Invertibility and Isomorphic Vector Spaces

Lecture 8

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Vector Spaces

Invertible Linear Maps

3.53 **Definition** ***invertible, inverse***

- A linear map $T \in \mathcal{L}(V, W)$ is called ***invertible*** if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity map on V and TS equals the identity map on W.

- A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an ***inverse*** of T (note that the first I is the identity map on V and the second I is the identity map on W).

Inverse

Inverse is unique.

3.54 Inverse is unique

An invertible linear map has a unique inverse.

Proof. Suppose $T \in \mathscr{L}(V, W)$ is invertible and S_1 and S_2 are inverses of T. Then

$$S_1 = S_1 I = S_1(TS_2) = (S_1 T)S_2 = (S_1 T)S_2 = IS_2 = S_2.$$

Thus $S_1 = S_2$.

Inverse

Now that we know that the inverse is unique, we can give it a notation.

3.55 **Notation** T^{-1}

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

- Can you name a few invertible linear maps?
- Is ST invertible? Now we are assuming that S, T are two linear maps properly defined so that the product makes sense.
- Invertibility is equivalent to injectivity and surjectivity.

Invertible Linear Map

The following result characterizes the invertible linear maps.

Proposition

A linear map is invertible if and only if it is injective and surjective.

Proof

Proof Suppose $T \in \mathscr{L}(V, W)$. We need to show that T is invertible if and only if it is injective and surjective.

First suppose T is invertible. To show that T is injective, suppose $u, v \in V$ and $Tu = Tv$. Then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v,$$

so $u = v$. Hence T is injective.

We are still assuming that T is invertible. Now we want to prove that T is surjective. To do this, let $w \in W$. Then $w = T(T^{-1}w)$, which shows that w is in the range of T. Thus range $T = W$. Hence T is surjective, completing this direction of the proof.

Proof

Now suppose T is injective and surjective. We want to prove that T is invertible. For each $w \in W$, define Sw to be the unique element of V such that $T(Sw) = w$ (the existence and uniqueness of such an element follow from the surjectivity and injectivity of T). Clearly $T \circ S$ equals the identity map on W.

To prove that $S \circ T$ equals the identity map on V, let $v \in V$. Then

$$T((S \circ T)v) = (T \circ S)(Tv) = I(Tv) = Tv.$$

This equation implies that $(S \circ T)v = v$ (because T is injective). Thus $S \circ T$ equals the identity map on V.

Proof

To complete the proof, we need to show that S is linear. To do this, suppose $w_1, w_2 \in W$. Then

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2.$$

Thus $Sw_1 + Sw_2$ is the unique element of V that T maps to $w_1 + w_2$. By the definition of S, this implies that $S(w_1 + w_2) = S(w_1) + S(w_2)$. Hence S satisfies the additive property required for linearity.

The proof of homogeneity is similar. Specifically, if $w \in W$ and $\lambda \in \mathbb{F}$, then $T(\lambda Sw) = \lambda T(Sw) = \lambda w$. Thus λSw is the unique element of V that T maps to λw. By the definition of S, this implies that $S(\lambda w) = \lambda Sw$. Hence S is linear, as desired.

Linear Maps That are NOT Invertible

Although many linear maps are invertible, there are many more linear maps that are NOT invertible.

Example

(a) The multiplication by x^2 linear map from $\mathscr{P}(\mathbb{F})$ to $\mathscr{P}(\mathbb{F})$ is not invertible because it is not surjective (1 is not in the range).

(b) The backward shift linear map from $\mathbb{F}^\infty$ to $\mathbb{F}^\infty$ is not invertible because it is not injective because it is not injective. [$(1, 0, 0, \cdots)$ is in the null space].

Isomorphic Vector Spaces

The next definition captures the idea of two vector spaces that are essentially the same, except for the names of the elements of the vector spaces.

3.58 **Definition** ***isomorphism, isomorphic***

- An ***isomorphism*** is an invertible linear map.
- Two vector spaces are called ***isomorphic*** if there is an isomorphism from one vector space onto the other one.

Isomorphism

Dimension shows whether vector spaces are isomorphic

Theorem

Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Proof

First suppose V and W are isomorphic finite-dimensional vector spaces. Thus there exists an isomorphism T from V onto W. Because T is invertible, we have null $T = \{0\}$ and range $T = W$. Thus $\dim \text{null} T = 0$ and $\dim \text{range} T = \dim W$. The formula

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

thus becomes the equation $\dim V = \dim W$, completing the proof in one direction.

Proof

To prove the other direction, suppose V and W are finite-dimensional vector spaces with the same dimension. Let $v_1, v_2, \cdots, v_n$ be a basis of V and $w_1, w_2, \cdots, w_n$ be a basis of W. Let $T \in \mathscr{L}(V, W)$ be defined by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n.$$

Then T is a well-defined linear map because $v_1, v_2, \cdots, v_n$ is a basis of V (see 3.5). Also, T is surjective because $w_1, \cdots, w_n$ spans W. T is injective and surjective. Hence V and W are isomorphic, as desired.

Isomorphism

Once bases have been fixed for V and W, $\mathscr{M}$ becomes a function from $\mathscr{L}(V,W)$ to $\mathbb{F}^{m,n}$. Notice that 3.36 and 3.38 show that $\mathscr{M}$ is a linear map. This linear map is actually invertible, as we now show.

3.60 $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are isomorphic

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$.

Proof. We already noted that $\mathscr{M}$ is linear. We need to prove that $\mathscr{M}$ is injective and surjective. Both are easy. We begin by injectivity. If $T \in \mathscr{L}(V,W)$ and $\mathscr{M}(T) = 0$, then $Tv_k = 0$ for $k = 1, 2, \cdots, n$. Because $v_1, v_2, \cdots, v_n$ is a basis of V, this implies that $T = 0$. Thus $\mathscr{M}$ is injective.

Proof.

To prove that $\mathscr{M}$ is surjective, suppose $A \in \mathbb{F}^{m,n}$. Let T be the linear map from V to W such that

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j, \text{ for } k = 1, 2, \cdots, n.$$

Obviously $\mathscr{M}(T)$ equals A, and thus the range of $\mathscr{M}$ equals $\mathbb{F}^{m,n}$.

$\dim \mathscr{L}(V, W)$

Now we can determine the dimension of the vector space of linear maps from one finite-dimensional vector space to another.

3.61 $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W).$$

Example

Example

Define

$$T:\mathscr{P}_3(\mathbb{R})\to\mathbb{R}^{2\times 2} \text{ by } T(f)=\begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}.$$

Show that T is linear and injective. And we know that $\dim\mathscr{P}_3(\mathbb{R})=\dim\mathbb{R}^{2\times 2}$, therefore, T is an isomorphism from $\mathscr{P}_3(\mathbb{R})$ onto $\mathbb{R}^{2\times 2}$.

Linear Maps Thought of as Matrix Multiplication

Previously we defined the matrix of a linear map, now we define the matrix of a vector.

3.62 **Definition** ***matrix of a vector,*** $\mathcal{M}(v)$

Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The ***matrix of*** v with respect to this basis is the n-by-1 matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \ldots, c_n$ are the scalars such that

$$v = c_1 v_1 + \cdots + c_n v_n.$$

The matrix $\mathscr{M}(v)$ of a vector $v \in V$ depends on the basis $v_1, \cdots, v_n$ of V, as well as on v.

Linear Maps Thought of as Matrix Multiplication

Recall that if A is an m by n matrix, then $A_{\cdot,k}$ denotes the k^{th} column of A, thought of as an m by 1 matrix. In the next result, $\mathscr{M}(Tv_k)$ is computed with respect to the basis $w_1, \cdots, w_m$ of W.

Proposition

3.64. Suppose $T \in \mathscr{L}(V,W)$ and $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W. Let $1 \leq k \leq n$. Then the kth column of $\mathscr{M}(T)$, which is denoted by $\mathscr{M}(T)_{\cdot,k}$, equals $\mathscr{M}(Tv_k)$.

3.65

The next result shows how the notions of the matrix of linear map, the matrix of a vector, and matrix multiplication fit together.

3.65 Linear maps act like matrix multiplication

Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v).$$

Operators

Linear maps from a vector space to itself are so important that they get a special name and special notation.

3.67 **Definition** ***operator,*** $\mathcal{L}(V)$

- A linear map from a vector space to itself is called an ***operator***.
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

The deepest and most important parts of linear algebra, as well as most the rest of this book, deal with operators.

Example

- A linear map is invertible if it is injective and surjective.
- For an operator, you might wonder whether injectivity alone, or surjectivity alone, is enough to imply invertibility.
- On infinite-dimensional vector spaces, neither condition alone implies invertibility, as illustrated by the next example:

3.68 **Example** ***neither injectivity nor surjectivity implies invertibility***

- The multiplication by x^2 operator on $\mathcal{P}(\mathbf{R})$ is injective but not surjective.
- The backward shift operator on $\mathbf{F}^\infty$ is surjective but not injective.

Injectivity is equivalent to surjectivity in finite dimensions

In view of the examples in 3.68, the next result is remarkable–it states that for operators on a finite-dimensional vector space, either injectivity or surjectivity alone implies the other condition. Often it is easier to check that an operator on a finite-dimensional vector space is injective, and then we get the surjectivity for free.

3.69

3.69 Injectivity is equivalent to surjectivity in finite dimensions

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is invertible;

(b) T is injective;

(c) T is surjective.

Example

The next example illustrates the power of the previous result. Although it is possible to prove the result in the example below without using linear algebra, the proof using linear algebra is cleaner and easier.

3.70 **Example** Show that for each polynomial $q \in \mathcal{P}(\mathbf{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ with $\big((x^2 + 5x + 7)p\big)'' = q$.

Solution Example 3.68 shows that the magic of 3.69 does not apply to the infinite-dimensional vector space $\mathcal{P}(\mathbf{R})$. However, each nonzero polynomial q has some degree m. By restricting attention to $\mathcal{P}_m(\mathbf{R})$, we can work with a finite-dimensional vector space.

Suppose $q \in \mathcal{P}_m(\mathbf{R})$. Define $T : \mathcal{P}_m(\mathbf{R}) \to \mathcal{P}_m(\mathbf{R})$ by

$$Tp = \big((x^2 + 5x + 7)p\big)''.$$

Example

Multiplying a nonzero polynomial by $(x^2 + 5x + 7)$ increases the degree by 2, and then differentiating twice reduces the degree by 2. Thus T is indeed an operator on $\mathcal{P}_m(\mathbf{R})$.

Every polynomial whose second derivative equals 0 is of the form $ax + b$, where $a, b \in \mathbf{R}$. Thus null $T = \{0\}$. Hence T is injective.

Now 3.69 implies that T is surjective. Thus there exists a polynomial $p \in \mathcal{P}_m(\mathbf{R})$ such that $\big((x^2 + 5x + 7)p\big)'' = q$, as desired.

Exercise 30 in Section 6.A gives a similar but more spectacular application of 3.69. The result in the exercise is quite difficult to prove without using linear algebra.

Homework Assignment 8

3.D: 1, 4, 7, 10, 12, 14, 18, 20.