$\mathbb{R}^n$ and $\mathbb{C}^n$; Definition of Vector Space (向量空间的定义)

Lecture 1

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Vector Spaces (向量空间)

Introduction

- Linear algebra is the study of linear maps on finite-dimensional vector spaces.
- In linear algebra, better theorems and more insight emerge if complex numbers are investigated along with real numbers.
- We will begin by introducing the complex numbers and their basic properties.
- We will generalize the examples of a plane and ordinary space to $\mathbb{R}^n$ and $\mathbb{C}^n$.
- We then will generalize to the notion of a vector space.
- Then our next topic will be subspaces, which play a role for vector spaces analogous to the role played by subsets for sets.

Complex Numbers (复数)

The idea is to assume we have a square root of -1, denoted i, that obeys the usual roles of arithmetic. Here are the formal definitions:

1.1 **Definition** ***complex numbers***

- A ***complex number*** is an ordered pair (a, b), where $a, b \in \mathbf{R}$, but we will write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbf{C}$:

$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$

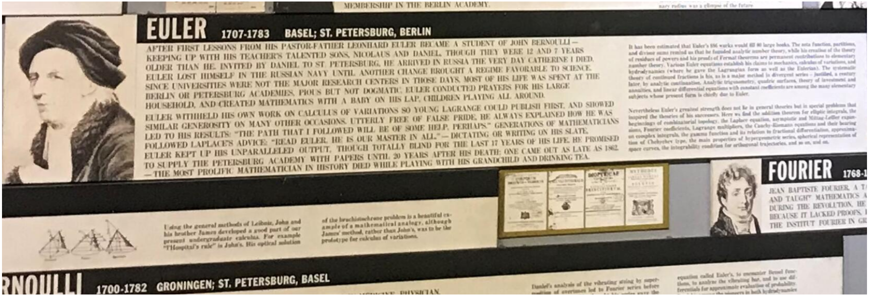
- ***Addition and multiplication*** on $\mathbf{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

here $a, b, c, d \in \mathbf{R}$.

Euler

The symbol i was first used to denote $\sqrt{-1}$ by Swiss mathematician Leonhard Euler in 1777.



(Courtesy: New York Hall of Science, August 11, 2019)

1.3 Properties of complex arithmetic

commutativity

$\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbf{C}$;

associativity

$(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$;

identities

$\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbf{C}$;

additive inverse

for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$;

distributive property

$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

1.5 Definition $-\alpha$, ***subtraction***, $1/\alpha$, ***division***

Let $\alpha, \beta \in \mathbf{C}$.

- Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0.$$

- ***Subtraction*** on $\mathbf{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

- For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1.$$

- ***Division*** on $\mathbf{C}$ is defined by

$$\beta/\alpha = \beta(1/\alpha).$$

Notation

So that we can conveniently make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation:

1.6 **Notation F**

Throughout this book, **F** stands for either **R** or **C**.

- The letter $\mathbb{F}$ is used because $\mathbb{R}$ and $\mathbb{C}$ are examples of what are called fields.
- Elements of $\mathbb{F}$ are called scalars.
- The word “scalar”, a fancy word for “number”, is often used when we want to emphasize that an object is a number, as opposed to a vector.

Lists

1.8 **Definition** ***list, length***

Suppose n is a nonnegative integer. A ***list*** of ***length*** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \ldots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

- Many mathematicians call a list of length n an n-tuple.
- Lists differ from sets in two ways: in lists, order matters and repetitions have meaning: in sets, order and repetitions are irrelevant.

$\mathbb{F}^n$

1.10 **Definition** $\mathbf{F}^n$

$\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$:

$$\mathbf{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \ldots, n\}.$$

For $(x_1, \ldots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} ***coordinate*** of $(x_1, \ldots, x_n)$.

- Addition in $\mathbb{F}^n$.
- Commutativity of addition in $\mathbb{F}^n$.
- Definition of 0 in $\mathbb{F}^n$.
- Additive inverse in $\mathbb{F}^n$.
- Scalar multiplication in $\mathbb{F}^n$.

Degression on Fields

Definition

A field is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all the properties listed in 1.3.

Example

Thus $\mathbb{R}$ and $\mathbb{C}$ are fields, as is the set of rational numbers along with the usual operations of addition and multiplication.

Example

Another example of a field is the set $\{0, 1\}$ with the usual operations of addition and multiplication except that $1 + 1$ is defined to equal 0.

addition, scalar multiplication

The motivation for the definition of a vector space comes from properties of addition and scalar multiplication in $\mathbb{F}^n$:

- Addition is commutative, associative, and has an identity.
- Every element has an additive inverse.
- Scalar multiplication is associative.
- Addition and scalar multiplication are connected by distributive properties.

1.18 **Definition** ***addition, scalar multiplication***

- An ***addition*** on a set V is a function that assigns an element $u+v \in V$ to each pair of elements $u, v \in V$.
- A ***scalar multiplication*** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

Vector Space: Definition

Definition

A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

(1) *Commutativity:* $u+v=v+u$ *for all* $u,v\in V$;

(2) *Associativity:* $(u+v)+w=u+(v+w)$ *and* $(ab)v=a(bv)$ *for all* $u,v,w\in V$ *and all* $a,b\in\mathbb{F}$;

(3) *Additive Identity: there exists an element* $0\in V$ *such that* $v+0=v$ *for all* $v\in V$;

(4) *Additive Inverse: for every* $v\in V$, *there exists* $w\in V$ *such that* $u+w=0$;

(5) *Multiplicative Identity:* $1v=v$ *for all* $v\in V$;

(6) *Distributive Properties:* $a(u+v)=au+av$ *and* $(a+b)v=av+bv$ *for all* $a,b\in\mathbb{F}$ *and all* $u,v\in V$.

One more definition

1.23 **Notation** $\mathbf{F}^S$

- If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$.
- For $f, g \in \mathbf{F}^S$, the ***sum*** $f + g \in \mathbf{F}^S$ is the function defined by

  $$(f + g)(x) = f(x) + g(x)$$

  for all $x \in S$.
- For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the ***product*** $\lambda f \in \mathbf{F}^S$ is the function defined by

  $$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$.

F^S **is a vector space.**

Elementary Properties of Vector Spaces

- vector, point
- real vector space, complex vector space

- Unique additive identity: A vector space has a unique additive identity.
- Unique additive inverse: Every element in a vector space has a unique additive inverse.
- Notation $-v, w-v$
- Notation V: For the rest of the book, V denotes a vector space over $\mathbb{F}$.
- The number 0 times a vector.
- A number times the vector 0.
- The number -1 times a vector.

Cancellation Law for Vector Addition

Theorem

If $x, y,$ and z are vectors in a vector space V such that $x+z=y+z$, then $x=y$.

Proof.

There exists a vector v in V such that $z+v=0$. Thus

$$\begin{aligned} x &= x+0 = x+(z+v) = (x+z)+v \\ &= (y+z)+v = y+(z+v) = y+0 = y. \end{aligned}$$

$\square$

Example

Example

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 2b_2)$$

and

$$c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Homework Assignment 1

1.A: 1, 3, 11, 12, 14.

1.B: 2, 3, 4, 5.