Linear Independence, Basis, and Dimension(线性无关, 基，维数)

Lecture 9

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Independence, Basis, and Dimension

Example

Consider the following system again:

$$Ax = b \text{ is } \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(a) The coefficient matrix has three rows and four columns, but the third row is only a combination of the first two. After elimination it becomes a zero row. It has no effect on the homogeneous problem $Ax = 0$.

(b) The four columns also fail to be independent, and the column space degenerates into a two dimensional plane.

(c) By themselves, the numbers m and n of an $m \times n$ matrix give an incomplete picture of the true size of a linear system.

Rank

- The important number that is beginning to emerge(the true size) is the rank r.
- The rank was introduced as the number of pivots in the elimination process. Equivalently, the final matrix U has r nonzero rows. This definition could be given to a computer.
- But it would be wrong to leave it there because the rank has a simple and intuitive meaning:

Definition

The rank counts the number of genuinely independent rows in the matrix A.

We want definitions that are mathematical rather than computational.

Goal

The goal of this section is to explain and use four ideas:

(a) Linear independence or dependence

(b) Spanning a subspace

(c) Basis for a subspace(a set of vectors)

(d) Dimension of a subspace (a number)

Steps

(a) The first step is to define linear independence. Given a set of vectors $v_1, \cdots, v_n$, we look at their combinations $c_1 v_1 + \cdots + c_n v_n$.

(b) The trivial combination, with all weights $c_i = 0$, obviously produces the zero vector: $0v_1 + \cdots + 0v_n = 0$.

(c) The question is whether this is the only way to produce zero. If so, the vectors are independent.

(d) If any other combination of the vectors gives zero, they are **dependent**.

Linear Independence

Definition

Suppose

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

only happens when

$$c_1 = c_2 = \cdots = c_k = 0.$$

Then the vectors $v_1, v_2, \cdots, v_k$ are linearly independent. If any c's are nonzero, the v's are linearly dependent. One vector is a combination of the others.

Remarks

1. Linear dependence is easy to visualize in three-dimensional space, when all vectors go out from the origin.
2. Two vectors are dependent if they lie on the same line.
3. Three vectors are dependent if they lie in the same plane.
4. A random choice of three vectors, without any special accident, should produce linear independence. Four vectors are always linearly dependent in $\mathbb{R}^3$.

Examples

1. Example 1 If $v_1 =$ zero vector; then the set is linearly dependent. We may choose $c_1 = 3$ and all other $c_i = 0$; this is a nontrivial combination that produces zero.
2. Example 2 The columns of the matrix

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent, since the second column is three times the first. The combination of columns with weights $-3, 1, 0, 0$ gives a column of zeros. The rows are also linearly dependent; row 3 is two times row 2 minus five times row 1. (This is the same as the combination of b_1, b_2, b_3 that had to vanish on the right-hand side in order for $Ax = b$ to be consistent. Unless $b_3 - 2b_2 + 5b_1 = 0$, the third equation would not become $0 = 0$.)

Example

Example

Example 3 The columns of this triangular matrix are linearly independent:

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}.$$

Look for a combination of the columns that makes zero: Solve $Ac = 0$. We have to show that c_1, c_2, c_3 are all forced to be zero. The only combination to produce the zero vector is the trivial combination. The nullspace of A contains only the zero vector $c_1 = c_2 = c_3 = 0$.

Theorem

Theorem

The columns of A are independent exactly when $N(A) = \{$zero vector$\}$.

A similar reasoning applies to the rows of A, which are also independent.

Independence

- The nonzero rows of any echelon matrix U must be independent.
- Furthermore, if we pick out the columns that contain the pivots.
  If A has been converted to its Row Echelon Form U, then the columns with pivots of U are guaranteed to be independent.

  The general rule is this:

Theorem

The r nonzero rows of an echelon matrix U and a reduced matrix R are linearly independent. So are the r columns that contain pivots.

Example 4 The columns $e_1, e_2, \cdots, e_n$ of the n by n identity matrix are independent. Most sets of four vectors in $\mathbb{R}^4$ are independent. Those e's might be the safest.

Linear Independence

- To check any set of vectors $v_1, v_2, \cdots, v_n$ for independence, put them in the columns of A.
- Then solve the system $Ac = 0$; the vectors are dependent if there is a solution other than $c = 0$.
- With no free variables (rank n), there is no nullspace except $c = 0$; the vectors are independent.
- If the rank is less than n, at least one free variable can be nonzero and the columns are dependent.

One case has special importance:

Theorem

A set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

Example 5

Example 5 These three columns in $\mathbb{R}^2$ cannot be independent.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}.$$

- To find the combination of the columns producing zero we solve $Ac = 0$:

$$A \to U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- If we give the value 1 to the free variable c_3, then back-substitution in $Uc = 0$ gives $c_2 = -1, c_1 = 1$.
- With these three weights, the first column minus the second plus the third equals zero: Dependence.

Spanning a subspace

Now we define what it means for a set of vectors to span a space. The column space of A is spanned by the columns. **Their combinations produce the whole space.**

Definition

If a vector space V consists of all linear combinations of $w_1, w_2, w_3, \cdots, w_l$, then these vectors span the space. Every vector v in V is some combination of the w's:

Every v comes from w's

$$v = c_1 w_1 + \cdots + c_l w_l$$

for some coefficients c_i.

Remarks

It is permitted that a different combination of w's could give the same vector v. The c's need not be unique, because the spanning set might be excessively large–it could include the zero vector, or even all vectors.

Examples

Example

Example 6 The vectors $w_1 = (1,0,0), w_2 = (0,1,0)$, and $w_3 = (-2,0,0)$ span a plane(the xy plane) in $\mathbb{R}^3$. The first two also span this plane, whereas w_1 and w_3 span only a line.

Example

Example 7

- The column space of A is exactly **the space that is spanned by its columns**. The row space is spanned by the rows. The definition is made to order. Multiplying A by any x gives a combination of the columns; it is a vector Ax in the column space.
- The coordinate vectors $e_1, e_2, \cdots, e_n$ coming from the identity matrix span $\mathbb{R}^n$. But the columns of other matrices also span $\mathbb{R}^n$.

Basis

To decide if b is a combination of the columns, we try to solve $Ax = b$. To decide if the columns are independent, we solve $Ax = 0$. **Spanning involves the column space, and independence involves the nullspace.** The coordinate vectors $e_1, e_2, \cdots, e_n$ span $\mathbb{R}^n$ and they are linearly independent. Roughly speaking, **no vectors in the set are wasted.** This leads to the crucial idea of a basis:

Definition

A basis for V is a sequence of vectors having two properties at once:

1. The vectors are linearly independent(not too many vectors).
2. They span the space V (not too few vectors).

Remarks

Remarks:

- There is one and only one way to write v as a combination of the basis vectors. Why? Can you prove it?
- A vector space has infinitely many different bases. Which one is the best?

Figure 2.4

Example 8

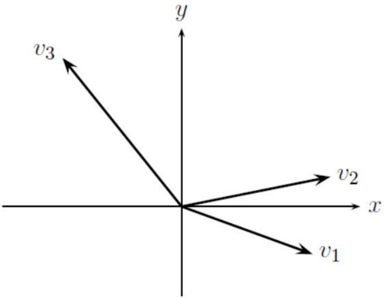

Figure 2.4: A spanning set v_1, v_2, v_3. Bases v_1, v_2 and v_1, v_3 and v_2, v_3.

Examples

The xy-plane in Figure 2.4 is just $\mathbb{R}^2$. The vector v_1 by itself is linearly independent, but it fails to span $\mathbb{R}^2$. The three vectors v_1, v_2, v_3 certainly span $\mathbb{R}^2$, but are not independent. Any two of these vectors, say, v_1 and v_2, have both properties–they span, and they are independent. So they form a basis. Notice again that a vector space does not have unique basis.

Example 9

Example 9 These four columns span the column space of U, but they are not independent:

$$\textbf{Echelon matrix} \quad U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are many possibilities for a basis, but we propose a specific choice: **The columns that contain pivots are a basis for the column space.**
- These columns are independent, they can span the column space.
- $C(U)$ is not the same as the column space of A, $C(A)$ before elimination–but the number of independent columns didn't change.
- To summarize: The columns of any matrix span its column space.

Dimension

A space has infinitely many different bases, but there is something common to all of these choices.

Definition

Any two bases for a vector space V contain the same number of vectors. This number, which is shared by all bases and expresses the number of “degrees of freedom” of the space, is the **dimension** of V.

Here is our first big theorem in linear algebra:

Theorem

If $v_1, v_2, \cdots, v_m$ and $w_1, w_2, \cdots, w_n$ are both bases for the same vector space, then $m = n$. The number of vectors is the same.

The dimension of a space is the number of vectors in every basis.

Proof.

Suppose there are more w's than v's ($n > m$). We will arrive at a contradiction. Since the v's form a basis, they must span the space. Every w_j can be written as a combination of the v's: If $w_1 = a_{11}v_1 + \cdots + a_{m1}v_m$, this is the first column of a matrix multiplication VA:

$$W = [w_1 \ w_2 \ \cdots \ w_n] = [v_1 \ v_2 \ \cdots \ v_m] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = VA$$

we don't know each a_{ij}, but we know the shape of A (it is m by n).The second vector w_2 is also a combination of the v's. The coefficients in that combination fill the second column of A. The key is that A has a row for every v and a column for every w. A is short, wide matrix,since $n > m$. There is a nonzero solution to $Ax = 0$. Then $VAx = 0$ which is $Wx = 0$. A combination of the w's gives zero! The w's could not be a basis–so we cannot have $n > m$. If $m > n$ we exchange the v's and w's and repeat the same steps. The only way to avoid a contradiction is to have $m = n$. □

Maximal independent set; minimal spanning set

Remark: In a subspace of dimension k, no set of more than k vectors can be independent, and no set of fewer than k vectors can span the space.

Theorem

Any linearly independent set in V can be extended to a basis, by adding more vectors if necessary. Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

Remarks:

- A basis is a maximal independent set. It cannot be made larger without losing independence.
- A basis is also a minimal spanning set. It cannot be made smaller and still span the space.

Two More Examples

Example

Find two independent vectors on the plane $x+2y-3z-t=0$ in $\mathbb{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Example

Example

Decide whether or not the following vectors are linearly independent, by solving $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Decide also if they span $\mathbb{R}^4$, by trying to solve

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Final Note

1. You must notice that the word “dimensional” is used in two different ways.
2. We speak about a four-dimensional vector, meaning a vector in $\mathbb{R}^4$.
3. Now we have defined a four-dimensional subspace; an example is the set of vectors in $\mathbb{R}^6$ whose first and last components are zero.
4. The members of this four-dimensional subspace are six-dimensional vectors like $(0,5,1,3,4,0)^T$.
5. NEVER use the terms “basis of a matrix” or “rank of a space”.
6. The dimension of the column space is equal to the rank of the matrix.

Homework Assignment 9

2.3: 1, 2, 7, 12, 13, 16, 24, 29, 33, 37.