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MA215 Probability Theory

Assignment 05

1. A r.v. X is said to follow the logarithmic distribution with parameter p $(0 < p < 1)$, if X has probability mass function

$$P(X = k) = A\left(\frac{p^k}{k}\right) \text{ for } k = 1, 2, 3, \ldots$$

for some appropriate constant A. Find the value of the constant A in terms of the parameter p.

According to the definition of p.m.f., $\sum_{k=1}^{\infty} P(X=k)=1$

$\Rightarrow A\sum_{k=1}^{\infty}\frac{p^k}{k}=1$ By Taylor expansion, $\sum_{k=1}^{\infty}\frac{p^k}{k}=-\ln(1-p)$ $(0<p<1)$

$\therefore A(-\ln(1-p))=1$ $\therefore A=-\frac{1}{\ln(1-p)}$

2. Suppose that independent trials, each having a probability p, $0 < p < 1$, of being a success, are performed until a success occurs. Let X equal the number of trials required. What are the possible values of X? Write down the p.m.f. of X.

When a success occurs in kth trials, the former (k-1) trials are all fail. So $X \in N^*$. p.m.f: $P(X=k)=(1-p)^{k-1}p$, $k \in N^*$

3. Suppose that independent trials, each having a probability p, $0 < p < 1$, of being a success, are performed until a total of r successes is accumulated, where $r \geqslant 1$ is a positive integer. Let X equal the number of trials required. What are the possible values of X? Write down the p.m.f. of X.

When k trials are required, it must have (r-1) successes in the former (k-1) trials and also succeed in kth trial.

So $X=r, r+1, r+2, \cdots$ p.m.f: $P(X=k)=C_{k-1}^{r-1}p^{r-1}(1-p)^{k-r}\cdot p = C_{k-1}^{r-1}p^{r}(1-p)^{k-r}$,

$k=r, r+1, r+2, \cdots$

4. Assuming that each dart has probability 0.2 of hitting its target, give the c.d.f. of the number of darts one should throw at the target to get the first successful hit. What is the c.d.f. of the number of throws required to get two hits? Finally what is the probability of at least one hit in n throws, and what is the smallest value of n for which this is greater than 0.9?

Suppose that X implies the number of darts when getting the first successful hit, Y implies the number of darts when getting two successful hits, Z implies n hits.

p.m.f: $P(X=k) = 0.8^{k-1} \times 0.2, \ k=1,2,3,\cdots$

$P(Y=k') = C_{k'-1}^{1} 0.2 \times 0.8^{k'-2} \times 0.2 = (k'-1) \times 0.8^{k'-2} \times 0.04, \ k'=2,3,4,\cdots$

c.m.f: $F_1(k) = \sum_{i=1}^{k} P(X=i) = (0.8^0 + 0.8^1 + \cdots + 0.8^{k-1}) \times 0.2 = \frac{1-0.8^k}{1-0.8} \times 0.2 = 1-0.8^k$

$k=1,2,3,\cdots$

$F_2(k') = \sum_{i=2}^{k'} P(Y=i) = 0.04\left(1 + 2\times 0.8^1 + \cdots + (k'-1)0.8^{k'-2}\right) \triangleq 0.04 S_{k'}$

$0.8 S_{k'} = 0.8^1 + 2\times 0.8^2 + \cdots + (k'-1)0.8^{k'-1}$

$\therefore -0.2 S_{k'} = (k'-1)0.8^{k'-1} - (0.8^{k'-2} + \cdots + 0.8 + 1) = (k'+4)0.8^{k'-1} - 5$

$\therefore F_2(k') = 1 - (k'+4)0.8^{k'-1} \times 0.2$

$P(\text{at least one hit}) = 1 - P(\text{none}) = 1 - 0.8^n$

when $P(\text{at least one hit}) > 0.9 \Rightarrow 1-0.8^n > 0.9 \Rightarrow n \ln 0.8 < \ln 0.1 \Rightarrow n > \frac{\ln 0.1}{\ln 0.8} \approx 10.3$

as $n \in N^*$, so the smallest value of n is 11.

5. The number of phone calls received at a certain residence in any period of c hours is a Poisson random variable with parameter $\lambda = 0.5c$.

(a) What is the probability that the phone rings during a given 15 minute period?

(b) How long a period must one wait for the probability of at least one phone call during that period to be at least 0.5?

(a) $c = 0.25$. $\lambda = 0.5c = 0.125$

By the definition of Poisson random variable, $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

The phone does not ring: $P(X=0) = e^{-\lambda} = e^{-0.125}$

So the phone rings $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-0.125} \approx 0.1175$

(b) $P(X \geq 1) \geq 0.5 \Rightarrow P(X=0) = e^{-\lambda} \leq 0.5 \Rightarrow \lambda \geq \ln 2$

$\Rightarrow c = 2\lambda \geq 2\ln 2 \approx 1.386$

$\therefore$ One must wait for about 1.386 hours.