The Vector Space of Linear Maps

Lecture 5

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Vector Spaces

Linear Maps (线性映射)

In this chapter we will frequently need another vector space, which we will call W, in addition to V. Thus our standing assumptions are as follows:

- $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$.
- V and W denote vector spaces over $\mathbb{F}$.

Now we are ready for one of the key definitions in Linear Algebra.

Definition

Definition

A linear map from V to W is a function $T : V \to W$ with the following properties:

(a) Additivity
$$T(u+v) = Tu + Tv$$
for all $u, v \in V$;

(b) Homogeneity
$$T(\lambda v) = \lambda Tv$$
for all $\lambda \in \mathbb{F}$ and all $v \in V$.

Loosely speaking, a linear transformation is a function from one vector space to another that preserves the vector space operations.

Linear Maps

Some mathematicians use the term linear transformation, which means the same as linear map. Note that for linear maps we often use the notation Tv as well as the more standard functional notation $T(v)$.

Notation $\mathscr{L}(V,W)$

3.3 Notation $\mathcal{L}(V, W)$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

Examples

Examples of Linear Maps:

(a) zero map, identity map

(b) rotation, reflection, and projection

(c) differentiation, integration

(d) multiplication by x^2

(e) backward shift

Example: Linear Maps

(1) from $\mathbb{R}^3$ to $\mathbb{R}^2$: Define $T \in \mathscr{L}(\mathbb{R}^3, \mathbb{R}^2)$ by

$$T(x,y,z) = (2x - y + 3z, 7x + 5y - 6z).$$

(2) from $\mathbb{F}^n$ to $\mathbb{F}^m$: Generalizing the previous example, let m and n be positive integers, let $A_{j,k} \in \mathbb{F}$ for $j = 1, \cdots, m$ and $k = 1, \cdots, n$, and define $T \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$ by

$$T(x_1, \cdots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ \cdots, \ A_{m,1}x_1 + \cdots + A_{m,n}x_n).$$

Actually every linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$ is of this form.

(3) Suppose $T : \mathscr{P}_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ is defined as follows:

$$T(p(x)) = \begin{pmatrix} p(1) - p(2) & 0 \\ 0 & p(0) \end{pmatrix}.$$

Verify that T is a linear map.

Linear Maps and Basis of Domain

(a) The existence part of the next result means that we can find a linear map that takes on whatever values we wish on the vectors in a basis.

(b) The uniqueness part of the next result means that a linear map is completely determined by its values on a basis.

3.5 Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

This result is extremely important.

Proof

First we show the existence of a linear map T with the desired property. Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n,$$

where $c_1, c_2, \cdots, c_n$ are arbitrary elements of $\mathbb{F}$. The list $v_1, \cdots, v_n$ is a basis of V, and thus the equation above does indeed define a function T from V to W.

For each j, taking $c_j = 1$ and the other c's equal to 0 in the equation above shows that $Tv_j = w_j$.

If $u, v \in V$ with $u = a_1v_1 + \cdots + a_nv_n$ and $v = c_1v_1 + \cdots + c_nv_n$, then

$$T(u + v) = Tu + Tv.$$

Proof

Let us be more precise.

$$\begin{aligned} T(u+v) &= T((a_1+c_1)v_1+\cdots+(a_n+c_n)v_n) \\ &= (a_1+c_1)w_1+\cdots+(a_n+c_n)w_n \\ &= (a_1w_1+\cdots+a_nw_n)+(c_1w_1+\cdots+c_nw_n) \\ &= Tu+Tv. \end{aligned}$$

Similarly, if $\lambda \in \mathbb{F}$ and $v = c_1v_1+\cdots+c_nv_n$, then

Proof

$$
\begin{aligned}
T(\lambda v) &= T(\lambda(c_1v_1 + \cdots + c_nv_n)) \\
&= \lambda c_1w_1 + \cdots + \lambda c_nw_n \\
&= \lambda(c_1w_1 + \cdots + c_nw_n).
\end{aligned}
$$

Thus T is a linear map from V to W.

Proof

To prove the uniqueness, now suppose that $T \in \mathscr{L}(V,W)$ and that $Tv_j = w_j$ for $j = 1, \cdots, n$. Let $c_1, \cdots, c_n \in \mathbb{F}$. The homogeneity of T implies that

$$T(c_j v_j) = c_j w_j \text{ for } j = 1, \cdots, n.$$

The additivity of T now implies that

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n.$$

Thus T is uniquely determined on span$(v_1, \cdots, v_n)$ by the equation above. Because $v_1, \cdots, v_n$ is a basis of V, this implies that T is uniquely determined on V.

Algebraic Operations on $\mathscr{L}(V, W)$

We begin by defining addition and scalar multiplication on $\mathscr{L}(V, W)$

3.6 **Definition** ***addition and scalar multiplication on*** $\mathcal{L}(V, W)$

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The ***sum*** $S + T$ and the ***product*** λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

$\mathscr{L}(V, W)$ is a vector space

3.7 $\mathcal{L}(V, W)$ is a vector space

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Product of Linear Maps

Usually it makes no sense to multiply together two elements of a vector space, but for some pairs of linear maps a useful product exists. We will need a third vector space, so for the rest of this section suppose U is a vector space over $\mathbb{F}$.

3.8 Definition ***Product of Linear Maps***

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the ***product*** $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$.

A Few Remarks

In other words, ST is just the usual composition $S \circ T$ of two functions, but when both functions are linear, most mathematicians write ST instead of $S \circ T$.

You should verify that ST is indeed a linear map from U to W whenever $T \in \mathscr{L}(U,V)$ and $S \in \mathscr{L}(V,W)$.

Note that ST is defined only when T maps into the domain of S.

Algebraic properties of products of linear maps

(a) **Associativity**:

$$(T_1T_2)T_3 = T_1(T_2T_3)$$

whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning that T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1).

(b) **Identity**:

$$TI = IT = T,$$

whenever $T \in \mathscr{L}(V, W)$ (the first I is the identity map on V, and the second I is the identity map on W).

(c) **Distributive properties**:

$$(S_1 + S_2)T = S_1T + S_2T \text{ and } S(T_1 + T_2) = ST_1 + ST_2$$

whenever $T,\ T_1,\ T_2 \in \mathscr{L}(U, V)$ and $S,\ S_1,\ S_2 \in \mathscr{L}(V, W)$.

Example and 3.11

Example

Suppose $D \in \mathscr{L}(\mathscr{P}(\mathbb{R}), \mathscr{P}(\mathbb{R}))$ is the differentiation map defined in Example 3.4 and $T \in \mathscr{L}(\mathscr{P}(\mathbb{R}), \mathscr{P}(\mathbb{R}))$ is the multiplication by x^2 map defined earlier in this section. Show that $TD \neq DT$.

Solution. We have

$$((TD)p)(x) = x^2 p'(x)$$

but

$$((DT)p)(x) = x^2 p'(x) + 2xp(x).$$

In other words, differentiating and then multiplying by x^2 is not the same as multiplying by x^2 and then differentiating.

Proposition

Proposition

***Linear maps take* 0 *to* 0** *Suppose* T *is a linear map from* V *to* W*. Then* $T(0) = 0$*.*

By additivity, we have

$$T(0) = T(0+0) = T(0) + T(0).$$

Add the additive inverse of $T(0)$ to each side of the equation above to conclude that $T(0) = 0$.

Definition

Definition

The following terms are also employed:

(1) **homomorphism** for linear map;

(2) **endomorphism** for linear operator;

(3) **monomorphism** (or embedding) for injective linear map;

(4) **epimorphism** for surjective linear map;

(5) **isomorphism** for bijective linear map;

(6) **automorphism** for bijective linear operator.

Homework Assignment

3.A: 3, 6, 7, 9, 10, 11, 13, 14.