Partitioned Matrices (分块矩阵的乘法)

Lecture 4+

Dept. of Math., SUSTech

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Introduction

Often it is useful to think of a matrix as being composed of a number of submatrices. A matrix C can be partitioned into smaller matrices by drawing horizontal lines between the rows and vertical lines between the columns. The smaller matrices are often referred to as blocks.

Example

Example. $C = \left[\begin{array}{ccc|cc} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{array}\right]$. If lines are drawn between the second and third rows and between the third and fourth columns, then C will be divided into four submatrices

$$C = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]$$

where

$$C_{11} = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 1 & 1 \end{bmatrix}, C_{12} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}, C_{21} = \begin{bmatrix} 3 & 3 & 2 \\ 4 & 6 & 2 \end{bmatrix}, C_{22} = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Block Multiplication I

Block Multiplication I. If A is an $m \times n$ matrix and B is an $n \times r$ matrix that has been partitioned into columns $(b_1, b_2, \cdots, b_r)$, then the block multiplication of A times B is given by

$$AB = (Ab_1, Ab_2, \cdots, Ab_r)$$

Block Multiplication II

Block Multiplication II. Let A be an $m \times n$ matrix. If we partition A into rows, and if B is an $n \times r$ matrix, the ith row of the product AB is determined by multiplying the ith row of A times B, then

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{a}_m \end{bmatrix}, AB = \begin{bmatrix} \vec{a}_1 B \\ \vec{a}_2 B \\ \cdot \\ \cdot \\ \cdot \\ \vec{a}_m B \end{bmatrix}.$$

Block Multiplication III

Let A be an $m \times n$ matrix and B and $n \times r$ matrix. It is often useful to partition A and B and express the product in terms of the submatrices of A and B.

Case 1. $A(B_1 \ B_2) = (AB_1 \ AB_2)$

In this case, B_1 is an $n \times t$ matrix and B_2 is an $n \times (r-t)$ matrix.

Case 2. $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$

Here A_1 is a $k \times n$ matrix and A_2 is an $(m-k) \times n$ matrix.

Block Multiplication IV

Case 3. $(A_1\ A_2)\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = A_1B_1 + A_2B_2$

Case 4. $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

Can you figure out the sizes of all the above submatrices?

In general, if the blocks have the proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication.

Examples

Example 1 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 2 & 2 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$. Partition A and B into four blocks and perform the block multiplication.

Example 2 Let A be an $n \times n$ matrix of the form

$$\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$$

where A_{11} is a $k \times k$ matrix($k < n$). Show that A is nonsingular if and only if A_{11} and A_{22} are nonsingular.

Inner Product

Given two vectors x and y in $\mathbb{R}^n$, it is possible to perform a matrix multiplication of the vectors if we transpose one of the vectors first.The matrix product $x^T y$ is the product of a row vector (a $1 \times n$ matrix) and a column vector (an $n \times 1$ matrix). The result is a 1×1 matrix, or simply a scalar. This type of product is referred to as a scalar product or an inner product.

Outer product

The outer product has special structure in that each of its rows is a multiple of y^T and each of its column vectors is a multiple of x.

Outer Product Expansion

It is also useful to consider a column vector times a row vector. The matrix product xy^T is a full $n \times n$ matrix. The product xy^T is referred to as the outer product of x and y. The outer product matrix has special structure in that each of its rows is a multiple of y^T and each of its column vectors is a multiple of x.

Example Compute $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2023}$.

One more example

Example Given

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$$

compute the outer product expansion of XY^T.

Homework Assignment 5(Suggested)

1.4: 45, 46, 47, 51, 52, 53, 55, 57, 58.