1. Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (a) Every operator on a complex vector space has a square root.

   (b) The orthogonal complement of any set is a subspace.

   (c) If $\lambda$ is an eigenvalue of a self-adjoint operator $T$, then $\lambda$ is a singular value of $T$.

   (d) Any linear operator on a finite-dimensional vector space has a Jordan form.

   (e) Let $T$ be a linear operator on a vector space $V$ such that $T$ has $n$ distinct eigenvalues, where $n = \dim(V)$. Then the degree of the minimal polynomial of $T$ equals $n$.

2. Define $T \in \mathcal{L}(\mathbb{C}^3)$ by $T(z_1, z_2, z_3) = (z_2 + z_3, z_1 + z_3, z_1 + z_2)$.

   (a) Find all the eigenvalues of $T$.

   (b) Find a basis for each of the eigenspaces of $T$.

   (c) Find the characteristic polynomial and minimal polynomial of $T$.

   (d) Find the trace and determinant of $T$.

   (e) Find the Jordan form of $T$.

3. Let $S = \{(1, 2, 3, -4), (-5, 4, 3, 2)\}$ in $\mathbb{R}^4$. Find an orthonormal basis of $S^\perp$.

4. Suppose $V$ is a real vector space. Prove that there exists $T \in \mathcal{L}(V)$ such that $T^2 = -I$ if and only if $V$ has even dimension.

5. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{\pi}^{-\pi} f(x)g(x)dx,$$

   let

$$V = \operatorname{span}(1, \cos x, \cos 2x, \cos 3x, \sin x, \sin 2x, \sin 3x).$$

   (a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that $D$ is normal but not self-adjoint.

   (b) Find an orthonormal basis of $V$ with respect to which $T$ has a block diagonal matrix such that each block is a $1 \times 1$ matrix or a $2 \times 2$ matrix of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

   with $b > 0$.

6. Let $V$ be a finite dimensional inner product space over $\mathbb{F}$, and let $v_1$ and $v_2$ be two linearly independent vectors in $V$ such that $||v_1|| = ||v_2||$. Define the **Householder operator** $H_u : V \to V$ by $H_u(v) = v - 2\langle v, u\rangle u$ for all $v \in V$. Prove the following results.

   (a) Let $u$ be a unit vector in $V$, $H_u$ is linear, and $H_u(v) = v$ if $v$ is orthogonal to $u$.

   (b) Let $u$ be a unit vector in $V$, $H_u$ is an isometry.

   (c) If $\mathbb{F} = \mathbb{C}$, then there exists a unit vector $u$ in $V$ and a complex number $\theta$ with $|\theta| = 1$ such that
   $$H_u(v_1) = \theta v_2.$$

   (d) If $\mathbb{F} = \mathbb{R}$, then there exists a unit vector $u$ in $V$ such that
   $$H_u(v_1) = v_2.$$