6. Prestabilization.

Assume that $\mathrm{sr}(B) < \infty$. By Theorem 4.4 (a), $\mathrm{wh}(\mathrm{GL}_n B) = \mathrm{K}_1(A, B)$ for $n \geq \mathrm{sr}(B)$. By Theorem 4.7, the kernel of

$$\text{(6.1)} \qquad \mathrm{wh} : \mathrm{GL}_n B \to \mathrm{K}_1(A, B)$$

is $\mathrm{E}_n(A, B)$ when $n \geq \mathrm{sr}B) + 1$. But what about the kernel of (6.1) in the case $n = \mathrm{sr}(B)$?

First we state know results without proofs. We start with the case $n = 1$. By Section 2, the kernel of (6.1) is $[\mathrm{GL}_1 D,\ \mathrm{GL}_1 D]$ when $B = A = D$ is a division ring and $n = 1 = \mathrm{sr}(D)$. The same proof gives that the kernel of (6.1) is $[\mathrm{GL}_1 A,\ \mathrm{GL}_1 A]$ when $B = A$ is a local ring and $n = 1 = \mathrm{sr}(A)$.

When $B = A = \mathrm{M}_2(\mathbf{Z}/2\mathbf{Z})$ and $n = \mathrm{sr}(A) = 1$, the kernel of (6.1) is bigger than $[\mathrm{GL}_1 A,\ \mathrm{GL}_1 A]$. In fact the kernel is

$$\mathrm{E}_2(\mathbf{Z}/2\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z}/2\mathbf{Z}) = \mathrm{GL}_2(\mathbf{Z}/2\mathbf{Z})$$

where the subgroup

$$[\mathrm{GL}_1 A, \mathrm{GL}_1 A] = [\mathrm{GL}_2(\mathbf{Z}/2\mathbf{Z}), \mathrm{GL}_2(\mathbf{Z}/2\mathbf{Z})]$$

has index 2.

When $B = A$ is the subring of all upper triangular matrices in $\mathrm{M}_2(\mathbf{Z}/2\mathbf{Z})$ and $n = 1 = \mathrm{sr}(A)$, then $[\mathrm{GL}_1 A,\ \mathrm{GL}_1 A]$ also has index 2 in the kernel.

The following theorem covers the last two examples.

Theorem 6.2 (Vaserstein [V9]). Suppose that $A/\mathrm{rad}(A)$ is the direct product of matrix rings over division rings. Then for any ideal B of A

$$\mathrm{K}_1(A, B) = \mathrm{GL}_1 B/\tilde{\mathrm{E}}_1(A, B)$$

where $\tilde{\mathrm{E}}_1(A, B)$ is the subgroup of $\mathrm{GL}_1 B$ generated by all $(1 + ab)(1 + ba)^{-1}$ with $a \in A, b \in B$, and $1 + ab \in \mathrm{GL}_1 B$.

Theorem 6.3 (Vaserstein [V128]). Under the conditions of Theorem 6.2, assume that A has no $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ nor $\mathrm{M}_2(\mathbf{Z}/2\mathbf{Z})$ as a factor ring. Then

$$\mathrm{K}_1 A = \mathrm{GL}_1 A/[\mathrm{GL}_1 A, \mathrm{GL}_1 A].$$

It is still an open problem whether

$$\mathrm{K}_1(A, B) = \mathrm{GL}_1 B/\tilde{\mathrm{E}}_1(A, B)$$

whenever $\mathrm{sr}(B) = 1$. Here is what is known for any such B :

Theorem 6.4 (Menal–Moncasi [MM] in the case $B = A$, Magurn-Vaserstein [MV] in general). If $\mathrm{sr}(B) = 1$, then

$$\mathrm{K}_1(A, B) = \mathrm{GL}_1 B/\mathrm{W}_1(A, B)$$

where $\mathrm{W}_1(A,B)$ is the subgroup of $\mathrm{GL}_1 B$ generated by all $(b+b'+bab')(b+b'+b'ab)^{-1}$ with $b, b'-1 \in B, a \in A$, and $b+b'+bab' \in \mathrm{GL}_1 B$.

Besides examples above, here is another additional condition on B which guarantees (Magurn–Vaserstein [MV]) that $\mathrm{W}_1(A,B) = \tilde{\mathrm{E}}_1(A,B)$:

(6.5) for every $\begin{pmatrix} 1+b \\ a \end{pmatrix} \in \mathrm{Un}_2 B$, there is $t \in \mathrm{GL}_1 B$ such that $a + t(1+b) \in \mathrm{GL}_1 B$.

The condition (6.5) with $B = A$ is known as the unit 1-stable range condition. If on top of this condition,

(6.6) for any $x, y \in A$ there is $a \in \mathrm{GL}_1 A$ such that $1 + ax, a - y \in \mathrm{GL}_1 A$,

then [MV]

$$\mathrm{W}_1(A,B) = \tilde{\mathrm{E}}_1(A,B) = [\mathrm{GL}_1 A, \mathrm{GL}_1 B].$$

The last statement with $B = A$ was also proved in [G] where the condition (6.6) was written in the following form:

for any $x, x' \in A$ there is $a \in \mathrm{GL}_1 A$ such that $x + 1/a, x' + a \in \mathrm{GL}_1 A$.

Theorem 6.12 below together with Theorem 6.4 give the following

Corollary 6.7. If $\mathrm{sr}(A) = 1$ and (6.6) holds then $\mathrm{K}_1(A,B) = [\mathrm{GL}_1 A, \mathrm{GL}_1 B]$ for every ideal B of A.

Now we outline some results with $n \geq 2$.

Theorem 6.8 (Vaserstein [V9], [V73, Theorem 11]). Assume that A is finitely generated as module over the centre whose space of maximal ideals is a finite union of subspaces of dimension $\leq d$. Then

$$\mathrm{K}_1(A,B) = \mathrm{GL}_{d+1} B / \tilde{\mathrm{E}}_{d+1}(A,B)$$

where $\tilde{\mathrm{E}}_{d+1}(A,B)$ is the subgroup of $\mathrm{GL}_{d+1} B$ generated by the mixed commutator subgroup $[\mathrm{E}_{d+1} A, \mathrm{GL}_{d+1} B]$ and all matrices $(1_{d+1} + XY)(1_{d+1} + YX)^{-1}$ with $X \in \mathrm{M}_n B$, $Y = \begin{pmatrix} y & 0 \\ 0 & 1_d \end{pmatrix}$, $y \in A, 1_{d+1} + XY \in \mathrm{GL}_n A$.

Theorem 6.9 (Bass [B1] in the case $n = 1$, Vaserstein [V73] in the case $n = 2$, van der Kallen [vdK] in the case $n \geq 3$). Assume that A is commutative and $\mathrm{sr}(A) = n$. Then

$$\mathrm{K}_1(A,B) = \mathrm{GL}_n B / \tilde{\mathrm{E}}_n(A,B)$$

where $\tilde{\mathrm{E}}_n(A,B)$ is defined as in Theorem 6.8.

A proof (in case $B = A$) involves covering $\mathrm{E}_{n+1}(A)$ by $XX'X$ where X, X' are as in the previous section but with additional conditions on the three factors.

Theorem 6.10 (Magurn–Vaserstein [MV]). Let A be a Banach algebra and $n = \mathrm{sr}(A) < \infty$. Then

$$\mathrm{K}_1(A,B) = \mathrm{GL}_n B / \mathrm{V}_n(A,B)$$

where $\mathrm{V}_n(A,B)$ is the subgroup of $\mathrm{GL}_n B$ generated by $(1_n + xy)(1_n + yx)^{-1}$ with $x \in \mathrm{M}_n B$, $y \in \mathrm{M}_n A$ and $1_n + xy \in \mathrm{GL}_n A$.

The conclusion of Theorem 6.10 was obtained in [MV] also for other classes of rings. For example, this conclusion was obtained under the following condition (twofold n-stable rank condition):

given unimodular $(n+1)$-rows (c, d) and (c', d') over A (where $d, d' \in A$) there is an n-row c'' over A such that both $c + dc''$ and $c' + d'c''$ are unimodular.

This condition is stronger thn the condition $\mathrm{sr}(A) \leq n$. But it follows from the condition of Theorem 6.8 when $n \geq 3, d+1$. On the other hand, it is unknown whether the conclusion can be obtained just under the stable rank condition $\mathrm{sr}(A) = n < \infty$.

The case of commutative ring with $\mathrm{sr}(A) = 2$ is of special interest because such A includes the rings of integers in number fields and, more generally, the ring of S-integers in global fields called Dedekind rings of arithmetic type by Bass-Milnor-Serre who computed $\mathrm{SK}_1(A, B)$ for the latter rings A.

This group $\mathrm{SK}_1(A, B)$ is always a factor group of the torsion subgroup μ of $\mathrm{GL}_1 A$. If B" is another nonzero ideal of A and $B' \subset B$, then the induced homomorphism $\mathrm{SK}_1(A, B') \to \mathrm{SK}_1(A, B)$ is surjective (this is because the ring B/B' is finite hence $\mathrm{sr}(B/B') = 1$), The group $\mathrm{SK}_1(A, B)$ is trivial unless A is the ring of integers in a totally imaginary number field and $B \neq A$ is a small (with respect to inclusions) nonzero ideal of A in which case $\mathrm{SK}_1(A, B)$ is isomorphic to the torsion subgroup of $\mathrm{GL}_1 A$ for sufficiently small B.

The proof uses the class field theory (norm residue symbols and the uniqueness of the reciprocity law) and a description of $\mathrm{SK}_1(A, B)$ via Mennicke symbols. The Mennicke symbol $[\,] : \mathrm{Um}_2 B \to \mathrm{SK}_1(A, B)$ is defined (for any commutative B) by

$$\begin{bmatrix} a \\ b \end{bmatrix} = \mathrm{wh} \begin{pmatrix} a & * \\ b & * \end{pmatrix} \in \mathrm{SK}_1(A, B)$$

where $\begin{pmatrix} a & * \\ b & * \end{pmatrix} \in \mathrm{SL}_2 B$. When $\mathrm{sr}(B) \leq 2$ (e.g., $\mathrm{sr}(A) = 2$), the group $\mathrm{SK}_1(A, B)$ consists of the symbols. For any commutative A, the symbols enjoy the following relations (discovered by Mennicke, Newman, Lam):

(**MS1**) $$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b + xa \end{bmatrix} = \begin{bmatrix} a + yb \\ b \end{bmatrix}$$

for $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathrm{Um}_2 B, x \in B, y \in A$;

(**MS2**) $$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a \\ b' \end{bmatrix} = \begin{bmatrix} a \\ bb' \end{bmatrix} \text{ and } \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a' \\ b \end{bmatrix} = \begin{bmatrix} aa' \\ b \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b \end{bmatrix}, \begin{bmatrix} a \\ b' \end{bmatrix} \in \mathrm{Um}_2 B$$

(some of these relation follows from others).

Bass-Milnor-Serre proved the triviality of $\mathrm{SK}_1(A, B)$ for most of Dedekind rings of arithmetic type using the above relations (Mennicke also did this partialy in an unpublished paper). However to prove that $\mathrm{SK}_1(A, B)$ is nontrivial and compute it, they needed to show that the relations are defining. For this purpose, they proved and used Theorem 6.6 with $n = \mathrm{sr}(A) = 2$) for Dedekind rings A.

Theorem 6.11. Let A be a commutative ring with 1 and $\mathrm{sr}(A) \le 2$ (e.g., A is a Dedekind ring). Then for any ideal B of A the group $\mathrm{SK}_1(A, B)$ has a presentation with the generating set $\mathrm{Um}_2 B$ and defining relations (MS1), (MS2).

Proof. It has two ingradients:

by Theorem 6.6, the group $\mathrm{SK}_1(A, B)$ is the orbits of the group $\tilde{\mathrm{E}}_2(A, B)$ on the set $\mathrm{Um}_2 B$;

computations of [BMS] or (in more detail) of [B2, Chapter VI] proving [MS1] and [MS2] for any commutative ring A and some computations for more special rings A (namely, Noetherian domains of Krull dimension ≤ 1 and for Dedekind rings).

The only thing we need to add is the following lemma which allows to use (MS2) to write a product of Mennicke symbols as a single symbol.

Lemma 6.12. Let A, B be as in Theorem 6.11. Let $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in \mathrm{Um}_2 B$.

Then there are $c_1, c_2, d_1, d_2 \in B$ such that

$$b_1 + d_1(a_1 + c_1 b_1) = b_2 + d_2(a_2 + c_2 b_2).$$

Proof. It is clear that $Aa_1 + Aa_2 + Ab_1 b_2 = A$. Since $\mathrm{sr}(A) \le 2$, there is $c \in A$ such that $A(a_1 + c_3 b_1 b_2) + A(a_2 + c_4 b_1 b_2) = A$. We set $c_1 = c_3 b_2 \in B$ and $c_2 = c_4 b_1 \in B$. Then $A(a_1 + c_1 b_1) + A(a_2 + c_2 b_2) = A$. We write $d_3(a_1 + c_1 b_1) + d_4(a_2 + c_2 b_2) = 1$ with $d_3, d_4 \in A$ and set $d_1 = (b_2 - b_1) d_3 \in B$ and $d_2 = (b_1 - b_2) d_4 \in B$. Then

$$b_1 + d_1(a_1 + c_1 b_1) = b_2 + d_2(a_2 + c_2 b_2)$$

QED.

Indeed, under the conditions of Lemma 6.12,

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} a_i + c_i b_i \\ b_i \end{bmatrix} = \begin{bmatrix} a_i + c_i b_i \\ b_i + d_i(a_i + c_i b_i) \end{bmatrix}$$

by (MS1) for $i = 1, 2$, hence

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} (a_1 + c_1 b_1)(a_2 + c_2 b_2) \\ b_i + d_i(a_i + c_i b_i) \end{bmatrix}$$

by (MS2) for $i = 1, 2$.

Example 6.13. [B2, p.338,714] Let $A = \mathbf{R}[x, y]/(x^2 + y^2 - 1)\mathbf{R}[x, y]$, the ring of polynomial functions on the cicle $x^2 + y^2 = 1$. Then A is a Dedikind ring (so $\mathrm{sr}(A) = 2$) and $\mathrm{SK}_1 A = \mathbf{Z}/2\mathbf{Z}$. This is shown in [B2] using the Mennicke symbols.

It is not clear whether $\tilde{\mathrm{E}}_2 A = \mathrm{E}_2 A$ for this A. However if we enlage A to the ring A' of all real-valued continuous (or smooth) functions, then A' is not Noetherian but $\mathrm{sr}(A') = 2$ and as mentioned in Section 3,

$$\mathrm{SK}_1 A' = \mathrm{SL}_n A')/\mathrm{E}_n A' = \pi_1(\mathrm{SL}_n \mathbf{R}) = \mathbf{Z}/2\mathbf{Z} \text{ for } n \geq 3,$$

while

$$\mathrm{SL}_2 A'/\mathrm{E}_2 A' = \pi_1(\mathrm{SL}_2 \mathbf{R}) = \mathbf{Z}$$

so $\tilde{\mathrm{E}}_2 A' \neq \mathrm{E}_2 A'$.

Note that for any subring A with 1 in any global field, $\mathrm{sr}(A) \leq 2$ and every nonzero ideal B of A is of finite index. Therefore, the principal convergence subgroups $\mathrm{SL}_n B$ with $B \neq 0$ are normal subgroups of finite index in $\mathrm{SL}_n A$ $(n \geq 2)$.

Moreover, it is not difficult to show that for $n \geq 3$ the subgroup $\mathrm{E}_n(A, B)$ with $B \neq 0$ are also normal subgroups of finite index and that every subgroup H of finite index in $\mathrm{SL}_n A$ contains a normal subgroup of finite index and hence some $\mathrm{E}_n(A, B)$ with $B \neq 0$. Therefore triviality of $\mathrm{SL}_n B/\mathrm{E}_n(A, B)$ for all B is equivalent to the "yes" answer to the following *congruence subgroup problem:* Does every subgroup of finite index in $\mathrm{SL}_n A$ contains $\mathrm{SL}_n B$ for some $B \neq 0$? Thus, Bass-Milnor-Serre solved the congruence subgroup problem for $\mathrm{SL}_n A$, $n \geq 3$. The answer is negative when A is the integers in a totally imaginary number field and positive otherwise (the later case was also done by Mennicke).

Later Serre solved the problem for $\mathrm{SL}_2 A$, and Vaserstein showed that $\mathrm{E}_2(A, B) = \tilde{\mathrm{E}}_2(A, B)$ for all B if $\mathrm{GL}_1 A$ is infinite. If $\mathrm{GL}_1 A$ is finite, then $\mathrm{E}_2(A, B)$ has infinite index in $\mathrm{SL}_2 A$ for sufficiently small nonzero ideal B of A. In characteristic 0, the total number of exceptions is finite; in positive characteristic, the exceptions are $B = A = F[x]$ with finite field F.

Geometric insight. Let use connect by an edge every two matrices which can be obtained from each other by a row addition operation. So we obtain a graph. The group $\mathrm{E}_n A$ is the connected component of 1_n in $\mathrm{GL}_n A$. Stabilization means any two matrices in $\mathrm{GL}_n A$ which can be connected in $\mathrm{GL}_{n+1} A$, can be connected in $\mathrm{GL}_n A$. Proofs involve "defermations" of paths. Prestabilization means that any two matrices in $\mathrm{GL}_n A$ which can be connected in $\mathrm{GL}_{n+k} A$, can be connected by short (four or five edges) segments, each segment starts and finish in $\mathrm{GL}_n A$. Proofs also involve "deformations" of paths.

We conclude this section with the observation that the condition (6.6) by itself implies that the group $\tilde{\mathrm{E}}_n(A, B)$ defined in Theorem 6.8 coincides with the mixed commutator subgroup $[\mathrm{GL}_1 A, \mathrm{GL}_n B]$ for every ideal B of A and every $n \geq 1$.

Here is a relative version of (6.6):

(6.14) for every $x \in B, y \in A$ there is $a \in \mathrm{GL}_1 A$ such that $1 + ax, a - y \in \mathrm{GL}_1 A$.

In the case $B = A$ the condition (6.14) coincides with the condition (6.6).

Theorem 6.15. Let A be an associative ring with 1, B an ideal of A. Assume the condition (6.14). Then $\tilde{\mathrm{E}}_n(A, B) = [\mathrm{GE}_n A, \mathrm{GL}_n B]$ for all $n \geq 1$.

Proof. Recall that $\mathrm{GE}_n A = \mathrm{GL}_1 A\ \mathrm{E}_n A$ is generated by elementary and diagonal matrices. Let $X = \begin{pmatrix} x & u \\ v & d \end{pmatrix} \in \mathrm{M}_n B$, where $x \in B, d \in \mathrm{M}_{n-1} B$. In the case $n = 1$, $X = x$ and d, u, v are missing.

Let $y \in A$. Set $Y = \begin{pmatrix} y & 0 \\ 0 & 1_{n-1} \end{pmatrix}$. In the case $n = 1$, $Y = y$

Set

$$\alpha_1 = 1_n + YX = \begin{pmatrix} 1 + yx & yu \\ v & d \end{pmatrix} \in 1_n + \mathrm{M}_n B$$

and

$$\alpha_4 = 1_n + XY = \begin{pmatrix} 1+xy & u \\ vy & d \end{pmatrix} \in 1_n + \mathrm{M}_n B.$$

Assume that $\alpha_4 = 1_n + XY \in \mathrm{GL}_n A$, hence $\alpha_1 = 1_n + XY \in \mathrm{GL}_n B$. We have to prove that $\alpha_4\alpha_1^{-1} \in [\mathrm{GL}_1 A,\ \mathrm{GL}_n B]$.

In the case $y \in \mathrm{GL}_1 A$, this is clear: $(1_n + XY)(1_n + YX)^{-1} = [1_n + XY, Y]$. In general case, we use the condition (6.9). We apply (6.9) to x, y above. When $n \geq 2$, these $x \in B, y \in A$ are arbitrary (e.g., when $n = 2$, take $u = x, v = -yx, d = -yx$). When $n = 1$, there is a condition on x, y, namely, $1 + xy \in \mathrm{GL}_1 A$, so the condition (6.14) can be relaxed.

By (6.14), there is $a \in \mathrm{GL}_1 A$ such taht $1 + ax, a - y \in \mathrm{GL}_1 A$.

Set $\varepsilon_1 = \begin{pmatrix} 1 & -(1+ax)^{-1}au \\ 0 & 1_{n-1} \end{pmatrix}$. By (6.9), there are $\delta_1, \delta_2 \in \mathrm{GL}_1 A$ such that $\delta_1 + \delta_2 = -(1+ax)^{-1}a$. Then

$$\varepsilon_1 = [\delta_1, \begin{pmatrix} 1 & u \\ 0 & 1_{n-1} \end{pmatrix}][\delta_2, \begin{pmatrix} 1 & u \\ 0 & 1_{n-1} \end{pmatrix}] \in [\mathrm{GL}_1 A, \mathrm{GL}_n B].$$

Set $\alpha_2 = \alpha_1\varepsilon_1 = \begin{pmatrix} 1+yx & (y-a)(1+xa)^{-1}u \\ v & d - v(1+xa)^{-1}au \end{pmatrix}$ We used that

$$y - (1+yx)(1+ax)^{-1}a = y(1+xa)(1+xa)^{-1} - (1+yx)a(1+xa)^{-1}$$
$$= (y-a)(1+xa)^{-1}.$$

Set

$$\alpha_3 = \begin{pmatrix} (1+xa)(y-a)^{-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix} \alpha_2 \begin{pmatrix} (1+ax)^{-1}(y-a) & 0 \\ 0 & 1_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1+xy & u \\ -v(1+ax)^{-1}(a-y) & d - v(1+ax)^{-1}au \end{pmatrix} = \alpha_2\varepsilon_2,$$

where

$$\varepsilon_2 = [\alpha_2^{-1}, (1+xa)(y-a)^{-1}](1+xa)(y-a)^{-1}(1+ax)^{-1}(y-a)$$
$$= [\alpha_2^{-1}, (1+xa)(y-a)^{-1}][1+xa, a][1+xa, (y-a)^{-1}] \in [\mathrm{GL}_1 A, \mathrm{GL}_n B].$$

We used that $x(1+ax)^{-1} = (1+xa)^{-1}x$ and

$$(a-y)^{-1}(1+yx)(1+ax)^{-1}(a-y) = 1 - x(1+ax)^{-1}(a-y),$$

hence

$$(1+xa)(a-y)^{-1}(1+yx)(1+ax)^{-1}(a-y) = 1 + xy.$$

Finally, set $\varepsilon_3 = \begin{pmatrix} 1 & 0 \\ -v(1+ax)^{-1}a & 1_{n-1} \end{pmatrix} = [\begin{pmatrix} 1 & 0 \\ v & 1_{n-1} \end{pmatrix}, \delta_1][\begin{pmatrix} 1 & 0 \\ v & 1_{n-1} \end{pmatrix}, \delta_2] \in [\mathrm{GL}_1 A, \mathrm{GL}_n B]$. Then $\varepsilon_3\alpha_3 = \alpha_4$.

Thus, $\alpha_4\alpha_1^{-1} = \varepsilon_3\alpha_1\varepsilon_1\varepsilon_2\alpha_1^{-1} \in [\mathrm{GE}_n A, \mathrm{GL}_n B]$. QED.

Problems.

1. Prove that the gropup $\mathrm{K}_1(A,B)/\mathrm{wh}(\mathrm{GL}_2(B)$ depends only on the ring B.

2. In the case $BB = 0$, show that $\mathrm{GL}_1 B$ is isomorphic to the additive group B while for $n \geq 2$,

$\mathrm{GL}_n B/\mathrm{E}_n(A,B) = \mathrm{K}_1(A,B)$

is the additive group B modulo all $ab - ba$ with $a \in A, b \in B$.

3. Show by an example that $\mathrm{K}_1(A,B)$ may depend on A.

4. (Kervaire) Show that the Mennicke symbol has the following property:

$\begin{bmatrix} a \\ bq \end{bmatrix} = \begin{bmatrix} b \\ aq \end{bmatrix}$ for any $a-1, b-1 \in B$, $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathrm{Um}_2 A$ and any $q \in B$.

5. Show that if A is commutative and $\mathrm{sr}(A) = 2$, then for any $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \in \mathrm{Um}_2 A$ there are $x, x', y, y' \in A$ such that $a + x(b + ya) = a' + x'(b' + y'a')$.

6. Let A be an associate ring with 1. Define $\tilde{\mathrm{E}}_n A$ to be a subgroup of $\mathrm{GL}_n A$ generated by all matrices of the form $(1_n + XY)(1_n + YX)^{-1}$ with

$$X \in \mathrm{M}_n A,\ Y = \begin{pmatrix} y & 0 \\ 0 & 1_{n-1} \end{pmatrix},\ y \in A, 1_n + XY \in \mathrm{GL}_n A.$$

Show that:

(a) $\tilde{\mathrm{E}}_n A$ contains $\mathrm{E}_n A$;

(b) $\tilde{\mathrm{E}}_n A$ contains every matrix of the form $1_n + vu$ where $v \in A^n$ is a column, u is an n-row, and $uv = 0$;

(c) $\tilde{\mathrm{E}}_n A = \tilde{\mathrm{E}}_n(A,A)$;

(d) $\tilde{\mathrm{E}}_n A \subset \mathrm{E}_{n+1} A$.

7. Let $\alpha = (\alpha_{i,j}) \in \mathrm{GL}_{n+1} A$ and $\beta = \alpha^{-1} = (\beta_{i,j}) \in \mathrm{GL}_{n+1} A$. Show that

$\sum_{i+2}^{n+1} A\alpha_{i,1} = A$

if and only if

$\sum_{i+2}^{n+1} A\beta_{i,1} = A$.

When A is commutative, the following stronger statement is true:

$$\sum_{i+2}^{n+1} A\alpha_{i,1} = \sum_{i+2}^{n+1} A\beta_{i,1}.$$

8. Show that the following two statements are equivalent:

(i) $\mathrm{sr}(A) \leq n$,

(ii) for every matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_{n+1} A$ (whre $a \in A, c \in A^n$) there is $v \in A^n$ such that $c + dv$ is unimodular.