李轩然 12312110

MA215 Probability Theory

Assignment 08

1. Suppose that the continuous random variable X has p.d.f

$$f_X(x) = \begin{cases} kx(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the constant k, and then find the non-zero range of Y and the p.d.f $f_Y(y)$ of Y when

(a) $Y = -3X + 3$;

(b) $Y = \frac{1}{X}$.

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^1 kx(1-x)\,dx = 1 \Rightarrow k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 1 \Rightarrow k = 6$$

$$\therefore f_X(x) = \begin{cases} 6x(1-x), & 0<x<1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) non-zero range of X is $(0,1)$, so $Y \in (0,3)$

$\because Y = -3X+3 \quad \therefore X = \frac{3-Y}{3} \quad \frac{dX}{dY} = -\frac{1}{3}$

$$f_Y(y) = f_X\left(\frac{3-y}{3}\right)\left|\frac{dX}{dY}\right|$$

when $y \in (0,3)$, $\quad f_Y(y) = 6 \cdot \frac{3-y}{3}\left(1-\frac{3-y}{3}\right) \cdot \frac{1}{3} = \frac{2}{9}y(3-y)$

$$\therefore f_Y(y) = \begin{cases} \frac{2}{9}y(3-y), & 0<y<3 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\because X \in (0,1) \quad \therefore Y = \frac{1}{X} \in (1, +\infty)$

$X = \frac{1}{Y}, \quad \left|\frac{dX}{dY}\right| = \frac{1}{Y^2}$

$$\therefore f_Y(y) = f_X\left(\frac{1}{y}\right)\left|\frac{dX}{dY}\right| = 6 \cdot \frac{1}{y}\left(1-\frac{1}{y}\right) \cdot \frac{1}{y^2} = \frac{6(y-1)}{y^4}, \quad y>1.$$

$$\therefore f_Y(y) = \begin{cases} \frac{6(y-1)}{y^4}, & y>1 \\ 0, & \text{otherwise} \end{cases}$$

2. Suppose that the random variable X has c.d.f.

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1-\cos(x)}{2}, & 0 \leqslant x \leqslant \pi, \\ 1, & x > \pi. \end{cases}$$

and that $Y = \sqrt{X}$. What is the non-zero range of Y? Find the c.d.f. $F_Y(y)$ of Y, and hence find the p.d.f of Y.

For $f_X(x) = \frac{dF_X(x)}{dx}$, $f_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2}\sin x, & 0 < x \leq \pi \\ 0, & x > \pi. \end{cases}$

So non-zero range of X is $(0, \pi]$ $\therefore Y = \sqrt{X} \in (0, \sqrt{\pi}]$

$F_Y(y) = P(Y \leq y) = P(X \leq y^2) = F_X(y^2) = \begin{cases} 0, & y \leq 0 \\ \frac{1-\cos y^2}{2}, & 0 < y \leq \sqrt{\pi} \\ 1, & y > \sqrt{\pi} \end{cases}$

$f_Y(y) = \frac{dF_Y(y)}{dy}$, when $0 < y \leq \sqrt{\pi}$, $f_Y(y) = \frac{d}{dy}\left(1 - \frac{1-\cos y^2}{2}\right) = \frac{1}{2}\sin y^2 \cdot 2y = y\sin y^2$

$\therefore f_Y(y) = \begin{cases} 0, & y \leq 0 \text{ or } y > \sqrt{\pi} \\ y\sin y^2, & 0 < y \leq \sqrt{\pi} \end{cases}$

3. Suppose that the two random variables X and Y have joint probability c.d.f. $F(x, y)$. Show that $F(x, y)$ possesses the following properties:

(a) For any fixed x, $F(x, y)$ is a non-decreasing function of y and, similarly, for any fixed y, $F(x, y)$ is a non-decreasing function of x.

(b) $F(x, y) \to 1$ when both $x \to +\infty$ and $y \to +\infty$.

(c) $F(x, y) \to 0$ when either $x \to -\infty$ or $y \to -\infty$.

(d) If $x_1 < x_2$ and $y_1 < y_2$, then

$$P(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1).$$

(a) Suppose $x_1 < x_2$, we want to get $F(x_1, y) \leq F(x_2, y)$

$F(x_2, y) - F(x_1, y) = P(X \leq x_2, Y \leq y) - P(X \leq x_1, Y \leq y) = P(x_1 < X \leq x_2, Y \leq y)$

$\because x_1 < x_2$ $\therefore P(x_1 < X \leq x_2, Y \leq y) \geq 0$. $\therefore F(x_1, y) \leq F(x_2, y)$

Using the same method, we know that when $y_1 < y_2$, $F(x, y_1) \leq F(x, y_2)$

So $F(x, y)$ is a non-decreasing function of both x and y.

(b) $\lim\limits_{x \to +\infty} F(x, y) = \lim\limits_{x \to +\infty} P((X \leq x) \cap (Y \leq y)) = P(Y \leq y)$

$\therefore \lim\limits_{x, y \to +\infty} F(x, y) = \lim\limits_{y \to +\infty} P(Y \leq y) = \lim\limits_{y \to +\infty} F(y) = 1$

(c) $\lim_{x\to-\infty} F(x,y) = \lim_{x\to-\infty} P((X\le x)\cap(Y\le y)) = P(\phi\cap Y\le y) = P(\phi)=0$

$\lim_{y\to-\infty} F(x,y) = \lim_{y\to-\infty} P((X\le x)\cap(Y\le y)) = P(X\le x\cap\phi) = P(\phi)=0$

(d) $P(x_1<X\le x_2,\ y_1<Y\le y_2) = P(X\le x_2,\ y_1<Y\le y_2) - P(X\le x_1,\ y_1<Y\le y_2)$

$= P(X\le x_2, Y\le y_2) - P(X\le x_2, Y\le y_1) - (P(X\le x_1, Y\le y_2) - P(X\le x_1, Y\le y_1))$

$= F(x_2,y_2) - F(x_2,y_1) - F(x_1,y_2) + F(x_1,y_1)$ $\square$

4. Suppose that the two discrete random variables X and Y have joint p.m.f. given by

Y	$Y=1$	$Y=2$	$Y=3$	$Y=4$
$X=1$	2/32	3/32	4/32	5/32
$X=2$	3/32	4/32	5/32	6/32

Obtain the marginal p.m.f. of X.

All the (X,Y) are independent

When $X=1$, $P_X(1) = \sum_{y=1}^{4} P_{X,Y}(1,y) = P_{X,Y}(1,1) + P_{X,Y}(1,2) + P_{X,Y}(1,3) + P_{X,Y}(1,4)$

$= \frac{2}{32} + \frac{3}{32} + \frac{4}{32} + \frac{5}{32} = \frac{7}{16}$

When $X=2$, $P_X(2) = \sum_{y=1}^{4} P_{X,Y}(2,y) = P_{X,Y}(2,1) + P_{X,Y}(2,2) + P_{X,Y}(2,3) + P_{X,Y}(2,4)$

$= \frac{3}{32} + \frac{4}{32} + \frac{5}{32} + \frac{6}{32} = \frac{9}{16}$

$\therefore$

X	$P_X(x)$
1	$\frac{7}{16}$
2	$\frac{9}{16}$