Orthonormal Bases

Lecture 17

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Inner Product Spaces

Orthonormal

6.23 **Definition** ***orthonormal***

- A list of vectors is called ***orthonormal*** if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.

- In other words, a list $e_1, \ldots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Examples

6.24 **Example** ***orthonormal lists***

(a) The standard basis in $\mathbf{F}^n$ is an orthonormal list.

(b) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is an orthonormal list in $\mathbf{F}^3$.

(c) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ is an orthonormal list in $\mathbf{F}^3$.

The norm of an orthonormal linear combination

Orthonormal lists are particularly easy to work with, as illustrated by the next result.

6.25 The norm of an orthonormal linear combination

If $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbf{F}$.

The result above has the following important corollary.

6.26 An orthonormal list is linearly independent

Every orthonormal list of vectors is linearly independent.

Orthonormal Basis

6.27 Definition ***orthonormal basis***

An ***orthonormal basis*** of V is an orthonormal list of vectors in V that is also a basis of V.

6.28 An orthonormal list of the right length is an orthonormal basis

Every orthonormal list of vectors in V with length $\dim V$ is an orthonormal basis of V.

6.29 Example Show that

$$\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right), \left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right), \left(-\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

is an orthonormal basis of $\mathbf{F}^4$.

Writing a vector as linear combination of orthonormal basis

6.30 Writing a vector as linear combination of orthonormal basis

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

Now that we understand the usefulness of orthonormal bases, how do we go about finding them? For example, does $\mathscr{P}_m(\mathbb{R})$, with inner product given by integration on $[-1, 1]$ [see 6.4(c)], have an orthonormal basis? the next result will lead to answers to these questions.

Gram-Schmidt Procedure

The algorithm used in the next proof is called the Gram-Schmidt Procedure. It gives a method for turning a linearly independent list into an orthonormal list with the same span as the original list.

6.31 Gram–Schmidt Procedure

Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $e_1 = v_1/\|v_1\|$. For $j = 2, \ldots, m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\|}.$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

for $j = 1, \ldots, m$.

Example

6.33 **Example** Find an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$, where the inner product is given by $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

Orthonormal Basis of $\mathscr{P}_2(\mathbb{R})$:

$$\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$$

Existence of orthonormal basis

Now we can answer the question about the existence of orthonormal bases.

6.34 Existence of orthonormal basis

Every finite-dimensional inner product space has an orthonormal basis.

Sometimes we need to know not only that an orthonormal basis exists, but also that every orthonormal list can be extended to an orthonormal basis. In the next corollary, the Gram-Schmidt procedure shows that such an extension is always possible.

6.35 Orthonormal list extends to orthonormal basis

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Upper-Triangular Matrices

Now that we are dealing with inner product spaces, we would like to know whether there exists an orthonormal basis with respect to which we have an upper triangular matrix.

6.37 Upper-triangular matrix with respect to orthonormal basis

Suppose $T \in \mathcal{L}(V)$. If T has an upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some orthonormal basis of V.

The next result is an important application of the result above.

6.38 Schur's Theorem

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V.

Linear Functionals on Inner Product Spaces

Recall:

6.39 Definition ***linear functional***

A ***linear functional*** on V is a linear map from V to $\mathbf{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbf{F})$.

Examples.

6.40 Example The function $\varphi: \mathbf{F}^3 \to \mathbf{F}$ defined by

$$\varphi(z_1, z_2, z_3) = 2z_1 - 5z_2 + z_3$$

is a linear functional on $\mathbf{F}^3$. We could write this linear functional in the form

$$\varphi(z) = \langle z, u \rangle$$

for every $z \in \mathbf{F}^3$, where $u = (2, -5, 1)$.

Example

6.41 **Example** The function $\varphi\colon \mathcal{P}_2(\mathbf{R}) \to \mathbf{R}$ defined by

$$\varphi(p) = \int_{-1}^{1} p(t)\big(\cos(\pi t)\big)\,dt$$

is a linear functional on $\mathcal{P}_2(\mathbf{R})$ (here the inner product on $\mathcal{P}_2(\mathbf{R})$ is multiplication followed by integration on $[-1, 1]$; see 6.33). It is not obvious that there exists $u \in \mathcal{P}_2(\mathbf{R})$ such that

$$\varphi(p) = \langle p, u \rangle$$

for every $p \in \mathcal{P}_2(\mathbf{R})$ [we cannot take $u(t) = \cos(\pi t)$ because that function is not an element of $\mathcal{P}_2(\mathbf{R})$].

Riesz Representation Theorem

If $u \in V$, then the map that sends v to $\langle v, u \rangle$ is a linear functional on V. The next result shows that every linear functional on V is of this form. Example 6.41 above illustrates the power of the next result because for the linear functional in that example, there is no obvious candidate for u.

6.42 Riesz Representation Theorem

Suppose V is finite-dimensional and φ is a linear functional on V. Then there is a unique vector $u \in V$ such that

$$\varphi(v) = \langle v, u \rangle$$

for every $v \in V$.

Example

6.44 **Example** Find $u \in \mathcal{P}_2(\mathbf{R})$ such that

$$\int_{-1}^{1} p(t)\big(\cos(\pi t)\big)\,dt = \int_{-1}^{1} p(t)u(t)\,dt$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Solution

Solution Let $\varphi(p) = \int_{-1}^{1} p(t)\big(\cos(\pi t)\big)\, dt$. Applying formula 6.43 from the proof above, and using the orthonormal basis from Example 6.33, we have

$$u(x) = \Big(\int_{-1}^{1} \sqrt{\tfrac{1}{2}}\big(\cos(\pi t)\big)\, dt\Big)\sqrt{\tfrac{1}{2}} + \Big(\int_{-1}^{1} \sqrt{\tfrac{3}{2}}\,t\big(\cos(\pi t)\big)\, dt\Big)\sqrt{\tfrac{3}{2}}\,x$$

$$+ \Big(\int_{-1}^{1} \sqrt{\tfrac{45}{8}}\big(t^2 - \tfrac{1}{3}\big)\big(\cos(\pi t)\big)\, dt\Big)\sqrt{\tfrac{45}{8}}\big(x^2 - \tfrac{1}{3}\big).$$

A bit of calculus shows that

$$u(x) = -\tfrac{45}{2\pi^2}\big(x^2 - \tfrac{1}{3}\big).$$

Final Note

Suppose V is finite-dimensional and φ a linear functional on V. Then 6.43 gives a formula for the vector u that satisfies $\varphi(v) = \langle v, u \rangle$ for all $v \in V$. Specifically, we have

$$u = \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n.$$

The right side of the equation above seems to depend on the orthonormal basis $e_1, \ldots, e_n$ as well as on φ. However, 6.42 tells us that u is uniquely determined by φ. Thus the right side of the equation above is the same regardless of which orthonormal basis $e_1, \ldots, e_n$ of V is chosen.

Homework Assignment 17

6.B: 1, 3, 5, 8, 11, 13, 15, 17.