Subspaces (子空间)

Lecture 2

Dept. of Math.

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Subspaces (子空间)

Subspace

By considering subspaces, we can greatly expand our examples of vector spaces.

Definition

A subset U of V is called a subspace of V if U is also a vector space(using the same addition and scalar multiplication as on V).

Example

$$\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{F}\}$$

is a subspace of $\mathbb{F}^3$.

Some mathematicians use the term linear subspace, which means the same as subspace.

Conditions for a subspace

The next result gives the easiest way to check whether a subset of a vector space is a subspace.

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

(1) *Additive Identity:* $0 \in U$.

(2) *Closed under addition:* $u, w \in U$ *implies* $u + w \in U$.

(3) *Closed under scalar multiplication:* $a \in \mathbb{F}$ *and* $u \in U$ *implies* $au \in U$.

The three conditions in the result above usually enable us to determine quickly whether a given subset of V is a subspace of V.

Examples

The three conditions in the result above usually enable us to determine quickly whether a given subset of V is a subspace of V.

(a) If $b \in \mathbf{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

(b) The set of continuous real-valued functions on the interval $[0,1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

(d) The set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^{\infty}$.

Subspaces: Typical Examples

(1) $\{0\}$ is the smallest subspace of V and V itself is the largest subspace.

(2) The subspaces of $\mathbb{R}^2$ are precisely $\{0\}, \mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin.

(3) The subspaces of $\mathbb{R}^3$ are precisely $\{0\}, \mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin. Can you figure out why as well?

Intersection of subspaces

Theorem

Any intersection of subspaces of a vector space V is a subspace of V.

Proof.

Let C be a collection of subspaces of V, and let W denote the intersection of subspaces in C.

Since every subspace contains the zero vector, $0 \in W$.

Let $a \in \mathbb{F}$ and $x, y \in W$. Then x and y are contained in each subspace in C.

Because each subspace in C is closed under addition and scalar multiplication, it follows that $x+y$ and ax are contained in each subspace in C.

Hence $x+y$ and ax are also contained in W, so that W is a subspace of V. □

Sums of Subspaces

The union of subspaces is rarely a subspace(see Exercise 12), which is why we usually work with sums rather than unions.

When dealing with vector spaces, we are usually interested only in subspaces as opposed to arbitrary subsets.

1.36 **Definition** ***sum of subsets***

Suppose $U_1, \ldots, U_m$ are subsets of V. The ***sum*** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}.$$

Remark

Sums of subspaces in the theory of vector spaces are analogous to unions of subsets in set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union.

Examples

1.37 **Example** Suppose U is the set of all elements of $\mathbf{F}^3$ whose second and third coordinates equal 0, and W is the set of all elements of $\mathbf{F}^3$ whose first and third coordinates equal 0:

$$U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, y, 0) \in \mathbf{F}^3 : y \in \mathbf{F}\}.$$

Then

$$U + W = \{(x, y, 0) : x, y \in \mathbf{F}\},$$

as you should verify.

1.38 **Example** Suppose that $U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$ and $W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$. Then

$$U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\},$$

as you should verify.

Sum of Subspaces

Proposition

Suppose $U_1, \cdots, U_m$ are subspaces of V. Then $U_1 + U_2 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \cdots, U_m$.

Proof.

It is easy to see that $0 \in U_1 + U_2 + \cdots + U_m$ and that $U_1 + U_2 + \cdots + U_m$ is closed under addition and scalar multiplication.
Therefore, $U_1 + U_2 + \cdots + U_m$ is a subspace of V.
Clearly $U_1, \cdots, U_m$ are all contained in $U_1 + U_2 + \cdots + U_m$. Conversely, every subspace of V containing $U_1, \cdots, U_m$ contains $U_1 + U_2 + \cdots + U_m$.
Thus $U_1 + U_2 + \cdots + U_m$ is the **smallest** subspace of V containing $U_1, \cdots, U_m$.

$\square$

Examples

Example

Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Example

Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Direct Sums (直和)

We will be especially interested in cases where each vector in $U_1 + U_2 + \cdots + U_m$ can be represented in the form above in only one way.

1.40 **Definition** ***direct sum***

Suppose $U_1, \ldots, U_m$ are subspaces of V.

- The sum $U_1 + \cdots + U_m$ is called a ***direct sum*** if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

Examples

1.41 **Example** Suppose U is the subspace of $\mathbf{F}^3$ of those vectors whose last coordinate equals 0, and W is the subspace of $\mathbf{F}^3$ of those vectors whose first two coordinates equal 0:

$$U = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\}.$$

Then $\mathbf{F}^3 = U \oplus W$, as you should verify.

1.42 **Example** Suppose U_j is the subspace of $\mathbf{F}^n$ of those vectors whose coordinates are all 0, except possibly in the j^{th} slot (thus, for example, $U_2 = \{(0, x, 0, \ldots, 0) \in \mathbf{F}^n : x \in \mathbf{F}\}$). Then

$$\mathbf{F}^n = U_1 \oplus \cdots \oplus U_n,$$

as you should verify.

Nonexample

The definition of direct sum requires that every vector in the sum has a unique representation as an appropriate sum. The symbol $\oplus$, which is a plus sign inside a circle, serves as a reminder that we are dealing with a special type of sum of subspaces–each element in the direct sum can be represented only way as a sum of elements from the specified subspaces.

Sometimes nonexamples add to our understanding as much as examples.

1.43 **Example** Let

$$U_1 = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\},$$
$$U_2 = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\},$$
$$U_3 = \{(0, y, y) \in \mathbf{F}^3 : y \in \mathbf{F}\}.$$

Show that $U_1 + U_2 + U_3$ is not a direct sum.

Solution

Solution Clearly $\mathbf{F}^3 = U_1 + U_2 + U_3$, because every vector $(x, y, z) \in \mathbf{F}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0),$$

where the first vector on the right side is in U_1, the second vector is in U_2, and the third vector is in U_3.

However, $\mathbf{F}^3$ does not equal the direct sum of U_1, U_2, U_3, because the vector $(0, 0, 0)$ can be written in two different ways as a sum $u_1 + u_2 + u_3$, with each u_j in U_j. Specifically, we have

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1)$$

and, of course,

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0),$$

where the first vector on the right side of each equation above is in U_1, the second vector is in U_2, and the third vector is in U_3.

Condition for a direct sum

Proposition

Suppose $U_1, \cdots, U_m$ are subspaces of V. Then $U_1 + U_2 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

Proof.

First suppose $U_1 + U_2 + \cdots + U_m$ is a direct sum. Then the definition of direct sum implies that the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.
Now suppose that the only way to write 0 as a sum $u_1 + \cdots + u_m$,where each u_j is in U_j, is by taking each u_j equal 0. To show that $U_1 + \cdots + U_m$ is a direct sum, let $v \in U_1 + \cdots + U_m$. We can write $v = u_1 + \cdots + u_m$, for some $u_1 \in U_1, \cdots, u_m \in U_m$. □

Proof

Proof.

(Continue)To show that this representation is unique, suppose we also have $v = v_1 + \cdots + v_m$, where $v_1 \in U_1, \cdots, v_m \in U_m$. Subtracting these two equations, we have

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

Because $u_1 - v_1 \in U_1, \cdots, u_m - v_m \in U_m$, the equation above implies that each $u_j - v_j$ equals 0. Thus $u_1 = v_1, \cdots, u_m = v_m$, as desired. □

Direct Sums of two subspaces

Proposition

Suppose U and W are subspaces of V. Then $U+W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof.

First suppose that $U+W$ is a direct sum. If $v \in U \cap W$, then $0 = v + (-v)$ where $v \in U$ and $-v \in W$. By the unique representation of 0 as the sum of a vector in U and a vector in W, we have $v = 0$. Thus $U \cap W = \{0\}$, completing the proof in one direction.

To prove the other direction, now suppose $U \cap W = \{0\}$. To prove that $U+W$ is a direct sum, suppose $u \in U, w \in W$, and $0 = u + w$. To complete the proof, we need only show that $u = w = 0$. The equation above implies that $u = -w \in W$. Thus $u \in U \cap W$. Hence $u = 0$, which by the equation above implies that $w = 0$, completing the proof. □

Final Note

- Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint, because both contain 0. So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals $\{0\}$.

Final Note

- Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint, because both contain 0. So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals $\{0\}$.
- The result above deals only with the case of two subspaces. When asking about a possible direct sum with more than two subspaces, it is NOT enough to test that each pair of the subspaces intersect only at 0. To see this, consider example 1.43. In that nonexample of a direct sum, we have

$$U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}.$$

However, $U_1 + U_2 + U_3$ is NOT a direct sum.

Homework Assignment 2

1.C: 7, 9, 14, 17, 19, 20, 22, 24.