Duality

Lecture 10 and 11

Dept. of Math., SUSTech

2024.03

Vector Spaces

Linear Functional

Linear functions into the scalar field $\mathbb{F}$ play a special role in linear algebra, and thus they get a special name:

Definition

A linear functional on V is a linear map from V to $\mathbb{F}$. In other words, a linear functional is an element of $\mathscr{L}(V,\mathbb{F})$.

Example

1. Define $\varphi : \mathbb{R}^3 \to \mathbb{R}$ by $\varphi(x,y,z) = 4x - 5y + 2z$. Then φ is a linear functional on $\mathbb{R}^3$.
2. Fix $(c_1, c_2, \cdots, c_n) \in \mathbb{F}^n$. Define $\varphi : \mathbb{F}^n \to \mathbb{F}$ by

$$\varphi(x_1, x_2, \cdots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

   Then φ is a linear functional on $\mathbb{F}^n$.

Linear Functionals

Example

1. Define $\varphi : \mathscr{P}(\mathbb{R}) \to \mathbb{R}$ by $\varphi(p) = 3p''(5) + 7p(4)$. Then φ is a linear functional on $\mathscr{P}(\mathbb{R})$.
2. Define $\varphi : \mathscr{P}(\mathbb{R}) \to \mathbb{R}$ by $\varphi(p) = \int_0^1 p(x)dx$. Then φ is a linear functional on $\mathscr{P}(\mathbb{R})$.

Linear Functionals

The vector space $\mathscr{L}(V,\mathbb{F})$ also gets a special name and special notation:

Definition

The dual space of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathscr{L}(V,\mathbb{F})$.

Dual Space

The dimension of V' is equal to the dimension of V:

3.95 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim V' = \dim V$.

The proof of this result follows directly from 3.61.

Dual Basis

In the following definition, 3.5 implies that each φ_j is well defined.

3.96 **Definition** ***dual basis***

If $v_1, \dots, v_n$ is a basis of V, then the ***dual basis*** of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Example

3.97 **Example** What is the dual basis of the standard basis $e_1, \dots, e_n$ of $\mathbf{F}^n$?

Solution For $1 \leq j \leq n$, define φ_j to be the linear functional on $\mathbf{F}^n$ that selects the j^{th} coordinate of a vector in $\mathbf{F}^n$. In other words,

$$\varphi_j(x_1, \dots, x_n) = x_j$$

for $(x_1, \dots, x_n) \in \mathbf{F}^n$. Clearly

$$\varphi_j(e_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Thus $\varphi_1, \dots, \varphi_n$ is the dual basis of the standard basis $e_1, \dots, e_n$ of $\mathbf{F}^n$.

Dual basis is a basis of the dual space

The next result shows that the dual basis is indeed a basis. Thus the terminology "dual basis" is justified.

3.98 Dual basis is a basis of the dual space

Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V'.

Proof

Proof Suppose $v_1, \ldots, v_n$ is a basis of V. Let $\varphi_1, \ldots, \varphi_n$ denote the dual basis.

To show that $\varphi_1, \ldots, \varphi_n$ is a linearly independent list of elements of V', suppose $a_1, \ldots, a_n \in F$ are such that

$$a_1\varphi_1 + \cdots + a_n\varphi_n = 0.$$

Now $(a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j) = a_j$ for $j = 1, \ldots, n$. The equation above thus shows that $a_1 = \cdots = a_n = 0$. Hence $\varphi_1, \ldots, \varphi_n$ is linearly independent.

Now 2.39 and 3.95 imply that $\varphi_1, \ldots, \varphi_n$ is a basis of V'. ∎

dual map

In the definition below, note that if T is a linear map from V to W then T' is a linear map from W' to V'.

3.99 **Definition** ***dual map***, T'

If $T \in \mathcal{L}(V, W)$, then the ***dual map*** of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.

- If $T \in \mathscr{L}(V, W)$ and $\varphi \in W'$, then $T'(\varphi)$ is defined above to be the composition of the linear maps φ and T.
- Thus $T'(\varphi)$ is indeed a linear map from V to $\mathbb{F}$; in other words, $T'(\varphi) \in V'$.
- Can you verify that T' is indeed a linear map from W' to V'?

Example

3.100 **Example** Define $D: \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Dp = p'$.

- Suppose φ is the linear functional on $\mathcal{P}(\mathbf{R})$ defined by $\varphi(p) = p(3)$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ given by

  $$\big(D'(\varphi)\big)(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = p'(3).$$

  In other words, $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ that takes p to $p'(3)$.

- Suppose φ is the linear functional on $\mathcal{P}(\mathbf{R})$ defined by $\varphi(p) = \int_0^1 p$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ given by

  $$\big(D'(\varphi)\big)(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = \int_0^1 p' = p(1) - p(0).$$

  In other words, $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ that takes p to $p(1) - p(0)$.

Algebraic properties of dual maps

The first two bullet points in the result below imply that the function that takes T to T' is a linear map from $\mathscr{L}(V,W)$ to $\mathscr{L}(W',V')$. In the third bullet point below, note the reversal of order from ST on the left to $T'S'$ on the right.

3.101 Algebraic properties of dual maps

- $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$.
- $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$ and all $T \in \mathcal{L}(V, W)$.
- $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and all $S \in \mathcal{L}(V, W)$.

annihilator

Our goal in this subsection is to describe null T' and range T' in terms of range T and null T. To do this, we will need the following definition.

3.102 Definition ***annihilator, U^0***

For $U \subset V$, the ***annihilator*** of U, denoted U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

Example

Suppose U is the subspace of $\mathscr{P}(\mathbb{R})$ consisting of all polynomial multiples of x^2. If φ is the linear functional on $\mathscr{P}(\mathbb{R})$ defined by $\varphi(p) = p'(0)$, then $\varphi \in U^0$.

Examples

3.104 **Example** Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbf{R}^5$, and let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbf{R}^5)'$. Suppose

$$U = \operatorname{span}(e_1, e_2) = \{(x_1, x_2, 0, 0, 0) \in \mathbf{R}^5 : x_1, x_2 \in \mathbf{R}\}.$$

Show that $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Annihilator is a subspace; The dimension of the the annihilator

3.105 The annihilator is a subspace

Suppose $U \subset V$. Then U^0 is a subspace of V'.

Proof Clearly $0 \in U^0$ (here 0 is the zero linear functional on V), because the zero linear functional applied to every vector in U is 0.

Suppose $\varphi, \psi \in U^0$. Thus $\varphi, \psi \in V'$ and $\varphi(u) = \psi(u) = 0$ for every $u \in U$. If $u \in U$, then $(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0$. Thus $\varphi + \psi \in U^0$.

Similarly, U^0 is closed under scalar multiplication. Thus 1.34 implies that U^0 is a subspace of V'. ∎

Dimension of the annihilator

You should construct the proof outlined in the paragraph above, even though a slicker proof is presented here.

3.106 Dimension of the annihilator

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof. Let $i \in \mathscr{L}(U, V)$ be the inclusion map defined by

$$i(u) = u, \text{ for } u \in U.$$

Thus i' is a linear map from V' to U'. The Fundamental Theorem of Linear Maps (3.22) applied to i' shows that

$$\dim \text{ range } i' + \dim \text{ null } i' = \dim V'.$$

Proof

However, null $i' = U^0$ (as can be seen by thinking about the definitions) and $\dim V' = \dim V$(by 3.95), we can rewrite the equation above as

$$\dim \text{ range } i' + \dim \ U^0 = \dim \ V.$$

If $\varphi \in U'$, then φ can be extended to a linear functional ϕ on V (see, for example, Exercise 11 in Section 3.A). The definition of i' shows that $i'(\phi) = \varphi$. Thus $\varphi \in$ range i', which implies that range $i' = U'$. Hence $\dim$ range $i' = \dim \ U' = \dim \ U$, and the displayed equation above becomes the desired result.

The null space of T'

The proof of part (a) of the result below does not use the hypothesis that V and W are finite-dimensional.

3.107 The null space of T'

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T' = (\operatorname{range} T)^0$;

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$.

T surjective is equivalent to T' injective

The next result can be useful because sometimes it is easier to verify that T' is injective than to show directly that T is surjective.

3.108 T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Proof. The map $T \in \mathscr{L}(V,W)$ is surjective if and only if range $T = W$, which happens if and only if $(\text{range } T)^0 = \{0\}$, which happens if and only if null $T' = \{0\}$ [By 3.107(a)], which happens if and only if T' is injective.

The range of T'

The range of T'

3.109 The range of T'

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$;

(b) $\operatorname{range} T' = (\operatorname{null} T)^0$.

T injective is equivalent to T' surjective

The next result should be compared to 3.108.

3.110 T injective is equivalent to T' surjective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T is injective if and only if T' is surjective.

Proof The map $T \in \mathcal{L}(V, W)$ is injective if and only if $\operatorname{null} T = \{0\}$, which happens if and only if $(\operatorname{null} T)^0 = V'$, which happens if and only if $\operatorname{range} T' = V'$ [by 3.109(b)], which happens if and only if T' is surjective. ■

The matrix of the Dual of a Linear Map

We define the transpose of a matrix

3.111 **Definition** ***transpose***, A^{t}

The ***transpose*** of a matrix A, denoted A^{t}, is the matrix obtained from A by interchanging the rows and columns. More specifically, if A is an m-by-n matrix, then A^{t} is the n-by-m matrix whose entries are given by the equation

$$(A^{\mathrm{t}})_{k,j} = A_{j,k}.$$

The Transpose of the Product of Matrices

The transpose of the product of matrices

3.113 The transpose of the product of matrices

If A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^{\mathrm{t}} = C^{\mathrm{t}} A^{\mathrm{t}}.$$

Matrices

3.114 The matrix of T' is the transpose of the matrix of T

Suppose $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(T') = \left(\mathcal{M}(T)\right)^{\mathrm{t}}$.

Proof. Let $A = \mathscr{M}(T)$ and $C = \mathscr{M}(T')$. Suppose $1 \le j \le m$ and $1 \le k \le n$. From the definition of $\mathscr{M}(T')$ we have

$$T'(\phi_j) = \sum_{r=1}^{n} C_{r,j}\varphi_r.$$

The left side of the equation above equals $\phi \circ T$. Thus applying both sides of the equation above to v_k gives

$$(\phi_j \circ T)(v_k) = \sum_{r=1}^{n} C_{r,j}\varphi_r(v_k) = C_{k,j}.$$

Matrices

We also have

$$(\phi_j \circ T)(v_k) = \phi_j(Tv_k) = \phi_j\left(\sum_{r=1}^{m} A_{r,k} w_r\right) = A_{j,k}.$$

Comparing the last line of the last two sets of equations, we have

$$C_{k,j} = A_{j,k}.$$

Thus $C = A^t$. In other words, $M(T') = (M(T))^t$, as desired.

Row Rank and Column Rank

3.115 **Definition** ***row rank, column rank***

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$.

- The ***row rank*** of A is the dimension of the span of the rows of A in $\mathbf{F}^{1,n}$.
- The ***column rank*** of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.

Rank

3.117 **Dimension of range T equals column rank of $\mathcal{M}(T)$**

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$.

Row rank equals column rank

3.118 **Row rank equals column rank**

Suppose $A \in \mathbf{F}^{m,n}$. Then the row rank of A equals the column rank of A.

Rank

The last result allows us to dispense the terms “row rank” and “column rank” and just use the simpler term “rank”

3.119 **Definition** ***rank***

The ***rank*** of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A.

Homework Assignment 10 and 11

3.F: 6, 8, 15, 22, 29, 31, 34, 36.