Introduction; Properties and Formulas of Determinants(行列式的定义和性质)

Lecture 17 and 18

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Determinants: Properties and Formulas

Introduction

Four of the main uses of determinants:

1. The test for invertibility. If the determinant of A is zero, then A is singular. If $\det(A) \neq 0$, then A is invertible.
2. The determinant of A equals the volume of a box in n-dimensional space. The edges of the box come from the rows of A.
3. The determinant gives a formula for each pivot.
4. The determinant measures the dependence of $A^{-1}b$ on each element of b.

Test for invertibility

Proposition

If the determinant of A is zero, then A is singular. If det $A \neq 0$, then A is invertible (and A^{-1} involves $1/\det A$).

Remarks:

- The most important application.
- The eigenvalue is defined to be the roots of the polynomial det $(A - \lambda I) = 0$.
- This is a fact that follows from the determinant formula, and not from a computer.

Volume

Proposition

The determinant of A equals the volume of a box in n-dimensional space.

Remarks:

- The edges of the box come from the rows of A.
- The columns of A would give an entirely different box with the same volume.

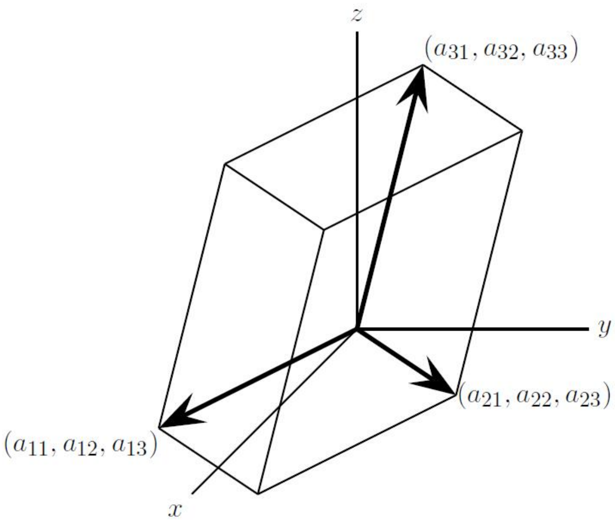

Figure 4.1: The box formed from the rows of A: volume $= |\text{determinant}|$.

The determinant gives a formula for each pivot.

Proposition

determinant $= \pm$ *(product of the pivots)*

Remarks:

- Regardless of the order of elimination, the product of the pivots remains the same apart from sign.
- In practice, if an abnormally small pivot is not avoided, is that it is soon followed by an abnormally large one. This brings the product back to normal but it leaves the numerical solution in ruins.

The dependence of $A^{-1}b$ on each element of b

- The determinant measures the dependence of $A^{-1}b$ on each element of b.
- If one parameter is changed in an experiment, or one observation is corrected, the "influence coefficient" in A^{-1} is a ratio of determinants.

How to define determinant?

The simple things about the determinant are not about the explicit formulas, but the properties it possesses. There are three most basic properties:

- det $I = 1$.
- the sign is reversed by a row exchange.
- the determinant is linear in each row separately.

Question:

How many exchanges does it take to change $VISA$ into $AVIS$? Is this permutation odd or even?

Properties 1-3

1. The determinant of the identity matrix is 1.

$$\det \quad I = 1 \quad \text{and} \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdots$$

2. The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3. The determinant depends linearly on the first row. Add vectors in row 1 and multiply by t in row 1:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}, \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Definition

1. We can refer to the determinant of a specific matrix by enclosing the array between vertical lines. Notice the two accepted notations for the determinant, $\det(A)$ and $|A|$.
2. Properties 1,2,3 are the defining properties of the determinant. Every property is a consequence of the first three. Which is literally to say that the determinant is now settled.
3. But that fact is not at all obvious. Therefore, we gradually use these rules to find the determinant of any matrix.

Properties 4-6

4. If two rows of A are equal, then $\det A = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

5. Subtracting a multiple of one row from another row leaves the same determinant.

$$\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. If A has a row of zeros, then $\det A = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$$

Properties 7-8

7. If A is triangular, then det A is the product $a_{11}a_{22}a_{33}\cdots a_{nn}$ of the diagonal entries. If the triangular A has 1's along the diagonal, then det $A = 1$.

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad, \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$$

8. If A is singular, then det $A = 0$. If A is invertible, then det $A \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. If A is nonsingular, elimination puts the pivots $d_1, d_2, \cdots, d_n$ on the main diagonal. We have a " product of pivots" formula for det A: $\det A = \pm d_1 d_2 \cdots d_n$.

Property 9

The main property is the product rule, which is also the most surprising.

9. The determinant of AB is the product of det A times det B.

$$|A||B| = |AB| \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right|.$$

A particular case of this rule gives the determinant of A^{-1}. It must be $\frac{1}{\det A}$.

Proof of Property 9

Assume A and B are nonsingular; otherwise AB is singular and the equation $\det AB = \det A \det B$ is easily verified. By rule 8, it becomes $0 = 0$.

Proof.

We prove that the ratio $d(A) = \det AB / \det B$ has properties 1,2,3. Then $d(A)$ must equal $\det A$. For example, $d(I) = \det B / \det B = 1$. rule 1 is satisfied. If two rows of A are exchanged , so are the two rows of AB, and the sign of d changes as required by rule 2. A linear combination of the first row of A gives the same linear combination in the first row of AB. The rule 3 for the determinant of AB, divided by the fixed quantity $\det B$, leads to rule 3 for the ration $d(A)$. Thus $d(A)$ coincides with $\det A$, which is our product formula. □

Property 10

10. The transpose of A has the same determinant as A itself: det $A^T = $ det A.

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = |A^T|$$

Again the singular case is separate; A is singular if and only if A^T is singular, and we have $0 = 0$. If A is nonsingular, then it allows the factorization $PA = LDU$, and we apply rule 9 for the determinant of a product:

$$\det P \det A = \det L \det D \det U. \tag{1}$$

Transposing $PA = LDU$ gives $A^T P^T = U^T D^T L^T$, and again by rule 9

$$\det A^T \det P^T = \det U^T \det D^T \det L^T. \tag{2}$$

$\det\ P = \det\ P^T$

We only need to prove $\det\ P = \det\ P^T$.

Since P is a permutation matrix, the inverse of P is P^T.

$$\det\ P^T P = \det\ I = 1.$$

Since $\det P = -1$ or 1, it follows that $\det\ P = \det\ P^T$. The products (1) and (2) are the same, and $\det\ A = \det\ A^T$.

Several Remarks

Remarks:

- This fact practically doubles our list of properties, because every rule that applied to the rows, can now be applied to the columns.
- It only remains to find a definite formula for the determinant, and to put that formula to use.

Examples

Example 1. Find the determinant of A.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Solution. 4.

Example 2. Find the determinant of A.

$$A = \begin{bmatrix} -2 & 5 & -1 & 3 \\ 1 & -9 & 13 & 7 \\ 3 & -1 & 5 & -5 \\ 2 & 8 & -7 & -10 \end{bmatrix}.$$

Solution. 312.

Computing the determinants

Proposition

If A is invertible, then $PA = LDU$ and det$P = \pm 1$. The product rule gives

$$\text{det}\, A = \pm\ \text{det}\, L\ \text{det}\, D\ \text{det}\, U = \pm(\text{product of the pivots})$$

The sign ± 1 depends on whether the number of row exchanges is even or odd. The triangular factors have det $L =$ det $U = 1$ and det $D = d_1 d_2 \cdots d_n$.

Example

Example

The $-1, 2, -1$ second difference matrix has pivots $\frac{2}{1}, \frac{3}{2}, \cdots$ in D:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} = LDU = L \begin{bmatrix} 2 & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \ddots & \\ & & & & \frac{n+1}{n} \end{bmatrix} U.$$

Its determinant is the product of its pivots. The numbers $2, 3, \cdots, n$ all cancel:

$$\det A = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = n+1.$$

Formulas for the Determinants

Consider 3 by 3 matrix, the determinant formula is pretty well known:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{array}$$

Our goal is to derive the above formulas directly from the three defining properties 1-3 of $\det A$.

- The nonzero terms have to come in different columns.
- There are $n!$ ways to permute the numbers $1, 2, \cdots, n$.
- If we consider $n = 3$:

Column numbers

$$(\alpha, \beta, \nu) = (1,2,3), (2,3,1), (3,1,2), (1,3,2), (2,1,3), (3,2,1).$$

The determinant of a 3×3 Matrix

The determinant of A is now reduced to six separate and much simpler determinants.

$$
\det A = \begin{aligned}
& a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} \\
& + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} \\
& + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}.
\end{aligned}
$$

Determinant of the permutation matrix P

To find the determinant of a permutation matrix P, we do row exchanges to transform it to the identity matrix, and each exchange reverses the sign of the determinant:

$$\det P = 1 \text{ or } -1 \quad \text{for an even or odd number of row exchanges.}$$

$$(1,3,2) \text{ is odd so } \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} = -1$$

$$(3,1,2) \text{ is even so } \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} = 1.$$

The Big Formula

For an $n \times n$ matrix, the sum is taken over all $n!$ permutations $(\alpha, \cdots, \gamma)$ of the numbers $(1, \cdots, n)$. The permutation gives the column numbers as we go down the matrix. The 1s appear in P at the same places where the a's appeared in A.

Definition

$$\det A = \sum_{\text{all } P's} (a_{1\alpha} a_{2\beta} \cdots a_{n\gamma}) \det P.$$

Cofactor of a_{11}:

$$C_{11} = \sum (a_{2\beta} \cdots a_{n\nu}) \det P = \det(\text{submatrix of } A)$$

Cofactors along row 1:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Expansion of $\det A$ in Cofactors

Definition

The determinant of A is a combination of any row i times its cofactors:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor C_{ij} is the determinant of M_{ij} with the correct sign:

$$C_{ij} = (-1)^{i+j} \det M_{ij}.$$

Remarks:

- We can define the determinant by induction on n.
- We can expand in cofactors of a column of A as well, as a consequence, $\det A = \det A^T$.

Example 3

Example

The 4 by 4 second difference matrix A_4 has only two nonzeros in row 1:

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

We can get a recursion relation: $\det A_n = 2(\det A_{n-1}) - \det A_{n-2}$. Therefore the determinant of the $-1, 2, -1$ matrix is:

$$\det A_n = 2(n) - (n-1) = n+1.$$

The answer $n+1$ agrees with the product of pivots at the start of this section.

The determinant of Vandermonde matrix

Find the determinant of the following Vandermonde Matrix:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ & \cdots & & \cdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

Solution. Induction on n.

$$\prod_{1 \le j < i \le n} (x_i - x_j).$$

Homework Assignment 17 and 18

4.2: 2, 3, 4, 6, 7, 10, 12, 14, 19, 25, 34, 35.

4.3: 1, 5, 7, 11, 16, 22, 25, 30, 34, 36.