1. Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (a) The empty set is a subspace of every vector space.

   (b) The dimension of $P_n(\mathbb{F})$ is n.

   (c) Let u, v, w be distinct vectors of a vector space V. If u, v, w is a basis for V, then $u+v+w, v+w, w$ is also a basis for V.

   (d) $P_n(\mathbb{F})$ is isomorphic to $P_m(\mathbb{F})$ if and only if $n = m$.

   (e) If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

2. Show that the set of solutions, V, to the system of linear equations

$$\begin{cases} x_1 - 2x_2 + x_3 & = 0 \\ 2x_1 - 3x_2 + x_3 & = 0 \end{cases}$$

   is a subspace of $\mathbb{R}^3$. Find a basis for this subspace, V.

3. State and prove the **Fundamental Theorem of Linear Maps**.

4. Suppose V is a finite dimensional inner product space and $T \in \mathcal{L}(V)$.

   (a) State the definition of **invariant subspace**.

   (b) Let U be a subspace of V. Show that U and $U^\perp$ are invariant under T if and only if $P_U T = T P_U$.

5. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (-3x_1 + 3x_2 - 2x_3,\ -7x_1 + 6x_2 - 3x_3,\ x_1 - x_2 + 2x_3).$$

   (a) Determine the eigenspace of T corresponding to each eigenvalue.

   (b) Find the Jordan form and a Jordan basis of T.

   (c) Find the minimal polynomial of T.

   (d) Compute trace T and det T.

6. Suppose T is a linear operator defined on $\mathbb{R}^4$ with $T^2 = -I$.

   (a) Show that the only eigenvalues of $T_{\mathbb{C}}$ are i and $-i$. Where $T_{\mathbb{C}}$ is the complexification of T.

   (b) Show that v is an eigenvector of $T_{\mathbb{C}}$ with respect to i if and only if $\bar{v}$ is an eigenvector of $T_{\mathbb{C}}$ with respect to $-i$, and hence show that there is a basis consisting of complex eigenvectors of $T_{\mathbb{C}}$ of the form $v_1, v_2, \bar{v}_1, \bar{v}_2$.

   (c) Show that there is a basis of $\mathbb{R}^4$ with respect to which T has the following matrix representation

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

7. Let T be a linear operator on a finite-dimensional complex vector space V, and let $\lambda_1,\ \lambda_2,\ \cdots,\lambda_k$ be the distinct eigenvalues of T. Let $S: V \to V$ be the mapping defined by

$$S(x) = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k,$$

where, for each i, v_i is the unique vector in $G(\lambda_i, T)$, such that $x = v_1 + v_2 + \cdots + v_k$.

(a) Prove that S is a diagonalizable linear operator on V.

(b) Let $N = T - S$. Prove that N is nilpotent and commutes with S, that is $SN = NS$.