Matrices

Lecture 7

Dept. of Math., SUSTech

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Vector Spaces

matrix

We know that if $v_1, v_2, \cdots, v_n$ is a basis of V and $T : V \to W$ is linear, then the values of $Tv_1, \cdots, Tv_n$ determine the values of T on arbitrary vectors in V.

As we will soon see, matrices are used as an efficient method of recording the values of the Tv_j's in terms of a basis of W.

The matrix $\mathscr{M}(T)$ of a linear map $T \in \mathscr{L}(V, W)$ depends on the basis $v_1, \cdots, v_n$ of V and the basis $w_1, w_2, \cdots, w_m$ of W, as well as on T.

Matrix Multiplication

Definition

Let m, n denote positive integers. An m-by-n matrix A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A. In other words, the first index refers to the row number and the second index refers to the column number.

Matrix of a Linear Map

Now we come to the key definition in this section.

Definition

Suppose $T \in \mathscr{L}(V,W)$ and $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W. The matrix of T with respect to these bases is the m-by-n matrix $\mathscr{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation

$$\mathscr{M}(T, (v_1, v_2, \cdots, v_n), (w_1, \cdots, w_m))$$

is used.

addition

3.35 **Definition** ***matrix addition***

The ***sum of two matrices of the same size*** is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \dots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \dots & C_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} + C_{1,1} & \dots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \dots & A_{m,n} + C_{m,n} \end{pmatrix}.$$

In other words, $(A + C)_{j,k} = A_{j,k} + C_{j,k}$.

The matrix of the sum of linear maps

In the following result, the assumption is that the same bases are used for all three linear maps $S+T, S$, and T.

Proposition

Suppose $S, T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

The verification of the result above is left to the reader.

3.37 **Definition** ***scalar multiplication of a matrix***

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \dots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \dots & \lambda A_{m,n} \end{pmatrix}.$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$.

scalar multiplication

Proposition

Suppose $\lambda \in \mathbb{F}$ *and* $T \in \mathscr{L}(V,W)$. *Then* $\mathscr{M}(\lambda T) = \lambda \mathscr{M}(T)$.

3.39 Notation $\mathbf{F}^{m,n}$

For m and n positive integers, the set of all m-by-n matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$.

Dimension

3.40 $\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space with dimension mn.

Matrix Multiplication

(1) Suppose, as previously, that $v_1, v_2, \cdots, v_n$ is a basis of V and $w_1, w_2, \cdots, w_m$ is a basis of W. Suppose also we have another vector space U and that $u_1, u_2, \cdots, u_p$ is a basis of U.

(2) Consider linear maps $T : U \to V$ and $S : V \to W$. The composition ST is a linear map from U to W. Does $\mathscr{M}(ST)$ equal $\mathscr{M}(S)\mathscr{M}(T)$? This question does not yet make sense, because we have not defined the product of two matrices.

(3) We will choose a definition of matrix multiplication that forces this question to have a positive answer.

Let's see how to do this. Suppose $\mathscr{M}(S) = A$ and $\mathscr{M}(T) = C$. For $1 \leq k \leq p$, we have

Matrix Multiplication: Motivation

$$
\begin{aligned}
(ST)u_k &= S\left(\sum_{r=1}^{n} C_{r,k} v_r\right) \\
&= \sum_{r=1}^{n} C_{r,k} S v_r \\
&= \sum_{r=1}^{n} C_{r,k} \sum_{j=1}^{m} A_{j,r} w_j \\
&= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j.
\end{aligned}
$$

Thus $\mathscr{M}(ST)$ is the m-by-p matrix whose entry in row j, column k, equals $\sum_{r=1}^{n} A_{j,r} C_{r,k}$.

Matrix Multiplication

Now we see how to define matrix multiplication so that the desired equation $\mathscr{M}(ST) = \mathscr{M}(S)\mathscr{M}(T)$ holds.

3.41 **Definition** ***matrix multiplication***

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then AC is defined to be the m-by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

In other words, the entry in row j, column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing.

The matrix of the product of linear maps

The proof of the result above is the calculation that was done as motivation before the definition of matrix multiplication.

3.43 The matrix of the product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

3.44 Notation $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m-by-n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

Matrix Multiplication

Our next result gives another way to think of matrix multiplication: The entry in row j, column k of AC (row j of A) times (column k of C).

3.47 Entry of matrix product equals row times column

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{j,k} = A_{j,\cdot}\, C_{\cdot,k}$$

for $1 \le j \le m$ and $1 \le k \le p$.

Matrix Multiplication

The next result gives yet another way to think of matrix multiplication. It states that column k of AC equals A times the column k of C.

3.49 Column of matrix product equals matrix times column

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

for $1 \leq k \leq p$.

Linear Combinations

3.52 Linear combination of columns

Suppose A is an m-by-n matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}.$$

In other words, Ac is a linear combination of the columns of A, with the scalars that multiply the columns coming from c.

Homework Assignment 7

3.C: 3, 4, 5, 6, 7, 12, 14, 15.