Invariant Subspaces(不变子空间)

Lecture 13

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Eigenvalues

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Introduction

Linear maps from one vector space to another vector space were the objects of study in Chapter 3. Now we begin our investigation of linear maps from a finite-dimensional vector space to itself. Their study constitutes the most important part of linear algebra.

Learning objectives for this Chapter:

1. invariant subspaces
2. eigenvalues, eigenvectors, and eigenspaces
3. each operator on a finite-dimensional complex vector space has an eigenvalue and an upper-triangular matrix with respect to some basis

Invariant Subspaces

- In this chapter we develop the tools that will help us understand the structure of operators. Recall that an operator is a linear map from a vector space to itself. Recall also that we denote the set of operators on V by $\mathscr{L}(V)$; in other words, $\mathscr{L}(V) = \mathscr{L}(V,V)$.
- Let's see how we might better understand what an operator looks like. Suppose $T \in \mathscr{L}(V)$. We have a direct sum decomposition

$$V = U_1 \oplus U_2 \cdots \oplus U_m$$

- Where each U_j is a proper subspace of V, then to understand the behavior of T, we need only understand the behavior of each $T|_{U_j}$; here $T|_{U_j}$ denotes the restriction of T to the smaller domain U_j. Dealing with $T|_{U_j}$ should be easier than dealing with T because U_j is a smaller vector space than V.

Eigenvalues and Eigenvectors

- However, if we intend to apply tools useful in the study of operators (such as taking powers), then we have a problem: $T|_{U_j}$ may not map U_j into itself; in other words, $T|_{U_j}$ may not be an operator on U_j.
- Thus we are led to consider only decomposition of V of the form above where T maps each U_j into itself.
- The notion of a subspace that gets mapped into itself is sufficiently important to deserve a name.

5.2 **Definition** ***invariant subspace***

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called ***invariant*** under T if $u \in U$ implies $Tu \in U$.

In other words, U is invariant under T if $T|_U$ is an operator on U.

Example

5.3 **Example** Suppose $T \in \mathcal{L}(V)$. Show that each of the following subspaces of V is invariant under T:

(a) $\{0\}$;

(b) V;

(c) null T;

(d) range T.

*The most famous unsolved problem in functional analysis is called the **invariant subspace problem**. It deals with invariant subspaces of operators on infinite-dimensional vector spaces.*

Remark:

1. Must an operator $T \in \mathscr{L}(V)$ have any invariant subspaces other than $\{0\}$ and V?
2. Later we will see that this question has an affirmative answer if V is finite-dimensional and $\dim V > 1$(for $\mathbb{F} = \mathbb{C}$) or $\dim V > 2$ (for $\mathbb{F} = \mathbb{R}$).

Invariant Subspaces

Although null T and range T are invariant under T, they do not necessarily provide easy answers to the question about the existence of invariant subspaces other than $\{0\}$ and V.

Example

Suppose that $T \in \mathscr{L}(\mathscr{P}(\mathbb{R}))$ is defined by $T(p) = p'$. Then $\mathscr{P}_4(\mathbb{R})$, which is a subspace of $\mathscr{P}(\mathbb{R})$, is invariant under T because if $p \in \mathscr{P}(\mathbb{R})$ has degree at most 4, then p' also has degree at most 4.

Eigenvalues and Eigenvectors

We will return later to a deeper study of invariant subspaces. Now we turn to an investigation of the simplest possible nontrivial invariant subspaces–invariant subspaces with dimension 1. The equation $Tv = \lambda v$, which we have just seen is intimately connected with 1-dimensional invariant subspace, is important enough that the vectors v and scalars λ satisfying it are given special names.

5.5 Definition ***eigenvalue***

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an ***eigenvalue*** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Remarks

The word eigenvalue is half-German, half-English. The German adjective eigen means “own” in the sense of characterizing an intrinsic property. Some mathematicians use the term characteristic value instead of eigenvalue.

Eigenvalue

The comments above show that T has a 1-dimensional invariant subspace if and only if T has an eigenvalue. In the definition above, we require that $v \neq 0$ because every scalar $\lambda \in \mathbb{F}$ satisfies $T0 = \lambda 0$.

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Then the following are equivalent:

(a) λ is an eigenvalue of T;

(b) $T - \lambda I$ is not injective;

(c) $T - \lambda I$ is not surjective;

(d) $T - \lambda I$ is not invertible.

Recall that $I \in \mathcal{L}(V)$ is the identity operator defined by $Iv = v$ for all $v \in V$.

Eigenvector

Eigenvector

5.7 **Definition** ***eigenvector***

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an ***eigenvector*** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Because $Tv = \lambda v$ if and only if $(T - \lambda I)v = 0$, a vector $v \in V$ with $v \neq 0$ is an eigenvector of T corresponding to λ if and only if $v \in \operatorname{null}(T - \lambda I)$.

5.8 **Example** Suppose $T \in \mathcal{L}(\mathbf{F}^2)$ is defined by

$$T(w, z) = (-z, w).$$

(a) Find the eigenvalues and eigenvectors of T if $\mathbf{F} = \mathbf{R}$.

(b) Find the eigenvalues and eigenvectors of T if $\mathbf{F} = \mathbf{C}$.

Solution.

(a) If $\mathbb{F} = \mathbb{R}$, then T is a counterclockwise rotation by 90° about the origin in $\mathbb{R}^2$. An operator has an eigenvalue if and only if there exists a nonzero vector in its domain that gets sent by the operator to a scalar multiple of itself. A 90° counterclockwise rotation of a nonzero vector in $\mathbb{R}^2$ obviously never equals a scalar multiple of itself. Conclusion: if $\mathbb{F} = \mathbb{R}$, then T has no eigenvalues (and thus has no eigenvectors).

(b)

(b) To find the eigenvalues of T, we must find the scalars such that $T(w,z) = \lambda(w,z)$ has some solution other than $w = z = 0$. The equation $T(w,z) = \lambda(w,z)$ is equivalent to the simultaneous equaitons

$$-z = \lambda w, w = \lambda z.$$

It follows that $-1 = \lambda^2$. The solutions to this equation are $\lambda = i$ and $\lambda = -i$. You should be able to verify that i and $-i$ are eigenvalues of T. Indeed, the eigenvectors corresponding to the eigenvalue i are the vectors of the form $(w, -wi)$, with $w \in \mathbb{C}$ and $w \neq 0$, and the eigenvectors corresponding to the eigenvalue $-i$ are the vectors of the form (w, wi) with $w \in \mathbb{C}$ and $w \neq 0$.

eigenvectors corresponding to distinct eigenvalues are linearly independent

Now we show that eigenvectors corresponding to distinct eigenvalues are linearly independent.

5.10 Linearly independent eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

Proof.

Proof. Suppose $v_1, \cdots, v_m$ is linearly dependent. Let k be the smallest positive integer such that $v_k \in \text{span}(v_1, v_2, \cdots, v_{k-1})$; the existence of k with this property follows from the Linear Dependence Lemma (2.21). Thus there exist $a_1, a_2, \cdots, a_{k-1} \in \mathbb{F}$ such that

$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1} \cdots\cdots (*).$$

Apply T to both sides of this equation, getting

$$\lambda_k v_k = a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}.$$

Solution

Multiply both sides of $(*)$ by λ_k and then subtract the equation above, getting

$$0 = a_1(\lambda_k - \lambda_1)v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1}.$$

Because we chose k to be the smallest positive integer satisfying $v_k \in \text{span}(v_1, v_2, \cdots, v_{k-1})$, $v_1, \cdots, v_{k-1}$ is linearly independent. Thus the equation above implies that all the a's are 0 (recall that λ_k is not equal to any of $\lambda_1, \cdots, \lambda_{k-1}$). However, this means that v_k equals 0 (see $(*)$), contradicting our hypothesis that v_k is an eigenvector. Therefore our assumption that $v_1, \cdots, v_m$ is linearly dependent was false.

Number of Eigenvalues

Number of eigenvalues.

The corollary below states that an operator can not have more distinct eigenvalues than the dimension of the vector space on which it acts.

5.13 Number of eigenvalues

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Proof. Let $T \in \mathscr{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T. Let $v_1, \cdots, v_m$ be corresponding eigenvectors. Then 5.10 implies that the list $v_1, \cdots, v_m$ is linearly independent. Thus $m \leq \dim\ V$ (see 2.23), as desired.

Restriction and Quotient Operators

If $T \in \mathscr{L}(V)$ and U is a subspace of V invariant under T, then U determines two other operators $T|_U \in \mathscr{L}(U)$ and $T/U \in \mathscr{L}(V/U)$ in a natural way, as defined below.

5.14 **Definition** $T|_U$ ***and*** T/U

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.

- The ***restriction operator*** $T|_U \in \mathcal{L}(U)$ is defined by
$$T|_U(u) = Tu$$
for $u \in U$.
- The ***quotient operator*** $T/U \in \mathcal{L}(V/U)$ is defined by
$$(T/U)(v + U) = Tv + U$$
for $v \in V$.

Example

- Suppose T is an operator on a finite-dimensional vector space V and U is a subspace of V invariant under T, with $U \neq \{0\}$ and $U \neq V$.
- In some sense, we can learn about T by studying the operators $T|_U$ and T/U, each of which is an operator on a vector space with smaller dimension than V. For example, proof 2 of 5.27 makes nice use of T/U.
- However, sometimes $T|_U$ and T/U do not provide enough information about T. In the next example, both $T|_U$ and T/U are 0 even though T is not the zero operator.

Example

5.15 **Example** Define an operator $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(x, y) = (y, 0)$. Let $U = \{(x, 0) : x \in \mathbf{F}\}$. Show that

(a) U is invariant under T and $T|_U$ is the 0 operator on U;

(b) there does not exist a subspace W of $\mathbf{F}^2$ that is invariant under T and such that $\mathbf{F}^2 = U \oplus W$;

(c) T/U is the 0 operator on $\mathbf{F}^2/U$.

Solution

Solution

(a) For $(x, 0) \in U$, we have $T(x, 0) = (0, 0) \in U$. Thus U is invariant under T and $T|_U$ is the 0 operator on U.

(b) Suppose W is a subspace of V such that $\mathbf{F}^2 = U \oplus W$. Because $\dim \mathbf{F}^2 = 2$ and $\dim U = 1$, we have $\dim W = 1$. If W were invariant under T, then each nonzero vector in W would be an eigenvector of T. However, it is easy to see that 0 is the only eigenvalue of T and that all eigenvectors of T are in U. Thus W is not invariant under T.

(c) For $(x, y) \in \mathbf{F}^2$, we have

$$\begin{aligned}(T/U)\big((x, y) + U\big) &= T(x, y) + U \\ &= (y, 0) + U \\ &= 0 + U,\end{aligned}$$

where the last equality holds because $(y, 0) \in U$. The equation above shows that T/U is the 0 operator.

Homework Assignment 13

5.A: 2, 14, 18, 22, 34, 36.