Euler's "De Partitio Numerorum"

by

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Abstract

Chapter 16 of Euler's Introductio in Analysin Infinitorum is titled De Partitio Numerorum. Our goal is to provide a survey of the impact of Euler's chapter on the study of partitions in the following 250 plus years. We ask the natural question: Do we still have anything to learn from Euler? The answer is "yes".

1 Introduction

In the 300 years since Euler's birth, mathematics has grown into a massive enterprise that plays a vital role in some of the most important projects in modern society. So it is natural in this anniversary year to look back and examine Euler's extensive impact on many branches of mathematics.

Twenty five years ago, in another Euler celebration, I wrote about Euler's Pentagonal Number Theorem [8], his proof of it, and what lessons might have been learned from a careful study of his proof.

For the Euler tercentenary, I will expand on my 1983 theme by considering Chapter 16 of Euler's Introductio in Analysin Infinitorum which was entitled De Partitio Numerorum, On the partition of numbers. John Blanton [16, Ch.

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16] has provided a full English translation which makes it possible now for everyone to follow Euler's still lucid introduction to the theory of partitions.

The theory of partitions has been intensively studied during several periods since Euler wrote. Indeed, early in the twentieth century, Euler's title "De Partitio Numerorum" was used by G. H. Hardy and J. E. Littlewood [22] in a series of path breaking papers, Some Problems of Partitio Numerorum, in analytic number theory. An early account of the history of partitions was provided in L. E. Dickson's History of the Theory of Numbers [15, Ch. III], and a more recent history is about to appear [10].

Euler's chapter on partitions begins with Section 297 and concludes after Section 331 with three pages of tables. In this paper, we shall divide the topics from Chapter XVI into four parts. First we consider the breakthrough beginning in which the role of infinite products and their expansions is presented. Next comes Euler's famous partition identity and the world that it opened. After this we should say a little more about the pentagonal number theorem.

Up to this point, we have mostly been mulling over the three centuries worth of accomplishments built upon Euler's insights. The final section of this paper derives from Sections 328–331 of Chapter XVI. As I began to reread these for the first time in twenty five years, I was struck by Euler's formulation of the ancient theorem that every positive integer is uniquely the sum or difference of distinct powers of 3. For example, $300 = 3^5 + 3^4 - 3^3 + 3^1$. Euler states this in terms of generating functions:

$$\prod_{n=0}^{\infty} (x^{-3^n} + 1 + x^{3^n}) = \sum_{n=-\infty}^{\infty} x^n. \tag{1.1}$$

My goodness! Here is an identity in one of the historically fundamental books of analysis, and it converges for **no** values of x. Oh well, let's just be modern about it and treat it as an identity in formal Laurent series; then everything is OK. With that comment, we can sweep our initial concern under the rug. I suggest that we not worry about justifying (1.1), but rather ask ourselves: Why have we thought so little about partition generating functions in which some of the partitions might have some negative parts? The final section of this paper will suggest that Euler's eye-catching identity (1.1) leads us to new and appealing discoveries. Euler has taught us another lesson.

2 Infinite Products and Partitions

The first three paragraphs of Chapter 16 as translated by John Blanton [16] give the full flavor of Euler's approach to partitions. They have not been improved upon in the intervening 250+ years.

297. Let the following expression be given:
$(1+x^\alpha z)(1+x^\beta z)(1+x^\gamma z)(1+x^\delta z)(1+x^\epsilon z)\cdots$. We ask about the form if the factors are actually multiplied. We suppose that it has the form $1+Pz+Qz^2+Rz^3+Sz^4+\cdots$, where it is clear that P is equal to the sum of the powers $x^\alpha+x^\beta+x^\gamma+x^\delta+x^\epsilon+\cdots$. Then Q is the sum of the products of the powers taken two at a time, that is Q is the sum of the different powers of x whose exponents are the sum of two of the different terms in the sequence $\alpha,\beta,\gamma,\delta,\epsilon,\zeta,\eta$, etc. In like manner R is the sum of powers of x whose exponents are the sum of three of the different terms. Further, S is the sum of powers of x whose exponents are the sum of four of the different terms in that same sequence $\alpha,\beta,\gamma,\delta,\epsilon$, etc., and so forth.

298. The individual powers of x which constitute the values of the letters P,Q,R,S, etc. have a coefficient of 1 if their exponents can be formed in only one way from $\alpha,\beta,\gamma,\delta$, etc. If the same exponent of a power of x can be obtained in several ways as the sum of two, three, or more terms of the sequence $\alpha,\beta,\gamma,\delta,\epsilon$, etc., then that power has a coefficient equal to the number of ways the exponent can be obtained. Thus, if in the value of Q there is found Nx^n, this is because n has N different ways of being expressed as a sum of two terms from the sequence α,β,γ, etc. Further, if in the expression of the given [*product*] the term Nx^nz^m occurs, this is because there are N different ways in [*the product that N*] can be a sum of m terms of the sequence $\alpha,\beta,\gamma,\delta,\epsilon,\zeta$, etc.

299. If the given product $(1+x^\alpha z)(1+x^\beta z)(1+x^\gamma z)(1+x^\delta z)\cdots$ is actually multiplied, then from the resulting expression it becomes immediately apparent how many different ways a given number can be the sum of any desired number of terms from the sequence $\alpha,\beta,\gamma,\delta,\epsilon,\zeta$, etc. For example if it is desired to know how many different ways the number n can be the sum of m terms of the given sequence, then we find the term x^nz^m, and its coefficient is the desired number.

With this introduction, Euler then goes to work on how to obtain expansions of such products as power series in z. His method boils down to the observation that if

$$f(z) = \prod_{n=1}^{\infty} (1 + zx^n), \tag{2.1}$$

then

$$\begin{aligned} f(z) &= (1+zx)(1+zx^2)(1+zx^3)\cdots \\ &= (1+zx)f(zx). \end{aligned} \tag{2.2}$$

Consequently if

$$f(z) = \sum_{n \geqq 0} A_n z^n, \tag{2.3}$$

then substituting (2.3) into (2.2) and comparing coefficients of z^n on each side reveals that $A_0 = 1$ and for $n > 0$

$$A_n = x^n A_n + x^n A_{n-1}. \tag{2.4}$$

Iteration of the recurrence for A_n contained in (2.4) shows that

$$A_n = \frac{x^{n(n+1)/2}}{(1-x)(1-x^2)\cdots(1-x^n)}\,. \tag{2.5}$$

This method and its consequences are developed by Euler through Section 324; in particular, a similar argument is applied to partitions with repeated parts to obtain

$$\prod_{n=1}^{\infty} \frac{1}{1 - zx^n} = 1 + \sum_{n=1}^{\infty} \frac{x^n z^n}{(1-x)(1-x^2)\cdots(1-x^n)}\,. \tag{2.6}$$

In addition, these formulae are used for the efficient construction of the tables of partitions at the end of the paper.

Euler's functional equation method which produced (2.3) and (2.5) has been extended in countless ways by Rogers and Ramanujan [30], Rogers [7], N. J. Fine [18] and others [3]. In Section 4, we shall return briefly to (2.2) when we note how Cayley used this very same functional equation to prove Euler's pentagonal number theorem.

3 Partition Identities

Euler began the vast subject of partition identities with these two simple paragraphs:

325. If the product $(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots$ is developed, we have the series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\cdots,$$

in which each coefficient indicates the number of different ways in which the exponent can be expressed as the sum of different numbers. For example, the number 9 can be expressed in the following eight ways as the sum of different numbers.

$$\begin{array}{rlrl} 9=9 & 9=6+2+1 & 9=8+1 & 9=5+4 \\ 9=7+2 & 9=5+3+1 & 9=6+3 & 9=4+3+2 \end{array}$$

326. In order that we may compare these two forms, we let

$$P=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\cdots$$

$$Q=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots$$

$$PQ=(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\cdots.$$

Since all of the factors of PQ are contained in P, when P is divided by PQ we obtain $\frac{1}{Q}=(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\cdots$. It follows that $Q=\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\cdots}$. If this rational function is developed in an infinite series, the coefficient of each term will indicate the number of different ways in which the exponent can be expressed as the sum of odd numbers. Since this series is the one which we considered in the preceding section, we have the following theorem.

The number of different ways a given number can be expressed as the sum of different whole numbers is the same as the number of ways in which that same number can be expressed as the sum of odd numbers, whether the same or different.

Just prior to this paragraph Euler listed the eight partitions of 9 into distinct parts: 9, $8+1$, $7+2$, $6+3$, $6+2+1$, $5+4$, $5+3+1$, $4+3+2$. The theorem just stated tells us that there are also exactly eight partitions of 9 into odd summands. Indeed, this is the case: 9, $7+1+1$, $5+3+1$, $5+1+1+1+1$, $3+3+3$, $3+3+1+1+1$, $3+1+1+1+1+1+1$, $1+1+1+1+1+1+1+1+1$.

Results of this nature did not garner much attention until the early part of the twentieth century when the Rogers-Ramanujan identities became famous. These two identities were originally discovered in 1894 by L. J. Rogers [28] and became famous in the romantic saga surrounding Ramanujan. The whole amazing story is recounted by G. H. Hardy [21].

First we state the Rogers-Ramanujan identities as series-product identities:

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{3.1}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{3.2}$$

where $(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$.

While neither Rogers nor Ramanujan commented on the relationship of these identities to partitions, P. A. MacMahon [26, Sec. VII, Ch. III] and I. Schur [31] noted that these results are equivalent to the following theorems on partitions.

Theorem 1. *The number of partitions of n into summands wherein the difference between any two parts is at least 2 equals the number of partitions of n into parts congruent to* $\pm 1 \pmod 5$.

Theorem 2. *The number of partitions of n into summands each larger than* 1 *wherein the difference between any two parts is at least 2 equals the number of partitions of n into parts congruent to* $\pm 2 \pmod 5$.

In the last few decades results of this nature have burgeoned in the literature. Several surveys have appeared (cf. Alder [1], Alladi [2], Andrews [4], [5, Ch. 7], and it is clear that the subject is far from exhausted.

4 The Pentagonal Number Theorem

Euler introduces his famous result in paragraph 323:

323. The columns of the table all have the same beginnings and continue to have some terms in common. From this we understand that if there were an infinite number of columns, the series would agree completely. These series arise from the function $\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\cdots}$. Because the series is recurrent, first we consider the denominator, since from it we obtain the scale of the relation. If the factors of the denominator are multiplied one by one, we obtain

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \cdots .$$

If we consider this sequence with some attention we will note that the only exponents which appear are of the form $\frac{3n^2 \pm n}{2}$ and that the sign of the corresponding term is negative when n is odd, and the sign is positive when n is even.

Euler eventually found a rigorous proof of this theorem. It is really the first result discovered in the vast and important theory of elliptic theta functions. Jacobi was the one who developed the general theory [25]. Indeed, Jacobi's triple product identity [5, p. 21, eq. (2.2.10)] [24, p. 282, Th. 352]

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}) \tag{4.1}$$

reduces to the pentagonal number theorem under the specialization $q \to q^{3/2}$, $z \to -q^{1/2}$.

Recently, K. Ono [27] has made this branch of the theory of partitions one of the most active areas of number theory today.

It is noteworthy that the functional equation (2.2) was precisely the result proved by A. Cayley [14] in order to show that the series

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(3n+1)/2}(1 + zq^{2n+1})(-zq; q)_n}{(q; q)_n}$$

is, in fact, equal to the infinite product

$$\prod_{n=1}^{\infty} (1 + zq^n)$$

If in these expressions we set $z = -1$, we obtain the pentagonal number theorem. Thus more than a century after Euler wrote De Partitio Numerorum, Cayley [14] found a proof of the pentagonal number theorem that has its genesis in Euler's approach.

It should be noted that Cayley's proof arose from a challenge posed by Sylvester [33] who had unearthed this generalization of the pentagonal number theorem by purely combinatorial arguments.

Finally Cayley's lovely proof forms the basis for the Rogers and Ramanujan proof [30] of the Rogers-Ramanujan identities and for much of the subsequent cornucopia of such results (cf. [4]).

5 Partitions With Some Negative Parts

We now come to the end of Chapter 16, De Partitio Numerorum. Euler's last two results are generating function versions of ancient theorems.

Theorem 3. *Every integer is uniquely a sum of distinct powers of* 2.

Theorem 4. *Every integer is uniquely a sum of distinct powers of three and their negatives.*

L. E. Dickson [15, p. 105] notes that these results are not new with Euler, and he cites earlier references in Leonardo Pisano and Michael Stifel.

However, in the spirit of **Introductio in analysin infinitorum**, Euler presents these assertions as generating function identities:

$$\prod_{n=0}^{\infty}(1+q^{2^n}) = \sum_{n=0}^{\infty} q^n, \tag{5.1}$$

and

$$\prod_{n=0}^{\infty}(q^{-3^n}+1+q^{3^n}) = \sum_{n=-\infty}^{\infty} q^n. \tag{5.2}$$

We have replaced Euler's x with q in light of literature cited in the rest of this section.

Now, of course, there is no problem with (5.1). Indeed, one can easily prove by mathematical induction that

$$\prod_{n=0}^{N-1}(1+q^{2^n}) = \sum_{n=0}^{2^N-1} q^n, \tag{5.3}$$

and then assuming $|q| < 1$, we deduce (5.1) as a limiting case of (5.3).

Equation (5.2) is more problematic. In analogy with (5.3), we may easily prove that

$$\prod_{n=0}^{N-1} (q^{-3^n} + 1 + q^{3^n}) = \sum_{j=-(3^N-1)/2}^{(3^N-1)/2} q^j \tag{5.4}$$

Now comes the tricky part. There is no complex value of q that will make (5.4) converge. Hence if we are to consider (5.2) as a limit of (5.4), it can only be in the sense of formal Laurent series.

This somewhat awkward interpretation of (5.2) is often what happens when the possibility of partitions with negative parts arises. For example, the following formula of Jacobi [5, p. 21, eq. (2.2.9)] is often attributed to Euler [23, p. 78, eq. (1.31)], [5, p. 21, just following (2.2.9)].

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n. \tag{5.5}$$

Now suppose we decide that in the series on the left (whose n-th term is the generating function for partitions with Durfee square of side n) we will replace the partitions generated by partitions in which all the parts are negative. I.e. formally we replace q by $1/q$. This can only be done formally because the series in question converges only for $|q| < 1$. The result of our replacement is

$$1+\sum_{n=1}^{\infty} \frac{q^{-n^2}}{(1-\frac{1}{q})^2(1-\frac{1}{q^2})^2 \cdots (1-\frac{1}{q^n})^2} = 1+\sum_{n=1}^{\infty} \frac{q^n}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}.$$

To add to the mystery [5, p. 19, Cor. 2.3, $c = t = q$, a and $b \to 0$]

$$1 + \sum_{n=1}^{\infty} \frac{q^n}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}, \tag{5.6}$$

and this series turns out to be the generating function for a new class of partitions first explored by Auluck [13].

The replacement of q by $1/q$ has been a useful technique in certain problems where the theory of partitions has been applied to statistical mechanics [6], [11], [12]. In these instances, the $q \to 1/q$ transformation was applied to polynomial generating functions where convergence questions were not initially an issue.

Now to be fair, Euler did not pull anything as unreasonable as our switch from (5.5) to (5.6). Indeed, (5.2) makes perfectly good sense formally. In terms of partitions, Euler is considering partitions into distinct powers of 3 where each part may appear with either a "+" or "−".

We shall call partitions in which parts may appear with + or − "signed partitions".

Theorem 5. *The number of ordinary partitions of $n, p(n)$, equals the number of signed partitions of n into distinct parts where each part larger than n is positive, each part not exceeding n is negative and there are equally many positive and negative parts.*

Remark. Let us consider the case $n = 6$ where $p(6) = 11$. The signed partitions in question are $7-1$, $8-2$, $9-3$, $10-4$, $11-5$, $12-6$, $8+7-5-4$, $8+7-6-3$, $9+7-6-4$, $9+8-6-5$, and $10+7-6-5$.

Actually Theorem 5 is an immediate corollary (with $N = n$) of the following more general result.

Theorem 6. *We denote by $p_N(n)$ the number of ordinary partitions of n with each part at most N. Then $p_N(n)$ equals the number of signed partitions of n into distinct parts where all parts $> N$ are positive, all parts $\leqq N$ are negative, and there are equally many positive and negative parts.*

Proof. As we easily deduce from (2.6)

$$\frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

is the generating function for partitions with exactly n parts. This current theorem also requires the use of the Gaussian polynomials:

$$\begin{bmatrix} N \\ m \end{bmatrix}_q = \frac{(q;q)_N}{(q;q)_m (q;q)_{N-m}},$$

where

$$(A;q)_m = \prod_{j=0}^{m-1} (1 - Aq^j).$$

The main fact required about the Gaussian polynomials is that

$$q^{j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix}_q$$

is the generating function for partitions with exactly j distinct parts each $\leqq N$ (cf. [26, p. 10]).

Finally we require the following limiting case of the q-Chu-Vandermonde summation [5, p. 37, eq. (3.3.10) with $n = h = N$, $k \to N - k$ and $m \to \infty$]

$$\begin{aligned} \frac{1}{(q;q)_N} &= \sum_{j=0}^{N} \frac{q^{j^2}}{(q;q)_j} \begin{bmatrix} N \\ j \end{bmatrix}_q \\ &= \sum_{j=0}^{N} \frac{q^{j(j+1)/2+Nj}}{(q;q)_j} q^{-j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix}_{q^{-1}} . \end{aligned}$$

We now observe that in this last expression first

$$\frac{q^{j(j+1)/2+Nj}}{(q;q)_j}$$

generates j positive distinct summands each $> N$ while

$$q^{-j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix}_{q^{-1}}$$

generates j negative distinct summands in the interval $[-N, -1]$. Our result follows immediately once we remark that $1/(q;q)_N$ is the generating function for $p_N(n)$. □

Readers familiar with the use of Frobenius symbols in the representation of partitions (cf. [9]) will have no difficulty recognizing that the above signed representation of the partitions of n may be obtained from the Frobenius symbols by adding $n+1$ to each entry in the top row and subtracting n from each entry in the bottom row.

This is just the beginning. Signed partitions fit naturally as interpretations of many classical identities.

Theorem 7. *Let $\mathcal{E}(m,n)$ denote the number of ordinary partitions of n with exactly m even parts none of which is repeated. Let $\mathcal{S}(m,n)$ denote the number of signed partitions of n in which each positive part is larger than the* *number of positive parts, the negative parts are all distinct each not exceeding the number of positive parts and m designates the excess of the number of positive parts over the number of negative parts.*

Remark. As an example, we consider the case $m = 2$, $n = 11$. The seven partitions enumerated by $\mathcal{E}(2,11)$ are 8+2+1, 6+4+1, 6+3+2, 6+2+1+1+1, $5+4+2$, $4+3+2+1+1$, and $4+2+1+1+1+1+1$. The seven signed partitions enumerated by $\mathcal{S}(2,11)$ are $8+3$, $7+4$, $6+5$, $4+4+4-1$, $5+4+4-2$, $5+5+4-3$ and $6+4+4-3$.

Proof. The techniques of the theory of partitions [5, Ch. 1] reveal directly that

$$\sum_{m,n\geqq 0} \mathcal{E}(m,n) z^m q^n = \frac{(-zq^2;q^3)_\infty}{(q;q^2)_\infty}\,, \tag{5.7}$$

and

$$\sum_{m,n\geqq 0} \mathcal{S}(m,n) z^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n} z^n (1+\frac{1}{zq})(1+\frac{1}{zq^2})\cdots(1+\frac{1}{zq^n})}{(q;q)_n}\,. \tag{5.8}$$

The identity of these two generating functions and consequently the completion of our proof hinges on the fact [5, p. 21, Cor. 2.7] that

$$\sum_{n=0}^{\infty} \frac{(-zq)_n q^{n(n+1)/2}}{(q;q)_n} = \frac{(-zq^2;q^2)_\infty}{(q;q^2)_\infty}\,. \tag{5.9}$$

The left-hand side of (5.9) is term by term equal to the series in (5.8) and the right-hand side of (5.9) is the product in (5.7). $\square$

With two examples of signed partition identities in hand, we see that in the latter case special attention is given to the number of positive parts. In the following, the number of positive parts in a signed partition is called the + **number**.

Theorem 8. *Let $\mathcal{G}_1(n)$ denote the number of signed partitions of n in which each positive part is even and $\geqq$ twice the + number, the negative parts are distinct and odd with each smaller than twice the + number. Let $\mathcal{G}_2(n)$ denote the number of ordinary partitions of n in which each part is $\equiv 1, 4$ or $7 \pmod 8$. Then $\mathcal{G}_1(n) = \mathcal{G}_2(n)$.*

Remark. In the case $n = 8$, $\mathcal{G}_1(8)$ enumerates 8, $4+4$, $6+6-1-3$ and $8+4-1-3$ while $\mathcal{G}_2(8)$ enumerates 7+1, $4+4$, $4+1+1+1+1$, and $1+1+1+1+1+1+1+1$.

Proof. This result is merely a variation on the first Göllnitz-Gordon identity [19], [20], [5, Sec. 7.4]

$$1 + \sum_{n=1}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{\substack{n=1 \\ n \equiv 1,4 \text{ or } 7 \pmod 8}}^{\infty} \frac{1}{1-q^n}. \tag{5.10}$$

Clearly the right-hand side of (5.10) is the generating function for $\mathcal{G}_2(n)$, and we can adjust the left-hand side as follows:

$$\begin{aligned}
& 1 + \sum_{n=1}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} \\
= \quad & 1 + \sum_{n=1}^{\infty} \frac{q^{\overbrace{2n+2n+\cdots+2n}^{n\text{times}}}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} (1+\frac{1}{q})(1+\frac{1}{q^3})\cdots(1+\frac{1}{q^{2n-1}}) \\
= \quad & \sum_{n \geqq 0} \mathcal{G}_1(n) q^n.
\end{aligned}$$

□

While the three preceding results all have natural interpretations purely in terms of ordinary partitions, (cf. [5, Chs. 2, 7]) we shall close this section with two results (among many possible choices) that seem to be most natural as identities involving signed partitions.

Theorem 9. *Let $S_1(n)$ denote the number of signed partitions of n with + number odd, each positive parts $\geqq$ (+ number $-1)/2$, with distinct negative parts each $<$ + number. Let $S_2(n)$ denote the number of ordinary partitions of n into parts not divisible by 4. Then $S_1(n) = S_2(n)$.*

Remark. In the case $n = 5$, $S_1(5)$ enumerates the signed partitions 5, 2+2+1, $2+2+2-1$, $3+1+1$, $3+2+1-1$, $4+1+1-1$ while $S_2(5)$ enumerates 5, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$.

Proof. Here we rely on an identity of L. J. Slater [32, p. 157, eqs. (11), (51), or (64)].

$$1+\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{n(n+1)}}{(q;q)_{2n+1}}=\prod_{\substack{n=1\\4\nmid n}}^{\infty}\frac{1}{1-q^n}. \tag{5.11}$$

We note in passing that this result may also be viewed as a limiting case of the q-analog of Gauss's theorem [5, p. 20, Cor. 2.4].

Clearly the right-hand side of (5.11) is the generating function for $S_2(n)$. Concerning the left-hand side, we see that

$$\begin{aligned}
&1+\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{n(n+1)}}{(q;q)_{2n+1}}\\
=\ &1+\sum_{n=1}^{\infty}\frac{q^{\overbrace{n+n+\cdots+n}^{2n+1\text{ times}}}}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}(1+\frac{1}{q})(1+\frac{1}{q^3})\cdots(1+\frac{1}{q^{2n-1}})\\
=\ &\sum_{n\geqq 0}S_1(n)q^n.
\end{aligned}$$

□

Finally we obtain a result that may be thought of as a cousin of the Rogers-Ramanujan identities, (3.1) and (3.2).

Theorem 10. *Let $R_1(n)$ denote the number of signed partitions of n with + number odd, each positive part $\geqq$ (+ number $-1)/2$ with distinct negative parts each $\leqq$ (+ number $-1)/2$. Let $R_2(n)$ denote the number of ordinary partitions of n into parts that are either odd or $\equiv \pm 2 \pmod{10}$. Then $R_1(n)=R_2(n)$.*

Remark. In the case $n=6$, $R_1(6)$ enumerates 6, $2+2+2$, $3+2+1$, $4+1+1$, $3+2+2-1$, $3+3+1-1$, $4+2+1-1$, $5+1+1-1$, while $R_2(6)$ enumerates $5+1$, $3+3$, $3+2+1$, $3+1+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$, and $1+1+1+1+1+1$.

Proof. We now must use a result of L. J. Rogers[29, p. 330(2), line 1](cf. L. J. Slater [32, p. 158, eq. (62)]) which relies on the Rogers-Ramanujan function $G(q)$ given in (3.1). Namely

$$\sum_{n\geqq 0}\frac{(-q;q)_n q^{n(3n+1)/2}}{(q;q)_{2n+1}}=\frac{G(q^2)}{(q;q^2)_\infty}. \tag{5.12}$$

Once we recognize that the right-hand side of (5.12) may be written as

$$\prod_{\substack{n=1 \\ n \text{ either} \\ \text{odd or } \equiv \pm 2 \pmod{10}}}^{\infty} \frac{1}{1-q^n},$$

we see that the right-hand side of (5.12) is the generating function for $R_2(n)$. Next we rewrite the left-hand side of (5.12)

$$\begin{aligned}
&\sum_{n \geqq 0} \frac{(-q;q)_n q^{n(3n+1)/2}}{(q;q)_{2n+1}} \\
= &\sum_{n \geqq 0} \frac{q^{\overbrace{n+n+\cdots+n}^{2n+1 \text{ times}}}}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} (1+\frac{1}{q})(1+\frac{1}{q^2})\cdots(1+\frac{1}{q^n}) \\
= &\sum_{n \geqq 0} R_1(n) q^n.
\end{aligned}$$

□

6 Conclusion

Any commentary on an aspect of Euler's work seems somewhat pretentious. In this case, we have restricted ourselves to one chapter in one book, one small star in the Eulerian galaxy. Yet even in this rather narrow study we gain some idea both of the subsequent impact of Euler's work, and also realize that Euler's free spirited elegance can inspire new ways of looking at old problems.

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