南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线 性 代 数 **开课单位:** 数 学 系

考试时长: 120 分钟 **命题教师:** 线性代数教学团队

题 号	1	2	3	4	5	6
分 值	15 分	15 分	20 分	20 分	10 分	20 分

本试卷共 (6) 大题, 满分 (100) 分. **请将所有答案写在答题本上**.

This exam includes **6** questions and the score is 100 in total. **Write all your answers on the examination book.**

本套试卷为 A 卷 Version A

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题. 每题只有一个选项是正确的.

(1) Suppose

$$\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Let $\beta_1 = \alpha_1, \beta_2 = \alpha_2 - k\beta_1, \beta_3 = \alpha_3 - l_1\beta_1 - l_2\beta_2$, and suppose $\beta_1, \beta_2, \beta_3$ are mutually orthogonal. Then ( )

(A) $l_1 = -\frac{5}{2},\ l_2 = \frac{1}{2}$.

(B) $l_1 = \frac{5}{2},\ l_2 = -\frac{1}{2}$.

(C) $l_1 = -\frac{5}{2},\ l_2 = -\frac{1}{2}$.

(D) $l_1 = \frac{5}{2},\ l_2 = \frac{1}{2}$.

已知

$$\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

记 $\beta_1 = \alpha_1, \beta_2 = \alpha_2 - k\beta_1, \beta_3 = \alpha_3 - l_1\beta_1 - l_2\beta_2$, 若 $\beta_1, \beta_2, \beta_3$ 两两正交, 则 ( )

(A) $l_1 = -\frac{5}{2},\ l_2 = \frac{1}{2}$.

(B) $l_1 = \frac{5}{2},\ l_2 = -\frac{1}{2}$.

(C) $l_1 = -\frac{5}{2},\ l_2 = -\frac{1}{2}$.

(D) $l_1 = \frac{5}{2},\ l_2 = \frac{1}{2}$.

(2) Suppose A and B are two $n \times n$ invertible matrices, then which of the following statements is always true? ( )

(A) $|A+B| = |A| + |B|$.

(B) $AB = BA$.

(C) $|AB| = |BA|$.

(D) $A+B$ is invertible, and $(A+B)^{-1} = A^{-1} + B^{-1}$.

设 A, B 为两个可逆的 n 阶矩阵, 下列哪个结论总是正确的? ( )

(A) $|A+B| = |A| + |B|$.

(B) $AB = BA$.

(C) $|AB| = |BA|$.

(D) $A+B$ 为可逆矩阵, 且 $(A+B)^{-1} = A^{-1} + B^{-1}$.

(3) Let A and B be two $n \times n$ real symmetric matrices. Which of the following statements is **false**? ( )

(A) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

(B) If B is positive semidefinite, then $b_{ii} \geq 0, i = 1, \cdots, n$, where b_{ii} is the ith diagonal entry of B.

(C) If λ is an eigenvalue of AB, then λ is an eigenvalue of BA.

(D) If x is an eigenvector of AB, then x is an eigenvector of BA.

设 A, B 为两个 $n \times n$ 实对称矩阵. 下列哪个结论是**错误**的? ( )

(A) 若 λ 为 A 的特征值, 则 λ^2 为 A^2 的特征值.

(B) 若 B 为半正定矩阵, 则 $b_{ii} \geq 0, i = 1, \cdots, n$, 其中 b_{ii} 为矩阵 B 的第 i 个对角元.

(C) 若 λ 为 AB 的特征值, 则 λ 也为 BA 的特征值.

(D) 若 x 为 AB 一个特征向量, 则 x 为 BA 的一个特征向量.

(4) Let

$$A = \begin{bmatrix} 1 & a & 1 \\ a & b & a \\ 1 & a & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If A and B are similar, then $a =$ ( )

(A) 1 or 2.

(B) 2.

(C) 1.

(D) 0.

设

$$A = \begin{bmatrix} 1 & a & 1 \\ a & b & a \\ 1 & a & 1 \end{bmatrix} \text{ 和 } B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

如果 A 和 B 相似, 则 $a =$ ( )

(A) 1 or 2.

(B) 2.

(C) 1.

(D) 0.

(5) Which of the following matrices is positive definite? ( )

(A) $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 8 \end{bmatrix}$.

(C) $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -3 \\ 2 & -3 & -4 \end{bmatrix}$.

(D) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

下列哪个矩阵是正定矩阵? ( )

(A) $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 8 \end{bmatrix}$.

(C) $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -3 \\ 2 & -3 & -4 \end{bmatrix}$.

(D) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

2. (15 points, 3 points each) Fill in the blanks. (共 15 分, 每小题 3 分) 填空题.

(1) Suppose the quadratic form $f(x, y, z) = x^2 + y^2 - tyz + 4z^2$ is positive definite, then the possible range of t is ___________.

如果二次型 $f(x, y, z) = x^2 + y^2 - tyz + 4z^2$ 为正定的, 则 t 的取值范围是 ___________.

(2) Suppose A is a 10×10 matrix, where all entries of A are 1. Then the biggest eigenvalue of A is ___________.

设 A 为一个 10×10 矩阵, A 的所有矩阵元素都为 1, 则 A 的最大的特征值为 ___________.

(3) Suppose

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ is an eigenvector of } A = \begin{bmatrix} 3 & 2 & -1 \\ a & -2 & 2 \\ 3 & b & -1 \end{bmatrix}.$$

Then $a =$ __________, $b =$ __________.

设

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{为矩阵} A = \begin{bmatrix} 3 & 2 & -1 \\ a & -2 & 2 \\ 3 & b & -1 \end{bmatrix} \text{的一个特征向量,}$$

则 $a =$ __________, $b =$ __________.

(4) Let $A = \begin{bmatrix} 103 & 100 & 204 \\ 199 & 200 & 395 \\ 301 & 300 & 600 \end{bmatrix}$. Then $|A| =$ __________.

设 $A = \begin{bmatrix} 103 & 100 & 204 \\ 199 & 200 & 395 \\ 301 & 300 & 600 \end{bmatrix}$, 则 $|A| =$ __________.

(5) Let

$$A = \begin{bmatrix} 1+t & 1+t & 1 \\ -t & -t & -1 \\ t & t-1 & 0 \end{bmatrix}, t \in \mathbb{R}.$$

If A is diagonalizable, then $t =$ __________.

设

$$A = \begin{bmatrix} 1+t & 1+t & 1 \\ -t & -t & -1 \\ t & t-1 & 0 \end{bmatrix}, t \in \mathbb{R}.$$

如果 A 为可对角化的, 则 $t =$ __________.

3. (20 points) Let

$$A = \begin{bmatrix} 0 & 7 & -6 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{bmatrix}.$$

(a) Find the eigenvalues of A and their corresponding eigenvectors.

(b) Find a matrix B such that $B^3 = A$.

(20 分) 设

$$A = \begin{bmatrix} 0 & 7 & -6 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{bmatrix}.$$

(a) 求矩阵 A 的所有特征值以及它们对应的特征向量.

(b) 求矩阵 B 使得 $B^3 = A$.

4. (20 points) Consider the quadratic form

$$f(x_1, x_2, x_3) = 3x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_1x_3 + 4x_2x_3 + 3x_3^2.$$

(a) Find the real symmetric matrix A such that $f(x) = x^T Ax$ for all $x = (x_1, x_2, x_3)^T$.

(b) The quadric surface defined by the equation $f(x) = 1$ is ___________.

(A) an ellipsoid (B) a hyperboloid of one sheet (C) a hyperboloid of two sheets (D) none of the above.

(c) Find an orthogonal matrix Q and a diagonal matrix Λ such that

$$Q^T AQ = \Lambda.$$

(20 分) 考虑二次型

$$f(x_1, x_2, x_3) = 3x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_1x_3 + 4x_2x_3 + 3x_3^2.$$

(a) 求实对称矩阵 A 使得 $f(x) = x^T Ax$ 对所有的 $x = (x_1, x_2, x_3)^T$ 成立.

(b) 由方程 $f(x) = 1$ 所定义的二次曲面是 ___________.

(A) 一个椭球面 (B) 一个单叶双曲面 (C) 一个双叶双曲面 (D) 以上都不是.

(c) 求一个正交矩阵 Q 和一个对角矩阵 Λ 使得

$$Q^T AQ = \Lambda.$$

5. (10 points) Let A, B be two $n \times n$ matrices. Let

$$C = \begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$

Prove that C is similar to a diagonal matrix if and only if both A and B are similar to diagonal matrices.

(10 分) 设 A, B 为两个 $n \times n$ 矩阵. 已知

$$C = \begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$

证明: C 相似于对角矩阵当且仅当 A 和 B 都相似于对角矩阵.

6. (20 points) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

(a) Find all the nonzero singular values σ_1, σ_2 of A.

(b) Find a Singular Value Decomposition of A. In other words, find U, Σ, V such that $A = U\Sigma V^T$, where U and V are orthogonal matrices and

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) If $A = U\Sigma V^T$, then $A^+ = V\Sigma^+ U^T$ is called the **pseudoinverse** of A, where

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Prove that A^+b is the minimum-length least-squares solution to $Ax = b$. The minimum-length least-squares solution to $Ax = b$ is the solution $\hat{x}$ to $A^TA\hat{x} = A^Tb$ and has the shortest length.

(d) Find the minimum-length least-squares solution of $Ax = b$.

(20 分) 设

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

(a) 求矩阵 A 的所有非零奇异值, σ_1, σ_2.

(b) 求矩阵 A 的一个奇异值分解. 换言之, 求矩阵 U, Σ, V 使得 $A = U\Sigma V^T$, 其中 U 和 V 为正交矩阵,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) 如果 $A = U\Sigma V^T$, 则称 $A^+ = V\Sigma^+ U^T$ 为矩阵 A 的伪逆, 其中

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

证明: A^+b 是在最小二乘意义下 $Ax = b$ 的最小长度解. 这里在最小二乘意义下 $Ax = b$ 的最小长度解 $\hat{x}$ 是指满足 $A^TA\hat{x} = A^Tb$ 且长度达到最小的解.

(d) 求 $Ax = b$ 在最小二乘意义下的最小长度解.