8. Commutative rings. Class $\mathbf{R1}(n)$ of rings.

Here we study subgroups of $\mathrm{GL}_n A$ which are normalized by $\mathrm{E}_n A$ in the case when $n \geq 3$ and A is commutative.

In the next lemma (due to Vaserstein [V34]), e_i's stand for standard basis of A^n, i.e., they are the columns of 1_n. The symbol ^{T} stands for transposition.

Lemma 8.1. Let A be commutative, $n \geq 1$, $v = (v_i) \in \mathrm{Um}_n A$, Then the left A-module

$$P = \{w \in (A^n)^T : wv = 0\}$$

is generated by the rows $w_{i,j} = (e_i v_j - e_j v_i)^T$.

Proof. Since v is unimodular, there is an n-row $u = (u_i)$ such that $uv = 1$. Then

$$A^n = vuA^n \oplus (1_n - vu)A^n$$

and

$$(A^n)^T = (A^n)^T vu \oplus (A^n)^T (1_n - vu).$$

Clearly, $P = (A^n)^T(1_n - vu)$, i.e., P is the row space of $1_n - vu$. So we have to prove that the rows of $1_n - vu$ are spanned by $w_{i,j}$.

For the i-th row we have $e_i^T(1_n - vu) = \sum_{j=1}^n u_j w_{i,j}$. QED.

Note that $w_{i,i} = 0$ and that the lemma is trivial for $n = 1$. The condition that v is unimodular cannot be dropped as the following example shows. Let $a, b \in A, ab = 0$, but both $a, b \neq 0$. For any $n \geq 1$ we set $v = e_1 b$ and $u = ae_1^T$. Then $uv = 0$ but u is not a linear combination of $w_{i,j}$.

Suslin proved the version of the lemma with the condition that v is unimodular is replaced by the condition that the sequence $b_1, \ldots, b_n$ is regular in the sense that b_k is not a zero divisor in the ring $A/\sum_{i=1}^{k-1} Ab_i$ for $k = 1, \ldots, n$. The proof is an easy induction on n. We do not use here this version. Suslin [Su1] also proved the following corollary in the case $A_0 = A$.

Corollary 8.2 Let A be an associative ring with 1 and A_0 a commutative subring. Then for $n \geq 3$, the group $\mathrm{E}_n A$ is normalized by $\mathrm{GL}_n A_0$.

Proof. Adding the identity 1 of A to the subring A_0, we can assume that $1 \in A_0$. We have to prove that $\beta = \alpha a^{i,j} \alpha^{-1} \in \mathrm{E}_n A$ for every elementary matrix $a^{i,j}$ in $\mathrm{GL}_n A$ and every matrix $\alpha \in \mathrm{GL}_n A_0$.

The matrix β has the form $1_n + vau$ where $v = \alpha e_i$, and $u = e_j^T \alpha^{-1}$. Note that $uv = 0$.

By Lemma 8.1, $u = \sum a_{i,j} w_{i,j}$ with $a_{i,j} \in A_0$.

Then $\beta = \prod(1_n + vaa_{i,j} w_{i,j})$ (factors commute), so it suffices to show that each factor $1_n + vaa_{i,j} w_{i,j}$ belongs to $\mathrm{E}_n A$. But this is a particular case (take $B = A; k \neq i, j$) of the following corollary of Lemma 3.5.

Proposition 8.3. Let A be an associative ring with unity, B an ideal of A, $n \geq 2$, $v = (v_i) \in A^n$, $u = (u_i) \in (B^n)^T$, and $uv = 0$. Assume that $1 + u_k v_k \in \mathrm{GL}_1 A$ for some index k. Then $1_n + vu \in \mathrm{E}_n(A, B)$.

Proof. Since all permutation matrices normalize $\mathrm{E}_n(A,B)$, we can assume that $k=n$. We write $u=(u',u_n)$ and $v=\begin{bmatrix} v' \\ v_n \end{bmatrix}$ with $u_n \in B$, $v_n \in A$. Set $d = 1+v_nu_n \in \mathrm{GL}_1 B$ and (see Lemma 3.5) $d' = 1+u_nv_n = 1-u'v' \in \mathrm{GL}_1 B$. Then

$$1_n + vu = \begin{pmatrix} 1_{n-1}+v'u' & v'u_n \\ v_nu' & d \end{pmatrix} = \begin{pmatrix} 1_{n-1} & v'u_nd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d' \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ d^{-1}v_nu' & 1 \end{pmatrix}$$

$$\in \mathrm{E}_n B \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mathrm{E}_n B$$

where

$$a = 1_{n-1} + v'u' - v'u_nd^{-1}v_nu' = 1_{n-1} + v'(1-v_nd^{-1}v_n)u' = 1_{n-1} + v'd^{-1}u'.$$

We have to prove that $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathrm{E}_n(A,B)$. Since $1+u'v'd'^{-1} = d'^{-1}$, we have $\begin{pmatrix} a & 0 \\ 0 & d' \end{pmatrix} \in \mathrm{E}_n(A,B)$ by Lemma 3.5. Also by Lemma 3.5, $\begin{pmatrix} 1_{n-1} & 0 \\ 0 & d^{-1}d' \end{pmatrix} \in \mathrm{E}_n(A,B)$.

So $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d' \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ 0 & d^{-1}d' \end{pmatrix}^{-1} \in \mathrm{E}_n(A,B)$. QED.

It is unknown whether the conclusion of Proposition 8.3 with $n \geq 3$ holds for commutative rings A under the assumption $uv=0$ only, without additional conditions such as : $1+u_kv_k \in \mathrm{GL}_1 A$; v is unimodular; $\mathrm{sr}(B) \leq n-1$.

Proposition 8.4 (Bass). Let A be an associative ring with unity, B an ideal of A. Assume that $\mathrm{E}_n A$ is normalized by $\mathrm{G}_n(A,B)$ and that $[\mathrm{E}_n A, \mathrm{GL}_n B] \subset \mathrm{E}_n(A,B)$. Then $[[\mathrm{E}_n A, \mathrm{E}_n A], \mathrm{G}_n(A,B)] \subset \mathrm{E}_n(A,B)$.

Proof. By our definition, $\mathrm{E}_n(A,B)$ is normal in $\mathrm{E}_n A$. By our first condition, $\mathrm{E}_n(A,B)$ is normal in $\mathrm{GL}_n B$. By the same condition, $\mathrm{E}_n A$ and $\mathrm{GL}_n B$ commute modulo $\mathrm{E}_n(A,B)$.

Let $\beta \in \mathrm{GL}_n(A,B)$. For any $\alpha \in \mathrm{E}_n A$ define

$$f(\alpha) = [\alpha^{-1}, \beta]\mathrm{E}_n(A,B) \in (\mathrm{GL}_n B \cap \mathrm{E}_n A)/\mathrm{E}_n(A,B).$$

This gives a homomorphism

$$f : \mathrm{E}_n A \to (\mathrm{GL}_n B \cap \mathrm{E}_n A)/\mathrm{E}_n(A,B)$$

into a commutative group, hence $f[\mathrm{E}_n A, \mathrm{E}_n A] = 1$. QED

Remark. In Corollary 4.18 we did not have the condition that $\mathrm{G}_n(A,B)$ normalizes $\mathrm{E}_n A$, but the group $\mathrm{GL}_n B/\mathrm{E}_n(A,B)$ was commutative.

Corollary 8.5. Let A be an associative ring with 1 and $n \geq 3$. Then:

(*a*)
$$[\mathrm{E}_n A, \mathrm{G}_n(A,B)] = \mathrm{E}_n(A,B)$$

for every commutative (as a ring) ideal B of A;

(b) if A is commutative,

$$\mathrm{E}_n(A,B) = [\mathrm{GL}_n A, \mathrm{E}_n B]$$

for every ideal B of A.

Proof. (a) By Lemma 1.3, Corollary 8.2, and Proposition 8.4, it suffices to show that

$$[\mathrm{E}_n A, \mathrm{GL}_n B] \subset \mathrm{E}_n(A,B).$$

Let $\alpha \in \mathrm{E}_n A$ and $\beta \in \mathrm{GL}_n B$. Let $A' = \{(a,a') \in A \times A : a - a' \in B\}$. We set $\alpha' = (\alpha,\alpha) \in \mathrm{E}_n A'$ and $\beta' = (\beta, 1_n) \in \mathrm{GL}_n B'$ where $B' = (B,0)$.

By Corollary 8.2, $[\alpha',\beta'] \in \mathrm{E}_n A'$. Clearly $[\alpha',\beta'] \in \mathrm{GL}_n B'$. Since A' is a semidirect product of its subring $\{(a,a) : a \in A\}$ and its ideal $B' = (B,0)$, with A'/B' isomorphic to the subring, $\mathrm{E}_n A' \cap \mathrm{GL}_n B' = \mathrm{E}_n(A',B')$.

Thus, $[\alpha',\beta'] = [[\alpha,\beta],1_n] \in \mathrm{E}_n(A',B')$, hence $[\alpha,\beta] \in \mathrm{E}_n(A,B)$.

(b) The equality means that $\mathrm{E}_n(A,B)$ is normal in $\mathrm{GL}_n A$. To prove this, consider any $\alpha \in \mathrm{GL}_n A$ and $\beta \in \mathrm{E}_n B$. Let $A', B', \alpha' = (\alpha,\alpha) \in \mathrm{GL}_n A'$ and $\beta' = (\beta, 1_n) \in \mathrm{GL}_n B'$ be as in the proof of (a) above. By Corollary 8.2 with $A = A_0 = A'$, $[\alpha',\beta'] \in \mathrm{E}_n A'$. Clearly $[\alpha',\beta'] \in \mathrm{GL}_n B'$. Since A' is a semidirect product of its subring $\{(a,a) : a \in A\}$ and its ideal $B' = (B,0)$, with A'/B' isomorphic to the subring, $\mathrm{E}_n A' \cap \mathrm{GL}_n B' = \mathrm{E}_n(A',B')$. Therefore

$$[\alpha',\beta'] = [[\alpha,\beta],1_n] \in \mathrm{E}_n A' \cap \mathrm{GL}_n B' = \mathrm{E}_n(A',B'),$$

hence $[\alpha,\beta] \in \mathrm{E}_n(A,B)$. QED

Proposition 8.6. Let A be an associative ring with unity, $n \geq 3$, H a subgroup of $\mathrm{GL}_n A$ which is normalized by $\mathrm{E}_n A$. Suppose that $u\alpha e_1 = 0$ for the first column αe_1 of a matrix $\alpha \in H$ and an n-row $u = (u_j)$ such that $u_1 \neq 0$ and $u_n = 0$. Then H contains non-trivial elementary matrix.

Proof. We set

$$u' = (u_1, \ldots, u_{n-1}), \beta = \begin{pmatrix} 1_{n-1} & 0 \\ u' & 1 \end{pmatrix} \in \mathrm{E}_n A, \gamma = [\beta^{-1}, \alpha^{-1}] = \beta^{-1}(1_n + vw) \in H$$

where $v = \alpha e_1$ is the first column of α^{-1} and $w = (0, w') = u\alpha$.

Note that the first column $\gamma e_1 = \beta e_1$, looks like $\begin{pmatrix} 1 \\ 0 \\ -u_1 \end{pmatrix}$. Therefore, γ is not central. So we are done by Proposition 1.10. QED

Remark. For commutative A, the proposition was proved by Willson ($n \geq 4$) and Golubchik ($n \geq 3$).

Theorem 8.7. Let A be an associative ring with 1, B an ideal of A, and $n \geq 3$.‘ Let H be a subgroup of $\mathrm{GL}_n B$. Assume that B is commutative.

Then the following two conditions are equivalent:

(a) H is normalized by $\mathrm{E}_n A$;

(b) $\mathrm{E}_n(A, B_0) \subset H \subset \mathrm{G}_n(A, B_0)$ for an ideal $B_0 \subset B$ of A.

Proof. (a) $\Rightarrow$ (b). Let H is normalized by $\mathrm{E}_n A$. Set $B_0 = \{b \in A : b^{1,2} \in H\}$. Since $H \subset \mathrm{GL}_n B, B_0 \subset B$. By (1.5), (1.11), B_0 is an ideal of A and $\mathrm{E}_n(A, B_0) \subset H$. We have to prove that $H \subset \mathrm{GL}_n(A, B_0)$.

Otherwise, the image H' of H in $\mathrm{GL}_n(A, B)$ is not central. We claim that H' contains $(b')^{1,2}$ with $0 \neq b' \in A/B$. To prove this we pick a non-central $\alpha' = (\alpha'_{i,j}) \in H'$.

If $\alpha'_{2,1} \neq 0$ we can use Proposition 8.5, with $u_1 = \alpha'_{2,1} \neq 0$, $u_2 = -\alpha_{11}, u_i = 0$ for $i \geq 3$. If $\alpha'_{2,1} = 0$ we can use Proposition 1.10.

So $H' \ni (a')^{1,2}$ with $0 \neq a' \in A/B$. In other words, $H \ni a^{1,2}\beta$ where $a \in A \setminus B$ and $\beta \in \mathrm{GL}_n B$. Now $H \ni [\alpha^{1,2}\beta, 1^{2,3}] = a^{1,3}\gamma$ with $\gamma \in [\mathrm{E}_n A, \mathrm{GL}_n B] = \mathrm{E}_n(A, B)$, hence $H \ni a^{1,3}$ which contradicts the assumption $a \in A \setminus B$.

(b) $\Rightarrow$ (a). This follows from Corollary 8.5. QED

The methods of this section were generalized to rings A which are finite as modules over their centers by Suslin [Su1]. We will prove the conclusion of Theorem 8.7 with $n \geq 3$ for these A using central localizations in Section 11. Golubchik proved the implication (a) $\Rightarrow$ (b) for PI-rings using non-central localizations. The implication (b)$\Rightarrow$ (a) for PI-rings is still an open problem. Recall that a PI-ring (polynomial identity ring) is an associative ring A with 1 such that there is a polynomial with integer coefficients in non-commuting indeterminates without constant term with one of coefficients equal 1 which vanishes on A. Examples include all rings which are finite as modules over their centers.

The ideal B in Theorem 8.7(b) is obviously unique. Namely, B consists of all $b \in A$ such that $b^{1.2} \in H$. Also B consists of the off-diagonal entries of all matrices in H.

Here is an easy corollary of Proposition 8.6.

Corollary 8.8. Let A be an associative ring with 1, $n \geq 3$, H a subgroup of $\mathrm{GL}_n A$ which is normalized by $\mathrm{E}_n A$, $\alpha = (\alpha_{i,j}) \in H$. Assume that α is not central and that one of diagonal entries $\alpha_{i,i}$ of α is zero or a zero divisor, i.e., either $\alpha_{i,i} z = 0$ or $z\alpha_{i,i} = 0$ for a nonzero $z \in A$. Then H contains non-trivial elementary matrix.

Proof. We assume that $z\alpha_{i,i} = 0$, the case $\alpha_{i,i} z = 0$ being similar. Without loss of generality, we can assume that $i = 1$. Now our conclusion follows from Proposition 8.6 by setting $u_1 = z$ and $u_i = 0$ for $i \geq 2$. QED.

Remark. Another way to state the standard classification of the subgroups H of $\mathrm{GL}_n A$ which are normalized by $\mathrm{E}_n A$ uses levels. Namely, for any ring A with 1 and any subgroup H of $\mathrm{GL}_n A$, it is clear that there is the largest ideal B of A such that $\mathrm{E}_n(A, B) \subset H$. We can call this ideal *the lower level* of H.

It is also clear that there is the least ideal B' of A such that $H \subset \mathrm{G}_n(A, B')$. We can call this ideal *the upper level* of H. The upper level always contains the lower level.

Theorem 8.7 asserts that if A is commutative and $n \geq 3$ then H is normalized by $\mathrm{E}_n A$ if and only if its lower level equals the upper level. Theorem 4.22 says the same when $n \geq \max(3, \mathrm{sr}(A) + 1)$. Noe we will try to extend the class of rings with the standard classification of the subgroups H of $\mathrm{GL}_n A$ which are normalized by $\mathrm{E}_n A$.

Classes R1(n) of rings

Note that the conclusion of Theorem 8.7 holds if we replace the condition that A is commutative by the condition that $\mathrm{sr}(A) \leq n - 1$, see Corollary 4.8 and Theorem 4.12.

Having in mind the unification of these two cases, i.e., an extension of these conclusions to a bigger class of rings (see preface), we will introduce classes $\mathrm{R1}(n)$ of rings, $n \geq 2$.

For any $n \geq 2$, let $\mathrm{R1}(n)$ be the class of associative rings B satisfying the following condition:

(8.9) for any associative ring A with 1 containing B as an ideal and any ideal $B_0 \subset B$ of A,

$$[\mathrm{E}_n A, \mathrm{GL}_n B_0] \subset \mathrm{E}_n(A, B_0).$$

By Corollary 4.7, $\mathrm{R1}(n)$ contains any ring B such that $\mathrm{sr}(B) \leq n-1$. By Corollary 8.5, $\mathrm{R1}(n)$ contains any commutative ring B provided that $n \geq 3$.

For any ring B with identity 1_B the condition (8.9) is reduced to the following simpler condition:

(8.10) for any ideal B_0 of B,

$$[\mathrm{E}_n B, \mathrm{GL}_n B_0] \subset \mathrm{E}_n(B, B_0).$$

This is because, for any associative ring A with 1 containing B as an ideal, B_0 is an ideal of A and $A = B \times A(1-1_B)$ so

$$[\mathrm{E}_n A, \mathrm{GL}_n B_0] = [\mathrm{E}_n B, \mathrm{GL}_n B_0]$$

and

$$\mathrm{E}_n(A, B_0) = \mathrm{E}_n(B, B_0).$$

Proposition 8.11. Let $n \geq 2$, and let B be an associative ring with identity 1_B. Assume that B belongs to $\mathrm{R1}(n)$. Then for any associative ring A with 1 containing B as an ideal and any ideal B_0 of B,

$$[\mathrm{E}_n A, \mathrm{G}_n(A, B_0)] \subset \mathrm{E}_n(A, B_0).$$

Therefore every subgroup H of $\mathrm{G}_n(A, B_0)$ containing $\mathrm{E}_n(A, B_0)$ is normalized by $\mathrm{E}_n A$.

Proof. We have a direct product decomposition $A = B \times A(1-1_B)$ of rings, hence B_0 is an ideal of A and the group $\mathrm{G}_n(A, B_0)$ decomposes into the direct product $\mathrm{GL}_n(B_0 \times C$ where C is the group of invertible scalar matrices over the center of the ring $A(1-1_B)$. So $[\mathrm{E}_n A, \mathrm{G}_n(A, B_0)] = [\mathrm{E}_n A, \mathrm{GL}_n B_0] \subset \mathrm{E}_n(A, B_0)$. QED.

Proposition 8.12. Let $n \geq 2$, and let B be an associative ring. Assume that B belongs to $\mathrm{R1}(n)$ Then for any associative ring A with 1 containing B as an ideal any ideal $B_0 \subset B$ of A such that $\mathrm{E}_n(A, B_0)$ is normalized by $\mathrm{G}_n(A, B)$ we have

$$[[\mathrm{E}_n A, \mathrm{E}_n A], \mathrm{G}_n(A, B_0)] \subset \mathrm{E}_n(A, B_0).$$

So every subgroup H of $\mathrm{G}_n(A, B_0)$ containing $\mathrm{E}_n(A, B_0)$ is normalized by $[\mathrm{E}_n A, \mathrm{E}_n A]$.

Proof. Let $\beta \in \mathrm{G}_n(A, B_0)$. For every $\alpha \in \mathrm{E}_n A$, $[\alpha, \beta] \in \mathrm{GL}_n B_0$. Set $f(\alpha)[\alpha, \beta]$ $\mathrm{E}_n(A, B_0) \in \mathrm{GL}_n B_0 / \mathrm{E}_n(A, B_0)$.

Since $[\mathrm{E}_n A, \mathrm{GL}_n B_0] \subset \mathrm{E}_n(A, B_0)$, $f : \mathrm{E}_n A \to \mathrm{GL}_n B_0 / \mathrm{E}_n(A, B_0)$ is a group morphism. Note that $[\alpha, \beta] \in H$, where $H = (\beta \mathrm{E}_n A \beta^{-1}) \mathrm{E}_n A$ and that $(\beta E_n(A, B_0)\beta^{-1} = E_n(A, B_0)$.

So the image of f belongs to a commutative subgroup $(H \cap \mathrm{GL}_n B_0)/\mathrm{E}_n(A, B_0)$. Thus, $f([\mathrm{E}_n A, \mathrm{E}_n A])$ is trivial. QED.

Proposition 8.13. Let A be an associative ring with 1, B' an ideal of A. Assume that $\mathrm{E}_n A$ is perfect, e.g., $n \geq 3$. Assume also that both B' and A/B' belong to R1(n) and that $\mathrm{E}_n(A, B')$ is normal in $\mathrm{GL}_n A$. Then A belongs to R1(n).

Proof. Let B_0 be an ideal of A. We have to prove that

$$[\mathrm{E}_n A, \mathrm{GL}_n B_0] \subset \mathrm{E}_n(A, B_0).$$

Let $\alpha \in \mathrm{E}_n A, \beta \in \mathrm{GL}_n B_0$. We have to prove that $[\alpha, \beta] \in \mathrm{E}_n(A, B_0)$.

Since A/B' belong to R1(n), we can write $[\alpha, \beta] = \gamma\delta$ with $\gamma \in \mathrm{E}_n(A, B_0)$ and $\delta \in \mathrm{GL}_n B'$. To find $\gamma \in \mathrm{E}_n(A, B_0)$ we lift the image of $[\alpha, \beta]$ in $\mathrm{E}_n(A/B', B_0/(B' \cap B_0))$. We can make it uniquely up to a matrix in $\mathrm{E}_n(A, B' \cap B_0)$. So matrix δ above is well-defined up to a matrix in $\mathrm{E}_n(A, B' \cap B_0)$. We fix β and denote by $f(\alpha) \in \mathrm{GL}_n B'/\mathrm{E}_n(A, B')$ the image of δ. Since conjugation by $\mathrm{E}_n A$ is trivial on $\mathrm{GL}_n B'/\mathrm{E}_n(A, B')$, f is a group morphism. We have to prove that f is trivial. Since $\mathrm{E}_n A$ is perfect it suffices to show that the image of f is commutative.

Clearly, $\delta \in H$ where $H = \mathrm{E}_n A[\mathrm{E}_n A, \mathrm{GL}_n A]$ is the normal subgroup of $\mathrm{GL}_n A$ generated by $\mathrm{E}_n A$. Since $\mathrm{E}_n(A, B')$ is normal in $\mathrm{GL}_n A$ and contains $[\mathrm{GL}_n B', \mathrm{E}_n A]$ (recall that $B' \in$ R1(n)), the group $(H \cap \mathrm{GL}_n B')/\mathrm{E}_n(A, B')$ is commutative. QED.

Corollary 8.14. Let A be an associative ring with 1, B' an ideal of A. Assume that $\mathrm{E}_n A$ is perfect, e.g., $n \geq 3$. Assume also that both B' and A/B' belong to R1(n) and that B' has identity. Then A belongs to R1(n).

Proof. Since B' has identity, $\mathrm{E}_n(A, B') = \mathrm{E}_n B'$ and $[\mathrm{GL}_n A, \mathrm{E}_n B'] = [\mathrm{GL}_n B', \mathrm{E}_n B']$ Since $B' \in$ R1(n), $[\mathrm{E}_n A, \mathrm{GL}_n B'] \subset \mathrm{E}_n(A, B')$, hence $[\mathrm{E}_n B', \mathrm{GL}_n B'] \subset \mathrm{E}_n(A, B') = \mathrm{E}_n B'$. So $\mathrm{E}_n(A, B')$ is normal in $\mathrm{GL}_n A$ and we can apply Proposition 8.12. QED.

Proposition 8.15 Let A be an associative ring with 1, B an ideal of A. Assume that $\mathrm{E}_n A$ is perfect, e.g., $n \geq 3$. Assume also that $\mathrm{GL}_n B$ normalizes $\mathrm{E}_n(A, B)$. Then the following four statements are equivalent:

(i) every subgroup H of $\mathrm{GL}_n B$ containg $\mathrm{E}_n(A, B)$ is noramlized by $\mathrm{E}_n A$;

(ii) every subgroup H of $\mathrm{G}_n(A, B)$ containg $\mathrm{E}_n(A, B)$ is noramlized by $\mathrm{E}_n A$;

(iii) $[\mathrm{E}_n A, \mathrm{GL}_n B] \subset \mathrm{E}_n(A, B)$;

(iv) $[\mathrm{E}_n A, \mathrm{G}_n(A, B)] \subset \mathrm{E}_n(A, B)$.

Proof. Since $\mathrm{GL}_n B] \subset \mathrm{G}_n(A, B)$,

(ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii).

It is also obvious that (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). By Proposition 8.11, (iii) $\Rightarrow$ (iv). So it remains to show that

(i) $\Rightarrow$ (iii).

Let $\alpha \in \mathrm{E}_n A$ and $\beta \in \mathrm{GL}_n B$. We have to prove that, assuming (i), $[\alpha, \beta] \in \mathrm{E}_n(A, B)$. Let H be the subgroup of $\mathrm{GL}_n B$ genrerated by β and $\mathrm{E}_n(A, B)$. Since $\mathrm{GL}_n B$ normalizes $\mathrm{E}_n(A, B)$. the subgroup $\mathrm{E}_n(A, B) \subset H$ is normal and $H/\mathrm{E}_n(A, B$ is a cyclic group.

By (i), H is normalized by $\mathrm{E}_n A$. Therefore we have a group morphism $f : \mathrm{E}_n A \to \mathrm{Aut}(H/\mathrm{E}_n(A, B))$ given by $h \mapsto \alpha h \alpha^{-1}$.

Note that the automorphism group $\mathrm{Aut}(H/\mathrm{E}_n(A,B))$ of the cyclic group $H/\mathrm{E}_n(A,B)$ is commutative. Namely, if $H/\mathrm{E}_n(A,B)$ is isomorphic to the additive group of $\mathbf{Z}/m\mathbf{Z}$ with an integer $m \geq 0$, then $\mathrm{Aut}(H/\mathrm{E}_n(A,B))$ is isomorphic to the muultiplicative group of the ring $\mathbf{Z}/m\mathbf{Z}$.

On the other hand, we have assumed that $\mathrm{E}_n A$ is perfect. So f is trivial, hence $[\alpha, \beta] \in \mathrm{E}_n(A,B)$. QED.

Problems.

1. Replace the condition that A is commutative in Corollary 8.5 and Theorem 8.7 by the weaker condition that $A/\mathrm{rad}(A)$ is commutative.

2. For any PI-ring and any $n \geq 3$ prove that every non-central subgroup H of $\mathrm{GL}_n A$ which is normalized by $\mathrm{E}_n A$ contains a nontrivial elementary matrix.

3. Let A be commutative, $n \geq 3, u = (u_i)$ an n-row over $A, v = (v_i)$ an n-column over A, and $uv = 0$. Show that

$$1_n + vbu \in \mathrm{E}_n A \text{ for all } b \in \sum_{i=1}^{n} (Au_i + Av_i).$$

4. Let A be commutative, $n \geq 3, u$ an n-row over A, v an n-column over A, and $uv = 0$. Assume that $\begin{pmatrix} u^T \\ v \end{pmatrix} \in \mathrm{Un}_{2n} A$. Show that for any ideal B of A

$$1_n + vbu \in \mathrm{E}_n(A,B) \text{ for all } b \in B.$$

5. Do Problem 4 with the condition $\begin{pmatrix} u^T \\ v \end{pmatrix} \in \mathrm{Um}_{2n} A$ replaced by the condition that A is a principal ideal ring.