)(1-x^{10})}$

(ii) the coefficient of term x^r of $G(x)$

(b) \$1 tokens: x

\$5 tokens: x^5

\$10 tokens: x^{10}

(i) the generating function is

$G'(x)=\frac{1}{1-(x+x^5+x^{10})}$

(ii) the coefficient of term x^r of $G'(x)$

Q. 2. How many positive integers less than 1,000,000 have the sum of their digits equal to 19? Please use combinations. [Note: Please keep your answer as $C(m,n)$ expressions.]

Suppose the integer is $(x_1x_2x_3x_4x_5x_6)_{10}=x_1\cdot 10^5+x_2\cdot 10^4+x_3\cdot 10^3+x_4\cdot 10^2+x_5\cdot 10^1+x_6\cdot 1$

then $x_1+x_2+x_3+x_4+x_5+x_6=19$, $0\le x_i\le 9$, $x_i\in\mathbb{Z}$ $(i=1,2,3,4,5,6)$

Suppose $y_i=x_i+1$ $(i=1,2,3,4,5,6)$, then $\sum_{i=1}^{6}y_i=25$, $1\le y_i\le 10$.

Only consider $y_i\ge 1$, then # of combinations will be C_{24}^{5}

Consider the case that $\exists\, k\in\{1,2,\cdots,6\}$, $x_k\ge 10$

k is unique for $\sum_{i=1}^{k}x_i=19<10+10$, and k has 6 choices

WLOG, assume $k=1$, then $\sum_{i=2}^{6}x_i\le 9\Rightarrow\sum_{i=2}^{k}y_i\le 14$

the # total cases will be $C_{13}^{4}+C_{12}^{4}+\cdots+C_5^4+C_4^4=C_{14}^{5}$

∴ the final answer will be $C_{24}^{5}-\cancel{6\times(C_{13}^{4}+C_{12}^{4}+\cdots+C_{4}^{4})}$ $6\times C_{14}^{5}$

Q. 3. Consider relation R on $\mathbf{Z} \times \mathbf{Z}$ defined by $((a,b),(c,d)) \in R$ if and only if $a+d = b+c$. Prove or disprove this relation is an equivalence relation.

Prove:

① reflexive: ~~$a=c, b=d$~~ when $(a,b)=(c,d)$. $a=c, b=d$ ~~that~~, we have $a+d=b+c$

so ~~$\forall$~~ for $\forall$ $((a,b),(a,b))$, $((a,b),(a,b)) \in R$. ~~$\therefore$~~ so R is reflexive

② symmetric: ~~$\forall$ when~~ for $\forall$ $((a,b),(c,d)) \in R$, $a+d=b+c$

$\therefore$ ~~$c+b=a$~~ $c+b=d+a$

$\therefore$ $((c,d),(a,b)) \in R$ $\therefore$ R is symmetric

③ transitive: for $\forall$ $((a_1,b_1),(c_1,d_1)) \in R$, $((c_1,d_1),(a_2,b_2)) \in R$

we have $a_1+d_1=b_1+c_1$, $c_1+b_2=a_2+d_1$

$\therefore$ $a_1+d_1+c_1+b_2 = b_1+c_1+a_2+d_1$

$\therefore$ $a_1+b_2=b_1+a_2$

$\therefore$ $((a_1,b_1),(a_2,b_2)) \in R$ $\therefore$ R is transitive.

$\therefore$ R is an equivalence relation.