Orthogonal Complements and Minimization Problems

Lecture 18

Dept. of Math., SUSTech

2024.04

Inner Product Spaces

Orthogonal Complements

We begin with the definition of Orthogonal Complements:

6.45 **Definition** ***orthogonal complement,*** $U^{\perp}$

If U is a subset of V, then the ***orthogonal complement*** of U, denoted $U^{\perp}$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^{\perp} = \{v \in V : \langle v, u \rangle = 0 \text{ for every } u \in U\}.$$

For example, if U is a line containing the origin in $\mathbb{R}^3$, then $U^{\perp}$ is the plane containing the origin that is perpendicular to U.

Orthogonal Complements

6.46 Basic properties of orthogonal complement

(a) If U is a subset of V, then $U^{\perp}$ is a subspace of V.

(b) $\{0\}^{\perp} = V$.

(c) $V^{\perp} = \{0\}$.

(d) If U is a subset of V, then $U \cap U^{\perp} \subset \{0\}$.

(e) If U and W are subsets of V and $U \subset W$, then $W^{\perp} \subset U^{\perp}$.

Orthogonal Complements

6.47 **Direct sum of a subspace and its orthogonal complement**

Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^{\perp}.$$

Proof. First we will show that $V = U + U^{\perp}$. To do this, suppose $v \in V$. Let $e_1, e_2, \cdots, e_m$ be an orthonormal basis of U. Obviously

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m + v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m.$$

Let $u = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$ and $w = v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m$

It can be verified that $u \in U$, $w \in U^{\perp}$, and $U \bigcap U^{\perp} = \{0\}$. Thus

$$V = U \oplus U^{\perp}.$$

Orthogonal Complements

6.50 Dimension of the orthogonal complement

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^{\perp} = \dim V - \dim U.$$

Proof. The formula for $\dim U^{\perp}$ follows immediately from 6.47 and 3.78.

6.51 The orthogonal complement of the orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}.$$

Orthogonal Projection

We now define an operator P_U for each finite-dimensional subspace of V.

6.53 **Definition** ***orthogonal projection,*** P_U

Suppose U is a finite-dimensional subspace of V. The ***orthogonal projection*** of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^{\perp}$. Then $P_U v = u$.

6.54 **Example** Suppose $x \in V$ with $x \neq 0$ and $U = \operatorname{span}(x)$. Show that

$$P_U v = \frac{\langle v, x\rangle}{\|x\|^2} x$$

for every $v \in V$.

6.55 Properties of the orthogonal projection P_U

Suppose U is a finite-dimensional subspace of V and $v \in V$. Then

(a) $P_U \in \mathcal{L}(V)$;

(b) $P_U u = u$ for every $u \in U$;

(c) $P_U w = 0$ for every $w \in U^\perp$;

(d) range $P_U = U$;

(e) null $P_U = U^\perp$;

(f) $v - P_U v \in U^\perp$;

(g) $P_U{}^2 = P_U$;

(h) $\|P_U v\| \leq \|v\|$;

(i) for every orthonormal basis $e_1, \ldots, e_m$ of U,

$$P_U v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m.$$

Minimization Problems

The following problem often arises: given a subspace U of V and a point $v \in V$, find a point $u \in U$ such that $\|v-u\|$ is as small as possible. The next proposition shows that this minimization problem is solved by taking $u = P_U v$.

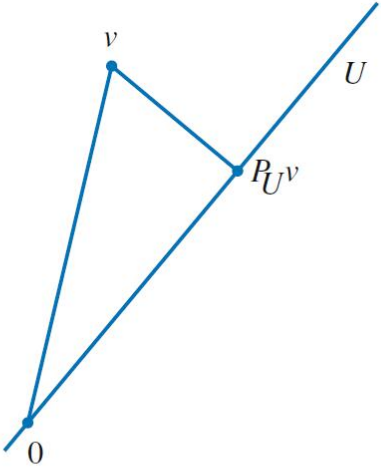
6.56 Minimizing the distance to a subspace

Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$. Then

$$\|v - P_U v\| \leq \|v - u\|.$$

Furthermore, the inequality above is an equality if and only if $u = P_U v$.

Minimizing the distance to a subspace



$P_U v$ is the closest point in U to v.

Example

The last result is often combined with the formula 6.55(i) to compute explicit solutions to minimization problems.

6.58 **Example** Find a polynomial u with real coefficients and degree at most 5 that approximates $\sin x$ as well as possible on the interval $[-\pi, \pi]$, in the sense that

$$\int_{-\pi}^{\pi} |\sin x - u(x)|^2 \, dx$$

is as small as possible. Compare this result to the Taylor series approximation.

Solution.

(a) Let $C_R[-\pi,\pi]$ denote the real inner product space of continuous real-valued functions on $[-\pi,\pi]$ with inner product

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

(b) Let $v \in C_R[-\pi,\pi]$ be the function defined by $v(x) = \sin x$. Let U denote the subspace of $C_R[-\pi,\pi]$ consisting of the polynomials with real coefficients and degree at most 5. Our problem can now be reformulated as follows:

Find $u \in U$ such that $||v-u||$ is as small as possible.

(c) $u(x)$ is given as follows (using 6.55(i)):

$$u(x) = 0.987862x - 0.155271x^3 + 0.00564312x^5.$$

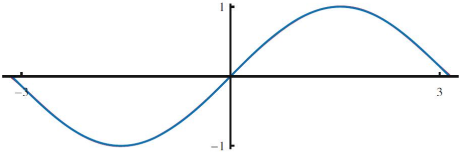
Solution.

(d) The polynomial u above is the best approximation to $\sin x$ on $[-\pi, \pi]$ using polynomials of degree at most 5.

(e) Here "best approximation" means in the sense of minimizing

$$\int_{-\pi}^{\pi} |\sin x - u(x)|^2 dx.$$

(f) To see how good this approximation is, the next figure shows the graphs of both $\sin x$ and our approximation $u(x)$ given by 6.60 over the interval $[-\pi, \pi]$.

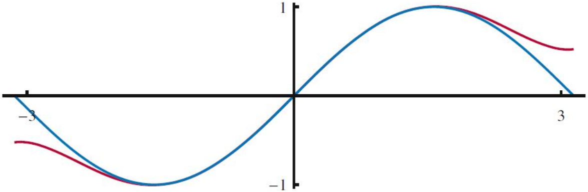


Graphs on $[-\pi, \pi]$ of $\sin x$ (blue) and its approximation $u(x)$ (red) given by 6.60.

Example

(g) Another well-known approximation to $\sin x$ by a polynomial of degree 5 is given by the Taylor polynomial

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$



Graphs on $[-\pi, \pi]$ *of* $\sin x$ *(blue) and the Taylor polynomial 6.61 (red).*

Homework Assignment 18

6.C: 5, 7, 8, 9, 11, 12, 14.