Chapter 5: Duality

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Outline

Primal and Dual Problems

Diet problem

How can a dietician design the most economical diet that satisfies the basic daily nutritional requirements for a good health? Assume that there are only two foods F_1 and F_2 and the daily nutrition required are N_1, N_2 and N_3. The unit cost of the foods and their nutrition values together with the daily requirement of each nutrition are given as below

	F_1	F_2	Daily Requirement
Cost	120	180	–
N_1	1	1	10
N_2	2	4	24
N_3	3	6	32

Let x_j , $j = 1, 2$ be the number of units of F_j that one should eat in order to minimize the cost and yet fulfill the daily nutrition requirement.

$$\begin{array}{rrrrr}
\text{Min} & 120x_1 & +180x_2 & & \\
\text{Subject to} & x_1 & +x_2 & \geq & 10 \\
 & 2x_1 & +4x_2 & \geq & 24 \\
 & 3x_1 & +6x_2 & \leq & 32 \\
 & x_1, & x_2 \geq & 0 &
\end{array}$$

Diet problem

In matrix form, we have

$$\begin{aligned} \text{Min} \quad & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{Subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where

$$\mathbf{c} = \begin{bmatrix} 120 \\ 180 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 24 \\ 32 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

Now let us look at the same problem from a pharmaceutical company's point of view. How can a pharmaceutical company determine the price for each unit of nutrient pill so as to maximize revenue, if a synthetic diet made up of nutrient pills of various pure nutrients is adopted? Thus we have three types of nutrient pills P_1, P_2 and P_3. Assume that each unit of P_i contains one unit of the N_i. Let u_i be the unit price of P_i, the problem is to maximize the total revenue u_0 from selling such a synthetic diet.

Diet problem

	F_1	F_2	Daily Requirement
Cost	120	180	–
N_1	1	1	10
N_2	2	4	24
N_3	3	6	32

Let x_j , $j = 1, 2$ be the number of units of F_j that one should eat in order to minimize the cost and yet fulfill the daily nutrition requirement.

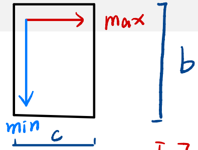
$$\begin{array}{rrrrcl} \text{Max} & 10u_1 & +24u_2 & +32u_3 & & \\ \text{Subject to} & u_1 & +2u_2 & +3u_3 & \leq & 120 \\ & u_1 & +2u_2 & +6u_3 & \leq & 180 \\ & u_1, & u_2, & u_3 & \geq & 0 \end{array}$$

In matrix form, the problem is:

$$\begin{array}{rl} \text{Max} & u_0 = \mathbf{b}^T \mathbf{u} \\ \text{Subject to} & \mathbf{A^T u} \leq \mathbf{c} \\ & \mathbf{u} \geq \mathbf{0} \end{array}$$

Primal and dual problems

对偶问题.



Definition

$c: n\times 1$ $m\times 1$

Let $\mathbf{x}$ and $\mathbf{c}$ be column n-vectors, $\mathbf{b}$ and $\mathbf{u}$ be column m-vectors and $\mathbf{A}$ be an m-by-n matrix. The primal and the dual problems can be defined as follows: $[\ \ ]_{m\times n}$

注: 需 C form

Primal	Dual
Max $\mathbf{c}^T\mathbf{x}$	Min $\mathbf{b}^T\mathbf{u}$
Subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$	Subject to $\mathbf{A}^T\mathbf{u} \geq \mathbf{c}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{u} \geq \mathbf{0}$

Example

Let the original (primal) problem be given by

$$c = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$$

$$\begin{array}{rrrrcl}
\text{Max} & x_1 & +4x_2 & +3x_3 & & \\
\text{Subject to} & 2x_1 & +2x_2 & +x_3 & \leq & 4 \\
 & x_1 & +2x_2 & +2x_3 & \leq & 6 \\
 & x_1, & x_2, & x_3 & \geq & 0
\end{array}$$

The dual problem is

$$\begin{array}{rrrcl}
\text{Min} & 4u_1 & +6u_2 & & \\
\text{Subject to} & 2u_1 & +u_2 & \geq & 1 \\
 & 2u_1 & +2u_2 & \geq & 4 \\
 & u_1 & +2u_2 & \geq & 3 \\
 & u_1 & & \geq & 0 \\
 & & u_2 & \geq & 0
\end{array}$$

Theorem

Theorem

The dual of the dual is the primal.

Proof: Transforming the dual into canonical form, we have

$$\begin{array}{rrcl} \text{Max} & u_0' = -\mathbf{b}^T\mathbf{u} & & \\ \text{Subject to} & -\mathbf{A}^T\mathbf{u} & \leq & -\mathbf{c} \\ & \mathbf{u} & \geq & \mathbf{0} \end{array}$$

Taking the dual of this problem, we have

$$\begin{array}{rrcl} \text{Min} & x_0' = -\mathbf{c}^T\mathbf{x} & & \\ \text{Subject to} & -\mathbf{A}\mathbf{x} & \geq & -\mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

which is the same as the primal problem.

Dual problem from standard form

- To obtain the dual of an LP problem in standard form:

$$\begin{array}{rrcl} \text{Max} & x_0 = \mathbf{c}^T\mathbf{x} & & \\ \text{Subject to} & \mathbf{A}\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

- we can first change it into canonical form:

$$\begin{array}{rrcl} \text{Max} & x_0 = \mathbf{c}^T\mathbf{x} & & \\ \text{Subject to} & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} \\ & -\mathbf{A}\mathbf{x} & \leq & -\mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

- its dual is given by

$$\begin{array}{rrcl} \text{Min} & u_0 = \mathbf{b}^T\mathbf{u}_1 - \mathbf{b}^T\mathbf{u}_2 & & \\ \text{Subject to} & \mathbf{A}^T\mathbf{u}_1 - \mathbf{A}^T\mathbf{u}_2 & \geq & \mathbf{c} \\ & \mathbf{u}_1, \mathbf{u}_2 & \geq & \mathbf{0} \end{array}$$

Dual problem from standard form

Letting $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, we finally have

$$
\begin{array}{rl}
\text{Min} & u_0 = \mathbf{b}^T\mathbf{u} \\
\text{Subject to} & \mathbf{A}^T\mathbf{u} \quad \geq \quad \mathbf{c} \\
 & \mathbf{u} \text{ free}
\end{array}
$$

The following is a general rule of the relationship between a dual pair.

$$
\begin{array}{ll|ll}
\text{Max} & \sum_{j=1}^{n} c_j x_j & \text{Min} & \sum_{i=1}^{m} u_i b_i \\
\text{Subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \cdots, k) & \text{Subject to} & u_i \geq 0 \quad (i = 1, 2, \cdots, k) \\
 & \sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = k+1, \cdots, m) & & u_i \text{ free} \quad (i = k+1 \cdots, m) \\
 & x_j \geq 0 \quad (j = 1, 2, \cdots, l) & & \sum_{i=1}^{m} u_j a_{ij} \geq c_j \quad (j = 1, 2, \cdots, l) \\
 & x_j \text{ free} \quad (j = l+1, \cdots, n) & & \sum_{i=1}^{m} u_i a_{ij} = c_j \quad (j = l+1, \cdots, n)
\end{array}
$$

Primal and dual

change sign: max costraint $\longrightarrow$ min variable

We observe from the above the following correspondence:

不区分原问题还是对偶问题，只区分 max、min.

Maximization problem		Minimization problem
Constraint		Variable
$\leq$	$\leftrightarrow$	≥ 0
$\geq$	$\leftrightarrow$	≤ 0
$=$	$\leftrightarrow$	unrestricted
Variable		Constraint
≥ 0	$\leftrightarrow$	$\geq$
≤ 0	$\leftrightarrow$	$\leq$
unrestricted	$\leftrightarrow$	$=$

$\sum_{j=1}^{n} a_{ij} x_j$ b_j u_j

x_j $\sum_{i=1}^{m} a_{ij} u_i$ c_i

$\max \ 5u_1 + 3u_2 + 8u_3$

s.t. $u_1 - u_2 + 4u_3 = 5$

$2u_1 + 5u_2 + 7u_3 \le 6$

$\Rightarrow u_1$ free

$u_2 \ge 0, \ u_3 \le 0.$

$\begin{bmatrix} 1 & 2 \\ -1 & 5 \\ 4 & 7 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Let us consider a primal given by

$$
\begin{array}{rrrcr}
\text{Min} & 5x_1 & +6x_2 & & \\
\text{Subject to} & x_1 & +2x_2 & = & 5 \\
 & -x_1 & +5x_2 & \geq & 3 \\
 & 4x_1 & +7x_2 & \leq & 8 \\
 & \multicolumn{4}{c}{x_1 \text{ free}, \ x_2 \geq 0}
\end{array}
$$

The dual problem is

$$
\begin{array}{rrrrcr}
\text{Max} & 5u_1 & +3u_2 & -8u_3 & & \\
\text{Subject to} & u_1 & -u_2 & \underline{-4u_3} & = & 5 \\
 & 2u_1 & +5u_2 & \underline{-7u_3} & \leq & 6 \\
 & u_1 & & & & \text{free} \\
 & & u_2, & \underline{u_3} & \underline{\geq} & \underline{0}
\end{array}
$$

Example

(Transportation Problem) Suppose that there are m sources that can provide materials to n destinations that require the materials. The following is called the costs and requirements table for the transportation problem.

	Destination				Supply
Origin	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
Demand	d_1	d_2	$\cdots$	d_n	

where c_{ij} is the unit transportation cost from origin i to destination j, s_i is the supply available from origin i and d_j is the demand required for destination j.

Example

The problem is to decide the amount x_{ij} to be shipped from i to j so as to minimize the total transportation cost while meeting all demands. That is

$c^T x$.

$$
\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{Subject to} \quad & \sum_{j=1}^{n} x_{ij} = s_i && (i = 1, 2, \cdots, m) \\
& \sum_{i=1}^{m} x_{ij} = d_j && (j = 1, 2, \cdots, n) \\
& x_{ij} \geq 0 && (i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, m)
\end{aligned}
$$

The dual is then given by

$$
\begin{aligned}
\text{Max} \quad & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\
\text{Subject to} \quad & u_i + v_j \leq c_{ij} && (i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, m) \\
& u_i, v_j \quad \text{free}
\end{aligned}
$$

Strong and Weak Duality

Weak duality

Theorem (Weak Duality Theorem)

Consider the following primal-dual pair.

$$
\begin{array}{llrcl}
(P) & Max & \mathbf{c}^T\mathbf{x} & & \\
 & Subject\ to & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} \\
 & & \mathbf{x} & \geq & \mathbf{0}
\end{array}
$$

$$
\begin{array}{llrcl}
(D) & Min & \mathbf{b}^T\mathbf{u} & & \\
 & Subject\ to & \mathbf{A}^\mathbf{T}\mathbf{u} & \geq & \mathbf{c} \\
 & & \mathbf{u} & \geq & \mathbf{0}
\end{array}
$$

If $\mathbf{x}$ *is a feasible solution (not necessarily basic) to the primal and* $\mathbf{u}$ *is a feasible solution (not necessarily basic) to the dual, then*

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u}.$$

$\cup$

$\cap$

Weak duality

Proof:
Since $\mathbf{x}$ is a feasible solution to the primal (P), we have $\mathbf{Ax} \leq \mathbf{b}$. As $\mathbf{u} \geq \mathbf{0}$, we have

$$\mathbf{u}^T\mathbf{Ax} \leq \mathbf{u}^T\mathbf{b} = \mathbf{b}^T\mathbf{u}. \tag{1}$$

Similarly, since $\mathbf{A}^T\mathbf{u} \geq \mathbf{c}$ and $\mathbf{x} \geq \mathbf{0}$, we have

借助中间量

$$\mathbf{x}^T\mathbf{A}^T\mathbf{u} \geq \mathbf{x}^T\mathbf{c}.$$

Taking the transpose and combining with (1), we get $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u}$.

Weak duality

As an immediate corollary, we have

注意是 unbounded

→ 没有最大值 → 不存在最小值.

Corollary

If the primal objective is unbounded, then the dual problem is infeasible.

也同理.

Corollary

If the dual objective is unbounded, then the primal problem is infeasible.

But, the converse of each corollary may not be true. Because, if one problem is infeasible, it is also possible for the other to be infeasible. This is illustrated via the following example.

Consider the following canonical primal-dual pair:

$$
\begin{array}{rrcr}
\text{Max} & x_1 + x_2 & & \\
\text{Subject to} & -x_1 + 2x_2 & \leq & -2 \\
 & x_1 - 2x_2 & \leq & -2 \\
 & x_1, x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{rrcr}
\text{Min} & -2u_1 - 2u_2 & & \\
\text{Subject to} & -u_1 + u_2 & \geq & 1 \\
 & 2u_1 - 2u_2 & \geq & 2 \\
 & u_1, u_2 & \geq & 0
\end{array}
$$

Case in primal and dual

We summarize the results in the following tables.

	Primal is feasible	Primal not feasible
Dual is feasible	both optimal solutions exist	Dual has unbounded solutions
Dual is not feasible	Primal has unbounded solutions	possible

		primal		
dual		infeasible	feasible bounded	unbounded
	infeasible	✓	x	✓
	feasible bounded	x	✓	x
	unbounded	✓	x	x

✓: possible; x impossible

Strong duality

Theorem (The Strong Duality Theorem)

A feasible solution $\mathbf{x}_0$ *to the primal is optimal if and only if there exists a feasible solution* $\mathbf{u}_0$ *to the dual such that*

$\Leftrightarrow$

$$\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{u}_0 \tag{2}$$

In particular, $\mathbf{u}_0$ *is an optimal solution to the dual.*

Strong duality

Proof:

"$\Rightarrow$"

For all feasible solutions $\mathbf{x}$ to the primal, by weak duality theorem, we have

$$\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{u}_0 = \mathbf{c}^T\mathbf{x}_0.$$

Thus $\mathbf{x}_0$ is an optimal solution to primal. Similarly, if $\mathbf{u}$ is any feasible solution to the dual, then

$$\mathbf{b}^T\mathbf{u} \ge \mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{u}_0.$$

Thus $\mathbf{u}_0$ is an optimal solution to the dual.

"$\Leftarrow$"

Let the primal be

$$\begin{array}{rrcl} \text{Max} & z = \mathbf{c}^T\mathbf{x} & & \\ \text{Subject to} & \mathbf{A}\mathbf{x} & \le & \mathbf{b} \\ & \mathbf{x} & \ge & \mathbf{0} \end{array}$$

Standardizing it, we have

$$\begin{array}{rrcl} \text{Max} & z = \mathbf{c}^T\mathbf{x} + \mathbf{c}_s^T\mathbf{x}_s & & \\ \text{Subject to} & \mathbf{A}\mathbf{x} + \mathbf{x}_s & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x}_s & \ge & \mathbf{0}, \end{array}$$

where $\mathbf{x}_s$ are all slack variables and $\mathbf{c}_s = \mathbf{0}$. Suppose that $\mathbf{x}^*$ is an optimal solution to the primal problem with basis matrix $\mathbf{B}$.

Strong duality

The proof will be complete if we can produce a feasible solution to (D), which has the same objective value. Consider $\mathbf{u}_0 \equiv (\mathbf{B}^{-1})^T \mathbf{c}_B$. Clearly, $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{u}_0 = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$. Thus, it only remains to be shown that $\mathbf{u}_0$ is a feasible solution to (D).
Consider the initial simplex tableau corresponding to the primal problem.

	$\mathbf{x}$	$\mathbf{x}_s$	RHS
$\mathbf{x}_B$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
z	$-\mathbf{c}^T$	$\mathbf{0}^T$	0

Now, because $\mathbf{B}$ is an optimal basis matrix, the optimal tableau will be as in the following table.

	$\mathbf{x}$	$\mathbf{x}_s$	RHS
$\mathbf{x}_B$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$
z	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{I} - \mathbf{0}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$

Strong duality

By the optimality of the primal solution, we have

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A} - \mathbf{c}^T \geq \mathbf{0}^T$$

and

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{I} - \mathbf{0}^T = \mathbf{c}_B^T \mathbf{B}^{-1} \geq \mathbf{0}^T$$

But, recall that $\mathbf{u}_0 \equiv (\mathbf{B}^{-1})^T \mathbf{c}_B$. Now, substituting into the above inequalities yields

$$\begin{aligned} \mathbf{A}^T \mathbf{u}_0 &\geq \mathbf{c} \\ \mathbf{u}_0 &\geq \mathbf{0} \end{aligned}$$

which are precisely the dual feasibility conditions. Thus $\mathbf{u}_0$ is dual feasible and $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{u}_0$.

Lessons from strong du

- **对偶变量** u_i 的值对应原问题中 **第 i 个松弛变量** 在最优 tableau 的目标函数行系数的 **相反数**（若原问题为最大化问题）。
  - 例如：原问题第 i 个约束的松弛变量为 x_{n+i}，则对偶变量 $u_i = -$(目标函数行中 x_{n+i} 的系数)。

- The dual feasibility conditions are precisely the same as primal optimality conditions.
- In an analogous manner, it can be shown that primal feasibility conditions are exactly the same as dual optimality conditions.
- The theorem provides a method for computing the values of the dual variables. That is, whereas the primal solution can be written as

$$\mathbf{x}_N = \mathbf{0}$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

$(B^{-1}\cancel{b})^T c_B = \cancel{b}^T u_0$

$$\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{u}_0$$

the dual solution is given by

$$\mathbf{u} = (\mathbf{B}^{-1})^T\mathbf{c}_B$$
$$\mathbf{u}_s = \mathbf{A}^T(\mathbf{B}^{-1})^T\mathbf{c}_B - \mathbf{c}$$

$A^T(u_B - u_s) = c.$

where $\mathbf{u}_s$ is the vector of dual surplus variables. Finally, the objective value of both problem is

$$z = \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{u} = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$$

Example

Let the primal problem be

$$\begin{aligned} \text{Max} \qquad & x_0 = 4x_1 + 3x_2 \\ \text{Subject to} \qquad & \\ & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 6 \\ 8 \\ 7 \\ 15 \\ 1 \end{bmatrix} \\ & x_1, x_2 \ge 0. \end{aligned}$$

Standardizing the problem, we have

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 7 \\ 15 \\ 1 \end{bmatrix}$$

Example

The optimal tableau is given by

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**b**
x_3	0	0	1	0	$1/2$	$-1/2$	0	2
x_2	0	1	0	0	$3/2$	$-1/2$	0	3
x_4	0	0	0	1	$3/2$	$1/2$	0	5
x_1	1	0	0	0	$-1/2$	$1/2$	0	4
x_7	0	0	0	0	$1/2$	$1/2$	1	4
x_0	0	0	0	0	$5/2$	$1/2$	0	25

Thus the optimal solution is $[x_1, x_2] = [4, 3]$ with $[x_3, x_4, x_5, x_6, x_7] = [2, 5, 0, 0, 4]$. From the x_0 row, we see that the optimal solution to the dual is given by

$$[u_1, u_2, u_3, u_4, u_5, u_6, u_7] = \left[0, 0, \frac{5}{2}, \frac{1}{2}, 0, 0, 0\right].$$

Example

Let us verify this by considering the dual. The dual of the primal is given by

$$\begin{aligned} \text{Min} \quad & u_0 = 6u_1 + 8u_2 + 7u_3 + 15u_4 + u_5 \\ \text{Subject to} \quad & \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$u_i \geq 0, \quad i = 1, 2, 3, 4, 5$$

Changing the minimization problem to a maximization problem and using simplex method, we obtain the optimal tableau for the dual:

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	**c**
u_4	$\frac{1}{6}$	$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
u_3	$-\frac{1}{2}$	$\frac{3}{2}$	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{5}{2}$
u_0	2	5	0	0	4	4	3	-25

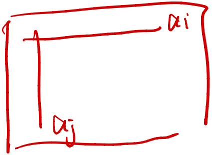
Example

Thus the optimal solution for the dual is $[u_1, u_2, u_3, u_4, u_5] = \left[0, 0, \frac{5}{2}, \frac{1}{2}, 0\right]$ with optimal surplus variables $[u_6, u_7] = [0, 0]$. Notice that the optimal solution to the primal is given by the reduced cost coefficients for u_6 and u_7, i.e. $[x_1, x_2] = [4, 3]$ and the optimal values of the primal slack variables are given by $[x_3, x_4, x_5, x_6, x_7] = [2, 5, 0, 0, 4]$.

The Complementary Slackness

The complementary slackness 互补松弛

- Given a primal-dual pair:



$$
\begin{array}{lll}
\text{(P)} & \text{Max} & \mathbf{c}^T\mathbf{x} \\
 & \text{Subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
 & & \mathbf{x} \geq \mathbf{0}
\end{array}
$$

$$
\begin{array}{lll}
\text{(D)} & \text{Min} & \mathbf{b}^T\mathbf{u} \\
 & \text{Subject to} & \mathbf{A}^\mathbf{T}\mathbf{u} \geq \mathbf{c} \\
 & & \mathbf{u} \geq \mathbf{0}
\end{array}
$$

- $\mathbf{a}^i$ represents the ith row of matrix $\mathbf{A}$, and $\mathbf{a}_j$ represents the jth column of matrix $\mathbf{A}$
- *complementary slackness conditions*:

Ax≤b中的第i行

$$
\begin{aligned}
u_i(\overline{b_i - \mathbf{a}^i\mathbf{x}}) &= 0, \quad \text{for all } i = 1, 2, \ldots, m \\
\underbrace{(\mathbf{a}_j^T\mathbf{u} - c_j)}_{\geq 0} x_j &= 0, \quad \text{for all } j = 1, 2, \ldots, n
\end{aligned}
$$

Equivalent forms of the complementary slackness conditions

(1)

① $\sum_{j=1}^{n} a_{ij} x_j = b_i. \Rightarrow x_{s,i} = 0$

$< b_i \Rightarrow u_i = 0, \; x_{s,i} \neq 0$

②

$$
\begin{aligned}
u_i x_{s,i} &= 0, \qquad \text{for all } i = 1, 2, \ldots, m \\
u_{s,j} x_j &= 0, \qquad \text{for all } j = 1, 2, \ldots, n
\end{aligned}
$$

where $x_{s,i}$ be the slack variable in primal constraint i and let $u_{s,j}$ be the surplus variable in dual constraint j.

(2)

$$
\mathbf{u}^T \mathbf{x}_s + \mathbf{u}_s^T \mathbf{x} = 0.
$$

This is because $\mathbf{x}$, $\mathbf{x}_s$, $\mathbf{u}$, and $\mathbf{u}_s$ are all nonnegative.

- $x_j > 0$ 时，对偶问题第 j 个约束 $\sum_{i=1}^{m} a_{ij} y_i = c_j$（即对偶约束取等号）；反之，若 $\sum_{i=1}^{m} a_{ij} y_i > c_j$，则 $x_j = 0$。
- $y_i > 0$ 时，原问题第 i 个约束 $\sum_{j=1}^{n} a_{ij} x_j = b_i$（即原约束取等号）；反之，若 $\sum_{j=1}^{n} a_{ij} x_j < b_i$，则 $y_i = 0$。

The complementary slackness

Theorem (Complementary Slackness)

Consider the following primal-dual pair that has been converted to standard form by adding the appropriate slack/surplus variables.

$$\begin{array}{lll} (P) & Max & \mathbf{c}^T\mathbf{x} \\ & Subject\ to & \mathbf{A}\mathbf{x}+\mathbf{x}_s=\mathbf{b} \\ & & \mathbf{x},\mathbf{x}_s \geq \mathbf{0} \end{array} \tag{3}$$

$$\begin{array}{lll} (D) & Min & \mathbf{b}^T\mathbf{u} \\ & Subject\ to & \mathbf{A}^T\mathbf{u}-\mathbf{u}_s=\mathbf{c} \\ & & \mathbf{u},\mathbf{u}_s \geq \mathbf{0} \end{array} \tag{4}$$

Let $(\mathbf{x}_0, \mathbf{x}_{s0})$ be feasible to (P) and $(\mathbf{u}_0, \mathbf{u}_{s0})$ be feasible to (D). Then $(\mathbf{x}_0, \mathbf{x}_{s0})$ is optimal to (P) and $(\mathbf{u}_0, \mathbf{u}_{s0})$ is optimal to (D) if and only if complementary slackness holds.

The complementary slackness

Proof

Because $(\mathbf{x}_0, \mathbf{x}_{s0})$ is feasible to (P), it follows from (3) that

$$\mathbf{A}\mathbf{x}_0 + \mathbf{x}_{s0} = \mathbf{b} \tag{5}$$

$$\mathbf{x}_0, \mathbf{x}_{s0} \geq \mathbf{0} \tag{6}$$

Similarly, $(\mathbf{u}_0, \mathbf{u}_{s0})$ is feasible to (D) implies that

$$\mathbf{A}^{\mathbf{T}}\mathbf{u}_0 - \mathbf{u}_{s0} \quad = \mathbf{c} \qquad \text{i.e.} \quad \mathbf{u}^{\mathbf{T}}\mathbf{A}_0 - \mathbf{u}_{s0}^T = \mathbf{c}^T \tag{7}$$

$$\mathbf{u}_0, \mathbf{u}_{s0} \quad \geq \mathbf{0} \tag{8}$$

Now, multiplying (5) by $\mathbf{u}_0^T$ and multiplying (7) by $\mathbf{x}_0$ yield the following:

$$\mathbf{u}_0^T\mathbf{A}\mathbf{x}_0 + \mathbf{u}_0^T\mathbf{x}_{s0} = \mathbf{u}_0^T\mathbf{b} = \mathbf{b}^T\mathbf{u}_0 \tag{9}$$

$$\mathbf{u}_0^T\mathbf{A}\mathbf{x}_0 - \mathbf{u}_{s0}^T\mathbf{x}_0 = \mathbf{c}^T\mathbf{x}_0 \tag{10}$$

Now, subtracting (10) from (9), we get

$$\mathbf{u}_0^T\mathbf{x}_{s0} + \mathbf{u}_{s0}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{u}_0 - \mathbf{c}^T\mathbf{x}_0 \tag{11}$$

Note that because all variables are nonnegative, it follows that the left side of (11) is zero if and only if complementary slackness holds. Thus, from (11), we see that $\mathbf{b}_0^T - \mathbf{c}^T\mathbf{x}_0 = 0$ if and only if complementary slackness holds. That is, the primal and dual solutions have the same objective value if and only if complementary slackness holds.

Example

Let the primal be given by

$$\begin{array}{rrcrcrcr} \text{Max} & x_1 & + & 4x_2 & + & 3x_3 & & \\ \text{Subject to} & 2x_1 & + & 2x_2 & + & x_3 & \leq & 4 \\ & x_1 & + & 2x_2 & + & 2x_3 & \leq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Its dual is

$$\begin{array}{rrcrcr} \text{Min} & 4u_1 & + & 6u_2 & & \\ \text{Subject to} & 2u_1 & + & u_2 & \geq & 1 \\ & 2u_1 & + & 2u_2 & \geq & 4 \\ & u_1 & + & 2u_2 & \geq & 3 \\ & & & u_1, u_2 & \geq & 0 \end{array}$$

Example

Initial Tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	2	2	1	1	0	4
x_5	1	2	2	0	1	6
x_0	-1	-4	-3	0	0	0

Optimal Tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_2	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1
x_3	-1	0	1	-1	1	2
x_0	2	0	0	1	1	10

Example

Thus the optimal primal solution is $\mathbf{x}^* = [0, 1, 2, 0, 0]$ and by the duality theorem, the optimal dual solution is $\mathbf{u}^* = [1, 1, 2, 0, 0]$. Let us check for the complementary slackness for these two dual solutions.

$$
\begin{array}{lllllll}
u_1^* > 0 & \Rightarrow & x_4^* = 0 & \Rightarrow & 2x_1^* + 2x_2^* + x_3^* = 4 & \text{i.e.} & 2(0) + 2(1) + 2 = 4 \\
u_2^* > 0 & \Rightarrow & x_5^* = 0 & \Rightarrow & x_1^* + 2x_2^* + 2x_3^* = 6 & \text{i.e.} & 0 + 2(1) + 2(2) = 6 \\
x_1^* = 0 & \Rightarrow & u_3^* \geq 0 & \Rightarrow & 2u_1^* + u_2^* \geq 1 & \text{i.e.} & 2(1) + 1 = 3 \geq 1 \\
x_2^* > 0 & \Rightarrow & u_4^* = 0 & \Rightarrow & 2u_1^* + 2u_2^* = 4 & \text{i.e.} & 2(1) + 2(1) = 4 \\
x_3^* > 0 & \Rightarrow & u_5^* = 0 & \Rightarrow & u_1^* + 2u_2^* = 3 & \text{i.e.} & 1 + 2(1) = 3
\end{array}
$$

Tutorial (exercise)

1. Show that $(x_1, x_2, x_3) = (\frac{5}{26}, \frac{5}{2}, \frac{27}{26})$ is an optimal solution to the following LPP, Please do not use the simplex method.

$$\begin{aligned} \text{maximize} \quad & z = 9x_1 + 14x_2 + 7x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + 3x_3 \leq 6 \\ & 5x_1 + 4x_2 + x_3 \leq 12 \\ & 2x_2 \leq 5 \\ & x_1, x_2, x_3 \text{ free} \end{aligned}$$

   Hint: formulate the dual problem and find the feasible solution.

2. Consider the following LPP

$$\begin{aligned} \text{minimize} \quad & z = 3x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 10 \\ & 3x_1 + 5x_2 \leq 26 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0. \end{aligned}$$

   Let w_i denote dual variable. By using the principle of complementary slackness, show that $w_1 = 0$ in any optimal solution of the dual problem.

3. Use excel to solve a linear programming problem.