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MA215 Probability Theory

Assignment 12

1. The covariance between X and Y, denoted by $\mathrm{Cov}(X,Y)$, is defined by

$$\mathrm{Cov}(X,Y) \triangleq E[(X-E(X))(Y-E(Y))].$$

Show that

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y).$$

Proof. $\mathrm{Cov}(X,Y) = E[(X-E(X))(Y-E(Y))] = E[XY - XE(Y) - YE(X) + E(X)E(Y)]$

$= E(XY) - E(XE(Y)) - E(YE(X)) + E[E(X)E(Y)]$

$\because E(X), E(Y)$ are constants. $\therefore \mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) - E(Y)E(X)$

$+ E(X)E(Y) = E(XY) - E(X)E(Y)$ $\square$

2. Let X be a discrete random variable with p.m.f.

$$P\{X=0\} = P\{X=1\} = P\{X=-1\} = \frac{1}{3}.$$

Define

$$Y = \begin{cases} 0, & \text{if } X \neq 0, \\ 1, & \text{if } X = 0. \end{cases}$$

(i) Show that $\mathrm{Cov}(X,Y) = 0$.

(ii) Find the joint p.m.f. of X and Y, and show that X and Y are not independent.

(i) $E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + (-1) \cdot P(X=-1) = 0$

$P(X=0) = \frac{1}{3}$. $P(X \neq 0) = \frac{2}{3}$

$\therefore E(Y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1) = \frac{1}{3}$

$\because$ In X and Y, at least one of them equals to 0. $\therefore E(XY) = 0$

$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0$

(ii) $P(X=0, Y=1) = P(X=0) = \frac{1}{3}$ $P(X=-1, Y=0) = P(X=-1) = \frac{1}{3}$. $P(X=1, Y=0) = P(X=1) = \frac{1}{3}$

$\therefore$ p.m.f: $P(X=x, Y=y) = \begin{cases} \frac{1}{3}, & (x,y) = (0,1) \\ \frac{1}{3}, & (x,y) = (-1,0) \\ \frac{1}{3}, & (x,y) = (1,0) \\ 0, & \text{otherwise} \end{cases}$

$P(Y=1)=\frac{1}{3}$, $P(Y=0)=\frac{2}{3}$

$\because P(X=0, Y=1)=\frac{1}{3}$, $P(X=0)P(Y=1)=\frac{1}{9}$. $\quad \therefore P(X=0, Y=1) \neq P(X=0)P(Y=1)$

$\therefore X, Y$ are not independent.

3. Show that the following conclusions are true:

(i) $\operatorname{Cov}(X, Y)=\operatorname{Cov}(Y, X)$;

(ii) $\operatorname{Cov}(X, X)=\operatorname{Var}(X)$;

(iii) $\operatorname{Cov}(aX, Y)=a\operatorname{Cov}(X, Y)$, where a is a constant;

(iv) $\operatorname{Cov}\left(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j\right)=\sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{Cov}(X_i, Y_j)$;

(v) If X is a random variable and C is a constant, then $\operatorname{Cov}(X, C)=0$.

(vi) Show that the following statements are true:

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)=\sum_{i=1}^{n} \operatorname{Var}(X_i)+\sum_{1 \leqslant i \neq j \leqslant n} \operatorname{Cov}(X_i, X_j),$$

or, equivalently,

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)=\sum_{i=1}^{n} \operatorname{Var}(X_i)+2 \sum_{1 \leqslant i<j \leqslant n} \operatorname{Cov}(X_i, X_j).$$

Further show that if $X_1, \ldots, X_n$ are pairwise independent (i.e., X_i and X_j are independent for $1 \leqslant i \neq j \leqslant n$), then we have

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)=\sum_{i=1}^{n} \operatorname{Var}(X_i).$$

(i) $\operatorname{Cov}(X,Y) = E[(X-E(X))(Y-E(Y))] = E[(Y-E(Y))(X-E(X))] = \operatorname{Cov}(Y,X)$

(ii) $\operatorname{Cov}(X,X) = E[(X-E(X))^2] = E(X^2) - 2E(XE(X)) + E(X)^2 = E(X^2) - E(X)^2 = \operatorname{Var}(X)$

(iii) $\operatorname{Cov}(aX,Y) = E[(aX-E(aX))(Y-E(Y))] = E[a(X-E(X))(Y-E(Y))]$

$= aE[(X-E(X))(Y-E(Y))] = a\operatorname{Cov}(X,Y)$

(iv) $\operatorname{Cov}(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j) = E[(\sum_{i=1}^{m} X_i - E(\sum_{i=1}^{m} X_i))(\sum_{j=1}^{n} Y_j - E(\sum_{j=1}^{n} Y_j))] = E(\sum_{i=1}^{m} X_i \sum_{j=1}^{n} Y_j)$

$- E(\sum_{i=1}^{m} X_i)E(\sum_{j=1}^{n} Y_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} E(X_i Y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} E(X_i)E(Y_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} \operatorname{Cov}(X_i, Y_j)$

(v) $\operatorname{Cov}(X,C) = E[(X-E(X))(C-E(C))] = E[(X-E(X))(C-C)] = 0$.

(vi) $\operatorname{Var}(\sum_{i=1}^{n} X_i) = E[(\sum_{i=1}^{n} X_i - E(\sum_{i=1}^{n} X_i))^2] = E[\sum_{i=1}^{n}(X_i - E(X_i))^2 + 2\sum_{1\le i<j\le n}(X_i - E(X_i))(X_j - E(X_j))]$

$= \sum_{i=1}^{n} E[(X_i - E(X_i))^2] + 2\sum_{1\le i<j\le n} E[(X_i - E(X_i))(X_j - E(X_j))]$

$= \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2\sum_{1\le i<j\le n} \operatorname{Cov}(X_i, X_j)$

if $X_1, \cdots, X_n$ are pairwise independent, then $\operatorname{Cov}(X_i, X_j) = 0$. $(i<j)$

$\therefore \operatorname{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \operatorname{Var}(X_i)$

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common expectation μ and common variance σ^2. Let $\bar{X}$ and S^2 be defined as follows.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Find $E[\bar{X}]$, $\mathrm{Var}(\bar{X})$, and $E\left[\frac{S^2}{n-1}\right]$.

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\cdot n\mu = \mu$$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2}\cdot n\delta^2 = \frac{\delta^2}{n}$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - X)^2$$

$$\therefore E\left(\frac{S^2}{n-1}\right) = \frac{1}{(n-1)^2} E\left(\sum_{i=1}^{n}(X_i - X)^2\right) = \frac{1}{(n-1)^2}\cdot(n-1)\delta^2 = \frac{\delta^2}{n-1}$$

5. Let I_A and I_B be the indicator variables for the events A and B. That is,

$$I_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases} \qquad I_B(\omega) = \begin{cases} 1, & \omega \in B, \\ 0, & \omega \notin B. \end{cases}$$

Show that

(i) $E[I_A] = P(A)$, $E[I_B] = P(B)$, $E[I_A I_B] = P(AB)$.

(ii) $\mathrm{Cov}(I_A, I_B) = P(AB) - P(A)P(B)$.

Proof. (i) $E(I_A) = 1\cdot P(A) + 0\cdot(1 - P(A)) = P(A)$

$E(I_B) = 1\cdot P(B) + 0\cdot(1 - P(B)) = P(B)$

$E(I_A I_B) = 1\cdot 1\cdot P(A\cap B) + 1\cdot 0 P(A\cap\neg B) + 0\cdot 1\cdot P(\neg A\wedge B) + 0\cdot 0\cdot P(\neg A\wedge\neg B) = P(AB)$

(ii) $\mathrm{Cov}(I_A, I_B) = E[(I_A - E(I_A))(I_B - E(I_B))] = E(I_A I_B) - E(I_A)E(I_B)$

$= P(AB) - P(A)P(B)$. $\square$

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common variance σ^2. Show that for any fixed i $(1 \leqslant i \leqslant n)$,

$$\mathrm{Cov}(X_i - \bar{X}, \bar{X}) = 0,$$

where $\bar{X}$ is the sample mean (i.e. $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$).

Proof. $\mathrm{Cov}(X_i - \bar{X}, \bar{X}) = \mathrm{Cov}(X_i, \bar{X}) - \mathrm{Cov}(\bar{X}, \bar{X})$

$\because \bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j \quad \therefore \mathrm{Cov}(X_i, \bar{X}) = \mathrm{Cov}\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j)$

$i = j$: $\mathrm{Cov}(X_i, X_j) = \mathrm{Var}(X_i) = \delta^2$, $i \neq j$: $\mathrm{Cov}(X_i, X_j) = 0$.

$\therefore \mathrm{Cov}(X_i, \bar{X}) = \frac{1}{n}(\delta^2 + 0 + \cdots + 0) = \frac{\delta^2}{n}$

$$Cov(\bar{X}, \bar{X}) = Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

$$\therefore \ Cov(X_i - \bar{X}, \bar{X}) = Cov(X_i, \bar{X}) - Cov(\bar{X}, \bar{X}) = 0 \qquad \square$$