1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Which one of the following statements must be true? ( )

(A) If A and B are $m \times n$ matrices and $Ax = 0$ has the same solution set as $Bx = 0$, then A and B have the same column space.

(B) If A is an $n \times n$ real symmetric positive definite matrix, then all the square submatrices of A have positive determinants.

(C) If real symmetric matrices A and B are congruent, then they are similar.

(D) If $AB = 0$ and A and B are not zero matrices, then A has linearly dependent columns and B has linearly dependent rows.

下列陈述一定正确的是? ( )

(A) 若 A 和 B 为 $m \times n$ 矩阵, 且 $Ax = 0$ 与 $Bx = 0$ 同解，则 A 和 B 具有相同的列空间.

(B) 若实对称矩阵 A 正定, 则它的所有正方形子矩阵的行列式都为正.

(C) 若实对称矩阵 A 和 B 相合 (也称合同), 则它们相似.

(D) 设 A, B 为满足 $AB = 0$ 的两个非零矩阵, 则 A 的列向量组线性相关, B 的行向量组线性相关.

(2) The plane curve defined by the equation $2x^2 - 8xy + 2y^2 = 1$ is ( )

(A) an ellipse.

(B) a hyperbola.

(C) a parabola.

(D) a union of two intersecting lines.

由方程 $2x^2 - 8xy + 2y^2 = 1$ 定义的平面曲线是 ( )

(A) 椭圆.

(B) 双曲线.

(C) 抛物线.

(D) 一对相交直线.

(3) Let A be an $n \times n$ real matrix. Suppose that for all column vectors $x \in \mathbb{R}^n$ we have $x^T Ax = 0$. Then ( )

(A) The determinant $|A|$ of A is 0.

(B) $A = 0$.

(C) The trace, trace(A), of A is 0.

(D) The only eigenvalue of A in $\mathbb{C}$ is 0.

设 A 为 $n \times n$ 实矩阵. 假设对任意列向量 $x \in \mathbb{R}^n$ 都有 $x^T Ax = 0$. 则 ( )

(A) A 的行列式 $|A|$ 为 0.

(B) $A = 0$.

(C) A 的迹, trace(A), 为 0.

(D) A 在 $\mathbb{C}$ 中唯一的特征值是 0.

(4) Let $n \geq 2$. Let A be an $n \times n$ real matrix of rank 1. Then ( )

(A) A is necessarily diagonalizable.

(B) A has only one nonzero column.

(C) The trace, trace(A), of A is nonzero.

(D) A has at least $n-1$ linearly independent eigenvectors.

设 $n \geq 2$, A 是秩为 1 的 $n \times n$ 实矩阵. 则 ( )

(A) A 一定可以对角化.

(B) A 只有一列是非零列.

(C) A 的迹, trace(A), 不为零.

(D) A 有至少 $n-1$ 个线性无关的特征向量.

(5) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The matrices A and B are ( )

(A) congruent and similar.

(B) congruent but not similar.

(C) similar but not congruent.

(D) neither similar nor congruent.

设 $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. 则 A 与 B ( )

(A) 合同且相似. (这里的“合同”也称“相合”.)

(B) 合同但不相似.

(C) 相似但不合同.

(D) 既不相似也不合同.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Evaluate the determinant: $\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} =$____________.

计算行列式: $\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} =$____________.

(2) Let $\mathbb{R}^{2\times 2}$ be the real vector space of 2×2 real matrices. Let V be the subspace of $\mathbb{R}^{2\times 2}$ spanned by the 4 matrices

$$A_1=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\ A_2=\begin{bmatrix}3 & 1\\ 1 & -3\end{bmatrix},\ A_3=\begin{bmatrix}5 & 1\\ 1 & -5\end{bmatrix},\ A_4=\begin{bmatrix}4 & 0\\ 0 & -4\end{bmatrix}.$$

Then $\dim V =$___________.

设 $\mathbb{R}^{2\times 2}$ 为所有 2×2 实矩阵构成的实向量空间. 令 V 为以下 4 个矩阵在 $\mathbb{R}^{2\times 2}$ 中张成 (也称"生成") 的子空间:

$$A_1=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\ A_2=\begin{bmatrix}3 & 1\\ 1 & -3\end{bmatrix},\ A_3=\begin{bmatrix}5 & 1\\ 1 & -5\end{bmatrix},\ A_4=\begin{bmatrix}4 & 0\\ 0 & -4\end{bmatrix}.$$

则 $\dim V =$___________.

(3) Let A be a 2×2 matrix and suppose α_1,α_2 are 2-dimensional linearly independent column vectors such that $A\alpha_1=0$, $A\alpha_2=4\alpha_1+2\alpha_2$. Then the eigenvalues of A are ___________.

设 A 是 2 阶方阵, α_1,α_2 是线性无关的二维列向量, 满足 $A\alpha_1=0$, $A\alpha_2=4\alpha_1+2\alpha_2$. 则 A 的所有特征值为 ___________.

(4) Suppose that the quadratic form $f(x_1,x_2,x_3)=x_1^2+ax_2^2+x_3^2+2bx_1x_2+2x_1x_3+2x_2x_3$ can be transformed by an orthogonal transformation $(x_1,x_2,x_3)^T=Q(y_1,y_2,y_3)^T$ to $y_2^2+4y_3^2$. Then $a=$ ___________, $b=$ ___________.

假定二次型 $f(x_1,x_2,x_3)=x_1^2+ax_2^2+x_3^2+2bx_1x_2+2x_1x_3+2x_2x_3$ 可由正交变换 $(x_1,x_2,x_3)^T=Q(y_1,y_2,y_3)^T$ 化为 $y_2^2+4y_3^2$. 则 $a=$ ___________, $b=$ ___________.

(5) Let A be a 3×3 matrix with eigenvalues -1, 0, 1. Suppose that

$$v_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad v_2=\begin{bmatrix}-1\\1\\0\end{bmatrix},\quad v_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

are eigenvectors belonging to the eigenvalues -1, 0, 1 respectively. Then $A^{2021}=$___________.

假设 A 为 3×3 矩阵, 其特征值为 -1, 0, 1. 假设

$$v_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad v_2=\begin{bmatrix}-1\\1\\0\end{bmatrix},\quad v_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

分别为属于特征值 -1, 0, 1 的特征向量. 则 $A^{2021}=$___________.

3. (10 points 本题共 10 分) Suppose $A=\begin{bmatrix}a & 1 & 0\\ 1 & a & -1\\ 0 & 1 & a\end{bmatrix}$ and $A^3=0$.

(a) Find $|A|$.

(b) Find the value of a.

(c) Show that $I - A$ is invertible. (Here I denotes the 3×3 identity matrix.)

(d) Find an invertible matrix X of order 3 such that $(I-A)^{-1}X = (X^{-1} - X^{-1}A)^{-1}A^2 + I$. (*Hint:* $X^{-1} - X^{-1}A = X^{-1}(I - A)$.)

设 $A = \begin{bmatrix} a & 1 & 0 \\ 1 & a & -1 \\ 0 & 1 & a \end{bmatrix}$, 且 $A^3 = 0$.

(a) 求 A 的行列式 $|A|$.

(b) 求 a 的值.

(c) 证明 $I - A$ 可逆. (这里 I 表示 3 阶单位矩阵.)

(d) 求一个 3 阶可逆矩阵 X 使得 $(I-A)^{-1}X = (X^{-1} - X^{-1}A)^{-1}A^2 + I$. (提示: $X^{-1} - X^{-1}A = X^{-1}(I - A)$.)

4. (10 points 本题共 10 分) Let

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) Find the determinant and the trace of H.

(b) Find all the singular values of H.

(c) Find a real number α such that $\mathrm{rank}(\alpha I_4 - H)$ is the smallest possible.

令

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) 求 H 的行列式和迹.

(b) 求 H 的所有奇异值.

(c) 求一个实数 α 使得 $\mathrm{rank}(\alpha I_4 - H)$ 达到最小可能的值.

5. (10 points 本题共 10 分) Suppose that the complex matrix $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$ is a Hermitian matrix.

(a) Find the value of α.

(b) Find all the complex eigenvalues of A.

(c) Find a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix.

(d) Let $B = A + A^T$, where A^T denotes the transpose of A. Show that B is a real symmetric matrix, and decide whether B is positive definite.

假设复矩阵 $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$ 是个埃尔米特矩阵.

(a) 求 α 的值.

(b) 求 A 的所有复特征值.

(c) 求一个酉矩阵 U 使得 $U^{-1}AU$ 为对角阵.

(d) 令 $B = A + A^T$, 其中 A^T 表示 A 的转置. 证明 B 是实对称阵, 并确定 B 是否正定.

6. (10 points 本题共 10 分) Let A be a 3×3 real matrix whose second and third columns are $(1, 0, 0)^T$ and $(2, 1, 0)^T$ respectively. Suppose that the QR factorization of A takes the form $A = QR$ with

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the values of x, y, z and a, b, c.

(b) Find the determinant $|A|$.

设 A 为 3 阶实方阵, 其第二列和第三列分别为 $(1, 0, 0)^T$ 和 $(2, 1, 0)^T$. 假设 A 的 QR 分解 $A = QR$ 满足

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad \text{及} \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) 求 x, y, z 和 a, b, c 的值.

(b) 求行列式 $|A|$.

7. (10 points 本题共 10 分) Let A be an $n \times n$ real symmetric positive definite matrix.

(a) Show that there exists an $n \times n$ invertible matrix R such that $A = R^T R$.

(b) Show that for all column vectors x, $y \in \mathbb{R}^n$,

$$(x^T Ay)^2 \leq (x^T Ax)(y^T Ay).$$

设 A 为 n 阶正定实对称矩阵.

(a) 证明: 存在 n 阶可逆阵 R 使得 $A = R^T R$.

(b) 证明: 对任意列向量 $x, y \in \mathbb{R}^n$ 都有

$$(x^T Ay)^2 \leq (x^T Ax)(y^T Ay).$$

8. (10 points 本题共 10 分) Let $A,\ B$ be two $n \times n$ real symmetric matrices.

(a) Suppose A is positive definite. Show that there exists an invertible $n \times n$ matrix C such that $C^T AC = I_n$ and $C^T BC$ is diagonal. (Here I_n denotes the $n \times n$ identity matrix).

(b) Suppose $B - A$ and A are positive semidefinite matrices. Show that: $\det B \geq \det A$.

设 $A,\ B$ 都为 n 阶实对称矩阵.

(a) 设 A 为正定实对称阵. 证明: 存在 n 阶可逆实矩阵 C 使得 $C^T AC = I_n$ 且 $C^T BC$ 是对角阵. (这里 I_n 为 n 阶单位阵).

(b) 设 $B - A$ 和 A 都是半正定矩阵. 证明: $\det B \geq \det A$.