The Four Fundamental Subspaces (四个基本子空间)

Lecture 10

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The Four Fundamental Subspaces

Introduction

- The previous section dealt with definitions rather than constructions. We know what a basis is, but not how to find one. Now, starting from an explicit description of a subspace, we would like to compute an explicit basis.
- Subspaces can be described in two ways. First, we may be given a set of vectors that span the space. (Example: The columns span the column space.)
- Second, we may be told which conditions the vectors in the space must satisfy. (Example: The nullspace consists of all vectors that satisfy $Ax = 0$.)
- The first description may include useless vectors (dependent columns). The second description may include repeated conditions (dependent rows.) We can not write a basis by inspection, and a systematic procedure is necessary.

Introduction

- The reader can guess what that procedure will be.
- When elimination on A produces an echelon matrix U or a reduced R, we will find a basis for each of the subspaces associated with A.
- Then we have to look at the extreme case of full rank.

Let A be an $m \times n$ matrix:

1. If rank $(A) = m$, then A is said to be of full row rank.
2. If rank $A = n$, then A is of full column rank.
3. If rank $A = m = n$, then A is of full rank.

Full Rank Matrices

For **full rank** matrices:

Theorem

When the rank is as large as possible, $r = n$ or $r = m$ or $r = m = n$, the matrix has a left-inverse B or a right-inverse C or a two-sided A^{-1}.

Four Fundamental Subspaces (四个基本子空间)

To organize the whole discussion, we take each of the four subspaces in turn. Two of them are familiar and the other two are new.

1. The column space of A is denoted by $C(A)$. Its dimension is the rank r.
2. The nullspace of A is denoted by $N(A)$. Its dimension is $n-r$.
3. The row space of A is the column space of A^T. It is $C(A^T)$, and it is spanned by the rows of A. Its dimension is also r.
4. The left nullspace of A is the nullspace of A^T. It contains all vectors y such that $A^Ty=0$, and it is written $N(A^T)$. Its dimension is $m-r$.

Our problem will be to connect the four spaces for U (after elimination) to the four spaces for A.

The Four Fundamental Subspaces

The point about the last two subspaces is that they come from A^T.

Theorem

The nullspace $N(A)$ and row space $C(A^T)$ are subspaces of R^n.
The left nullspace $N(A^T)$ and column space $C(A)$ are subspaces of R^m.

- Describe the four subspaces associated with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Find the dimension and a basis for the fundamental subspaces for

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{came from} \quad A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

For novelty, we take the four subspaces in a more interesting order.

3. The Row Space of A

For every echelon matrix U and R, with r pivots and r nonzero rows:

- The r nonzero rows of an echelon matrix U and a reduced matrix R are linearly independent. So are the r columns that contain pivots.
- **The nonzero rows are a basis, and the row space has dimension r.** That makes it easy to deal with the original matrix A.

Theorem

*The row space of A has the same dimension r as the row space of U, and it has the same bases, because **the row spaces of A and U (and R) are the same**.*

The reason is that each elementary operation leaves the row space unchanged. The rows of U are combinations of the original rows in A.

2. The Nullspace of A

The nullspace of A is the same as the nullspace of U and R. Only r of equations $Ax = 0$ are independent. Choosing the $n-r$ "special solutions" to $Ax = 0$ provides a definite basis for the nullspace:

Theorem

The nullspace $N(A)$ has dimension $n-r$. The "special solutions" are a basis–each free variable is given the value 1, while the other free variables are 0. Then $Ax = 0$ or $Ux = 0$ or $Rx = 0$ gives the pivot variables by back-sustitution.

This is exactly the way we have been solving $Ux = 0$. The nullspace is also called the **kernel** of A, and its dimension $n-r$ is the **nullity** （零度）.

1. The Column Space of A

The column space is sometimes called the **range**. This is consistent with the usual idea of the range, as the set of all possible values $f(x)$; x is in the domain and $f(x)$ is in the range. In our case the function is $f(x) = Ax$.

Theorem

The dimension of the column space $C(A)$ equals the rank r, which also equals the dimension of the row space: ***The number of independent columns equals the number of independent rows****. A basis for $C(A)$ is formed by the r columns of A that correspond, in U, to the columns containing pivots.*

Row Rank Equals Column Rank

Theorem

The row space and the column space have the same dimension:

Row Rank = Column Rank.

Proof. Let us consider

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \end{bmatrix}.$$

Suppose the row rank of A is r, and the column rank of A is r'. We need to show that $r = r'$. Let's first show that $r \leq r'$. Let $v_1, v_2, \cdots, v_s$ be the rows of A, and without loss of generality, we can assume that $v_1, v_2, \cdots, v_r$ is a maximal linearly independent subset of $v_1, v_2, \cdots, v_s$.

Proof.

Since $v_1, v_2, \cdots, v_r$ are linearly independent, then

$$x_1 v_1 + x_2 v_2 + \cdots + c_r v_r = 0$$

has only the zero solution, which is equivalent to say that the following system of linear equations

$$Bx = 0 : \begin{cases} a_{11}x_1 + a_{21}x_2 + \cdots + a_{r1}x_r = 0 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{r2}x_r = 0 \\ \cdots\cdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{rn}x_r = 0 \end{cases}$$

has only the zero solution.

Proof.

It follows that the row rank of the coefficient matrix of the previous system

$$B=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ a_{12} & a_{22} & \cdots & a_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{rn} \end{bmatrix}$$

is greater or equal to r (Otherwise, $Bx=0$ will have nonzero solutions, which is contradictory to the assumption that $Bx=0$ has only the zero solution). Therefore, we can find r linearly independent rows of the above matrix, without loss of generality, we assume that

$$(a_{11},a_{21},\cdots,a_{r1}),(a_{12},a_{22},\cdots,a_{r2}),\cdots,(a_{1r},a_{2r},\cdots,a_{rr})$$

are linearly independent.

Proof.

If we add more components to these vectors, the new vectors

$$(a_{11}, a_{21}, \cdots, a_{r1}, \cdots, a_{s1}), \cdots, \cdots, (a_{1r}, a_{2r}, \cdots, a_{rr}, \cdots, a_{sr})$$

are linearly independent as well. Note that these vectors are the column vectors of A, they are linearly independent, therefore the column rank r' of A is greater or equal to r, in other words, $r' \geq r$.

Similarly, we can prove that $r' \leq r$. In conclusion, $r' = r$.

4. The left nullspace of A= the nullspace of A^T

- If A is an m by n matrix, then A^T is n by m. Its nullspace is a subspace of $\mathbb{R}^m$; the vector y has m components. Written as $y^T A = [0 \cdots 0]$.
- The dimension of this nullspace $N(A^T)$ is easy to find. For any matrix, **the number of pivot variables plus the number of free variables must match the total number of columns.**
  The rank of A plus the nullity of A equals n:

$$\text{dimension of } C(A) + \text{dimension of } N(A) = \text{number of columns.}$$

- This law applies equally to A^T, which has m columns. A^T is just as good a matrix as A. But the dimension of its column space is also r, so

Theorem

The left nullspace $N(A^T)$ has dimension $m - r$.

The Four Fundamental Subspaces

The $m-r$ solutions to $y^TA=0$ are hiding somewhere in elimination. The rows of A combine to produce the $m-r$ zero rows of U. Start from $PA=LU$, or $L^{-1}PA=U$. The last $m-r$ rows of the invertible matrix $L^{-1}P$ must be a basis of y's in the left nullspace–because they multiply A to give the zero rows in U.

Now we know the dimensions of the four spaces. We can summarize them in a table, and it even seems fair to advertise them as the Fundamental Theorem of Linear Algebra.

Fundamental Theorem of Linear Algebra

Theorem

(Fundamental Theorem of Linear Algebra, Part I)

1. $C(A)$= *Column space of* A*; dimension* r*.*
2. $N(A)$= *Nullspace of* A*; dimension* $n-r$*.*
3. $C(A^T)$= *Row space of* A*; dimension* r*.*
4. $N(A^T)$ = *Left nullspace of* A*; dimension* $m-r$*.*

Example 1

$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has $m = n = 2$, and rank $r = 1$.

1. The column space contains all multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The second column is in the same direction and contributes nothing new.
2. The nullspace contains all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. This vector safisfies $Ax = 0$.
3. The row space contains all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We write it as a column vector since strictly speaking it is in the column space of A^T.
4. The left nullspace contains all multiples of $y = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Example 1: Continue

In this example all four subspaces are lines. That is an accident, coming from $r = 1$ and $n - r = 1$ and $m - r = 1$. Figure 2.5 shows that two pairs of lines are perpendicular. That is no accident!

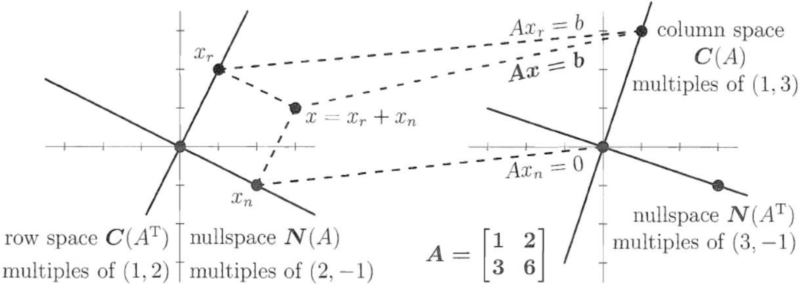

Figure 2.5: The four fundamental subspaces (lines) for the singular matrix A.

Remarks

If you change the last entry of A from 6 to 7, all the dimensions are different.

Left Inverse and Right Inverse

We know that if A has a left-inverse ($BA = I$) and a right-inverse ($AC = I$), then the two inverses are equal. An inverse exists only when the rank is as large as possible.

The rank always satisfies $r \leq m$ and also $r \leq n$. An $m \times n$ matrix can not have more than m independent rows or n independent columns. Only a square matrix can have both $r = n$ and $r = m$, and therefore only a square matrix can achieve both existence and uniqueness. Only a square matrix has a two-sided inverse.

Existence and Uniqueness

- **Existence**: Full row rank $r = m$. $Ax = b$ has at least one solution x for every b if and only if the columns span $\mathbb{R}^m$. Then A has a right-inverse C such that $AC = I_m (m$ by $m)$. This is possible only if $m \leq n$.
- **Uniqueness**: Full column rank $r = n$. $Ax = b$ has at most one solution x for every b if and only if the columns are linearly independent. Then A has an n by m left-inverse B such that $BA = I_n$. This is possible only if $m \geq n$.

formulas for the best left and right inverses

There are simple formulas for the best left and right inverses, if they exist:

One-sided inverses $B = (A^TA)^{-1}A^T$ and $C = A^T(AA^T)^{-1}$.

Example

Consider a simple 2 by 3 matrix of rank 2:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

Find a right inverse of A.

Invertibility

- A rectangular matrix cannot have both existence and uniqueness. If m is different from n, we can not have $r = m$ and $r = n$.
- A square matrix is the opposite. If $m = n$, we cannot have one property without the other.
- The condition for invertibility is full rank: $r = m = n$. In this case, we cannot have one property without the other. A matrix has a left-inverse if and only if it has a right-inverse. There is only one inverse. Existence implies uniqueness and uniqueness implies existence, when the matrix is square.
- Each of these conditions is a necessary and sufficient test:
  1. The columns span $\mathbb{R}^n$, so $Ax = b$ has exactly one solution for every b.
  2. The columns are independent, so $Ax = 0$ has only the solution $x = 0$.

Invertibility

The list can be made much longer, especially if we look ahead to later chapters. Every condition is equivalent to every other, and ensures that A is invertible.

3. The rows of A span $\mathbb{R}^n$.
4. The rows are linearly independent.
5. Elimination can be completed: $PA = LDU$, with all n pivots.
6. The determinant of A is not zero.
7. Zero is not an eigenvalue of A.
8. A^TA is positive definite.

Vandermonde matrix

Given any values $b_1, b_2, \cdots, b_n$, there exists a polynomial of degree $n-1$ interpolating these values: $P(t_i) = b_i$. The point is that we are dealing with a square matrix; the number n of coefficients in $P(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$ matches the number of equations:

$$\textbf{Interpolation } P(t_i) = b_i : \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Vandermonde Matrix

The **Vandermonde matrix** is n by n and full rank. $Ax = b$ always has solution–a polynomial can be passed through any b_i at distinct points t_i. Later we shall actually find the determinant of A; it is not zero.

Matrices of Rank 1

- Every matrix of rank 1 has the simple form $A = uv^T$=column times row.
- The row space and column space are lines–the easiest case.
- Rank 1 matrix can be written as the product of a column vector and a row vector as the following example shows:

$$A = (column)(row) \quad \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}.$$

the product of a 4 by 1 matrix and a 1 by 3 matrix is a 4 by 3 matrix. This product has rank 1. At the same time, the columns are all multiples of the same column vector; the column space shares the same dimension $r = 1$ and reduces to a line.

Two More Examples

Example

A is an m by n matrix of rank r. Suppose there are right-hand sides b for which $Ax = b$ has no solution.

(a) What inequalities ($<$ or $\leq$) must be true between m, n, and r?

(b) How do you know that $A^T y = 0$ has a nonzero solution?

Example

If $AB = 0$, the columns of B are in the nullspace of A. If those vectors are in $\mathbb{R}^n$, prove that $\text{rank}(A) + \text{rank}(B) \leq n$.

Homework Assignment 10

2.4: 3, 5, 6, 9,14,18, 27, 33, 35, 38.