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MA215 Probability Theory

Assignment 07

1. Find the following values by using the Statistical Tables:

   (a) $F(-1.72)$, $F(-1.723)$, $F(0.48)$ and $F(1.234)$, where $F(x)$ is the c.d.f. of the standard normal random variable.

   (b) Find x such that $F(x) = 0.546$, where $F(x)$ is the c.d.f. of the standard normal random variable. Similarly find y such that $F(y) = 0.258$.

(a) The p.d.f. of the standard normal random variable is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < +\infty$$

$$\therefore F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

According to the table, $F(-1.72) \approx 0.043$ $F(-1.723) \approx 0.042$

$F(0.48) \approx 0.684$ $F(1.234) \approx 0.891$.

(b) $x \approx 0.10$. $y \approx -0.65$

2. Assume that heights of children in a certain age group average are normally distributed, i.e., $X \sim N(\mu, \sigma^2)$, where $\mu = 58.4$ inches and with $\sigma = 2.9$ inches.

   (a) What proportion of children are between 57 and 61 inches tall?

   (b) What is the number c such that 90% of the children's height in a certain age group average is less than c?

(a) Suppose that $X' = \frac{X-\mu}{\delta}$, then $X \sim N(\mu, \delta^2) \Rightarrow X' \sim N(0,1)$

So X' is standard normal random variable

When $57 \le X \le 61$, $-0.483 \le X' \le 0.897$

$\therefore P(-0.483 \le X' \le 0.897) = F(0.897) - F(-0.483)$

According to the table, $P(-0.483 \le X' \le 0.897) \approx 0.816 - 0.316 = 0.500$

So about 50% of children are between 57 and 61 inches tall.

(b) $P(X' < c) = 0.9$ According to the table, $c \approx 1.28$

$\because X' = \frac{X-\mu}{\delta}$, $\therefore X = \delta X' + \mu < 58.4 + 2.9 \times 1.28 \approx 62.112$

So c is about 62.112 inches.

3. Suppose $X \sim N(\mu, \sigma^2)$ and let $Y = \exp(X) = e^X$.

(a) What are all possible values of Y?

(b) Obtain the probability density function of Y.

(a) X follows normal distribution, so $-\infty < X < +\infty$

$\lim_{x \to -\infty} Y = 0$. $\lim_{x \to +\infty} Y = +\infty$ $\therefore Y \in (0, +\infty)$

(b) CDF of Y: $P(Y \le y) = P(e^x \le y) = P(X \le \ln y)$ $\because X \sim N(\mu, \delta^2)$ $\therefore P(X \le \ln y) = \Phi\left(\frac{\ln y - \mu}{\delta}\right)$

So PDF of Y: $f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} \Phi\left(\frac{\ln y - \mu}{\delta}\right) = \varphi\left(\frac{\ln y - \mu}{\delta}\right) \cdot \frac{1}{y} \cdot \frac{1}{\delta}$

φ is the PDF of the standard normal distribution, $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$\therefore f_Y(y) = \frac{1}{y\delta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\delta}\right)^2}$, $y > 0$

4. Suppose $X \sim N(\mu, \sigma^2)$ and let $Y = aX + b$ where a and b are two constants and the constant a is not zero.

(a) What are all possible values of Y?

(b) Obtain the probability density function of Y.

(c) Explain Y is also normally distributed. What are the parameters of Y?

(a) X follows a normal distribution, so $-\infty < X < +\infty$

$\therefore -\infty < Y = aX + b \, (a \neq 0) < +\infty$ $\therefore Y \in (-\infty, +\infty)$

(b) PDF of X: $f_X(x) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(x-\mu)^2}{2\delta^2}}$

using transformation formula: $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|} \cdot \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{1}{2\delta^2}\left(\frac{y-b-a\mu}{a}\right)^2}$

$f_Y(y) = \frac{1}{|a|\sqrt{2\pi\delta^2}} e^{-\frac{1}{2}\left(\frac{y-(a\mu+b)}{\delta|a|}\right)^2}$

(c) Because Y is a linear transformation of X, so Y is also normally distributed.

$E(Y) = E(aX+b) = aE(X) + b = a\mu + b$

$Var(Y) = Var(aX+b) = a^2 Var(X) = a^2\delta^2$

$\therefore Y \sim N(a\mu+b, a^2\delta^2)$

5. Suppose $X \sim N(0, 1)$ and let $Y = X^2$.

(a) What are all possible values of Y?

(b) Obtain the probability density function of Y.

(a) $X \sim N(0,1)$ $\therefore -\infty < X < +\infty$ $\therefore Y = x^2 \geq 0$ $Y \in [0, +\infty)$

(b) CDF of Y: $F_Y(y) = P(Y \le y) = P(x^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$

$F_Y(y) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\Phi(\sqrt{y})$ (Φ is CDF of the standard normal distribution)

PDF of Y: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(2\Phi(\sqrt{y})) = 2\varphi(\sqrt{y}) \frac{d}{dy}(\sqrt{y}) = \frac{\varphi(\sqrt{y})}{\sqrt{y}}$

$\varphi(\sqrt{y}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}$

$\therefore f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}, & y>0 \\ 0, & y=0 \end{cases}$

6. Suppose $Y \sim N(0,1)$. Let $-\infty < a < b < +\infty$ and $m = \frac{1}{2}\max\{a^2, b^2\}$. Show that

$$(b-a)e^{-m} \leqslant \sqrt{2\pi} P\{a \leqslant Y \leqslant b\} \leqslant b-a.$$

Proof. $\because Y \sim N(0,1)$ $\therefore P(a \le Y \le b) = \Phi(b) - \Phi(a)$

According to Lagrange: $\exists\, c \in (a,b)$, $\Phi'(c)(b-a) = \Phi(b) - \Phi(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}(b-a)$

$\therefore \sqrt{2\pi} P(a \le Y \le b) = (b-a) e^{-\frac{c^2}{2}}$

$\because m = \frac{1}{2}\max\{a^2, b^2\}$. $\therefore \frac{c^2}{2} \le m$. $\therefore \sqrt{2\pi} P(a \le Y \le b) \ge (b-a)e^{-m}$

$\frac{c^2}{2} \ge 0$. $\therefore \sqrt{2\pi} P(a \le Y \le b) \le (b-a)e^{0} = b-a$ $\square$

7. Suppose $Y \sim N(0,1)$. Show that for any $y > 0$, we have

$$\frac{1}{y} - \frac{1}{y^3} \leqslant \sqrt{2\pi} e^{\frac{y^2}{2}} P\{Y \geqslant y\} \leqslant \frac{1}{y}.$$

Hint: First show that for any $y > 0$,

$$e^{-\frac{y^2}{2}}\left(\frac{1}{y} - \frac{1}{y^3}\right) = \int_y^{+\infty} e^{-\frac{x^2}{2}}\left(1 - \frac{3}{x^4}\right) dx.$$

Proof. $P(Y \ge y) = 1 - P(Y \le y) = 1 - \Phi(y) = \int_y^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_y^{+\infty} e^{-\frac{x^2}{2}} dx$

For any $x \in [y, +\infty)$, $e^{-\frac{x^2}{2}}(1 - \frac{3}{x^4}) \le e^{-\frac{x^2}{2}}$

$\therefore \int_y^{+\infty} e^{-\frac{x^2}{2}} dx \ge \int_y^{+\infty} e^{-\frac{x^2}{2}}(1 - \frac{3}{x^4}) dx = e^{-\frac{y^2}{2}}(\frac{1}{y} - \frac{1}{y^3})$

$\therefore \sqrt{2\pi} e^{\frac{y^2}{2}} P(Y \ge y) \ge \frac{1}{y} - \frac{1}{y^3}$

$y \cdot \int_y^{+\infty} e^{-\frac{x^2}{2}} dx = \int_y^{+\infty} y e^{-\frac{x^2}{2}} dx \le \int_y^{+\infty} x e^{-\frac{x^2}{2}} dx = -e^{-\frac{x^2}{2}}\Big|_y^{+\infty} = -\lim_{x\to\infty} e^{-\frac{x^2}{2}} + e^{-\frac{y^2}{2}} \le e^{-\frac{y^2}{2}}$

$\therefore \int_y^{+\infty} e^{-\frac{x^2}{2}} dx \le \frac{1}{y} e^{-\frac{y^2}{2}}$ $\therefore \sqrt{2\pi} e^{\frac{y^2}{2}} P(Y \ge y) \le \frac{1}{y}$ $\square$