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MA215 Probability Theory

Assignment 14

1. Suppose that the m.g.f. of X is $M_X(t) = \frac{2}{\sqrt{4-t}}$, $(t < 4)$.

   (i) Find $E(X)$, $E(X^2)$ and Var (X).

   (ii) Suppose that X and Y are independent and both with this m.g.f. (i.e., $M_X(t) = M_Y(t) = \frac{2}{\sqrt{4-t}}$). Find the m.g.f. of of $X+Y$ and identify the distribution of $X+Y$.

(i) $E(X) = \frac{d}{dt}E(e^{tX})\Big|_{t=0} = \frac{d}{dt}M_X(t)\Big|_{t=0} = \frac{d}{dt}\frac{2}{\sqrt{4-t}}\Big|_{t=0} = \frac{1}{(4-t)^{\frac{3}{2}}}\Big|_{t=0} = \frac{1}{8}$

$E(X^2) = \frac{d^2}{dt^2}M_X(t)\Big|_{t=0} = \frac{d}{dt}\frac{1}{(4-t)^{\frac{3}{2}}}\Big|_{t=0} = \frac{3}{2}(4-t)^{-\frac{5}{2}}\Big|_{t=0} = \frac{3}{64}$

$Var(X) = E(X^2) - (E(X))^2 = \frac{3}{64} - \frac{1}{64} = \frac{1}{32}$

(ii) $\because$ X, Y are independent $\therefore M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = \frac{4}{4-t}$

$\therefore$ X+Y follows Gamma distribution, and $\lambda = \frac{1}{2}$, $k=2$

$\therefore X+Y \sim Gamma(2, \frac{1}{2})$

2. (a) If the m.g.f. of X is $M_X(t) = \frac{a^2}{a^2-t^2}$, then find the kth moment $E(X^k)$, $k \in \mathbb{N}_+$.

   (b) Suppose the m.g.f. of X can be expressed as a power series

$$M_X(t) = \sum_{k=0}^{\infty} a_k t^k = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$$

   and assume further that $a_0 = 1$, $a_1 = 3$ and $a_2 = 7$. Find $E(X)$ and Var (X).

(a) $E(X^k) = \frac{d^k}{dt^k}M_X(t)\Big|_{t=0} = \frac{d^k}{dt^k} \cdot \frac{a^2}{a^2-t^2}\Big|_{t=0} = 0$

(b) $E(X) = \frac{d}{dt}M_X(t)\Big|_{t=0} = a_1 = 3$

$E(X^2) = \frac{d^2}{dt^2}M_X(t)\Big|_{t=0} = 2a_2 = 14$

$Var(X) = E(X^2) - (E(X))^2 = 5$

3. Suppose the m.g.f. of X has the Maclaurin series

$$M_X(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots .$$

Find the variance and the third central moment $E[(X - E(X))^3]$ of X in terms of a_1, a_2, and a_3.

$E[(X-E(X))^3] = E(X^3 - 3X^2E(X) + 3XE(X)^2 - E(X)^3) = E(X^3) - 3E(X^2)E(X) + 2E(X)^3$

$E(X) = \frac{d}{dt} M_X(t)\big|_{t=0} = a_1 \qquad E(X^2) = \frac{d^2}{dt^2} M_X(t)\big|_{t=0} = 2a_2$

$E(X^3) = \frac{d^3}{dt^3} M_X(t)\big|_{t=0} = 6a_3$

$\therefore E[(X-E(X))^3] = 6a_3 - 6a_1a_2 + 2a_1^3$

4. Let $X_1, X_2, \cdots, X_n$ be i.i.d., each having the normal distribution with parameters μ and σ^2.

(i) Find the m.g.f.s of the sample sum $S_n = \sum_{i=1}^n X_i$ and sample average $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

(ii) What are the distributions of these two random variables?

(i) We already know that $X_1, \cdots, X_n \sim N(\mu, \delta^2)$, then

$M_{X_1}(t) = \cdots = M_{X_n}(t) = E(e^{tX_i}) = e^{\mu t + \frac{1}{2}\delta^2 t^2}$

$\because X_1, \cdots, X_n$ are i.i.d.

$\therefore M_{S_n}(t) = \prod_{i=1}^n M_{X_i}(t) = (e^{\mu t + \frac{1}{2}\delta^2 t^2})^n = e^{\mu n t + \frac{1}{2}\delta^2 n t^2}$

$M_{\bar{X}_n}(t) = E(e^{t \cdot \frac{S_n}{n}}) = M_{S_n}(\frac{t}{n}) = e^{\mu n \cdot \frac{t}{n} + \frac{1}{2}\delta^2 n \cdot (\frac{t}{n})^2} = e^{\mu t + \frac{1}{2n}\delta^2 t^2}$

(ii) $S_n, \bar{X}_n$ both follow the normal distribution

$S_n \sim N(\mu n, n\delta^2)$, $\bar{X}_n \sim N(\mu, \frac{\delta^2}{n})$

5. Suppose X is a discrete random variable taking values of non-negative integers (or subset of non-negative integers) with p.m.f $\{p_k; k \geqslant 0\}$. Define the probability generating function (p.g.f.) of X, denoted by $\prod_X(t)$, as

$$\prod_X(t) = E\left(t^X\right).$$

(i) Write down the form $\prod_X(t)$ in terms of the p.m.f. $\{p_k; k \geqslant 0\}$.

(ii) Investigate the problem as how to get $E(X)$ and $\mathrm{Var}(X)$ by using $\prod_X(t)$.

(iii) Find the p.g.f. of the Binomial Random Variable X with parameter n and p.

(iv) Find the p.g.f. of the Poisson Random Variable X with parameter λ.

(i) $\Pi_X(t) = E(t^X) = \sum_{k=0}^{\infty} P_k t^k$

(ii) $\because \frac{d}{dt} t^X = X t^{X-1}$ $\quad \therefore E(X) = \frac{d}{dt} \Pi_X(t)\Big|_{t=1} = \sum_{k=0}^{\infty} k P_k \cdot t^{k-1}\Big|_{t=1} = \sum_{k=0}^{\infty} k P_k$

$= \Pi_X(k^{\frac{1}{k}})$

$E(X^2 - X) = \frac{d^2}{dt^2} \Pi_X(t)\Big|_{t=1} = \sum_{k=0}^{\infty} k(k-1) P_k \cdot t^{k-2}\Big|_{t=1} = \sum_{k=0}^{\infty} k(k-1) P_k$

$\therefore E(X^2) = E(X^2 - X) + E(X) = \sum_{k=0}^{\infty} k^2 P_k$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \sum_{k=0}^{\infty} k^2 P_k - (\sum_{k=0}^{\infty} k P_k)^2 = \Pi_X(k^{\frac{2}{k}}) - (\Pi_X(k^{\frac{1}{k}}))^2$

(iii) p.m.f: $P_k = C_n^k p^k (1-p)^{n-k}$, $k = 0, 1, 2, \cdots, n$

p.g.f: $\Pi_X(t) = E(t^X) = \sum_{k=0}^{n} C_n^k p^k (1-p)^{n-k} t^k = (pt + (1-p))^n$

(iv) p.m.f: $P_k = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \cdots$

p.g.f. $\Pi_X(t) = \sum_{k=0}^{\infty} P_k t^k = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{(\lambda t)^k}{k!} = e^{\lambda(t-1)} \cdot \sum_{k=0}^{\infty} e^{-(\lambda t)} \cdot \frac{(\lambda t)^k}{k!} = e^{\lambda(t-1)}$