Geometry of Linear Equations and Matrices(线性方程的几何和矩阵简介)

Lecture 2 and 3

Dept. of Math.

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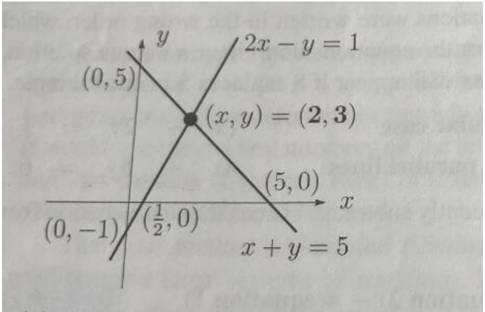
Solving Linear Equations

Introduction

We first study a system of 2 equations in 2 unknowns

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

We look at the system by rows and by columns separately.



(a) **Lines meet at $x = 2, y = 3$**

Row Picture

We first look at the system by rows:

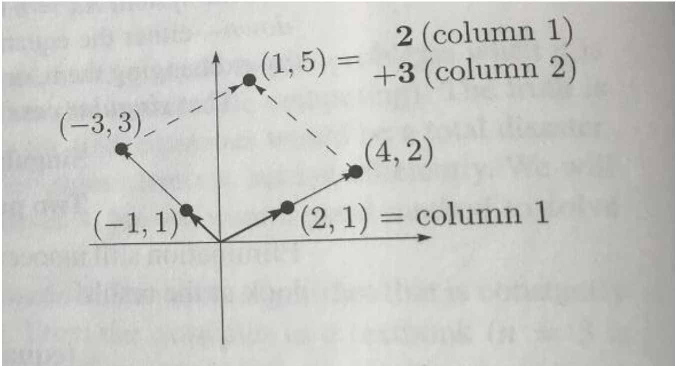
- The equation $2x - y = 1$ is represented by a straight line in the xy-plane.
- The second equation $x + y = 5$ produces a second line.
- The point of intersection lies on both lines. It is the only solution to both equations, and therefore $(x, y) = (2, 3)$ is the unique solution to the above system.

A second approach looks at the columns of the linear system. The two separated equations are really one vector equation:

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Column Picture

See the following picture:



(b) **Columns combine with 2 and 3**

The problem is to find the combination of the column vectors on the left side that produces the vector on the right side.

The $n = 3$ case

Now let's study the following system, which involves 3 equations and 3 unknowns:

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

Remarks:

- Each equation describes a plane in three dimensions.
- In three dimensions a line requires two equations.

Once the geometry of this system is clear, what can you say regarding the general case when $n > 3$? In other words, how does the row picture extend into n dimensions?

The row picture

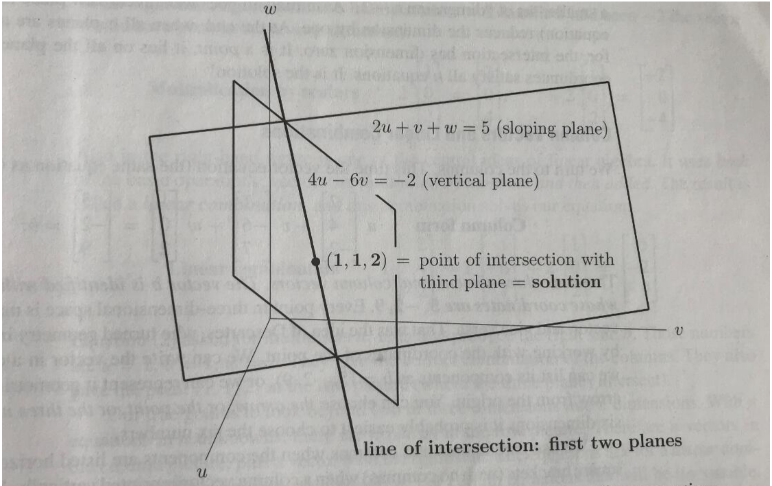


Figure 1.3 The row picture: three intersecting planes from three linear equations.

Column Vectors

The previous system could be written as follows:

$$u\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$$

- This is the column form for the linear system involving three-dimensional column vectors.
- Two operations show up in the above equation: addition of vectors and multiplication by a scalar(a number).

Vector Addition and Scalar Multiplication

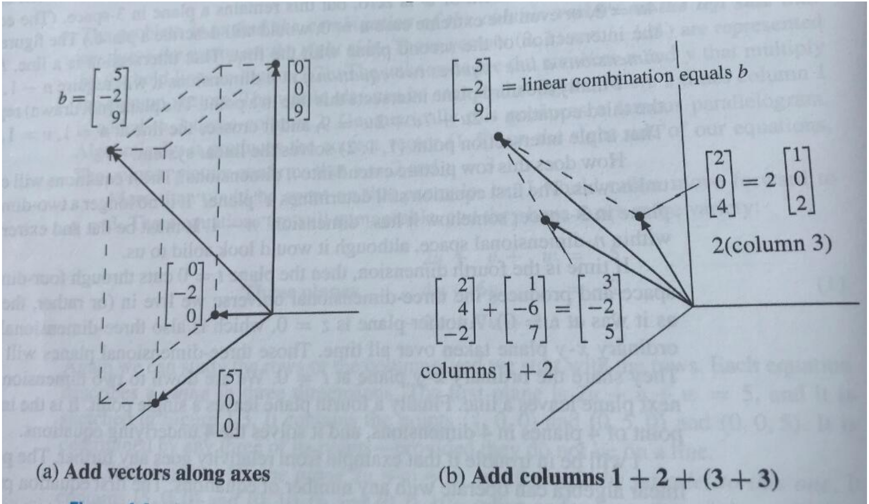


Figure 1.4 The column picture: linear combination of columns equals b.

Linear Combinations (线性组合)

When vectors are multiplied by numbers and added, and the result is called a linear combination. Continuing with the previous system, we can check that

$$1\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

This means that the multipliers u, v, w that produce the right side b are $(1, 1, 2)$.

To sum up:

Row Picture: Intersection of planes;

Column Picture: Combination of Columns.

The n dimensional case

If the n planes have no point in common, or infinitely many points, then the n columns lie in the same plane. For $n = 3$, you can check the following figure:

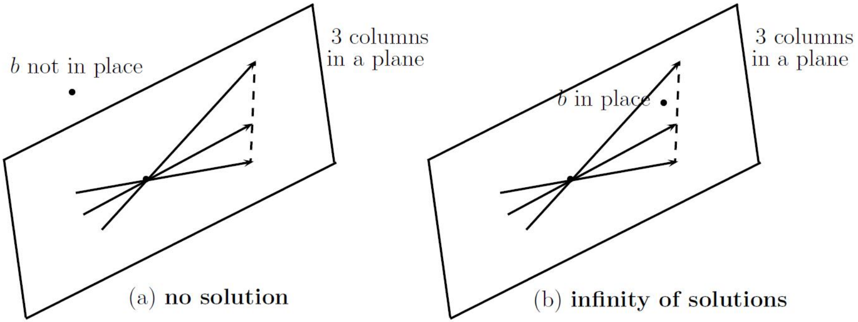


Figure 1.6: Singular cases: b outside or inside the plane with all three columns.

Number of Solutions

Now we classify the systems of linear equations with respect to the number of solutions.

1. No solution.
   Question: How do you know that one system has no solution?
2. Only one solution.
   Question: How to get the unique solution?
3. Infinitely many solutions.
   Question: How to write out all the solutions?

Singular Cases

Three Planes

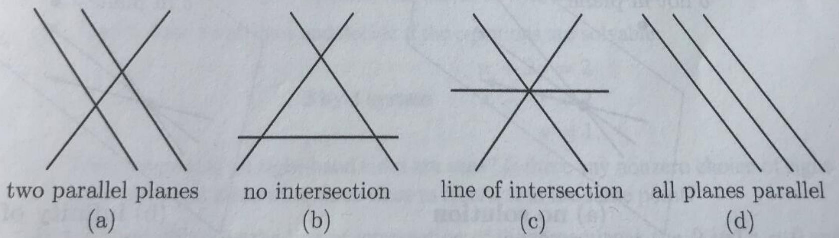


Figure 1.5 Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

The most common difficulty is shown in Figure 1.5b. From the end view the planes form a triangle.

Reduced Row Echelon Form and One Example

A matrix is said to be in Reduced Row Echelon Form if

1. The matrix is in row echelon form.
2. The first nonzero entry in each row is the only nonzero entry in its column.

The matrix

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

is NOT in the Reduced Row Echelon Form (简化阶梯形矩阵). The matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

is in the Reduced Row Echelon Form.

Example

Example

For what values of λ does the linear system

$$\begin{cases} x_1 - x_2 + x_3 = 2 \\ x_1 - 2x_2 + 2x_3 = \frac{1}{2}\lambda + 2 \\ x_1 + x_2 + (\lambda^2 - 3\lambda + 1)x_3 = \lambda^2 \end{cases}$$

1. have only one solution and solve for it;
2. have no solution and why;
3. have infinitely many solutions.

Row Echelon Form

$$\bar{A} = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & -2 & 2 & \frac{1}{2}\lambda + 2 \\ 1 & 1 & \lambda^2 - 3\lambda + 1 & \lambda^2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & -1 & 1 & \frac{1}{2}\lambda \\ 0 & 2 & \lambda^2 - 3\lambda & \lambda^2 - 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -\frac{1}{2}\lambda \\ 0 & 0 & \lambda^2 - 3\lambda + 2 & \lambda^2 + \lambda - 2 \end{bmatrix}$$

Solution

1. If $\lambda \neq 1$ and $\lambda \neq 2$, the system has a unique solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - \frac{1}{2}\lambda \\ (\lambda+2)/(\lambda-2) - \frac{1}{2}\lambda \\ (\lambda+2)/(\lambda-2) \end{bmatrix}$$

2. If $\lambda = 2$, then the system has no solution.
3. If $\lambda = 1$, then the system has infinitely many solutions, and all the solutions are given by

$$x = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Matrices

A matrix is an arrangement of mn elements with m rows and n columns, denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 4 & 1 \\ 3 & 2 & 2 \\ 9 & 7 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 3 & 9 & 4 \end{bmatrix}$$

Matrix Size

- The size of a matrix is specified in terms of the number of rows and the number of columns.
- If we denote the number of rows of a matrix as m, and the number of columns of a matrix as n, then the size of the matrix is $m \times n$.

$$m\left\{ \overbrace{\begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}^{n} \right.$$

- **THE ORDER IS IMPORTANT: rows $\times$ columns !**

Matrix Notation

Given

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots & \cdots & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}.$$

They have the same shape, hence we can define the addition and subtraction by adding and subtracting the corresponding entries.

Matrix: Addition and Subtraction

More precisely, we have

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ & \dots & \dots & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

$$A-B=\begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ & \dots & \dots & \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{bmatrix}$$

Scalar Multiplication

Scalar Multiplication:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ & \cdots & \cdots & \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Multiplication of a Matrix and a Vector

Write system of linear equations in matrix form $Ax = b$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

1. The right-hand side b is the column vector of "inhomogeneous term".
2. The left-hand side is A times x.
3. This multiplication will de defined exactly so as to reproduce the original system.

Combination of Columns

Ax by columns:

$$2\begin{bmatrix}1\\3\\1\end{bmatrix}+5\begin{bmatrix}1\\0\\1\end{bmatrix}+0\begin{bmatrix}6\\3\\4\end{bmatrix}=\begin{bmatrix}7\\6\\7\end{bmatrix}.$$

1. Every product Ax can be found using whole columns as in equation above.
2. **Therefore, Ax is a combination of the columns of A.**
3. The coefficients are the components of x.

Individual Entries

Sigma Notation: The ith component of Ax is

$$\sum_{j=1}^{n} a_{ij}x_j.$$

1. The length of the rows must match the length of x.
2. An $m \times n$ matrix multiplies an n-dimensional vector.
3. Summations are simpler than writing everything out in full, but matrix notation is better.
4. Row times column is fundamental to all matrix multiplications. From two vectors it produces a single number. This number is called the **inner product** of the two vectors.

The Matrix Form of One Elimination Step

Question: How can you use matrix to describe the operations that are carried out during elimination?

Elementary Matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 3 \\ -2 & 5 & 6 \\ 9 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ (-2)\times 5 - 2 & (-2)\times 1 + 5 & (-2)\times 3 + 6 \\ 9 & 4 & 1 \end{bmatrix}$$

Identity Matrix and Elementary Matrix (单位矩阵和初等矩阵)

Proposition

1. *The identity matrix I, with 1s on the diagonal and 0s everywhere else, leaves every vector unchanged.*
2. *The elementary matrix E_{ij} subtracts l times row j from row i. This E_{ij} includes $-l$ in row i, column j.*

(a) Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then $Ib = b$.

(b) Let $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}$. Then $E_{31}b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - lb_1 \end{bmatrix}$.

Matrix Multiplication (矩阵乘法)

Now we multiply two matrices A and B.

Definition

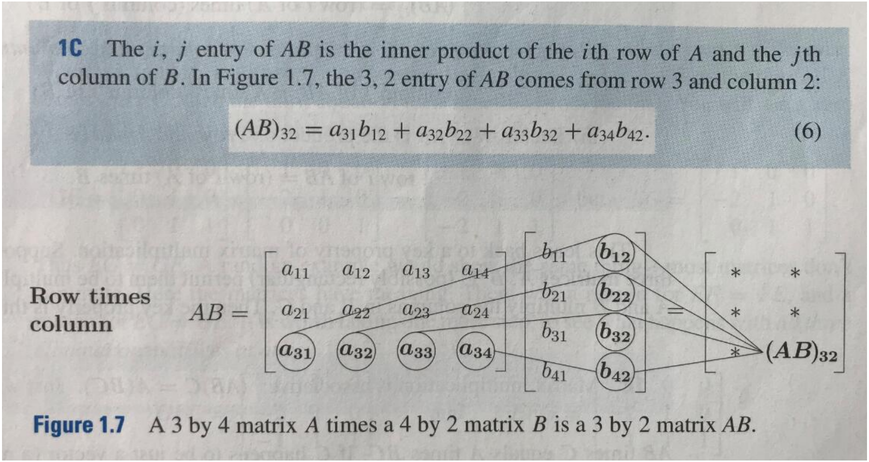
The number of columns in A has to equal the number of rows in B. We define the multiplication as follows: The (i,j) entry of AB is the inner product of the i th row of A and the j th column of B.

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Example

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 0 \\ 4 & 8 & 0 \end{bmatrix}$$

A 3×4 matrix A times a 4 by 2 matrix B



1C The i, j entry of AB is the inner product of the ith row of A and the jth column of B. In Figure 1.7, the 3, 2 entry of AB comes from row 3 and column 2:

$$(AB)_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}. \tag{6}$$

Row times column

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \\ * & (AB)_{32} \end{bmatrix}$$

Figure 1.7 A 3 by 4 matrix A times a 4 by 2 matrix B is a 3 by 2 matrix AB.

Remark: We write AB when the matrices have nothing special to do with elimination. Our earlier example was EA, because of the elementary matrix E.

Matrix Multiplication

We now summarize three different ways to look at matrix multiplication:

Proposition

(i) *Each entry of AB is the product of a row and a column:*

$$(AB)_{ij} = (\textit{row } i \textit{ of } A) \textit{ times} (\textit{column } j \textit{ of } B)$$

(ii) *Each column of AB is the product of a matrix and a column:*

$$\textit{column } j \textit{ of } AB = A \textit{ times } (\textit{column } j \textit{ of } B)$$

(iii) *Each row of AB is the product of a row and a matrix:*

$$\textit{row } i \textit{ of } AB = (\textit{row } i \textit{ of } A) \textit{times } B.$$

Associativity (结合律)

Suppose the shapes of three matrices A, B, C (possibly rectangular) permit them to be multiplied. The rows in A and B multiply the columns in B and C. Then the key property is this:

Proposition

Matrix multiplication is associative:

$$(AB)C = A(BC).$$

Just write ABC.

Proof of the associative property

Let

$$A=(a_{ij})_{sn}, B=(b_{jk})_{nm}, C=(c_{kl})_{mr}.$$

Let $V=AB=(v_{ik})_{sm}, W=BC=(w_{jl})_{nr}$, where

$$v_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}(i=1,2,\cdots,s;k=1,2,\cdots,m),$$

$$w_{jl}=\sum_{k=1}^{m} b_{jk}c_{kl}(j=1,2,\cdots,n;l=1,2,\cdots,r).$$

Let

$$(AB)C=VC,$$

then the (i,l) entry of VC is

$$\sum_{k=1}^{m} v_{ik}c_{kl}=\sum_{k=1}^{m}\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)c_{kl}=\sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}b_{jk}c_{kl}, \tag{1}$$

Proof

Since $A(BC) = AW$, then the (i,l) entry of AW is

$$\sum_{j=1}^{n} a_{ij} w_{jl} = \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{m} b_{jk} c_{kl} \right) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ij} b_{jk} c_{kl}. \tag{2}$$

It follows from (1) and (2) that $(AB)C = A(BC)$. In other words, matrix multiplication is associative.

See also Exercise 18.

Remarks:

1. The shapes of A, B, C must match properly.
2. If C happens to be a vector, this is the requirement $(EA)x = E(Ax)$.

Distributivity and Commutativity

There are two more properties to mention–one property that matrix multiplication has, and another which it does not have.

Proposition

Matrix operations are distributive:

$$A(B+C) = AB + AC \text{ and } (B+C)D = BD + CD$$

Proposition

Matrix multiplication is not commutative:
Usually $FE \neq EF$.

Matrix multiplication is not commutative: Example

Example

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(a) What are AB and BA?

(b) Do they commute?

(c) Why?

Example 4

Example

Suppose E subtracts twice the first equation from the second. Suppose F is the matrix from the next step, to add row 1 to row 3.

1. These two matrices do commute and the product does both steps at once, why?
2. In either order, EF or FE, this changes rows 2 and 3 using row 1.

Example 5

Example

Suppose E is the same but G adds row 2 to row 3.

(a) Now the order makes a difference.

(b) That is $GFE \neq EFG$.

(c) Can you figure out why as well?

Homework Assignment

1.2: 1, 3, 4, 6, 9, 14, 17,19, 20, 21.

1.4: 1, 2, 6, 10, 13, 15, 19, 24, 44, 53.