SIMILARITY OF MATRICES OVER A COMMUTATIVE ARTINIAN LOCAL RING

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INTRODUCTION

Let R be a commutative artinian local ring with identity e and maximal ideal $J(R) = J$, R_m the ring of $m \times m$ matrices over R, R_m^* its multiplicative group. If the matrices $A, B \in R_m$ are similar, i.e., if there exists a matrix $t \in R_m^*$ such that $T^{-1}AT = B$, we shall write $A \approx B$. The main problem considered in this paper is:

Problem 1. Given two matrices $A, B \in R_m$, determine whether or not they are similar.

The solution of this problem would also provide a solution of the corresponding problem over any commutative artinian ring with identity since such a ring is a direct sum of local rings. However, the given problem is very hard and far from a solution. Equally far from a solution are the following two problems which are connected with the first one and which aim at the construction of canonical representatives for the classes into which the ring R_m splits under the relation $\approx$.

Problem 2. Given a matrix $A \in R_m$, determine if it is similar to a reducible matrix $\mathrm{Diag}\,(A_1, A_2)$.

Assume that $F(x) \in R[x]$ is a monic (i.e., the highest coefficient is e) polynomial: $F(x) = x^m - c_{m-1}x^{m-1} - \ldots - c_0$. The companion matrix of F(x) is the matrix

$$S(F) = \begin{pmatrix} 0 & e & 0 \ldots 0 \\ \cdots & & \cdots\cdots \\ 0 & . & . \; . \; . \; . \; e \\ c_0 & . & . \; . \; . \; c_{m-1} \end{pmatrix}.$$

We shall say that the matrix $A \in R_m$ is normal if it is similar to a block-diagonal matrix $\mathrm{Diag}(S(F_1), \ldots, S(F_t))$, where the $F_i(x)$ are monic polynomials from R[x].

Problem 3. To determine whether a given matrix $A \in R_m$ is normal or not.

The known results connected with the solution of these problems can be divided into two groups. The first group contains results which highlight the complexity of the problems in particular classes of matrices in R_m by reducing them to other unsolved problems. Most frequently, problem 1 is reduced to the problem of solving a system of two similarity equations for matrices over a field. The latter task will for brevity be referred to as the matrix pair problem; it has for a long time been a reference point for unsolved algebraic problems, and has also been called the "wild" problem.

The second group contains results which allow the solution of the above problems in particular classes of matrices. For a given polynomial $F(x) \in R[x]$ we denote by $\mathcal{V}(F, R_m)$ the class of matrices $A \in R_m$ such that $F(A) = 0$. In [1] it is shown that if $n \geq 2$ problem 1 in the classes $\mathcal{V}(x^3, (\mathbf{Z}/p^n)_{3m})$ and $\mathcal{V}(x^{p^2} - e(\mathbf{Z}/p^n)_{4m})$ includes the matrix pair problem for $m \times m$ matrices over the field $\bar{R} = R/J$. In the same paper it is shown that the class of matrices $\in (\mathbf{Z}/p^n)_{4m}$ with $A^2 \equiv 0 \pmod p$ has a similar property. In [2] it is shown that if R is finite, char $R = 2^n$ and J(R) is not a principal ideal then problem 1 in the class $\mathcal{V}(x^2 - e, R_m)$ contains the matrix pair problem for $m \times m$ matrices over the field $\bar{R}$.

A large number of papers deal with problem 1 in the class $\mathcal{V}(x^2 - e, (\mathbf{Z}/2^n)_m)$. [2] contains a survey of the literature on this topic and gives the simplest solution of the problem on the basis of particular solutions of problems 2 and 3.

Let $a \in R$, $F(x) \in R[x]$, $A \in R_m$; then $\bar{a}$, $\bar{F}$, $\bar{A}$ denote respectively the images of these elements over the field $\bar{R}$ under the natural homomorphism $R \to \bar{R}$. The polynomial $F(x) \in R[x]$ is called a strong invariant for matrix similarity over R or simply a strong invariant, if for any two matrices $A, B \in \mathcal{V}(F(x), R_m)$ the condition $A \approx B$ is equivalent to $\bar{A} \approx \bar{B}$. In [3]

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