Math 302 | Take-Home Final | Winter 2020

Name: Alex Stroebel **Due:** April 27, 2020

This assignment is a part of your final exam. When you hand in your work, please include this paper (signature required below). Failure to submit the assignment on time will result in a loss of all possible points.

This assignment consists of the following problems from the textbook:

- Section 3.6 #6a
- Section 4.2 #14
- Section 5.2 #13
- Section 6.3 #5
- Section 8.1 #5d

You should write complete solutions to these problems. Any time it asks for justification, it should be a formal proof or specific counterexample. Each problem will be graded out of 6 points for a total of 30 points (see rubric on reverse side). These will count for 30 out of a total of 100 points on your final exam.

Since this is part of an exam, you may not discuss it with anyone else; however, you are allowed to use some resources. Here is a complete list of materials you are allowed to consult:

- The book
- The screencasts associated with the book
- Your notes from in-class activities
- Me (Prof Mcclurkin)

By signing below, I, Alex Stroebel (print name), certify that I only consulted the book and associated videos, my notes, and Prof McClurkin. I did NOT discuss the exam with anyone else before submitting.

Alex Stroebel (Signature) April 27, 2020 (Date)

Please submit this paper WITH YOUR EXAM.

Rubric

I will be grading each of these problems out of 6 points, split up across three areas: readability, validity, fluency. I have taken this rubric from a nationally-used rubric recommended by the Mathematical Association of America.

Category	0	1	2
Readability	Informal or nonmathematical language is used for most of the proof. Most notation/symbols are used incorrectly.	Mostly proper mathematical language and notation is used, but there are some usage errors that obscure understanding.	Proper mathematical language is used everywhere or nearly everywhere. Only small notation errors that do not obscure meaning.
Validity	Significantly inaccurate or irrelevant statements in definitions, theorems, and techniques. Important information not present.	Mostly accurate statements in definitions, theorems, and techniques. May include some unjustified statements, or a lack of connection between a statement and its justification.	Completely correct statements in theorems, definitions, and techniques, with clear connections between statements and their justifications.
Fluency	Little to no coherent flow of ideas, or a listing of facts with no connection to the proof at hand.	Somewhat coherent, but with unjustified appeals to intuition. Some improperly justified statements and one or two major jumps in logic.	Correct and complete solution, with a clear flow from each sentence to the next. Elegant and concise.

Take Home Final Exam
Intro to Modern Mathematics

Alex Stroebel

April 27, 2020

3.6 #6a

Lemma 1. If c^2 is even, c is even

Proof of Lemma. Let c be an odd integer. There exists some integer s such that $c = 2s + 1$. Notice

$$\begin{aligned} c^2 &= (2s+1)^2 \\ &= 4s^2 + 4s + 1 \\ &= 2(2s^2 + 2) + 1 \end{aligned}$$

Since integers are closed under addition and multiplication, $(2s^2 + 2)$ is an integer. Thus c^2 is odd. □

Lemma 2. If c^2 is odd, c is odd

Proof of Lemma. Let c be an even integer. There exists some integer s such that $c = 2s$. Notice

$$\begin{aligned} c^2 &= (2s)^2 \\ &= 4s^2 \\ &= 2(2s^2) \end{aligned}$$

Since integers are closed under addition and multiplication, $(2s^2)$ is an integer. Thus c^2 is even. □

Proposition. Let a and b be natural numbers such that $a^2 = b^3$. If a is even, then 4 divides a.

Proof. Let $a, b \in \mathbb{N}$ such that $a^2 = b^3$ and a is even. By definition, there exists some $n \in \mathbb{N}$ such that $a = 2n$. Either b is ever or odd.

Case 1: Suppose b is even. There exists some integer m such that $b = 2m$. Notice

$$\begin{aligned} a^2 &= b^3 \\ (2n)^2 &= (2m)^3 \\ 2^2 \cdot n^2 &= 2^3 \cdot m^3 \\ n^2 &= 2m^3 \end{aligned}$$

Thus n^2 is even. By Lemma 1, n is even; so there exists some integer k such that $n = 2k$. Notice.

$$a = 2n = 2(2k) = 4k$$

Thus a is divisable by 4.

Case 2: Suppose b is odd. There exists some integer m such that $b = 2m + 1$. Notice

$$\begin{aligned} a^2 &= b^3 \\ (2n)^2 &= (2m+1)^3 \\ 2^2 \cdot n^2 &= 8m^3 + 12m^2 + 6m + 1 \\ 2(2n^2) &= 2(4m^3 + 6m^2 + 3m) + 1 \end{aligned}$$

Thus we have an even number equals an odd number. This is a contradiction, thus b is not odd. This implies b is even meaning a is dividable by 4. $\square$

4.2 #14

Is the following proposition true or false? Justify your conclusion.

Proposition. For each natural number n, $\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{7n}{6}\right)$ is a natural number.

Proof. Let n be a natural number. Notice

$$\begin{aligned} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{7n}{6}\right) &= \frac{1}{6}\left(2n^3 + 3n^2 + 7n\right) \\ &= \frac{n}{6}(2n^2 + 3n + 7) \\ &= \frac{n}{6}[(2n^2 + 3n + 1) + 6] \\ &= \frac{n}{6}[(2n^2 + 3n + 1) + 6] \\ &= \frac{n}{6}[(2n + 1)(n + 1) + 6]. \end{aligned}$$

Lets take a closer look at the expression $n[(2n+1)(n+1)+6]$; If this expression is equivalent to $0 \pmod 6$, it is divisable by 6 and therefore, $\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{7n}{6}\right)$ is a natural number.

$$n[(2n+1)(n+1)+6] \equiv n(2n+1)(n+1) \qquad \pmod 6.$$

There are six cases

Case 1: $n \equiv 1 \pmod 6$

$$n(2n+1)(n+1) \equiv 1(2+1)(1+1) \equiv (3)(2) \equiv 6 \equiv 0 \pmod 6.$$

Case 2: $n \equiv 2 \pmod 6$

$$n(2n+1)(n+1) \equiv 2(2\cdot 2+1)(2+1) \equiv 2(5)(3) \equiv (6)(5) \equiv 0(5) \equiv 0 \pmod 6.$$

Case 3: $n \equiv 3 \pmod 6$

$$n(2n+1)(n+1) \equiv 3(7)(4) \equiv 12(7) \equiv 0(7) \equiv 0 \pmod 6.$$

Case 4: $n \equiv 4 \pmod 6$

$$n(2n+1)(n+1) \equiv 4(9)(5) \equiv 36(5) \equiv 0(5) \equiv 0 \pmod 6.$$

Case 5: $n \equiv 5 \pmod 6$

$$n(2n+1)(n+1) \equiv 5(11)(6) \equiv 5(11)(0) \equiv 0 \quad (\text{mod } 6.)$$

Case 1: $n \equiv 0 \pmod 6$

$$n(2n+1)(n+1) \equiv 0(2n+1)(n+1) \equiv 0 \pmod 6.$$

Thus $n[(2n+1)(n+1)+6] \equiv 0 \pmod 6$ and therefore $\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{7n}{6}\right)$ is a natural number. □

5.2 #13

Let A, B, and C be subsets of a universal set U. Are the following propositions true or false? Justify your conclusions.

1. If $A \cap C \subseteq B \cap C$ then $A \subseteq B$.
   Solution: False. Let $A = \{1, 4\}$, $B = \{1, 2\}$, and $C = \{1, 2, 3\}$.

$$A \cap C = \{1\} \subseteq \{1, 2\} = B \cap C$$

   But

$$A = \{1, 4\} \not\subseteq \{1, 2\} = B$$

2. If $A \cup C \subseteq B \cup C$ then $A \subseteq B$.
   Solution: False Let $A = \{1\}$, $B = \{2\}$, and $C = \{1, 3\}$.

$$A \cup C = \{1, 3\} \subseteq \{1, 2, 3\} = B \cup C$$

   But

$$A = \{1, \} \not\subseteq \{2\} = B$$

3. If $A \cup C = B \cup C$ then $A = B$.
   Solution: False Let $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2, 3\}$.

$$A \cup C = \{1, 2, 3\} = \{1, 2, 3\} = B \cup C$$

   But

$$A = \{1\} \not\subseteq \{2\} = B$$

4. If $A \cap C = B \cup C$ then $A = B$.
   Solution: False Let $A = \{1, 2, 3\}$, $B = \{2\}$, and $C = \{1, 2\}$.

$$A \cap C = \{1, 2\} = \{1, 2\} = B \cup C$$

   But

$$A = \{1, 2, 3\} \not\subseteq \{2\} = B$$

5. If $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$.

   Proof. Let $A, B, C \subseteq U$ where U is some universal set where $A \cup C = B \cup C$ and $A \cap C = B \cap C$.

   $\Longrightarrow$) Let $x \in A$.

   Case 1: Suppose $x \in C$. Since $x \in A$ and $x \in C$, $x \in A \cap C$. Since $A \cap C = B \cap C$, $x \in B \cap C$. Thus $x \in B$.

Case 2: Suppose $x \notin C$. Since $x \in A$, $x \in A \cup C$. Since $A \cup C = B \cup C$, $x \in B \cup C$. Since $x \notin C$, $x \in B$.

$\Longleftarrow$) Let $x \in B$.

Case 1: Suppose $x \in C$. Since $x \in B$ and $x \in C$, $x \in B \cap C$. Since $B \cap C = A \cap C$, $x \in A \cap C$. Thus $x \in A$.

Case 2: Suppose $x \notin C$. Since $x \in B$, $x \in B \cup C$. Since $B \cup C = A \cup C$, $x \in A \cup C$. Since $x \notin C$, $x \in A$. $\square$

6.3 #5

Let $s : \mathbb{N} \to \mathbb{N}$, where for each $n \in \mathbb{N}, s(n)$ is the sum of the distinct natural number divisors of n. This is the **sum of the divisors function** that was introduced in Preview Activity 2 from Section 6.1. Is s an injection? Is s a surjection? Justify your conclusions.
Solution:s is not an injection. $s(6) = s(11) = 12$. s also is not a surjection. There is no preimage for $s(x) = 2$ since $s(n) > n$ and $s(1)$ and $s(2)$ do not equal 2.

8.1 #5d

For $a = 12628$ and $b = 21361$, use the Euclidean Algorithm to find $\gcd(a, b)$ and to write $\gcd(a, b)$ as a linear combination of a and b. That is, find integers m and n such that $d = am + bd$.
Solution: Using the Euclidean algorithm yeilds

$$\begin{aligned}
21361 &= 12628 + 8733 \\
12628 &= 8733 + 3895 \\
8733 &= 2(3895) + 943 \\
3895 &= 4(943) + 123 \\
943 &= 7(123) + 82 \\
123 &= 82 + 41 \\
82 &= 2(41) + 0
\end{aligned}$$

Now going back up the chain

$$\begin{aligned}
41 &= 123 - 82 \\
&= 123 - [943 - 7(123)] \\
&= -943 + 8(123) \\
&= -943 + 8[3895 - 4(943)] \\
&= 8(3895) - 33(943) \\
&= 8(3895) - 33[8733 - 2(3895)] \\
&= -33(8733) + 74(3895) \\
&= -33(8733) + 74(12628 - 8733) \\
&= 74(12628) - 107(8733) \\
&= 74(12628) - 107(21361 - 12628) \\
&= -107(21361) + 181(12628)
\end{aligned}$$