Eigenvalues and Eigenvectors (特征值和特征向量)

Lecture 21

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Eigenvalues and Eigenvectors

Introduction

- $Ax = b$ and $Ax = \lambda x$.
- Determinants give a transition from $Ax = b$ and $Ax = \lambda x$. In both cases the determinant leads to a "formal solution": to Cramer's rule for $x = A^{-1}b$, and to the polynomial $\det(A - \lambda I)$, whose roots will be the eigenvalues.
- All matrices are square.

Introduction

Consider the coupled pair of equations

$$\frac{dv}{dt} = 4v - 5w, v = 8 \text{ at } t = 0$$

$$\frac{dw}{dt} = 2v - 3w, w = 5 \text{ at } t = 0$$

This is an initial-value problem. It is easy to write the system in matrix form. Let the unknown vector be $u(t)$, with initial value $u(0)$, the coefficient matrix is A:

$$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, u(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}.$$

The two coupled equations can be written as a vector equation:

$$\frac{du}{dt} = Au \text{ with } u = u(0) \text{ at } t = 0$$

This is the basic statement of the problem.

Initial Value Problem

Note that it is a first-order equation–no higher derivatives–and it is linear in the unknowns. It also has constant coefficients: the matrix A is independent of time.

How do we find $u(t)$? For one unknown:

$$\frac{du}{dt} = au \text{ with } u = u(0) \text{ at } t = 0.$$

The solution to this equation is the one thing you need to know:

$$u(t) = e^{at}u(0).$$

Notice the behavior of u for large times. The equation is unstable if $a > 0$, neutrally stable if $a = 0$, or stable if $a < 0$.

Eigenvalue Problem and Eigenvalue Equation

For two unknowns: We look for solutions with the same exponential dependence on t just found in the scalar case

$$v(t) = e^{\lambda t} y, w(t) = e^{\lambda t} z$$

or in vector notation:

$$u(t) = e^{\lambda t} x.$$

This is the whole key to differential equations $du/dt = Au$:

Look for pure exponential solutions.

The Solutions of $Ax = \lambda x$

Substituting $v(t) = e^{\lambda t}y$ and $w(t) = e^{\lambda t}z$ into the equation, we find

$$\lambda e^{\lambda t}y = 4e^{\lambda t}y - 5e^{\lambda t}z$$
$$\lambda e^{\lambda t}z = 2e^{\lambda t}y - 3e^{\lambda t}z.$$

The factor $e^{\lambda t}$ is common to every term, and can be removed. This cancellation is the reason for assuming the same exponent λ for both unknowns; it leaves

$$\text{Eigenvalue Problem}: \quad \begin{aligned} 4y - 5z &= \lambda y \\ 2y - 3z &= \lambda z \end{aligned}$$

Eigenvalue equation:

$$Ax = \lambda x$$

Eigenvalue Problem

Consider

$$(A - \lambda I)x = 0.$$

The number λ is an eigenvalue of the matrix, and the vector is the associated eigenvector. Our goal is to find the eigenvalues and eigenvectors, λ's and x's, and to use them.

Proposition

The vector x is in the nullspace of $A - \lambda I$. The number λ is chosen so that $A - \lambda I$ has a nontrivial nullspace.

Eigenvalues and Eigenvectors

We are interested in the particular values λ for which there is a nonzero eigenvector x. To be of any use, the nullspace of $A - \lambda I$ must contain vectors other than zero. In short, $A - \lambda I$ must be singular.

Definition

The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det\,(A - \lambda I) = 0.$$

This is the characteristic equation. Each λ is associated with eigenvectors x:

$$(A - \lambda I)x = 0 \text{ or } Ax = \lambda x.$$

Solving $Ax = \lambda x$

- In our example, we shift A by λI to make it singular, and hence $\det\,(A - \lambda I) = (\lambda^2 - \lambda - 2) = 0$.
- $\lambda^2 - \lambda - 2$ is the *characteristic polynomial*. Its roots, where the determinant is zero, are the eigenvalues.
- There are two eigenvalues -1 and 2, because a quadratic has two roots.
- The values $\lambda = -1$ and $\lambda = 2$ lead to a solution of $Ax = \lambda x$. A matrix with zero determinant is singular.
- There must be nonzero vectors x in its nullspace.

Solving $Ax = \lambda x$

Now, let's find the eigenvectors corresponding to the eigenvalues.

$$\lambda_1 = -1 : (A - \lambda_1 I)x = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\lambda_2 = 2 : (A - \lambda_2 I)x = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Eigenvector for λ_1 is any nonzero multiple of

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The second eigenvector for λ_2 is any nonzero multiple of

$$x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Eigenvalue Problem Again

A bit more discussion about differential equations:

- Pure exponential solutions to $du/dt = Au$:

$$u(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, u(t) = e^{2t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

- Complete solution and superposition:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2.$$

- Initial condition and Solution:

$$c_1 x_1 + c_2 x_2 = u(0).$$

The constants are $c_1 = 3$ and $c_2 = 1$, and the solution to the original equation is: $v(t) = 3e^{-t} + 5e^{2t}, w(t) = 3e^{-t} + 2e^{2t}$.

Solving $Ax = \lambda x$

Steps in solving $Ax = \lambda x$:

1. Compute the determinant of $A - \lambda I$. With λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.
2. Find the roots of this polynomial. The n roots are the eigenvalues of A.
3. For each eigenvalue solve the equation $(A - \lambda I)x = 0$. Since the determinant is zero, there are solutions other than $x = 0$. Those are the eigenvectors.

Example

Example Let

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Find all the eigenvalues and their corresponding eigenvectors of A.

Solution.

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 & -1 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ -1 & 0 & -1 & 2-\lambda \end{vmatrix} = \lambda(\lambda-2)^2(\lambda-4).$$

The eigenvalues are $0,2,2,4$. The eigenvectors of A can be found by solving $Ax=\lambda x$ for each λ accordingly.

Examples

Example 1 For diagonal matrices, the eigenvalues are sitting along the main diagonal.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ has } \lambda_1 = 3 \text{ with } x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda_2 = 2 \text{ with } x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Example 2 The eigenvalues of a projection matrix are 1 or 0!

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ has } \lambda_1 = 1 \text{ with } x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 = 0 \text{ with } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Example 3

Example 3 The eigenvalues are on the main diagonal when A is triangular.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = (1-\lambda)(\frac{3}{4}-\lambda)(\frac{1}{2}-\lambda).$$

The determinant is just the product of the diagonal entries. It is zero if $\lambda = 1, \lambda = \frac{3}{4}$ or $\lambda = \frac{1}{2}$; the eigenvalues were already sitting along the main diagonal.

LU is not suited to the purpose of finding the eigenvalues

- Converting A to an upper-triangular matrix U, we obtain the Gaussian factorization $A = LU$. The eigenvalues may be visible on the diagonal, but they are **NOT** the eigenvalues of A. We now need to transform A into a diagonal or triangular matrix without changing its eigenvalues.
- For most matrices, there is no doubt that the eigenvalue problem is computationally more difficult than $Ax = b$. For a 5 by 5 matrix, $\det(A - \lambda I)$ involves λ^5. Galois and Abel proved that there can be no algebraic formula for the roots of a fifth-degree polynomial.

Sum and Product

Theorem

The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of } A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}.$$

Furthermore, the product of the n eigenvalues equals the determinant of A.

For a 2 by 2 matrix, the trace and determinant tell us everything:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has trace } a+d, \text{and determinant } ad-bc.$$

The eigenvalues are then given by the quadratic formula.

2 by 2 matrices

There should be no confusion between the diagonal entries and the eigenvalues. Normally the pivots, diagonal entries, and eigenvalues are completely different. And for a 2 by 2 matrix, the trace and determinant tell us everything:

$$\det(A-\lambda I) = \det \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (\text{trace})\lambda + \text{determinant}$$

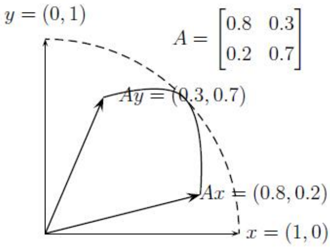
The eigenvalues are

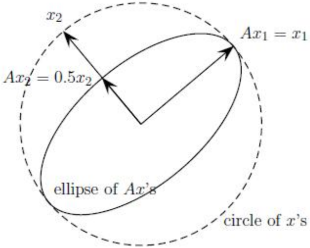
$$\lambda = \frac{\text{trace} \pm [(\text{trace})^2 - 4\det]^{\frac{1}{2}}}{2}.$$

Those two λ's add up to the trace; Exercise 9 gives $\sum \lambda_i = \text{trace}$ for all matrices.

Eigshow

There is a MATLAB demo (just type eigshow), displaying the eigenvalue problem for a 2 by 2 matrix.





Only certain special numbers λ are eigenvalues, and only certain special vectors x are eigenvectors.

Homework Assignment 21

5.1: 1, 5, 8, 9, 10, 14, 18, 25, 29, 35.