Products and Quotients of Vector Spaces

Lecture 9

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Vector Spaces

Products of Vector Spaces

As usual when dealing with more than one vector space, all the vector spaces in use should be over the same field.

Definition

Suppose $V_1, \cdots, V_m$ are vector spaces over $\mathbb{F}$.

(a) *The product $V_1 \times \cdots \times V_m$ is defined by*

$$V_1 \times \cdots \times V_m = \{(v_1, \cdots, v_m) : v_1 \in V_1, \cdots, v_m \in V_m\}.$$

(b) *Addition on $V_1 \times \cdots \times V_m$ is defined by*

$$(u_1, \cdots, u_m) + (v_1, \cdots, v_m) = (u_1 + v_1, \cdots, u_m + v_m).$$

(c) *Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by*

$$\lambda(v_1, \cdots, v_m) = (\lambda v_1, \cdots, \lambda v_m).$$

Product Space is a Vector Space

The next result should be interpreted to mean that the product of vector spaces is a vector space with the operations of addition and scalar multiplication as defined above:

Theorem

Suppose $V_1, \cdots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

(a) The proof of the result above is left to the reader.

(b) Note that the additive identity of $V_1 \times V_2 \times \cdots V_m$ is $(0, \cdots, 0)$, where 0 in the jth slot is the additive identity of V_j.

(c) The additive inverse of $(v_1, v_2, \cdots, v_m) \in V_1 \times \cdots \times V_m$ is $(-v_1, \cdots, -v_m)$.

Example

3.74 Example Is $\mathbf{R}^2 \times \mathbf{R}^3$ equal to $\mathbf{R}^5$? Is $\mathbf{R}^2 \times \mathbf{R}^3$ isomorphic to $\mathbf{R}^5$?

Solution Elements of $\mathbf{R}^2 \times \mathbf{R}^3$ are lists $\big((x_1, x_2), (x_3, x_4, x_5)\big)$, where $x_1, x_2, x_3, x_4, x_5 \in \mathbf{R}$.

Elements of $\mathbf{R}^5$ are lists $(x_1, x_2, x_3, x_4, x_5)$, where $x_1, x_2, x_3, x_4, x_5 \in \mathbf{R}$.

Although these look almost the same, they are not the same kind of object. Elements of $\mathbf{R}^2 \times \mathbf{R}^3$ are lists of length 2 (with the first item itself a list of length 2 and the second item a list of length 3), and elements of $\mathbf{R}^5$ are lists of length 5. Thus $\mathbf{R}^2 \times \mathbf{R}^3$ does not equal $\mathbf{R}^5$.

The linear map that takes a vector $\big((x_1, x_2), (x_3, x_4, x_5)\big) \in \mathbf{R}^2 \times \mathbf{R}^3$ to $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5$ is clearly an isomorphism of $\mathbf{R}^2 \times \mathbf{R}^3$ onto $\mathbf{R}^5$. Thus these two vector spaces are isomorphic.

In this case, the isomorphism is so natural that we should think of it as a relabeling. Some people would even informally say that $\mathbf{R}^2 \times \mathbf{R}^3$ equals $\mathbf{R}^5$, which is not technically correct but which captures the spirit of identification via relabeling.

Dimension of a Product Space

3.76 Dimension of a product is the sum of dimensions

Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

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Proof Choose a basis of each V_j. For each basis vector of each V_j, consider the element of $V_1 \times \cdots \times V_m$ that equals the basis vector in the j^{th} slot and 0 in the other slots. The list of all such vectors is linearly independent and spans $V_1 \times \cdots \times V_m$. Thus it is a basis of $V_1 \times \cdots \times V_m$. The length of this basis is $\dim V_1 + \cdots + \dim V_m$, as desired. ■

Products and Direct Sums

In the next result, the map Γ is surjective by the definition of $U_1 + \cdots + U_m$. Thus the last word in the result below should be changed from “injective” to “invertible”.

3.77 Products and direct sums

Suppose that $U_1, \ldots, U_m$ are subspaces of V. Define a linear map $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ by

$$\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m.$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective.

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$$\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m.$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective.

Proof The linear map Γ is injective if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0. Thus 1.44 shows that Γ is injective if and only if $U_1 + \cdots + U_m$ is a direct sum, as desired. ■

direct sum and dimensions

3.78 A sum is a direct sum if and only if dimensions add up

Suppose V is finite-dimensional and $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m.$$

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Proof The map Γ in 3.77 is surjective. Thus by the Fundamental Theorem of Linear Maps (3.22), Γ is injective if and only if

$$\dim(U_1 + \cdots + U_m) = \dim(U_1 \times \cdots \times U_m).$$

Combining 3.77 and 3.76 now shows that $U_1 + \cdots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m,$$

as desired. ■

Quotients of Vector Spaces

We begin our approach to quotient spaces by defining the sum of a vector and a subspace.

3.79 Definition $v + U$

Suppose $v \in V$ and U is a subspace of V. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}.$$

Example

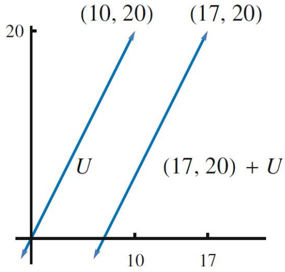
3.80 **Example** Suspose

$$U = \{(x, 2x) \in \mathbf{R}^2 : x \in \mathbf{R}\}.$$

Then U is the line in $\mathbf{R}^2$ through the origin with slope 2. Thus

$$(17, 20) + U$$

is the line in $\mathbf{R}^2$ that contains the point $(17, 20)$ and has slope 2.



affine subset

3.81 **Definition** ***affine subset, parallel***

- An ***affine subset*** of V is a subset of V of the form $v + U$ for some $v \in V$ and some subspace U of V.
- For $v \in V$ and U a subspace of V, the affine subset $v + U$ is said to be ***parallel*** to U.

3.82 **Example** ***parallel affine subsets***

- In Example 3.80 above, all the lines in $\mathbf{R}^2$ with slope 2 are parallel to U.
- If $U = \{(x, y, 0) \in \mathbf{R}^3 : x, y \in \mathbf{R}\}$, then the affine subsets of $\mathbf{R}^3$ parallel to U are the planes in $\mathbf{R}^3$ that are parallel to the xy-plane U in the usual sense.
  Important: With the definition of ***parallel*** given in 3.81, no line in $\mathbf{R}^3$ is considered to be an affine subset that is parallel to the plane U.

quotient space

3.83 Definition ***quotient space, V/U***

Suppose U is a subspace of V. Then the ***quotient space*** V/U is the set of all affine subsets of V parallel to U. In other words,

$$V/U = \{v + U : v \in V\}.$$

3.84 Example ***quotient spaces***

- If $U = \{(x, 2x) \in \mathbf{R}^2 : x \in \mathbf{R}\}$, then $\mathbf{R}^2/U$ is the set of all lines in $\mathbf{R}^2$ that have slope 2.
- If U is a line in $\mathbf{R}^3$ containing the origin, then $\mathbf{R}^3/U$ is the set of all lines in $\mathbf{R}^3$ parallel to U.
- If U is a plane in $\mathbf{R}^3$ containing the origin, then $\mathbf{R}^3/U$ is the set of all planes in $\mathbf{R}^3$ parallel to U.

V/U

Our next goal is to make V/U into a vector space. To do this, we will need the following result.

3.85 **Two affine subsets parallel to U are equal or disjoint**

Suppose U is a subspace of V and $v, w \in V$. Then the following are equivalent:

(a) $v - w \in U$;

(b) $v + U = w + U$;

(c) $(v + U) \cap (w + U) \neq \varnothing$.

We prove the result as follows: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

Quotient Space

3.86 Definition *addition and scalar multiplication on V/U*

Suppose U is a subspace of V. Then ***addition*** and ***scalar multiplication*** are defined on V/U by

$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$

for $v, w \in V$ and $\lambda \in \mathbf{F}$.

Quotient Space is a Vector Space

3.87 Quotient space is a vector space

Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

Proof: Quotient space is a vector space

Proof.

(a) The potential problem with the definitions above of addition and scalar multiplication on V/U is that the representation of an affine subset parallel to U is not unique. Specifically, suppose $v, w \in V$. Suppose also that $\hat{v}, \hat{w} \in V$ are such that $v + U = \hat{v} + U$ and $w + U = \hat{w} + U$. To show that the definition of addition on V/U given above makes sense, we must show that $(v + w) + U = (\hat{v} + \hat{w}) + U$.

(b) By 3.85, we have

$$v - \hat{v} \in U \text{ and } w - \hat{w} \in U.$$

Because U is a subspace of V and thus is closed under addition, this implies that $(v - \hat{v}) + (w - \hat{w}) \in U$. Thus $(v + w) - (\hat{v} + \hat{w}) \in U$. Using 3.85 again, we see that

$$(v + w) + U = (\hat{v} + \hat{w}) + U,$$

as desired. Thus the definition of addition on V/U makes sense.

Proof

(c) Similarly, suppose $\lambda \in \mathbb{F}$. Because U is a subspace of V and thus is closed under scalar multiplication, we have $\lambda(v - \hat{v}) \in U$. Thus $\lambda v - \lambda \hat{v} \in U$. Hence 3.85 implies that $(\lambda v) + U = (\lambda \hat{v}) + U$. Thus the definition of scalar multiplication on V/U makes sense.

(d) Now that addition and scalar multiplication have been defined on V/U, the verification that these operations make V/U into a vector space is straightforward and is left to the reader. Note that the additive identity of V/U is $0 + U$ (which equals U) and that the additive inverse of $v + U$ is $(-v) + U$.

Quotient Map

The next concept will give us an easy way to compute the dimension of V/U.

Definition

Suppose U is a subspace of V. The quotient map π is the linear map $\pi : V \to V/U$ defined by

$$\pi(v) = v + U$$

for $v \in V$.

The reader should verify that π is indeed a linear map. Although π depends on U as well as V, these spaces are left out of the notation because they should be clear from the context.

Dimension

Proposition

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U.$$

Proof Let π be the quotient map from V to V/U. From 3.85, we see that $\operatorname{null}\pi = U$. Clearly $\operatorname{range}\pi = V/U$. The Fundamental Theorem of Linear Maps (3.22) thus tells us that

$$\dim V = \dim U + \dim V/U,$$

which gives the desired result. ■

Linear Map

Each linear map T on V induces a linear map $\tilde{T}$ on $V/(\text{null } T)$, which we now define:

Definition

Suppose $T \in \mathscr{L}(V,W)$. Define $\tilde{T} : V/(\textit{null } T) \to W$ by

$$\tilde{T}(v + \textit{null } T) = Tv.$$

To show that the definition of $\tilde{T}$ makes sense, suppose $u, v \in V$ are such that $u + \text{null } T = v + \text{null } T$. By 3.85, we have $u - v \in \text{null } T$. Thus $T(u-v) = 0$. Hence $Tu = Tv$. Thus the definition of $\tilde{T}$ indeed makes sense.

3.91

Theorem

Suppose $T \in \mathscr{L}(V,W)$. Then

(a) $\tilde{T}$ *is a linear map from $V/(\text{null}\, T)$ to W;*

(b) $\tilde{T}$ *is injective;*

(c) *range $\tilde{T}$ = range T;*

(d) *$V/(\text{null}\, T)$ is isomorphic to range T.*

Homework Assignment 9

3.E: 6, 10, 11, 12, 13, 17, 20.