The ranks and cranks of partitions moduli 2, 3 and 4.

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ABSTRACT: *We discuss inequalities between the rank counts, N(r,m,n), and between the crank counts, M(r,m,n), for m = 2, 3 and 4 and state three conjectures.*

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§1 **Introduction**

A partition $\pi = (\pi_0, \pi_1, \cdots, \pi_{k-1})$ is a finite, (weakly) descending sequence of positive integers (the *parts* of π). Thus π_0 is the largest part of π. $\#\pi = k$, is the *length* of π and $\mathrm{w}(\pi) = \pi_0 + \pi_1 + \cdots + \pi_{k-1}$ is the *weight* of π. If $\mathrm{w}(\pi) = n$, π is a partition *of* n. In 1944 Dyson [5] defined the *rank* of a partition, π, by

$$\mathrm{rank}(\pi) := \pi_0 - \#\pi$$

and set

$$N(m, n) := \#\{\pi : \mathrm{w}(\pi) = n, \mathrm{rank}(\pi) = m\}$$

$$N(r,m,n) := \#\{\pi : \mathrm{w}(\pi) = n, \mathrm{rank}(\pi) \equiv r \bmod m\}.$$

Noting that $\mathrm{rank}(\pi) = -\mathrm{rank}(\overline{\pi})$ (where $\overline{\pi}$ denotes the *conjugate* [1, p.7] of π), it follows that

$$N(m,n) = N(-m,n) \text{ and } N(r,m,n) = N(-r,m,n).$$

Dyson observed that several relations appeared to hold among the $N(r,m,n)$, when $m = 5$ and 7 and his observations; these were shown to be universally valid by Atkin and Swinnerton-Dyer [4]. Some 35 years later, Garvan defined the crank for certain *vector partitions* and he and Andrews subsequently defined

$$\mathrm{crank}(\pi) := \begin{cases} \pi_0, & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0, \end{cases}$$

where $\mu(\pi)$ denote the number of ones in π and $\nu(\pi)$ denotes the number of parts of π larger than $\mu(\pi)$. Following Dyson's suggestion [5], they set, for $n > 1$,

$$M(m,n) = \#\{\pi : \mathrm{w}(\pi) = n, \mathrm{crank}(\pi) = m\}$$

$$M(r,m,n) = \#\{\pi : \mathrm{w}(\pi) = n, \mathrm{crank}(\pi) \equiv r \bmod m\}.$$

We suppose the rank and the crank of the empty partition of 0 are each 0 and that

$$M(1,1) = M(-1,1) = 1, M(0,1) = -1 \text{ and } M(m,1) = 0, m \neq \pm 1, 0.$$

So the numbers $M(m,n)$ are the numbers $N_V(m,n)$ defined by Garvan [7,8,9].

We take z and q to be complex variables with $z \neq 0$ and $|q| < 1$ and we will use the familiar notation:

$$(z;q)_n := \prod_{k=0}^{n-1}(1 - zq^k),$$
$$(z;q)_\infty := \prod_{k=0}^{\infty}(1 - zq^k).$$

For future reference, we note that

$$\frac{1}{(-q;q)_{2n}} = \frac{(q;q^2)_n}{(q^{2n+2};q^2)_n} \tag{1.1}$$

and

$$\frac{1}{(-q;q)_{2n+1}} = \frac{(q;q^2)_{n+1}}{(q^{2n+2};q^2)_{n+1}} \tag{1.2}$$

It is not difficult to see that the generating function of the numbers $N(m,n)$ is given by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(zq;q)_k(z^{-1}q;q)_k} \tag{1.3}$$
$$= 1 + \sum_{k=1}^{\infty} \frac{z^{k-1}q^k}{(z^{-1}q;q)_k} \tag{1.4}$$

and we also have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \tag{1.5}$$

In (1.3), k marks the size of the Durfee square [1, pp.27,28] of a partition and, in the alternative expression (1.4), k is the size of the largest part. The generating function for the crank (1.5) was given by Garvan [7,8,9]

It is shown in [10] that

$$N(0,2,2n) < N(1,2,2n) \text{ if } n \geq 1 \quad \text{and} \quad N(1,2,2n+1) < N(0,2,2n+1) \quad \text{ if } n \geq 0. \tag{1.6}$$

The proof given in [10] of (1.6) is combinatorial (bijective) in nature and consists of the construction of maps

$$\{\text{partitions of } 2n \text{ of even rank}\} \to \{\text{partitions of } 2n \text{ of odd rank}\}$$

$$\{\text{partitions of } 2n+1 \text{ of odd rank}\} \to \{\text{partitions of } 2n+1 \text{ of even rank}\}$$

that are injective, but not surjective.

Setting $z = -1$ in (1.3), we see that

$$\sum_{n=0}^{\infty} \big(N(0,2,n) - N(1,2,n)\big) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} =: f(q),$$

where $f(q)$ is one of the third-order mock theta functions [11]. Thus (1.6) is the statement that the signs of the coefficients of $f(q)$ are $+,+,-,+,-,\ldots$ (alternating thereafter), or, equivalently, that the signs of the coefficients in $f(-q)$ are $+,-,-,-,\ldots$ (and thereafter all negative).

In fact, (1.6) has a straightforward algebraic derivation which we include, since it foreshadows our later arguments. Setting $z = -1$ in (1.4), we have

$$f(q) = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^k}{(-q;q)_k} \tag{1.7}$$

and so

$$f(-q)=1-\sum_{k=1}^{\infty}\frac{q^k}{(q;-q)_k}$$

which, by (1.1) and (1.2),

$$=1-\left\{\sum_{k=1}^{\infty}\frac{q^{2k}(-q;q^2)_k}{(q^{2k+2};q^2)_k}+\sum_{k=1}^{\infty}\frac{q^{2k-1}(-q;q^2)_k}{(q^{2k-2};q^2)_k}\right\}.$$

The coefficients of the terms of each sum in the brackets are clearly positive and this settles (1.6). $\square$

A number of inequalities between the $N(r,m,n)$ and between the $M(r,m,n)$ were found by Garvan [7,8,9] when $m=5,7$ and Ekin [6] gave some inequalities between the $M(r,11,n)$. Here we establish some inequalities between the $M(r,m,n)$ and between the $N(r,m,n)$ when $m=2,3$ and 4. We also state a number of conjectures.

§2 m = 2

The numbers $M(r,2,n)$ satisfy inequalities that are the reverse of those for the rank (1.6). We prove

Theorem 1. *For all $n\geq 0$,*

$$M(0,2,2n)>M(1,2,2n)$$

$$M(1,2,2n+1)>M(0,2,2n+1).$$

Proof.

By (1.5), we have

$$\sum_{n=0}^{\infty}\big(M(0,2,n)-M(1,2,n)\big)q^n = \frac{(q;q)_\infty}{(-q;q)_\infty^2} =: g(q), \tag{2.1}$$

say, and we want to show that the coefficient of q^n in $g(q)$ is positive/negative according as n is even/odd. So we need to show that the coefficients of $g(-q)$ are all positive. But

$$\begin{aligned} g(-q) &= \frac{(-q;-q)_\infty}{(q;-q)_\infty^2} \\ &= (-q;q^2)_\infty^3(q^2;q^2)_\infty \end{aligned}$$

which, by Jacobi's Triple Product Identity,

$$= (-q;q^2)_\infty \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Since every positive integer is the sum of a perfect square and an odd number, the coefficients of $g(-q)$ are all positive. $\square$

§3 m = 3

We have no solid facts about the case $m = 3$ and merely present two conjectures. We first note that, setting $z = e^{2\pi i/3}$ in (1.3) gives

$$\begin{aligned} \sum_{n\geq 0}\Big(N(0,3,n)-N(1,3,n)\Big)q^n &= \sum_{n\geq 0}\frac{q^{n^2}}{(1+q+q^2)\dots(1+q^n+q^{2n})} \\ &= \sum_{n\geq 0}\frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} =: \gamma(q), \end{aligned}$$

where $\gamma(q)$ is one of the sixth-order Mock Theta functions [3]. Also, setting $z = e^{2\pi i/3}$ in (1.2) we have

$$\sum_{n\geq 0}\Big(M(0,3,n) - M(1,3,n)\Big)q^n = \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}.$$

Computer evidence suggests the following:

Conjecture 1. *For all $n > 0$*

$$N(0,3,3n) < N(1,3,3n), \tag{3.1}$$

$$N(0,3,3n+1) > N(1,3,3n+1), \tag{3.2}$$

$$N(0,3,3n+2) < N(1,3,3n+2). \tag{3.3}$$

Conjecture 2. *For all n,*

$$M(0,3,3n) > M(1,3,3n), \tag{3.4}$$

$$M(0,3,3n+1) < M(1,3,3n+1), \tag{3.5}$$

$$M(0,3,3n+2) \leq M(1,3,3n+2), \quad \textit{if } n \neq 1, \tag{3.6}$$

with strict inequality in (3.6), if $n \neq 4, 5$.

These Conjectures, Conjecture 2, in particular, seem to be related to the Borwein conjectures [2]. We have no proofs of any one of (3.1)-(3.6).

§4 m = 4

Setting $z = i$ in (1.3) gives

$$\sum_{n=0}^{\infty}\Big(N(0,4,n) - N(2,4,n)\Big)q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n} =: \phi(q), \tag{4.1}$$

which is one of the third-order mock theta functions [11]. We will prove

Theorem 2.

$$N(0,4,n) = N(2,4,n),\ \textit{for } n = 2, 8, 10 \textit{ and } 26,\ \textit{while, for other } n, \tag{4.2}$$

$$N(0,4,n) > N(2,4,n),\ \textit{if } n \equiv 0, 1 \bmod 4, \tag{4.3}$$

$$N(0,4,n) < N(2,4,n),\ \textit{if } n \equiv 2, 3 \bmod 4. \tag{4.4}$$

Proof. Set $\alpha(n) := N(0,4,n) - N(2,4,n)$. Then, with $\phi(q) = \sum_{n=0}^{\infty}\alpha(n)q^n$, we will show that

$$\alpha(n) = \begin{cases} 0, & n = 2, 8, 10, 26, \\ > 0, & n \equiv 0, 1 \bmod 4, \quad n \neq 8, \\ < 0, & n \equiv 2, 3 \bmod 4, \quad n \neq 2, 10, 26. \end{cases}$$

We first note, by expanding the series for $\phi(q)$, that $\alpha(n) = 0$ for $n = 2, 8, 10, 26$, thus verifying (4.2).

The q-binomial Theorem [1; Theorem 3.3, p.36] states that

$$(z;q)_n = \sum_{i=0}^{n}(-1)^i z^i q^{i(i-1)/2}\begin{bmatrix} n \\ i \end{bmatrix}$$

and so we have

$$
\begin{aligned}
\phi(q) &= \sum_{n=0}^{\infty} \frac{q^{4n^2}(q^2;q^4)_n}{(q^{4n+4};q^4)_n} + \sum_{n=0}^{\infty} \frac{q^{4n^2+4n+1}(q^2;q^4)_{n+1}}{(q^{4n+4};q^4)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^{4n+4};q^4)_n} \sum_{j=0}^{n} (-1)^j q^{2j^2} \begin{bmatrix} n \\ j \end{bmatrix}_{q^4} + \\
&\quad + \sum_{n=0}^{\infty} \frac{q^{4n^2+4n+1}}{(q^{4n+4};q^4)_{n+1}} \sum_{j=0}^{n+1} (-1)^j q^{2j^2} \begin{bmatrix} n+1 \\ j \end{bmatrix}_{q^4}. \qquad (4.5)
\end{aligned}
$$

But the coefficients of $\begin{bmatrix} n \\ j \end{bmatrix}$ are nonnegative (since $\begin{bmatrix} n \\ j \end{bmatrix}$ is the generating function for partitions into $n-j$ or fewer parts all no bigger than j) and (4.5) shows that $\alpha(m) \geq 0$ (≤ 0), when $m \equiv 0, 1 \bmod 4$ ($\equiv 2, 3 \bmod 4$).

Now the first few terms of $\phi(q)$ are:

$$
\begin{aligned}
1 + \frac{q(1-q^2)}{1-q^4} + \frac{q^4(1-q^2)}{1-q^8} + \frac{q^9(1-q^2)(1-q^6)}{(1-q^8)(1-q^{12})} + \frac{q^{16}(1-q^2)(1-q^6)}{(1-q^{12})(1-q^{16})} \\
+ \frac{q^{25}(1-q^2)(1-q^6)(1-q^{10})}{(1-q^{12})(1-q^{16})(1-q^{20})} + \frac{q^{36}(1-q^2)(1-q^6)(1-q^{10})}{(1-q^{16})(1-q^{20})(1-q^{24})}.
\end{aligned}
$$

We see that the term $\frac{q(1-q^2)}{1-q^4}$ guarantees that $\alpha(m) > 0$, if $m \equiv 1 \bmod 4$, and $\alpha(m) < 0$, if $m \equiv 3 \bmod 4$. The term $\frac{q^4(1-q^2)}{1-q^8}$ means that $\alpha(m) > 0$ if $m \equiv 4 \bmod 8$ and $\frac{q^{16}(1-q^2)(1-q^6)}{(1-q^{12})(1-q^{16})}$ means that $\alpha(m) > 0$ if $m \equiv 0 \bmod 8$ and $m \neq 8$. Hence $\alpha(m) > 0$ if $m \equiv 0 \bmod 4$ and $m \neq 8$. Finally, the term $\frac{q^4(1-q^2)}{1-q^8}$ guarantees $\alpha(m) < 0$ if $m \equiv 6 \bmod 8$, the term $\frac{q^{16}(1-q^2)(1-q^6)}{(1-q^{12})(1-q^{16})}$ guarantees $a(m) < 0$ if $m \equiv 2 \bmod 16$ and $m \geq 18$ and the term $\frac{q^{36}(1-q^2)(1-q^6)(1-q^{10})}{(1-q^{16})(1-q^{20})(1-q^{24})}$ guarantees $a(m) < 0$ if $m \equiv 10 \bmod 16$ and $m \geq 42$. So $a(m) < 0$ if $m \equiv 2 \bmod 4$ and $m \neq 2, 10, 26$. This completes the proofs of (4.3) and (4.4). $\square$

Setting $z = i$ in (1.5) we have

$$\sum_{n=0}^{\infty}\Big(M(0,4,n) - M(2,4,n)\Big)q^n = \frac{(q;q)_\infty}{(-q^2;q^2)_\infty} = \frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^4;q^4)_\infty}. \qquad (4.6)$$

Again, there seem to be inequalities among the $M(0,4,n)$ and $M(2,4,n)$ that are periodic mod 4 and computer evidence suggests

Conjecture 3. *For $n \neq 5$*

$$M(0,4,n) \geq M(2,4,n), \textit{ if } n \equiv 0,3 \bmod 4, \qquad (4.7)$$

$$M(0,4,n) \leq M(2,4,n), \textit{ if } n \equiv 1,2 \bmod 4 \qquad (4.8)$$

the inequalities being strict if $n \neq 11, 15, 21$. We have no proof of either (4.7) or (4.8). (In fact, $M(0,4,5) - M(2,4,5) = 1$, which suggests that this conjecture, if true, may be hard to prove.)

Now we have, by (2.1),

$$\sum_{n=0}^{\infty}(M(0,4,n) + M(2,4,n) - 2M(1,4,n))q^n = \sum_{n=0}^{\infty}(M(0,2,n) - M(1,2,n))q^n = \frac{(q;q)_\infty}{(-q;q)_\infty^2}$$

and, with (4.6), we have

$$\begin{aligned} 2\sum_{n=0}^{\infty}(M(0,4,n) - M(1,4,n))q^n &= \sum_{n=0}^{\infty}\Big(M(0,4,n) + M(2,4,n) - 2M(1,4,n)\Big) + \Big(M(0,4,n) - M(2,4,n)\Big) \\ &= (q;q)_\infty\left\{\frac{1}{(-q;q)_\infty^2} + \frac{1}{(-q^2;q^2)_\infty}\right\} =: \alpha(q), \end{aligned}$$

say. Now

$$
\begin{aligned}
\alpha(-q) &= (-q;-q)_\infty \left\{ \frac{1}{(q;-q)_\infty^2} + \frac{1}{(-q^2;q^2)_\infty} \right\} \\
&= (-q;q^2)_\infty (q^2;q^2)_\infty \Big\{ (-q;q^2)_\infty^2 + (q^2;q^4)_\infty \Big\} \\
&= (-q;q^2)_\infty^2 (q^2;q^2)_\infty \Big\{ (-q;q^2)_\infty + (q;q^2)_\infty \Big\}
\end{aligned}
$$

But

$$
(-q;q^2)_\infty^2 (q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2}
$$

has non-negative coefficients and

$$
(-q;q^2)_\infty + (q;q^2)_\infty = 2\sum_{n=0}^{\infty} a(n) q^n,
$$

where $a(n)$ is the number of partitions of n into an even number of different odd numbers (taking $a(0) = 1$) . Thus

$$
\alpha(-q) = \sum_{n=-\infty}^{\infty} q^{n^2} \sum_{n=0}^{\infty} a(n) q^n = (1 + 2q + 2q^4 + \ldots)(1 + q^4 + q^6 + 2q^8 + \ldots)
$$

has non-negative coefficients. It is easy to see that $a(n) > 0$ for even $n > 2$ and it follows that the coefficients of q^n in $\alpha(-q)$ are positive for $n > 3$.

In just the same way, we see that, if

$$
\beta(q) := \sum_{n=0}^{\infty} \Big(M(2,4,n) - M(1,4,n) \Big) q^n = \frac{1}{2} (q;q)_\infty \left\{ \frac{1}{(-q;q)_\infty^2} - \frac{1}{(-q^2;q^2)_\infty} \right\},
$$

then

$$\beta(-q) = \frac{1}{2}(-q;q^2)_\infty^2(q^2;q^2)_\infty\left\{(-q;q^2)_\infty - (q;q^2)_\infty\right\} = \sum_{n=-\infty}^{\infty} q^{n^2}\sum_{n=0}^{\infty} b(n)q^n,$$

where $b(n)$ is the number of partitions of n into an odd number of distinct odd parts ($b(0) = 0$). We see that the coefficients of q^n in $\beta(-q)$ are positive for $n > 0$ and we have proved

Theorem 3.

(i) $M(0,4,2n) > M(1,4,2n)$, *for* $n \neq 1$,

(ii) $M(0,4,2n-1) < M(1,4,2n-1)$, *for* $n \neq 2$,

(iii) $M(2,4,2n) > M(1,4,2n)$, *for* $n > 0$,

(iv) $M(2,4,2n-1) < M(1,4,2n-1)$, *for* $n > 0$. □

If $f(q) = \sum_{n=0}^{\infty} a_n q^n$ and $g(q) = \sum_{n=0}^{\infty} b_n q^n$ are power series in q, we write $f(q) \preceq g(q)$ to mean $a_n \leq b_n$ for all n. We now prove

Theorem 4.

$$N(0,4,2n) < N(1,4,2n), (\textit{for all } n \geq 1), \tag{4.9}$$

$$N(0,4,2n-1) > N(1,4,2n-1), (\textit{for all } n \geq 1), \tag{4.10}$$

$$N(2,4,2n) < N(1,4,2n), (\textit{for all } n \geq 1), \tag{4.11}$$

$$N(2,4,2n-1) > N(1,4,2n-1), (\textit{for all } n \geq 2). \tag{4.12}$$

Proof.

We note first that

$$1+\sum_{k=0}^{\infty} q^{2k+1}(-q;q^2)_k = (-q;q^2)_\infty = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q^2;q^2)_k}, \tag{4.13}$$

since each of these expressions is the generating function of partitions into distinct odd parts.

We have, by (1.7) and (4.1),

$$\begin{aligned}
2\sum_{n=1}^{\infty}(N(0,4,n)-N(1,4,n))q^n &= \sum_{n=1}^{\infty}(N(0,4,n)+N(2,4,n)-2N(1,4,n))q^n + \sum_{n=1}^{\infty}(N(0,4,n)-N(2,4,n))q^n \\
&= \sum_{n=1}^{\infty}(N(0,2,n)-N(1,2,n))q^n + \sum_{n=1}^{\infty}(N(0,4,n)-N(2,4,n))q^n \\
&= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^k}{(-q;q)_k} + \sum_{k=1}^{\infty}\frac{q^{k^2}}{(-q^2;q^2)_k} \\
&= f_1(q)+\phi_1(q),
\end{aligned}$$

say (where we have written $f_1(q)$ and $\phi_1(q)$ for $f(q)-1$ and $\phi(q)-1$, respectively). To prove (4.9) and (4.10) we must show that the coefficients of $f_1(-q)+\phi_1(-q)$ are negative for $n \geq 1$.

Now

$$\phi_1(-q) = \sum_{k=1}^{\infty}(-1)^k\frac{q^{k^2}}{(-q^2;q^2)_k}$$

$$\preceq \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q^2;q^2)_k}$$
$$= -1 + (-q;q^2)_\infty$$

by (4.13), and

$$\begin{aligned}
f_1(-q) &= -\sum_{k=1}^{\infty} \frac{q^k}{(q;-q)_k} \\
&= -\left(\sum_{k=1}^{\infty} \frac{q^{2k-1}}{(q;-q)_{2k-1}} + \sum_{k=1}^{\infty} \frac{q^{2k}}{(q;-q)_{2k}}\right)
\end{aligned}$$

which, by (1.1) and (1.2),

$$\begin{aligned}
&= -\left(\sum_{k=1}^{\infty} q^{2k-1} \frac{(-q;q^2)_k}{(q^{2k};q^2)_k} + \sum_{k=1}^{\infty} q^{2k} \frac{(-q;q^2)_k}{(q^{2k+2};q^2)_k}\right) \\
&\preceq -\left(\sum_{k=1}^{\infty} q^{2k-1}(-q;q^2)_{k-1} + \sum_{k=1}^{\infty} q^{2k}(1+q)\right) \\
&= 1 - (-q;q^2)_\infty - \sum_{k=1}^{\infty} q^{2k}(1+q),
\end{aligned}$$

by (4.13). So $f_1(-q) + \phi_1(-q) \preceq -\sum_{k\geq 2} q^k$, showing that (4.9) and (4.10) hold for $n \geq 2$. But $N(0,4,1) = 1$ and $N(1,4,1) = 0$, which completes the proofs of (4.9) and (4.10). (4.11) and (4.12) may be proved in the same way, using

$$2\sum_{n=1}^{\infty}\Big(N(2,4,n) - N(1,4,n)\Big)q^n = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^k}{(-q;q)_k} - \sum_{k=1}^{\infty}\frac{q^{k^2}}{(-q^2;q^2)_k}. \qquad \square$$

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