MA202 Complex Analysis, Midterm Exam

Name:

ID:

Problem 1. [15 pts]

(i) What is the definition of a holomorphic function on a domain $\Omega \subseteq \mathbb{C}$?

(ii) Determine whether $z^3, |z|^3$ are holomorphic functions on $\mathbb{C}$.

Problem 2. [15 pts] Let f be a holomorphic function on a connected open set $\Omega \subseteq \mathbb{C}$. Show that if $|f|$ is constant, then f is constant.

Problem 3. [20 pts] Show that for all $w \in \mathbb{C}$, there is

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x w} \mathrm{d}x = e^{-\pi w^2}.$$

Problem 4. [20 pts] Let n be an integer $\geqslant 2$ and α a real number such that $n > 1 + \alpha > 0$. Evaluate the integral

$$\int_0^{\infty} \frac{x^\alpha}{1 + x^n} \mathrm{d}x.$$

Problem 5. [15 pts] Find the number of zeros, counting multiplicity, of the polynomial $z^8 - 7z^3 + 2z + 1$ in the annulus $1 < |z| < 2$.

Problem 6. [15 pts] Let $f(z)$ be a holomorphic function on the annulus

$$\Omega := \{z \in \mathbb{C} : r_1 < |z| < r_2\}.$$

Show that there exists complex numbers $\{a_n\}_{n \in \mathbb{Z}}$ such that

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n,$$

where the right hand side is absolutely and uniformly convergent on any closed annulus $\Omega' := \{z \in \mathbb{C} : r_1' \leqslant |z| \leqslant r_2'\}$ contained in Ω.

南方科技大学
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Course Name: Complex analysis **Dept.:** Mathematics
Exam Duration: 110 mins

Question No.	1	2	3	4	5	6	7	8	9	10
Score										

This exam paper contains 7 questions and the score is 100 in total.
(Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

1. (15 points) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex valued function which is written as

$$f(x, y) = u(x, y) + iv(x, y),$$

where u, v are continuously differentiable.

(a) Let $g(z) = \overline{f(\bar{z})}$, show that g is holomorphic $\boxed{\text{if and only if}}$ $\boxed{f}$ is holomorphic.

(b) Suppose $u(x, y) = xy - x + y$, find all possible v such that f is holomorphic. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ∂u

2. (10 points) Let $f(z)$ be a continuous function defined in the unit disk $D_1 = \{z \mid |z| < 1\}$. Assuming $f(z)^5$ and $f(z)^7$ are holomorphic, show that $f(z)$ itself is holomorphic.

3. (15 points) Consider the meromorphic function $f(z) = \frac{1}{e^{z^2}+1}$.

(a) Find all the poles of $f(z)$.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series expression of $f(z)$ centered at 0, find the radius of convergence and explain your answer.

4. (15 points) Consider the strip $S = \{z \mid 0 < \mathrm{Im}(z) < 1\}$. Let $f : S \to \mathbb{C}$ be a holomorphic function on S such that it extends continuously to the closure $\bar{S}$ and has real values on the boundary.

(a) Show that there is an entire function F whose restriction to S is f. (Hints: use Schwarz reflection principle)

(b) Assuming $f : \bar{S} \to \mathbb{C}$ is bounded, show that f is a constant function. bounded + entire = cor

5. (15 points) For each of the following functions $f(z)$, determine the type of singularity, and compute the residue at the point z if it is a pole.

(a)

$$f(z) = \frac{e^z - e^{-z}}{z^3(e^z + e^{-z})}, \quad \text{at } z = -\frac{\pi i}{2}.$$

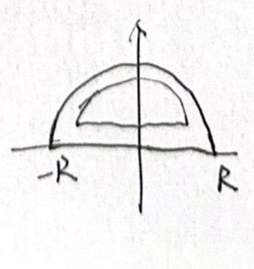
(b)

$$f(z) = \sin\left(\frac{1}{z}\right), \quad \text{at } z = 0.$$

6. (10 points) Calculate the following integral: $\cos\theta = \sqrt{2}$?

$x+iy$

$$\int_0^{2\pi} \frac{1}{\sqrt{2} - \cos(\theta)} d\theta.$$



$e^{i\theta}$

7. (20 points) Consider a function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{\sin(z)}{z}.$$

(a) Show that f has a removable singularity at 0 and find the value that f can be extended continuously to 0.

(b) Show that $f(z) \neq 0$ for $|z| \leq 3$.

(c) Show that for any holomorphic function g defined in a neighborhood of the closed unit disk $\{z \mid |z| \leq 1\}$, we have the integral formula

$$g(z) = \frac{1}{2\pi i} \int_{C_1} \frac{g(w)}{\sin(w - z)} dw$$

for $|z| < 1$. Here C_1 is the unit circle oriented in counterclockwise direction.

COMPLEX ANALYSIS (H) MIDTERM EXAM

Instructions

- Allotted time: 4:20-6:10pm
- Partial marks will be awarded for correct reasoning

(1) True or False? No need to justify your answer.

i An entire function that does not take on any real values is constant.

ii The Bessel function, defined by the power series

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}$$

where r is a positive integer, is entire.

iii All values of r^i, for $r \in \mathbb{R} \smallsetminus \{0\}$, lie on the unit circle.

iv A continuous function defined on the closed unit disk can be uniformly approximated on the closed unit disk by a sequence of holomorphic functions.

(30 marks)

(2) Evaluate, where C is the positively oriented circle centered at the origin with radius 1

$$\int_C \frac{z^4 + 3z^2 + 1}{z^{16}} dz$$

(10 marks)

(3) Evaluate

$$\int_0^{+\infty} \frac{\cos(x)}{x^2 + b^2} dz$$

when $b > 0$.

(20 marks)

(4) Let γ be the closed curve parameterised by $e^{\pi i t}$, $t \in [0, 4]$. What is the winding number of γ around the origin?

(5 marks)

(5) Suppose $c \in \mathbb{C}$ satisfies $|c| > e$. Calculate the number (with multiplicities) of solutions of the equation $e^z = cz^n$ for $|z| < 1$.

(10 marks)

(6) Give an example of a subset of $\mathbb{C}$ on which a branch of the multivalued function $\log(1-z^2)$ can be defined. It is enough to draw the subset, no need to give a formula.
For $z = e^{i\frac{\pi}{4}}$, list all values of $\log(1-z^2)$.

(15 marks)

(7) Let $f(z)$ be a function holomorphic on Ω, where Ω is open and contains the closed unit disk $\mathbb{D}$ centered on the origin. Show that if $|f(z^2)| \geq |f(z)|$ for all z in the interior of $\mathbb{D}$, then f is constant.

(10 marks)

MA202 Complex Analysis, Final Exam
Name:
ID:

Problem 1. [15 pts]
(i) What is the definition of a meromorphic function on a domain $\Omega \subseteq \mathbb{C}$?
(ii) State the Riemann mapping theorem.

Problem 2. [10 pts] Calculate the integral $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} \mathrm{d}x$, $a \in \mathbb{R}^{\times}$.

Problem 3. [10 pts] Calculate the integral $\int_0^{\infty} \frac{\sin x}{x} \mathrm{d}x$.

Problem 4. [15 pts] Calculate the integral

$$\int_0^{2\pi} \log|1 - ae^{i\theta}| \mathrm{d}\theta, \quad |a| \neq 1$$

Problem 5. [15 pts] Find an infinite product formula of the function $\cos z$ in terms of its zeros.

Problem 6. [15 pts] Let $f(z)$ be an entire function and n a non-negative integer. Show that if $\lim\limits_{z\to\infty} \frac{f(z)}{z^n}$ exists and is nonzero, then $f(z)$ is a polynomial of degree n.

Problem 7. [10 pts] Let $\mathbb{D}$ denote the open unit disk and f be a holomorphic function on $\mathbb{D}$. Show that the diameter

$$d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$$

satisfies the inequality $d \geqslant 2|f'(0)|$.

Problem 8. Suppose f and g are holomorphic in a region containing the disc $|z| \leqslant 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leqslant 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show the following:

(i) [5pts] $f_\epsilon(z)$ has a unique zero z_ϵ in $|z| \leqslant 1$ if ϵ is sufficiently small.
(ii) [5pts] The map $\epsilon \to z_\epsilon$ is holomorphic when ϵ is sufficiently smally.

Q_1: $f(z) = \dfrac{z\log(z)}{(z+1+i)^2}$ $\quad -\pi < \mathrm{Arg} < \pi$ (20分)

a) 求留数 ($z=-1-i$ 处)

b) 问 如果 branch 不同，留数是否改变

Q_2:

10. 寻找下面函数的 Hadamard 乘积：(20分) a问 of Ch5 10 原题

ps. 老师会给公式

(a) $e^z - 1$;

(b) $\cos\pi z$.

Q_3:

10. 令 $F: \mathbb{H} \to \mathbb{C}$ 是一个全纯函数，它满足 (20分)

$$|F(z)| \leqslant 1 \text{ 和 } F(i) = 0.$$

a. 证明 $\varphi(z): \mathbb{H} \to \mathbb{D}$ conformal $\varphi(z) = \dfrac{i-z}{i+z}$ (Ch8 例 1原题)

b. 证明：

$$|F(z)| \leqslant \left|\frac{z-i}{z+i}\right| \quad \text{对于所有 } z \in \mathbb{H}.$$ (Ch8 作业 10题原题)

Q_4: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(A) = 1$ $\quad \varphi_A(z) = \dfrac{ai+b}{ci+d}$ $\quad A: \mathbb{H} \to \mathbb{H}$ (20分)

a. 求 A，使 $\varphi_A(z) = 1+2i$

b. 求所有 A，使 $\varphi_A(i) = i$

Q_5 $f(z) = e^z + 5z^3$ (25分)

① 求 函数 e^z 在 unit disc $\overline{\mathbb{D}}$ 上的最大模

② 用 Rouché 定理 说明 inside $\mathbb{D}$ 有 3 个零点

(bonus 5分) ③判断②中零点是否 distinct