Solutions – Linear Algebra Spring 2023

June 11, 2023

Grading: June 12, 2023 2:00pm–10:00

15' Question 1:

(1) D (2) C (3) D (4) D (5) B

15' Question 2: (1) (-4,4) (2) 10 (3) -2, 6 (4) 2000 (5) 0.

20' Question 3:

(a) eigenvalues / eigenvectors

$\lambda_1 = 1$ $\begin{bmatrix} 3 \\ 1 \\ \frac{2}{3} \end{bmatrix}$

$\lambda_2 = -1$ $\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$

$\lambda_3 = 2$ $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

$$S = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 1 & 2 \\ \frac{2}{3} & 2 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & -3 \\ -\frac{1}{6} & -\frac{1}{6} & 1 \\ -\frac{2}{3} & \frac{4}{3} & 1 \end{bmatrix}$$

(b) $$S^{-1}AS = \Lambda = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 2 \end{bmatrix}$$

Let $$D = \Lambda^{1/3} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & \sqrt[3]{2} \end{bmatrix}$$

$$B = S^{-1}DS = \begin{bmatrix} -\frac{16}{3} - \frac{8}{3}\sqrt[3]{2} & -\frac{11}{3} + \frac{16}{3}\sqrt[3]{2} & -14 + 4\sqrt[3]{2} \\ \frac{5}{3} - \frac{4}{3}\sqrt[3]{2} & \frac{5}{3} + \frac{8}{3}\sqrt[3]{2} & -4 + 2\sqrt[3]{2} \\ \frac{4}{3} - \frac{2}{3}\sqrt[3]{2} & \frac{7}{6} + \frac{4}{3}\sqrt[3]{2} & -4 + \sqrt[3]{2} \end{bmatrix}.$$

20' Question 4:

(a) $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$.

(b) A

(c) $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$

10' Question 5:

($\Rightarrow$)

Let $x_1, x_2, \cdots, x_n$ be a maximal set of linearly independent eigenvectors of A and $y_1, y_2, \cdots, y_n$ be a maximal set of linearly independent eigenvectors of B.

Set

$$\hat{x}_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad \hat{x}_2 = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad \cdots, \quad \hat{x}_n = \begin{bmatrix} x_n \\ 0 \end{bmatrix},$$

$$\hat{y}_1 = \begin{bmatrix} 0 \\ y_1 \end{bmatrix}, \quad \hat{y}_2 = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}, \quad \cdots, \quad \hat{y}_n = \begin{bmatrix} 0 \\ y_n \end{bmatrix}.$$

Then $\hat{x}_1, \cdots, \hat{x}_n, \hat{y}_1, \cdots, \hat{y}_n$ is a maximal set of linearly independent eigenvectorsof C.

($\Leftarrow$) similar argument as "$\Rightarrow$".

20' Question 6:

(a) $\sigma_1 = 2, \quad \sigma_2 = 1.$

(b)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt6} & -\frac{1}{\sqrt3} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt6} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt2} \\ \frac{2}{\sqrt6} & \frac{1}{\sqrt3} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{2}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}}_{V^T}.$$

(c)

$$\| Ax - b \| = \| U\Sigma V^T x - b \|$$

$$= \| \Sigma V^T x - U^T b \|$$

$y = V^T x, \; b' = U^T b$

$$= \| \Sigma y - b' \|$$

minimum length solution to $\Sigma y = b'$ is $y = \Sigma^+ b'$
$= \Sigma^+ U^T b$

minimum length solution to $A^TA\hat{x} = A^Tb$ is

$$x^+ = V\Sigma^+U^Tb = A^+b.$$

x^+ is in the row space of A.

(d)

$$x^+ = V\Sigma^+U^Tb = \begin{bmatrix} 1 \\ \frac12 \\ \frac12 \end{bmatrix}.$$