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MA215 Probability Theory

Assignment 03

1. Show that if the conditional probabilities exist, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Proof. When $n=2$, $P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$ $(*)$ $\Leftrightarrow P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}$

which is the definition of the conditional probability, so it's true.

Assume that the conclusion is true when $n=k$.

Then when $n=k+1$, according to $(*)$,

$$P(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) = P(A_1 \cap A_2 \cap \cdots \cap A_k)P(A_{k+1} | \bigcap_{i=1}^{k} A_i)$$

$\because P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1) \cdots P(A_k|A_1 \cap \cdots \cap A_{k-1})$

$\therefore P(A_1 \cap A_2 \cap \cdots \cap A_{k+1}) = P(A_1)P(A_2|A_1) \cdots P(A_k|A_1 \cap \cdots \cap A_{k-1})P(A_{k+1}|A_1 \cap \cdots \cap A_k)$

so when $n=k+1$, the conclusion is still true.

So for any $n \geq 2$, $n \in N^*$, $P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$ $\square$

2. Urn A has 3 red balls and 2 white balls, and urn B has 2 red balls and 5 white balls. A fair coin is tossed; if it hands heads up, a ball is drawn from urn A and otherwise a ball is drawn from urn B.

(a) What is the probability that a red ball is drawn?

(b) If a red ball is drawn, what is the probability that the coin landed heads up?

(a) Event A implies the coin heads up, and B implies the coin doesn't head up. Event A_r implies a red ball is drawn from urn A, and B_r implies a red ball is drawn from urn B.

Event R implies a red ball is drawn.

Then $P(A) = P(B) = \frac{1}{2}$. $P(A_r) = \frac{3}{5}$. $P(B_r) = \frac{2}{7}$

$$P(R) = P(AA_r) + P(BB_r) = P(A)P(A_r) + P(B)P(B_r) = \frac{31}{70}$$

(b) $P(A|R) = \frac{P(AR)}{P(R)} = \frac{P(AAr)}{P(R)} = \frac{P(A)P(Ar)}{P(R)} = \frac{21}{31}$

3. Urn A has 4 red, 3 blue and 2 green balls. Urns B has 2 red, 3 blue and 4 green balls. A ball is drawn from urn A and put into urn B and then a ball is drawn from urn B.

   (a) What is the probability that a red ball is drawn from urn B?

   (b) If a red ball is drawn from urn B, what is the probability that a red ball was drawn from urn A?

(a) Event X implies that the ball drawn from urn A is red

Y_1 implies that X is true and a red ball is drawn from urn B.

Y_2 implies that X is false and a red ball is drawn from urn B.

So. the probability that a red ball is drawn from urn B.

is $P(B) = P(X)P(Y_1) + P(\neg X)P(Y_2) = \frac{4}{9} \times \frac{3}{10} + (1 - \frac{4}{9}) \times \frac{2}{10} = \frac{11}{45}$

(b) $P(X|B) = \frac{P(XB)}{P(B)} = \frac{P(X)P(Y_1)}{P(B)} = \frac{6}{11}$

4. There are 3 cabinets A, B, and C, each of which has 2 drawers. Each drawer contains 1 coin; A has 2 gold coins, B has 2 silver coins and C has 1 gold and 1 silver coin. Take a experiment as a cabinet is chosen at random, one drawer is opened and a silver coin has found. What is the probability that the other drawer in that cabinet contains a silver coin?

Suppose event X implies that a silver coin has found.

$P(A) = P(B) = P(C) = \frac{1}{3}$

Because there's no silver coin in A, $P(X|A) = 0$

$P(X|B) = 1$. $P(X|C) = \frac{1}{2}$

So, $P(X) = P(X|A)P(A) + P(X|B)P(B) + P(X|C)P(C) = \frac{1}{2}$

According to Bayce's rule, $P(B|X) = \frac{P(X|B) \cdot P(B)}{P(X)} = \frac{2}{3}$

$P(C|X) = \frac{P(X|C)P(C)}{P(X)} = \frac{1}{3}$

Only cabinet B can find a silver coin in the other drawer.

$\therefore$ $P = P(B|X) = \frac{2}{3}$.

5. If B is an event with $P(B) > 0$, show that the set function $Q(A) = P(A \mid B)$ is a probability measure. Thus, we can use the following formulas in lectures

$$P(A \cup C \mid B) = P(A \mid B) + P(C \mid B) - P(A \cap C \mid B),$$
$$P(A^c \mid B) = 1 - P(A \mid B).$$

Proof. By definition, $P(\Omega)=1$, $P(A|B) \geq 0$.

$\therefore Q(\Omega)=1$ and $Q(A) \geq 0$

Assume A and C are disjoint, then $A \cap C = \phi$.

$\therefore P(A \cup C|B) = P(A|B) + P(C|B) - P(A \cap C|B) = P(A|B) + P(C|B)$

$\Rightarrow Q(A \cup C) = Q(A) + Q(C)$

Assume $A_1, A_2, \cdots, A_n, \cdots$ are mutually disjoint. When $n=2$,

$Q(A_1 \cup A_2) = Q(A_1) + Q(A_2)$ is certainly true.

Suppose that $n=k$, $Q(\bigcup_{n=1}^{k} A_n) = \sum_{n=1}^{k} Q(A_n)$

When $n=k+1$, $Q(\bigcup_{n=1}^{k+1} A_n) = P(\bigcup_{n=1}^{k} A_n \cup A_{k+1} | B) = P(\bigcup_{n=1}^{k} A_n | B) + P(A_{k+1}|B) = \sum_{n=1}^{k+1} Q(A_n)$

So, $Q(\bigcup_{n=1}^{k} A_n) = \sum_{n=1}^{k} P(A_n)$ for any k, $k \to +\infty$, then $Q(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$

So Q is a probability measure. $\square$

6. Show that if A, B, and C are mutually independent, then $A \backslash B$ and C are independent and $A \cup B$ and C are independent.

Proof. According to Theorem 2.6.1, A and B are independent $\Rightarrow$ A and B^c are independent $\Rightarrow P(A \cap B^c) = P(A)P(B^c) = P(A)(1-P(B))$

$\therefore P(A \backslash B) = P(A \cap B^c) = P(A)P(B^c)$

$P((A \backslash B) \cap C) = P(A \cap B^c \cap C) = P(A \cap C \cap B^c)$

$\because$ A and C are independent, so $P(A \cap C) = P(A)P(C)$

$\because$ B and C are independent, so $P(A \cap C \cap B^c) = P(A \cap C)P(B^c) = P(A)P(C)P(B^c)$

$\Rightarrow P((A \backslash B) \cap C) = P(A \backslash B)P(C)$ $\therefore$ $A \backslash B$ and C are independent.

$\because$ A and B are independent, so $P(A \cap B) = P(A)P(B)$

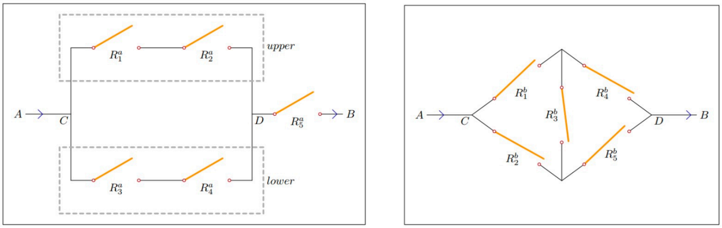
$\therefore P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C)) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C) = P(A \cup B)P(C)$

$\therefore$ $A \cup B$ and C are independent. $\square$

7. The probability of the closing of the ith relay in the circuits shown is given by p_i; $i = 1, 2, 3, 4, 5$. If all relays function independently, what is the probability that a current flows between A and B for the respective circuits? (see the next page for the two cases of circuit)



(a) circuit a

(b) circuit b

Figure 1: Figure of Problem 7

For (a), basically it has 2 paths: $R_1R_2R_5$ and $R_3R_4R_5$

So $P_a = P_1P_2P_5 + P_3P_4P_5 - P_1P_2P_3P_4P_5$

For (b). basically is has 4 paths: R_1R_4. $R_1R_3R_5$. R_2R_5 and $R_2R_3R_4$

$$
\begin{aligned}
\text{So } P_b = {} & P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4 - P_1P_2P_3P_5 \\
& - P_1P_2P_3P_4P_5 - P_2P_3P_4P_5 + P_1P_2P_3P_4P_5 \times 4 - P_1P_2P_3P_4P_5 \\
= {} & P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4 - P_1P_2P_3P_5 \\
& - P_2P_3P_4P_5 + 2P_1P_2P_3P_4P_5
\end{aligned}
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