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MA215 Probability Theory

Assignment 11

1. Let $Y = e^X$ with $X \sim N(\mu, \sigma^2)$. Use the following two methods to obtain $E(Y)$.

   (a) First obtain the p.d.f. $f_Y(y)$ of Y, and then find $E(Y)$ by using $f_Y(y)$.

   (b) Find $E(Y)$ directly by viewing Y as a function of X and then using the formula of getting the expected value of a function of the random variable X.

(a) $Y = e^X$, $X = \ln Y$, $\frac{dX}{dY} = \frac{1}{Y}$

$\therefore$ p.d.f.: $f_Y(y) = \left|\frac{dx}{dy}\right| \cdot f_X(\ln y) = \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(\ln y-\mu)^2}{2\delta^2}}$, $y>0$

$\therefore$ p.d.f: $f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi\delta^2}\cdot y} e^{-\frac{(\ln y-\mu)^2}{2\delta^2}}, & y>0 \\ 0, & \text{otherwise} \end{cases}$

$$E(Y) = \int_0^\infty y f_Y(y)\,dy = \int_0^\infty \frac{1}{\sqrt{2\pi\delta^2}} \cdot e^{-\frac{(\ln y-\mu)^2}{2\delta^2}} dy = \frac{1}{\sqrt{2\pi\delta^2}} \int_{-\infty}^{+\infty} e^{\ln y - \frac{(\ln y-\mu)^2}{2\delta^2}} \cdot d(\ln y)$$
$$= e^{\mu + \frac{\delta^2}{2}}$$

(b) $E(Y) = E(e^X) = \int_{-\infty}^{+\infty} e^x \cdot f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\delta^2}} e^{x - \frac{(x-\mu)^2}{2\delta^2}} dx = e^{\mu+\frac{\delta^2}{2}}$

2. (a) Suppose the random variable X obeys the uniformly distribution over interval $[a, b]$. Find $E(X^2)$ and $E(X^2) - [E(X)]^2$.

   (b) Suppose $X \sim N(\mu, \sigma^2)$. Find $E(X^2)$ and $E(X^2) - [E(X)]^2$.

(a) $f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases}$

$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{b^2-a^2}{2} = \frac{a+b}{2}$

$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_a^b \frac{x^2}{b-a} dx = \frac{a^2+ab+b^2}{3}$

$E(X^2) - [E(X)]^2 = \frac{(b-a)^2}{12}$

(b) $\because X \sim N(\mu, \delta^2)$ $\therefore E(X^2) = Var(X) + [E(X)]^2 = \delta^2 + \mu^2$

$\therefore E(X^2) - [E(X)]^2 = Var(X) = \delta^2$

3. (a) The p.d.f. of X is given by

$$f_X(x) = \begin{cases} \frac{1}{x(\ln 3)}, & 1 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(X)$, $E(X^2)$, and $E(X^3)$.

(b) Use the results of part (a) to determine $E(X^3 + 2X^2 - 3X + 1)$.

(a) $E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_1^3 \frac{1}{\ln 3}\,dx = \frac{2}{\ln 3}$

$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_1^3 \frac{x}{\ln 3}\,dx = \frac{4}{\ln 3}$

$E(X^3) = \int_{-\infty}^{+\infty} x^3 f_X(x)\,dx = \int_1^3 \frac{x^2}{\ln 3}\,dx = \frac{26}{3\ln 3}$

(b) $E(X^3+2X^2-3X+1) = \int_{-\infty}^{+\infty} (x^3+2x^2-3x+1)\cdot\frac{1}{x\ln 3}\,dx = E(X^3)+2E(X^2)-3E(X)$

$+ \int_1^3 \frac{1}{x\ln 3}\,dx = \frac{32}{3\ln 3} + 1$

4. The p.d.f. X is given by

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 < x \leqslant 1, \\ \frac{1}{2}, & 1 < x \leqslant 2, \\ \frac{3-x}{2}, & 2 < x \leqslant 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the expectation of $Y = X^2 - 5X + 3$.

$E(Y) = E(X^2-5X+3) = \int_{-\infty}^{+\infty} (x^2-5x+3) f_X(x)\,dx = \int_0^1 \frac{1}{2}(x^3-5x^2+3x)\,dx + \int_1^2 \frac{1}{2}(x^2-5x+3)\,dx$

$+ \int_2^3 \frac{3-x}{2}\cdot(x^2-5x+3)\,dx = \frac{1}{24} + (-\frac{13}{12}) + \frac{3}{2} = \frac{11}{24}$

5. The two continuous random variables X and Y have joint p.d.f.

$$f(x,y) = \begin{cases} x+y, & 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $E[(X+Y)^2]$.

$E(X^2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,y)\,dy\,dx = \int_0^1\int_0^1 (x^3+x^2y)\,dy\,dx = \frac{5}{12}$

$E(Y^2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2 f(x,y)\,dx\,dy = \int_0^1\int_0^1 (xy^2+y^3)\,dy\,dx = \frac{5}{12}$

$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f(x,y)\,dx\,dy = \int_0^1\int_0^1 (x^2y+xy^2)\,dx\,dy = \frac{1}{3}$

$E[(X+Y)^2] = E(X^2) + 2E(XY) + E(Y^2) = \frac{3}{2}$