8. PARTITIONS

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Abstract. While Leibniz appears to be the first person to consider the partitioning of integers into sums, Euler was the first person to make truly deep discoveries. J. J. Sylvester was the next researcher to make major contributions. The twentieth century saw an explosion of research with monumental contributions from Rogers, Hardy, MacMahon, Ramanujan and Rademacher.

1. Introduction.

Leibniz was apparently the first person to ask about partitions. In a 1674 letter [47, p. 37] he asked J. Bernoulli about the number of "divulsions" of integers. In modern terminology, he was asking the first question about the number of partitions of integers. He observed that there are three partitions of 3 $(3, 2+1$, and $1+1+1)$ as well as five of 4 $(4, 3+1, 2+2, 2+1+1$ and $1+1+1+1)$. He then went on to observe that there are seven partitions of 5 and eleven of 6. This suggested that the number of partitions of any n might always be a prime; however, this 'exemplum memorabile fallentis inductionis' is found out once one computes the fifteen partitions of 7. So even this first tentative exploration of partitions suggested a problem still open today: Are there infinitely many integers n for which the total number of partitions of n is prime? (Put your money on "yes.")

From this small beginning we are led to a subject with many sides and many applications: The Theory of Partitions. The starting point is precisely that of Leibniz put in modern notation.

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Let $p(n)$ denote the number of ways in which n can be written as a sum of positive integers. A reordering of summands is not counted as a new partition; so $2+1+1$, $1+2+1$, and $1+1+2$ are considered the same partition of 4.

As Leibniz noted,

$$p(3)=3,\ p(4)=5,\ p(5)=7,\ p(6)=11,\ p(7)=15.$$

A number of questions can be asked about $p(n)$. How fast does it grow? What is its parity? Does it have special arithmetic properties? Are there efficient ways for computing $p(n)$? Is $p(n)$ prime infinitely often?

To give some order to an account of these questions, we organize the subject around the contributions of the great partition theorists: Euler, Sylvester, MacMahon, Rogers, Hardy, Ramanujan and Rademacher. Each of these played a seminal role in the development of one or more themes in the history of partitions.

2. Euler and Generating Functions.

Leonhard Euler was asked in a letter from Ph. Naudé [29] to solve the problem of partitioning a given integer n into a given number of parts m. In particular, Naudé asked how many partitions there are of 50 into seven distinct parts?

The correct answer, 522, is not likely to be obtained by writing out all the ways of adding seven distinct positive integers to get 50. To solve this problem Euler introduced generating functions. Following Euler's lead, but using modern notation, we let $D(m,n)$ denote the number of partitions of n into m parts. Then

$$\begin{aligned}\sum_{m,n\geqq 0} D(m,n)z^m q^n &= (1+zq^1)(1+zq^2)(1+zq^3)\cdots \\ &= \prod_{j=1}^{\infty}(1+zq^j)\,. \qquad (2.1)\end{aligned}$$

This identity becomes clear as we consider what happens when we multiply the terms on the right together. A typical term is

$$(zq^{i_1})(zq^{i_2})\cdots(zq^{i_j}) = z^j q^{i_1+i_2+\cdots+i_j}$$

which arises precisely from the partition with j distinct parts $i_1 + i_2 + \cdots + i_j$.

Noting that

$$\prod_{j=1}^{\infty}(1+zq^j) = (1+zq)\prod_{j=1}^{\infty}(1+(zq)q^j)\,,$$

we are able to obtain a functional equation for the generating function of $D(m,n)$:

$$\sum_{m,n\geqq 0} D(m,n)z^m q^n = (1+zq)\sum_{m,n\geqq 0} D(m,n)z^m q^{n+m}\,.$$

Comparing the coefficients of $z^m q^n$ on both sides, we find that

$$D(m,n) = D(m,n-m) + D(m-1,n-m)\,. \tag{2.2}$$

This equation allows easy computation of the values of $D(m,n)$. Indeed, the following table is easily extended to include $D(7,50) = 522$:

$m\backslash n$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1
2	0	0	0	1	1	2	2	3
3	0	0	0	0	0	0	1	1

Euler was naturally led from Naudé's question to an even more fundamental one. What is the generating function for $p(n)$, the total number of partitions of n? Here he applied the same principle that was so effective in computing $D(m,n)$ – namely,

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n)q^n =&(1+q^1+q^{1+1}+q^{1+1+1}+q^{1+1+1+1}+\cdots)\\
&\times(1+q^2+q^{2+2}+q^{2+2+2}+q^{2+2+2+2}+\cdots)\\
&\times(1+q^3+q^{3+3}+q^{3+3+3}+q^{3+3+3+3}+\cdots)\\
&\vdots\\
=&\prod_{n=1}^{\infty}(1+q^n+q^{n+n}+q^{n+n+n}+q^{n+n+n+n}+\cdots)\\
=&\prod_{n=1}^{\infty}(1+q^n+q^{2n}+q^{3n}+q^{4n}+\cdots)\\
=&\prod_{n=1}^{\infty}\frac{1}{1-q^n}\,.
\end{aligned} \tag{2.3}$$

At this point, Euler realized that a power series expansion for $\prod_{n=1}^{\infty}(1-q^n)$ would be essential for simplifying the computation of $p(n)$. He discovered empirically that

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^n) &= 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}\cdots \\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n-1)/2}\,. \end{aligned} \tag{2.4}$$

Many years after his empirical discovery, Euler managed to provide a proof of this himself; a modern exposition of his proof is given in [10]. This formula is now known as Euler's Pentagonal Number Theorem. We shall examine Fabian Franklin's proof of it [31] in the next section.

Combining the pentagonal number theorem with the generating function for $p(n)$, we see that

$$\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}\right)\sum_{n=0}^{\infty}p(n)q^n = 1\,.$$

Comparing coefficients of q^N on both sides of this last identity, Euler found the following recurrence for $p(N)$: $p(0)=1$, and

$$p(N)-p(N-1)-p(N-2)+p(N-5)+p(N-7)-\cdots = 0, N>0.$$

No one has ever found a more efficient algorithm for computing $p(N)$. It computes a full table of values of $p(n)$ for $n \leqq N$ in time $O(N^{3/2})$.

Euler's introduction of generating functions was certainly the most important innovation in the entire history of partitions. Almost every discovery in partitions owes something to Euler's beginnings. Extensive accounts of the use of generating functions in the theory of partitions can be found in [5, Ch. 13], [9, Chs. 1 and 2], [37, Ch. 7], [41, Ch.19] and [56, §7, pp. 29–69]. In addition, a paper-by-paper summary of the entire history of partitions up to 1918 is found in Volume II of Dickson's *History of the Theory of Numbers* [27].

3. Sylvester and the Intrinsic Study of Partitions

From the mid-eighteenth century until the mid-nineteenth century, little happened of significance in the study of partitions. Of course, lots was happening in mathematics. Subjects such as the theory of complex variables and the theory of elliptic functions were born, and these would turn out to affect partitions profoundly. Great mathematicians such as Legendre, Gauss, Cauchy and others would make discoveries in their explications of Euler's work.

In the century between 1750 and 1850, the primary focus of research concerned explicit formulae for $p_k(n)$, the number of partitions of n into at most k parts. P. Paoli, A. DeMorgan, J. F. W. Herschel, T. Kirkman and H. Warburton each studied $p_k(n)$ for small fixed values of k, and each produced a number of explicit formulas. We shall say more about this in Section 5.

However, J. J. Sylvester is the next mathematician to provide some truly new insight. In his magnum opus, A Constructive Theory of Partitions, Arranged in Three Acts, An Interact, and an Exodion [70], Sylvester began with these words:

> "In the new method of partitions it is essential to consider a partition as a definite thing, which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of parts shall be according to their order of magnitude."

After considering several ways that a partition may be actually given some sort of geometric image, he asserted that the partition $5+5+4+3+3$, "... may be represented much more advantageously by the figure

$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & \\ * & * & * & & \\ * & * & * & & \end{array} \quad ."$$

This representation Sylvester called the Ferrers graph of the partition, after N. M. Ferrers.

He noted immediately that one can count nodes in columns instead of rows. In the above instance this produces the partition $5+5+5+3+2$. The two partitions produced from such a graph are called *conjugates*. Thus within the first two pages of this 83-page paper, Sylvester inaugurated a brand new approach to partitions.

To appreciate the value of this new line of thought, we present Fabian Franklin's proof [31] of Euler's pentagonal number theorem. Franklin was one of Sylvester's students at Johns Hopkins University, and his proof illustrates the power of Sylvester's idea.

We begin by noting (as did Legendre) that the pentagonal number theorem can be reformulated purely as an assertion about partitions. If we set $z = -1$ in (2.1), we see that

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{m,n\geqq 0}(-1)^m D(m,n)q^n \,. \tag{3.1}$$

Thus the coefficient of q^n in (3.1) is the difference between the number of partitions of n into an even number of *distinct* parts (say, $\Delta_e(n)$) and the number of partitions of n into an odd number of *distinct* parts (say, $\Delta_0(n)$). So, Euler's pentagonal number theorem, (2.4), is equivalent to the assertion that

$$\Delta_e(n) - \Delta_0(n) = \begin{cases} (-1)^j, & \text{if } n = j(3j\pm 1)/2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Franklin's idea for proving (3.2) was to find a one-to-one mapping between the partitions of n with an even number of parts (all distinct) and the partitions of n with an odd number of parts (all distinct). Of course, for (3.2) to be valid, this mapping must run aground on an occasional exceptional case.

We proceed to examine partitions π with distinct parts. We define $s(\pi)$ to be the smallest summand in π, and $\sigma(\pi)$ to be as the length of the longest sequence of consecutive integers appearing in π containing the largest part of π. For example, if π is $9+8+7+5+4+2$, then $s(\pi) = 2$ and $\sigma(\pi) = 3$. We can provide images of

$s(\pi)$ and $\sigma(\pi)$ when we look at the Ferrers graph of π

$$\begin{array}{llllllllll}
* & * & * & * & * & * & * & * & * & \\
* & * & * & * & * & * & * & * & & \sigma(\pi) \\
* & * & * & * & * & * & * & & & \\
* & * & * & * & * & & & & & \\
* & * & * & * & & & & & & \\
* & * & & & & & & & & \\
s(\pi) & & & & & & & & &
\end{array}$$

Franklin then defines a transformation of partitions with distinct parts:

Case 1. $s(\pi) \leqq \sigma(\pi)$.

In the Ferrers graph of π, move the nodes in $s(\pi)$ to a position parallel to those indicated above defining $\sigma(\pi)$. Thus the transformed partition arising when this move is applied to $9+8+7+5+4+2$ is $10+9+7+5+4$ – namely

$$\begin{array}{llllllllll}
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & \\
* & * & * & * & * & * & * & & & \\
* & * & * & * & * & & & & & \\
* & * & * & * & & & & & &
\end{array}$$

2. $s(\pi) > \sigma(\pi)$.

In the Ferrers graph of π, move the nodes in $\sigma(\pi)$ so that they form the smallest part of the transformed partition. Thus if we consider $10+9+7+5+4$, we see that $\sigma(\pi) = 2$, $s(\pi) = 4$, and the transformed partition is $9+8+7+5+4+2$.

Note that Franklin's map changes the parity of the number of parts, and it appears not only to be one-to-one, but indeed to be an involution. If it really were all of these things, then we would have just proved that the right-hand side of (3.2) must always be 0.

However, Franklin's map gets into trouble whenever the two sets of nodes in the Ferrers graph defining $s(\pi)$ and $\sigma(\pi)$ respectively are not disjoint. If they are disjoint, Franklin's map gives no problems. Indeed if $s(\pi) < \sigma(\pi)$ in Case 1, or $s(\pi) > \sigma(\pi) + 1$ in Case 2, everything still works.

Exceptional case 1. $s(\pi) = \sigma(\pi)$, and the defining sets for $s(\pi)$ and $\sigma(\pi)$ are not disjoint. For example,

$$\begin{array}{llllllll}
* & * & * & * & * & * & * & \\
* & * & * & * & * & * & & \sigma(\pi) = j \\
* & * & * & * & * & & & \\
* & * & * & * & & & & \\
\multicolumn{4}{l}{s(\pi) = j} & & & &
\end{array}$$

Clearly we cannot do the transformation required in Case 1. Thus the partition with j distinct parts

$$j + (j + 1) + (j + 2) + \cdots + (2j - 1) = j(3j - 1)/2$$

has no image under Franklin's mapping.

Exceptional case 2. $s(\pi) = \sigma(\pi) + 1$ and the defining set for $s(\pi)$ and $\sigma(\pi)$ are not disjoint. For example.

$$\begin{array}{lllllll}
* & * & * & * & * & * & \\
* & * & * & * & * & & \sigma(\pi) = j \\
* & * & * & * & & & \\
\multicolumn{4}{l}{s(\pi) = j + 1} & & &
\end{array}$$

Now we cannot do the transformation required in Case 2. Thus the partition with j distinct parts

$$(j + 1) + (j + 2) + \cdots + (2j) = j(3j + 1)/2$$

has no image under Franklin's mapping.

We conclude that $\Delta_e(n) - \Delta_0(n)$ is 0, except when n is either $j(3j - 1)/2$ or $j(3j + 1)/2$ in which case it is $(-1)^j$. In other words, we have proved (3.2) and consequently (2.4).

Hans Rademacher termed Franklin's work the first major achievement of American mathematics. It may be viewed as the starting point for a variety of deep combinatorial studies of partitions.

Isaai Schur's first proof of the (unknown to him) Rogers-Ramanujan identities [64; §3] owes much to Franklin. More recently, building on Schur's work, A.Garsia and S. Milne in 1981 produced a pure one-to-one correspondence proof of the Rogers-Ramanujan identities [32]. The Rogers-Ramanujan identities form a major topic in Section 6.

Perhaps the most striking achievement of this nature is the 1985 proof by D. Zeilberger and D. Bressoud [74] of the q-Dyson conjecture – namely if

$$\prod_{1 \leqq i < j \leqq n} \prod_{h=0}^{a_i - 1} \left(1 - \frac{x_i q^h}{x_j}\right)\left(1 - \frac{x_j q^{h+1}}{x_i}\right)$$

is fully expanded, then the terms involving only powers of q but no x_is sum up to the polynomial

$$\frac{(1-q)(1-q^2)(1-q^3)\cdots(1-q^{a_1+a_2+\cdots+a_n})}{\prod_{h=1}^{n}\{(1-q)(1-q^2)\cdots(1-q^{a_i})\}}\,.$$

Further accounts of these ideas and related work can be found in Andrews [6] and [11; Chs. 6 and 7].

4. Partitions Representing Other Mathematical Objects.

It should not be surprising that partitions have a life outside their own intrinsic interest. After all, it is clear that whenever some set of n objects is grouped into subsets in which only the size of each subset is of significance, then the object of interest is a partition of n. Thus when Cauchy [23] first studied what in effect were the conjugacy classes of the symmetric group in 1845, their enumeration required classification by sizes of cycles. Thus there are $p(n)$ conjugacy classes of the symmetric group S_n.

Classical invariant theory, as practised by Cayley [25], Sylvester [69], MacMahon [52; Vol. 2, Ch. 18] and others, bulges with the theory of partitions. For example, Stroh [67] solved a problem considered extensively by Cayley, Sylvester and

MacMahon when he showed that the generating function for perpetuants of a given degree θ (> 2) is

$$\frac{q^{2^{\theta-1}-1}}{(1-q^2)(1-q^3)\cdots(1-q^\theta)} .$$

Most important in this period was the realization that this external use of partitions was mutually beneficial. For example, from the work of Cauchy [22], Gauss [33], and others, it was determined that for non-negative integers n and m,

$$\begin{bmatrix} n+m \\ n \end{bmatrix} := \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+m})}{(1-q)(1-q^2)\cdots(1-q^m)}$$

is, in fact, a reciprocal polynomial in q of degree nm. Indeed, the coefficient of q^j is the number of partitions of j into at most m parts each n. These polynomials have come to be called *Gaussian polynomials*, or *q-binomial coefficients*.

Empirically one finds they are all unimodal; in fact, the coefficients of q^j form a nondecreasing sequence for $j \leqq \frac{nm}{2}$ and a nonincreasing sequence for $j \geqq \frac{nm}{2}$. For example,

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4 ,$$
$$\begin{bmatrix} 7 \\ 3 \end{bmatrix} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12} .$$

While this is an appealing oddity viewed merely as a fact about partitions, it was of great importance in invariant theory. Witness Sylvester's unbounded enthusiasm as he introduces his proof [68]:

> I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. It is the more necessary that this should be done, because the theory has been supposed to lead to false conclusions, and its correctness has consequently been impugned. But, of the two suppositions that might be made to account for the observed discrepancy between the supposed consequences of the theorem and ascertained facts—one that the theory is false and the reasoning applied to it correct, the other that the theorem is true but that an error was committed in drawing certain deductions from it (to which one might add a third, of the theorem and the reasoning upon it being both erroneous)—the wrong alternative was chosen. An error was committed in reasoning out certain supposed consequences of the theorem; but the theorem itself is perfectly true, as I shall show by an argument so irrefragable that it must be considered for ever hereafter safe from all doubt or cavil. It lies as the basis of the investigations begun by Professor Cayley in his *Second Memoir on Quantics*, which it has fallen to my lot, with no small labour and contention of mind, to lead to a happy issue, and thereby to advance the standards of the Science of Algebraical Forms to the most advanced point that has hitherto been reached. The stone that was rejected by the builders has become the chief corner-stone of the building.

Now the Gaussian polynomials were observed by Dickson [28, p. 49] also to count the total number of vector spaces of dimension n over a finite field of q elements (q is now not a real or complex variable but a prime power). Of course, Dickson proved this directly; however, Donald Knuth was the first person to point out exactly how the underlying partitions fit in [45].

Namely, Knuth considered all possible canonical bases for the k-dimensional subspaces of V_n, the vector space of dimension n over $GF(q)$. Letting $(x_1, \ldots, x_n)$ ($x_i \in GF(q)$) denote the elements of V_n, we see that a canonical basis for some subspace U consists of m vectors of the form

$$u_i = (u_{i1}, \ldots, u_{in}) \qquad (1 \le i \le k)$$

satisfying

$$u_{in_i} = 1\,, \qquad u_{ij} = 0 \text{ for } j > n_i, \text{ and } u_{ln_i} = 0 \text{ for } l < i\,.$$

for $1 \leqq i \leqq k$, with $n \geqq n_1 > n_2 > \cdots > n_k \geqq 1$. He then gave an example that revealed everything. Take $n = 9$, $k = 4$, $n_1 = 8$, $n_2 = 5$, $n_3 = 3$, $n_4 = 2$, then the canonical basis of U is

$$u_1 = (u_{11}, 0, 0, u_{14}, 0, u_{16}, u_{17}, 1, 0)\,,$$

$$u_2 = (u_{21}, 0, 0, u_{24}, 1, 0, 0, 0, 0)\,,$$

$$u_3 = (u_{31}, 0, 1, 0, 0, 0, 0, 0, 0)\,,$$

$$u_4 = (u_{41}, 1, 0, 0, 0, 0, 0, 0, 0)\,.$$

The pattern of the u_{ij} – namely,

$$\begin{array}{cccc} * & * & * & * \\ * & * & & \\ * & & & \\ * & & & \end{array}$$

is the Ferrers graph of the partition $4+2+1+1$. So each u_{ij} appearing above may be filled in $q^8 = q^{4+2+1+1}$ ways. By this means we see clearly the correspondence with the partitions with at most k parts each being at most $n-k$, and Knuth's observation is clear.

Near the turn of the century, the Revd. Alfred Young, in a series of papers [73] on invariant theory, introduced partitions and variations thereof (now called *Young tableaux*) in what would come to be called the representation theory of the symmetric group (see [66]).

In the last half of the twentieth century, applications mushroomed. J. W. B. Hughes [44] developed applications (both ways) between Lie algebras and partitions. J. Lepowsky et al. [48] showed how to interpret and prove the Rogers-Ramanujan identities in Lie algebras, and a number of applications of partitions have arisen in physics (cf. [18]). Perhaps the most satisfying (and most surprising) was Rodney Baxter's solution of the Hard Hexagon model [16]; in simple terms, this work says that the Rogers-Ramanujan identities are crucial in studying the behavior of liquid helium on a graphite plate [17]. It is fitting to close this section by noting that K. O'Hara [54] discovered a purely partition–theoretic proof of Sylvester's theorem that the Gaussian polynomials are unimodal.

5. Asymptotics.

One of the greatest surprises in the history of partitions was the formula due to Hardy, Ramanujan [39] and Rademacher [57] for $p(n)$. However it was not the first formula found for partition functions.

Cayley [24] and Sylvester [69] (anticipated by J. Herschel [43]) gave a number of formulas for $p_k(n)$ with small k, where $p_k(n)$ is the number of partitions of n into at most k parts (or, by conjugation, the number of partitions of n into parts each at most k). For example,

$$p_2(n) = \lfloor (n+1)/2 \rfloor \tag{5.1}$$

and

$$p_3(n) = \left\{ (n+3)^2/12 \right\} , \tag{5.2}$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x and $\{x\}$ is the nearest integer to x.

Such results are fairly easy to prove using nothing more powerful than the binomial series. For example,

$$\begin{aligned}
\sum_{n=0}^{\infty} p_3(n) q^n &= \frac{1}{(1-q)(1-q^2)(1-q^3)} \\
&= \tfrac{1}{6}(1-q)^{-3} + \tfrac{1}{4}(1-q)^{-2} + \tfrac{1}{4}(1-q^2)^{-1} + \tfrac{1}{3}(1-q^3)^{-1} \\
&= \tfrac{1}{6}\sum_{n=0}^{\infty} \binom{n+2}{2} q^n + \tfrac{1}{4}\sum_{n=0}^{\infty}(n+1)q^n + \tfrac{1}{4}\sum_{n=0}^{\infty} q^{2n} + \tfrac{1}{3}\sum_{n=0}^{\infty} q^{3n} \\
&= \sum_{n=0}^{\infty} \left(\frac{(n+3)^2}{12} - \tfrac{1}{3} \right) q^n + \tfrac{1}{4}\sum_{n=0}^{\infty} q^{2n} + \tfrac{1}{3}\sum_{n=0}^{\infty} q^{3n} \, .
\end{aligned} \tag{5.3}$$

So $p_3(n)$ must be an integer that is in absolute value within $\frac{1}{3}$ of $(n+3)^2/12$. Hence (5.2) is valid.

An extensive account of such results and their history is given in [35].

Now Hardy and Ramanujan, in perhaps their most important joint paper [39], found an asymptotic series for $p(n)$. The simplest special case of their result is the assertion that, as $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \, e^{\pi\sqrt{\frac{2n}{3}}} , \tag{5.4}$$

a result found independently by Uspensky [71] a few years later.

In 1937, Rademacher [57] improved the formula of Hardy and Ramanujan so that a convergent infinite series was found for $p(n)$, namely,

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{\frac{1}{2}}\right)}{\left(x - \frac{1}{24}\right)^{\frac{1}{2}}} \right]_{x=n}, \tag{5.5}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i s(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \left\lfloor \frac{\mu}{k} \right\rfloor - \frac{1}{2} \right) \left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2} \right).$$

Now this is not one of those mathematical formulas that elicits the response: "Just as I expected!" Indeed, it is really astounding. The proof consists of sheer wizardry. The contributions of both Hardy and Ramanujan were summarized as follows by J. E. Littlewood [50]

> We owe the theorem to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him. Ramanujan's genius did have this one opportunity worthy of it.

Rademacher [57] also proved that if the series in (5.5) is truncated after N terms, the error is bounded in absolute value by

$$\frac{2\pi^2}{9\sqrt{3}} \, e^{\frac{\pi}{N+1}\sqrt{\frac{2n}{3}}} \, \frac{1}{N^{\frac{1}{2}}},$$

which clearly tends to 0 as $N \to \infty$. Indeed this error is quite crude, suggesting that 26 terms of the series are required to get within one unit of the correct value of $p(200) = 3972999029388$. Actually after 8 terms are evaluated the result is within 0.004 of the correct answer [61, p.284].

It would be hard to overstate the impact of this result on number theory in the 20^{th} century. After Ramanujan's death in 1920, Hardy and Littlewood published a series of papers (see [42]) with the title *Some problems of Partitio Numerorum* (originally used by Euler for Chapter 16 of [30]). These papers gave deep results on sums of squares, Waring's problem, twin primes, the Goldbach conjecture, etc. After Rademacher made his contribution to $p(n)$, he and many of his school used his refinements and alternative work using Ford circles [58] to prove a myriad of asymptotic theorems for the coefficients of modular forms and related functions.

6. Ramanujan.

The history of partitions is filled with starts and stops. Euler's penetrating study of partitions stood for more than a hundred years before others made significant advances. One of the strangest stories surrounds two easily understood partition theorems, the *Rogers-Ramanujan identities*.

First Rogers-Ramanujan Identity. *The partitions of n into summands that differ from each other by at least 2 are equinumerous with the partitions into parts of the forms $5m+1$ and $5m+4$.*

Second Rogers-Ramanujan Identity. *The partitions of n into summands each larger than 1 which differ from each other by at least 2 are equinumerous with the partitions into parts of the forms $5m+2$ and $5m+3$.*

We may express these theorems as identities of the related generating functions:

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}\ ,$$

and

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}\ .$$

These latter two identities appeared in 1894 in a paper by L. J. Rogers that appeared in the *Proceedings of the London Mathematical Society*, hardly an obscure journal. The paper was entitled, *Second Memoir on the Expansion of Certain Infinite Products* [63]. Apparently the mathematical public had lost interest somewhere in the first memoir on the *Expansion of Some Infinite Products* [61]. In any event, the paper was quite forgotten when nineteen years later an unknown Indian clerk, S. Ramanujan, sent these identities to G. H. Hardy. At first glance, these formulas look very much like ones that Euler had found, connected with the products we described in Section 2. However, Hardy found that he was completely unable to prove them. He communicated them to Littlewood, MacMahon and Perron, and no one could prove them. No one thought of sending them to Rogers. MacMahon was the person who saw the series as generating functions for the two partition theorems, and he stated [51; Vol. 2, p. 33]:

> This most remarkable theorem has been verified as far as the coefficient of q^{89} by actual expansion so that there is practically no reason to doubt its truth; but it has not yet been established.

Let us now turn to Hardy's account of the moment of illumination [40, p. 91]:

> The mystery was solved, trebly, in 1917. In that year Ramanujan, looking through old volumes of the *Proceedings of the London Mathematical Society*, came accidentally across Rogers's paper. I can remember very well his surprise, and the admiration which he expressed for Rogers's work. A correspondence followed in the course of which Rogers was led to a considerable simplication of his original proof. About the same time I. Schur, who was then cut off from England by the war, rediscovered the identities again. Schur published two proofs, one of which is "combinatorial" and quite unlike any other proof known.

Hardy wrote these words in 1940. For a while it seemed that these results were isolated curiosities. Fourteen years later, basing his remarks on the work of Lehmer [46] and Alder [1], Rademacher [59; p. 73] was to say: "It can be shown that there can be no corresponding identities for moduli higher than 5." This turned out to be false. In 1963, B. Gordon [34] made the first step toward a full exploration of theorems of this type.

Gordon's Theorem. *Let $A_{k,a}(n)$ denote the number of partitions of n into parts not congruent to* 0, $\pm a \pmod{2k+1}$. *Let $B_{k,a}(n)$ denote the number of partitions* *of n into parts of the form $b_1 + b_2 + \cdots + b_j$, where $b_i \geqq b_{i+1}$ and $b_i - b_{i+k-1} \geqq 2$ and at most $a-1$ of the b_i are 1. Then for $0 < a \leqq k$ and each $n \geqq 0$,*

$$A_{k,a}(n) = B_{k,a}(n)\,.$$

Gordon's theorem led to an explosion of results ([2], [7], [8]). D. Bressoud made the first major combinatorial breakthroughs in the study of partition identities, culminating in [20]. In recent years, K. Alladi and his collaborators [3] have found a substantial combinatorial theory of weighted words related to such results.

Ramanujan made many contributions to mathematics which are carefully surveyed by Hardy in his book [40]. However we should not fail to mention one more aspect of Ramanujan's discoveries; he discovered a number of divisibility properties of $p(n)$. Most notable are the following three [61, p. 210]:

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod 7$$

There are many other such results. Ramanujan conjectured infinite families of congruences, and these conjectures were proved (after removing some false ones) by Watson [72] and Atkin [15]. Most recently K. Ono (in [55], among many other papers) has given us a picture of the depth and scope of these problems.

7. Other Types of Partitions.

In Section 4, we alluded to some variations of partitions. The first mild variation is composition. A *composition* is a partition wherein different orders of summands count as different compositions. Thus there are eight partitions of 4, namely, $4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1$. In 1876, Cayley [26] proved the first non-trivial theorem on compositions

Cayley's Theorem. *Let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$ (these are the famous Fibonacci numbers). The number of compositions of n not using 1's is F_{n-1}.*

P. A. MacMahon gave compositions their name and scrutinized them thoroughly in [52; Vol. 1, Ch.5]. As he developed his study of partitions, we can observe him almost blindly stumbling onto another form of partitions, plane partitions [52, pp. 1075–1080]. Up to now, partitions have been linear or single-fold sums of integers: $n = \sum_{i=1}^{j} a_i$ $(a_i \geqq a_{i+1})$. However, MacMahon found many intriguing properties associated with plane (or two-dimensional) partitions:

$$n = \sum_{i,j \geqq 1} a_{ij} \text{ , where } a_{i,j} \geqq a_{i,j+1} \text{ and } a_{i,j} \geqq a_{i+1,j} \, .$$

Usually plane partitions are most easily understood when pictured as an array in the fourth quadrant. For example the six plane partitions of 3 are

$$\begin{array}{llllll} 3, & 21, & 2, & 111, & 11, & 1 \\ & & 1 & & 1 & 1 \\ & & & & & 1 \end{array}$$

Much to his surprise, MacMahon discovered [50, Vol. 1, p. 1071] (and proved seventeen years later [50, Vol. 1, Ch. 12]) that

$$\sum_{n=0}^{\infty} M(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \, .$$

MacMahon at first believed that comparably interesting discoveries awaited higher dimensional partitions. However, he noted that such hopes were in vain [52, Vol. 1, p. 1168].

Many subsequent discoveries have been made for plane partitions. A beautiful development of recent work has been given by Richard Stanley, in two papers [65] and an excellent book [66].

8. Further Leads to the History

A short chapter like this must slight much of the history of partitions. Many favourite topics have received little or no attention. So in this final section we mention some historical sources where one may get a more detailed treatment of various aspects of the history of partitions.

First and foremost is Chapter III of Volume 2 of L. E. Dickson's *History of the Theory of Numbers* [27]. It cites every paper on partitions known up to 1916. H. Ostmann's *Additive Zahlentheorie*, Volume 1, Chapter 7 [56] contains a fairly full account of progress in the first half of the twentieth century. Reviews of all papers on partitions from 1940 to 1983 can be found in [49] and [38], refer to Chapter P. In as much as MacMahon is a seminal and lasting influence in partitions, you should examine Volume 1 of his Collected Papers [52, Vol. 1]. It is organized with the same chapter headings as used in MacMahon's *Combinatory Analysis* [51], and each chapter is introduced with some history and a bibliography of work since MacMahon's death in 1930.

Besides these major works, there have been a number of survey articles with extensive histories. H. Gupta provided a general survey in [36]. Partition identities are handled in [2], [4], [7], [12]. Richard Stanley gave a history of plane partitions in [65]. Applications in physics are discussed by Berkovich and McCoy in [18] (see [11, Ch. 8]).

Finally there are books that have some of the history of partitions. Andrews [9] is devoted entirely to partitions, and the Notes sections concluding each chapter have extensive historical references. Bressoud [21] has recently published the history of the alternating sign matrix conjecture, an appealing and well told tale that is tightly bound up with the theory of partitions. Ramanujan's amazing contributions to partitions (as well as many other aspects of number theory) have been chronicled by G. H. Hardy [40], and most thoroughly by B. Berndt in five volumes [19]. Books with chapters on partitions include Gupta [37, Chs. 7–10], Hardy and Wright [41, Ch. 19], Macdonald [53, Ch.1, Section 1], Rademacher [59, Parts I and III], Rademacher [60, Chs. 12–14] and Stanley [66; Ch. 7].

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