Multipartitions: Congruences and Identities

by

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Abstract

The concept of a multipartition of a number, which has proved so useful in the study of Lie algebras, is studied for its own intrinsic interest. Following up on the work of Atkin, we shall present an infinite family of congruences for $P_k(n)$, the number of k-component multipartitions of n. We shall also examine the enigmatic Tri-Pentagonal Number Theorem and show that it implies a theorem about tri-partitions. Building on this latter observation, we examine a variety of multipartition identities connecting them with mock theta functions and the Rogers-Ramanujan identites.

1 Introduction

In 1882, J. J. Sylvester [25] broke new ground in the theory of partitions. Throughout most of the nineteenth century, partitions of integers were viewed primarily as an auxiliary aid in the theory of invariants. Sylvester's monumental paper [25] revealed that partitions were themselves interesting mathematical objects with a surprising rich arithmetic/combinatorial theory.

Today we find multipartitions in an analogous situation. We use M. Fayers definition of a multipartition [15, p. 4].

A *multipartition of* n *with* r *components* is an r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. If r is understood, we shall

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just call this a multipartition of n. As with partitions, we write the unique multipartition of 0 as $\emptyset$, and if λ is a multipartition of n then we write $|\lambda| = n$.

In many papers on the representation theory of Lie algebras (e.g. [12], [15]), some with applications to physics [11], we see multipartitions playing an important auxiliary role. It is not the case that the generating functions for multipartitions have not been examined previously. Indeed Atkin [10], Cheema et al [13], Gupta et al. [18], [19] and others have examined congruences and other arithmetic properties of $P_k(n)$, where

$$\sum_{n=0}^{\infty} P_k(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{-k}. \tag{1.1}$$

We see immediately by Euler's standard argument [3, Ch. 1] that $P_k(n)$ is the number of k component multipartitions of n. Atkin [10] has supplied the most extensive list of congruences for $P_k(n)$ building on his success in proving Ramanujan's "11^n" conjecture [9]. In Section 2, we prove the following.

Theorem 1. *For every prime $p > 3$, there are $(p+1)/2$ values of b in the interval $[1, p]$ for which*

$$P_{p-3}(pn+b) \equiv 0 \pmod{p}$$

for all $n \geqq 0$.

When $p = 5$, the b values are 2, 3, 4; for $p = 7$, they are 2, 4, 5, 6, and for $p = 11$ they are 2, 4, 5, 7, 8, 9.

The proof of Theorem 1 requires only the original elementary method of Ramanujan as extended in [8].

Section 3 leads us to the genesis of our study. Namely, what is the combinatorial significance of the tri-pentagonal number theorem [6, eq. (1.3)]?

$$\sum_{i,j,k \geqq 0} \frac{q^{i^2+j^2+k^2}}{(q;q)_{i+j-k}(q;q)_{i+k-j}(q;q)_{j+k-1}} \tag{1.2}$$
$$= \frac{\sum_{n,m,p=-\infty}^{\infty} (-1)^{n+m+p} q^{n(3n-1)/2+m(3m-1)/2+p(3p-1)/2+nm+np+mp}}{\prod_{n=1}^{\infty}(1-q^n)^3}.$$

We use the standard notation [17, eq. (1.2.5)]

$$(a;q)_n = \prod_{j=0}^{n-1} (1-aq^j),$$

and for future reference [17, eq. (1.2.4)]

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$

This leads to a vast array of multipartition identities which are further explored in Section 4. Section 5 examines the relationship of several generalizations of classical partitions to multipartitions, and Section 6 and 7 conclude with an incomplete account of the further possibilities for multipartitions.

2 Congruences

The main object of this section is to prove Theorem 1, and, by so doing, to suggest that this is the tip of a congruential iceberg. To prove Theorem 1 we must recall a slightly weakened version of the main result in [8].

Extended Ramanujan Congruence Theorem *Let $p > 3$ be a prime. Let $0 < a < p$ and b be integers with $-a$ a quadratic nonresidue mod p. Let $\{\alpha_n\}_{n=-\infty}^{\infty}$ be any doubly infinite sequence of integers. Then there exists an integer $c = c_p(a, b)$ such that the coefficient of q^{pN} for any $N > 0$ in*

$$\frac{\sum_{n=-\infty}^{\infty} \alpha_n q^{a\binom{n}{2}+bn+c}}{(q;q)_\infty^{p-3}}$$

is divisible by p. The number c is $\overline{8}(a(2b\overline{a}-1)^2+1)$ modulo p, and $\overline{m}$ denotes the multiplicative inverse of m modulo p.

To deduce Theorem 1, we note that for any specific n, say $n = \nu$, we may select the $\{\alpha_n\}_{n=-\infty}^{\infty}$ by $\alpha_n = 0$ if $n \neq \nu$, $\alpha_\nu = 1$. Thus the coefficient of q^{pN} in

$$\frac{q^{a\binom{\nu}{2}+b\nu+c}}{(q;q)_{p-3}} = \sum_{n \geqq 0} P_{p-3}\left(n - a\binom{\nu}{2} - b\nu - c\right) q^n$$

is divisible by p. So the proof of Theorem 1 is reduced to showing that the exponents $a\binom{n}{2} + bn + c$ run over $(p+1)/2$ different residue classes modulo p. To this end we consider the congruence

$$a\binom{n}{2} + bn + c \equiv r \pmod{p}.$$

This congruence is equivalent to

$$(n+\overline{2}(\overline{a}b-1))^2 \equiv 2\overline{a}r - \overline{4}(\overline{a}b-1)^2 \pmod{p}.$$

Now as n runs over a complete residue system modulo p, the left side of this last congruence takes the value of each of the $(p-1)/2$ quadratic residues twice and 0 once for a total of $(p+1)/2$ different values modulo p. For each of these values there is a unique value of r which is what was necessary to prove. ■

One of the most appealing aspects of the theory of congruences for $p(n)(= P_1(n))$ concerns the "crank". The crank of a partition is defined as follows:

Definition. For a partition π, let $l(\pi)$ denote the largest part of π, $\omega(\pi)$ denote the number of ones in π, and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The crank $c(\pi)$ is give by

$$c(\pi) = \begin{cases} l(\pi) & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

In [7], it is shown that the generating function for $c(m, n)$, the number of partitions of n with crank m is given by

$$\sum_{n \geqq 0} \sum_{m=-\infty}^{\infty} c(m,n) z^m q^n = \frac{(q;q)_\infty}{(zq, z^{-1}q; q)_\infty}.$$

The results in [16] and [7] together reveal the following combinatorial interpretation of the Ramanujan congruence

$$p(5n+4) \equiv 0 \pmod{5}.$$

Namely, if $C(j, M, n)$ denotes the number of partitions of n with crank $\equiv j \pmod{M}$, then for $0 \leqq j \leqq 4$

$$C(j, 5, 5n+4) = \frac{1}{5}\, p(5n+4).$$

We can define for bipartitions a bi-crank as the sum of the cranks of each component, and, following what has gone before, we can define $B(j, M, n)$ to be the number of bipartitions of n with bi-crank congruent to j mod M.

Theorem 2. *For $0 \leqq j \leqq 4$,*

$$B(j,5,5n+3) = \frac{1}{5}\,p_2(5n+3).$$

Proof. Garvan [16, eq. (1.30) and Sec. 3] proved the following formula from Ramanujan's Lost Notebook [23, p. 20]

$$\frac{(q;q)_\infty}{(\xi q;q)_\infty(\xi^{-1}q;q)_\infty} = A(q^5) - q(\xi+\xi^{-1})^2 B(q^5) + q^2(\xi^2+\xi^{-2})C(q^5) - q^3(\xi+\xi^{-1})D(q^5),$$

where ξ is any primitive fifth root of unity, and

$$A(q) = \frac{(q^5;q^5)_\infty G(q)^2}{H(q)},$$

$$B(q) = (q^5;q^5)_\infty G(q),$$

$$C(q) = (q^5;q^5)_\infty H(q),$$

$$D(q) = \frac{(q^5;q^5)_\infty H(q)^2}{G(q)},$$

$$G(q) = \frac{1}{(q,q^4;q^5)_\infty},$$

$$H(q) = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

We note for future reference that

$$A(q)D(q) = B(q)C(q). \tag{2.1}$$

Hence

$$\sum_{j=0}^{4}\sum_{n=0}^{\infty} B(j,5,n)\xi^j q^n = \left(\frac{(q;q)_\infty}{(\xi q,\xi^{-1}q;q)_\infty}\right)^2$$
$$= (A(q^5) - q(\xi+\xi^{-1})^2 B(q^5) + q^2(\xi^2+\xi^{-2})C(q^5) - q^3(\xi+\xi^{-1})D(q^5))^2.$$

Therefore

$$\begin{aligned}
&\sum_{j=0}^{4}\sum_{n=0}^{\infty} B(j,5,5n+3)\xi^j q^{5n+3} \\
&= -2q^3((\xi+\xi^{-1})A(q^5)D(q^5) + (\xi+\xi^{-1})^2(\xi^2+\xi^{-2})B(q^5)C(q^5)) \\
&= -2q^3A(q^5)D(q^5)(\xi+\xi^{-1})(1+(\xi+\xi^{-1})(\xi^2+\xi^{-2})) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (2.1))} \\
&= -2q^3A(q^5)D(q^5)(\xi+\xi^{-1})(1+\xi^3+\xi^4+\xi+\xi^2) \\
&= 0.
\end{aligned}$$

Consequently all of the $B(j,5,5n+3)$ must be identical in value otherwise we would have constructed a new polynomial of degree $\leqq 4$ satisfied by ξ different from $1+\xi+\xi^2+\xi^3+\xi^4$ which is **the** irreducible polynomial of which ξ is a root, a contradiction.

Finally since all the $B(j,5,5n+3)$ are equal, each must be equal to $P_2(5n+3)/5$. $\square$

It should be noted that K. Mahlburg [21] has proved amazing congruence theorems for the crank, and this suggests that it may be possible to do much more with the bi-crank and further extensions of the crank to multipartitions.

3 The Triple Theta Series

We devote this section to

$$S_3(q) = \frac{\sum_{n_1,n_2,n_3=-\infty}^{\infty}(-1)^{n_1+n_2+n_3}q^{n_1^2+n_2^2+n_3^2+\binom{n_1+n_2+n_3}{2}}}{(q)_\infty^3}, \tag{3.1}$$

the theta series side of (1.2). Our object will be to obtain as succinct as possible representation of $S_3(q)$ as a linear combination of infinite products.

Theorem 3.

$$S_3(-q) = \frac{1}{(-q;-q)_\infty(q^2;q^2)_\infty}\sum_{n,j=-\infty}^{\infty}(-1)^j q^{\binom{n}{2}+n^2-4nj+6j^2}. \tag{3.2}$$

Proof. Using the notation

$$[z^m]\sum_{n=0}^{\infty} a_n z^n = a_m, \tag{3.3}$$

we see that

$$S_3(q) = [z^0]\,\frac{\left(\sum_{n=-\infty}^{\infty}(-z)^n q^{n^2}\right)^3 \sum_{m=-\infty}^{\infty} z^{-m} q^{\binom{m}{2}}}{(q)_\infty^3}$$

$$= \frac{(q^2;q^2)_\infty^3}{(q)_\infty^2}\,[z^0](zq;q^2)_\infty^3(z^{-1}q;q^2)_\infty^3(-z^{-1})_\infty(-zq)_\infty$$

(by [3, eq. (2.2.10)])

$$= \frac{(q^2;q^2)_\infty^3}{(q)_\infty^2}\,[z^0](zq;-q)_\infty(-z^{-1};-q)_\infty(zq;q^2)_\infty(z^{-1}q;q^2)_\infty(z^2q^2;q^4)_\infty(z^{-2}q^2;q^4)_\infty$$

$$= \frac{(q^2;q^2)_\infty^3}{(q)_\infty^2}\,[z^0]\frac{\sum_{n=-\infty}^{\infty} z^{-n}(-q)^{\binom{n}{2}}\ \sum_{m=-\infty}^{\infty}(-z)^m q^{m^2}\ \sum_{j=-\infty}^{\infty}(-1)^j z^{2j} q^{2j^2}}{(-q;-q)_\infty(q^2;q^2)_\infty(q^4;q^4)_\infty}$$

(by [3, eq. (2.2.10)])

$$= \frac{1}{(q)_\infty(q^2;q^2)_\infty}\sum_{n,j=-\infty}^{\infty}(-q)^{\binom{n}{2}}(-1)^{n-2j}q^{(n-2j)^2+2j^2}(-1)^j\,.$$

Now replacing q by $-q$ in the above and simplifying, we arrive at the desired identity (3.2). ☐

Theorem 4. *With $S_3(q)$ as defined in (3.1),*

$$S_3(q) = T_1(q) - q^4T_2(q) + 2q^3T_3(q),$$

where

$$T_1(q) = \frac{(q^{28},q^{32},q^{60};q^{60})_\infty}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^3;q^6)_\infty(q^2;q^4)_\infty},$$
$$T_2(q) = \frac{(q^8,q^{52},q^{60};q^{60})_\infty}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^3;q^6)_\infty(q^2;q^4)_\infty},$$

and

$$T_3(q) = \frac{(q^3;q^3)_\infty(q^{12},q^{48},q^{60};q^{60})_\infty}{(q)_\infty(q^2;q^2)_\infty(q^6;q^{12})_\infty}\,.$$

Proof. Moving the infinite products to the left side of (3.2) and replacing j by $3j+\nu$ ($\nu=-1,0,1$), we see that

$$
\begin{aligned}
&S_3(-q)(-q;-q)_\infty(q^2;q^2)_\infty \\
&= \sum_{j,n=-\infty}^{\infty}\sum_{\nu=-1}^{1}(-1)^{j+\nu}q^{n(3n-1)/2-4n(3j+\nu)+6(3j+\nu)^2} \\
&= \sum_{j,n=-\infty}^{\infty}\sum_{\nu=-1}^{1}(-1)^{j+\nu}q^{6\nu^2-4n\nu+20j\nu+n(3n-1)/2+30j^2-2j} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(where we have shifted } n \text{ to } n+4j) \\
&= -\sum_{j,n=-\infty}^{\infty}(-1)^j q^{n(3n+7)/2+30j^2-22j+6} \\
&\quad + \sum_{j,n=-\infty}^{\infty}(-1)^j q^{n(3n-1)/2+30j^2-2j} \\
&\quad - \sum_{j,n=-\infty}^{\infty}(-1)^j q^{n(3n-9)/2+30j^2+18j+6} \\
&= -q^4\sum_{j,n=-\infty}^{\infty}(-1)^j q^{n(3n+1)/2+30j^2-22j} \\
&\quad + \sum_{j,n=-\infty}^{\infty}(-1)^j q^{n(3n-1)/2+30j^2-2j} \\
&\quad - q^3\sum_{j,n=-\infty}^{\infty}(-1)^j q^{3n(n-1)/2+30j^2+18j} \\
&= \frac{(q^3;q^3)_\infty(-q)_\infty}{(-q^3;q^3)_\infty}\Big((q^{28},q^{32},q^{60};q^{60})_\infty - q^4(q^8,q^{52},q^{60};q^{60})_\infty\Big) \\
&\quad - \frac{2q^3(q^6;q^6)_\infty}{(q^3;q^6)_\infty}(q^{12},q^{48},q^{60};q^{60})_\infty \qquad\qquad \text{(by[3, eq. (2.2.10)]).}
\end{aligned}
$$

Now if we replace q by $-q$ in the above and isolate $S_3(q)$, we find that we have the result stated in Theorem 4. $\square$

4 The Triple q-Series

The genesis of this paper lies in the desire to understand the left-hand side of (1.2) in some partition-theoretic sense. We define

$$W_3(n) = \sum_{i,j,k \geqq 0} \frac{q^{i^2+j^2+k^2}}{(q)_{i-j+k}(q)_{j-k+i}(q)_{k-i+j}} . \tag{4.1}$$

To this end, we change the indices of summation to m, n, r with

$$\begin{aligned} m &= i - j + k \\ n &= j - k + i \\ r &= k - i + j, \end{aligned}$$

and we deduce directly that

$$\begin{aligned} i &= \frac{m+n}{2} \\ j &= \frac{n+r}{2} \\ k &= \frac{r+m}{2} \end{aligned}$$

This one-to-one transformation is admissable in triples of positive integers if and only if m, n and r are all of the same parity.

Additionally under this transformation

$$i^2 + j^2 + k^2 = m\left(\frac{m+n}{2}\right) + n\left(\frac{n+r}{2}\right) + r\left(\frac{r+n}{2}\right).$$

Hence

$$W_3(n) = \sum_{m,n,r \geqq 0}{}' \frac{q^{m\left(\frac{m+n}{2}\right)+n\left(\frac{n+r}{2}\right)+r\left(\frac{r+n}{2}\right)}}{(q)_m(q)_n(q)_r} , \tag{4.2}$$

where $\sum'$ means that the indices m, n, r must all have the same parity.

We now recall a basic fact about partition generating functions (cf. [3, Ch. 1], [20, Ch. XIX]):

$$\frac{q^{An}}{(q)_n} \qquad (A > 0)$$

is the generating function for partitions into exactly n parts each $\geqq A$.

The above observations now allow us to interpret (4.1) in terms of tri-partitions. We say a multipartition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ has **equiparity** provided each of $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ has an even number of parts or each has an odd number. In a multipartition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ we say that λ_{i+1} is the **next** component to λ_i with the convention that λ_1 is next to λ_k (i. e. $\lambda_{k+1} = \lambda_1$). We say that a multipartition of equiparity has parts of **average size** if each part in component λ_i is at least as large as the average of the number of parts in λ_i and λ_{i+1}.

Finally we define $B_3(n)$ to be the number of tripartitions of equiparity and parts of average size.

Theorem 5.

$$\sum_{n=0}^{\infty} B_3(n)q^n = W_3(q) = \sum_{i,j,k \geqq 0} \frac{q^{i^2+j^2+k^2}}{(q)_{i-j+k}(q)_{j-k+i}(q)_{k-i+j}}.$$

Proof. Inspection of (4.2) reveals that $W_3(q)$ is indeed the generating function in question. □

The obvious relevance of tri-partitions with equiparity and average size suggests that we apply these ideas to other classes of partitions. We apply them to bi-partitions in the next section.

5 Bi-partitions with parts of average size

In treating bi-partitions, it turns out that we must keep track of whether the parity implied by equiparity is even or odd.

We define $B_{2,e}(n)$ (resp. $B_{2,0}(n)$) to be the number of bi-partitions of even (resp. odd) equiparity with parts of average size.

Theorem 6.

$$B_{2,e}(n) - B_{2,0}(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ A_{2,2}\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

where $A_{2,2}(n)$ is the Rogers-Ramanujan partition function, the number of ordinary partitions of n into parts $\equiv \pm 1 \pmod 5$.

Proof.

$$\sum_{n=0}^{\infty}(B_{2,e}(n)-B_{2,0}(n))q^n$$

$$=\frac{1}{2}\sum_{n,m\geqq 0}\frac{q^{n\binom{n+m}{2}+m\binom{n+m}{2}}}{(q)_n(q)_m}\Big((-1)^n+(-1)^m\Big)$$

$$=\sum_{n,m\geqq 0}\frac{(-1)^n q^{n\binom{n+m}{2}+m\binom{n+m}{2}}}{(q)_n(q)_m}$$

$$=\sum_{n,m\geqq 0}\frac{(-1)^n q^{\frac{n^2}{2}+\frac{m^2}{2}+mn}}{(q)_n(q)_m}$$

$$=\sum_{n\geqq 0}\frac{q^{2n^2}}{(q^2;q^2)_n}$$

(by [27, p. 46])

$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^{10n-2})(1-q^{10n-8})}$$

(by [3, eq. (7.1.7)])

$$=\sum_{n=0}^{\infty}A_{2,2}(n)q^{2n}.$$

Comparison of the extremes in the above string of equations yields the assertion of our theorem. □

It is natural to ask why we examined the excess of even equiparity over odd equiparity. Why not just consider the full count of such partitions? Of course this can be done; however the resulting generating functions reduce to series such as

$$\sum_{j=0}^{\infty}\frac{(-1)^j q^{j^2}}{(q)_{2j}}$$

which appear not to have the interesting qualities associated with the Rogers-Ramanujan identities or related formulae.

6 Further Multipartition identities

Once we have introduced the idea of multipartitions with average size, it is natural to ask seemingly simpler questions. For example, let us consider $C_k(n)$ the number of k-component multipartitions in which each part in the i^{th} component ($1 \leqq i \leqq k$) is larger than the number of parts in the next component.

Lemma 7.

$$\sum_{j\geqq 0} (q^{j+1})_j q^j = \sum_{n=0}^{\infty} (-1)^n q^{n(15n+7)/2} + \sum_{n=0}^{\infty} (-1)^n q^{n(15n+13)/2\ +1} + \sum_{n=0}^{\infty} (-1)^n q^{n(15n+17)/2\ +2} + \sum_{n=0}^{\infty} (-1)^n q^{n(15n+23)/2\ +4}.$$

Proof. $\sum_{j\geqq 0} q^j (q^{j+1})_j$

$$= \sum_{j\geqq 0} q^j \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} (-1)^h q^{\binom{h+1}{2}+jh}$$

(by [3, eq. (3.3.6)])

$$= \sum_{h,j\geqq 0} \begin{bmatrix} j+h \\ h \end{bmatrix} (-1)^h q^{\binom{h+1}{2}+(j+h)h+j+h}$$

$$= \sum_{h\geqq 0} \frac{(-1)^h q^{3h(h+1)/2}}{(q^{h+1})_{h+1}}$$

(by [3, eq. (3.3.7)])

$$= \sum_{n=0}^{\infty} (-1)^n q^{n(15n+7)/2} + \sum_{n=0}^{\infty} (-1)^n q^{n(15n+13)/2\ +1} + \sum_{n=0}^{\infty} (-1)^n q^{n(15n+17)/2\ +2} + \sum_{n=0}^{\infty} (-1)^n q^{n(15n+23)/2\ +4}$$

(by [24, eq. A(8), p. 333]),

which is the assertion of the Lemma. □

Theorem 8.

$$\sum_{n\geqq 0} C_2(n)q^n = \frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{n(15n+7)/2}\left(1+q^{3n+1}+q^{5n+2}+q^{8n+4}\right)\right).$$

Proof. Clearly

$$\begin{aligned}
\sum_{n\geqq 0} C_2(n)q^n &= \sum_{i,j\geqq 0} \frac{q^{i(j+1)+j(i+1)}}{(q)_i(q)_j} \\
&= \sum_{j\geqq 0} \frac{q^j}{(q)_j(q^{2j+1})_\infty} \qquad\qquad \text{(by [3, eq. (2.2.5)])} \\
&= \frac{1}{(q)_\infty}\sum_{j\geqq 0} q^j(q^{j+1})_j,
\end{aligned}$$

and we replace the final sum by the expression in Lemma 7, and the result follows. □

For our next result, we recall one of Ramanujan's fifth order mock theta functions [23, p. 131]

$$\chi_1(q) = \sum_{i\geqq 0} \frac{q^i}{(q^{i+1})_{i+1}}.$$

Theorem 9. $\sum_{n\geqq 0} C_3(n)q^n = \frac{\chi_1(q)}{(q)_\infty}$.

Proof. Clearly

$$\begin{aligned}
\sum_{n\geqq 0} C_3(n)q^n &= \sum_{i,j,k\geqq 0} \frac{q^{i(j+1)+j(k+1)+k(i+1)}}{(q)_i(q)_j(q)_k} \\
&= \sum_{i,j\geqq 0} \frac{q^{ij+i+j}}{(q)_i(q)_j(q^{j+i+1})_\infty} \\
&\qquad\qquad\qquad\qquad \text{(by [3, eq. (2.2.5)])} \\
&= \frac{1}{(q)_\infty}\sum_{i,j\geqq 0} \begin{bmatrix} j+i \\ i \end{bmatrix} q^{ij+i+j} \\
&= \frac{1}{(q)_\infty}\sum_{i\geqq 0} \frac{q^i}{(q^{i+1})_{i+1}}
\end{aligned}$$

(by [3, eq. (3.3.7)])

$$= \frac{\chi_1(q)}{(q)_\infty}.$$

□

Theorem 10. $\sum_{n\geqq 0} C_4(n)q^n = \frac{1}{(q)_\infty} \sum_{i,j\geqq 1} \frac{q^{ij-1}}{(q)_{i+j-1}}$.

Proof. As previously,

$$\sum_{n\geqq 0} C_4(n)q^n = \sum_{i,j,k,m\geqq 0} \frac{q^{i(j+1)+j(k+1)+k(m+1)+m(i+1)}}{(q)_i(q)_j(q)_k(q)_m}$$

$$= \sum_{i,j,k\geqq 0} \frac{q^{ij+jk+i+j+k}}{(q)_i(q)_j(q)_k(q^{k+i+1})_\infty}$$

(by [3, eq. (2.2.5)])

$$= \frac{1}{(q)_\infty} \sum_{i,j,k\geqq 0} \begin{bmatrix} k+i \\ i \end{bmatrix} \frac{1}{(q)_j} q^{ij+jk+i+j+k}$$

$$= \frac{1}{(q)_\infty} \sum_{i,k\geqq 0} \begin{bmatrix} k+i \\ i \end{bmatrix} \frac{q^{i+k}}{(q^{i+k+1})_\infty}$$

$$= \frac{1}{(q)_\infty^2} \sum_{i\geqq 0} q^i (q)_i \sum_{k\geqq 0} \frac{(q^{i+1})_k^2 q^k}{(q)_k}$$

$$= \frac{1}{(q)_\infty^2} \sum_{i\geqq 0} q^i (q)_i \frac{(q^{i+1})_\infty (q^{i+2})_\infty}{(q)_\infty} \sum_{j\geqq 0} \frac{q^{(i+1)_j}}{(q^{i+2})_j}$$

$$= \frac{1}{(q)_\infty} \sum_{i,j\geqq 0} \frac{q^{ij+i+j}}{(q)_{j+i+1}}$$

$$= \frac{1}{(q)_\infty} \sum_{i,j\geqq 1} \frac{q^{ij-1}}{(q)_{i+j-1}}.$$

□

Theorem 11.

$$\sum_{n \geqq 0} C_5(n) q^n = \frac{1}{(q)_\infty^2} \sum_{m=1}^{\infty} \sigma(m) q^{m-1}, = e^{-13\pi i\tau/6} \eta^{-2}(\tau)(1 - E_2(\tau))/24,$$

where $\sigma(m)$ *is the sum of the divisors of* m, $q = e^{2\pi i\tau}$, $\eta(\tau)$ *is Dedekind's* η*-function given by*

$$\eta(\tau) = q^{1/24}(q)_\infty,$$

and $E_2(\tau)$ *is the Eisenstein series given by*

$$E_2(\tau) = 1 - 24 \sum_{m=1}^{\infty} \sigma(m) q^m.$$

Proof. $\sum_{n \geqq 0} C_5(n) q^n$

$$\begin{aligned}
&= \sum_{i,j,k,\ell,m \geqq 0} \frac{q^{ij+jk+k\ell+\ell m+mi+i+j+k+\ell+m}}{(q)_i (q)_j (q)_k (q)_\ell (q)_m} \\
&= \sum_{i,k,m \geqq 0} \frac{q^{mi+i+k+m}}{(q)_m (q)_k (q)_i (q^{i+k+1})_\infty (q^{k+m+1})_\infty} && \text{(by [3, eq. (2.2.5)])} \\
&= \frac{1}{(q)_\infty} \sum_{k,m \geqq 0} \frac{q^{k+m}}{(q)_m (q^{k+m+1})_\infty} \sum_{i \geqq 0} \frac{(q^{k+1})_i q^{i(m+1)}}{(q)_i} \\
&= \frac{1}{(q)_\infty} \sum_{k,m \geqq 0} \frac{q^{k+m} (q^{m+k+2})_\infty}{(q)_m (q^{k+m+1})_\infty (q^{m+1})_\infty} && \text{(by [3, eq. (2.2.1)])} \\
&= \frac{1}{(q)_\infty^2} \sum_{k,m \geqq 0} \frac{q^{k+m}}{(1 - q^{k+m+1})} \\
&= \frac{1}{(q)_\infty^2} \sum_{N=0}^{\alpha} \frac{q^N (N+1)}{(1 - q^{N+1})} \\
&= \frac{q^{-}1}{(q)_\infty^2} \sum_{m=1}^{\infty} \frac{m q^m}{1 - q^m} \\
&= \frac{q^{-1}}{(q)_\infty^2} \sum_{m=1}^{\infty} \sigma(m) q^m.
\end{aligned}$$

□

Corollary 12. $\sum_{m=1}^{\infty} \sigma(m) q^m \equiv q(q)_\infty^2 H(q)^5 \pmod 5$ *where* $H(q) = 1/((q^2;q^5)_\infty (q^3;q^5)_\infty)$.

Proof. In examining the quintuple series generating function for $C_5(n)$, we see that cyclic permutation of the five indices leaves each term unaltered. Since there are no nontrivial subgroups of the cyclic group of order 5, we see that modulo 5

$$\begin{aligned}\sum_{n=0}^{\infty} C_5(n) q^n &\equiv \sum_{n=0}^{\infty} \frac{q^{5n^2+5n}}{(q)_n^5} \qquad \pmod 5\\ &\equiv \sum_{n=0}^{\infty} \frac{q^{5n^2+5n}}{(q^5;q^5)_n} = H(q^5) \qquad \text{(by[3, eq. (7.1.7)]).}\end{aligned}$$

The result now follows from Theorem 11. □

7 Another Family of Mulitpartition Identities

The partition function, $C_k(n)$, is the number of multipartitions of n in which each part in the i^{th} component is larger than the number of parts in the next component. In this section we consider, $D_k(n)$, the number of multipartitions of n with k components in which each part in the i^{th} component is larger that the number of parts in the next component with the exception of the k^{th} component (which now does **NOT** have to have its parts larger than the number of elements in the first component).

Theorem 13. $\sum_{n \geqq 0} D_2(n) q^n = \frac{1}{(1-q)(q)_\infty}$.

Proof. $\sum_{n \geqq 0} D_2(n) q^n = \sum_{i,j \geqq 0} \frac{q^{i(j+1)+j}}{(q)_i (q)_j} = \frac{1}{(q)_\infty} \sum_{i \geqq 0} q^i = \frac{1}{(1-q)(q)_\infty}$. □

Theorem 14. $\sum_{n \geqq 0} D_3(n) q^n = \frac{q^{-1}}{(q)_\infty^2} - \frac{q^{-1}}{(q)_\infty}$.

Proof. $\sum_{n\geqq 0} D_3(n)q^n$

$$
\begin{aligned}
&= \sum_{i,j,k\geqq 0} \frac{q^{i(j+1)+j(k+1)+k}}{(q)_i(q)_j(q)_k}\\
&= \frac{1}{(q)_\infty} \sum_{i,j\geqq 0} \frac{q^{i(j+1)+j}}{(q)_i}\\
&= \frac{1}{(q)_\infty} \sum_{i\geqq 0} \frac{q^i}{(q)_{i+1}}\\
&= \frac{q^{-1}}{(q)_\infty} \sum_{i\geqq 1} \frac{q^i}{(q)_i}\\
&= \frac{q^{-1}}{(q)_\infty} \left(\frac{1}{(q)_\infty} - 1 \right).
\end{aligned}
$$

□

Theorem 15. *$\sum_{n\geqq 0} D_4(n)q^n = \frac{q^{-1}\Delta(q)}{(q)^2_\infty}$, where $\Delta(q) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)}$ is the generating function for the number of divisors of n.*

Proof. $\sum_{n\geqq 0} D_4(n)q^n$

$$
\begin{aligned}
&= \sum_{i,j,k,\ell\geqq 0} \frac{q^{i(j+1)+j(k+1)+k(\ell+1)+\ell}}{(q)_i(q)_j(q)_k(q)_\ell}\\
&= \frac{1}{(q)_\infty} \sum_{i,j,k\geqq 0} \frac{q^{i(j+1)+j(k+1)+k}}{(q)_i(q)_j}\\
&= \frac{1}{(q)_\infty} \sum_{i,j} \frac{q^{i(j+1)+j}}{(q)_i(q)_{j+1}}\\
&= \frac{1}{(q)^2_\infty} \sum_{j\geqq 0} \frac{q^j}{(1-q^{j+1})}\\
&= \frac{q^-1}{(q)^2_\infty} \Delta(q).
\end{aligned}
$$

□

Theorem 16. $\sum_{n\geqq 0} D_5(n)q^n = \frac{1}{(q)^2_\infty} \sum_{k\geqq 1} \frac{q^{k-1}}{(q)_k(1-q^k)}$.

Proof. $\sum_{n\geqq 0} D_5(n)q^n$

$$= \sum_{i,j,k,\ell,m\geqq 0} \frac{q^{i(j+1)+j(k+1)+k(\ell+1)+\ell(m+1)+m}}{(q)_i(q)_j(q)_k(q)_\ell(q)_m}$$

$$= \frac{1}{(q)_\infty} \sum_{i,j,k\geqq 0} \frac{q^{ij+jk+i+j+k}}{(q)_i(q)_j(q)_{k+1}}$$

$$= \frac{1}{(q)_\infty^2} \sum_{j,k\geqq 0} \frac{q^{jk+j+k}}{(q)_{k+1}}$$

$$= \frac{1}{(q)_\infty} \sum_{k\geqq 0} \frac{q^k}{(q)_{k+1}(1-q^{k+1})}\,.$$

□

Theorem 17. $\sum_{n\geqq 0} D_6(n)q^n = \frac{q^{-1}}{(q)_\infty^3} \sum_{h\geqq 0} \frac{1-(q^{h+1})_\infty}{1-q^{h+1}}$.

Proof. $\sum_{n\geqq 0} D_6(n)q^n$

$$= \sum_{i,j,k,h,m,n\geqq 0} \frac{q^{i(j+1)+j(h+1)+k(n+1)+h(m+1)+m(n+1)+n}}{(q)_i(q)_j(q)_k(q)_h(q)_m(q)_n}$$

$$= \frac{1}{(q)_\infty^2} \sum_{j,k,h,m\geqq 0} \frac{q^{jk+kh+hm+j+k+h+m}}{(q)_k(q)_h}$$

$$= \frac{q^{-1}}{(q)_\infty^2} \sum_{k,h\geqq 0} \frac{q^{(k+1)(h+1)}}{(q)_{k+1}(q)_{h+1}}$$

$$= \frac{q^{-1}}{(q)_\infty^2} \sum_{k,h\geqq 1} \frac{q^{kh}}{(q)_k(q)_h}$$

$$= \frac{q^{-1}}{(q)_\infty} \sum_{h\geqq 1} \frac{1}{(q)_h} \left(\frac{1}{(q^h)_\infty} - 1\right)$$

$$= \frac{q^{-1}}{(q)_\infty^2} \sum_{h\geqq 1} \frac{1-(q^h)_\infty}{1-q^h}\,.$$

□

8 Interpretations of Multiple q-Series Identities

There are numerous theorems identifying multiple q-series including [1], [2], [4], [5], [22]. Whether this approach using multipartitions will provide really new insight concerning these identities awaits future investigations. In this section, we choose one example from [4] which has not been given a partition-theoretic interpretation previously.

We define $W(n)$ to be the number of tripartitions $\lambda_1 + \lambda_2 + \lambda_3$ of n for which the following three conditions are met (here $\#(\lambda_i)$ is the number of parts in λ_i)

(i) each part in λ_1 is $\geqq \#(\lambda_1)$

(ii) each part in λ_2 is $\geqq \#(\lambda_1) + \#(\lambda_2)$

(iii) each part in λ_3 is $\geqq 2(\#(\lambda_1) + \#(\lambda_2) + \#(\lambda_3))$.

We define $V(n)$ to be the number of bi-partitions $\lambda_1 + \lambda_2$ wherein all parts of both components are $\equiv \pm 1 \pmod 5$.

Theorem 18. *For each $n \geqq 0$, $W(n) = V(n)$.*

Proof. In [4, eq. (1.7)], it is shown that

$$\sum_{n,m,r \geqq 0} \frac{q^{(n+m)^2+(r+m)^2+nr}}{(q)_n(q)_m(q)_r} = \frac{1}{(q,q^4;q^5)_\infty^2} .$$

Clearly [3, eq. (7.1.6)]

$$\sum_{n=0}^{\infty} V(n)q^n = \frac{1}{(q,q^4;q^5)_\infty^2} ,$$

and

$$\sum_{n,m,r \geqq 0} \frac{q^{(n+m)^2(r+m)^2+nr}}{(q)_n(q)_m(q)_r} = \sum_{n,m,r \geqq 0} \frac{q^{n^2+r(r+n)+m(2m+2r+2n)}}{(q)_n(q)_m(q)_r} = \sum_{n \geqq 0} W(n)q^n,$$

and our result follows. $\square$

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