Null Space and Range

Lecture 6

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Vector Spaces

Null Spaces and Injectivity

We will learn about two subspaces that are intimately connected with each linear map. We begin with the set of vectors that get mapped to 0.

3.12 **Definition** ***null space***, null T

For $T \in \mathcal{L}(V, W)$, the ***null space*** of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0:

$$\operatorname{null} T = \{v \in V : Tv = 0\}.$$

Some mathematicians use the term kernel instead of null space. The word "null" means zero. Thus the term "null space" should remind you of the connection to 0.

Examples

(a) If T is the zero map from V to W, in other words if $Tv = 0$ for every $v \in V$, then null $T = V$.

(b) Suppose $\varphi \in \mathscr{L}(\mathbb{C}^3, \mathbb{C})$ defined by $\varphi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$. Then null $\varphi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$. A basis of null φ is $(-2, 1, 0), (-3, 0, 1)$.

(c) Suppose $D \in \mathscr{L}(\mathscr{P}(\mathbb{R}), \mathscr{P}(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. The only functions whose derivative equals the zero function are the constant functions. Thus the null space of D equals the set of constant functions.

Examples

(d) Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the multiplication by x^2 map defined by

$$(Tp)(x) = x^2 p(x).$$

The only polynomial p such that $x^2 p(x) = 0$ for all $x \in \mathbb{R}$ is the zero polynomial. Thus null$T = \{0\}$.

(e) Suppose $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ is the backward shift defined by

$$T(x_1, x_2, \cdots) = (x_2, x_3, \cdots).$$

Clearly $T(x_1, x_2, \cdots)$ equals 0 if and only if $x_2, x_3, \cdots$ are all 0. Thus in this case we have

$$\text{null } T = \{(a, 0, 0, \cdots) : a \in \mathbb{F}\}.$$

Null Space

The next result shows that the null space of each linear map is a subspace of the domain. In particular, 0 is in the null space of every linear map.

3.14 The null space is a subspace.

Proposition

Suppose $T \in \mathscr{L}(V, W)$. Then null T is a subspace of V.

As we will soon see, for a linear map the next definition is closely connected to the null space.

Definition

A function $T : V \to W$ is called injective if $Tu = Tv$ implies $u = v$.

Injectivity

Injectivity is equivalent to null space equals $\{0\}$

Proposition

Let $T \in \mathscr{L}(V,W)$. Then T is injective if and only if null $T = \{0\}$.

Proof. First suppose T is injective. We want to prove that null $T = \{0\}$. We already know that $\{0\} \subset$ null T (by 3.11). To prove the inclusion in the other direction, suppose $v \in$ null T. Then

$$T(v) = 0 = T(0).$$

Because T is injective, the equation above implies that $v = 0$. Thus we can conclude that null $T = \{0\}$, as desired.

Proof.

To prove the implication in the other direction, now suppose null $T = \{0\}$. We want to prove that T is injective. To do this, suppose $u,\ v \in V$ and $Tu = Tv$. Then

$$0 = Tu - Tv = T(u - v).$$

Thus $u - v$ is in null T, which equals $\{0\}$. Hence $u - v = 0$, which implies that $u = v$. Hence T is injective, as desired.

Range and Surjectivity

Now we give a name to the set of outputs of a function

3.17 **Definition** ***range***

For T a function from V to W, the ***range*** of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv : v \in V\}.$$

Some mathematicians use the word image, which means the same as range.

Examples

(a) If T is the zero map from V to W, in other words if $Tv = 0$ for every $v \in V$, then range $T = \{0\}$.

(b) Suppose $T \in \mathscr{L}(\mathbb{R}^2, \mathbb{R}^3)$ is defined by $T(x,y) = (2x, 5y, x+y)$, then the range $T = \{(2x, 5y, x+y) : x, y \in \mathbb{R}\}$. A basis of range T is $(2,0,1), (0,5,1)$.

(c) Suppose $D \in \mathscr{L}(\mathscr{P}(\mathbb{R}), \mathscr{P}(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Because for every polynomial $q \in \mathscr{P}(\mathbb{R})$ there exists a polynomial $p \in \mathscr{P}(\mathbb{R})$ such that $p' = q$, the range of D is $\mathscr{P}(\mathbb{R})$.

Proposition

The next result shows that the range of each linear map is a subspace of the vector space into which it is being mapped.

Proposition

If $T \in \mathscr{L}(V, W)$, then range T is a subspace of W.

Surjectivity

Definition

A function $T : V \to W$ is called surjective if its range equals W.

Many mathematicians use the term onto, which means the same as surjective.

Example

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_5(\mathbb{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_4(\mathbb{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbb{R})$, which is now the vector space into which S maps.

Fundamental Theorem of Linear Maps

The next result is so important that it gets a dramatic name.

3.22 Fundamental Theorem of Linear Maps

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof. Let $u_1, u_2, \cdots, u_m$ be a basis of null T; thus $\dim$ null $T = m$. The linearly independent list $u_1, u_2, \cdots, u_m$ can be extended to a basis

$$u_1, \cdots, u_m, v_1, \cdots, v_n.$$

of V (by 2.33). Thus $\dim V = m + n$.

Proof

To complete the proof, we need only to show that range T is finite-dimensional and $\dim$ range $T = n$. We will do this by proving that $Tv_1, \cdots, Tv_n$ is a basis of range T.

Let $v \in V$. Because $u_1, \cdots, u_m, v_1 \cdots, v_n$ spans V, we can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n,$$

where the a's and b's are in $\mathbb{F}$. Applying T to both sides of this equation, we get

$$Tv = b_1 Tv_1 + \cdots + b_n Tv_n,$$

where the terms of the form Tu_j disappeared because each u_j is in null T. The last equation implies that $Tv_1, \cdots, Tv_n$ spans range T. In particular, range T is finite-dimensional.

Proof.

To show $Tv_1, \cdots, Tv_n$ is linearly independent, suppose $c_1, c_2, \cdots, c_n \in \mathbb{F}$ and $c_1 Tv_1 + \cdots + c_n Tv_n = 0$. Then $T(c_1 v_1 + \cdots + c_n v_n) = 0$. Hence $c_1 v_1 + \cdots + c_n v_n \in$ null T. Because $u_1, \cdots, u_m$ spans null T, we can write $c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + c_m u_m$, where the d's are in $\mathbb{F}$. This equation implies that all the c's and d's are 0. Thus $Tv_1, \cdots, Tv_n$ is linearly independent and hence is a basis of range T, as desired.

3.23 & 3.24

A map to a smaller dimensional space is not injective

3.23 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

A map to a larger dimensional space is not surjective

3.24 A map to a larger dimensional space is not surjective

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Example

Example

Rephrase in terms of a linear map the question of whether a homogeneous system of linear equations has a nonzero solution.

Solution. Fix positive integers m and n, and let $A_{j,k} \in \mathbb{F}$ for $j = 1, 2, \cdots, m$ and $k = 1, \cdots, n$. Consider the homogeneous system of linear equations

$$\sum_{k=1}^{n} A_{1,k} x_k = 0$$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k = 0.$$

Solution

Obviously $x_1 = \cdots = x_n = 0$ is a solution of the system of equations above; the question here is whether any other solutions exist.

Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \cdots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \cdots, \sum_{k=1}^{n} A_{m,k} x_k \right)$$

The equation $T(x_1, x_2, \cdots, x_n) = 0$ (the zero is the additive identity in $\mathbb{F}^m$, namely, the list of length m of all 0's) is the same as the homogeneous system of linear equations above.

Homogeneous, in this context, means that the constant term on the right side of each equation is zero.

Systems of linear equations

Thus we want to know if null T is strictly bigger than $\{0\}$. In other words, we can rephrase our question about nonzero solutions as follows: What condition ensures that T is not injective?

3.26 Homogeneous system of linear equations

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Example

Example

Consider the question of of whether an inhomogeneous system of linear equations has no solutions for some choice of the constant terms. Rephrase this question in terms of a linear map.

Solution. Fix positive integers m and n, and let $A_{j,k} \in \mathbb{F}$ for $j = 1, 2, \cdots, m$ and $k = 1, \cdots, n$. for $c_1, c_2 \cdots, c_n \in \mathbb{F}$, consider the system of linear equations

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k = c_m.$$

Solution

The question here is whether there is some choice of $c_1, c_2, \cdots, c_m \in \mathbb{F}$ such that no solution exists to the system above.

Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \cdots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \cdots, \sum_{k=1}^{n} A_{m,k} x_k \right)$$

The equation $T(x_1, x_2, \cdots, x_n) = (c_1, c_2, \cdots, c_m)$ is the same as the system of equations above .

Here we want to know if range $T \neq \mathbb{F}^m$.

Inhomogeneous system of linear equations

3.29 Inhomogeneous system of linear equations

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Homework Assignment 6

3.B: 3, 10, 18, 21, 23, 25, 27, 28.