Chapter 1 Preliminary: Baby Set Theory

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Outline I

Outline

1.1 Basic concepts

1. Definition: A set is any collection of objects.

“object” $\Rightarrow$ “element”

2. Notations and representations:

(1) “Listing”: $\{a, b, c\}$, say, denoted by $A = \{a, b, c\}$. Also, $\{0, 1, 2, 3, \ldots\}$, etc.

(2) “Function form”: $\{x : x^2 = 1\}$ or $\{x \mid x^2 = 1\}$.

(3) “Venn diagram”: $\bigcirc$

1.1 Basic concepts

3. Some special notations:

(1) "$\in$" means "belongs to" : "1 is an element of a set B" is usually denoted by $1 \in B$.

"$\notin$" means "does not belong to".

Hence if $A = \{a, b, c\}$, then $a \in A, b \in A, c \in A$, but $d \notin A$.

Note that we view $\{1, 2, 1, 3\}$ and $\{1, 2, 3\}$, for example, as the same set.

(2) "$\forall$" means "for every" or "for all".

(3) "$\exists$" means "there exists".

4. Some special sets:

(1) Empty set, denoted by $\emptyset$: No element!

(2) Singleton, $\{1\}$, say: Only one element and this element is “1”.

(3) Universal set, denoted by Ω: The totality of objects under consideration.

5. Set of sets:

A set is any collection of objects and so an element of a set can be a set itself.

For example, if $A = \{1, -1\}$ and $B = \{a, b, c\}$, then

$$D = \{A, B, \emptyset, 5, cats, dogs\}$$

is a set.

So, $\{a, \{a, b\}\}$ is a set of <u>two</u> elements (Not 3 and these two elements are $\{a\}$ and $\{a, b\}$).

Also, $E = \{a, b, c, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ is a set of <u>7</u> elements.

Note also that $\emptyset$ and $\{\emptyset\}$ have different meanings:

$\emptyset$ is the empty set: No element.

$\{\emptyset\}$ is a singleton, with only <u>one element</u> and <u>this element</u> is $\emptyset$.

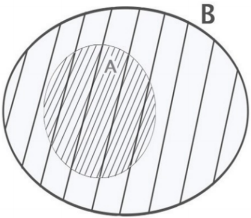
1.2 Subsets of a set

1. Definition: If any element of a set A is also an element of a set B, then we say A is a subset of B.

We also say that A is contained in B or B contains A.

We use the notation $A \subseteq B$ or $B \supseteq A$ to indicate that B contains A:

$$\forall x \in A \quad \Rightarrow \quad x \in B \tag{1.1}$$



1.2 Subsets of a set

2. Equality of sets:

If $A \subseteq B$ and $B \subseteq A$, we say A and B are equal and denote it by $A = B$, i.e., "$A = B$" means

$$\forall x \in A \quad \Rightarrow \quad x \in B, \tag{1.2}$$

and

$$\forall x \in B \quad \Rightarrow \quad x \in A. \tag{1.3}$$

Note that, essentially, (1.2) and (1.3) are the only way to show the two sets are equal.

1.2 Subsets of a set

3. Simple facts:

(1) $\forall$ set A, $A \subseteq A$;

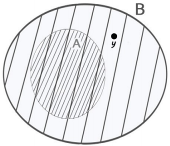
(2) $\forall$ set A, $\emptyset \subseteq A$ (convention);

(3) $\forall$ set A, $A \subseteq \Omega$ (under the consideration).

4. Proper subset:

If $A \subseteq B$ and $A \neq B$, we say that A is a proper subset of B. In other words, A is a proper subset of B means:

$$\forall x \in A \Rightarrow x \in B \quad \text{and} \quad \exists y \in B \text{ such that } y \notin A.$$



1.3 Operations of sets

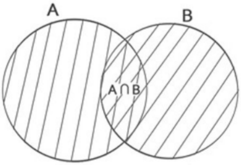
1. Union ($\cup$):

(1) Definition:

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

i.e. the set of elements that belong to either A or B.

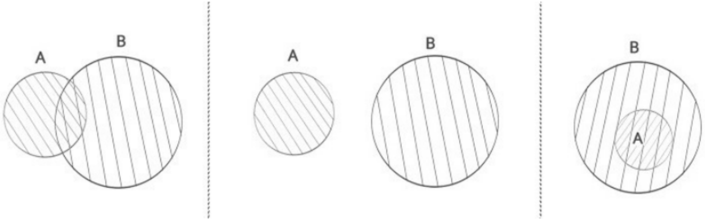
(2) Diagram:



$A \cup B$ is the shaded region.

1.3 Operations of sets

(3) Three possible cases:



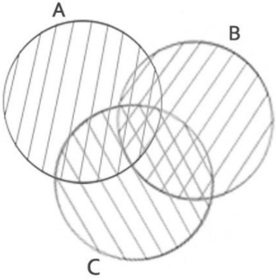
In the last case $A \subseteq B$ and so $A \cup B = B$.

1.3 Operations of sets

(4) Union of finitely many, or even infinitely many of sets:

For example, we may define the union of three sets:

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}$$



1.3 Operations of sets

(5) Laws:

$$
\begin{aligned}
A \cup B &= B \cup A && \text{(Commutative law)}\\
(A \cup B) \cup C &= A \cup (B \cup C) && \text{(Associative law)}\\
A \cup A &= A && \text{(Absorbing law)}\\
A \cup \emptyset &= A && (\because \emptyset \subseteq A)\\
A \cup \Omega &= \Omega && (\because A \subseteq \Omega)
\end{aligned}
$$

These laws can be easily proved.

1.3 Operations of sets

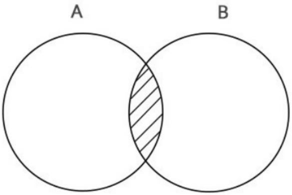
2. Intersection ($\cap$):

(1) Definition:

$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$

i.e. the set of elements that belong to both A and B.

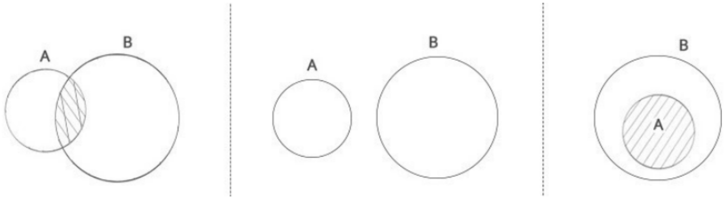
(2) Diagram:



$A \cap B$ is the shaded region.

1.3 Operations of sets

(3) Three possible cases:



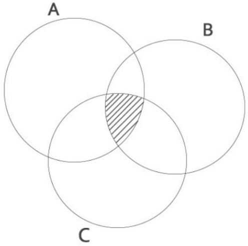
In the second case $A \cap B = \emptyset$.

In the third case $A \subset B$ and thus $A \cap B = A$.

(4) Intersection of finitely many, or even infinitely many of sets:

For example, we may define the intersection of three sets:

$$A \cap B \cap C = \{x : x \in A \text{ and } x \in B \text{ and } x \in C\}$$



$A \cap B \cap C$ is the shaded region.

Questions: What are $A \cap B$, $B \cap C$, and $A \cap C$?

(5) Laws:

$$
\begin{aligned}
A \cap B &= B \cap A && \text{(Commutative law)} \\
(A \cap B) \cap C &= A \cap (B \cap C) && \text{(Associative law)} \\
A \cap A &= A && \text{(Absorbing law)} \\
A \cap \emptyset &= \emptyset && (\because \emptyset \subseteq A) \\
A \cap \Omega &= A && (\because A \subseteq \Omega)
\end{aligned}
$$

These laws can be also easily proved.

(5) Distribution Laws:

For the operations of union and intersection, we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \tag{1.4}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{1.5}$$

Try to prove (1.4) and (1.5) yourself !!!

1.3 Operations of sets

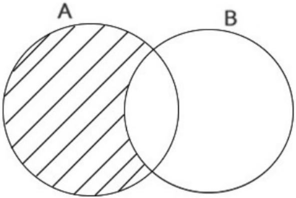
3. Difference ($\setminus$) (or just $-$):

(1) Definition:

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\},$$

i.e. the set of elements that belong to A but do not belong to B.

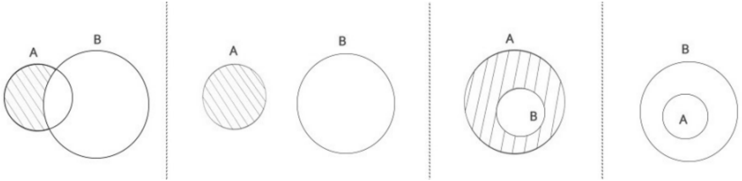
(2) Diagram:



$A \setminus B$ is the shaded region.

1.3 Operations of sets

(3) Four possible cases:



4. Complement (A^c):

(1) Definition: The difference of the universal set Ω and A is called the complement of A and denoted by A^c, i.e.

$$A^c = \{x : x \notin A\}$$

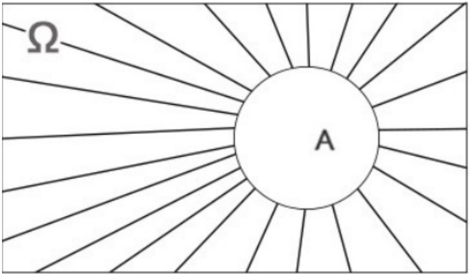
Note: universal set must be specified before talking the complement.

$$A^c = \Omega \setminus A = \{x : x \in \Omega \text{ and } x \notin A\} = \{x : x \notin A\}$$

since $x \in \Omega$ is always true.

1.3 Operations of sets

(2) Diagram:



A^c is the shaded region.

1.3 Operations of sets

(2) Laws:

$$
\begin{aligned}
(A^c)^c &= A, &\quad (A^c)^c &= \{x : x \notin A^c\} = \{x : x \in A\}, \\
\emptyset^c &= \Omega, &\quad \emptyset^c &= \{x : x \notin \emptyset\} = \Omega, \\
\Omega^c &= \emptyset, &\quad \Omega^c &= \{x : x \notin \Omega\} = \emptyset.
\end{aligned}
$$

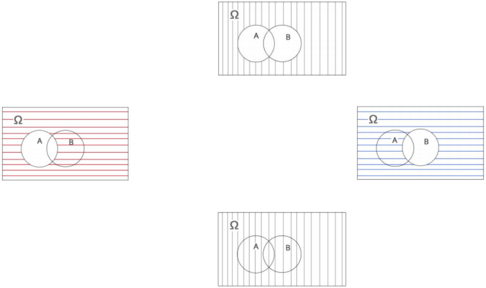
1.3 Operations of sets

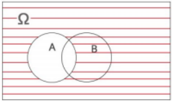
(3) De Morgan's Laws: (Important !!!)

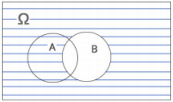
$$(A \cup B)^c = A^c \cap B^c, \tag{1.6}$$

$$(A \cap B)^c = A^c \cup B^c. \tag{1.7}$$

See the following diagrams: (Continued on the next slide)







Try to prove (1.6) and (1.7) yourself.

5. Operations of a family of sets

(1) Union: For the union of finitely many sets, we usually write it as

$$A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \cup_{i=1}^n A_i.$$

Suppose we have a sequence of sets $A_1, A_2, A_3, A_4, \cdots$. Then the union of this sequence of sets is defined as the elements that belong to at least one of $A_k, (k = 1, 2, 3, \cdots)$ and denoted as $\bigcup_{k=1}^{\infty} A_k$, i.e.,

$$\cup_{k=1}^{\infty} A_k = A_1 \cup A_2 \cup A_3 \cup \cdots = \{x : \exists\ k \text{ such that } x \in A_k\}.$$

Similarly we may define the union of any family of sets as

$$\cup_{i \in \mathcal{I}} A_i = \{x : \exists\ i \in \mathcal{I} \text{ such that } x \in A_i\}.$$

Operations of a family of sets

(2) Intersection:

Similarly, suppose we have a sequence of sets $A_1, A_2, A_3, A_4, \cdots$, then the intersection of this sequence of sets is defined as the elements that belong to each of $A_k, (k = 1, 2, 3, \cdots)$ and denoted as

$$\bigcap_{k=1}^{\infty} A_k = A_1 \cap A_2 \cap A_3 \cap \cdots = \{x : x \in A_k, \forall\ k\}.$$

Operations of a family of sets

(3) De Morgan's Laws:

$$\left(\bigcap_{k=1}^{\infty} A_k\right)^c = \bigcup_{k=1}^{\infty} A_k^c, \quad \left(\bigcup_{k=1}^{\infty} A_k\right)^c = \bigcap_{k=1}^{\infty} A_k^c, \tag{1.8}$$

$$\left(\bigcap_{k\in\mathcal{I}} A_k\right)^c = \bigcup_{k\in\mathcal{I}} A_k^c, \quad \left(\bigcup_{k\in\mathcal{I}} A_k\right)^c = \bigcap_{k\in\mathcal{I}} A_k^c, \tag{1.9}$$

where $\mathcal{I}$ is any index set. Try to prove (1.8)–(1.9) yourself.

Outline

2.1 Cartesian product

1. Ordered Pairs: A pair is called ordered if $(a, b) = (c, d)$ implies $a = c$ and $b = d$. In other words, usually, $(a, b) \neq (b, a)$.

2. Cartesian Product: Suppose A and B are two sets. Then the Cartesian product of A and B, denoted by $A \times B$, is defined to be the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

   Note that, usually, $A \times B \neq B \times A$.

3. Example:

$$A = \{1, 2\} \text{ and } B = \{2, 3, 4\},$$

then

$$\begin{aligned} A \times B &= \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\}, \\ B \times A &= \{(2,1),(2,2),(3,1),(3,2),(4,1),(4,2)\}. \end{aligned}$$

4. General cases:

   Ordered triple (a, b, c);

   Ordered n-tuple $(a_1, a_2, \cdots, a_n)$;

   Suppose $A_1, A_2, \cdots, A_n$ are sets, then the Cartesian product of $A_1, A_2, \cdots, A_n$, denoted by $A_1 \times A_2 \times A_3 \times \cdots \times A_n$, is the set of all ordered n-tuple, $(a_1, a_2, \cdots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \cdots,\ a_n \in A_n$, i.e.,

   $$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_1 \in A_1, \cdots, a_n \in A_n\}.$$

   If all the A_i are the same, then we write it as A^n, i.e.,

   $$A^n = A \times A \times \cdots \times A.$$

5. More examples: $\mathbb{R} = (-\infty, +\infty)$,

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) : a \in \mathbb{R} \text{ and } b \in \mathbb{R}\}.$$

Also, $\mathbb{R}^3$, $\mathbb{R}^n$, etc.

1. Basic Questions

(1) Problems: To discuss the "size" or "number" of sets. Try to answer the following questions: For two sets A and B,

"Do A and B have the same size?"

"Does A have more elements than B?"

In particular, answer the above questions for infinite sets.

For example, let $N = \{1, 2, 3, \cdots\}$ and $E = \{2, 4, 6, \cdots\}$, more elements in N? (E is a proper subset of N)

The problem is: how to compare?

(2) Idea: From finite case to infinite case:

One-to-One correspondence between the elements.

2.2 Cardinal number of sets

(3) Definition: Two sets A and B are said to have the same cardinal number iff there exists a bijection between A and B.

The cardinal number of A is denoted by $Card(A)$.

So, $Card(A) = Card(B)$ iff $\exists$ a bijection between the elements of A and B.

We say $Card(A) \leq Card(B)$ iff there exists a bijection between A and a subset of B.

Furthermore,

$$Card(A) < Card(B) \Longleftrightarrow Card(A) \leq Card(B) \text{ and } Card(A) \neq Card(B).$$

2.2 Cardinal number of sets

(4) More Questions:

(i) Does any infinite set have a cardinal number?

(ii) If "Yes", then comparable for any two sets? i.e., Does the following statement always hold true?

"For any two sets A and B, either $Card(A) \leq Card(B)$ or $Card(B) \leq Card(A)$".

(iii) If again "Yes" (in fact, this is equivalent to the so called "Axiom of Choice", then does there exist a "smallest infinity"?

If "Yes", which one? $\aleph_0$ say.

2.2 Cardinal number of sets

(iv) If "Yes" again, then does there exists a set A such that $Card(A) > \aleph_0$?

If yes, which one?

(v) "Largest infinity" ?

If No, "continued"? In particular, "the second smallest set"?

For Question (i) and (ii), <u>we shall say "Yes"</u>, but $\cdots$

For the answers for other questions, see below.

2. Countable sets:

(1) A set B is called countable if the $Card(B)$ is the same as the cardinal number of the natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$.

In other words, B is countable iff there exists a bijection between the elements of B and $\mathbb{N} = \{1, 2, 3, \cdots\}$.

The cardinal number of a countable set is denoted by $\aleph_0$.

(2) Properties:

Theorem 1.5.1. A set B is countable iff all the elements of B can be written as a sequence, i.e., $B = \{x_1, x_2, x_3, \cdots\}$.

Proof. By definition ($\exists$ a bijection between B and $\mathbb{N}$).

Theorem 1.5.2. If both A and B are countable, then so is $A \cup B$.

Moreover, if $A_1, A_2, \cdots, A_n$ are all countable, then so is $\cup_{i=1}^{n} A_i$.

Even more, if $A_1, A_2, A_3, \cdots$ are all countable, then so is $\cup_{k=1}^{\infty} A_k$.

Proof. Just need to prove the last statement, since the former two are more easy. By Theorem 1.5.1, all the elements of $A_1, A_2, A_3, \cdots$ can be written as sequences:

$$
\begin{aligned}
A_1 &: \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \cdots\} \\
A_2 &: \{a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \cdots\} \\
A_3 &: \{a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \cdots\} \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Now all the elements of $\cup_{k=1}^{\infty} A_k$ can be written as a sequence as well, for example,

$$\{a_{11}, \underbrace{a_{12}, a_{21}}, \overbrace{a_{13}, a_{22}, a_{31}}, \cdots\}$$

(diagonal elements).

Theorem 1.5.3. (the "smallest infinite set") Any infinite set contains a countable subset.

Proof. $A \neq \emptyset$. So we can choose $a_1 \in A_1$.

A infinite set, so $A \setminus \{a_1\} \neq \emptyset$ and we can choose $a_2 \in A \setminus \{a_1\}$.

In general, after choosing $a_1, a_2, \cdots, a_n$, then

$$A \setminus \{a_1, a_2, \cdots, a_n\} \neq \emptyset$$

(otherwise A is a finite set), we can therefore choose

$$a_{n+1} \in A \setminus \{a_1, a_2, \cdots, a_n\}.$$

So we can <u>extract a sequence</u> from A.

The meaning of Theorem 1.5.3:

For any infinite set A, A has a subset which is countable and thus there exist a bijection between the countable set and a subset of A

$$\Longrightarrow Card(A) \geq \aleph_0$$

i.e., $\aleph_0$ is the smallest cardinal number among the infinite sets.

Theorem 1.5.4. If $A_1, A_2, \cdots, A_n$ are all countable sets, then so is the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$. In particular, if A is countable, then so is A^n.

Proof. Similar to the proof of Theorem 1.5.2.

Warning: In Theorem 1.5.2, we have shown that if $A_1, A_2, A_3, \cdots$ is a sequence of countable sets, then so is $\cup_{k=1}^{\infty} A_k$. However, we can not say, $A_1 \times A_2 \times A_3 \times \cdots$ is also countable, see later.

(3) Examples of countable sets:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \cdots\}, \qquad E = \{2, 4, 6, 8, \cdots\}$$

Direct proof: $n \leftrightarrow 2n$. Same "size"? !! Astonishing?

More "sparse" examples:

$$F = \{10, 100, 1000, 10000, \cdots\}.$$

Even

$$G = \{10, 10^{10}, 10^{10^{10}}, 10^{10^{10^{10}}}, \cdots\}.$$

Can you image how sparse the G is? But any way, G is countable.

On the other hand, more "dense" examples?

Theorem 1.5.5. The set of all rational number is countable.

Proof. Just consider the positive (>0) rational numbers (Why? See Theorem 1.5.2).

Note that each positive rational number, r say, can be written as $\frac{m}{n}$, where both m and n are positive integers. Hence, all the positive rational numbers can be listed as follows

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \ldots$$
$$\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \ldots$$
$$\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots$$
$$\ldots\ldots\ldots$$

So, the set of all positive rational numbers is countable.

- How about the set of **all irrational numbers?**
- Interesting question! See later.
- If it were, then all real numbers $\mathbb{R} = (-\infty, +\infty)$ would be also countable (by Theorem 1.5.2).
- Before considering irrational numbers, let us first give another more "dense" example.

2.2 Cardinal number of sets

Theorem 1.5.6. The set of all algebraic numbers is countable.

Note: An algebraic number is a real number that is the root of some polynomial with integer coefficients.

Any rational number must be an algebraic number ($r = \frac{m}{n}$ is the root of $nx - m = 0$).

Many irrational numbers are also algebraic number. For example, $\sqrt{2}$ is the root of $x^2 - 2 = 0$. In fact, most of irrational numbers you know are algebraic numbers.

Now, does there exist a non-algebraic (transcendental) number??

Prove Theorem 1.5.6 by yourself.

3. Cardinal Number c:

Question: Does there exist any set A such that $Card(A) > \aleph_0$??

(1) Definition: The cardinal number of the set $[0,1]$ is denoted by c.

$[0,1] = \{x : 0 \leq x \leq 1\}$ is, of course, infinite and thus $c \geq \aleph_0$.

(2) The question is whether $c = \aleph_0$ or not !!

Theorem 1.5.7. The set $[0,1]$ is not countable. Hence $c > \aleph_0$.

Proof. Recall each real number in $[0,1]$ can be written as the form of $0.b_1b_2b_3\cdots$, where each b_i is a positive integer of $\{0,1,2,\cdots,9\}$.

Now, suppose $[0,1]$ is countable, then it can be written as a sequence (see Theorem 1.5.1) $\{x_1,x_2,x_3,\cdots\}$ say. Assume

$$\begin{aligned}
x_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots a_{1n}\cdots \\
x_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots a_{2n}\cdots \\
x_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots a_{3n}\cdots \\
&\cdots\cdots\cdots \\
x_n &= 0.a_{n1}a_{n2}a_{n3}a_{n4}\cdots a_{nn}\cdots \\
&\cdots\cdots\cdots
\end{aligned} \tag{2.1}$$

(remember all of the numbers in $[0,1]$ have been listed in the above), where a_{ij} are all one of the numbers $\{0,1,2,3,4,5,6,7,8,9\}$.

Now we define a number, x^*, say, as

$$x^* = 0.a_{*1}a_{*2}a_{*3}\cdots a_{*n}\cdots ,$$

where $a_{*1} \neq a_{11}, a_{*2} \neq a_{22}, \ldots, a_{*n} \neq a_{nn}, \cdots$ and all a_{*k}, $k \geq 1$, take value in $\{0,1,2,\cdots,9\}$.

Surely $x^* \in [0,1]$, but x^* is not be listed in (2.1) (since it equals neither of the x_n). This is a contradiction.

(3) Properties:

(i) $\forall i = 1, 2, \cdots, n,\ Card(A_i) = c \ \Rightarrow\ Card(\cup_{i=1}^{n} A_i) = c.$

(ii) $\forall i = 1, 2, \cdots,\ Card(A_i) = c \ \Rightarrow\ Card(\cup_{i=1}^{\infty} A_i) = c.$

(iii) $\forall i = 1, 2, \cdots, n,\ Card(A_i) = c$

$$\Rightarrow\ Card(A_1 \times A_2 \times A_3 \times \cdots \times A_n) = c.$$

In particular, $Card(A) = c \Rightarrow Card(A^n) = c.$

All these properties can be easily proven. But we shall omit the proofs here.

2.2 Cardinal number of sets

(4) Examples:

The following sets all have the cardinal number c:

$$[0,1]; \quad (0,1); \quad \mathbb{R} = (-\infty, +\infty); \quad \mathbb{R}_+ = [0, \infty); \quad \mathbb{R}^n.$$

Conclusion: The cardinal number of the set of irrational numbers is c.

The proof is given at the end of slides.

Conclusion: The cardinal number of the set of transcendental numbers is c.

Easily to be proved.

4. Maximal Cardinal Number? No!!

Can easily prove that: **There is no maximal cardinal number.**

But we shall ignore the proof here, since it involves introducing a concept of the so-called "power set".

5. Continuum Hypothesis:

(1) Question: Does there exist a cardinal number κ, say, such that

$$\aleph_0 < \kappa < c \text{ ??}$$

This is the famous Continuum Hypothesis (C.H).

(C.H. states: <u>No such kind of κ.</u>)

2.2 Cardinal number of sets

(2) Historical Notes:

George Cantor (1845-1918)
Between 1874-1897: Cantor published many papers on set theory.

Cantor conjectured that the continuum hypothesis is true.

David Hilbert later published a proof. But, unfortunately, the proof is incorrect – recognised by himself.

In 1939, Gödel proved that C.H. could not be disproved on the basis of our axioms for set theory.

In 1963, Paul Cohen proved that C.H. could not be proved on the basis of our axioms for set theory.

2.2 Cardinal number of sets

6. Remark on the term "countable"

$$\underbrace{\textit{finite set} - \textit{countable set}}_{\textit{countable (denumerable)}} - \textit{uncountable set}$$

7. Some Remarks for thinking:

(1) True for the following statement? Why?

"If there exists a bijection between the set A and a subset of B, then

$$Card(A) < Card(B),$$

<u>even a proper subset of the set B</u>."

2.2 Cardinal number of sets

(2) Meaning of the "there exists a bijection"

Does it mean: "we can find the exact form"?

(3) Is the following set countable?

"The set of all the sequences with 0 and 1 values only"

Not countable! Think why? (Binary digit $\cdots$).

2.2 Cardinal number of sets

Theorem. The cardinal number of the set of irrational numbers is c.

Proof. Denote by $\mathbb{R}$ and $\mathbb{Q}$ the set of real numbers and rational numbers respectively. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \left(\mathbb{Q} \cup (\mathbb{Q} + \sqrt{2})\right), \\ g(x), & \text{if } x \in \mathbb{Q} \cup (\mathbb{Q} + \sqrt{2}), \end{cases}$$

where $g : \mathbb{Q} \cup (\mathbb{Q} + \sqrt{2}) \to \mathbb{Q} + \sqrt{2}$ is a bijection. (Such a g exists since both the domain and the range of g are countable.)

The range of f is

$$\left(\mathbb{R} \setminus \left[\mathbb{Q} \cup (\mathbb{Q} + \sqrt{2})\right]\right) \bigcup \left[\mathbb{Q} + \sqrt{2}\right] = \mathbb{R} \setminus \mathbb{Q},$$

which means the map f is surjective.

Try to prove that f is injective by yourself.

Therefore the map f is bijective $\Rightarrow$ $Card(\mathbb{R} \setminus \mathbb{Q}) = Card(\mathbb{R}) = c$.