MA215 Probability Theory

Assignment 15

1. Suppose that a discrete random variable X has finite kth moment, i.e., $E(|X|^k) < \infty$ ($k > 0$, but k may not be a positive integer). Show that for any $\varepsilon > 0$,

$$P\{|X| \geqslant \varepsilon\} \leqslant \frac{E(|X|^k)}{\varepsilon^k}.$$

2. Suppose that $\{X_1, X_2, \cdots, X_n, \cdots\}$ is a sequence of independent r.v.s (not necessarily with the same distribution), each with finite (but not necessarily with the same) mean and uniformly bounded variance by $M < \infty$ (i.e., $\mathrm{Var}\,(X_i) \leqslant M\ \forall i \geqslant 1$). Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean. Show that for any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} P\left\{|\bar{X}_n - E\bar{X}_n| > \varepsilon\right\} = 0.$$

3. Suppose that $\{X_1, X_2, \cdots\}$ is a sequence of i.i.d. r.v.s with common mean 0 and variance 16. Let n be sufficiently large and $Y = X_1 + X_2 + \cdots + X_n$. Estimate the value of $P\{26.08 < Y \leqslant 44.12\}$ when $n = 100$. (Hint: using the central limit theorem and normal approximation method.)

4. Let $\{X_n\}_{n\geqslant 1}$ be a sequence of i.i.d. random variables with a common normal distribution $N(20, 25)$. For positive integer n, let

$$Y = \sum_{i=1}^{6} X_i, \quad W_n = \sum_{i=1}^{n} X_i, \quad \bar{X}_n = \frac{W_n}{n}.$$

   (i) Find the distributions of $Y,\ \bar{X}_n,\ W_n$.

   (ii) Find the probability that the random variable $\bar{X}_{30}$ is between 19 and 21, i.e., find $P(19 < \bar{X}_{30} < 21)$. Also, find the probability that W_{30} is greater than 650, i.e., find $P(W_{30} > 650)$.

   (iii) Are Y and W_{30} uncorrelated or correlated? Use details to illustrate your statements.

   (iv) Let $Y_n = \frac{\bar{X}_n - 20}{5\sqrt{n}}$. Use the moment generating functions to find the asymptotic distributions of $\bar{X}_n$ and Y_n as $n \to +\infty$.

5. Let X and Y are independent Poisson random variables with parameter 1. Use the moment generating functions to show $X + Y \sim \mathit{Poisson}(2)$. Then use the central limit theorem to show the following statement:

$$\lim_{n\to+\infty} e^{-n}\left[1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right] = \frac{1}{2}.$$