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MA215 Probability Theory

Assignment 16

1. Suppose that X and Y have joint p.d.f. $f(x,y)$. Let $f_Y(y)$ be the marginal p.d.f. of Y. Show that for $f_Y(y)>0$,

$$\lim_{\varepsilon\to 0+} P\{X\leqslant x \mid y<Y\leqslant y+\varepsilon\} = \int_{-\infty}^{x} \frac{f(u,y)}{f_Y(y)}du.$$

(This is why we define $P\{X\leqslant x \mid Y=y\} \triangleq \int_{-\infty}^{x}\frac{f(u,y)}{f_Y(y)}du$ and call $f_{X|Y}(x\mid y)\triangleq \frac{f(x,y)}{f_Y(y)}$ the conditional probability density function of X, given that $Y=y$.)

Proof. $P\{X\leq x \mid y<Y\leq y+\varepsilon\} = \dfrac{P\{X\leq x, y<Y\leq y+\varepsilon\}}{P\{y<Y\leq y+\varepsilon\}}$

$P\{X\leq x, y<Y\leq y+\varepsilon\} = \int_{-\infty}^{x}\int_{y}^{y+\varepsilon} f(u,v)\,dv\,du$ $\quad P\{y<Y\leq y+\varepsilon\} = \int_{y}^{y+\varepsilon} f_Y(v)\,dv$

for $\varepsilon>0$ is very small $\therefore \int_{y}^{y+\varepsilon} f_Y(v)dv \approx \varepsilon f_Y(y)$

$(\varepsilon\to 0^+)$

$\int_{y}^{y+\varepsilon} f(u,v)dv \approx \varepsilon f(u,y)$

$\therefore P\{X\leq x \mid y<Y\leq y+\varepsilon\} = \dfrac{\int_{-\infty}^{x}\varepsilon f(u,y)\,du}{\varepsilon f_Y(y)} = \dfrac{\int_{-\infty}^{x} f(u,y)du}{f_Y(y)}$

$\therefore \lim\limits_{\varepsilon\to 0^+} P\{X\leq x \mid y<Y\leq y+\varepsilon\} = \int_{-\infty}^{x} \dfrac{f(u,y)}{f_Y(y)}\,du$ $\square$

2. 设 (X,Y) 服从圆域 $G: x^2+y^2\leqslant 1$ 上的均匀分布. 求条件概率密度$f_{X|Y}(x\mid y)$.

$\because (X,Y)$ 在 $G: x^2+y^2\leq 1$ 上均匀分布 $\therefore f_{X,Y}(x,y) = \frac{1}{\pi}$ $(x^2+y^2\leq 1)$

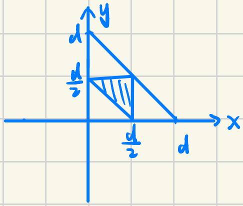
$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{X,Y}(x,y)\,dx = \frac{2}{\pi}\sqrt{1-y^2}$ $(-1\leq y\leq 1)$

$\therefore f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{1}{2\sqrt{1-y^2}}$ $(-\sqrt{1-y^2}\leq x\leq \sqrt{1-y^2})$

3. 将长度为 d 的一根木棒任意截去一段, 再将剩下的木棒任意截为两段. 求这三段木棒能构成三角形的概率.

设截成三段的长度分别为 $x, y, d-x-y$ $(0<x<d, 0<y<d)$

若能构成三角形，则 $\begin{cases} x+y>\frac{d}{2} \\ d-x>\frac{d}{2} \\ d-y>\frac{d}{2}\end{cases} \Rightarrow \begin{cases} x+y>\frac{d}{2} \\ x,y<\frac{d}{2}\end{cases}$



显然 (X,Y) 在区域 G: $\{(x,y) \mid x>0, y>0, x+y<d\}$ 上均匀分布

区域 F: $\{(x,y) \mid 0<x<\frac{d}{2}, 0<y<\frac{d}{2}, \frac{d}{2}<x+y<d\}$

$$\therefore P(\text{构成三角形}) = \frac{f_F(x,y)}{f_G(x,y)} = \frac{S_F}{S_G} = \frac{\frac{1}{8}d^2}{\frac{1}{2}d^2} = \frac{1}{4}$$

4. 假设在某个系统中, 元件和备用件的平均寿命都是 μ. 如果元件失效, 系统自动用其备件替代, 但替换出错的概率为 p. 求整个系统的平均寿命.

设系统的平均寿命为 X. 则 $X = \begin{cases} \mu, & \text{替换出错} \\ 2\mu, & \text{替换不出错} \end{cases}$

$$\therefore E(X) = p \cdot \mu + (1-p) \cdot 2\mu = (2-p)\mu.$$