Inverses and Transposes (矩阵的逆和转置)

Lecture 5

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Inverses and Tranposes (逆和转置)

Introduction

- The inverse of an n by n matrix is another n by n matrix. The inverse of A is written as A^{-1}(and pronounced "A inverse"). The fundamental property is simple: If you multiply by A and then multiply by A^{-1}, you are back where you started.
- Not all matrices have inverses.
- An inverse is impossible when Ax is zero and x is nonzero. Then A^{-1} would have to get back from $Ax = 0$ to x. No matrix can multiply that zero vector Ax and produce a nonzero vector x.
- Our goals are to define the inverse matrix and compute it and use it, when A^{-1} exists–and then to understand which matrices don't have inverses.

Inverse Matrix (逆矩阵)

Definition

The inverse of A is a matrix B such that $BA = I$ and $AB = I$. There is at most one such B, and it is denoted by A^{-1}: $A^{-1}A = I$ and $AA^{-1} = I$. If A has an inverse, then A is said to be invertible.

- The inverse exists if and only if elimination produces n pivots.
- The matrix A can not have two different inverses.
- If A is invertible, the one and only solution to $Ax = b$ is $x = A^{-1}b$.
- Suppose there is a nonzero vector x such that $Ax = 0$. Then A can not have an inverse.
- Invertibility of a 2×2 matrix.
- A diagonal matrix has an inverse provided no diagonal entries are zero.

The inverse of a 2 by 2 invertible matrix

If $ad - bc \neq 0$, then the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

You can verify this fact very quickly by multiplying these two matrices out. Actually, AA^{-1} and $A^{-1}A$ are both equal to the identity matrix. In the future, we will call $ad - bc$ the determinant of matrix A.

The inverses come in reverse order

Proposition

A product AB of invertible matrices is inverted by $B^{-1}A^{-1}$:

$$\textbf{\textit{Inverse of}}\ AB : (AB)^{-1} = B^{-1}A^{-1}.$$

A similar rule holds with three or more matrices:

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

and

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1},$$

given that $A, B, C, A_1, \cdots, A_k$ are invertible.

The calculation of A^{-1}: The Gauss-Jordan Method (高斯-约旦方法)

Consider the equation $AA^{-1} = I$. If it is taken a column at a time, that equation determines each column of A^{-1}.

Example

Compute A^{-1} if

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Note: The determinant is the product of the pivots. It enters at the end when the rows are divided by the pivots.

Solution

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}.$$

Another Example

Example

Let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 4 \\ 2 & -1 & 0 \end{bmatrix}.$$

Find A^{-1}.

Elementary Matrices

Theorem

If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Definition

A matrix B is row equivalent to a matrix A if there exists a finite sequence $E_1, E_2, \cdots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A.$$

In other words, B is equivalent to A if B can be obtained from A by a finite number of row operations.

Invertible=Nonsingular(n pivots)

Ultimately we want to know which matrices are invertible and which are not. This question is so important that it has many answers. Each of the first five chapters will give a different(but equivalent) test for invertibility.

Proposition

Let A be an $n \times n$ matrix. The following are equivalent:

(a) A *is nonsingular.*

(b) $Ax = 0$ *has only the trivial solution 0.*

(c) A *is row equivalent to* I.

Remarks:

1. A 1-sided inverse of a square matrix is automatically a 2-sided inverse.
2. Every nonsingular matrix is invertible.
3. If A is invertible, it has n pivots.

The Transpose Matrix (转置矩阵)

We need one more matrix, and fortunately it is much simpler than the inverse. The transpose of A is denoted by A^T. Its columns are taken directly from the rows of A–the ith row of A becomes the ith column of A^T.

Definition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix B defined by

$$b_{ji} = a_{ij}$$

for $j = 1, 2, \cdots, n$ and $i = 1, 2, \cdots, m$. The transpose of A is denoted by A^T.

Algebraic Rules for Transposes

There are five basic algebraic rules involving transposes:

Proposition

Algebraic Rules for Transposes

1. $(A^T)^T = A$
2. $(\alpha A)^T = \alpha A^T$
3. $(A+B)^T = A^T + B^T$
4. $(AB)^T = B^T A^T$
5. *The transpose of* A^{-1} *is* $(A^{-1})^T = (A^T)^{-1}$

Symmetric Matrices (对称矩阵)

With the above rules established, we can introduce a special class of matrices, probably the most important class of all.

Definition

A symmetric matrix is a matrix that equals its own transpose: $A^T = A$.

Symmetric Matrices：

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{4}\begin{bmatrix} 8 & -2 \\ -2 & 1 \end{bmatrix}.$$

Remarks:

- The matrix is necessarily square.
- A symmetric matrix need not be invertible.

Symmetric Products R^TR, RR^T, and LDL^T

Multiplying any matrix R by R^T gives a symmetric matrix. The transpose of R^TR is $R^T(R^T)^T$, which is R^TR.

LU misses the symmetry but LDL^T captures it perfectly.

Theorem

Suppose $A = A^T$ can be factored into $A = LDU$ without row exchanges. Then U is the transpose of L. ***The symmetric factorization becomes $A = LDL^T$.***

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage. The smaller matrices stay symmetric as elimination proceeds, and we can work with half of the matrix! The work of elimination is reduced from $n^3/3$ to $n^3/6$. There is no need to store entries from both sides of the diagonal, or to store both L and U.

Homework Assignment

1.6: 1, 2, 6, 11, 17, 19, 21, 40, 45, 49.