1. Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (a) The sum of two positive operators on a finite-dimensional complex inner product space is positive.

   (b) Let V be a 5-dimensional vector space and $T \in \mathcal{L}(V)$. Then there exists a 3-dimensional subspace U of V invariant under T.

   (c) Any polynomial of degree n with leading coefficients $(-1)^n$ is the characteristic polynomial of some linear operator.

   (d) If x, y, and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$.

   (e) Every normal operator is diagonalizable.

2. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ is defined by

$$T(x_1, x_2, x_3) = (2x_1,\ x_2 - x_3, x_2 + x_3).$$

   (a) Determine the eigenspace of T corresponding to each eigenvalue.

   (b) Find the Jordan form and a Jordan basis of T.

   (c) Find the minimal polynomial of T.

   (d) Find the trace of T, trace T.

   (e) Find the determinant of T, det T.

3. Suppose V is a finite-dimensional inner product space, $T \in \mathcal{L}(V)$ is normal, and U is a subspace of V that is invariant under T. Show that $U^\perp$ is invariant under T.

4. Let T be a linear operator on a finite-dimensional vector space V, and let v be a nonzero vector in V. The subspace

$$U = \text{span}\ (\{v, Tv, T^2v, \cdots\})$$

   is called the T-cyclic subspace of V generated by v.

   (a) Show that U is a finite-dimensional invariant subspace of V.

   (b) Let $k = \dim U$. Show that $\{v, Tv, T^2v, \cdots, T^{k-1}v\}$ is a basis for U.

   (c) If $a_0v + a_1Tv + a_2T^2v + \cdots + a_{k-1}T^{k-1}(v) + T^k(v) = 0$, show that the characteristic polynomial of $T|_U$ is

$$f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k).$$

   (d) Let $g(t)$ be the characteristic polynomial of T, show that $g(T) = 0$, where 0 is the zero operator. That is, T "satisfies" it characteristic equation.

5. If $\mathbb{F} = \mathbb{C}$, show that T is an isometry if and only if T is normal and $|\lambda| = 1$ for every eigenvalue λ of T.

6. Let $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$ be the polynomial spaces with inner products defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx,\ f, g \in \mathcal{P}_2(\mathbb{R}).$$

Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$ be the linear operator defined by

$$T(f(x)) = f'(x).$$

(a) Find orthonormal bases $\{v_1,\ v_2,\ v_3\}$ for $\mathcal{P}_2(\mathbb{R})$ and $\{u_1,\ u_2\}$ for $\mathcal{P}_1(\mathbb{R})$.

(b) Find $p \in \mathcal{P}_1(\mathbb{R})$ that makes

$$\int_{-1}^{1} |x^5 - p(x)|^2 dx$$

as small as possible.

(c) Find the singular values σ_1, σ_2 of T such that $T(v_i) = \sigma_i u_i,\ i = 1, 2$, and $T(v_3) = 0$.

7. Let V be a real inner product space. A function $f : V \to V$ is called a **rigid motion** if

$$||f(x) - f(y)|| = ||x - y||$$

for all $x,\ y \in V$. For example, any **isometry** on a finite-dimensional real inner product space is a **rigid motion**. Another class of rigid motions are the translations. A function $g : V \to V$, where V is a real inner product space, is called a **translation** if there exists a vector $v_0 \in V$ such that $g(v) = v + v_0$ for all $v \in V$. Let $f : V \to V$ be a rigid motion on a finite-dimensional real inner product space V, show that there exists a unique isometry T on V and a unique translation g on V such that $f = g \circ T$.