EULER'S PENTAGONAL NUMBER THEOREM AND THE ROGERS-FINE IDENTITY

GEORGE E. ANDREWS AND JORDAN BELL

Abstract. Euler discovered the pentagonal number theorem in 1740 but was not able to prove it until 1750. He sent the proof to Goldbach and published it in a paper that finally appeared in 1760. Moreover, Euler formulated another proof of the pentagonal number in his notebooks theorem around 1750. Euler did not publish this proof or communicate it to his correspondents, probably because of the difficulty of clearly presenting it with the notation at the time. In this paper we show that the method of Euler's unpublished proof can be used to give a new proof of the celebrated Rogers-Fine identity.

1. Introduction

Euler's pentagonal number theorem is the assertion that for $|q| < 1$,

$$\prod_{j=1}^{\infty}(1-q^j) = 1 + \sum_{j=1}^{\infty}(-1)^j\left(q^{j(3j-1)/2} + q^{j(3j+1)/2}\right). \tag{1}$$

Euler's work on the pentagonal number theorem and his notebook entries on it are described in [5]. The contents of Euler's notebooks are described in several papers in the collection [6].

Euler's published proof of the pentagonal number theorem is closely detailed in [5, pp. 327–339]. Let

$$A_N = \sum_{n=1}^{\infty} q^{N(n-1)}(1-q^N)\cdots(1-q^{n+N-1}).$$

Euler uses the identity

$$\prod_{n=1}^{\infty}(1-a_n) = 1 - a_1 - \sum_{n=2}^{\infty} a_n(1-a_1)\cdots(1-a_{n-1})$$

to get

$$\prod_{n=1}^{\infty}(1-q^n) = 1 - q - \sum_{n=2}^{\infty} q^n(1-q)\cdots(1-q^{n-1})$$

and so $\prod_{n=1}^{\infty}(1-q^n) = 1-q-q^2A_1$. Euler proves that $A_N = 1-q^{2N+1}-q^{3N+2}A_{N+1}$, and then iterates this to obtain the pentagonal number theorem. Andrews [3] gives a modern exposition of Euler's proof which shows that the proof can readily be extended to prove a more general result.

Euler's unpublished proof of the pentagonal number theorem was substantially more complicated than his published proof. The unpublished proof is described

Date: September 21, 2010.

in [5, pp. 340–345]. It proceeds in the following way. Let $S = \prod_{j=1}^{\infty}(1-q^j)$, let $S_N = 1 + \sum_{j=1}^{N} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{(1-q)\cdots(1-q^j)}$, and let

$$(2) \quad S_{M,N} = 1 + \sum_{j=1}^{M-1} (-1)^j \left(q^{j(3j-1)/2} + q^{j(3j+1)/2} \right) + \sum_{j=M}^{N} \frac{(-1)^j q^{(M-1)j + \frac{j(j+1)}{2}}}{(1-q^M)\cdots(1-q^j)}.$$

It is not difficult to show that $S = S_\infty$ [2, p. 19, Corollary 2.2]. Euler then proves that $S_\infty = S_{2,\infty}$ and that $S_{M,\infty} = S_{M+1,\infty}$ for all $M \geq 1$. Thus $S_\infty = S_{\infty,\infty}$, and therefore $S = S_{\infty,\infty} = 1 + \sum_{j=1}^{\infty}(-1)^j \left(q^{j(3j-1)/2} + q^{j(3j+1)/2} \right)$, which is the pentagonal number theorem.

Euler thought it to be illuminating to prove the same thing in multiple ways. In [7], a paper on spherical trigonometry where he proves some known results using the calculus of variations, Euler says "& ensuite il est toujours utile de parvenir par des routes differentes aux memes vérités, puisque notre esprit ne manque pas d'en tirer de nouveaux éclaircissemens." ["And then, it is always useful to arrive at the same truths by different routes, because our minds will not fail to draw from them new enlightenment."] [7, p. 224]

For $j \geq 1$, define

$$(a)_j = (1-a)(1-aq)\cdots(1-aq^{j-1})$$

and

$$(a)_0 = 1.$$

Further, let

$$(a)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

We define

$$F(a,b,t) = \sum_{j=0}^{\infty} \frac{(aq)_j t^j}{(bq)_j}.$$

The Rogers-Fine identity is

$$(3) \qquad F(a,b,t) = \sum_{j=0}^{\infty} \frac{(aq)_j (\frac{atq}{b})_j b^j t^j q^{j^2} (1-atq^{2j+1})}{(bq)_j (t)_{j+1}}.$$

The Rogers-Fine identity has had a variety of applications to q-series and the theory of partitions; see [1], [4] and [8]. It is interesting to note that Ramanujan must have known this identity even though he never wrote it down in full. There are numerous instances of it in the Lost Notebook. Indeed, a full chapter of [4, Chap. 9] is devoted to special cases that Ramanujan discovered.

The Rogers-Fine identity is proved by Rogers [9, pp. 334–335] and Fine [8, p. 15, Eq. 14.1]. The Rogers-Fine identity is proved combinatorially by Andrews [1, pp. 572–574].

Fine [8, p. 2, Eq. 4.1] shows that

$$F(a,b,t) = \frac{1-atq}{1-t} + \frac{(1-aq)(b-atq)}{(1-bq)(1-t)} tqF(aq,bq,tq),$$

and proves the Rogers-Fine by iterating this result.

2. An Eulerian treatment of Rogers-Fine

Let us define

$$F(a,b,t,N)=\sum_{j=0}^{N}\frac{(aq)_j t^j}{(bq)_j}.$$

We will derive a functional equation for $F(a,b,t,N)$ like Fine's [8, p. 2, Eq. 4.1] for $F(a,b,t)$.

$$\begin{aligned}
&F(a,b,t,N)-\frac{1-atq}{1-t}-\frac{(1-aq)(b-atq)}{(1-bq)(1-t)}tqF(aq,bq,tq,N-1)\\
&=\sum_{j=0}^{N}\frac{(aq)_j t^j}{(bq)_j}-\frac{1-atq}{1-t}-\frac{(1-aq)(b-atq)tq}{(1-bq)(1-t)}\sum_{j=0}^{N-1}\frac{(aq^2)_j t^j q^j}{(bq^2)_j}\\
&=\sum_{j=0}^{N}\frac{(aq)_j t^j}{(bq)_j}-\frac{1-atq}{1-t}-\frac{b-atq}{1-t}\sum_{j=0}^{N-1}\frac{(aq)_{j+1}t^{j+1}q^{j+1}}{(bq)_{j+1}}\\
&=\sum_{j=0}^{N}\frac{(aq)_j t^j}{(bq)_j}-\frac{1-atq}{1-t}-\frac{b-atq}{1-t}\sum_{j=1}^{N}\frac{(aq)_j t^j q^j}{(bq)_j}\\
&=\sum_{j=0}^{N}\frac{(aq)_j t^j}{(bq)_j}-\frac{1-atq}{1-t}-\frac{b-atq}{1-t}\sum_{j=0}^{N}\frac{(aq)_j t^j q^j}{(bq)_j}+\frac{b-atq}{1-t}\\
&=\sum_{j=0}^{N}\frac{(aq)_j t^j}{(bq)_j}\Big(1-\frac{(b-atq)q^j}{1-t}\Big)+\frac{b-1}{1-t}.
\end{aligned}$$

We define

$$R_{1,N}(a,b,t)=\sum_{j=0}^{N}\frac{(aq)_j t^j}{(bq)_j}\Big(1-\frac{(b-atq)q^j}{1-t}\Big)+\frac{b-1}{1-t}.$$

One checks that

$$R_{1,1}(a,b,t)=\frac{-t^2(aq)_2}{(1-t)(bq)_1}.$$

We assume that for some $N\geq 1$,

$$R_{1,N}(a,b,t)=\frac{-t^{N+1}(aq)_{N+1}}{(1-t)(bq)_N}. \tag{4}$$

Then

$$
\begin{aligned}
&R_{1,N+1}(a,b,t)\\
=&\frac{(aq)_{N+1}t^{N+1}}{(bq)_{N+1}}\Big(1-\frac{(b-atq)q^{N+1}}{1-t}\Big)+R_{1,N}(a,b,t)\\
=&\frac{(aq)_{N+1}t^{N+1}}{(bq)_{N+1}}\Big(1-\frac{(b-atq)q^{N+1}}{1-t}\Big)-\frac{t^{N+1}(aq)_{N+1}}{(1-t)(bq)_N}\\
=&\frac{(aq)_{N+1}t^{N+1}}{(bq)_N}\Big(\frac{1}{1-bq^{N+1}}-\frac{(b-atq)q^{N+1}}{(1-t)(1-bq^{N+1})}-\frac{1}{1-t}\Big)\\
=&\frac{(aq)_{N+1}t^{N+1}}{(bq)_N}\cdot\frac{-t+atq^{N+2}}{(1-t)(1-bq^{N+1})}\\
=&\frac{-(aq)_{N+2}t^{N+2}}{(1-t)(bq)_{N+1}}.
\end{aligned}
$$

Therefore by mathematical induction, (4) holds for all N.

Lemma 1. *For $N \geq 1$,*

$$F(a,b,t,N)=\frac{1-atq}{1-t}+\frac{(1-aq)(b-atq)}{(1-bq)(1-t)}tqF(aq,bq,tq,N-1)+R_{1,N}(a,b,t).$$

This is a partial sum version of Fine's [8, p. 2, Eq. 4.1]; Fine's functional equation is obtained by taking $N=\infty$ here.

Next for $N \geq M \geq 1$ we define

$$
\begin{aligned}
(5)\quad S_{M+1,N}(a,b,t) \quad = \quad &\sum_{j=0}^{M-1}\frac{(aq)_j(\frac{atq}{b})_j b^j t^j q^{j^2}(1-atq^{2j+1})}{(bq)_j(t)_{j+1}}\\
&+\frac{(aq)_M(\frac{atq}{b})_M t^M b^M q^{M^2}}{(bq)_M(t)_M}F(aq^M,bq^M,tq^M,N-M).
\end{aligned}
$$

We show in the following lemma that the series (5) specializes to the series $S_{M,N}$ (2) that is used in Euler's proof.

Lemma 2. $S_{M,N}=\lim_{t\to 0}S_{M,N}\Big(\frac{1}{t},1,t\Big)$.

Proof. Putting $a = \frac{1}{t}$ and $b = 1$ in the definition (5) gives us

$$\begin{aligned}
S_{M,N}\Big(\frac{1}{t},1,t\Big) &= \sum_{j=0}^{M-2} \frac{(\frac{q}{t})_j (q)_j t^j q^{j^2}(1-q^{2j+1})}{(q)_j (t)_{j+1}} \\
&\quad + \frac{(\frac{q}{t})_{M-1}(q)_{M-1} t^{M-1} q^{(M-1)^2}}{(q)_{M-1}(t)_{M-1}} F\Big(\frac{q^{M-1}}{t}, q^{M-1}, tq^{M-1}, N+1-M\Big) \\
&= \sum_{j=0}^{M-2} \frac{(\frac{q}{t})_j t^j q^{j^2}(1-q^{2j+1})}{(t)_{j+1}} \\
&\quad + \frac{(\frac{q}{t})_{M-1} t^{M-1} q^{(M-1)^2}}{(t)_{M-1}} F\Big(\frac{q^{M-1}}{t}, q^{M-1}, tq^{M-1}, N+1-M\Big) \\
&= \sum_{j=0}^{M-2} \frac{(\frac{q}{t})_j t^j q^{j^2}(1-q^{2j+1})}{(t)_{j+1}} \\
&\quad + \frac{(\frac{q}{t})_{M-1} t^{M-1} q^{(M-1)^2}}{(t)_{M-1}} \sum_{j=0}^{N+1-M} \frac{(\frac{q^M}{t})_j (tq^{M-1})^j}{(q^M)_j}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{t\to 0} S_{M,N}\Big(\frac{1}{t},1,t\Big) &= \sum_{j=0}^{M-2} (-1)^j q^{j(j+1)/2} q^{j^2}(1-q^{2j+1}) \\
&\quad + (-1)^{M-1} q^{\frac{(M-1)M}{2}} q^{(M-1)^2} \sum_{j=0}^{N+1-M} (-1)^j \frac{q^{Mj} q^{\frac{(j-1)j}{2}} q^{(M-1)j}}{(q^M)_j}
\end{aligned}$$

The first term of the right-hand side is equal to

$$1 - (-1)^{M-1} q^{\frac{(M-1)(3M-2)}{2}} + \sum_{j=1}^{M-1} (-1)^j \Big(q^{\frac{j(3j-1)}{2}} + q^{\frac{j(3j+1)}{2}}\Big).$$

The second term of the right-hand side is equal to

$$\sum_{j=M-1}^{N} \frac{(-1)^j q^{(M-1)j+\frac{j^2+j}{2}}}{(q^M)_{j-M+1}} = (-1)^{M-1} q^{\frac{(M-1)(3M-2)}{2}} + \sum_{j=M}^{N} \frac{(-1)^j q^{(M-1)j+\frac{j^2+j}{2}}}{(q^M)_{j-M+1}}.$$

Therefore

$$\begin{aligned}
\lim_{t\to 0} S_{M,N}\Big(\frac{1}{t},1,t\Big) &= 1 + \sum_{j=1}^{M-1} (-1)^j \Big(q^{\frac{j(3j-1)}{2}} + q^{\frac{j(3j+1)}{2}}\Big) \\
&\quad + \sum_{j=M}^{N} \frac{(-1)^j q^{(M-1)j+\frac{j^2+j}{2}}}{(q^M)_{j-M+1}}.
\end{aligned}$$

□

Define $R_{M,N}(a,b,t)$ by

$$F(a,b,t,N) = S_{M+1,N}(a,b,t) + R_{M,N}(a,b,t). \tag{6}$$

Lemma 3.

$$R_{M+1,N}(a,b,t) = -\frac{t^{N+1}b^M q^{M(N+1)}(aq)_{N+1}(\frac{atq}{b})_M}{(t)_{M+1}(bq)_N} + R_{M,N}(a,b,t).$$

Proof. First,

$$S_{M+2,N}(a,b,t) + R_{M+1,N}(a,b,t) = S_{M+1,N}(a,b,t) + R_{M,N}(a,b,t).$$

We have to find $R_{M+1,N}(a,b,t) - R_{M,N}(a,b,t)$.

$$\begin{aligned}
&R_{M+1,N}(a,b,t) - R_{M,N}(a,b,t)\\
=&\sum_{j=0}^{M-1} \frac{(aq)_j(\frac{atq}{b})_j b^j t^j q^{j^2}(1-atq^{2j+1})}{(bq)_j(t)_{j+1}}\\
&+\frac{(aq)_M(\frac{atq}{b})_M t^M b^M q^{M^2}}{(bq)_M(t)_M} F(aq^M, bq^M, tq^M, N-M)\\
&-\sum_{j=0}^{M} \frac{(aq)_j(\frac{atq}{b})_j b^j t^j q^{j^2}(1-atq^{2j+1})}{(bq)_j(t)_{j+1}}\\
&-\frac{(aq)_{M+1}(\frac{atq}{b})_{M+1} t^{M+1} b^{M+1} q^{(M+1)^2}}{(bq)_{M+1}(t)_{M+1}} F(aq^{M+1}, bq^{M+1}, tq^{M+1}, N-M-1)\\
=&\frac{(aq)_M(\frac{atq}{b})_M t^M b^M q^{M^2}}{(bq)_M(t)_M} F(aq^M, bq^M, tq^M, N-M)\\
&-\frac{(aq)_M(\frac{atq}{b})_M b^M t^M q^{M^2}(1-atq^{2M+1})}{(bq)_M(t)_{M+1}}\\
&-\frac{(aq)_{M+1}(\frac{atq}{b})_{M+1} t^{M+1} b^{M+1} q^{(M+1)^2}}{(bq)_{M+1}(t)_{M+1}} F(aq^{M+1}, bq^{M+1}, tq^{M+1}, N-M-1).
\end{aligned}$$

Now, by Lemma 1 and (4),

$$\begin{aligned}
&F(aq^M, bq^M, tq^M, N-M)\\
=&\frac{1-atq^{2M+1}}{1-tq^M} + \frac{(1-aq^{M+1})(b-atq^{M+1})tq^{2M+1}}{(1-bq^{M+1})(1-tq^M)} F(aq^{M+1}, bq^{M+1}, tq^{M+1}, N-M-1)\\
&-\frac{t^{N-M+1}q^{MN-M^2+M}(aq^{M+1})_{N-M+1}}{(1-tq^M)(bq^{M+1})_{N-M}}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&R_{M+1,N}(a,b,t)-R_{M,N}(a,b,t)\\
=&\frac{(aq)_M(\frac{atq}{b})_M t^M b^M q^{M^2}(1-atq^{2M+1})}{(bq)_M(t)_{M+1}}-\frac{(aq)_M(\frac{atq}{b})_M t^{N+1} b^M q^{MN+M}(aq^{M+1})_{N-M+1}}{(bq)_M(t)_{M+1}(bq^{M+1})_{N-M}}\\
&-\frac{(aq)_M(\frac{atq}{b})_M b^M t^M q^{M^2}(1-atq^{2M+1})}{(bq)_M(t)_{M+1}}\\
=&\frac{(aq)_M(\frac{atq}{b})_M t^M b^M q^{M^2}(1-atq^{2M+1})}{(bq)_M(t)_{M+1}}-\frac{(aq)_{N+1}(\frac{atq}{b})_M t^{N+1} b^M q^{MN+M}}{(bq)_N(t)_{M+1}}\\
&-\frac{(aq)_M(\frac{atq}{b})_M t^M b^M q^{M^2}(1-atq^{2M+1})}{(bq)_M(t)_{M+1}}\\
=&-\frac{(aq)_{N+1}(\frac{atq}{b})_M t^{N+1} b^M q^{MN+M}}{(bq)_N(t)_{M+1}},
\end{aligned}
$$

which completes the proof. □

Now using this lemma we shall prove the following theorem. The theorem gives an expression for $R_{M,N}(a,b,t)$ which reveals that $R_{M,\infty}(a,b,t)=0$.

Theorem 4. *For $N \geq M \geq 2$,*

$$
\begin{aligned}
R_{M,N}(a,b,t) \quad = \quad & -\frac{t^{N+1}(aq)_{N+1}}{(bq)_N}\sum_{j=1}^{M-2}\frac{b^{j-1}q^{(j-1)(N+1)}(\frac{atq}{b})_{j-1}}{(t)_j}\\
& -\frac{t^{N+1}(aq)_{N+1}q^{(M-2)(N+1)}b^{M-2}(\frac{atq}{b})_{M-2}}{(bq)_N(t)_M}\\
& \cdot\Big((1+bq^{N+1})-tq^{M-1}(1+aq^{N+1})\Big).
\end{aligned}
\tag{7}
$$

Proof. Lemma 3 and (4) give

$$
\begin{aligned}
R_{2,N}(a,b,t) \quad &= \quad -\frac{t^{N+1}bq^{N+1}(aq)_{N+1}(1-\frac{atq}{b})}{(1-t)(1-tq)(bq)_N}-\frac{t^{N+1}(aq)_{N+1}}{(1-t)(bq)_N}\\
&= \quad -\frac{t^{N+1}(aq)_{N+1}}{(t)_2(bq)_N}\Big((1+bq^{N+1})-tq(1+aq^{N+1})\Big),
\end{aligned}
$$

so the theorem holds for $M=2$.

We assume now that for some $M \geq 2$, (7) holds for all $N \geq M$. Using Lemma 3 we get

$$\begin{aligned}
&R_{M+1,N}(a,b,t)\\
&= -\frac{t^{N+1}b^M q^{M(N+1)}(aq)_{N+1}(\frac{atq}{b})_M}{(t)_{M+1}(bq)_N}\\
&\quad -\frac{t^{N+1}(aq)_{N+1}}{(bq)_N}\sum_{j=1}^{M-2}\frac{b^{j-1}q^{(j-1)(N+1)}(\frac{atq}{b})_{j-1}}{(t)_j}\\
&\quad -\frac{t^{N+1}(aq)_{N+1}q^{(M-2)(N+1)}b^{M-2}(\frac{atq}{b})_{M-2}}{(bq)_N(t)_M}\Big((1+bq^{N+1})-tq^{M-1}(1+aq^{N+1})\Big)\\
&= -\frac{t^{N+1}b^M q^{M(N+1)}(aq)_{N+1}(\frac{atq}{b})_M}{(t)_{M+1}(bq)_N}\\
&\quad +\frac{t^{N+1}(aq)_{N+1}b^{M-2}q^{(M-2)(N+1)}(\frac{atq}{b})_{M-2}}{(bq)_N(t)_{M-1}}\\
&\quad -\frac{t^{N+1}(aq)_{N+1}}{(bq)_N}\sum_{j=1}^{M-1}\frac{b^{j-1}q^{(j-1)(N+1)}(\frac{atq}{b})_{j-1}}{(t)_j}\\
&\quad -\frac{t^{N+1}(aq)_{N+1}q^{(M-2)(N+1)}b^{M-2}(\frac{atq}{b})_{M-2}}{(bq)_N(t)_M}\Big((1+bq^{N+1})-tq^{M-1}(1+aq^{N+1})\Big)\\
&= -\frac{t^{N+1}b^M q^{M(N+1)}(aq)_{N+1}(\frac{atq}{b})_M}{(t)_{M+1}(bq)_N}\\
&\quad -\frac{t^{N+1}(aq)_{N+1}}{(bq)_N}\sum_{j=1}^{M-1}\frac{b^{j-1}q^{(j-1)(N+1)}(\frac{atq}{b})_{j-1}}{(t)_j}\\
&\quad -\frac{t^{N+1}(aq)_{N+1}b^{M-1}q^{(M-1)(N+1)}(\frac{atq}{b})_{M-1}}{(bq)_N(t)_M}.
\end{aligned}$$

We combine the first and third terms in the above expression to get

$$\begin{aligned}
&R_{M+1,N}(a,b,t)\\
&= -\frac{t^{N+1}(aq)_{N+1}}{(bq)_N}\sum_{j=1}^{M-1}\frac{b^{j-1}q^{(j-1)(N+1)}(\frac{atq}{b})_{j-1}}{(t)_j}\\
&\quad -\frac{t^{N+1}(aq)_{N+1}b^{M-1}q^{(M-1)(N+1)}(\frac{atq}{b})_{M-1}}{(t)_{M+1}(bq)_N}\Big((1+bq^{N+1})-tq^{M}(1+aq^{N+1})\Big),
\end{aligned}$$

which is (7) with M replaced by $M+1$. This proves the theorem. □

Now we produce an expression for the difference of $S_{M+1,N}(a,b,t)$ and $S_{M,N}(a,b,t)$. We will specialize this to obtain Euler's expression for the difference of $S_{M+1,N}$ and $S_{M,N}$ [5, pp. 344–345].

Theorem 5.

$$S_{M+1,N}(a,b,t) - S_{M,N}(a,b,t) = \frac{t^{N+1}(aq)_{N+1}(\frac{atq}{b})_{M-1}b^{M-1}q^{(M-1)(N+1)}}{(bq)_N(t)_M}.$$

Proof. Subtract an instance of (6) at $M-1$ from an instance at M.

$$S_{M+1,N}(a,b,t) - S_{M,N}(a,b,t) = R_{M-1,N}(a,b,t) - R_{M,N}(a,b,t)$$

By Lemma 3,

$$R_{M-1,N}(a,b,t) - R_{M,N}(a,b,t) = \frac{t^{N+1}b^{M-1}q^{(M-1)(N+1)}(aq)_{N+1}(\frac{atq}{b})_{M-1}}{(t)_M(bq)_N},$$

proving the claim. □

Therefore in particular,

$$\begin{aligned} S_{M+1,N} - S_{M,N} &= \lim_{t\to 0}\Big(S_{M+1,N}\Big(\frac{1}{t},1,t\Big) - S_{M,N}\Big(\frac{1}{t},1,t\Big)\Big) \\ &= \frac{(-1)^{N+1}q^{\frac{N(N+1)}{2}+MN}}{(1-q^M)\cdots(1-q^N)} \end{aligned}$$

So $S_{M+1,\infty} = S_{M,\infty}$ for $M \geq 2$. But by (6),

$$\lim_{t\to 0} F\Big(\frac{1}{t},1,t,\infty\Big) = \lim_{t\to 0} S_{M+1,\infty}\Big(\frac{1}{t},1,t\Big)$$

for all $M \geq 2$. Therefore

$$(q)_\infty = \sum_{j=0}^{\infty} \frac{(-1)^j q^{j(j+1)/2}}{(q)_j} = \lim_{t\to 0} F\Big(\frac{1}{t},1,t,\infty\Big) = S_{\infty,\infty},$$

the pentagonal number theorem (1).

Finally we are prepared to deduce the Rogers-Fine identity. By Theorem 5, $S_{M+1,\infty}(a,b,t) = S_{M,\infty}(a,b,t)$. Thus $S_{2,\infty}(a,b,t) = S_{\infty,\infty}(a,b,t)$. By Theorem 4, $F(a,b,t,\infty) = S_{2,\infty}(a,b,t)$. Therefore $F(a,b,t,\infty) = S_{\infty,\infty}(a,b,t)$. That is,

$$\sum_{j=0}^{\infty} \frac{(aq)_j t^j}{(bq)_j} = \sum_{j=0}^{\infty} \frac{(aq)_j(\frac{atq}{b})_j b^j t^j q^{j^2}(1-atq^{2j+1})}{(bq)_j(t)_{j+1}},$$

the Rogers-Fine identity (3).

3. Conclusion

In this paper we have extended the method of Euler's unpublished proof of the pentagonal number theorem to prove the Rogers-Fine identity. What we have done is similar to the work in [3], which extends the method of Euler's published proof of the pentagonal number theorem to prove something stronger.

One true lesson to be drawn is the importance of studying the work of the masters, even those like Euler who died more than two centuries ago. The cleverness of Euler's unpublished, second proof of the pentagonal number theorem allows us to discern facets of the Rogers-Fine identity that had gone unnoticed. We hope that this proof may suggest combinatorial intricacies of the Rogers-Fine identity that might be revealed by this newly exposed, yet old, approach.

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Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: andrews@math.psu.edu

Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada

E-mail address: jordan.bell@gmail.com