考试科目: 线 性 代 数 精 讲 **开课单位:** 数 学 系
考试时长: 120 分钟

题 号	1	2	3	4	5	6	7
分 值	30 分	15 分	15 分	10 分	10 分	10 分	10 分

本试卷共 (7) 大题, 满分 (100) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

This test includes 7 questions. Write ***all your answers*** on the examination book.

Please put away all books, calculators, cell phones and other devices. Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

Unless otherwise noted, vector spaces are over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1. (30 points, 6 points each) Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (1) If $T : V \to W$ is a linear map and $v_1, v_2, \cdots, v_n$ are vectors in V, and $Tv_1, Tv_2, \cdots, Tv_n$ is linearly independent, then $v_1, v_2, \cdots, v_n$ is linearly independent.

   (2) Suppose that $p_0, p_1, \cdots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(-1) = 0$ for all j, then $p_0, p_1, \cdots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$.

   True. Let $U = \{p \in \mathcal{P}_m(\mathbb{R}) : p(-1) = 0\}$. U is a subspace of $\mathcal{P}_m(\mathbb{F})$. Also U is not equal to the full space $\mathcal{P}_m(\mathbb{R})$, because not every polynomial satisfies $p(-1) = 0$. So the dimension of U is strictly less than the dimension of $\mathcal{P}_m(\mathbb{R})$, which is $m+1$. Hence $\dim(U) \leq m$, and we can not have $m+1$ linearly independent vectors in U.

   (3) Let T be a linear operator defined on a 3 dimensional real vector space, then T always has an eigenvalue.

   (4) If v_1, v_2, and v_3 are eigenvectors of T such that $v_3 = v_1 + v_2$, then all three vectors have the same eigenvalue.

(5) Let V to be a complex vector space and U_1, U_2, U_3 be its subspaces with intersection $U_1 \cap U_2 \cap U_3 = \{\mathbf{0}\}$. Then $U_1 + U_2 + U_3$ is a direct sum.

2. If U_1 and U_2 are subspaces of a finite-dimensional vector space. Prove that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

3. Define $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

4. Define V^4 by

$$V^4 = V \times V \times V \times V.$$

Prove that V^4 and $\mathcal{L}(\mathbb{F}^4, V)$ are isomorphic vector spaces.

5. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Proof. We assume that $ST = I$. This implies that $TSTv = Tv$ for every $v \in V$. Equivalently, we can say $TSw = w$ for every $w \in \text{range } (T)$.

Finally, we claim that range $T = V$. If range $(T) \neq V$, then $\dim(\text{range } T) < \dim V$, and then $\dim(\text{range } (ST)) \leq \dim(\text{range } T)$, because range (ST) is the range of S as a linear map applied to range (T) (and dimension can not increase by applying a linear map). But $ST = I$, so range $(ST) = V$.

So it must be the case that $V = \text{range } (T)$, and $TSw = w$ for every $w \in V$.

6. Consider the standard basis $1, x, x^2$ of $V = \mathcal{P}_2(\mathbb{R})$. Let $\varphi_1, \varphi_2, \varphi_3$ be the corresponding dual basis of $\mathcal{P}_2(\mathbb{R})'$. Let $\varphi : V \to \mathbb{R}$ be the linear function

$$f(x) \mapsto f(2) + \int_0^1 f(x)dx.$$

Find the coefficients a, b, c for which $\varphi = a\varphi_1 + b\varphi_2 + c\varphi_3$.

7. Let V be a vector space that is generated by $v_1, v_2, \cdots, v_n$, and let $u_1, u_2, \cdots, u_m$ be a linearly independent list of vectors in V. Show that $m \leq n$ and there exists a subset H of $v_1, v_2, \cdots, v_n$ containing exactly $n - m$ vectors such that $H \cup \{u_1, u_2, \cdots, u_m\}$ generates V.

Proof. The proof is by mathematical induction on m. The induction begins with $m = 0$; for in this case, taking $H = \{v_1, v_2, \cdots, v_n\}$ gives the desired result.

Now suppose that the result is true for some integer $m \geq 0$. We prove that the result is true for $m + 1$. Let $u_1, u_2, \cdots, u_{m+1}$ be a linearly independent subset of V consisting of $m+1$ vectors. It follows that $u_1, u_2, \cdots, u_m$ is linearly independent, and so we may apply the induction hypothesis to conclude that $m \leq n$ and that there is a subset $v_1, v_2, v_3, \cdots, v_{n-m}$(without loss of generality) of $\{v_1, v_2, \cdots, v_n\}$ such that

$$\operatorname{span}(v_1, v_2, v_3, \cdots, v_{n-m}, u_1, u_2, \cdots, u_m) = V.$$

Thus there exist scalars $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_{n-m}$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + \cdots + b_{n-m}v_{n-m} = u_{m+1}$$

Note that $n - m > 0$, let u_{m+1} be a linear combination of $u_1, u_2, \cdots, u_m$, which contradicts the assumption that $u_1, u_2, \cdots, u_m, u_{m+1}$ is linearly independent. Hence $n > m$; that is, $n \geq m + 1$. Moreover, some b_i, say b_1, is nonzero, for otherwise we obtain the same contradiction.

Solving the above equation for v_1 gives

$$v_1 = -\frac{a_1}{b_1}u_1 - \frac{a_2}{b_1}u_2 - \cdots - \frac{a_m}{b_1}u_m - \frac{1}{b_1}u_{m+1} - \frac{b_2}{b_1}v_2 - \cdots - \frac{b_{n-m}}{b_1}v_{n-m}.$$

Let $H = \{v_2, \cdots, v_{n-m}\}$. Then $v_1 \in \operatorname{span}(u_1, u_2, u_3, \cdots, u_{m+1}, v_2, \cdots, v_{n-m})$. It follows that

$$\operatorname{span}(u_1, u_2, u_3, \cdots, u_{m+1}, v_2, \cdots, v_{n-m}) = V.$$

Since H is a subset of $v_1, v_2, \cdots, v_n$ that contains $n-(m+1)$ vectors, the result holds for $m + 1$. This completes the proof.

(Bonus) Let V be a finite-dimensional complex vector space and let S and T be linear operators on V such that $ST = TS$. Prove that if S and T can each be diagonalized, then there is a basis for V which simultaneously diagonalizes S and T.

Proof. Let S be the set of $n \times n$ diagonalizable matrices over a field $\mathbb{F}$ which commute with each other. Let $V = \mathbb{F}^n$. Suppose T is a maximal subset of S such that there exists a decomposition of

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

where V_i is a nonzero eigenspace for each element of T such that for $i \neq j$, there exists an element of T with distinct eigenvalues on V_i and V_j.

We claim that $T = S$. If not, there exists an $N \in S - T$. Since N commutes with all the elements of T, $NV_i \subset V_i$. Indeed, there exists a function $f_i : T \to \mathbb{F}$ such that $v \in V_i$, if and only if $Mv = f_i(M)v$ for all $M \in T$. Now if $v \in V_i$ and $M \in T$,

$$MNv = NMv = Nf_i(M)v = f_i(M)Nv$$

so $Nv \in V_i$. Since N is diagonalizable on V, it is diagonalizable on V_i. This means we can decompose each V_i into eigenspaces $V_{i,j}$ for N with distinct eigenvalues. Hence, we have a decomposition of the right sort for $T \cup N$,

$$V = \oplus_i \oplus_j V_{i,j}.$$

Hence, $T = S$. We may now make a basis for V by choosing a basis for V_i and taking the union. Then A will be the change of basis matrix.