Assignment 5

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11. Consider a hereditary set system $M = (S, I)$ (see section 1.64 in the CLRS text). Show that if for every possible collection of positive weights on the elements of S, it is the case that the natural greedy algorithm (given on page 440 of the CLRS text) returns the maximum weight independent set, then M is a matroid.

Answer: Assume M is not a matroid, then M must not have exchange property. Let A and B be elements of I, where $|A| < |B|$ and $|B| = |A| + 1$. Since M is not a matroid, it means for $x \in B - A$, $A \cup \{x\} \notin I$. Let w_1 be the weight for elements of $A \cap B$, w_2 be the weight for elements of $A - B$, w_3 be the weight for elements of $B - A$, and w_4 be the weight of elements for the remaining elements. Let $w_1 > w_2 > w_3 > w_4$ (since the question already says M is a hereditary set system and for every possible collection of positive weights of the elements of S).

Since natural greedy algorithm is being used, it will first pick elements in $A \cup B$, then pick elements in $A - B$. However, since M does not have exchange property, the algorithm won't pick any element in $B - A$. We assume A is not maximum weight independent set, so it means greedy has to keep picking elements from the remaining elements which have weight w_4. Let m be the size of the maximum independent set. Since greedy returns the maximum weight independent set, the weight of the returning result is larger than the weight of maximum independent set that contains B. Therefore , we have $|A \cup B|w_1 + |A - B|w_2 + (m - |A|)w_4 > |A \cup B|w_1 + |B - A|w_2 + (m - |B|)w_4$. Since we assume that $|B| = |A| + 1$, we can get $|A - B| + 1 = |B - A|$. Using this, we can rearranged the equation to $(w_2 - w_3)|B - A| > w_2 - w_4$. Only when $w_3 < w_4$ then this equation can work. This contradict to our assumption. So we can conclude that if the natural greedy algorithm returns the maximum weight independent set, then M is a matroid.