# Solving $Ax = 0$ and $Ax = b$ (part 2)

Lecture 8

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# Complete Solution

# Solving $Ax = b$, $Ux = c$, and $Rx = d$

The case $b \neq 0$ is quite different from $b = 0$. The row operations on $A$ must act also on the right-hand side(on $b$). We begin with letters $(b_1, b_2, b_3)$ to find the solvability condition– for $b$ lie in the column space.

For the original example $Ax = b = (b_1, b_2, b_3)$, apply to both sides the operations that led from $A$ to $U$. The result is an upper triangular system $Ux = c$:

$$Ux = c \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}.$$

# Remarks

(a) The equations are inconsistent unless $b_3 - 2b_2 + 5b_1 = 0$.

(b) The dependent columns, the second and the fourth, are exactly the ones without pivots.

# Solving $Ax = b$, $Ux = c$, and $Rx = d$: Continue

For a specific example with $b_3 - 2b_2 + 5b_1 = 0$, choose $b = (1, 5, 5)$:

$$Ax = b \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

The complete solution is as follows

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

# Remark

Remark:

Every solution to $Ax = b$ is the sum of one particular solution and a solution to $Ax = 0$:

$$x_{\textbf{complete}} = x_{\textbf{particular}} + x_{\textbf{nullspace}}$$

# Rank

Elimination reveals the pivot variables and free variables. If there are $r$ pivots, there are $r$ pivot variables and $n-r$ free variables. That important number $r$ will be given a name–it is the rank of the matrix.

## Definition

*The rank of $A$ is the number of pivots. This number is $r$.*

# Theorem

## Theorem

*Suppose elimination reduces $Ax = b$ to $Ux = c$ and $Rx = d$, with $r$ pivot rows and $r$ pivot columns.* ***The rank of those matrices is*** *$r$. The last $m - r$ rows of $U$ and $R$ are zero, so there is a solution only if the last $m - r$ entries of $c$ and $d$ are also zero.*

*The complete solution is $x = x_p + x_n$. One particular solution $x_p$ has all free variables zero. Its pivot variables are the first $r$ entries of $d$, so $Rx_p = d$.*

*The nullspace solution $x_n$ are combinations of $n - r$* ***special solutions****, with one free variable equal to $1$. The pivot variables in that special solution can be found in the corresponding column of $R$ (with sign reversed).*

# Another Worked Example

There are several remarks regarding the previous theorem:

1. You see how the rank $r$ is crucial. It counts the pivot rows in the "row space" and the pivot columns in the column space.
2. There are $n-r$ special solutions in the nullspace.
3. There are $m-r$ solvability conditions on $b$ or $c$ or $d$.

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The $3$ by $4$ matrix $A$ has rank $2$:

$$Ax = b \text{ } is \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

# Another Worked Example:Continue

$$Ax = b \text{ is } \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We follow the following steps to solve this system:

1. Reduce $[A \quad b]$ to $[U \quad c]$, to reach a triangular system $Ux = c$.
2. Find the condition on $b_1, b_2, b_3$ to have a solution.
3. Describe the column space of $A$: Which plane in $\mathbb{R}^3$?
4. Describe the nullspace of $A$: Which special solutions in $\mathbb{R}^4$?

# Remarks

5. Find a particular solution to $Ax = (0, 6, -6)$ and the complete $x_p + x_n$.
6. Reduce $[U \quad c]$ to $[R \quad d]$: Special solutions from $R$ and $x_p$ from $d$.

Now let us work out the details on blackboard together!

# Two More Examples

## Example

Describe the set of attainable right-hand sides $b$(in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on $b$ that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

## Example

Suppose $A$ and $B$ are $n$ by $n$ matrices, and $AB = I$. Prove from rank$(AB) \leq$ rank$(A)$ that the rank of $A$ is $n$. So $A$ is invertible and $B$ must its two-sided inverse. Therefore, $BA = I$.

# Homework Assignment 8

2.2: 2, 5, 7, 10, 12, 15, 24, 33, 45, 63.