李轩然 12312110

MA215 Probability Theory

Assignment 09

1. Suppose that the continuous r.v.s X and Y have joint p.d.f

$$f(x,y)=\begin{cases} x+y, & 0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1,\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the joint c.d.f. of the random vector (X,Y).

(b) Find the marginal p.d.f.s of X and Y.

(c) Find $P(X>Y)$.

(d) Find $P(X\leqslant 0.5)$.

(a) When $x\leq 0$, or $y\leq 0$, $f(x,y)=0$, $\therefore F(X,Y)=0$

when $x\geq 1, y\geq 1$, $\because f(x,y)=0, (x\geq 1, y\geq 1)$

$\therefore F(x,y)=\int_0^1\int_0^1 (x+y)\,dx\,dy=1$

When $0<x<1, 0<y<1$, $F(x,y)=\int_0^x dx\int_0^y (x+y)\,dy=\int_0^x(\frac{y^2}{2}+xy)\,dx$

$=\frac{1}{2}(xy^2+x^2y)$

When $0<x<1, y\geq 1$, $F(x,y)=\int_0^x dx\int_0^1 (x+y)\,dy=\frac{1}{2}(x+x^2)$

When $0<y<1, x\geq 1$. $F(x,y)=\int_0^y dy\int_0^1 (x+y)\,dx=\frac{1}{2}(y+y^2)$

$$\therefore F(x,y)=\begin{cases} 0, & x\leq 0 \text{ or } y\leq 0\\ \frac{1}{2}(x+x^2), & 0<x<1, y\geq 1\\ \frac{1}{2}(y+y^2), & x\geq 1, 0<y<1\\ \frac{1}{2}(xy^2+x^2y), & 0<x,y<1\\ 1, & x\geq 1, y\geq 1\end{cases}$$

(b) $0\leq x\leq 1$, $f_X(x)=\int_{-\infty}^0 f(x,y)\,dy+\int_0^1 f(x,y)\,dy+\int_1^{+\infty}f(x,y)\,dy=\int_0^1 f(x,y)\,dy$

$=\int_0^1 (x+y)\,dy=x+\frac{1}{2}$

$$\therefore f_X(x)=\begin{cases} x+\frac{1}{2}, & 0\leq x\leq 1\\ 0, & \text{otherwise}\end{cases}$$

using the same method, $f_Y(y)=\begin{cases} y+\frac{1}{2}, & 0\leq y\leq 1\\ 0, & \text{otherwise}\end{cases}$

(c) $P(X>Y)=F(X>Y)=\int_0^1 dx\int_0^x (x+y)\,dy=\int_0^1\frac{3}{2}x^2\,dx=\frac{x^3}{2}\Big|_0^1=\frac{1}{2}$

(d) $P(X\leq 0.5)=F(X\leq 0.5)=\int_{-\infty}^{+\infty}dy\int_{-\infty}^{0.5}(x+y)\,dx=\int_0^1 dy\int_0^{0.5}(x+y)\,dx=\int_0^1(\frac{1}{8}+\frac{y}{2})\,dy=\frac{3}{8}$

2. Find the joint p.d.f.s of the two r.v.s X and Y whose joint c.d.f. is given by

$$F(x,y) = \begin{cases} \left(1-e^{-x}\right)\left(1-e^{-y}\right), & x>0, y>0, \\ 0, & \text{otherwise.} \end{cases}$$

Also use the joint probability density to determine $P(1<X<3, 1<Y<2)$.

When $x \le 0$ or $y \le 0$, $F(x,y)=0$ $\quad \therefore f_{(X,Y)}(x,y)=0$

otherwise $f_{(X,Y)}(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x \partial y} = e^{-(x+y)}$ $\quad (x>0, y>0)$

$\therefore$ Joint p.d.f: $f_{(X,Y)}(x,y) = \begin{cases} e^{-(x+y)}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

$$\begin{aligned} P(1<X<3, 1<Y<2) &= P(X<3, 1<Y<2) - P(X\le 1, 1<Y<2) \\ &= P(X<3, Y<2) - P(X<3, Y\le 1) - (P(X\le 1, Y<2) - P(X\le 1, Y\le 1)) \\ &= F(3,2) - F(3,1) - F(1,2) + F(1,1) = e^{-2}(1-e^{-1})(1-e^{-2}) \end{aligned}$$

3. Consider a circle of radius R and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen, it follows, since (X,Y) is equally likely to be near each point in the circle, that the joint p.d.f. of X and Y is given by

$$f(x,y) = \begin{cases} c, & \text{if } x^2+y^2 \leqslant R^2, \\ 0, & \text{if } x^2+y^2 > R^2. \end{cases}$$

for some value of c.

(a) Determine the constant c.

(b) Find the marginal p.d.f.s of X and Y.

(c) Compute the probability that the distance from the origin of the point selected is not greater that a $(0 \leqslant a \leqslant R)$.

(d) Are X and Y independent? Specify your reasons clearly.

(a) Suppose that $B = \{(x,y) \mid x^2+y^2 \le R^2\}$

$$\therefore \lim_{\substack{x\to+\infty \\ y\to+\infty}} F(x,y) = \iint_B f(x,y)\,dxdy + \iint_{B^c} f(x,y)\,dxdy = \iint_B c\,dxdy + \iint_{B^c} 0\,dxdy$$
$$= \pi R^2 c = 1. \qquad \therefore c = \frac{1}{\pi R^2}$$

(b) $f(x,y) = \begin{cases} \frac{1}{\pi R^2}, & x^2+y^2 \le R^2 \\ 0, & x^2+y^2 > R^2 \end{cases}$

$$\therefore f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{-\infty}^{-\sqrt{R^2-x^2}} 0\,dy + \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy + \int_{\sqrt{R^2-x^2}}^{+\infty} 0\,dy$$
$$= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2} \qquad (x^2<R^2)$$

the same way, $f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\,dx=\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{1}{\pi R^2}\,dx=\frac{2\sqrt{R^2-y^2}}{\pi R^2}$ $(y^2<R^2)$

$\therefore$ $f_X(x)=\begin{cases}\frac{2\sqrt{R^2-x^2}}{\pi R^2}, & -R<x<R\\ 0, & \text{otherwise}\end{cases}$, $f_Y(y)=\begin{cases}\frac{2\sqrt{R^2-y^2}}{\pi R^2}, & -R<y<R\\ 0, & \text{otherwise}\end{cases}$

(c) We need to find $P(X^2+Y^2\le a^2)$ $(0\le a\le R)$

Suppose that $C=\{(x,y)\,|\,x^2+y^2\le a^2\}$ $S(C)=\pi a^2$

$\therefore$ $P(X^2+Y^2\le a^2)=\iint_C f(x,y)\,dxdy=\iint_C \frac{1}{\pi R^2}\,dxdy=\frac{a^2}{R^2}$

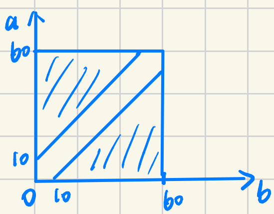
(d) X and Y are not independent. When $x^2+y^2\le R^2$,

$f_X(x)\cdot f_Y(y)=\frac{4\sqrt{(R^2-x^2)(R^2-y^2)}}{\pi^2 R^4}\ne\frac{1}{\pi R^2}=f_{(X,Y)}(x,y)$

4. A man and a woman decide to meet at a certain location. If each person independently arrives at a time uniformly distributed between 12 noon and 1 p.m., find the probability that the first to arrive has to wait longer than 10 minutes.

Suppose that the man arrive at 12:a, the woman arrive at 12:b $(0\le a,b\le 60)$

So the condition is: $|a-b|\ge 10.$ $\Rightarrow$ $a\ge b+10$ or $a\le b-10$



According to the map,

$P(|a-b|\ge 10)=\frac{S_{\text{阴}}}{S_{\square}}=\frac{2\times\frac{1}{2}\times 50^2}{60^2}=\frac{25}{36}$