9. Topological rings.

Let A be a topological ring, i.e., an associative ring endowed with a topology such that both subtraction and multiplication are continuous maps $A \times A \to A$.

For each n, the matrix ring $\mathrm{M}_n A$ is a topological ring with topology of direct product of n^2 copies of A. The group $\mathrm{GL}_n A$ is a topological group with topology induced by the embedding $\alpha \mapsto (\alpha, \alpha^{-1}) \in \mathrm{M}_n A \times \mathrm{M}_n A$. (When A has no 1, we consider here $\mathrm{GL}_n A$ as a subset of $\mathrm{M}_n A$ with the group operation $(x, y) \mapsto x + y + xy$.)

Note that the topology on $\mathrm{GL}_n A$ can be stronger than that induced by the embedding $\mathrm{GL}_n A \subset \mathrm{M}_n A$. For instance, this is the case when A is the ring of adeles and $\mathrm{GL}_1 A$ is the group of ideles.

Lemma 9.1. Let A be a topological ring such that $\mathrm{GL}_1 A$ is open in A. Let $n \geq 1$ and L (resp. U) be the group of all lower (resp. upper) triangular matrices in $\mathrm{GL}_n A$ with invertible diagonal entries. Then the set LU is open in $\mathrm{GL}_n A$.

Proof. Let X be the set of $\alpha \in \mathrm{M}_n A$ such that all entries of both $\alpha - 1_n$ and $\alpha^{-1} - 1_n$ belong to $\mathrm{GL}_1 A - 1$. This set is open. We show that $X \subset LU$ by induction on n. The case $n = 1$ is trivial. Let $n \geq 2$ and $\alpha \in X$. We write $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\alpha^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $a, a' \in \mathrm{M}_{n-1} A$, $d, d' \in A$. Since $d' \in \mathrm{GL}_1 A$ and

$$\begin{pmatrix} 1_{n-1} & -b'd'^{-1} \\ 0 & 1 \end{pmatrix} \alpha^{-1} \begin{pmatrix} 1_{n-1} & 0 \\ -d'^{-1}c' & 1 \end{pmatrix} = \begin{pmatrix} a' - b'd'^{-1}c' & 0 \\ 0 & 1 \end{pmatrix},$$

we have $a' - b'd'^{-1}c' \in \mathrm{GL}_{n-1} A$.

The matrix

$$\begin{pmatrix} a' - b'd'^{-1}c' & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1_{n-1} & 0 \\ d'^{-1}c' & 1 \end{pmatrix} \alpha \begin{pmatrix} 1_{n-1} & b'd'^{-1} \\ 0 & 1 \end{pmatrix}$$

has the form $\begin{pmatrix} a & 0 \\ 0 & * \end{pmatrix}$, hence $a \in \mathrm{GL}_{n-1} A$ (and $d'^{-1}c' = -ca^{-1}, b'd'^{-1} = -c'^{-1}b$). By the induction hypothesis $a = \beta\gamma$ with an upper and lower triangular matrices β and γ in $\mathrm{GL}_{n-1} A$, both with invertible diagonal entries. So

$$\alpha = \begin{pmatrix} 1_{n-1} & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 1_{n-1} & a^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ \alpha^{-1}c & 1 \end{pmatrix} \begin{pmatrix} \gamma & ba^{-1} \\ 0 & * \end{pmatrix} \in LU$$

has the desired form. QED.

Corollary 9.2. Under the conditions of Lemma 9.1, $\mathrm{GE}_n A = \begin{pmatrix} \mathrm{GL}_1 A & 0 \\ 0 & 1_{n-1} \end{pmatrix} \mathrm{E}_n A$ is an open subgroup of $\mathrm{GL}_n A$.

Proof. Since the diagonal matrices normalize all elementary matrices, $\mathrm{GE}_n A = \begin{pmatrix} \mathrm{GL}_1 A & 0 \\ 0 & 1_{n-1} \end{pmatrix} \mathrm{E}_n A$ is a subgroup. By the Whitehead lemma (1.5), $\mathrm{GE}_n A$ contains all the diagonal matrices. So $H \supset UL$. QED.

Examples of topological rings includes normed $\mathbf{R}$-algebras. Namely, a normed $\mathbf{R}$-algebra is an associative $\mathbf{R}$-algebra with unity with a function $|| \ || : A \to \mathbf{R}$ (real-valued norm) such that $||a+b|| \le ||a|| + ||b||, ||ab|| \le ||a|| ||b||, ||ar|| = ||a|| |r|$ for $a, b \in A$ and $r \in \mathbf{R}$ and $||a|| \ne 0$ if $a \ne 0$. The topology is generated by the balls $\{b \in A : ||b-a|| < \epsilon\}$ with $a \in A, 0 < \epsilon \in \mathbf{R}$.

A normed algebra is called *special* (Dayton, Milnor, Swan) if $\{a \in A : ||a-1|| < 1\} \subset \mathrm{GL}_1 A$.

Examples include all Banach algebras (which are complete) and the rings of bounded real or complex smooth functions on any smooth manifold with the norm being the supremum of absolute value.

We call a topological ring *special* if $\mathrm{GL}_1 A$ is open in A and A is connected. Thus, every special normed algebra is special topological ring.

Lemma 9.3. Let A be a special topological ring. Then

(9.4) for every $a \in A$ there are $x_1, \ldots, x_m \in A$ such that $1 + ax_k \in \mathrm{GL}_1 A$ for all k and $\sum x_k = 1$.

Proof. Since $\mathrm{GL}_1 A$ is open and contains 1 and the map $x \mapsto ax$ is continuous, there is an open neighborhood U of 0 such that $1 + ax \in \mathrm{GL}_1 A$ for all $x \in U$. Since A is connected, the additive subgroup generated by U (which is an open subgroup) is A. QED.

For every topological group G, let G^0 denote the connected component of the neutral element. Since G^0 is normal, the set $\pi_0(G)$ of all connected components is a group, $\pi_0(G) = G/G^0$.

Theorem 9.5. Let A be a special topological ring and $n \ge 2$. Then:

(a) $\mathrm{E}_n(A, B)$ is normal in $\mathrm{GL}_n A$;

(b) $\mathrm{E}_n A = [\mathrm{GE}_n A, \mathrm{E}_n A] = [\mathrm{GE}_n A, \mathrm{GE}_n A]$;

(c) $\mathrm{E}_n A \subset \mathrm{GL}_n^0 A \subset \mathrm{GE}_n A$;

(d)if $n \ge 3, \mathrm{E}_n A = [\mathrm{GL}_n^0 A, \mathrm{GL}_n^0 A]$;

(e) if A is commutative, $\mathrm{SL}_n^0 A = \mathrm{E}_n A$, so $\pi_0(\mathrm{SL}_n A) = \mathrm{SL}_n A / \mathrm{E}_n A$.

Proof. By Lemma 9.3, the parts (a) and (b) are contained in Theorem 9.7 below.

(c) Since A is connected, $\mathrm{E}_n A \subset \mathrm{Gl}_n^0 A$. By Corollary 9.2, $\mathrm{GL}_n^0 A \subset \mathrm{GE}_n A$.

(d) By (b) and (c), $[\mathrm{GL}_n^0 A, \mathrm{GL}_n^0 A] \subset \mathrm{E}_n A$. When $n \ge 3$, $\mathrm{E}_n A$ is perfect.

(e) For every n and any commutative A, $\mathrm{GE}_n A \cap \mathrm{SL}_n A = \mathrm{E}_n A$. So (e) follows from (c). QED.

Corollary 9.6. Let X is compact space, $A = \mathbf{R}^X$ (resp., $\mathbf{C}^X$) the ring of all continuous functions $X \to \mathbf{R}$ (resp. $X \to \mathbf{C}$ with the norm equal to the maximum of absolute value. Then $\mathrm{SL}_n A / \mathrm{E}_n A = \pi_0$ $(\mathrm{SL}_n A)$, the set of homotopy classes of continuous maps $X \to \mathrm{SL}_n \mathbf{R}$ (resp., the set of homotopy classes of continuous maps $X \to \mathrm{SL}_n \mathbf{C}$) for all n.

Proof. Note that $\mathrm{SL}_n A$ is the set of all continuous maps $X \to \mathrm{SL}_n \mathbf{R}$ (resp., $X \to \mathrm{SL}_n \mathbf{C}$) and use Theorem 9.5. QED.

Theorem 9.7. Let A be an associative ring with unity satisfying the condition (9.4). Then for any ideal B of A:

(a) $\mathrm{E}_n(A, B)$ contains all matrices of the form $1_n + vu$ where v is an n-column over A, u is an n-row over B, and $uv = 0$; in particular $\mathrm{E}_n(A, B)$ is normal in $\mathrm{GL}_n A$;

(b) $\mathrm{E}_n(A, B) = [\mathrm{GE}_n A, \mathrm{E}_n B]$;

(c) $\mathrm{E}_n(A,B) \supset [\mathrm{E}_n A, \mathrm{G}_n(A,B)]$.

Proof. (a) We write $u=(u_j), v=(v_i)$. By the condition (9.4), there are $x_k \in A$ such that $1+u_1v_1x_k \in \mathrm{GL}_1 A$ for all k and $\sum x_k = 1$. Then $1_n + vu = \prod(1_n + vx_k u)$. By Lemma 3.5 (with $m=n=1$), $1+v_1x_ku_1 \in \mathrm{GL}_1 A$ for all k. So $1_n + vx_ku \in \mathrm{E}_n(A,B)$ by Proposition 8.3 (with $k=1$).

The second statement in (a) follows from the first one. Indeed we have to prove that $\alpha b^{i,j}\alpha^{-1} \in \mathrm{E}_n(A,B)$ for any elementary matrix $1_n + b^{i,j} \in \mathrm{E}_n B$ and any $\alpha \in \mathrm{GL}_n A$. We have $\alpha b^{i,j}\alpha^{-1} = 1_n + vbw$, where w is the j-th row of α^{-1} and v is the i-th column of α, so $bwv = wv = 0$.

(b) Since all diagonal matrices normalize all elementary matrices,

$$\mathrm{E}_n(A,B) \supset [\mathrm{GE}_n A, \mathrm{E}_n B]$$

for every ring A and ideal B. When $n \geq 3$, the converse follows from (1.11) (also for an arbitrary A).

To handle the case $n=2$ we use the condition (9.4), with $a=1$. There are $x_i \in A$ such that $1+x_i \in \mathrm{GL}_1 A$ and $\sum x_i = 1$. Then for any $b \in B$, $b^{1,2} = \prod (x_i b)^{1,2}$ and $(x_ib)^{1,2} = [\begin{pmatrix} 1+x_i & 0 \\ 0 & 1 \end{pmatrix}, b^{1,2}] \in [\mathrm{GE}_2 A, \mathrm{E}_2 B]$.

Similarly, $B^{2,1} \subset [\mathrm{GE}_2 A, \mathrm{E}_2 B]$, hence $\mathrm{E}_2(A,B) \subset [\mathrm{GE}_2 A, \mathrm{E}_2 B]$.

(c) The trick with the double ring A' does not work because we do not have the condition (9.4) for A'. Also (9.4) does not imply that $\mathrm{E}_2 A$ is perfect. So we do direct computations.

Let $a^{i,j}$ be an elementary matrix in $\mathrm{E}_n A$ and $\beta \in \mathrm{G}_n(A,B)$. We have to prove that $[a^{i,j}, \beta] \in \mathrm{E}_n(A,B)$. Since both $\mathrm{E}_n(A,B)$ and $\mathrm{G}_n(A,B)$ are normalized by all permutation matrices, we can assume that $(i,j) = (n,1)$.

Then $[\ a^{i,j}, \beta] = a^{i,j}(1_n - vzu)$, where v is the last column of β and u is the first row of β^{-1}. We write $u = (u_j) = (u', u_n)$ and $v = (v_i) = \begin{pmatrix} v' \\ v_n \end{pmatrix}$. By (9.4), there are $x_i \in A$ such that $1 + u_n v_n a x_i \in \mathrm{GL}_1 A$ for all i and $\sum x_k = 1$.

We want to prove that $a^{n,1}$ and β commute modulo the normal subgroup $\mathrm{E}_n(A,B)$ of $\mathrm{GL}_n A$. Since $a^{i,j} = \prod (ax_i)^{n,1}$, it suffices to show that each $(ax_i)^{n,1}$ and β commute modulo $\mathrm{E}_n(A,B)$.

We have

$$\alpha = \alpha_{i,j} = [\beta, (-ax_i)^{i,j}] = (1_n - vax_ku)(ax_k)^{n,1}$$

with $\alpha_{n,n} = 1 + v_n a x_k u_n \in \mathrm{GL}_1 A$ because $1 + u_n v_n a x_i \in \mathrm{GL}_1 A$. Write $w = -ax_ku = (w', w_n)$. Note that $wv = uv = 0$.

Now by row addition operations over B we eliminate all non-diagonal entries of the last column and then of the last row of the matrix α. We obtain the following block-diagonal matrix α' (which belongs to $\mathrm{E}_n(A,B)$ if and only if α does): $\alpha' = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, where $d = \alpha_{n,n} = 1 + v_n w_n$ and

$$a = 1_{n-1} + v'w' - v'w_n d^{-1} v_n w' = 1_{n-1} + v'(1 - w_n d^{-1} v_n) w'$$

$$= 1_{n-1} + v'd'^{-1}w' = (1_{n-1} - v'w')^{-1}$$

with $d' = 1 + w_n v_n = 1 - w'v'$. By Lemma 3.5, $\alpha' \in \mathrm{E}_n(A, B)$ QED.

Theorem 9.8. Let A be an associative ring with unity satisfying the condition (9.4), $n \geq 3$, and H a subgroup of $\mathrm{GL}_n A$. Then the following two conditions are equivalent:

(a) H is normalized by $\mathrm{E}_n A$;

(b) $\mathrm{E}_n(A, B) \subset H \subset \mathrm{G}_n(A, B)$ for an ideal B of A.

Proof. (a) $\Rightarrow$ (b). Let H is normalized by $\mathrm{E}_n A$. Set $B = \{b \in A : b^{1,2} \in H\}$. By (1.5), (1.11), B is an ideal of A and $\mathrm{E}_n(A, B) \subset H$. We have to prove that $H \subset \mathrm{GL}_n(A, B)$.

Otherwise, the image H' of H in $\mathrm{GL}_n(A, B)$ is not central. Notice the H' is normalized by $\mathrm{E}_n A'$. We claim that H' contains $(b')^{1,2}$ with $0 \neq b' \in A/B$. To prove this we pick a non-central $\alpha' = (\alpha'_{i,j}) \in H'$.

If $\alpha'_{2,1} = 0$ we can use Proposition 1.10.

Let now $\alpha'_{2,1} \neq 0$. By Lemma 4.9, α' and $1^{1,2}$ do not commute modulo the center $\mathrm{G}(A', 0)$. Consider the (1,1)-entry v_1 of α' and the (2,1)-entry u_1 of α'^{-1}. The property (9.4) is inherited by the factor ring A'. So there are $x_k \in A'$ such that $1 + u_1 v_1 x_k \in \mathrm{GL}_1 A'$ for all k and $\sum x_k = 1$. Recall that α' and $1^{1,2} = \prod (x_k)^{1,2}$ do not commute modulo $\mathrm{G}(A', 0)$. So $\beta' = [\alpha', (x_k)^{1,2}]$ is a non-central matrix in H' for some k. The (1,1)-entry of β' is $1 + v_1 x_k u_1$, and it belongs to $\mathrm{GL}_1 A'$ by Lemma 3.5. By Proposition 4.11, $\mathrm{E}_n b' A \subset H'$ for a nonzero $b' = b + B \in A' = A/B$.

Therefore $\alpha_1 = b^{1,3}\beta_1 \in H$ with $b \in A \setminus B$ and $\beta_1 \in \mathrm{GL}_n B$. Taking commutator of α_1 with $1^{3,2}$ and using that $[\beta_1, 1^{3,2}] \in \mathrm{E}_n(A, B)$ by Theorem 9.7 (c) , we obtain that $b^{1,2} \in H$ which contradicts the definition of B.

(b) $\Rightarrow$ (a). This follows from Theorem 9.7 (c) and is true for all $n \geq 2$. QED

By Theorem 9.7b, every associative ring A with 1 satisfying the condition (9.4) belongs to the class $\mathrm{R1}(n)$ for every $n \geq 2$.

For any $n \geq 2$,, we define $\mathrm{R2}(n)$ to be the class of associative rings B satisfying the following condition:
(9.9) for any associative ring A with 1 containing B as an ideal and any ideal $B_0 \subset B$ of$A$,

$$[\mathrm{GL}_n A, \mathrm{E}_n B_0] \subset \mathrm{E}_n(A, B_0),$$

i.e., $\mathrm{E}_n(A, B_0)$ is a normal subgroup of $\mathrm{GL}_n A$.

By Theorems 3.8 and 4.7, $\mathrm{R2}(n)$ contains any ring B such that $\mathrm{sr}(B) \leq n - 1$. Also $\mathrm{R2}(n)$ contains any commutative ring B with identity provided that $n \geq 3$. Indeed, for any ring B with identity 1_B the condition (9.9) is reduced to the following simpler condition:
(9.10) for any ideal B_0 of B,

$$[\mathrm{GL}_n B, \mathrm{E}_n B_0] \subset \mathrm{E}_n(B, B_0),$$

i.e., $\mathrm{E}_n(B, B_0)$ is a normal subgroup of $\mathrm{GL}_n B$.

Now when $n \geq 3$ and B is a commutative ring with identity, B belongs to $\mathrm{R2}(n)$ by Corollary 8.5, the second equality.

By Theorem 9.7a, every associative ring A with 1 satisfying the condition (9.4) belongs to the class $\mathrm{R2}(n)$ for every $n \geq 2$.

Proposition 9.11. Let$A$ be an associative ring with 1, $n \geq 2$. Assume that $\mathrm{E}_n A$ is perfect, e.g., $n \geq 3$. Let $B_k \subset B_{k-1} \subset \cdots \subset B_1 = A$ be a chain of ideals in A. Assume that $B_i/B_{i+1} \in \mathrm{R1}(n)$ for $1 \leq i \leq k-1$ and that $B_i/B_{i+1} \in \mathrm{R2}(n)$ for $1 \leq 2 \leq k-1$. Then $A \in \mathrm{R1}(n.)$

Proof. When $k = 1$, this follows from Proposition 8.13. This form a basis for induction. By the induction hypothesis $A/B_k \in \mathrm{R1}(n.)$ By Proposition 8.13, $A \in \mathrm{R1}(n.)$ QED.

Topological stable rank

For a topological ring A with 1, Rieffel [Ri] defines the right (resp., left) topological stable rank, $\mathrm{rtsr}(A)$ (resp., $\mathrm{ltsr}(A)$) as the least n such that $\mathrm{Um}_n A$ is dense in A^n (resp. the set of unimodular n-rows over A is dense in the set of all n-rows over A).

It is easy to see that $\mathrm{Um}_n A$ is dense in A^n for all $n \geq \mathrm{rtsr}(A)$ (similar fact holds for $\mathrm{ltsr} A$). We give here a proof following Riefel [Ri, Corollary 2.14] where A is a Banach algebra.

Theorem 9.12. Let A is a topological ring with 1 and $\mathrm{GL}_1 A$ is open in A. Then $\mathrm{sr}(A) \leq \min(\mathrm{ltsr}(A), \mathrm{rtsr}(A))$.

Proof. Since the stable rank is right-left symmetric (see Proposition 4.12), it suffices to prove that $\mathrm{sr}(A) \leq \mathrm{ltsr}(A)$. Let $n > \mathrm{ltsr}(A)$, $a_i, b_i \in A$, and $\sum_{i=1}^{n} a_i b_i = 1$. We have to prove that there are $c_i \in A$ such that $\sum_{i=1}^{n-1} A(b_i + c_i b_n) = A$.

Since the unimodular rows are dense in the all $(n-1)$-rows and $\mathrm{GL}_1 A$ is open in A, there are $a'_i \in A$ such that $\sum_{i=1}^{n-1} a'_i A = A$ and $d := a_n b_n + \sum_{i=1}^{n-1} a'_i b_i \in \mathrm{GL}_1 A$.

Then there are $c_i \in A$ such that $-a_n = \sum_{i=1}^{n-1} a'_i c_i$. We have $\sum_{i=1}^{n-1} a'_i (b_i + c_i b_n) = d \in \mathrm{GL}_1 A$. Therefore $\sum_{i=1}^{n-1} A(b_i + c_i b_n) = A$. QED.

Problems.

1. Show that a field A of three elements satisfies (9.4) but (a) in Theorem 9.8 does not imply (b) in the case $n = 2$.

2. Show that every finitely dimensional $\mathbf{R}$-algebra A has a norm such that it becomes a special normed algebra. Hint: Consider A as an Euclidean space and define the norm of $a \in A$ as the norm of the left of multiplication by a on A.

3. Let A be the $\mathbf{R}$-algebra generated by $1, x, y$ with defining relations $xy + yx = x^2 = y^2 = 0$. Show that A is 4-dimensional over R. Show that (a) in Theorem 9.8 does not imply (b) in the case $n = 2$. Hint: take $H = E_2(A, B)$ where $B = \mathbf{R}x + \mathbf{R}y$.

4. If A satisfy (9.4), then every ideal B of A satisfies the following condition:

(9.9) for any $b, b' \in B$ there are $x_k \in B$ such that $1 + b x_k \in \mathrm{GL}_1 B$ for all k and $\sum x_k = b'$.

Prove Theorem 9.8 with the condition (9.4) on A replaced by the condition (9.9) on B.

5. Let X be a topological space, $A = \mathbf{R}^X$ the ring of all continuous functions $X \to \mathbf{R}$, and A_0 is the subring of bounded functions. We observed above that A_0 is a special normed algebra and hence satisfies (9.4). Show that $\mathrm{SL}_n A/\mathrm{E}_n A = \mathrm{SL}_n(A_0)/\mathrm{E}_n(A_0)$ for all n.

6. Let A_0 be as in Problem 5, and let A' be a dense subalgebra of A_0 such that $\mathrm{GL}_1 A'$ is open in A' (the topologies on A' and $\mathrm{GL}_1 A'$ are induced be embeddings $A \subset A_0$ and $\mathrm{GL}_1 A' \subset \mathrm{GL}_1 A_0$. Show that $\mathrm{SL}_n A'/\mathrm{E}_n A' = \mathrm{SL}_n(A_0)/\mathrm{E}_n(A_0)$ for all n.

7. Let A be a special topological ring. Show that $\mathrm{E}_2 A$ is perfect under any of the following conditions:

(a) $2A = A$;

(b) $\{\gamma^2 : \gamma \in \mathrm{GL}_1 A\}$ is open in A;

(c) (open problem) no additional conditions are imposed.

8. Let A be a special topological ring, X a topological space, A^X the ring of all continuous functions $X \to A$. We introduce the topology on A^X with the neighborhoods of 0 being $\{f : f(x) \in U \ \forall x \in X\}$ where U runs over all open neighborhoods of 0 in A. Let A_0^X be the connected component of 0 in A^X. Prove that A_0^X is a special topological ring. Show that when A is commutative and X is compact, then $\mathrm{SL}_n A_0^X/\mathrm{E}_n A_0^X = \pi_0(\mathrm{SL}_n A^X)$ (for every n). When A is arc-wise connected, so is $\mathrm{E}_n A$, and $\pi_0(\mathrm{SL}_n A^X)$ is the set of the homotopy classes of continuous maps $X \to \mathrm{SL}_n A$ (for every n).

9. (Exercise 1 on p. 315 of [Bo]) Let A is a Hausdorff topological ring. Show that its center is closed.

10. (Exercise 11 on p. 317 of [Bo]) A topological ring A with 1 is *Gelfand* if $\mathrm{GL}_1 A$ is open in A and the topology on $\mathrm{GL}_1 A$ coincides with the induced topology (so $x \mapsto 1/x$ is a continuous function on the subset $\mathrm{GL}_1 A \subset A$).

Show that every topological Hausdorff division ring is a Gelfand ring.

Show that for a Gelfand ring A, the matrix rings $\mathrm{M}_n A$ are Gelfand.

For Problems 11–15, let A and A_0 be as in Problem 5.

11. Show that if $A \neq A_0$ (so X is not compact) then in the topology of uniform convergence on X, $\mathrm{GL}_1 A$ is open in A but A is non connected, namely, A_0 is the connected component 0 in A.

12. Show that if $A \neq A_0$ then in the topology of uniform convergence on every compact, A is connected, but $\mathrm{GL}_1 A$ is not open in A.

13. Show that if A contains a function which is unbounded on a connected subspace of X then A does not satisfy (9.4) and therefore no topology on A makes it a special topological ring.

14. Prove that $\mathrm{E}_2 A$ is normal in $\mathrm{GL}_2 A$ and that the group $\mathrm{GL}_2 A/\mathrm{E}_2 A$ is commutative.

15. Prove the conclusion of Theorem 9.8 for A.

16. Prove the condition (9.4) for the ring A of adels.