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MA215 Probability Theory

Assignment 10

1. Suppose a player plays the following gambling games which is known as the wheel of fortune. The player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, $i = 1, 2, 3$; then the player wins i units; on the other hand, if the number bet by the player does not appear on any of the dies, then the player loses 1 unit. Is this game fair to the player?

Suppose that the player get X units

$X \in \{-1, 1, 2, 3\}$

$P(X=3) = (\frac{1}{6})^3 = \frac{1}{216}$ $\quad P(X=2) = C_3^1 (\frac{1}{6})^2 (\frac{5}{6}) = \frac{15}{216}$

$P(X=1) = C_3^2 \frac{1}{6} \cdot (\frac{5}{6})^2 = \frac{75}{216}$ $\quad P(X=-1) = (\frac{5}{6})^3 = \frac{125}{216}$

$\therefore$

X	-1	1	2	3
P	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

$\therefore E(X) = -1 \times \frac{125}{216} + 1 \times \frac{75}{216} + 2 \times \frac{15}{216} + 3 \times \frac{1}{216} = -\frac{17}{216} < 0$

$\therefore$ The game is not fair.

2. Suppose the r.v. X takes non-negative integer values only. Show that

$$E(X) = \sum_{n=0}^{\infty} P(X > n) = \sum_{n=1}^{\infty} P(X \geqslant n).$$

Proof. $\because X \in N$.

$\therefore \sum_{n=0}^{\infty} P(X>n) = (P(X=1) + P(X=2) + \cdots + P(X=n) + \cdots) + (P(X=2) + P(X=3) + \cdots + P(X=n) + \cdots)$

$+ \cdots + (P(X=n) + P(X=n+1) + \cdots) = P(X=1) + 2P(X=2) + \cdots + nP(X=n) + \cdots$

$= \sum_{n=1}^{\infty} nP(X=n) = E(X)$

同理，$\sum_{n=1}^{\infty} P(X \geqslant n) = P(X \geqslant 1) + \cdots + P(X \geqslant n) + \cdots = (P(X=1) + P(X=2) + \cdots + P(X=n) + \cdots) +$

$(P(X=2) + P(X=3) + \cdots + P(X=n) + \cdots) + \cdots + (P(X=n) + P(X=n+1) + \cdots)$

$= \sum_{n=1}^{\infty} nP(X=n) = E(X)$. $\square$

3. (a) Suppose the r.v. X obeys the uniformly distribution over $[a, b]$. Find $E(X)$.

   (b) Suppose the r.v. X obeys the general Γ distribution with parameters λ and α where $\lambda > 0, \alpha > 0$. Write down the p.d.f. of this general Γ random variable and the analytic form of the Γ function $\Gamma(\alpha)$ for $\alpha > 0$ and hence find the $E(X)$ of this general Γ random variable.

   (c) Suppose $Y = X^2$ where X is normally distributed with parameters μ and σ^2. Obtain the p.d.f. of Y and then find $E(Y)$.

(a) p.d.f. $f(x)=\begin{cases}\frac{1}{b-a}, & a\le x\le b\\ 0, & \text{otherwise}\end{cases}$

$\therefore E(x)=\int_{-\infty}^{+\infty} xf(x)\,dx=\int_a^b \frac{x}{b-a}\,dx=\frac{1}{2}\frac{x^2}{(b-a)}\Big|_a^b=\frac{a+b}{2}$

(b) p.d.f. $f(x)=\begin{cases}\frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}, & x\ge 0\\ 0, & \text{otherwise}\end{cases}$

$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\,dt$

$E(X)=\int_{-\infty}^{+\infty} xf(x)\,dx=\int_0^{+\infty}\frac{\lambda^\alpha x^\alpha e^{-\lambda x}}{\Gamma(\alpha)}\,dx=\int_0^{+\infty}\frac{(\lambda x)^\alpha e^{-\lambda x}}{\lambda\Gamma(\alpha)}\,d(\lambda x)$

$=\frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)}=\frac{\alpha\Gamma(\alpha)}{\lambda\Gamma(\alpha)}=\frac{\alpha}{\lambda}$

(c) $X\sim N(\mu,\delta^2)$. $\therefore$ p.d.f. $f_X(x)=\frac{1}{\sqrt{2\pi}\delta}e^{-\frac{(x-\mu)^2}{2\delta^2}}$, $-\infty<x<+\infty$

$\because Y=X^2$ $\therefore X=\ln Y$, $f_Y(y)=\left|\frac{1}{y}\right|\cdot f_X(\ln y)=\begin{cases}\frac{1}{\sqrt{2\pi}\delta y}\cdot e^{-\frac{(\ln y-\mu)^2}{2\delta^2}}, & y\ge 0\\ 0, & \text{otherwise.}\end{cases}$

$E(X^2)=Var(X)+E(X)^2=\delta^2+\mu^2$

$\therefore E(Y)=\delta^2+\mu^2$

4. (a) Suppose that the two discrete r.v.s X and Y have joint p.m.f. given by

X	$Y=1$	$Y=2$	$Y=3$	$Y=4$
$X=1$	2/32	3/32	4/32	5/32
$X=2$	3/32	4/32	5/32	6/32

Obtain $E(X)$ and $E(Y)$.

(b) Suppose that the two continuous r.v.s X and Y have joint p.d.f.

$$f(x,y)=\begin{cases}x+y, & 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1,\\ 0, & \text{otherwise.}\end{cases}$$

Find $E(X)$ and $E(Y)$.

(a)

X	1	2
P	$\frac{7}{16}$	$\frac{9}{16}$

Y	1	2	3	4
P	$\frac{5}{32}$	$\frac{7}{32}$	$\frac{9}{32}$	$\frac{11}{32}$

$\frac{x^3}{3}+\frac{x^2}{4}$

$\therefore E(X)=1\times\frac{7}{16}+2\times\frac{9}{16}=\frac{25}{16}$, $E(Y)=1\times\frac{5}{32}+2\times\frac{7}{32}+3\times\frac{9}{32}+4\times\frac{11}{32}=\frac{45}{16}$

(b) $f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy=\int_0^1(x+y)\,dy=(xy+\frac{y^2}{2})\Big|_0^1=x+\frac{1}{2}$ $(0\le x\le 1)$

$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\,dx=y+\frac{1}{2}$ $(0\le y\le 1)$

$\therefore E(X)=\int_{-\infty}^{+\infty} xf_X(x)\,dx=\int_0^1(x^2+\frac{x}{2})\,dx=\frac{7}{12}$, $E(Y)=\int_{-\infty}^{+\infty} yf_Y(y)\,dy=\int_0^1(y^2+\frac{y}{2})\,dy=\frac{7}{12}$