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MA215 Probability Theory

Assignment 13

1. (a) Suppose X is an continuous r.v. with p.d.f. $f_X(x)$. For any real value $-\infty < t < +\infty$, define a real-valued function, denoted by $M_X(t)$, as $M_X(t) = E(e^{tX})$. Further assume that $M_X(t)$ is well-defined for any $-\infty < t < +\infty$.
   (i) Write down the integration form of $M_X(t)$.
   (ii) If X is non-negative, show that $M_X(t)$ is a nondecreasing function of t.
   (iii) If X is non-negative, show that

$$\text{if } t < 0 \text{ then } 0 \leqslant M_X(t) \leqslant 1 \text{ and } M_X(0) = 1.$$

   (iv) If $Y = aX + b$ where a and b are two constants. Show that

$$M_Y(t) = e^{bt} M_X(at).$$

   (v) Suppose X and Y are two independent continuous r.v.s. Show that

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

(i) $M_X(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx$

(ii) $M_X'(t) = \dfrac{dE(e^{tX})}{dt} = \dfrac{d}{dt}\int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{+\infty} x e^{tx} f_X(x)\,dx$

$f_X(x) \geqslant 0$. $\because x \geqslant 0$. $\therefore x e^{tx} \geqslant 0$ $\therefore M_X'(t) \geqslant 0$

$\therefore M_X(t)$ is a nondecreasing function of t.

(iii) $M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx$ $\quad \because t<0,\ x \geqslant 0.$ $\therefore e^{tx} \leq 1$

$\therefore M_X(t) \leq \int_{-\infty}^{+\infty} f_X(x)\,dx = \lim_{x\to+\infty} F(x) = 1$

$\because e^{tx} \geqslant 0,\ f_X(x) \geqslant 0.$ $\quad \therefore M_X(t) \geqslant 0$

$M_X(0) = \int_{-\infty}^{+\infty} f_X(x)\,dx = \lim_{x\to+\infty} F(x) = 1$

(iv) $M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = \int_{-\infty}^{+\infty} e^{atx+bt} f_X(x)\,dx = e^{bt}\int_{-\infty}^{+\infty} e^{atx} f_X(x)\,dx$

$= e^{bt} E(e^{atX}) = e^{bt} M_X(at)$

(v) $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX} \cdot e^{tY}) = \int_{-\infty}^{+\infty} e^{tx} \cdot e^{ty} f_{X,Y}(x,y)\,dx\,dy$

$\because X, Y$ are independent $\quad \therefore f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$\therefore \int_{-\infty}^{+\infty} e^{tx} \cdot e^{ty} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{+\infty} e^{tx} \cdot e^{ty} f_X(x)\,dx \cdot f_Y(y)\,dy = \int_{-\infty}^{+\infty} e^{tx} \cdot f_X(x)\,dx \cdot$

$\int_{-\infty}^{+\infty} e^{ty} \cdot f_Y(y)\,dy = E(e^{tX}) E(e^{tY}) = M_X(t) \cdot M_Y(t)$

(b) Suppose X is a discrete r.v. with p.m.f. $p_k = P(X = x_k)$, $k \geqslant 1$. For any real value $-\infty < t < +\infty$, define a real-valued function, denoted by $M_X(t)$, as $M_X(t) = E(e^{tX})$. Further assume that $M_X(t)$ is well-defined for any $t \in \mathbb{R}$.

(i) Write down the series form of $M_X(t)$.

(ii) If X is non-negative, show that $M_X(t)$ is a nondecreasing function of t.

(iii) If X is non-negative, show that

$$\text{if } t < 0 \text{ then } 0 \leqslant M_X(t) \leqslant 1 \text{ and } M_X(0) = 1.$$

(iv) If $Y = aX + b$ where a and b are two constants. Show that

$$M_Y(t) = e^{bt} M_X(at).$$

(v) Suppose X and Y are two independent discrete r.v.s. Show that

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

(i) $M_X(t) = E(e^{tX}) = \sum_{k=1}^{\infty} e^{tx_k} p_k$

(ii) $M_X'(t) = \frac{d}{dt} \sum_{k=1}^{\infty} e^{tx_k} p_k = \sum_{k=1}^{\infty} x_k e^{tx_k} p_k$ $\quad \because x_k, p_k \geqslant 0. \quad \therefore M_X'(t) \geqslant 0$

$\therefore M_X(t)$ is a nondecreasing function of t.

(iii) $\because t<0,\ x_k \geqslant 0. \quad \therefore 0 < e^{tx_k} \leqslant 1$

$\therefore 0 \leqslant M_X(t) \leqslant \sum_{k=1}^{\infty} p_k = 1 \qquad M_X(0) = \sum_{k=1}^{\infty} p_k = 1.$

(iv) $M_Y(t) = E(e^{(ax+b)t}) = \sum_{k=1}^{\infty} e^{bt} \cdot e^{atx_k} p_k = e^{bt} \sum_{k=1}^{\infty} e^{atx_k} p_k = e^{bt} E(e^{atX}) = e^{bt} M_X(at)$

(v) $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX} \cdot e^{tY}) = \sum_{k=1}^{\infty} e^{tx_k} \cdot e^{ty_k} \cdot P(X=x_k, Y=y_k)$

$\because X, Y$ are independent. $\quad \therefore P(X=x_k, Y=y_k) = P(X=x_k) P(Y=y_k)$

$\therefore M_{X+Y}(t) = \sum_{k=1}^{\infty} e^{tx_k} P(X=x_k) \sum_{k=1}^{\infty} e^{ty_k} P(Y=y_k) \quad (P(X=x_i)P(Y=y_j)=0\ (i \neq j))$

$= E(e^{tX}) E(e^{tY}) = M_X(t) M_Y(t)$

2. Find the m.g.f. of

(i) the discrete random variable X with $P(X = 4) = 1$;

(ii) the Bernoulli random variable with parameter p $(0 < p < 1)$, and then applying the properties of m.g.f. to find the m.g.f. of the Binomial random variable with parameter p $(0 < p < 1)$ and n where n is a positive integer;

(iii) the Poisson random variable with parameter $\lambda > 0$;

(iv) the Geometric random variable with parameter p $(0 < p < 1)$, and then applying the properties of m.g.f. to find the m.g.f. of the Negative Binomial random variable with parameter p and r where r is a positive integer.

(v) the continuous random variable Y with probability density function

$$f_Y(y) = \begin{cases} 2y, & 0 \leqslant y \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

(vi) the random variable $X \sim U[a, b]$ $(-\infty < a < b < +\infty)$.

(vii) the exponential random variable with parameter $\lambda > 0$, and then applying the properties of m.g.f. to find the m.g.f. of the Gamma random variable with parameter $\lambda > 0$ and m where m is a positive integer;

(viii) the general Gamma random variable with parameter $\lambda > 0$ and α, where $\alpha > 0$ may NOT be a positive integer;

(ix) the standard normal random variable $Z \sim N(0, 1)$; Define $X = \mu + \sigma Z$ for real numbers μ, σ with $\sigma > 0$, use the properties of m.g.f. $M_Z(t)$ to find the m.g.f. $M_X(t)$ of X.

(i) $M_X(t) = \sum_{k=1}^{1} e^{tx_k} P(X=x_k) = e^{4t} P(X=4) = e^{4t}$

(ii) Bernoulli: $P(X=1)=p$. $P(X=0)=1-p$

$\therefore M_X(t) = \sum_{k=1}^{2} e^{tx_k} P(X=x_k) = 1 \cdot (1-p) + e^t \cdot p = pe^t - p + 1$

Binomial: $P(X=k) = C_n^k p^k (1-p)^{n-k}$, $k=0,1,\cdots,n$

$\therefore M_X(t) = \sum_{k=1}^{n+1} e^{tx_k} \cdot C_n^{x_k} p^{x_k} (1-p)^{n-x_k} = \sum_{k=0}^{n} C_n^k (pe^t)^k (1-p)^{n-k} = (pe^t + 1 - p)^n$

(iii) $p_k = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k=0,1,2,\cdots$

$\therefore M_X(t) = \sum_{k=0}^{\infty} e^{tk} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{\lambda(e^t-1)}$

(iv) Geometric: $P(X=k) = (1-p)^{k-1} p$, $k=1,2,3,\cdots$

$\therefore M_X(t) = \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} p = pe^t \sum_{k=1}^{\infty} (e^t(1-p))^{k-1} = pe^t \cdot \frac{1}{1-e^t(1-p)} = \frac{pe^t}{1-(1-p)e^t}$

$X \sim NB(p,r) \Rightarrow P(X=k) = C_{k-1}^{r-1} p^r (1-p)^{k-r}$, $k=r, r+1, r+2, \cdots$

$\therefore M_X(t) = \sum_{k=r}^{\infty} e^{tk} C_{k-1}^{r-1} p^r (1-p)^{k-r} = \sum_{k=r}^{\infty} C_{k-1}^{r-1} \frac{((1-p)e^t)^k p^r}{(1-p)^r} = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r$

(v) $M_Y(t) = E(e^{tY}) = \int_{-\infty}^{+\infty} e^{ty} f_Y(y)\,dy = \int_0^1 2ye^{ty}\,dy = \frac{2}{t}\left(ye^{ty} - \frac{1}{t}e^{ty}\right)\Big|_0^1 = \frac{2}{t}e^t - \frac{2}{t^2}e^t + \frac{2}{t^2}$

(vi) $f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$

$\therefore M_X(t) = E(e^{Xt}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx = \int_a^b \frac{1}{b-a} e^{tx}\,dx = \frac{e^{bt} - e^{at}}{(b-a)t}$

(vii) $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

$\therefore M_X(t) = \int_{-\infty}^{+\infty} e^{tx} \cdot f_X(x)\,dx = \int_0^{+\infty} \lambda \cdot e^{(t-\lambda)x}\,dx = -\frac{e^{-(\lambda-t)x}}{\lambda - t}\Big|_0^{+\infty} = \frac{\lambda}{\lambda - t}$, $t < \lambda$

when $t \ge \lambda$, $M_X(t)$ does not exist.

Gamma: $f_Y(y) = \frac{\lambda^m y^{m-1} e^{-\lambda y}}{(m-1)!}$, $y \ge 0$

$\therefore M_Y(t) = \int_0^{+\infty} e^{ty} \cdot \frac{\lambda^m y^{m-1} e^{-\lambda y}}{(m-1)!}\,dy = \left(\frac{\lambda}{\lambda - t}\right)^m$, $t < \lambda$

(viii) $f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$, $x \ge 0$, $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\,dt$

$\therefore M_X(t) = \int_0^{\infty} e^{tx} f_X(x)\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\lambda-t)x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \frac{\lambda^\alpha}{(\lambda-t)^\alpha}$, $t < \lambda$

(ix) $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$M_Z(t) = \int_{-\infty}^{+\infty} e^{tz} f_Z(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2 - 2tz}{2}}\,dz = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{t^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{(z-t)^2}{2}}\,dz = e^{\frac{t^2}{2}}$

$\therefore M_X(t) = E(e^{t(\mu+\sigma Z)}) = e^{t\mu} \cdot M_Z(t\sigma) = e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} = e^{t\mu + \frac{t^2\sigma^2}{2}}$

3. Suppose that the m.g.f. of a r.v. X is given by $M_X(t) = e^{3(e^t-1)}$. What is the probability $P(X=0)$? Also, find $E(X)$ and $\operatorname{Var}(X)$. (Hint: You do not need to do any detailed calculations. Just find what the distribution of the r.v. X is and then use the known results to answer this question.)

Notice that if $Y \sim \text{Poisson}(\lambda)$, then $M_Y(t) = E(e^{tY}) = \sum_{k=0}^{\infty} e^{tk} \cdot e^{-\lambda} \frac{\lambda^k}{k!}$

$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{\lambda(e^t-1)}$

$\because M_X(t) = e^{3(e^t-1)}$ $\quad \therefore X \sim \text{Poisson}(3)$

$\therefore P(X=0) = \frac{3^0 \cdot e^{-3}}{0!} = e^{-3}$

$E(X) = \sum_{k=0}^{\infty} k \cdot e^{-3} \cdot \frac{3^k}{k!} = \sum_{k=1}^{\infty} e^{-3} \frac{3^k}{(k-1)!} = 3 \cdot \sum_{k=0}^{\infty} P(X=k) = 3$

$E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot e^{-3} \cdot \frac{3^k}{k!} = 3 \cdot \sum_{k=1}^{\infty} k \cdot e^{-3} \cdot \frac{3^{k-1}}{(k-1)!} = 3 \cdot \sum_{k=0}^{\infty} (k+1) e^{-3} \cdot \frac{3^k}{k!}$

$= 3 \cdot \left(\sum_{k=0}^{\infty} k e^{-3} \cdot \frac{3^k}{k!} + \sum_{k=0}^{\infty} e^{-3} \cdot \frac{3^k}{k!} \right) = 3(E(X)+1) = 12$

$\therefore \operatorname{Var}(X) = E(X^2) - E(X)^2 = 3$