李轩然
12312110

MA215 Probability Theory

Assignment 02

1. Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

   (a) List the sample space Ω.

   (b) List the elements that make up the following events:

   (1) $A =$ the sum of the two values is at least 5;

   (2) $B =$ the value for the first die is higher than the value of the second;

   (3) $C =$ the first value is 4.

   (c) List the elements of the following events:

   (1) $A \cap C$;

   (2) $B \cup C$;

   (3) $A \cap (B \cup C)$.

1. (a) Suppose that the event P_{ij} implies that the first dice's face value is i, and the second one is j. ($i, j \in \{1,2,3,4,5,6\}$)

$\therefore \Omega = \{P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_{21}, P_{22}, P_{23}, P_{24}, P_{25}, P_{26}, P_{31}, P_{32}, P_{33}, P_{34}, P_{35}, P_{36}, P_{41}, P_{42}, P_{43}, P_{44}, P_{45}, P_{46}, P_{51}, P_{52}, P_{53}, P_{54}, P_{55}, P_{56}, P_{61}, P_{62}, P_{63}, P_{64}, P_{65}, P_{66}\}$

(b) (1) $A = \{P_{ij} \mid i, j \in \{1,2,3,4,5,6\};\ i+j \geq 5\}$

$= \{P_{14}, P_{15}, P_{16}, P_{23}, P_{24}, P_{25}, P_{26}, P_{32}, P_{33}, P_{34}, P_{35}, P_{36}, P_{41}, P_{42}, P_{43}, P_{44}, P_{45}, P_{46}, P_{51}, P_{52}, P_{53}, P_{54}, P_{55}, P_{56}, P_{61}, P_{62}, P_{63}, P_{64}, P_{65}, P_{66}\}$

(2) $B = \{P_{ij} \mid i, j \in \{1,2,3,4,5,6\};\ i > j\}$

$= \{P_{21}, P_{31}, P_{32}, P_{41}, P_{42}, P_{43}, P_{51}, P_{52}, P_{53}, P_{54}, P_{61}, P_{62}, P_{63}, P_{64}, P_{65}\}$

(3) $C = \{P_{4j} \mid j \in \{1,2,3,4,5,6\}\}$

$= \{P_{41}, P_{42}, P_{43}, P_{44}, P_{45}, P_{46}\}$

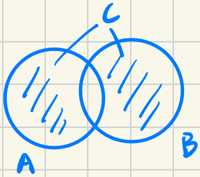
(c) (1) $A \cap C = \{P_{41}, P_{42}, P_{43}, P_{44}, P_{45}, P_{46}\}$

(2) $B \cup C = \{P_{21}, P_{31}, P_{32}, P_{41}, P_{42}, P_{43}, P_{51}, P_{52}, P_{53}, P_{54}, P_{61}, P_{62}, P_{63}, P_{64}, P_{65}, P_{44}, P_{45}, P_{46}\}$

(3) $A \cap (B \cup C) = \{P_{32}, P_{41}, P_{42}, P_{43}, P_{51}, P_{52}, P_{53}, P_{54}, P_{61}, P_{62}, P_{63}, P_{64}, P_{65}, P_{44}, P_{45}, P_{46}\}$

2. Let A and B be two arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

2.



According to Venn graph,

$$C = A\cup B - A\cap B = (A\cup B)\cap (A\cap B)^c$$

3. Suppose A and B are two events such that $A \subset B$. Show that

$$P(B\backslash A) = P(B) - P(A).$$

3. Proof. $P(B\backslash A) = P(B\cap A^c)$

By Property 4. $P(B\cap A^c) = P(B) + P(A^c) - P(B\cup A^c)$

$\because A\subset B \quad \therefore A^c \supset B^c. \quad \therefore B\cup A^c \supset B\cup B^c = \Omega.$ that's to say $B\cup A^c = \Omega$

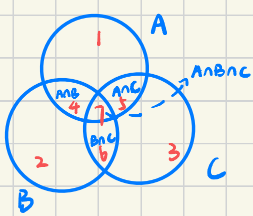
$\therefore P(B\cup A^c) = 1$

$\therefore P(B\cap A^c) = P(B) + 1 - P(A) - 1 = P(B) - P(A) \quad \therefore P(B\backslash A) = P(B) - P(A) \quad \square$

4. Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability and the propositions proved in the Lecture Notes.

$$\begin{aligned} P(A\cup B\cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A\cap B) - P(A\cap C) - P(B\cap C) \\ & + P(A\cap B\cap C). \end{aligned}$$

4. Proof. (a)



According to the Venn graph,

$$\begin{aligned} & P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C) \\ = {} & P_1+P_4+P_5+P_7+P_2+P_4+P_6+P_7+P_3+P_5+P_6+P_7 \\ & -P_4-P_7-P_5-P_7-P_6-P_7+P_7 \\ = {} & P_1+P_2+P_3+P_4+P_5+P_6+P_7 = P(A\cup B\cup C) \end{aligned}$$

(b) By Property 4. $P(A\cup B) = P(A)+P(B)-P(A\cap B)$

$$\begin{aligned} \therefore P(A\cup B\cup C) & = P(A\cup B)+P(C)-P((A\cup B)\cap C) = P(A\cup B)+P(C)-P((A\cap C)\cup(B\cap C)) \\ & = P(A)+P(B)-P(A\cap B)+P(C)-(P(A\cap C)+P(B\cap C)-P(A\cap B\cap C)) \\ & = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C) \quad \square \end{aligned}$$

5. Suppose that $\{A_n; n \geqslant 1\}$ is a sequence of events which may not be disjoint. Show that the following sub-additive property is true:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} P(A_n).$$

Also, for any $k \geqslant 2$, we have

$$P\left(\bigcup_{n=1}^{k} A_n\right) \leqslant \sum_{n=1}^{k} P(A_n).$$

In particular, for any two events A and B, we have $P(A \cup B) \leqslant P(A) + P(B)$.

5. Proof. When $k=2$, we already know that $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

For $P(A\cap B) \geqslant 0.$, $P(A\cup B) \leqslant P(A) + P(B)$. take equal if and only if A and B are disjoint.

Assume that when $k=n$, the conclusion is true: $P(\bigcup_{k=1}^{n} A_k) \leqslant \sum_{k=1}^{n} P(A_k)$

When $k=n+1$, $P(\bigcup_{k=1}^{n+1} A_k) = P(\bigcup_{k=1}^{n} A_k \cup A_{n+1}) = P(\bigcup_{k=1}^{n} A_k) + P(A_{n+1}) - P(\bigcup_{k=1}^{n} A_k \cap A_{n+1})$

$\leqslant P(\bigcup_{k=1}^{n} A_k) + P(A_{n+1}) \leqslant \sum_{k=1}^{n} P(A_k) + P(A_{n+1}) = \sum_{k=1}^{n+1} P(A_k)$

the conclusion is still true.

So for any $k \geqslant 2$, $P(\bigcup_{n=1}^{k} A_n) \leqslant \sum_{n=1}^{k} P(A_n)$

$\therefore \lim_{k\to\infty} P(\bigcup_{n=1}^{k} A_n) \leqslant \lim_{k\to\infty} \sum_{n=1}^{k} P(A_n) \Rightarrow P(\bigcup_{n=1}^{\infty} A_n) \leqslant \sum_{n=1}^{\infty} P(A_n)$

take equal if and only if for any $i \neq j$, $A_i \cap A_j = \phi$ $\square$

6. Suppose $\{A_i; 1 \leqslant i \leqslant n\}$ are events.

(1) Show that the following inclusion-exclusion formula is true.

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j\leqslant n} P(A_i \cap A_j) + \sum_{i<j<k\leqslant n} P(A_i \cap A_j \cap A_k)$$
$$- \cdots + (-1)^{n-1} P(A_1 \cap A_2 \ldots A_n).$$

(2) Write down this formula for cases of $n=2, n=3, n=4$, and $n=5$ clearly.

6. (1) Proof. When $n=2$, $P(A_1\cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$ is definitely true.

Assume that when $n=k$, the conclusion is true.

When $n=k+1$, we have $P(\bigcup_{i=1}^{k+1} A_i) = P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1}) - P(\bigcup_{i=1}^{k} A_i \cap A_{k+1})$

$= \sum_{i=1}^{k} P(A_i) - \sum_{i<j\leqslant n} P(A_i \cap A_j) + \sum_{i<j<k'\leqslant n} P(A_i \cap A_j \cap A_{k'}) - \cdots + (-1)^{k+1} P(A_1 \cap A_2 \cap \cdots \cap A_k)$

$+ P(A_{k+1}) - P(\bigcup_{i=1}^{k} (A_i \cap A_{k+1}))$

$$= \sum_{i=1}^{k} P(A_i) - \sum_{i<j\le k} P(A_i \cap A_j) + \sum_{i<j<k'\le k} P(A_i \cap A_j \cap A_{k'}) - \cdots + (-1)^{k+1} P(A_1 \cap A_2 \cap \cdots \cap A_k)$$
$$+ P(A_{k+1}) - \left[\sum_{i=1}^{k} P(A_i \cap A_{k+1}) - \sum_{i<j\le k} P(A_i \cap A_j \cap A_{k+1}) + \cdots + (-1)^{k+1} P(A_1 \cap \cdots \cap A_k \cap A_{k+1})\right]$$
$$= \sum_{i=1}^{k+1} P(A_i) - \sum_{i\le j\le k+1} P(A_i \cap A_j) + \cdots + (-1)^{k+2} P(A_1 \cap A_2 \cap \cdots \cap A_{k+1})$$

so the conclusion when $n=k+1$ is still true.

So for any $n \ge 2$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j\le n} P(A_i \cap A_j) + \sum_{i<j<k'\le n} P(A_i \cap A_j \cap A_{k'}) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$ $\square$

(2) $n=2$: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$n=3$: $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$

$n=4$: $P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1) + P(A_2) + P(A_3) + P(A_4) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_4) - P(A_3 \cap A_4) + P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_2 \cap A_3 \cap A_4)$

$n=5$: $P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_4) - P(A_3 \cap A_4) - P(A_1 \cap A_5) - P(A_2 \cap A_5) - P(A_3 \cap A_5) - P(A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4) + P(A_1 \cap A_2 \cap A_5) + P(A_1 \cap A_3 \cap A_5) + P(A_1 \cap A_4 \cap A_5) + P(A_2 \cap A_3 \cap A_5) + P(A_2 \cap A_4 \cap A_5) + P(A_3 \cap A_4 \cap A_5) - P(A_1 \cap A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_2 \cap A_3 \cap A_5) - P(A_1 \cap A_2 \cap A_4 \cap A_5) - P(A_1 \cap A_3 \cap A_4 \cap A_5) - P(A_2 \cap A_3 \cap A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$

7. (i) If $\{A_n; n \geqslant 1\}$ is an increasing sequence of events, i.e. for all $n \geqslant 1$, $A_n \subset A_{n+1}$, show that $\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$.

(ii) If $\{A_n; n \geqslant 1\}$ is a decreasing sequence of events, i.e. for all $n \geqslant 1$, $A_n \supset A_{n+1}$, show that $\lim_{n\to\infty} P(A_n)\big) = P\left(\bigcap_{n=1}^{\infty} A_n\right)$.

7. Proof. (i) Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $\cdots$, $B_n = A_n \setminus A_{n-1}$ $(n \ge 2)$, $\cdots$

then for any $i \ne j$, B_i and B_j are disjoint.

$A_1 = B_1$. $A_2 = B_1 \cup B_2$, $\cdots$, $A_n = \bigcup_{k=1}^{n} B_k$ $(n \ge 2)$, $\cdots$

Obviously, $B_n = A_n \setminus A_{n-1} \subset A_n$ for any $n \ge 2$. $B_1 = A_1 \subset A_1$, thus $\bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} A_n$

Assume that $x \in \bigcup_{n=1}^{\infty} A_n$, then $\exists k \ge 1$, $x \in A_k$

$\because A_k = \bigcup_{n=1}^{k} B_n \subset \bigcup_{n=1}^{\infty} B_n$ $\therefore x \in \bigcup_{n=1}^{\infty} B_n \Rightarrow \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$

Thus, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$

Because $\{B_n\}$ are disjoint, we have $P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n)$

So $P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n) = \lim_{k\to\infty} P\left(\bigcup_{n=1}^{k} B_n\right) = \lim_{k\to\infty} P(A_k) = \lim_{n\to\infty} P(A_n)$ $\square$

(ii) Define $C_1 = A_1^c$, $C_2 = A_2^c \setminus A_1^c$, $\cdots$ $C_n = A_n^c \setminus A_{n-1}^c$ $(n \geq 2)$, $\cdots$

then for any $i \neq j$, C_i and C_j are disjoint.

$A_1 = C_1^c$. $A_2 = C_1^c \cap C_2^c$, $\cdots$, $A_n = \bigcap_{k=1}^{n} C_k^c$ $(n \geq 2)$, $\cdots$

Obviously, $C_n = A_n^c \setminus A_{n-1}^c \subset A_n^c$ for any $n \geq 2$, $C_1 = A_1^c \subset A_1^c$, thus $\left(\bigcap_{n=1}^{\infty} C_n^c\right)^c = \bigcup_{n=1}^{\infty} C_n \subset \bigcup_{n=1}^{\infty} A_n^c = \left(\bigcap_{n=1}^{\infty} A_n\right)^c$

Assume that $x \in \left(\bigcap_{n=1}^{\infty} A_n\right)^c$, then $\exists k \geq 1$, $x \in A_k^c$

$\because A_k = \bigcap_{n=1}^{k} C_n^c \supset \bigcap_{n=1}^{\infty} C_n^c$ $\therefore A_k^c \subset \left(\bigcap_{n=1}^{\infty} C_n^c\right)^c$ $\Rightarrow \left(\bigcap_{n=1}^{\infty} A_n\right)^c \subset \left(\bigcap_{n=1}^{\infty} C_n^c\right)^c$

Thus, $\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \left(\bigcap_{n=1}^{\infty} C_n^c\right)^c \Rightarrow \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} C_n^c$

Because $\{C_n\}$ are disjoint, we have $P\left(\bigcup_{n=1}^{\infty} C_n\right) = P\left(\left(\bigcap_{n=1}^{\infty} C_n^c\right)^c\right) = \sum_{n=1}^{\infty} P(C_n)$

So $P\left(\bigcap_{n=1}^{\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} C_n^c\right) = \lim_{k \to \infty} P\left(\bigcap_{n=1}^{k} C_n^c\right) = \lim_{k \to \infty} P(A_k) = \lim_{n \to \infty} P(A_n)$ $\square$