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MA215 Probability Theory

Assignment 06

1. Suppose a probability density function $f(x)$ takes the form of

$$f(x) = cx^3, \quad (0 < x < 1).$$

   (a) Find the value of the constant c.

   (b) Sketch $f(x)$.

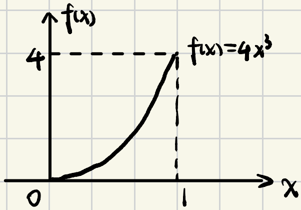
   (c) Obtain the cumulative distribution function $F(x)$.

   (d) Find $P(0.25 < X < 0.75)$.

(a) By definition of P.D.F, $\because 0<x<1 \quad \therefore \int_0^1 f(x)\,dx = 1 \Rightarrow \frac{1}{4}cx^4\Big|_0^1 = 1$

$\therefore c=4$

(b)



(c) $F(x) = P(X\le x) = \int_0^x f(t)\,dt = x^4 \quad (0<x<1)$

$$\therefore F(x) = \begin{cases} 0, & x\le 0 \\ x^4, & 0<x<1 \\ 1, & x\ge 1 \end{cases}$$

(d) $P(0.25<X<0.75) = P(0.25<X\le 0.75) = P(X\le 0.75) - P(X\le 0.25) = F(0.75) - F(0.25)$

$= 0.75^4 - 0.25^4 = 0.3125$

2. A continuous random variable X is said to have a *memoryless* property if

$$P(X > s+t \mid X > t) = P(X > s)$$

is true for all $s > 0$ and $t > 0$. Show that any exponential random variable has the memoryless property.

Proof. Suppose we have an exponential random variable which P.D.F is given by: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x\ge 0 \\ 0, & x<0 \end{cases}$

We only consider $x\ge 0$. $f(x) = \lambda e^{-\lambda x}$

$P(X>s) = \int_s^{+\infty} f(x)\,dx = -e^{-\lambda x}\Big|_s^{+\infty} = e^{-\lambda s}$, so as $P(X>t) = e^{-\lambda t}$

$P(X>s+t) = e^{-\lambda(s+t)}$

$P(X>s+t \mid X>t) = \dfrac{P(X>s+t \cap X>t)}{P(X>t)} \quad \because s>0 \quad \therefore \{X \mid X>s+t\} \supseteq \{X \mid X>t\}$

$\therefore P(X>s+t \wedge X>t) = P(X>s+t)$ $\therefore P(X>s+t \mid X>t) = \frac{P(X>s+t)}{P(X>t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$

$= e^{-\lambda s} = P(X>s)$

So it has the memoryless property. $\square$

3. For a certain type of electrical component, the lifetime X (in thousands of hours) has an exponential distribution with rate parameter $\lambda = 0.5$. What is the probability that a new component will last longer than 1000 hours? If a component has already lasted 1000 hours, what is the probability that it will last at least 1000 hours more?

The P.D.F: $f(x) = \lambda e^{-\lambda x}$ $(x \geqslant 0)$, so the probability is given by: $P(X \geqslant t) = e^{-\lambda t}$

For 1000 hours, $t=1$, we have: $P(X \geqslant 1) = e^{-0.5\times 1} = e^{-0.5} \approx 0.6065$

So the probability that a new component will last longer than 1000 hours is about 0.6065.

The probability that will last 1000 hours more is $P(X \geqslant 2 \mid X \geqslant 1)$

Suppose $s=t=1000$ hours $=1$ By Problem 2, we know $P(X \geqslant s+t \mid X \geqslant s) = P(X \geqslant t)$

$\therefore P(X \geqslant 2 \mid X \geqslant 1) = P(X \geqslant 1) = e^{-0.5} \approx 0.6065$

4. The number of phone calls received at a certain residence in any period of t hours is a Poisson random variable with parameter $\lambda = \mu t$ for some $\mu > 0$. What is the probability that no calls are received during a period of t hours? Denoting by T the time (in hours) at which the first call after time zero is received, write down an expression for $P(T \leqslant t)$. What is the name of the distribution of the random variable T?

The probability of receive k calls is: $P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

When there's no calls, $k=0$, $P(X=0) = e^{-\lambda} = e^{-\mu t}$

From 0 to t time, there's at least one call, so $P(T \leq t) = 1 - P(X=0)$

$= 1 - e^{-\mu t}$

According to the C.D.F, T is exponentially distributed,

$T \sim \text{Exponential}(\mu)$

5. The Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ has cumulative distribution function

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}, \quad x > 0.$$

(a) Find the median of the distribution in terms of the parameters α and β (The median of a random variable X is the value m such that $P(X \leqslant m) = 0.5$).

(b) From the Weibull distribution function given above, derive an expression for the corresponding probability density function.

(a) Suppose that the median of the distribution is $X = m$

then $P(X \leq m) = F(m) = 0.5 \Rightarrow F(m) = 1 - \exp\{-(\frac{m}{\alpha})^{\beta}\} = 0.5$

$\Rightarrow \exp\{-(\frac{\alpha}{m})^{\beta}\} = 0.5 \Rightarrow \beta \ln\frac{\alpha}{m} = \ln(-\ln 0.5) = \ln\ln 2 \Rightarrow m = \frac{\alpha}{(\ln 2)^{1/\beta}}$

(b) P.D.F: $f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}(1 - \exp\{-(\frac{\alpha}{x})^{\beta}\} = \exp\{-(\frac{\alpha}{x})^{\beta}\} \cdot (-\beta)(\frac{\alpha}{x})^{\beta-1} \cdot (-\frac{\alpha}{x^2})$

$= \frac{\beta x^{\beta}}{x^{\beta+1}} \exp\{-(\frac{\alpha}{x})^{\beta}\}$, $x > 0$