Vector Spaces and Subspaces (向量空间和子空间)

Lecture 6

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Vector Spaces

Introduction

Recall that:

- The space $\mathbb{R}^n$ consists of all column vectors with n components.
- $\mathbb{R}^2$ is represented by the usual xy plane.
- The three components of a vector in $\mathbb{R}^3$ give a point in three-dimensional space.

The valuable thing for linear algebra is that the extension to n dimensions is so straightforward. For a vector in $\mathbb{R}^7$, we just need the seven components, even if the geometry is hard to visualize.

Addition and Scalar Multiplication

Within all vector spaces, we can:

- add any two vectors;
- multiply all vectors by scalars.

In other words, we can take linear combinations.

- Addition obeys the commutative law $x+y=y+x$; there is a "zero vector" satisfying $0+x=x$; and there is a "vector" satisfying $-x+x=0$.
- Eight properties are fundamental, see the following page.
- A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers.
- Normally our vectors belong to one of the spaces $\mathbb{R}^n$; they are ordinary vectors. The formal definition allows other things to be "vectors"–provided that addition and scalar multiplication are **all right**.

Definition

A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers. There are eight fundamental properties:

1. $x+y=y+x$ *(commutativity of addition).*
2. $x+(y+z)=(x+y)+z$ *(associativity of addition).*
3. *There is a unique "zero vector" such that* $x+0=x$ *for all* x*.*
4. *For each* x *there is a unique vector* $-x$ *such that* $x+(-x)=0$*.*
5. $1x=x$*.*
6. $(c_1c_2)x=c_1(c_2x)$*.*
7. $c(x+y)=cx+cy$*.*
8. $(c_1+c_2)x=c_1x+c_2x$*.*

The elements $x+y$ *and* cx *are called the sum of* x *and* y *and the product of* c *and* x*, respectively.*

Examples

Example 1 The infinite-dimensional space $\mathbb{R}^\infty$. Its vectors have infinitely many components, as in $x = (1,2,1,2,\cdots)$. The laws for $x+y$ and cx stay unchanged.

Example 2 The space of 3 by 2 matrices. In this case, the “vectors” are matrices. We can add two matrices, and $A+B=B+A$, and there is a zero matrix, and so on. This space is almost the same as $\mathbb{R}^6$. (The six components are arranged in a rectangle instead of a column.) Any choice of m and n would give, as a similar example, the vector space of all m by n matrices.

Example 3

Example 3 The space of functions $f(x)$. Here we admit all functions f that are defined on a fixed interval, say $0 \leq x \leq 1$. The space includes $f(x) = x^2,\ g(x) = \sin x$, their sum $(f+g)(x) = x^2 + \sin x$, and all multiples like $3x^2$ and $-\sin x$. The vectors are functions, and the dimension is somehow a larger infinity than for $\mathbb{R}^\infty$.

Additional Properties of Vector Spaces

The following theorem states five more fundamental properties of vector spaces.

Theorem

If V is a vector space and x, y, and z are elements of V, then

1. $0x = 0$.
2. $(-1)x = -x$.
3. *If* $x + y = x + z$, *then* $y = z$.
4. $\beta 0 = 0$ *for each scalar* β.
5. *If* $\alpha x = 0$, *then either* $\alpha = 0$ *or* $x = 0$.

Example

Example

Let $\mathbb{R}^+$ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by

$$\alpha \circ x = x^{\alpha}$$

for each $x \in \mathbb{R}^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y \quad \text{for all} \quad x, y \in \mathbb{R}^+.$$

Show that $\mathbb{R}^+$ is a vector space with these operations.

Thus for this system, the scalar product of -3 times $\frac{1}{2}$ is given by $-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8$ and the sum of 2 and 5 is given by $2 \oplus 5 = 2 \cdot 5 = 10$.

Subspace

Consider any plane through the origin in three dimensional space $\mathbb{R}^3$, it is a vector space in its own right, and it is actually inside the original space $\mathbb{R}^3$. This example suggests us to study *subspace*:

Definition

A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space:
(i) If we add any vectors x and y in the subspace, $x+y$ is in the subspace.
(ii) If we multiply any vector x in the subspace by any scalar c, cx is in the subspace.

In other words, Linear combinations stay in the subspace.

Remarks

- Notice our emphasis on the word **space**. A subspace is a subset that is “closed” under addition and scalar multiplication.
- The zero vector belongs to every subspace.
- The distinction between a subset and a subspace: in a subspace, when you add vectors and multiply by scalars, without leaving the space. However, a subset does not generally have that property.

Remarks

- The smallest subspace contains only the the zero vector. It is a zero dimensional vector space.
- If the original space is $\mathbb{R}^3$, then the possible subspaces are easy to describe: $\mathbb{R}^3$ itself, any plane through the origin, any line through the origin, or the origin alone.
- The distinction between a subset and a subspace is made clear by examples.

Examples

Example

Example 4 All vectors in $\mathbb{R}^2$ whose components are positive or zero. This subset is not a subspace. This subset is the first quadrant of the $x-y$ plane; the coordinates satisfy $x \geq 0$ and $y \geq 0$. It is not a subspace, even though it contains zero and addition does leave us within the subset. Rule (ii) is violated, since if the scalar -1 and the vector $[1 \quad 1]$, the multiple $cx = [-1 \quad -1]$ is in the third quadrant instead of the first.

Example

Example 5 Lower Triangular Matrices and Symmetric Matrices. Start from the vector space of 3 by 3 matrices. One possible subspace is the set of lower triangular matrices. Another is the set of symmetric matrices.

The Column Space of A

The incredible thing is that the plane that is spanned by the column vectors of a coefficient matrix of system of linear equations is actually a subspace! It is the column space of A, which is denoted by $C(A)$.

Definition

The column space of a matrix A contains all linear combinations of the columns of A. It is a subspace of $\mathbb{R}^m$.

Theorem and Remarks

Theorem

The system $Ax = b$ is solvable if and only if the vector b can be expressed as a combination of the columns of A. Then b is in the column space.

(a) The attainable right-hand sides b are all combinations of the columns of A.

(b) Geometrical meaning: $Ax = b$ can be solved if and only if b lies in the plane that is spanned by the column vectors.

The Column Space is a subspace

Consider $Ax = b$ as follows:

$$Ax = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \left(= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

We can describe all combinations of the two columns geometrically:

(a) $Ax = b$ can be solved if and only if b lies in the plane that is spanned by the two column vectors (Figure 2.1). This is the thin set of attainable b.

(b) If b lies off the plane, then it is not a combination of the two columns. In that case $Ax = b$ has no solution.

(c) What is important is that this plane is not just a subset of $\mathbb{R}^3$; it is a subspace. It is the column space of A, consisting of all combinations of the columns.

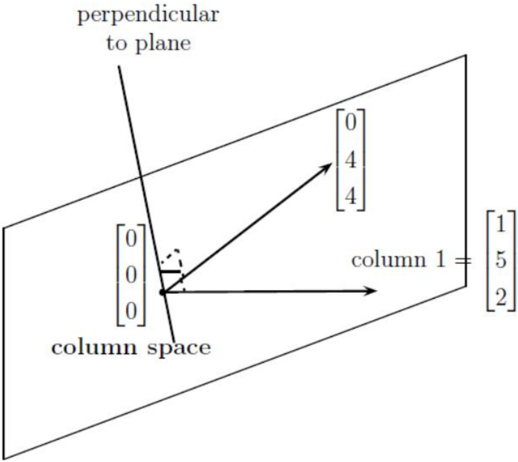

Figure 2.1: The column space $C(A)$, a plane in three-dimensional space.

Column Space $C(A)$ of an $m \times n$ matrix A is a subspace

It can be readily checked as follows:

- Suppose b and b' lie in the column space, so that $Ax = b$ for some x and $Ax' = b'$ for some x'. Then $A(x + x') = b + b'$, so that $b + b'$ is also a combination of the columns. The column space of all attainable vectors b is closed under addition.
- If b is in the column space $C(A)$, so is any multiple cb. If some combination of columns produces b (say $Ax = b$), then multiplying that combination by c will produce cb. In other words, $A(cx) = cb$.

For another matrix A, the dimensions in Figure 2.1 may be very different.

Examples

- zero matrix.
- nonsingular matrix.
- $C(A)$ can be somewhere between the zero space and the whole space. Together with its perpendicular space, it gives one of our two approaches to understand $Ax = b$.

The Nullspace of A

Definition

The solutions to $Ax = 0$ form a vector space–the nullspace of A.
The nullspace of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$. It is subspace of $\mathbb{R}^n$, just as the column space was a subspace of $\mathbb{R}^m$.

Can you check the requirements (i) and (ii) as well?

Nullspace: Example

Consider the following example again:

$$Ax = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \left(= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Only the solutions to a homogeneous equation $(b = 0)$ form a subspace.
- The nullspace contains only the vector $(0, 0)$.
- The columns of the coefficient matrix are independent.(A key concept that comes soon).

Nullspace: Another Example

If we add one more column to the previous matrix, which is a combination of the first two columns, then we immediately see that the column space stays the same. However, the Nullspace is quite different.

Example

If we add an extra column to the previous matrix, what is its new nullspace?

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

Two More Examples

Example

Let **P** be the plane in 3-space with equation $x+2y+z=6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to **P**? Are **P** and $\mathbf{P}_0$ subspaces of $\mathbb{R}^3$?

Example

If we add an extra column b to a matrix A, then the column space gets larger unless ________________. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax=b$ solvable exactly when the column space doesn't get larger by including b?

Example

Example

Let $V = \mathbb{R}^{n\times n}$ be the set of all $n \times n$ matrices.

(a) Verify that V is a vector space.

(b) Let W be the subset of V consisting of all $n \times n$ matrices such that $\mathrm{Tr}(A) = 0$. Show that W is a subspace of V.

(c) Can you find the dimension of W?

Further Remarks

We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all four subspaces that are intimately related to each other and to A–the column space of A, the nullspace of A, and their two perpendicular spaces.

Homework Assignment 6

2.1: 2, 3, 4, 14, 22, 24, 28.