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MA215 Probability Theory

Assignment 15

1. Suppose that a discrete random variable X has finite kth moment, i.e., $E(|X|^k) < \infty$ ($k > 0$, but k may not be a positive integer). Show that for any $\varepsilon > 0$,

$$P\{|X| \geqslant \varepsilon\} \leqslant \frac{E(|X|^k)}{\varepsilon^k}.$$

Proof. $\because$ X is discrete, and $|X| \geqslant \varepsilon \Rightarrow \frac{|X|^k}{\varepsilon^k} \geqslant 1$

$\therefore P\{|X| \geqslant \varepsilon\} = \sum_{|x|\geqslant\varepsilon} f(x) \leqslant \sum_{|x|\geqslant\varepsilon} \frac{|x|^k}{\varepsilon^k} f(x) = \frac{1}{\varepsilon^k} \sum_{|x|\geqslant\varepsilon} |x|^k f(x) \leqslant \frac{1}{\varepsilon^k} \sum_{x=-\infty}^{+\infty} |x|^k f(x)$

$= \frac{1}{\varepsilon^k} E(|X|^k)$ $\therefore$ for any $\varepsilon > 0$, $P\{|X| \geqslant \varepsilon\} \leqslant \frac{E(|X|^k)}{\varepsilon^k}$ $\square$

2. Suppose that $\{X_1, X_2, \cdots, X_n, \cdots\}$ is a sequence of independent r.v.s (not necessarily with the same distribution), each with finite (but not necessarily with the same) mean and uniformly bounded variance by $M < \infty$ (i.e., $\operatorname{Var}(X_i) \leqslant M$ $\forall i \geqslant 1$). Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean. Show that for any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} P\left\{|\bar{X}_n - E\bar{X}_n| > \varepsilon\right\} = 0.$$

Proof. $E(\bar{X}_n) = E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n} E(\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$

$\operatorname{Var}(\bar{X}_n) = \operatorname{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{Var}(X_i) \leqslant \frac{1}{n^2} \cdot \sum_{i=1}^{n} M = \frac{M}{n}$

By Chebyshev's inequality, $P(|\bar{X}_n - E(\bar{X}_n)| > \varepsilon) \leqslant \frac{\operatorname{Var}(\bar{X}_n)}{\varepsilon^2} \leqslant \frac{M}{n\varepsilon^2}$

$\because \lim_{n\to\infty} \frac{M}{n\varepsilon^2} = 0$, $P(|\bar{X}_n - E(\bar{X}_n)| > \varepsilon) \geqslant 0$.

$\therefore \lim_{n\to\infty} P(|\bar{X}_n - E(\bar{X}_n)| > \varepsilon) = 0$. $\square$

3. Suppose that $\{X_1, X_2, \cdots\}$ is a sequence of i.i.d. r.v.s with common mean 0 and variance 16. Let n be sufficiently large and $Y = X_1 + X_2 + \cdots + X_n$. Estimate the value of $P\{26.08 < Y \leqslant 44.12\}$ when $n = 100$. (Hint: using the central limit theorem and normal approximation method.)

$E(X_i) = 0$, $\operatorname{Var}(X_i) = 16 \Rightarrow \sigma = 4$, $\mu = 0$

According to Central Limit Theorem, $Z_n = \frac{Y - n\mu}{\sigma\sqrt{n}} = \frac{Y}{40} \sim N(0,1)$

$\therefore P\{26.08 < Y \leqslant 44.12\} = P(\frac{Y}{40} \leqslant 1.103) - P(\frac{Y}{40} \leqslant 0.652)$

$= \Phi(1.103) - \Phi(0.652) \approx 0.121$

4. Let $\{X_n\}_{n\geqslant 1}$ be a sequence of i.i.d. random variables with a common normal distribution $N(20,25)$. For positive integer n, let

$$Y=\sum_{i=1}^{6} X_i, \quad W_n=\sum_{i=1}^{n} X_i, \quad \bar{X}_n=\frac{W_n}{n}.$$

(i) Find the distributions of $Y,\ \bar{X}_n,\ W_n$.

(ii) Find the probability that the random variable $\bar{X}_{30}$ is between 19 and 21, i.e., find $P(19<\bar{X}_{30}<21)$. Also, find the probability that W_{30} is greater than 650, i.e., find $P(W_{30}>650)$.

(iii) Are Y and W_{30} uncorrelated or correlated? Use details to illustrate your statements.

(iv) Let $Y_n=\frac{\bar{X}_n-20}{5\sqrt{n}}$. Use the moment generating functions to find the asymptotic distributions of $\bar{X}_n$ and Y_n as $n\to+\infty$.

(i) $E(Y)=6\times 20=120$. $Var(Y)=6\times 25=150$

$\therefore$ Y follows a normal distribution: $Y\sim N(120,150)$

$E(\bar{X}_n)=\frac{20n}{n}=20$, $Var(\bar{X}_n)=\frac{25n}{n^2}=\frac{25}{n}$

$\therefore$ $\bar{X}_n$ follows a normal distribution: $\bar{X}_n\sim N(20,\frac{25}{n})$

$E(W_n)=20n$, $Var(W_n)=25n$.

$\therefore$ W_n follows a normal distribution: $W_n\sim N(20n,25n)$

(ii) $n=30$, $\bar{X}_{30}\sim N(20,\frac{5}{6})$ $\therefore$ $\mu=20$, $\delta=\sqrt{\frac{5}{6}}$

By Central Limit Theorem, $Z_{30}=\dfrac{\bar{X}_{30}-20}{\sqrt{\frac{5}{6}}}\sim N(0,1)$

$P(19<\bar{X}_{30}<21)=P(Z_{30}\leq\frac{21-20}{\sqrt{\frac{5}{6}}})-P(Z_{30}\leq 19-20/\sqrt{\frac{5}{6}})=\Phi(\sqrt{\frac{6}{5}})-\Phi(-\sqrt{\frac{6}{5}})\approx 0.724$

$W_{30}\sim N(600,750)$ $\therefore$ $Z_{30}'=\dfrac{W_{30}-600}{\sqrt{750}}\sim N(0,1)$

$P(W_{30}>650)=P(Z_{30}'>\frac{650-600}{\sqrt{750}})=1-\Phi(\frac{10}{\sqrt{30}})\approx 0.034$

(iii) $Cov(Y,W_{30})=Cov(\sum_{i=1}^{6}X_i,\sum_{i=1}^{30}X_i)=\sum_{i=1}^{6}Cov(X_i,X_i)=6Var(X_i)=150\neq 0$.

$\therefore$ Y and W_{30} are correlated.

(iv) $n\to+\infty$, $\frac{25}{n}\to 0$, by Central Limit Theorem, $\bar{X}_n\xrightarrow{P}20$, $(n\to+\infty)$

$\therefore$ $Y_n=\dfrac{\bar{X}_n-20}{5\sqrt{n}}\xrightarrow{P}0$ or $Y_n=\dfrac{\bar{X}_n-20}{\sqrt{\frac{25}{n}}}/5=\dfrac{Z_n}{5}$, $Z_n\sim N(0,1)$

$\therefore$ $Y_n\xrightarrow{D}N(0,\frac{1}{25})$ $(n\to+\infty)$

5. Let X and Y are independent Poisson random variables with parameter 1. Use the moment generating functions to show $X+Y \sim Poisson(2)$. Then use the central limit theorem to show the following statement:

$$\lim_{n\to+\infty} e^{-n}\left[1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}\right]=\frac{1}{2}.$$

Proof. $X, Y \sim \text{Poisson}(1)$ $\therefore M_X(t) = M_Y(t) = E(e^{tX}) = E(e^{tY}) = e^{e^t-1}$

$\because X, Y$ are independent $\therefore M_{X+Y}(t) = M_X(t)\cdot M_Y(t) = e^{2(e^t-1)}$

$\therefore X+Y \sim \text{Poisson}(2)$

$P(X=k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$ Consider $X \sim \text{Poisson}(n)$. then $S_n = \sum_{k=0}^{n} e^{-n}\cdot\frac{n^k}{k!}$

$n\to+\infty$, $S_n \to 1$

By Central Limit Theorem, $n\to+\infty$, Poisson(n) is close to $N(n,n)$

$P(X=k) \approx \frac{1}{\sqrt{2\pi n}} e^{-\frac{(k-n)^2}{2n}}$, $\sum_{k=0}^{n} P(X=k) \sim 1-\frac{1}{2} = \frac{1}{2}$

$\therefore \lim_{n\to\infty} e^{-n}[1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}] = \frac{1}{2}$. $\square$