

BASIS PARTITION POLYNOMIALS, OVERPARTITIONS AND THE ROGERS-RAMANUJAN IDENTITIES

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Dedicated with admiration to my friend, Richard Askey

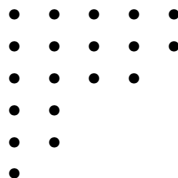
ABSTRACT. In this paper, a common generalization of the Rogers-Ramanujan series and the generating function for basis partitions is studied. This leads naturally to a sequence of polynomials, called BsP-polynomials. In turn, the BsP-polynomials provide simultaneously a proof of the Rogers-Ramanujan identities and a new, more rapidly converging series expansion for the basis partition generating function. Finally the basis partitions are identified with a natural set of overpartitions.

1. INTRODUCTION

The late Hansraj Gupta [8] introduced the concept of basis partitions. Basis partitions are defined in terms of successive ranks [6] or the “rank vector” of a partition.

Namely, each partition, π , of a positive integer contains a largest square of nodes in its Ferrers graph. This square is called the Durfee square. If the Durfee square has side d , we define the i^{th} rank r_i of π ($1 \leq i \leq d$) as the difference between the number of nodes in the i^{th} row of the Ferrers graph of π and the number in the i^{th} column. The rank vector for π is (r_1, r_2, \dots, r_d) .

For example, if π is the partition $5 + 5 + 4 + 2 + 2 + 1$, then its Ferrers graph is:



Its rank vector is $(-1, 0, 1)$

Gupta [8] showed that for every rank vector, \vec{r} , there is a smallest integer that has a partition with rank vector \vec{r} , and that partition is unique. This partition is called the basis partition for \vec{r} . We let $B(n)$ denote the number of basis partitions of n .

For example, the basis partition for $(-1, 0, 1)$ is $4 + 4 + 4 + 2 + 1$.

In [11], Nolan, Savage and Wilf showed that

$$(1.1) \quad \sum_{n=0}^{\infty} B(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q)_n}{(q; q)_n},$$

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where

$$(1.2) \quad (A; q)_n = (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}).$$

Hirschhorn [10] gave a new proof of (1.1) and related basis partitions to the Rogers-Ramanujan series from the first Rogers-Ramanujan identity [5, p. 113]:

$$(1.3) \quad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}$$

Our central object is to study

$$(1.4) \quad G(a, x; q) := \sum_{n=0}^{\infty} \frac{a^n q^{n^2} (x; q)_n}{(q; q)_n}.$$

Following the work of Nola, Savage and Wilf [11] and of Hirschhorn [10], Alladi had in 2007 considered $G(-1, -zq; q)$ and had interpreted the power of z as representing the signature of a basis partition (namely the number of different parts below the Durfee square); he then studied basis partitions combinatorially [1] with emphasis on the signature.

Notice that if we set $x = -q$ and set $a = 1$ in (1.4), we get the series in (1.1), and if we set $x = 0$ and $a = 1$ we get the series in (1.3). We want to find an identity for $G(a, x; q)$ which both leads directly to the Rogers-Ramanujan identities and also provides a new representation of the series in (1.1).

We shall prove

Theorem 1.

$$(1.5) \quad G(a, x; q) = \frac{1}{(aq; q)_{\infty}} \left(1 + \sum_{n=1}^{\infty} \frac{(aq; q)_{n-1} (1 - aq^{2n}) (-1)^n q^{n(3n-1)/2} a^n B_n(a, x)}{(q; q)_n} \right)$$

where

$$(1.6) \quad B_n(a, x) = \sum_{j=0}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix} (x; q)_j a^j x^{n-j} q^{nj},$$

and

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{cases} \frac{(q; q)_n}{(q; q)_j (q; q)_{n-j}} & \text{if } 0 \leq j \leq n \\ 0 & \text{if } j < 0 \text{ or } j > n. \end{cases}$$

It is immediate from (1.6) that $B_n(a, 0) = a^n q^{n^2}$, which implies, by setting $x = 0$ in (1.5),

Corollary 2.

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q; q)_n} = \frac{1}{(aq; q)_{\infty}} \left(1 + \sum_{n=0}^{\infty} \frac{(aq; q)_{n-1} (1 - aq^{2n}) (-1)^n a^{2n} q^{n(5n-1)/2}}{(q; q)_n} \right).$$

Corollary 2 is one of the standard identities (e.g. see [9, ch. 19]) used to deduce both Rogers-Ramanujan identities.

We shall call the $B_n(a, x)$, the Basis Partition polynomials, abbreviated BsP-Polynomial.

The BsP-polynomials have a variety of representations which we explore in Section 3.

In section 4, we prove that the basis partitions of n are in one-to-one correspondence with the overpartitions of n in which there are no overlined 1's, the difference between all parts is at least 2, and greater than 2 between parts where the larger is overlined. This is very much analogous to the Göllnitz-Gordon Theorem for ordinary partitions [5, p. 114].

2. PROOF OF THEOREM 1.

We recall the weak form of Bailey's lemma [4, Th.2, p. 273]:

If

$$(2.1) \quad \beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}},$$

then

$$(2.2) \quad \sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq; q)_{\infty}} \sum_{r=0}^{\infty} a^r q^{r^2} \alpha_r,$$

Furthermore the relation (2.1) can be inverted [4, p. 278, eq (4.1)]:

$$(2.3) \quad \alpha_n = \frac{(-1)^n q^{\binom{n}{2}} (a; q)_n (1 - aq^{2n})}{(q; q)_n (1 - a)} \sum_{j=0}^n (q^{-n}; q)_j (aq^n; q)_j q^j \beta_j.$$

Consequently comparing (2.2) with (1.5) we see that to establish (1.5) we need only show that

$$(2.4) \quad B_n(a, x) = \sum_{j=0}^n \frac{(q^{-n}; q)_j (aq^n; q)_j (x; q)_j q^j}{(q; q)_j}$$

Now by [7, p. 242, eq. (III.13)], with b replaced by d/b and then $d = 0$, $e \rightarrow \infty$] we see that

$$(2.5) \quad \sum_{j=0}^n \frac{(q^{-n}; q)_j (c; q)_j (b; q)_j q^j}{(q; q)_j} = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (c; q)_j b^j c^{n-j},$$

and if we set $c = x$, $b = aq^n$ in (2.5) we deduce (2.4). \square

3. PROPERTIES OF $B_n(a, x)$

There are several representations of $B_n(a, x)$ as well as a second order linear recurrence which we include in the following results.

Theorem 3.

$$(3.1) \quad B_n(a, x) = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (aq^n; q)_{n-j} x^{n-j} a^j q^{nj}$$

$$(3.2) \quad = \sum_{r,s \geq 0} \begin{bmatrix} n \\ n, r, n-r-s \end{bmatrix} (-1)^{n+r+s} (aq^n)^{n-s} x^{n-r} q^{\binom{n-r-s}{2}},$$

where

$$\begin{bmatrix} n \\ n, r, n-r-s \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix} = \frac{(q; q)_n}{(q; q)_r (q; q)_s (q; q)_{n-r-s}}$$

Proof. From (2.4) it is clear that $B_n(a, x)$ is symmetric in x and aq^n . Hence (3.1) is valid. To prove (3.2), we first prove

$$(3.3) \quad \sum_{j=0}^n \frac{(q^{-n}; q)_j (c; q)_j (b; q)_j q^j}{(q; q)_j} = \sum_{r, s \geq 0} \begin{bmatrix} n \\ r, s \end{bmatrix} (-1)^{n+r+s} c^{n-r} b^{n-s} q^{\binom{n-r-s}{2}}$$

This is proved as follows

$$(3.4) \quad \begin{aligned} \sum_{j=0}^n \frac{(q^{-n}; q)_j (c; q)_j (b; q)_j q^j}{(q; q)_j} &= \sum_{j=0}^n \frac{(q^{-n}; q)_j q^j}{(q; q)_j} \sum_{r \geq 0} \begin{bmatrix} j \\ r \end{bmatrix} (-1)^r c^r q^{\binom{r}{2}} \sum_{s \geq 0} \begin{bmatrix} j \\ s \end{bmatrix} (-1)^s b^s q^{\binom{s}{2}} \\ &= \sum_{r \geq 0} \frac{(-1)^r c^r q^{\binom{r}{2}}}{(q; q)_r} \sum_{s \geq 0} \frac{(-1)^s b^s q^{\binom{s}{2}}}{(q; q)_s} \sum_{j \geq 0} \frac{(q^{-n}; q)_j (q; q)_j q^j}{(q; q)_{j-r} (q; q)_{j-s}} \end{aligned}$$

In light of the fact that the inner sum is symmetric in r and s , we can without loss of generality assume that $r \geq s$. Hence

$$(3.5) \quad \begin{aligned} \sum_{j \geq 0} \frac{(q^{-n}; q)_j (q; q)_j q^j}{(q; q)_{j-r} (q; q)_{j-s}} &= \frac{(q^{-n}; q)_r (q; q)_r q^r}{(q; q)_{r-s}} \sum_{j \geq 0} \frac{(q^{-n+r}; q)_j (q^{r+1}; q)_j q^j}{(q; q)_j (q^{r-s+1}; q)_j} \\ &= \frac{(q^{-n}; q)_r (q; q)_r q^r}{(q; q)_{r-s}} \frac{q^{(r+1)(n-r)} (q^{-s}; q)_{n-r}}{(q^{r-s+1}; q)_{n-r}} \end{aligned}$$

(by [7, p. 11, eq. (1.5.3)]).

Substituting (3.5) in (3.4) and simplifying, we find

$$\begin{aligned} \sum_{j=0}^n \frac{(q^{-n}; q)_j (c; q)_j (b; q)_j q^j}{(q; q)_j} &= \sum_{r, s \geq 0} \frac{(q; q)_n (-1)^{n+r+s} q^{\binom{s+r-n}{2}} c^r b^s}{(q; q)_{n-r} (q; q)_{n-s} (q; q)_{r+s-n}} \\ &= \sum_{r, s \geq 0} \frac{(q; q)_n (-1)^{n+r+s} q^{\binom{n-r-s}{2}} c^{n-r} b^{n-s}}{(q; q)_r (q; q)_s (q; q)_{n-r-s}} \end{aligned}$$

(where $r \rightarrow n-r$ and $s \rightarrow n-s$), which is (3.2). \square

Theorem 4.

$$(3.6) \quad (x-1)B_n(a, x) = A_n B_{n+1}(a, x) - (A_n - C_n)B_n(a, x) - C_n B_{n-1}(a, x)$$

where

$$A_n = \frac{(1 - aq^n)}{(1 - aq^{2n})(1 - aq^{2n+1})},$$

and

$$C_n = \frac{(1 - q^n)a^2 q^{3n-1}}{(1 - aq^{2n-1})(1 - aq^{2n})}$$

Proof. This follows immediately from [7, p. 186, Ex. 7.10] with b replaced by $\frac{b}{aq}$, then a and c are set to 0 and then b is replaced by a new a . \square

We conclude this section with the following identity for the generating function for $B(n)$ given in (1.1).

Corollary 5.

$$(3.7) \quad \sum_{n=0}^{\infty} B(n)q^n := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q)_n}{(q; q)_n} = \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{n=1}^{\infty} (1+q^n)(-1)^n q^{n(3n-1)/2} b_n \right)$$

where b_n is given by $b_0 = 1, b_1 = q^2$, and for $n > 0$,

$$(3.8) \quad (1 - q^{2n-1})b_{n+1} = -(1 + q^n)(q + q^n - 2q^{2n} - q^{2n+1} - q^{2n+2} + q^{3n} + q^{4n+1})b_n + q^{3n-1}(1 - q^{2n+1})b_{n-1},$$

Proof. In Theorem 1, set $a = 1$, $x = -q$, then simplify and invoke Theorem 4 to determine the recurrence for $b_n = B_n(1, -q)$. \square

4. OVERPARTITIONS

Overpartitions are partitions in which one part of each size *may* be overlined. For example, the eight overpartitions of 3 are 3, $\overline{3}$, 2 1, 2 $\overline{1}$, $\overline{2}$ 1, $\overline{2}$ $\overline{1}$, 1 1 1, $\overline{1}$ 1 1. It is really a matter of taste whether one interprets basis partitions in terms of overpartitions as we now do, or as weights which are powers of 2 attached to Rogers-Ramanujan partitions (cf. Alladi [1], Hirschhorn [10]).

Theorem 6. *The number of basis partitions of n equals the number of overpartitions of n in which there are no overlined 1's, the difference between all parts is at least 2 and more than 2 when the larger part is overlined.*

Proof. Let us consider

$$(4.1) \quad \begin{aligned} G(a, -q; q) - G(aq, -q; q) &= \sum_{n=0}^{\infty} \frac{a^n q^{n^2} (-q; q)_n (1 - q^n)}{(q; q)_n} \\ &= \sum_{n=1}^{\infty} \frac{a^n q^{n^2} (-q; q)_n}{(q; q)_{n-1}} \\ &= \sum_{n=1}^{\infty} \frac{a^{n+1} q^{n^2+2n+1} (-q; q)_n (1 + q^{n+1})}{(q; q)_n} \\ &= aqG(aq^2, -q; q) + aq^2G(aq^3, -q; q). \end{aligned}$$

So if we define

$$G(a, -q; q) := \sum_{n, m \geq 0} g(m, n) a^m q^n,$$

then clearly,

$$(4.2) \quad g(m, n) = \begin{cases} 1 & \text{if } m = n = 0 \\ 0 & \text{if } m < 0 \text{ or } n < 0 \\ 0 & \text{if } m = 0 \text{ or } n = 0 \text{ but not both} \end{cases}$$

and by (4.1)

$$(4.3) \quad g(m, n) = g(m, n - n) + g(m - 1, n - 2m + 1) + g(m - 1, n - 3m + 1)$$

Clearly $g(m, n)$ is uniquely defined for all integers m and n by (4.2) and (4.3).

Now let $\gamma(m, n)$ denote the number of overpartitions of n with m parts with the proviso that there are no overlined ones, the difference between parts is at least 2, and greater than 2 if the smaller part is overlined.

The fact that $\gamma(m, n)$ satisfies (4.2) is immediate once we note that the empty partition of zero explains $\gamma(0, 0) = 1$.

Next consider (4.3). We split the partitions enumerated by $\gamma(m, n)$ up into three classes: (i) no 1 and no $\bar{2}$, (ii) a 1, (iii) a $\bar{2}$.

We transform the partitions in class (i) by subtracting 1 from each part. This reveals that there is a bijection between these partitions and $\gamma(m, n - m)$.

We transform the partitions in (ii) by deleting the 1 and subtracting 2 from each remaining part. The resulting partitions are just those enumerated by $\gamma(m - 1, n - 2(m - 1) - 1)$.

Finally we transform the partition in (iii) by deleting the $\bar{2}$ and subtracting 3 from each remaining part. The resulting partitions are precisely those enumerated by $\gamma(m - 1, n - 3(m - 1) - 2)$.

The above transformations are uniquely reversible. Hence we see that (4.3) holds for $\gamma(m, n)$.

Consequently,

$$g(m, n) = \gamma(m, n),$$

and so

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q)_n}{(q; q)_n} = \sum_{n \geq 0} \left(\sum_{m \geq 0} \gamma(m, n) \right) q^n;$$

Thus our theorem is proved. \square

5. CONCLUSION

The BsP-Polynomials are actually specializations of the Al-Salam and Carlitz polynomials [3]. Unfortunately, while (2.4) reveals that BsP-polynomials to be a limiting case of the Big q -Jacobi polynomials [7, p. 167, eq. (7.3.10)], the limiting process destroys orthogonality on the real line.

The series

$$G(1, -q; q^2) = \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q; q^2)_n}{(q^2; q^2)_n}$$

also has combinatorial interest. Alladi [2] has noted that this is the generating function for partitions into parts each ≥ 2 and differing by ≥ 4 with strict inequality if either part is odd. He has shown how these special partitions play an interesting role in the study of partitions with nonrepeating odd parts and related basis partitions.

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