MacMahon's Dream

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Abstract

We shall provide an account of MacMahon's development of a calculational, analytic method designed to produce the generating function for plane partitions. His efforts did not turn out as he had hoped, and he had to spend nearly twenty years finding an alternative treatment. This paper provides an account of our retrieval of MacMahon's original dream of using Partition Analysis to treat plane partitions in general.

1 Introduction

Major P. A. MacMahon's collected papers fill two large volumes [22] and [23]. Among these are seven lengthy works entitled, "Memoir on the theory of the partitions of numbers, I-VII."

The first of these [18], also [22, pp. 1026–1080], appeared in 1895 when MacMahon was President of the London Mathematical Society. It was 65 pages long and was mostly a leisurely account of what MacMahon termed partitions of multipartite numbers.

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Multipartite numbers are in modern parlance n-tuples of non-negative integers. For example, $(7, 5, 0, 3)$ is a 4-partite number. Partitions of $(7, 5, 0, 3)$ are direct sums of 4-tuples of non-negative integers that add to $(7, 5, 0, 3)$. For example,

$$(7, 5, 0, 3) = (4, 1, 0, 2) + (2, 3, 0, 0) + (1, 1, 0, 1),$$

or

$$= (3, 3, 0, 3) + (4, 2, 0, 0).$$

MacMahon considers a variety of combinatorial and geometrical aspects of such partitions. Of special interest is the classical representation of ordinary, or unipartite partitions in "Sylvester-graphs" (today called Ferrers graphs). For example, the unipartite partition of 29 given by

$$7 + 7 + 5 + 4 + 4 + 2,$$

has the graphical representation

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & & &
\end{array}$$

where each row of nodes represents the corresponding part of the partition.

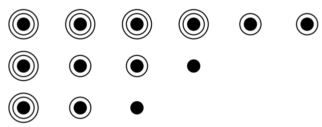
MacMahon then notes [22, p. 1058] that if one has a multipartite partition in which the Ferrers graph of each part contains the Ferrers graph of the next (called "the subjacent succession of lines" by him), then one may produce a three-dimensional analog of the Ferrers graph. Thus if we start with the "regularised" (i.e., the entries of the tuples involved are weakly decreasing) multipartite partition

$$\begin{aligned}
(16, 8, 6) &= (6, 4, 3) + (6, 3, 2) + (4, 1, 1) \\
&= A + B + C,
\end{aligned}$$

we may regard each of the parts as a unipartite partition, and the respective Ferrers graphs are

$$\begin{array}{lll}
A & B & C \\
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & &
\end{array}
&
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & &
\end{array}
&
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & & & \\
\bullet & & &
\end{array}
\end{array}$$

As MacMahon says [22, p. 1058], “it is clear that we may pile B upon A, and then C upon B & A, and thus form a three dimensional graph of the partition”



In subsequent papers, MacMahon will refer to this three-dimensional graph as representing the plane partition of 30 given by

$$\begin{matrix} 6 & 4 & 3 \\ 6 & 3 & 2 \\ 4 & 1 & 1 \end{matrix}$$

He next determines that there are three such partitions of 2, six of 3 and thirteen of 4. This leads him to the following conjecture [22, pp. 1064–1065];

> “The enumeration of the three-dimensional graphs that can be formed with a given number of nodes, corresponding to the regularised partitions of all multi-partite numbers of given content, is a weighty problem. I have verified to a high order that the generating function of the complete system is
>
> $$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-4}\cdots \textit{ ad inf.},$$
>
> and, so far as my investigations have proceeded, everything tends to confirm the truth of this conjecture.”

In Section 2, we shall look at MacMahon’s efforts to develop a calculus (later to be named Partition Analysis) that he hoped would allow him to prove his conjecture. In Section 3 we sketch our proof of all of MacMahon’s conjectures. We conclude with a brief account of our discoveries made using our Mathematica implementation of Partition Analysis, the package Omega which is freely available at [24].

2 Partition Analysis – the beginning

MacMahon [22, p. 1068] begins with some very simple problems. As an example, he considers plane partitions that only have 1’s and 2’s as parts and have only two columns. E.g.,

$$\begin{array}{cc} 2 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 1 \\ 1 & \\ 1 & \end{array} \tag{1}$$

The generating function for such partitions is

$$\sum_{\substack{n_1,n_2,m_1,m_2 \geqq 0 \\ n_1 \geqq n_2 \\ n_1+m_1 \geqq n_2+m_2}} x^{2n_1+2n_2+m_1+m_2}.$$

Here n_1 counts the number of 2's in the first column, n_2 the number in the second; m_1 counts the number of 1's in the first column, m_2 the number in the second.

He then utilizes an idea (traceable back to Cayley in invariant theory [22, p. 1142]) of coding the inequalities on the indices by considering

$$\sum_{n_1,n_2,m_1,m_2 \geqq 0} x^{2n_1+2n_2+m_1+m_2} a^{n_1+m_1-n_2-m_2} b^{n_1-n_2},$$

where all terms with negative exponents on either a or b will be thrown out and in all other terms a and b are set to 1. This device immediately allows all the series to be summed by the geometric series to

$$\begin{aligned} \frac{1}{(1-xa)\left(1-\frac{x}{a}\right)(1-abx^2)\left(1-\frac{x^2}{ab}\right)} &= \frac{1}{(1-xa)\left(1-\frac{x}{a}\right)(1-x^4)} \\ &\quad \times \left\{ \frac{1}{1-abx^2} + \frac{\frac{x^2}{ab}}{1-\frac{x^2}{ab}} \right\}. \end{aligned}$$

Now the second term inside the $\{\ \}$ has only negative powers of b and so can be dropped from consideration. The first term has only positive powers of b, and so we may set $b = 1$ in this term. Thus we have reduced the problem to considering

$$\begin{aligned}
&\frac{1}{(1-xa)\left(1-\frac{x}{a}\right)(1-x^4)(1-ax^2)}\\
&=\frac{1}{(1-x^4)\left(1-\frac{x}{a}\right)(1-x)}\left\{\frac{1}{1-ax}-\frac{x}{1-ax^2}\right\}\\
&=\frac{1}{(1-x^4)(1-x)}\\
&\times\left\{\frac{1}{1-x^2}\left(\frac{1}{1-ax}+\frac{\frac{x}{a}}{1-\frac{x}{a}}\right)-\frac{x}{1-x^3}\left(\frac{1}{1-ax^2}+\frac{\frac{x}{a}}{1-\frac{x}{a}}\right)\right\}.
\end{aligned}$$

As in the elimination of b, this reduces to

$$\begin{aligned}
&\frac{1}{(1-x^4)(1-x)}\left\{\frac{1}{(1-x^2)(1-x)}-\frac{x}{(1-x^3)(1-x^2)}\right\}\\
&\qquad\qquad=\frac{1}{(1-x)(1-x^2)^2(1-x^2)},
\end{aligned}$$

a result which, as MacMahon points out, is not obvious [22, p. 1068].

Now two things are clear. First, one must streamline this method which is cumbersome even in this simple example, and one must somehow introduce a simple notation for the deletion of terms with negative exponents on a and b. MacMahon turns his attention to these requirements in [19], and finally in his magnum opus [21, Vol. II, Sec. VIII], he has reduced the above treatment to the following. First he defines the omega operator [21, Vol. II, p. 92]

$$\underset{\geqq}{\Omega}\sum_{n_1,\ldots,n_j=-\infty}A(n_1,n_2,\ldots,n_j)\lambda_1^{n_1}\lambda_2^{n_2}\ldots\lambda_j^{n_j}=\sum_{n_1,n_2,\ldots,n_j\geqq 0}A(n_1,n_2,\ldots,n_j),$$

where $A(n_1,n_2,\ldots,n_j)$ is generally some rational function of variables like x, y or z.

Then MacMahon prepares a list of valid omega evaluations [21, Vol. II, p. 102] including

$$\underset{\geqq}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-\frac{z}{\lambda})}=\frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)}. \qquad (2)$$

Hence

$$
\begin{aligned}
&\sum_{\substack{n_1,n_2,m_1,m_2 \geqq 0 \\ n_1 \geqq n_2 \\ n_1+m_1 \geqq n_2+m_2}} x^{2n_1+2n_2+m_1+m_2} \\
= \; & \underset{\geqq}{\Omega} \frac{1}{(1-x\lambda_2)\left(1-\frac{x}{\lambda_2}\right)(1-\lambda_1\lambda_2 x^2)\left(1-\frac{x^2}{\lambda_1\lambda_2}\right)} \\
= \; & \underset{\geqq}{\Omega} \frac{1}{(1-x\lambda_2)\left(1-\frac{x}{\lambda_2}\right)(1-\lambda_2 x^2)(1-x^4)} \\
& \text{(by (2) with } x,\ y,\ z \text{ replaced by } \lambda_2 x^2, 0, \tfrac{x^2}{\lambda_2}, \text{ resp.)} \\
= \; & \frac{1}{(1-x^4)} \underset{\geqq}{\Omega} \frac{1}{(1-x\lambda_2)(1-x^2\lambda_2)\left(1-\frac{x}{\lambda_2}\right)} \\
= \; & \frac{1}{(1-x^4)} \cdot \frac{(1-x^4)}{(1-x)(1-x^2)(1-x^2)(1-x^3)} \\
& \text{(by (2) with } x,\ y,\ z \text{ replaced by } x, x^2, x, \text{ resp.)} \\
= \; & \frac{1}{(1-x)(1-x^2)^2(1-x^3)}.
\end{aligned}
$$

MacMahon clearly hoped to hone this tool into one that could prove his conjectures on plane partitions. Clearly the problems can all be set up in the language of his Partition Analysis. However, he was unable to develop this machinery adequately. Sadly he sets up the general problem [21, Vol. II, p. 186], but is forced to conclude: "Our knowledge of the Ω operation is not sufficient to enable us to establish the final form of result."

In the next section, we describe the work in [12] where we have overcome MacMahon's difficulties.

3 Partition Analysis — the dream

In our efforts to make MacMahon's dream come true the Omega package [24] has played a decisive role. Remarkably, MacMahon had already been aware of the algorithmic essence of Partition Analysis; see Section VIII of [21, Vol. 2, pp. 111–114] describing the "method of Elliott". However, 90 years before computer algebra systems emerged he was confined to use his technique essentially in the form of a table look-up method. After the first achievements of revitalizing Partition Analysis, [1] and [2], we have pursued the project of replacing MacMahon's transformation and elimination rules for his omega operator by a deterministic algorithmic procedure. Subsequently we have implemented these algorithms in the *Mathematica* system and called the corresponding package "Omega". For a description of this work and for a variety of new applications we refer to [3] – [12].

As an illustration we show how the example discussed in Section 2 can be treated with Omega in a fully automatic fashion. We initialize by loading the package

```
In[1]:=Omega2.m
```

Following MacMahon's terminology, the first step is to compute the "crude generating function". To this end one has only to input the problem in a form which is very close to the usual mathematical syntax. (All summation parameters are assumed to be non-negative, if not specified otherwise.)

In[2]:= Crude = OSum[$x^{2n_1+2n_2+m_1+m_2}, \{n_1 \geq n_2, n_1+m_1 \geq n_2+m_2\}, \lambda$]

Out[2]= $\underset{\lambda_1,\lambda_2}{\underset{\geq}{\Omega}} \frac{1}{(1-\frac{x}{\lambda_2})(1-\frac{x^2}{\lambda_1\lambda_2})(1-x\lambda_2)(1-x^2\lambda_1\lambda_2)}$

Finally the elimination of the λ variables is carried out by the procedure call

In[3]:= OR[Crude]
Eliminating λ_1...
Eliminating λ_2...

Out[3]= $\frac{1}{(1-x)(1-x^2)^2(1-x^3)}$

Note: During the computation the package tells the user in which order the elimination of the λ variables is carried out.

To present a brief account of how MacMahon's dream has come true we need a couple of definitions.

Given an $r \times c$ matrix $X = (x_{i,j})$ we define

$$p_{r,c}(X) := \sum_{(a_{i,j}) \in M_{r,c}} x_{1,1}^{a_{1,1}} \cdots x_{1,c}^{a_{1,c}} \cdots x_{r,1}^{a_{r,1}} \cdots x_{r,c}^{a_{r,c}}$$

where $M_{r,c}$ consists of all $r \times c$ matrices over non-negative integers $a_{i,j}$ such that $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} \geq a_{i+1,j}$. Hence $p_{r,c}(X)$ is the generating function for all plane partitions with at most r rows and c columns. In $p_{r,c}(X)$, setting all the $x_{i,j}$ to x produces the corresponding enumerative generating function which we denote by $q_{r,c}(x)$. The limiting case $r, c \to \infty$ corresponds to MacMahon's original conjecture [22, pp. 1064 – 1065] cited in Section 1, namely

$$q_{\infty,\infty}(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k}. \tag{3}$$

In [18] MacMahon also considered the case where r and c are set to concrete integers. His computations led him to conjecture that

$$q_{r,c}(x) = \sum_{n=0}^{\infty} P_{r,c}(n) x^n = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{1-x^{i+j-1}} \tag{4}$$

where $P_{r,c}(n)$ denotes the number of plane partitions of n with at most r rows and c columns. Obviously, for $r, c \to \infty$ this turns into (3).

In our attempt to give a possible explanation of why MacMahon had failed to prove (4) with his method, we first have to describe how Partition Analysis would work on such problems in principle.

The usual heuristics approach to prove (4) by means of Partition Analysis would be as follows: one tries to proceed by mathematical induction with respect to one of the free parameters, e.g., with respect to c with r fixed. To this end, one applies Partition Analysis to special instances of the problem in order to guess a pattern for the induction step from c to $c+1$. But in many applications it turns out that the enumerative generating function does not provide sufficient information into the mechanism of the induction. In such situations one often can overcome this problem by considering the full generating function, i.e., the generating function that constructs all the objects in question; see the various examples given in [3] – [12].

To illustrate this point let us consider $p_{r,c}(X)$ with $r = c = 3$. The case $q_{3,3}(x)$ where all the $x_{i,j}$ in $p_{3,3}(X)$ are set to x, causes no computational problem at all:

In[4]:=OSum[$x^{a_{11}+a_{12}+a_{13}+a_{21}+a_{22}+a_{23}+a_{31}+a_{32}+a_{33}}$,
$\{a_{11} \geq a_{12}, a_{12} \geq a_{13}, a_{21} \geq a_{22}, a_{22} \geq a_{23}, a_{31} \geq a_{32}, a_{32} \geq a_{33},$
$a_{11} \geq a_{21}, a_{21} \geq a_{31}, a_{12} \geq a_{22}, a_{22} \geq a_{32}, a_{13} \geq a_{23}, a_{23} \geq a_{33}\}$,λ]

Out[4]= $$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\Omega_{\geq}} \frac{1}{(1-x\lambda_1\lambda_7)(1-\frac{x\lambda_5}{\lambda_8})(1-\frac{x\lambda_3\lambda_8}{\lambda_7})} \times \frac{1}{(1-\frac{x\lambda_2\lambda_9}{\lambda_1})(1-\frac{x\lambda_6}{\lambda_5\lambda_{10}})(1-\frac{x\lambda_4\lambda_{10}}{\lambda_3\lambda_9})(1-\frac{x\lambda_{11}}{\lambda_2})(1-\frac{x}{\lambda_6\lambda_{12}})(1-\frac{x\lambda_{12}}{\lambda_4\lambda_{11}})}$$

In[5]:= OR[%4]

Out[5]= $$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^3(1-x^4)^2(1-x^5)}$$

Despite the fact that the Omega package confirms (4) within a fraction of a second, a closer inspection shows that in order to exhibit an induction pattern for a Partition Analysis proof of (4) the setting $x_{i,j} = 0$ is much too restrictive.

So let us have a look at the full generating function $p_{3,3}(X)$. The crude generating function comes out in perfect analogy to Out[4] above:

$$\texttt{In[6]:=OSum[}x_{11}^{a_{11}} x_{12}^{a_{12}} x_{13}^{a_{13}} x_{21}^{a_{21}} x_{22}^{a_{22}} x_{23}^{a_{23}} x_{31}^{a_{31}} x_{32}^{a_{32}} x_{33}^{a_{33}},$$
$$\{a_{11} \geq a_{12}, a_{12} \geq a_{13}, a_{21} \geq a_{22}, a_{22} \geq a_{23}, a_{31} \geq a_{32}, a_{32} \geq a_{33},$$
$$a_{11} \geq a_{21}, a_{21} \geq a_{31}, a_{12} \geq a_{22}, a_{22} \geq a_{32}, a_{13} \geq a_{23}, a_{23} \geq a_{33}\}, \lambda]$$

$$\texttt{Out[6]=} \underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\Omega_{\geq}} \frac{1}{(1-x_{11}\lambda_1\lambda_7)(1-\frac{x_{31}\lambda_5}{\lambda_8})(1-\frac{x_{21}\lambda_3\lambda_8}{\lambda_7})}$$
$$\times \frac{1}{(1-\frac{x_{12}\lambda_2\lambda_9}{\lambda_1})(1-\frac{x_{32}\lambda_6}{\lambda_5\lambda_{10}})(1-\frac{x_{22}\lambda_4\lambda_{10}}{\lambda_3\lambda_9})(1-\frac{x_{13}\lambda_{11}}{\lambda_2})(1-\frac{x_{33}}{\lambda_6\lambda_{12}})(1-\frac{x_{23}\lambda_{12}}{\lambda_4\lambda_{11}})}$$

The computation of the full generating function takes another couple of seconds:

```
In[7]:= OR[%6]
```

$\texttt{Out[7]=}$ $(1 - x_{11}^2 x_{12} x_{21} - x_{11}^2 x_{12} x_{13} x_{21} - \cdots + x_{11}^{14} x_{12}^{12} x_{13}^{7} x_{21}^{12} x_{22}^{7} x_{23}^{2} x_{31}^{7} x_{32}^{2})/$
$((1 - x_{11})(1 - x_{11}x_{12})(1 - x_{11}x_{12}x_{13})(1 - x_{11}x_{21})(1 - x_{11}x_{12}x_{21})$
$(1-x_{11}x_{12}x_{13}x_{21})(1-x_{11}x_{12}x_{21}x_{22})(1-x_{11}x_{12}x_{13}x_{21}x_{22})(1-x_{11}x_{12}x_{13}x_{21}x_{22}x_{23})$
$(1 - x_{11}x_{21}x_{31})(1 - x_{11}x_{12}x_{21}x_{31})(1 - x_{11}x_{12}x_{13}x_{21}x_{31})(1 - x_{11}x_{12}x_{21}x_{22}x_{31})$
$(1 - x_{11}x_{12}x_{13}x_{21}x_{22}x_{31})(1 - x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31})(1 - x_{11}x_{12}x_{21}x_{22}x_{31}x_{32})$
$(1 - x_{11}x_{12}x_{13}x_{21}x_{22}x_{31}x_{32})(1 - x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31}x_{32})$
$(1 - x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31}x_{32}x_{33}))$

However, the problem arising in this case consists in the complexity of the resulting rational function; namely, in order to display the numerator polynomial

$$1 - x_{11}^2 x_{12} x_{21} - x_{11}^2 x_{12} x_{13} x_{21} - \cdots + x_{11}^{14} x_{12}^{12} x_{13}^{7} x_{21}^{12} x_{22}^{7} x_{23}^{2} x_{31}^{7} x_{32}^{2}$$

in fully explicit form, one would need more than 30 printed pages!

Summarizing, the coding of the full generating function $p_{r,c}(X)$ in terms of the omega operator is straight-forward and has been carried out already by MacMahon [21, Vol. II, p. 92]. But without computer algebra he did not succeed in overcoming the computational difficulties when trying to obtain the beautiful product side of (3) , resp. (4), with omega evaluation. Essentially the problem is this: when specifying all the $x_{i,j}$ to x, the underlying algebraic structure gets lost entirely. If all the $x_{i,j}$ are kept, the computational complexity soon gets out of hand.

Consequently, we used the Omega package in a heuristic search to find a substitution for the x_{ij} which, on one side, provides more insight into the underlying Partition Analysis induction pattern than $q_{r,c}(x)$, and on the other side, for which the elimination of the λ_i results in a more feasible rational function than for the general $p_{r,c}(X)$.

Finally, after various attempts our strategy turned out to be successful. More precisely, we found that the substitution

$$x_{ij} \to z_{j-i} \tag{5}$$

has all the properties desired. First, the elimination of the λ_i results in a rational function that factors nicely for all choices of $r$and c. For instance, for $r = c = 3$,

In[8]:=OSum[$z_0^{a_{11}} z_1^{a_{12}} z_2^{a_{13}} z_{-1}^{a_{21}} z_0^{a_{22}} z_1^{a_{23}} z_{-2}^{a_{31}} z_{-1}^{a_{32}} z_0^{a_{33}}$,
$\{a_{11} \geq a_{12}, a_{12} \geq a_{13}, a_{21} \geq a_{22}, a_{22} \geq a_{23}, a_{31} \geq a_{32}, a_{32} \geq a_{33},$
$a_{11} \geq a_{21}, a_{21} \geq a_{31}, a_{12} \geq a_{22}, a_{22} \geq a_{32}, a_{13} \geq a_{23}, a_{23} \geq a_{33}\}, \lambda]$

Out[8]= $$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\underset{\geq}{\Omega}} \frac{1}{(1-z_0\lambda_1\lambda_7)(1-\frac{z_{-2}\lambda_5}{\lambda_8})(1-\frac{z_{-1}\lambda_3\lambda_8}{\lambda_7})}$$
$$\times \frac{1}{(1-\frac{z_1\lambda_2\lambda_9}{\lambda_1})(1-\frac{z_{-1}\lambda_6}{\lambda_5\lambda_{10}})(1-\frac{z_0\lambda_4\lambda_{10}}{\lambda_3\lambda_9})(1-\frac{z_2\lambda_{11}}{\lambda_2})(1-\frac{z_0}{\lambda_6\lambda_{12}})(1-\frac{z_1\lambda_{12}}{\lambda_4\lambda_{11}})}$$

In[9]:=OR[%8]

Out[9]= $1/((1 - z_0)(1 - z_{-1}z_0)(1 - z_{-2}z_{-1}z_0)(1 - z_0z_1)(1 - z_{-1}z_0z_1)$
$(1 - z_{-2}z_{-1}z_0z_1)(1 - z_0z_1z_2)(1 - z_{-1}z_0z_1z_2)(1 - z_{-2}z_{-1}z_0z_1z_2))$

Second, and more importantly, in this situation MacMahon's method of Partition Analysis works in a way that allows to set up an elementary induction proof for the corresponding plane partition result which originally is due to Emden Gansner. His theorem [16, Thm. 4.2] not only generalizes (4) but also Stanley's trace theorem [26, Thm. 2.2] which was also conjectured by MacMahon in [18]. In order to state Gansner's theorem we need a couple of definitions.

Let $\pi = (a_{i,j})$ be an $r\times c$ matrix over non-negative integers $a_{i,j}$ such that $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} \geq a_{i+1,j}$; i.e., π represents a plane partition of $n := \Sigma_{ij} a_{ij}$ with at most r rows and c columns. For any integer k we define the k-trace t_k of π by $t_k := \Sigma a_{i,j}$ where the sum runs over all i, j such that $k = j - i$. E.g., the traces of the plane partition of 30 in Section 1 are: $t_{-2} = 4$, $t_{-1} = 7$,$t_0 = 10$, $t_1 = 6$, and $t_2 = 3$.

If $T_{r,c}(t_{-r+1}, ..., t_{-1}; t_0, ..., t_{c-1}; n)$ denotes the number of plane partitions of n with at most r rows and c columns, and with k-trace t_k,$-r + 1 \leq k \leq c - 1$, Gansner's theorem reads as follows:

$$\sum_{n=0}^{\infty}\sum_{t_{-r+1}=0}^{\infty}\dots\sum_{t_{c-1}=0}^{\infty} T_{r,c}\left(t_{-r+1},...,t_{-1};t_0,...,t_{c-1};n\right) \times z_{-r+1}^{t_{-r+1}}\cdots z_{-1}^{t_{-1}} z_0^{t_0}\cdots z_{c-1}^{t_{c-1}} x^n = \prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1}{1-z_{-i+1}z_{-i+2}\cdots z_{j-1}x^{i+j-1}}.$$

Obviously, $z_{-r+1} = z_{-r+2} = ... = z_{c-1} = 1$ gives (4); setting all $z_k = 1$, except z_0, gives Stanley's trace theorem [26, Thm. 2.2].

It is immediate that $p_{r,c}(X)$ for $X = (x_{i,j})$ with $x_{ij} := z_{j-i}$ can be rewritten as the multiple series in Gansner's theorem. Our Partition Analysis proof of the fact that it finds the product representation above can be found in [12].

Summarizing, we want to note that our proof in [12] uses only basic power series arithmetic; essentially it proceeds by complete induction involving recursively defined rational functions. So our Partition Analysis approach is completely different from Gansner's original proof which is based on a combinatorial bijection of Burge [14]. This bijection is one of those variations of the Schensted-Knuth correspondence which Burge derived in order to give combinatorial proofs for a collection of Schur function identities due to D.E. Littlewood.

4 Conclusion

The implementation of MacMahon's Partition Analysis in the Omega package has provided the exploratory tool for our dozen papers on this topic [1] to [12]. It is important to point out that there have been a number of parallel and complementing projects that can be viewed as having goals similar to MacMahon's. An incomplete list would include: (1) LattE [15], an implementation of the work of A. Barvinok and J. Pommersheim [13], (2) the MAPLE package designed by J. Stembridge [27] to implement the discoveries of R. Stanley [25] which in turn were based on another MacMahon paper [20]. Recently G. Xiu [28] has made contributions based on his work on partial fractions.

In the future we hope to explore further with Omega. Also we are modifying Omega to treat problems in which not only linear Diophantine inequalities are considered but also divisibility properties of the summands are treated. Toward this goal, we have developed an extension of Partition Analysis that allows us to treat the Göllnitz-Gordon partition functions.

It is perhaps fitting to close with J. W. L. Glaisher's evaluation [17] of [18] (printed with permission of the Royal Society):

> "I don't fancy the paper very much, but it must be printed. I don't care much for a paper on very technical mathematics being published in the Phil. Trans. unless there is something very striking in it. However, it is one of a series, and they are in deep water now and cannot go on much farther. I have made my report because there is no more to be said than that it should be published (though the interesting results are the conjectural ones!), the balance being on that side."

How fortunate we are that Glaisher's lack of enthusiasm did not cause him to recommend against [18]. Also we can congratulate Glaisher on his recognition of the significance of MacMahon's conjectures.

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