Gaussian Elimination(高斯消元法)

Lecture 1

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Gaussian Elimination

1. Basic Concepts: Linear Equations and Solution Set
2. Gaussian Elimination
3. Potential Problems
4. Complexity
5. Homework Assignment 1

Introduction

Probably the most important problem in mathematics is that of solving a system of linear equations. Linear systems arise in applications to such areas as business, economics, sociology, ecology, demography, genetics, electronics, engineering, and physics. Therefore, it seems appropriate to begin our lecture with solving linear equations.

Two equations in 2 unknowns:

$$\begin{cases} 1x + 2y = 3 \\ 4x + 5y = 6. \end{cases}$$

We will describe two ways, namely, elimination and determinants, to solve the above equation.

Linear Equations

A linear equation（线性方程） in n unknowns is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, a_3, \cdots, a_n$ and b are real numbers and $x_1, x_2, \cdots, x_n$ are variables. A system of m equations in n unknowns （线性方程组）is then a system of the form

$$\begin{cases} a_{11}x_1 + & a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + & a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ & \cdots\cdots, \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

One Example

In the following we will use two methods, namely elimination and determinants, to solve system:

$$\begin{cases} 1x + 2y = 3 \\ 4x + 5y = 6. \end{cases}$$

I. Elimination.
II. Determinants.

The unknowns are x and y. Let us do it together now, and you are strongly suggested to work out several similar problems yourself later. When we are done, compare these two approaches.

Solution, Consistency, Solution Set(解，相容，解集)

Definition

By a solution of an $m \times n$ system, we mean an ordered n-tuple of numbers $(x_1, x_2, \cdots, x_n)$ that satisfies all the equations of the system.

In our previous example, the solution set is $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

Definition

*If a linear system has no solution, we say that the system is **inconsistent**（不相容）. If the system has at least one solution, we say that it is **consistent**（相容）. The set of all solutions of a linear system is called the **solution set**（解集） of the system.*

Four Aspects in Chapter 1

In Chapter 1, we will explain four aspects:

(1) Linear equations lead to geometry of planes.

(2) Matrix Notation. Write the n unknowns as a vector x and the n equations as $Ax = b$.

(3) Singular Cases. No solution or infinitely many solutions.

(4) Number of elimination steps. Rough count.

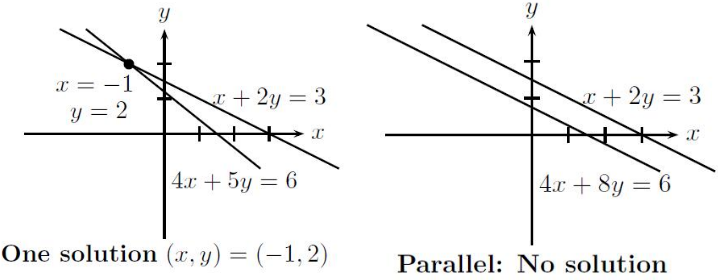
LU

The final result of this chapter will be an elimination algorithm that is about as efficient as possible.

$$PA = LU.$$

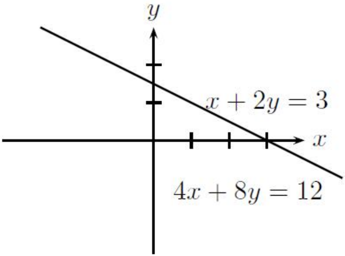
Geometry of Linear Equations

1. One Solution and No Solution



Geometry of Linear Equations

2. Infinitely many solutions



Whole line of solutions

Nonsingular and Singular

Nonsingular case has one solution. Singular cases have none or too many.

This singular case has infinitely many solutions. When elimination breaks down, we want to find every possible solution.

Equivalent System

Definition

If two systems of linear equations have the same solution set, they are said to be equivalent.

There are three operations that can be used on a system to obtain an *equivalent system*:

I. The order in which any two equations are written may be interchanged;
II. Both sides of an equation may be multiplied by the same nonzero real number;
III. A multiple of one equation may be added to(or subtracted from) another.

Example

Example

In the following, we will solve the linear system:

$$\begin{cases} 2u + \ \ v + \ \ w = 5 \\ 4u - 6v \qquad = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

Remark: It can be easily seen that subtracting multiples of the first equation from the other equations, we will get a new system of equations involving the same variables which has the same solution set as the original system, these two systems are called equivalent systems.

Triangular System

- We can use operations I and III to obtain an equivalent "strictly triangular system" if an $n \times n$ system has exactly one solution.
- For the previous system, when the elimination process is complete, at least in the "forward" direction, we obtain:

$$\begin{cases} 2u + \ \ v + \ \ w = 5 \\ \quad -8v - 2w = -12 \\ \qquad\qquad 1w = 2 \end{cases}$$

Forward elimination produced the **pivots** $2, -8, 1$. It subtracted multiples of each row from the rows beneath.

Back-Substitution

(1) The above system is solved backward, bottom to top.

(2) The last equation gives $w = 2$.

(3) Substituting into the second equation, we find $v = 1$.

(4) Then the first equation gives $u = 1$.

(5) This process is called back-substitution.

(6) Thus, the solution of the system is

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Matrix Notation

A matrix（矩阵） is an arrangement of mn elements with m rows and n columns, denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 4 & 1 \\ 3 & 2 & 2 \\ 9 & 7 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 3 & 9 & 4 \end{bmatrix}$$

Augmented matrix

One good way to write down the forward elimination steps in the previous example is to include the right-side as an extra column (called **augmented matrix**)（增广矩阵）. There is no need to copy u and v and w and $=$ at every step, so we are left with the bare minimum:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

At the end is the triangular system, ready for back-substitution.

Matrix Notation

For the system:

$$Ax = b : \begin{cases} x_1 + 2x_2 + \ \ x_3 = 3 \\ 3x_1 - \ \ x_2 - 3x_3 = -1 \\ 2x_1 + 3x_2 + \ \ x_3 = 4 \end{cases}$$

Coefficient matrix（系数矩阵）：
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

Augmented matrix（增广矩阵）：

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$$

Elementary Row Operations (初等行变换)

Corresponding to the three operations used to obtain equivalent systems, the following row operations may be applied to the augmented matrix:

I. Interchange two rows.

II. Multiply a row by a nonzero real number.

III. Replace a row by its sum with a multiple of another row.

Another Example

Example

Solve the system: $$\begin{cases} \quad\;\; -x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 = -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 = 3 \end{cases}$$

(a) Row exchange is needed, why?

(b) The procedure of solving this system of linear equations will be demonstrated on the blackboard.

Example

Solve the following system: $\begin{cases} 2x_1 + \ \ x_2 + \ \ x_3 = 1 \\ \ \ x_1 + 2x_2 + \ \ x_3 = 2 \\ \ \ x_1 + \ \ x_2 + 2x_3 = 3 \end{cases}$.

Row Echelon Form（行阶梯型矩阵）

A matrix is said to be in *row echelon form*

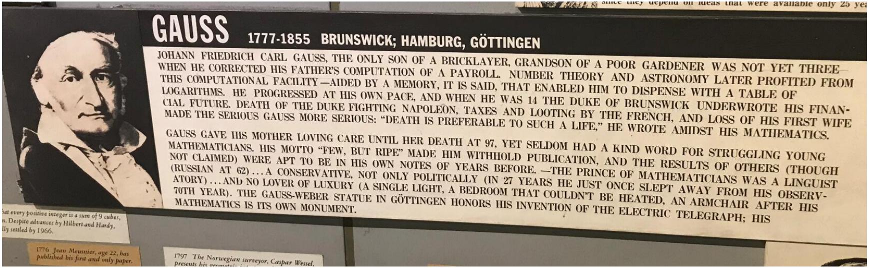
(1) The pivots are the first nonzero entries in their rows.

(2) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.

(3) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

Gaussian Elimination （高斯消元法）

The process of using row operations I, II, III to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian Elimination.**



(Courtesy: New York Hall of Science, *August 11, 2019*)

Johann Carl Friedrich Gauss

While still in his teens Gauss settled the 2000-year-old question by constructing a regular polygon of 17 sides with ruler and compass; determined precisely which regular polygons are so constructible, developed the method of least squares, and proved the law of quadratic reciprocity that had baffled Euler and Legendre. His "Disquisitiones Arithmeticae" of 1801 is the most important book on number theory ever written, even though the Gaussian integers and their bearing on biquadratic reciprocity were still in the future.

An expert in geodesy, Gauss was the first to study properties of curved surfaces as they would be measured by observers confined to the surface. This "intrinsic geometry," with its profound concepts of Gaussian curvature and bending invariants, underlies Riemannian geometry and the geometric interpretation of relativity. The systematic theory of conformal mapping is also due to Gauss.

Contributions to applied mathematics include an improved theory of perturbations in astronomy; the Gauss-Jacobi quadrature; the theory of errors; magnetic field theory; and the divergence theorem (anticipated, however, by Ostrogradski). Although Gauss was early recognized as the greatest living mathematician, his true stature became evident only after his death. His diary and letters show him familiar with such things as the elliptic modular function, theta series, complex integrals, the essential idea of quaternions, and non-Euclidean geometry, long before these were rediscovered by others.

Sometimes referred to as the Princeps mathematicorum (Latin, "the foremost of mathematicians") and "greatest mathematician since antiquity". Gauss had an exceptional influence in many fields of mathematics and science.

The Breakdown of Elimination

If a zero appears in a pivot position, elimination has to stop–either temporarily or permanently.

$$\begin{cases} u + v + w = \_\_\_ \\ 2u + 2v + 5w = \_\_\_ \\ 4u + 6v + 8w = \_\_\_ \end{cases} \rightarrow \begin{cases} u + v + w = \_\_\_ \\ 3w = \_\_\_ \\ 2v + 4w = \_\_\_ \end{cases}$$

$$\rightarrow \begin{cases} u + v + w = \_\_\_ \\ 2v + 4w = \_\_\_ \\ 3w = \_\_\_ \end{cases}$$

The system is now triangular, and back-substitution will solve it.

The Breakdown of Elimination

Singular (incurable)

$$\begin{cases} u + v + w = \_\_\_ \\ 2u + 2v + 5w = \_\_\_ \\ 4u + 4v + 8w = \_\_\_ \end{cases} \to \begin{cases} u + v + w = \_\_\_ \\ 3w = \_\_\_ \\ 4w = \_\_\_ \end{cases}$$

There is no exchange of equations that can avoid zero in the second pivot position.

The Cost of Elimination

How many separate arithmetical operations does elimination require, for n equations in n unknowns?

(a) First of all, it takes n operations for every zero we achieve, and there are $n-1$ rows underneath the first one.

(b) Therefore, the first stage of elimination needs $n(n-1)$ operations.

(c) The total number of operations is computed as follows:

$$(1^2+\cdots+n^2)-(1+2+\cdots+n)=\frac{n^3-n}{3}.$$

(d) If n is at all large, a good estimate for the number of operations is $\frac{1}{3}n^3$.

Homework Assignment 1

1.3: 1, 3, 6, 10, 13, 19, 24, 28, 31.