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MA215 Probability Theory

Assignment 04

1. For each of the following statements, say whether true or false. For false statements, give the correct version of the statement.

   (i) $P(A \cap B) = P(A) \times P(B)$ if A, B are independent.

   (ii) $P(A \cup B) = P(A) + P(B)$ if A, B are independent.

   (iii) In a sequence of n independent identical trials, each of which results in either "success" or "failure", with probability θ of success, the number of successes follows a Bernoulli distribution.

(i) true

(ii) false. $P(A \cup B) = P(A) + P(B)$ if A, B are disjoint.

(iii) false. Because of n trials, it follows a Binomial distribution, not Bernoulli distribution.

2. In five independent tosses of an unbiased coin, find

   (i) the probability that the total number of heads is even;

   (ii) the probability that there are exactly five heads.

   (Note: zero is also a even number.)

Suppose X implies the number of heads in five tosses.

Then, X follows a binomial distribution, $X \sim B(n, p)$, and $\begin{cases} n=5 \\ p=0.5 \end{cases}$

$X \in \{0,1,2,3,4,5\}$

(i) So $P(X=0) = C_5^0 (\frac{1}{2})^0 \cdot (\frac{1}{2})^5 = \frac{1}{32}$

$P(X=2) = C_5^2 (\frac{1}{2})^2 \cdot (\frac{1}{2})^3 = \frac{10}{32}$

$P(X=4) = C_5^4 (\frac{1}{2})^4 \cdot (\frac{1}{2})^1 = \frac{5}{32}$

$\therefore P(X \text{ is even}) = P(X=0) + P(X=2) + P(X=4) = \frac{1}{2}$

(ii) $P(X=5) = C_5^5 (\frac{1}{2})^5 \cdot (\frac{1}{2})^0 = \frac{1}{32}$

3. A discrete random variable X has possible values -2, 1, 3, 4 with probabilities satisfying

$$P(X = -2) = P(X = 1) = 2P(X = 3) = 3P(X = 4).$$

Find the probability mass function and the (cumulative) distribution function of X, and graph them both.

Suppose that $P(X=-2)=P(X=1)=2P(X=3)=3P(X=4)=P$.

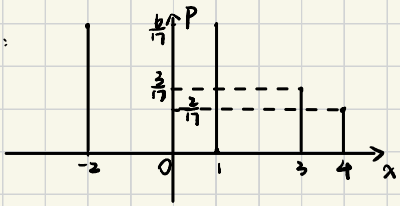
$\because X\in\{-2,1,3,4\}$ $\therefore P(X=-2)+P(X=1)+P(X=3)+P(X=4)=\frac{17}{6}P=1$

$\therefore P=\frac{6}{17}$. $P(X=-2)=P(X=1)=\frac{6}{17}$ $P(X=3)=\frac{3}{17}$, $P(X=4)=\frac{2}{17}$

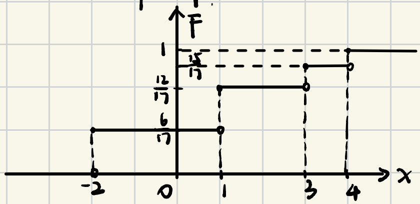
Probability mass function: $P(-2)=\frac{6}{17}$, $P(1)=\frac{6}{17}$, $P(3)=\frac{3}{17}$ $P(4)=\frac{2}{17}$

Cumulative distribution function:
$$F(x)=\begin{cases}0, & x<-2\\ \frac{6}{17}, & -2\le x<1\\ \frac{12}{17}, & 1\le x<3\\ \frac{15}{17}, & 3\le x<4\\ 1, & x\ge 4\end{cases}$$

Graph:



p.m.f.



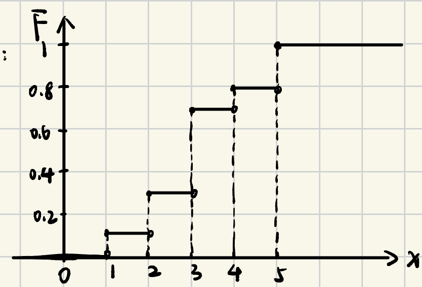
C.D.F.

4. The following table shows the probability mass function of a discrete random variable X. Plot the (cumulative) distribution function of this random variable.

k	1	2	3	4	5
$P(X=k)$	0.1	0.2	0.4	0.1	0.2

C.D.F.:
$$F(x)=\begin{cases}0, & x<1\\ 0.1, & 1\le x<2\\ 0.3, & 2\le x<3\\ 0.7, & 3\le x<4\\ 0.8, & 4\le x<5\\ 1, & x\ge 5\end{cases}$$

Plot:



5. Suppose $F(x)$ is the c.d.f. of a random variable X. Show that $F(x)$ has the following properties:

(i) $0 \leqslant F(x) \leqslant 1$;

(ii) $F(x)$ is an increasing function of x, i.e., $F(x) \leqslant F(y)$ for any $x < y$;

(iii) $\lim_{x \to +\infty} F(x) = 1$; $\quad \lim_{x \to -\infty} F(x) = 0$;

(iv) Show that $F(x)$ is a right-continuous function of $x \in R$: (Just show that if a sequence of real numbers $x_n \downarrow x$, then $\lim_{n \to \infty} F(x_n) = F(x)$).

Proof. (i) According to the definition of probability measure: $0 \le P(X \le x) \le 1$

$F(x) = P(X \le x) \quad \therefore 0 \le F(x) \le 1$

(ii) For any $x<y$, $F(y) - F(x) = P(X \le y) - P(X \le x) = P(x < X \le y)$

$\because x<y \quad \therefore x < X \le y$ is valid $\quad \therefore P(x < X \le y) \ge 0 \Rightarrow F(y) \ge F(x)$

$\therefore F(x)$ is an increasing function of x.

(iii) $\lim\limits_{x \to +\infty} F(x) = \lim\limits_{x \to +\infty} P(X \le x) = P(X \le +\infty) = 1$

$\lim\limits_{x \to -\infty} F(x) = \lim\limits_{x \to -\infty} P(X \le x) = P(X \le -\infty) = 0$

(iv) $F(x_n) = P(X \le x_n)$

As $n \to +\infty$, $x_n \downarrow x$, $P(X \le x_n) \to P(X<x) + P(X=x)$

$\therefore \lim\limits_{n \to \infty} F(x_n) = P(X<x) + P(X=x) = F(x)$

$\therefore F(x)$ is right-continuous.