1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题. 每题只有一个选项是正确的.

(1) Let A, B, C be $n \times n$ matrices with B invertible and $AB = C$. Which of the following must be true? ( )

(A) The row spaces of A and C are the same.

(B) The null spaces of A and C are the same.

(C) The column spaces of A and C are the same.

(D) The determinants of A and C are the same.

设 A, B, C 为 $n \times n$ 矩阵, 其中 B 可逆且 $AB = C$. 下列陈述一定正确的是 ( )

(A) A 和 C 的行空间相同.

(B) A 和 C 的零空间相同.

(C) A 和 C 的列空间相同.

(D) A 和 C 的行列式相同.

(2) Let P be a 5×5 permutation matrix. Which of the following is **false**? ( )

(A) P is an orthogonal matrix.

(B) P must have real eigenvectors.

(C) There always exists an invertible real matrix Q such that $Q^{-1}PQ$ is diagonal.

(D) The equation $Px = 0$ has only zero solution.

设 P 为 5×5 置换矩阵. 下列陈述错误的是 ( )

(A) P 是正交矩阵.

(B) P 一定有实特征向量.

(C) 存在可逆的实矩阵 Q 使得 $Q^{-1}PQ$ 为对角阵.

(D) 方程 $Px = 0$ 仅有零解.

(3) Let A be an $n \times n$ real symmetric matrix. Which of the following statements must be true? ( )

(A) A must have n distinct eigenvalues.

(B) Some of the complex eigenvalues of A need not be real.

(C) Any n linearly independent eigenvectors of A are pairwise orthogonal.

(D) There is an orthogonal matrix Q, such that Q^TAQ is diagonal.

设 A 为 $n \times n$ 实对称矩阵. 则下列陈述一定正确的是 ( )

(A) A 一定有 n 个互不相同的特征值.

(B) A 的一些复特征值可能不是实数.

(C) A 的任意 n 个线性无关的特征向量两两正交.

(D) 存在正交矩阵 Q, 使得 Q^TAQ 为对角矩阵.

(4) Let

$$\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \beta_1 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix},\ \beta_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

If γ can be written as a linear combination of α_1, α_2, and γ can also be written as a linear combination of β_1, β_2, then γ has the form

(A) $k \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}, k \in \mathbb{R}.$

(B) $k \begin{bmatrix} 3 \\ 5 \\ 10 \end{bmatrix}, k \in \mathbb{R}.$

(C) $k \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, k \in \mathbb{R}.$

(D) $k \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, k \in \mathbb{R}.$

已知向量

$$\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},\ \beta_1 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix},\ \beta_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

若 γ 既可由 α_1, α_2 线性表示, 也可由 β_1, β_2 线性表示, 则 γ 形如

(A) $k \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}, k \in \mathbb{R}.$

(B) $k \begin{bmatrix} 3 \\ 5 \\ 10 \end{bmatrix}, k \in \mathbb{R}.$

(C) $k \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, k \in \mathbb{R}.$

(D) $k \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, k \in \mathbb{R}.$

(5) Which of the following matrices is congruent to the identity matrix? ( )

(A) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$

(B) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 8 \end{bmatrix}$.

(C) $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -3 \\ 2 & -3 & -4 \end{bmatrix}$.

(D) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

下列矩阵中合同于单位阵的是 ( )

(A) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 8 \end{bmatrix}$.

(C) $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -3 \\ 2 & -3 & -4 \end{bmatrix}$.

(D) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

2. (15 points, 3 points each) Fill in the blanks. (共 15 分, 每小题 3 分) 填空题.

(1) Let A be a 2×2 matrix, which has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $A^2(\mathbf{v}_1 - \mathbf{v}_2) = 2\mathbf{v}_1 + \mathbf{v}_2$. Then $\det(A^4) =$ __________.

设 A 为 2×2 矩阵, 它有两个线性无关的特征向量 $\mathbf{v}_1$ 和 $\mathbf{v}_2$ 满足 $A^2(\mathbf{v}_1 - \mathbf{v}_2) = 2\mathbf{v}_1 + \mathbf{v}_2$. 则 $\det(A^4) =$ __________.

(2) The singular values of the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$ are __________.

矩阵 $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$ 的奇异值是 __________.

(3) Let A be a 3×3 matrix which has eigenvalues -1, 0, 1. Suppose that $(A + aI_3)A(A - bI_3) = 0$, where I_3 is the 3×3 identity matrix. Then $a =$ __________, $b =$ __________.

设 A 为 3×3 矩阵, 它以 $-1, 0, 1$ 为特征值. 假设 $(A + aI_3)A(A - bI_3) = 0$, 其中 I_3 为 3×3 单位矩阵. 则 $a =$ __________, $b =$ __________.

(4) If $A = \begin{bmatrix} 0 & 0 & 1 \\ x & 1 & 2x-3 \\ 1 & 0 & 0 \end{bmatrix}$ is diagonalizable, then $x =$ __________.

假设矩阵 $A = \begin{bmatrix} 0 & 0 & 1 \\ x & 1 & 2x-3 \\ 1 & 0 & 0 \end{bmatrix}$ 可对角化, 则 $x =$ __________.

(5) Let A be a 4×4 symmetric matrix such that $A^2 + A = 0$. Suppose that A has rank 3. A diagonal matrix that is similar to A is __________.

假设 4×4 对称矩阵 A 满足 $A^2 + A = 0$. 假设 A 的秩为 3. 与 A 相似的一个对角阵是 __________.

3. (20 points) Let A_n be the $n \times n$ matrix

$$A_n = \begin{bmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{bmatrix}$$

(a) Find constants b, c such that the sequence $\det(A_n)$ satisfies

$$\det(A_n) = b \cdot \det(A_{n-1}) + c \cdot \det(A_{n-2}) \quad \text{for all } n \geq 3\,.$$

(b) Find a matrix B such that $\mathbf{x}_n = B\mathbf{x}_{n-1}$ for $n \geq 3$, where $\mathbf{x}_n = \begin{bmatrix} \det(A_n) \\ \det(A_{n-1}) \end{bmatrix}$.

(c) For $a^2 = \frac{3}{16}$, find an expression for $\det(A_n)$ for all $n \geq 3$.

(20 分) 设 A_n 为以下 $n \times n$ 矩阵:

$$A_n = \begin{bmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{bmatrix}$$

(a) 求常数 b, c 使得数列 $\det(A_n)$ 满足

$$\text{对任意 } n \geq 3, \quad \det(A_n) = b \cdot \det(A_{n-1}) + c \cdot \det(A_{n-2}).$$

(b) 找出一个矩阵 B 使得 $\mathbf{x}_n = B\mathbf{x}_{n-1}$ 对所有 $n \geq 3$ 成立, 其中 $\mathbf{x}_n = \begin{bmatrix} \det(A_n) \\ \det(A_{n-1}) \end{bmatrix}$.

(c) 假设 $a^2 = \frac{3}{16}$. 对于任意正整数 $n \geq 3$, 求出 $\det(A_n)$ 的表达式.

4. (20 points) Suppose $\alpha,\ \theta \in (0, \pi/2)$.

(a) Compute $A_n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^n$ for all $n \geq 1$.

(b) Find a singular value decomposition (SVD) of A_n for each $n \geq 1$.

(c) Show that the matrix A_1 is symmetric if and only if $\alpha = \theta$.

(*Hint: the formula* $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$ *may be useful.*)

(d) Prove that if A_1 is symmetric, then A_n is positive definite for every $n \geq 1$.

(20 分) 设 $\alpha,\ \theta \in (0, \pi/2)$.

(a) 对所有 $n \geq 1$, 计算 $A_n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^n$.

(b) 对每个 $n \geq 1$, 求 A_n 的一个奇异值分解 (SVD).

(c) 证明: 矩阵 A_1 是对称矩阵当且仅当 $\alpha = \theta$.

(提示: 公式 $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$ 可能会有用.)

(d) 证明: 如果 A_1 是对称阵, 那么对每一个 $n \geq 1$, 矩阵 A_n 是正定的.

5. (10 points) Consider the quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 2x_3^2 - 4x_1x_3$.

   (a) Find the symmetric matrix A such that $f(x) = x^T Ax$ for all $x = (x_1, x_2, x_3)^T$, and find an orthogonal matrix Q such that $Q^T AQ$ is a diagonal matrix.

   (b) The quadric surface defined by the equation $f(x, y, z) = 2023$ is ____________.

   (A) a hyperboloid of one sheet (B) a hyperboloid of two sheets (C) an ellipsoid (D) none of the above.

   (10 分) 考虑二次型 $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 2x_3^2 - 4x_1x_3$.

   (a) 求对称矩阵 A 使得对任意 $x = (x_1, x_2, x_3)^T$ 均有 $f(x) = x^T Ax$, 再求一个正交矩阵 Q 使得 $Q^T AQ$ 为对角阵.

   (b) 由方程 $f(x, y, z) = 2023$ 定义的二次曲面是 ____________.

   (A) 一个单叶双曲面 (B) 一个双叶双曲面 (C) 一个椭球面 (D) 以上都不是.

6. (20 points) For any $a = (a_1, \cdots, a_n)^T \in \mathbb{R}^n$, put $\|a\| = \sqrt{a_1^2 + \cdots + a_n^2}$. Let $x, y \in \mathbb{R}^n$ be nonzero vectors.

   (a) Show that if there is an orthogonal matrix S such that $Sx = y$, then $\|x\| = \|y\|$.

   (b) Let N be the null space $N(x^T)$ of the $1 \times n$ matrix x^T. Show that $\dim N = n - 1$.

   (c) Let $\alpha_2, \cdots, \alpha_n$ be a basis of N. Show that the system $\alpha_1 := x, \alpha_2, \cdots, \alpha_n$ is linearly independent.

   (d) Let A be the matrix with $\alpha_1, \alpha_2, \cdots, \alpha_n$ as its columns. Let $A = QR$ be a factorization with Q orthogonal and R upper triangular. Write $R = (r_{ij})$. Show that $|r_{11}| = \|x\|$.

   (e) Prove that if $\|x\| = \|y\|$, then there exists an orthogonal matrix S such that $Sx = y$.

   (20 分) 对任意 $a = (a_1, \cdots, a_n)^T \in \mathbb{R}^n$, 令 $\|a\| = \sqrt{a_1^2 + \cdots + a_n^2}$. 设 $x, y \in \mathbb{R}^n$ 为非零向量.

   (a) 证明: 如果存在正交矩阵 S 使得 $Sx = y$, 则 $\|x\| = \|y\|$.

   (b) 设 N 为 $1 \times n$ 矩阵 x^T 的零空间 $N(x^T)$. 证明: $\dim N = n - 1$.

   (c) 设 $\alpha_2, \cdots, \alpha_n$ 为 N 的一组基. 证明: 向量组 $\alpha_1 := x, \alpha_2, \cdots, \alpha_n$ 是线性无关的.

   (d) 设 A 是以 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 为列的矩阵. 设 $A = QR$ 为一种分解式, 其中 Q 是正交矩阵, R 是上三角矩阵. 记 $R = (r_{ij})$. 证明: $|r_{11}| = \|x\|$.

   (e) 证明: 如果 $\|x\| = \|y\|$, 那么存在正交矩阵 S 使得 $Sx = y$.