PARTITIONS WITH FIXED DIFFERENCES BETWEEN LARGEST AND SMALLEST PARTS

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ABSTRACT. We study the number $p(n,t)$ of partitions of n with difference t between largest and smallest parts. Our main result is an explicit formula for the generating function $P_t(q) := \sum_{n\ge1} p(n,t)\, q^n$. Somewhat surprisingly, $P_t(q)$ is a rational function for $t>1$; equivalently, $p(n,t)$ is a quasipolynomial in n for fixed $t>1$. Our result generalizes to partitions with an arbitrary number of specified distances.

A *partition* of a positive integer n is an integer k-tuple $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k > 0$, for some k, such that

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \, .$$

The integers $\lambda_1, \lambda_2, \dots, \lambda_k$ are the *parts* of the partition. Enumeration results on integer partitions form a classic body of mathematics going back to at least Euler, including numerous applications throughout mathematics and some areas of physics; see, e.g., [2]. We are interested in the counting function

$$p(n,t) := \#\text{partitions of } n \text{ with difference } t \text{ between largest and smallest parts.}$$

It is immediate that

$$p(n,0) = d(n)$$

where $d(n)$ denotes the number of divisors of n. Charmingly, $p(n,1)$ equals the number of *non*divisors of n:

$$p(n,1) = n - d(n) \, ,$$

which can be explained bijectively by the fact that the partitions counted by $p(n,0)+p(n,1)$ contain exactly one sample with k parts, for each $k = 1, 2, \dots, n$ [1, Sequence A049820], or by the generating function identity

$$\sum_{n\ge1} p(n,1)\, q^n = \sum_{m\ge1} \frac{q^m}{1-q^m} \, \frac{q^{m+1}}{1-q^{m+1}} = \frac{q}{(1-q)^2} - \sum_{m\ge1} \frac{q^m}{1-q^m}$$

(the last equation follows from a few elementary operations on rational function). An even less obvious instance of our partition counting function is

$$p(n,2) = \binom{\lfloor \frac{n}{2} \rfloor}{2} \, , \tag{1}$$

as observed by Reinhard Zumkeller in 2004 [1, Sequence A008805]. (It is not clear to us where in the literature this formula first appeared, though specific values of $p(n,k)$ are well represented in

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[1], where Sequences A000005, A049820, A008805, A128508, and A218567–A218573 give the first values of $p(n,k)$ for fixed $k = 0, 1, \ldots, 10$, and Sequence A097364 paints a general picture of $p(n,t)$.)

We remark that $p(n,2)$ is arithmetically quite different from $p(n,0)$ and $p(n,1)$: namely, $p(n,2)$ is a *quasipolynomial*, i.e., a function that evaluates to a polynomial when n is restricted to a fixed residue class modulo some (minimal) positive integer, the *period* of the quasipolynomial. (For $p(n,2)$ this period is 2.) Equivalently, the accompanying generating function evaluates to a rational function all of whose poles are roots of unity. (See, e.g., [3, Chapter 4] for more on quasipolynomials and their rational generating functions.) Our goal is to prove closed formulas for these generating functions

$$P_t(q) := \sum_{n \ge 1} p(n,t)\, q^n .$$

Theorem 1. *For $t > 1$,*

$$P_t(q) = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}}{(1-q^t)^2(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q^2)} + \frac{q^t}{(1-q^t)(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q)} .$$

Written in terms of the usual shorthand $(q)_m := (1-q)(1-q^2)\cdots(1-q^m)$, Theorem 1 says

$$P_t(q) = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} + \frac{q^t}{(1-q^{t-1})(q)_t} .$$

Thus $P_t(q)$ is rational for $t > 1$, and so $p(n,t)$ is a quasipolynomial in n, with period $\operatorname{lcm}(1, 2, \ldots, t)$. For example, for $t = 2$, Theorem 1 gives

$$P_2(q) = \frac{q^4}{(1-q)^3(1+q)^2}$$

which confirms (1). The rational generating function given by Theorem 1 in the case $t = 3$ simplifies to

$$P_3(q) = \frac{q^5 + q^6 + q^7 - q^8}{(1-q)^4(1+q)^2(1+q+q^2)^2}$$

which translates to the partition counting function

$$p(n,3) = \frac{1}{108} \times \begin{cases} n^3 - 18n & \text{if } n \equiv 0 \bmod 6, \\ n^3 - 3n + 2 & \text{if } n \equiv 1 \bmod 6, \\ n^3 - 30n + 52 & \text{if } n \equiv 2 \bmod 6, \\ n^3 + 9n - 54 & \text{if } n \equiv 3 \bmod 6, \\ n^3 - 30n + 56 & \text{if } n \equiv 4 \bmod 6, \\ n^3 - 3n - 2 & \text{if } n \equiv 5 \bmod 6 \end{cases}$$

$$= \begin{cases} m(2m^2 - 1) & \text{if } n = 6m, \\ m(2m^2 + 1) & \text{if } n = 6m + 1, \\ m(2m^2 + 2m - 1) & \text{if } n = 6m + 2, \\ m(2m^2 + 3m + 2) & \text{if } n = 6m + 3, \\ (m-1)(2m^2 - 1) & \text{if } n = 6m - 2, \\ m^2(2m - 1) & \text{if } n = 6m - 1. \end{cases}$$

Using this explicit form of $p(n,3)$, one easily affirms a conjecture about the recursive structure of $p(n,3)$ given in [1, Sequence A128508] in the positive.

The corresponding data for $t=4$ is

$$P_4(q)=\frac{q^6+q^7+2q^8-q^{11}-q^{12}+q^{13}}{(1-q)^5(1+q)^3(1+q^2)^2(1+q+q^2)^2}$$

and

$$p(n,4)=\frac{1}{6912}\times\begin{cases}3n^4+20n^3-24n^2-288n & \text{if } n\equiv 0 \bmod 12,\\ 3n^4+20n^3-78n^2-492n+547 & \text{if } n\equiv 1 \bmod 12,\\ 3n^4+20n^3-24n^2+48n-208 & \text{if } n\equiv 2 \bmod 12,\\ 3n^4+20n^3-78n^2-1260n+3699 & \text{if } n\equiv 3 \bmod 12,\\ 3n^4+20n^3-24n^2+480n-3584 & \text{if } n\equiv 4 \bmod 12,\\ 3n^4+20n^3-78n^2-492n+35 & \text{if } n\equiv 5 \bmod 12,\\ 3n^4+20n^3-24n^2-720n+3888 & \text{if } n\equiv 6 \bmod 12,\\ 3n^4+20n^3-78n^2-492n+115 & \text{if } n\equiv 7 \bmod 12,\\ 3n^4+20n^3-24n^2+480n-4096 & \text{if } n\equiv 8 \bmod 12,\\ 3n^4+20n^3-78n^2-1260n+4131 & \text{if } n\equiv 9 \bmod 12,\\ 3n^4+20n^3-24n^2+48n+304 & \text{if } n\equiv 10 \bmod 12,\\ 3n^4+20n^3-78n^2-492n-397 & \text{if } n\equiv 11 \bmod 12\end{cases}$$

$$=\begin{cases}9m^4+5m^3-\frac12(m^2+m) & \text{if } n=12m,\\ 9m^4+8m^3-m & \text{if } n=12m+1,\\ 9m^4+11m^3+\frac12(7m^2+m) & \text{if } n=12m+2,\\ 9m^4+14m^3+\frac12(11m^2-3m) & \text{if } n=12m+3,\\ 9m^4+17m^3+\frac12(21m^2+7m) & \text{if } n=12m+4,\\ 9m^4+20m^3+14m^2+3m & \text{if } n=12m+5,\\ 9m^4+23m^3+\frac12(41m^2+13m)+1 & \text{if } n=12m+6,\\ 9m^4+26m^3+\frac12(51m^2+19m)+1 & \text{if } n=12m+7,\\ 9m^4+29m^3+\frac12(67m^2+35m)+3 & \text{if } n=12m+8,\\ 9m^4+32m^3+40m^2+19m+3 & \text{if } n=12m+9,\\ 9m^4+35m^3+\frac12(99m^2+61m)+7 & \text{if } n=12m+10,\\ 9m^4+38m^3+\frac12(115m^2+73m)+8 & \text{if } n=12m+11.\end{cases}$$

Proof of Theorem 1. We will use the usual shorthand

$$(A)_m:=(1-A)(1-A\,q)\cdots(1-A\,q^{m-1}),$$

the q-binomial coefficient

$$\begin{bmatrix}A\\B\end{bmatrix}:=\frac{(q)_A}{(q)_B(q)_{A-B}},$$

as well as *Heine's transformation* (see, e.g., [2, p. 38])

$$\sum_{m\geq 0}\frac{(a)_m(b)_m\,z^m}{(q)_m(c)_m}=\frac{(\frac{c}{b})_\infty(bz)_\infty}{(c)_\infty(z)_\infty}\sum_{j\geq 0}\frac{(\frac{abz}{c})_j(b)_j(\frac{c}{b})^j}{(q)_j(bz)_j}\,. \tag{2}$$

We start with the natural generating function for $p(n,t)$:

$$\begin{aligned}
P_t(q) &= \sum_{m\ge1} \frac{q^m}{1-q^m}\,\frac{1}{1-q^{m+1}}\cdots\frac{1}{1-q^{m+t-1}}\,\frac{q^{m+t}}{1-q^{m+t}} = q^t\sum_{m\ge1}\frac{q^{2m}(q)_{m-1}}{(q)_{m+t}} = q^{t+2}\sum_{m\ge0}\frac{q^{2m}(q)_m}{(q)_{m+t+1}}\\
&= \frac{q^{t+2}}{(q)_{t+1}}\sum_{m\ge0}\frac{(q)_m(q)_m\,q^{2m}}{(q)_m(q^{t+2})_m} \overset{(2)}{=} \frac{q^{t+2}(q^{t+1})_\infty(q^3)_\infty}{(q)_{t+1}(q^{t+2})_\infty(q^2)_\infty}\sum_{j\ge0}\frac{(q^{-t+2})_j(q)_j\,q^{j(t+1)}}{(q)_j(q^3)_j}\\
&= \frac{q^{t+2}}{(q)_t}\sum_{j=0}^{t-2}\frac{(q^{-t+2})_j\,q^{j(t+1)}}{(q^2)_{j+1}} = \frac{q^{t+2}}{(q)_t}\sum_{j=0}^{t-2}\frac{(1-q^{t-2})(1-q^{t-3})\cdots(1-q^{t-j-1})(-1)^j q^{2j+\binom{j+1}{2}}}{(q^2)_{j+1}}\\
&= \frac{q^{t+2}(1-q)}{(1-q^t)(1-q^{t-1})}\sum_{j=0}^{t-2}\frac{(-1)^j q^{2j+\binom{j+1}{2}}}{(q)_{j+2}(q)_{t-j-2}} = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t}\sum_{j=0}^{t-2}\begin{bmatrix} t\\ j+2\end{bmatrix}(-1)^j q^{\binom{j+3}{2}}\,.
\end{aligned}$$

Thus, by the q-binomial theorem (see, e.g., [2, p. 36])

$$\begin{aligned}
P_t(q) &= \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t}\sum_{j=2}^{t}\begin{bmatrix} t\\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}} = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t}\left((q)_t - 1 + q\begin{bmatrix} t\\ 1\end{bmatrix}\right)\\
&= \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} + \frac{q^t}{(1-q^{t-1})(q)_t}\,. \qquad \square
\end{aligned}$$

Next we shall generalize Theorem 1 by considering *partitions with specified distances.* Let $p(n,t_1,t_2,\dots,t_k)$ be the number of partitions of n such that, if σ is the smallest part then $\sigma+t_1+t_2+\cdots+t_k$ is the largest part and each of $\sigma+t_1$, $\sigma+t_1+t_2$, …, $\sigma+t_1+t_2+\cdots+t_{k-1}$ appear as parts. We consider the related generating function

$$P_{t_1,\dots,t_k}(q) := \sum_{n\ge1} p(n,t_1,t_2,\dots,t_k)\,q^n.$$

We note that when $k=1$ this is simply $P_t(q)$ from above. Let $t := t_1+t_2+\cdots+t_k$ and $T := kt_1+(k-1)t_2+\cdots+2t_{k-1}+t_k$.

Theorem 2. *For $t>k$,*

$$P_{t_1,\dots,t_k}(q) = \frac{(-1)^k q^{T-\binom{k+1}{2}}\left(\sum_{j=0}^k \begin{bmatrix} t\\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}} - (q)_t\right)}{\begin{bmatrix} t-1\\ k\end{bmatrix}(1-q^t)(q)_t}\,.$$

Proof. Again we start with the natural generating function

$$\begin{aligned}
P_{t_1,\dots,t_k}(q) &= \sum_{m\ge1}\frac{q^m\,q^{m+t_1}\,q^{m+t_1+t_2}\cdots q^{m+t_1+t_2+\cdots+t_k}}{(1-q^m)(1-q^{m+1})\cdots(1-q^{m+t_1+t_2+\cdots+t_k})} = \sum_{m\ge1}\frac{q^{(k+1)m+T}}{(q^m)_{t+1}}\\
&= \sum_{m\ge1}\frac{q^{(k+1)m+T}(q)_{m-1}}{(q)_{m+t}} = q^{T+k+1}\sum_{m\ge0}\frac{q^{(k+1)m}(q)_m}{(q)_{m+t+1}} = \frac{q^{T+k+1}}{(q)_{t+1}}\sum_{m\ge0}\frac{(q)_m(q)_m\,q^{(k+1)m}}{(q)_m(q^{t+2})_m}\\
&\overset{(2)}{=} \frac{q^{T+k+1}(q^{t+1})_\infty(q^{k+2})_\infty}{(q)_{t+1}(q^{k+1})_\infty(q^{t+2})_\infty}\sum_{j\ge0}\frac{(q^{k+1-t})_j(q)_j q^{(t+1)j}}{(q)_j(q^{k+2})_j}\\
&= \frac{q^{T+k+1}(q)_k}{(q)_t}\sum_{j=0}^{t-k-1}\frac{(q^{-(t-k+1)})_j q^{(t+1)j}}{(q)_{j+k+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^{T+k+1}(q)_k}{(q)_t} \sum_{j=0}^{t-k-1} \frac{(1-q^{t-k-1})(1-q^{t-k-2})\cdots(1-q^{t-k-j})(-1)^j q^{\binom{j}{2}-j(t-k-1)+(t+1)j}}{(q)_{j+k+1}} \\
&= \frac{q^{T+k+1}(q)_k}{(q)_t} \sum_{j=0}^{t-k-1} \frac{(q)_{t-k-1}(-1)^j q^{\binom{j+1}{2}+j(k+1)}}{(q)_{j+k+1}(q)_{j-k-j-1}} \\
&= \frac{q^{T+k+1}(q)_k(q)_{t-k-1}}{(q)_t(q)_t} \sum_{j=0}^{t-k-1} \begin{bmatrix} t \\ j+k+1 \end{bmatrix} (-1)^j q^{\binom{j+k+2}{2}-\binom{k+2}{2}} \\
&= \frac{q^{T+k+1}(q)_k}{\begin{bmatrix} t-1 \\ k \end{bmatrix}(1-q^t)(q)_t} \sum_{j=0}^{t-k-1} \begin{bmatrix} t \\ j+k+1 \end{bmatrix} (-1)^j q^{\binom{j+k+2}{2}-\binom{k+2}{2}} \\
&= \frac{q^{T-\binom{k+1}{2}}(-1)^{k+1}}{\begin{bmatrix} t-1 \\ k \end{bmatrix}(1-q^t)(q)_t} \sum_{j=k+1}^{t} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} \\
&= \frac{q^{T-\binom{k+1}{2}}(-1)^{k}}{\begin{bmatrix} t-1 \\ k \end{bmatrix}(1-q^t)(q)_t} \left(\sum_{j=0}^{k} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} - (q)_t \right). \qquad \square
\end{aligned}
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