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MA215 Probability Theory

Assignment 01

1. Provide a strict proof for the following set relations.

   (1) $B \backslash A = B \cap A^c$;

   (2) $(A \backslash B) \cap C = (A \cap C) \backslash (B \cap C)$;

   (3) $(\bigcup_{k=1}^{\infty} A_k)^c = \bigcap_{k=1}^{\infty} A_k^c$;

   (4) $(\bigcap_{k=1}^{\infty} A_k)^c = \bigcup_{k=1}^{\infty} A_k^c$;

   (5) $A \cup (\bigcap_{k=1}^{\infty} B_k) = \bigcap_{k=1}^{\infty} (A \cup B_k)$;

   (6) $A \cap (\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} (A \cap B_k)$.

   As the generalizations of (3) to (6) we have the following general De Morgan's Laws and Distributive laws: For any index set I, we have

   (7) $(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} (A_i)^c$

   (8) $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} (A_i)^c$

   (9) $A \cup (\bigcap_{i \in I} B_i) = \bigcap_{i \in I} (A \cup B_i)$;

   (10) $A \cap (\bigcup_{i \in I} B_i) = \bigcup_{i \in I} (A \cap B_i)$.

Proof. (1) Suppose that the universal set is Ω, then $A = \Omega \backslash A^c$

For every element $x \in B \backslash A$. which equal to $x \in B$ and $x \notin A$

$\Leftrightarrow$ $x \in B$ and $x \notin \Omega \backslash A^c$. $\Leftrightarrow$ $x \in B$ and $x \in A^c$. $\Leftrightarrow$ $x \in B \cap A^c$

vice versa. So $B \backslash A \subseteq B \cap A^c$, $B \backslash A \supseteq B \cap A^c$. $\Rightarrow$ $B \backslash A = B \cap A^c$

(2) For every element $x \in (A \backslash B) \cap C$, which equal to:

$x \in A$, $x \notin B$ and $x \in C$. $\xRightarrow[B \cap C \subseteq B]{}$ $x \in A \cap C$. $x \notin B \cap C$

$\Rightarrow$ $x \in (A \cap C) \backslash (B \cap C)$, so $(A \backslash B) \cap C \subseteq (A \cap C) \backslash (B \cap C)$

For every element $y \in (A \cap C) \backslash (B \cap C)$ $\Rightarrow$ $y \in A \cap C$ and $y \notin B \cap C$

$\Rightarrow$ $y \in A$ and $y \in C$. however, $y \notin B \cap C$ $\Rightarrow$ $y \notin B$

$\Rightarrow$ $y \in (A \backslash B) \cap C$. so $(A \backslash B) \cap C \supseteq (A \cap C) \backslash (B \cap C)$

$\therefore$ $(A \backslash B) \cap C = (A \cap C) \backslash (B \cap C)$

(3) Suppose that the universal set is Ω.

$$(\bigcup_{k=1}^{\infty} A_k)^c = \Omega \backslash \bigcup_{k=1}^{\infty} A_k = \bigcap_{k=1}^{\infty} (\Omega \backslash A_k) = \bigcap_{k=1}^{\infty} A_k^c$$

(4) $$(\bigcap_{k=1}^{\infty} A_k)^c = \Omega \backslash \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (\Omega \backslash A_k) = \bigcup_{k=1}^{\infty} A_k^c$$

(5) For any $x \in A \cup (\bigcap_{k=1}^{\infty} B_k)$, $x \in A$ or $x \in \bigcap_{k=1}^{\infty} B_k$, that's to say, if $x \notin A$,

then $x \in B_1$ and $x \in B_2$ and $\cdots$ $x \in B_n$ and $\cdots$

$\Leftrightarrow x \in A \cup B_1$ and $x \in A \cup B_2$ and $\cdots$ $x \in A \cup B_n$ and $\cdots \Leftrightarrow x \in \bigcap_{k=1}^{\infty} (A \cup B_k)$

vice versa. so $A \cup (\bigcap_{k=1}^{\infty} B_k) = \bigcap_{k=1}^{\infty} (A \cup B_k)$.

(6) For any $x \in A \cap (\bigcup_{k=1}^{\infty} B_k)$, $x \in A$ and $x \in \bigcup_{k=1}^{\infty} B_k$, that's to say, when $x \in A$, only have to exist $i \in N^*$, $x \in B_i \Leftrightarrow$ only have to exist $i \in N^*$, $x \in A \cap B_i$. $\Leftrightarrow x \in \bigcup_{k=1}^{\infty} (A \cap B_k)$. vice versa. so $A \cap (\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} (A \cap B_k)$

(7) $(\bigcup_{i \in I} A_i)^c = \{x \mid x \notin \bigcup_{i \in I} A_i\}$, and $x \notin \bigcup_{i \in I} A_i$ means x is not in any A_i, $i \in I$.

$\therefore$ $x \notin \bigcup_{i \in I} A_i$ if and only if x is not in A_i for all $i \in I$.

$\bigcap_{i \in I} (A_i)^c = \{x \mid x \in (A_i)^c, i \in I\}$, and $x \in (A_i)^c$ means $x \notin A_i$

$\therefore$ $x \in (A_i)^c$, $i \in I$ if and only if $x \notin A_i$ for all $i \in I$.

So $(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} (A_i)^c$.

(8) $(\bigcap_{i \in I} A_i)^c = \{x \mid x \notin \bigcap_{i \in I} A_i\}$, and $x \notin \bigcap_{i \in I} A_i$ means x is not in at least one A_i.

$\therefore$ $x \notin \bigcap_{i \in I} A_i$ if and only if x is not in at least one A_i.

$\bigcup_{i \in I} (A_i)^c = \{x \mid x \in (A_i)^c$ for some $i \in I\}$, and $x \in (A_i)^c$ means $x \notin A_i$.

$\therefore$ $x \in \bigcup_{i \in I} (A_i)^c$ if and only if x is not in at least one A_i

So $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} (A_i)^c$

(9) For any $x \in A \cup (\bigcap_{i \in I} B_i)$: if $x \in A$, then $A \in A \cup B_i$ for all $i \in I$

if $x \in \bigcap_{i \in I} B_i$, then $x \in B_i$ for all $i \in I$, so $x \in A \cup B_i$ for all $i \in I$

$\therefore$ $A \cup (\bigcap_{i \in I} B_i) \subseteq \bigcap_{i \in I} (A \cup B_i)$

For any $x \in \bigcap_{i \in I} (A \cup B_i)$. $x \in A \cup B_i$ for all $i \in I$. If $x \notin A$, then $x \in B_i$ for all $i \in I$

so $x \in \bigcap_{i \in I} B_i$; else $x \in A$ $\quad \therefore x \in A \cup (\bigcap_{i \in I} B_i)$

$\therefore$ $A \cup (\bigcap_{i \in I} B_i) \supseteq \bigcap_{i \in I} (A \cup B_i)$ $\quad \therefore A \cup (\bigcap_{i \in I} B_i) = \bigcap_{i \in I} (A \cup B_i)$

(10) For any $x \in A \cap (\bigcup_{i \in I} B_i)$, $x \in A$ and $x \in \bigcup_{i \in I} B_i \Rightarrow x \in A$ and $x \in B_i$ for some $i \in I$

$\therefore$ $x \in A \cap B_i$ for some $i \in I$. $\quad \therefore A \cap (\bigcup_{i \in I} B_i) \subseteq \bigcup_{i \in I} (A \cap B_i)$

For any $x \in \bigcup_{i \in I} (A \cap B_i)$, $x \in A \cap B_i$ for some $i \in I$.

$\therefore$ $x \in A$ and $x \in B_i$ for some $i \in I \Rightarrow x \in A$ and $x \in \bigcup_{i \in I} B_i$

$\therefore$ $A \cap (\bigcup_{i \in I} B_i) \supseteq \bigcup_{i \in I} (A \cap B_i)$ $\quad \therefore A \cap (\bigcup_{i \in I} B_i) = \bigcup_{i \in I} (A \cap B_i)$

2. A sequence of sets $\{A_1, A_2, \ldots, A_n, \ldots\}$ is called increasing if

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots.$$

Similarly, a sequence of sets $\{A_1, A_2, \ldots, A_n, \ldots\}$ is called decreasing if

$$A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots.$$

Show that

(i) If $\{A_n; n \geqslant 1\}$ is an increasing set sequence, then for any $n \geqslant 1$,

$$\bigcup_{k=1}^{n} A_k = A_n, \quad \text{and} \quad \lim_{n\to\infty} A_n = \bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n.$$

(ii) If $\{A_n; n \geqslant 1\}$ is a decreasing set sequence, then for any $n \geqslant 1$,

$$\bigcap_{k=1}^{n} A_k = A_n, \quad \text{and} \quad \lim_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n.$$

Proof. (1) $\because \{A_n\}$ is an increasing set sequence, $\therefore A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$

$\therefore A_1 \cup A_2 = A_2$, $\bigcup_{k=1}^{3} A_k = A_3$, $\cdots$, $\bigcup_{k=1}^{n} A_k = A_n$, A_n contains all the elements of $A_1, A_2, \cdots, A_n$

$\therefore$ When $n \to \infty$, $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$, which equal to $\bigcup_{k=1}^{\infty} A_k$, because any element in $\bigcup_{k=1}^{\infty} A_k$ is in some A_n, and A_n contains all elements of A_k, $k \le n$

(2) $\because \{A_n\}$ is a decreasing set sequence. $\therefore A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$

For any $x \in A_n$, $\because A_{n-1} \supset A_n$ $\therefore x \in A_{n-1}$ $\cdots$ $x \in A_1$. $\therefore x \in \bigcap_{k=1}^{n} A_k$, vice versa

$\therefore \bigcap_{k=1}^{n} A_k = A_n$ When $n \to \infty$, $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$, which equal to $\bigcap_{k=1}^{\infty} A_k$, because any element in $\bigcap_{k=1}^{n} A_k$ is in all A_n for sufficiently large n, and A_n contains all the elements of A_k, $k \ge n$

3. Show that if $A_1, A_2, \ldots, A_n$ are all countable sets, then so is the n-tuple Cartesian product

$$A_1 \times A_2 \times \cdots \times A_n.$$

In particular, if A is a countable set, then so is A^n.

Proof. Because $A_1, \cdots, A_n$ are all countable, so we can suppose that for any $i \in \{1, 2, \cdots, n\}$, $A_i = \{a_{i1}, a_{i2}, \cdots\}$

then $A_1 \times A_2 \times \cdots \times A_n = \{(a_{11}, a_{21}, \cdots, a_{n1}), (a_{11}, a_{21}, \cdots, a_{n2}) \cdots,$
$(a_{11}, a_{21}, \cdots, a_{(n-1)2}, a_{n1}), \cdots, (a_{12}, a_{21}, \cdots, a_{n1}), \cdots (a_{1n}, a_{2n}, \cdots, a_{nn}), \cdots\}$

it means we can write $A_1 \times A_2 \times \cdots \times A_n$ as a sequence.

$\therefore A_1 \times A_2 \times \cdots \times A_n$ is countable. $\square$

4. Suppose that the three sets A, B and C have the relationship $A \subset B \subset C$ and that $\text{Card}(A) = \text{Card}(C)$, then

$$\text{Card}(A) = \text{Card}(B) = \text{Card}(C),$$

where $\text{Card}(A)$ denotes the cardinal number of the set A etc.

Proof. Because $A \subset B$, every element $a \in A$ is also in B

So we can find an injective function $f: A \to B$ as follows:

$f(a) = a$, $a \in A$. The function f shows that no element in A is also not in B, which means $\text{Card}(A) \le \text{Card}(B)$

Because $B \subset C$, we also know that $\text{Card}(B) \le \text{Card}(C)$

So $\text{Card}(A) \le \text{Card}(B) \le \text{Card}(C)$

$\because \text{Card}(A) = \text{Card}(C)$ $\quad \therefore \text{Card}(A) = \text{Card}(B) = \text{Card}(C)$. $\square$

5. Show that the set $[0, 1]$ is not countable.

Proof. Every real number in $[0,1]$ can be written as $0.b_1b_2\cdots b_n\cdots$, where each $b_i \in \{0, 1, \cdots, 9\}$

Suppose $[0,1]$ is countable, then it can be written as a sequence $\{x_1, x_2, x_3, \cdots\}$

Assume: $x_1 = 0.a_{11}a_{12}a_{13}\cdots a_{1n}\cdots$

$x_2 = 0.a_{21}a_{22}a_{23}\cdots a_{2n}\cdots$

$\vdots$

$x_n = 0.a_{n1}a_{n2}\cdots a_{nn}\cdots$

$\vdots$

where a_{ij} are all one of the numbers $\{0, 1, 2, \cdots, 9\}$

Define a number x^*, $x^* = 0.a_{*1}a_{*2}\cdots a_{*n}\cdots$

where $a_{*1} \ne a_{11}, a_{*2} \ne a_{22}, \cdots, a_{*n} \ne a_{nn}$, and all $a_{*i} \in \{0, 1, 2, \cdots, 9\}$

Definitely, $x^* \in [0,1]$, BUT $x^* \notin \{x_1, x_2, \cdots, x_n, \cdots\}$.

This is a contradiction. So $[0,1]$ is not countable. $\square$

6. Show that the Cardinal number of the real number R is equal to the cardinal number of the open unit internal $(0, 1)$.

Proof. First, we find a bijective function between R and $(-1,1)$

Suppose $f(x)=\frac{x}{\sqrt{1+x^2}}$,

i) Assume $x_1, x_2 \in R$, $f(x_1)=f(x_2) \Rightarrow \frac{x_1}{\sqrt{1+x_1^2}}=\frac{x_2}{\sqrt{1+x_2^2}} \Rightarrow x_1^2(1+x_2^2)=x_2^2(1+x_1^2)$

$\Rightarrow x_1^2=x_2^2 \Rightarrow x_1=\pm x_2$, because $(-1,1)$ is symmetric about 0.

$f(x)$ is injective.

ii) Given any $y\in(-1,1)$. we find x, where $\frac{x}{\sqrt{1+x^2}}=y$

$\Rightarrow y^2(1+x^2)=x^2 \Rightarrow x^2=\frac{y^2}{1-y^2} \Rightarrow x=\pm\frac{y}{\sqrt{1-y^2}}$, definitely $x\in R$

So $f(x)$ is surjective.

According to i), ii), $f(x)$ is bijective.

Then we find a bijective function between $(-1,1)$ and $(0,1)$

Suppose $g(y)=\frac{y+1}{2}$, easy to prove $g(y)$ is both injective and surjective.

As a result, $f\circ g$ is a bijection between R and $(0,1)$

So R and $(0,1)$ have same cardinal number. $\square$

7. Suppose $\{A_n; n = 1, 2, \ldots\}$ is an increasing sequence of sets. Define $B_1 = A_1, B_2 = A_2\backslash A_1$, and in general, $B_n = A_n\backslash A_{n-1} (n \geqslant 2)$. Show that

(i) $\{B_n; n \geqslant 1\}$ are disjoint.

(ii) For any $k \geqslant 1$, $\bigcup_{n=1}^{k} B_n = A_k$.

(iii) $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

Proof. (i) To prove $\{B_n; n\geq 1\}$ are disjoint, we need to prove that $B_i\cap B_j=\emptyset$ for $i\neq j$

When $i<j$, for $B_i=A_i\setminus A_{i-1}$, $B_j=A_j\setminus A_{j-1}$, $i<j$, we have:

$B_i\cap B_j=(A_i\setminus A_{i-1})\cap(A_j\setminus A_{j-1})$

Because $\{A_n; n=1,2\cdots\}$ is increasing, $A_i\subseteq A_j$, $A_{i-1}\subseteq A_{j-1}$

$A_j\cap(A_j\setminus A_{j-1})=A_j\setminus A_{j-1}$. When $i<j-1$, $A_i\subseteq A_{j-1}\Rightarrow A_i\cap(A_j\setminus A_{j-1})=\emptyset$

$\therefore B_i\cap B_j=\emptyset$

When $i>j$, since $A_i\supseteq A_j$, $A_{i-1}\supseteq A_{j-1}$, we have $B_i\cap B_j=(A_i\setminus A_{i-1})\cap(A_j\setminus A_{j-1})$

because $A_j\subseteq A_i$, $A_{j-1}\subseteq A_{i-1}$, we have $(A_i\setminus A_{i-1})\cap(A_j\setminus A_{j-1})$

$=(A_j\setminus A_{j-1})\setminus(A_i\setminus A_{i-1})=\emptyset$ $\therefore B_i\cap B_j=\emptyset$ Thus, $\{B_n\}_{n\geq 1}$ is disjoint.

(ii) $k=1$, definitely $A_1 = B_1$

$k=2$, $\bigcup_{n=1}^{2} B_n = B_1 \cup B_2 = A_1 \cup (A_2 \setminus A_1) = A_2 \Rightarrow \bigcup_{n=1}^{2} B_n = A_2$

Assume that when $k=m$, $\bigcup_{n=1}^{m} B_n = A_m$.

For $k=m+1$, $\bigcup_{n=1}^{m+1} B_n = (\bigcup_{n=1}^{m} B_n) \cup B_{m+1} = A_m \cup B_{m+1} = A_m \cup (A_{m+1} \setminus A_m) = A_{m+1}$

So for any $k>1$, $\bigcup_{n=1}^{k} B_n = A_k$.

(iii) we already know that: $\bigcup_{n=1}^{k} B_n = A_k$

so we get: $\bigcup_{k=1}^{\infty} (\bigcup_{n=1}^{k} B_n) = \bigcup_{k=1}^{\infty} A_k$ Because $A_k \subseteq \bigcup_{k=1}^{\infty} A_k$,

$\bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} B_n$. Thus: $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. $\square$

8. Let S be the set of all the sequences with elements 0 and 1 only. Is S countable or not? Prove your conclusion.

S is not countable.

Proof. Define a number $x \in [0,1]$, $S = \{(a_1, a_2, \cdots, a_n, \cdots) \mid a_1, \cdots, a_n \cdots \in \{0,1\}\}$

$x = 0.a_1 a_2 \cdots a_n \cdots$. If we think x in binary, then $0.a_1 \cdots a_n \cdots$, $a_i \in \{0,1\}$, $i = 1, 2, \cdots, n, \cdots$ contain all the real number in $[0,1)$, which has same cardinal with $[0,1]$. As a result, we can find a bijective function between S and $[0,1]$, so they have same cardinal number, which is uncountable. $\square$