Bases and Dimension

Lecture 4

Dept. of Math.

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Vector Spaces

Basis

In the last section, we discussed linearly independent lists and spanning lists. Now we bring these concepts together.

Definition

A basis of V is a list of vectors in V that is linearly independent and spans V.

Examples

(a) The list $(1,0,\cdots,0),(0,1,0,\cdots,0),\cdots,(0,\cdots,0,1)$ is a basis of $\mathbb{F}^n$, called the standard basis of $\mathbb{F}^n$.

(b) The list $(1,2),(3,5)$ is a basis of $\mathbb{F}^2$.

(c) The list $(1,2,-4),(7,-5,6)$ is linearly independent in $\mathbb{F}^3$ but is not a basis of $\mathbb{F}^3$ because it does not span $\mathbb{F}^3$.

(d) The list $(1,2),(3,5),(4,13)$ spans $\mathbb{F}^2$ but is not a basis of $\mathbb{F}^2$ because it is not linearly independent.

(e) The list $(1,1,0),(0,0,1)$ is a basis of $\{(x,x,y)\in\mathbb{F}^3 : x,y\in\mathbb{F}\}$.

(f) The list $(1,-1,0),(1,0,-1)$ is a basis of

$$\{(x,y,z)\in\mathbb{F}^3 : x+y+z=0\}.$$

(g) The list $1,z,z^2,\cdots,z^m$ is a basis of $\mathscr{P}_m(\mathbb{F})$.

Criterion for basis

The next result helps explain why bases are useful. Recall that "uniquely" means "in only one way".

2.29 **Criterion for basis**

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

2.30
$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

The proof is essentially a repetition of the ideas of that led us to the definition of linear independence.

Properties

For a given spanning list, some(possibly none) of the vectors in it can be discarded so that the remaining list is linearly independent:

2.31 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space.

Every finite-dimensional vector space has a basis

The next result, which is an easy corollary of the previous result, tells us that every finite-dimensional vector space has a basis.

2.32 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Next result is in some sense a dual of 2.31, which said that every spanning list can be reduced to a basis.

Linearly Independent List Extends to a Basis

Now we show that given any linearly independent list, we can adjoin some additional vectors(this includes the possibly of adjoining no additional vectors) so that the extended list is still linearly independent but also spans the space.

2.33 Linearly independent list extends to a basis

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Direct Sum

As an application of the result above, we now show that every subspace of a finite-dimensional vector space can be paired with another subspace to form a direct sum that $V = U \oplus W$.

2.34 Every subspace of V is part of a direct sum equal to V

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Dimension

Although we have been discussing finite-dimensional vector spaces, we have not yet defined the dimension of such an object. Basis length does not depend on basis:

2.35 Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length.

Dimension

Now that we know that any two bases of a finite-dimensional vector space have the same length, we can formally define the dimension of such spaces.

2.36 **Definition** ***dimension***, $\dim V$

- The ***dimension*** of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

Dimension

Every subspace of a finite-dimensional vector space is finite-dimensional(by 2.26) and so has a dimension. The next result gives the expected inequality about the dimension of a subspace.

2.38 Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Every linearly independent list with the right length is a basis:

2.39 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Example

2.40 Example Show that the list $(5, 7), (4, 3)$ is a basis of $\mathbf{F}^2$.

Solution This list of two vectors in $\mathbf{F}^2$ is obviously linearly independent (because neither vector is a scalar multiple of the other). Note that $\mathbf{F}^2$ has dimension 2. Thus 2.39 implies that the linearly independent list $(5, 7), (4, 3)$ of length 2 is a basis of $\mathbf{F}^2$ (we do not need to bother checking that it spans $\mathbf{F}^2$).

2.41 Example Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbf{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbf{R}) : p'(5) = 0\}.$$

Solution

Solution Clearly each of the polynomials $1, (x-5)^2$, and $(x-5)^3$ is in U.

Suppose $a, b, c \in \mathbf{R}$ and

$$a + b(x-5)^2 + c(x-5)^3 = 0$$

for every $x \in \mathbf{R}$. Without explicitly expanding the left side of the equation above, we can see that the left side has a cx^3 term. Because the right side has no x^3 term, this implies that $c = 0$. Because $c = 0$, we see that the left side has a bx^2 term, which implies that $b = 0$. Because $b = c = 0$, we can also conclude that $a = 0$.

Thus the equation above implies that $a = b = c = 0$. Hence the list $1, (x-5)^2, (x-5)^3$ is linearly independent in U.

Thus $\dim U \geq 3$. Because U is a subspace of $\mathcal{P}_3(\mathbf{R})$, we know that $\dim U \leq \dim \mathcal{P}_3(\mathbf{R}) = 4$ (by 2.38). However, $\dim U$ cannot equal 4, because otherwise when we extend a basis of U to a basis of $\mathcal{P}_3(\mathbf{R})$ we would get a list with length greater than 4. Hence $\dim U = 3$. Thus 2.39 implies that the linearly independent list $1, (x-5)^2, (x-5)^3$ is a basis of U.

Spanning list of the right length is a basis

Spanning list of the right length is a basis:

2.42 Spanning list of the right length is a basis

Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Dimension of a Sum

2.43 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof Let $u_1, u_2, \cdots, u_m$ be a basis of $U_1 \bigcap U_2$; thus $\dim(U_1 \bigcap U_2) = m$. Because $u_1, u_2, \cdots, u_m$ is a basis of $U_1 \bigcap U_2$, it is linearly independent in U_1. Hence this list can be extended to a basis $u_1, u_2, \cdots, u_m, v_1, \cdots, v_j$ of U_1 (by 2.33). Thus $\dim(U_1) = m + j$. Also extend $u_1, u_2, \cdots, u_m$ to a basis $u_1, u_2, \cdots, u_m, w_1, \cdots, w_k$ of U_2 (by 2.33). Thus $\dim(U_2) = m + k$. We will show that $u_1, u_2, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k$ is a basis of $U_1 + U_2$.

Proof

This will complete the proof, because then we will have $\dim(U_1+U_2) = m+j+k = (m+j)+(m+k)-m = \dim U_1 + \dim U_2 - \dim(U_1 \bigcap U_2)$.

Clearly span$(u_1, u_2, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k)$ contains U_1 and U_2 and hence equals U_1+U_2. So to show that this list is a basis of U_1+U_2 we need only show that it is linearly independent. To prove this, suppose

$$a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kv_k = 0.$$

where all the a's, b's, and c's are scalars.

We need to prove that all the a's, b's, and c's equal 0. The equation above can be rewritten as

$$c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j.$$

which shows that $c_1w_1 + \cdots + c_kw_k \in U_1$.

Proof

All the w's are in U_2, so this implies that $c_1 w_1 + \cdots + c_k w_k \in U_1 \cap U_2$. Because $u_1, \cdots, u_m$ is a basis of $U_1 \cap U_2$, we can write

$$c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots + d_m u_m$$

for some choice of scalars $d_1, d_2, \cdots, d_m$. But $u_1, \cdots, u_m, w_1, \cdots, w_k$ is linearly independent, so the last equation implies that all the c's (and d's) equal 0. Thus our original equation involving the a's, b's, and c's becomes

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0.$$

Because the list $u_1, \cdots, u_m, v_1, \cdots, v_j$ is linearly independent, this equation implies that all the a's and b's are 0. We now know that all the a's, b's, and c's equal 0, as desired.

Homework Assignment 4

2.B: 4, 6, 8.

2.C: 7, 9, 15, 16, 17.