Applications of Determinants

Lecture 19 and 20

Dept. of Math., SUSTech

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Formulas for the Determinant; Applications of Determinants

Computation of A^{-1}

The 2 by 2 case shows how cofactors go into A^{-1}:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}.$$

Cofactor matrix C is transposed

$$A^{-1} = \frac{C^T}{\det A}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \det A \end{bmatrix}$$

Note: The above C should be transposed.

The critical question is: Why do we get zeros off the diagonal?

Computation of A^{-1}

If we combine the entries a_{1j} from row 1 with cofactors C_{2j} for row 2, why is the result zero?

$$a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n} = 0.$$

- We are computing the determinant of a new matrix B with a new row 2.
- The first row of A is copied into the second row of B. Then B has two equal rows, and $\det B = 0$.

Example

Example

The inverse of a sum matrix is a difference matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{has } A^{-1} = \frac{C^T}{\det A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The minus signs enter because cofactors always include $(-1)^{i+j}$.

The Solution of $Ax = b$.

Cramer's Rule: The j th component of $x = A^{-1}b$ is just $C^T b$ divided by $\det\ A$. There is a famous way in which to write the answer $(x_1, x_2, \cdots, x_n)$:

The jth component of $x = A^{-1}b$ is the ratio : $x_j = \dfrac{\det\ B_j}{\det\ A}$, where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}.$$

Can you prove this rule?

Example

Example

The solution of

$$x_1 + 3x_2 = 0$$
$$2x_1 + 4x_2 = 6$$

has 0 and 6 in the first column for x_1 and in the second column for x_2:

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9, \; x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{6}{-2} = -3.$$

- The denominators are always $\det A$.
- For 1000 equations Cramer's Rule would need 1001 determinants.

The Volume of a Box

The determinant equals the volume of a box. We first consider right-angled box, which has orthogonal rows:

$$AA^T = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} \text{column } 1 & \cdots & \text{column } n \end{bmatrix} = \begin{bmatrix} l_1^2 & & 0 \\ & \ddots & \\ 0 & & l_n^2 \end{bmatrix}.$$

The l_i's are the lengths of the rows (the edges), and the zeros off the diagonal come because the rows are orthogonal. Using the product and transposing rules,

$$l_1^2 l_2^2 \cdots l_n^2 = \det(AA^T) = (\det A)(\det A^T) = (\det A)^2.$$

The square root of this equation says that the absolute value of the determinant of A equals the volume.

Not Right-Angle Case

It the angles are not 90°, the volume is **NOT** the product of the lengths. However, by rule 5, like the following figure shows that we can change the parallelogram to rectangle, where it is already that volume = |determinant|.

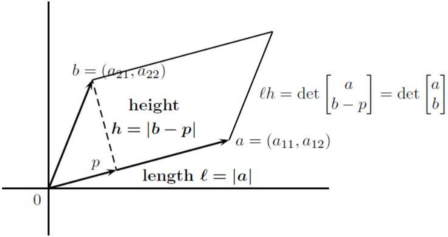

Figure 4.2: Volume (area) of the parallelogram = ℓ times $h = |\det A|$.

In n dimensions, the idea is the same. The Gram-Schmidt process produces orthogonal rows, with volume = |determinant|.

Volume = |Determinant|

We know that

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \det \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = 1.$$

These determinants give volumes–or areas, since we are in two dimensions–drawn in Figure 4.3. The parallelogram has unit base and unit height; its area is also 1.

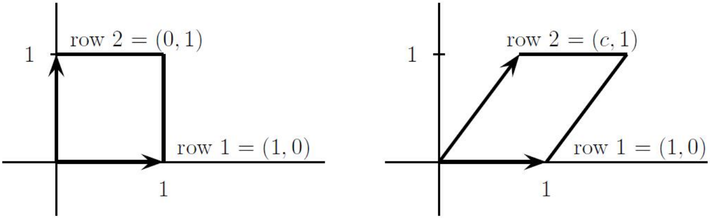

Figure 4.3: The areas of a unit square and a unit parallelogram are both 1.

A Formula for the Pivots

Elimination on A includes elimination on A_2:

$$A = \begin{bmatrix} a & b & e \\ c & d & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} a & b & e \\ 0 & \frac{ad-bc}{a} & \frac{af-ec}{a} \\ g & h & i \end{bmatrix}$$

Actually it is not just the pivots, but the entire upper-left corners of L, D, and U, that are determined by the upper-left corner of A:

$$A = LDU = \begin{bmatrix} 1 & & \\ \frac{c}{a} & 1 & \\ * & * & 1 \end{bmatrix} \begin{bmatrix} a & & \\ & \frac{ad-bc}{a} & \\ & & * \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} & * \\ & 1 & * \\ & & 1 \end{bmatrix}$$

Formulas for pivots

What we see in the first two rows and columns is exactly the factorization of the corner submatrix A_2. This is a general rule if there are no row exchanges:

Proposition

If A is factored into LDU, the upper left corners satisfy $A_k = L_k D_k U_k$. For every k, the submatrix A_k is going through a Gaussian elimination of its own.

$$LDU = \begin{bmatrix} L_k & 0 \\ B & C \end{bmatrix} \begin{bmatrix} D_k & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} U_k & F \\ 0 & G \end{bmatrix} = \begin{bmatrix} L_k D_k U_k & L_k D_k F \\ B D_k U_k & B D_k F + CEG \end{bmatrix}.$$

Formula for pivots

Comparing the last matrix with A, the corner $L_kD_kU_k$ coincides with A_k. Then:

$$\det\ A_k = \det\ L_k \det\ D_k \det\ U_k = \det\ D_k = d_1 d_2 \cdots d_k.$$

The product of the first k pivots is the determinant of A_k. This is the same rule we know already for the whole matrix. Since the determinant of A_{k-1} will be given by $d_1 d_2 \cdots d_{k-1}$, we can isolate each pivot d_k as a ratio of determinants:

$$\frac{\det A_k}{\det A_{k-1}} = \frac{d_1 d_2 \cdots d_k}{d_1 d_2 \cdots d_{k-1}} = d_k$$

Formulas for pivots

Multiplying together all the individual pivots, we recover

$$d_1 d_2 \cdots d_n = \frac{\det A_1}{\det A_0} \frac{\det A_2}{\det A_1} \cdots \frac{\det A_n}{\det A_{n-1}} = \frac{\det A_n}{\det A_0} = \det A.$$

The pivot entries are all nonzero whenever the numbers $\det\ A_k$ are all nonzero:

Proposition

Elimination can be completed without row exchanges (so $P = I$ and $A = LU$), if and only if the leading submatrices $A_1, A_2, \cdots, A_n$ are all nonsingular.

The determinant of a permutation matrix

The determinant of a permutation matrix P was the only questionable point in the big formula. Independent of the particular row exchanges linking P to I, is the number of exchanges always even or always odd?

If so, its determinant is well defined by rule 2 as either 1 or -1.

- An even number of exchanges can never produce the natural order, beginning with $(3,2,1)$.
- Let N count the pairs in which the lager number comes first. Every exchange alters N by an odd number.

Determinant of a permutation matrix P

- Certainly $N = 0$ for the natural order $(1,2,3)$. The order $(3,2,1)$ has $N = 3$ since all pairs $(3,2),(3,1)$, and $(2,1)$ are wrong.
- We will show that every exchange alters N by an odd number. Then to arrive at $N = 0$ (the natural order) takes a number of exchanges having the same evenness and oddness as N.
- When neighbors are exchanged, N changes by $+1$ or -1.
- Any exchange can be achieved by an odd number of exchanges of neighbors. This will complete the proof; an odd number of odd numbers is odd.

One Example

- To exchange the first and fourth entries below, which happen to be 2 and 3, we use five exchanges (an odd number) of neighbors:

$$(\mathbf{2},1,4,\mathbf{3}) \to (1,\mathbf{2},4,\mathbf{3}) \to (1,4,\mathbf{2},\mathbf{3}) \to (1,4,\mathbf{3},\mathbf{2})$$
$$\to (1,\mathbf{3},4,\mathbf{2}) \to (\mathbf{3},1,4,\mathbf{2}).$$

- We need $l-k$ exchanges of neighbors to move the entry in place k to place l.
- Then $l-k-1$ exchanges move the originally in place l (and now found in place $l-1$) back down to place k.
- Since $(l-k)+(l-k-1)$ is odd, the proof is complete.

Example

Example

Find the following determinant:

$$|A| = \begin{vmatrix} 1 & 4 & -1 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 2 & 3 & 11 \\ 3 & 0 & 9 & 2 \end{vmatrix}.$$

Further Examples

Example

Suppose the entries of A satisfy

$$a_{ij} = -a_{ji} \quad (i,j = 1,2,\cdots,n).$$

Find $\det(A)$ if n is odd.

Solution. 0.

Example

Find

$$|A| = \begin{vmatrix} a_1+b_1 & b_1+c_1 & c_1+a_1 \\ a_2+b_2 & b_2+c_2 & c_2+a_2 \\ a_3+b_3 & b_3+c_3 & c_3+a_3 \end{vmatrix}.$$

Example

Example

Find the determinant:

$$|A| = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix}.$$

Solution. Add all the columns to the first column, and factor out an common factor in the first column. Then add -1 times the first row to the rows beneath it to obtain a determinant of an upper triangular matrix, where its determinant can be found by taking the product of the diagonal entries. The determinant of A is $[a+(n-1)b](a-b)^{n-1}$.

Example

Example

Find the determinant:

$$|A| = \begin{vmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{vmatrix}.$$

Solution.

$$D_n = \begin{cases} \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, & \alpha \neq \beta; \\ \frac{(n+1)a^n}{2^n}, & \alpha = \beta. \end{cases}$$

Where $\alpha+\beta = a, \alpha\beta = bc$.

一些题目

(1) 设 $A=(a_{ij})$ 是三阶非零矩阵，$|A|$ 为 A 的行列式，A_{ij} 为 a_{ij} 的代数余子式，若 $a_{ij}+A_{ij}=0(i,j=1,2,3)$, 则 $|A|=$____.

(2) $$\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \underline{\qquad\qquad}$$

(3) 设 $A=(a_{ij})$ 为三阶矩阵, A_{ij} 为元素 a_{ij} 的代数余子式, 若 A 的每行元素的和均为 2, 且 $|A|=3$, 则 $A_{11}+A_{21}+A_{31}=$____.

(4) n 阶行列式 $$\begin{vmatrix} 2 & 0 & \cdots & 0 & 2 \\ -1 & 2 & \cdots & 0 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & \cdots & -1 & 2 \end{vmatrix} = \underline{\qquad}.$$

Homework Assignment 19 and 20

4.4: 2, 3, 5, 6, 16, 22, 23, 29, 44.