PARTITIONS WITH EARLY CONDITIONS

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Abstract. In an earlier paper, partitions in which the smaller parts were required to appear at least k-times were considered. Some of those results were tied up with Rogers-Ramanujan type identities and mock theta functions. By considering more general conditions on initial parts we are led to natural explanations of many more identities contained in Slater's compendium of 130 Rogers-Ramanujan identities.

1. Introduction

In 1886, J. J. Sylvester [15] posed a couple of problems in the Educational times that are precursors to the study undertaken here. We reproduce the problems in their entirety:

"Definition. If, in any arrangement of integers, each of the numbers 1,2,3,... up to any odd number (unity inclusive), say $2i-1$, occurs once or any odd number of times, but the even number following, say $2i$, does not occur any odd number of times, the arrangement is said to be flushed; if such kind of sequence does not occur, it is said to be unflushed.

"1. Requred to prove, that if any number be partitioned in every possible way, the number of unflushed partitions containing an odd number of parts is equal to the number of unflushed partitions containing an even number of parts.

"Ex.gr.: The total partitions of 7 are
7; 6, 1; 5,2; 5, 1, 1; 4, 3; 4, 2, 1; 4, 1, 1, 1; 3, 3, 1; 3, 2, 2; 3, 2, 1, 1; 2, 2, 2, 1; 3, 1, 1, 1, 1; 2, 2, 1, 1, 1; 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1.
Of these, 6, 1; 4, 1, 1, 1; 3, 3, 1; 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1 alone are flushed. Of the remaining unflushed partitions, five contain an odd number of parts, and five an even number.

"Again, the total partitions of 6 are
6; 5, 1; 4, 2; 4, 1, 1; 3, 3; 3, 2, 1; 2, 2, 2; 3, 1, 1, 1; 2, 2, 1, 1; 2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1; of which 5, 1; 3, 2, 1; 3, 1, 1, 1 alone are flushed. Of the remainder, four contain an odd and four an even number of parts.

"N.B.—This transcendental theorem compares singularly with the well-known algebraical one, that the total number of the permuted partitions of a number with an odd number of parts is equal to the same of the same with an even number.

"2. Required to prove that the same proposition holds when any odd number is partitioned without repetitions in every possible way."

Sylvester did not publish solutions to these problems. In 1970, solutions to both problems were published [1] and the generating function for flushed partitions

Partially supported by National Science Foundation Grant DMS-0801184.

(corrected) was revealed as

$$\frac{\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1-q^n)}{(q;q)_\infty},$$

where

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}).$$

The solutions of Sylvester's problems involved generating functions. It is completely unknown whether this was Sylvester's approach and how he came upon flushed partitions in the first place.

Sylvester's flushed partitions suggest a more extensive study of partitions subject to variations on the following three constraints which we shall call the *Sylvester constraints*:

(1) Some of the smaller parts are required to appear a specified number of times (e.g. in the case of flushed partitions, an odd number of times).
(2) Immediately following the parts considered in (1) there may be one or two special parts (e.g. in the case of flushed partitions, the first integer appearing an even number of times is even).
(3) The larger parts are constrained differently if at all (e.g. in the case of flushed partitions there are no constraints).

In the subsequent decades of the 20th century, N. J. Fine appears to have been the only one to consider questions of this type. In lectures at Penn State, he observed that the conjugates of partitions into distinct parts are "partitions without gaps," i.e. partitions in which every integer smaller than the largest part is also a part. For ex1067-Z1-1999ample, here are the partitions of 6 into distinct parts paired with their conjugates:

$$\begin{array}{rr} 6 & 1+1+1+1+1+1 \\ 5+1 & 2+1+1+1+1 \\ 4+2 & 2+2+1+1 \\ 3+2+1 & 3+2+1 \end{array}$$

Fine also noted in his book [7, p. 57] that one of Ramanujan's third order mock theta functions

$$\begin{aligned} \psi(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} \\ &= \sum_{n=0}^{\infty} \beta(n) q^n, \end{aligned}$$

where $\beta(n)$ is the number of partitions of n into *odd* parts where each odd integer smaller than the largest part must also be a part.

In 2009, the theme initiated by Sylvester was further developed in a paper titled "Partitions with initial repetitions" [5].

Definition. A partition with initial k-repetitions is a partition in which if any j appears at least k times as a part, then each positive integer less than j appears k times as a part.

As noted in [5, Th. 1], partitions with initial k-repetitions fit naturally into an expanded version of the Glaisher/Euler theorem [2, Cor 1.3, p.6].

Theorem 1. *The number of partitions of n with initial k-repetitions equals the number of partitions of n into parts not divisible by $2k$ and also equals the number of partitions of n in which no part is repeated more than $2k-1$ times.*

This idea was further developed in [5] and sets the stage for the results in this paper.

Definition. Let $F_e(n)$ (resp. $F_o(n)$) denote the number of partitions of n in which no odd (resp. no even) parts are repeated and no odd part (resp. even part) is smaller than a repeated even part (resp. odd part), and if an even (resp. odd) part is repeated then each smaller even (resp. odd) positive integer is also a repeated part.

Theorem 2. *$F_e(n)$ equals the number of partitions of n into parts $\not\equiv 0, \pm 2 \pmod 7$.*

This result follows immediately from the second Rogers-Selberg identity [14, p. 155, eq. (32)]

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(-q^{2n+1};q)_\infty}{(q^2;q^2)_n} = \prod_{\substack{n=1 \\ n\not\equiv 0 \pm 2 \ (\mathrm{mod}\ 7)}}^{\infty} \frac{1}{1-q^n}.$$

Theorem 3. $\sum_{n=0}^{\infty} F_o(n) = (-q;q)_\infty f(q^2)$, *where $f(q)$ is one of Ramanujan's seventh order mock theta functions* [12, p. 355]

$$f(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n;q)_n}.$$

Our object in this paper is to apply the Sylvester constraints to various other Rogers-Ramanujan type identities found by Slater [14], (cf. [12, Appendix A]). In each instance odds and evens will be subject to different restrictions. Interchanging the roles of odds and evens (as was done in passing from Theorem 2 to Theorem 3) has an interesting outcome. Sometimes mock theta functions (cf. [16]) arise (cf. (2.7), (2.8) and Section 4), and sometimes other Rogers-Ramanujan type identities arise (cf. Section 3).

In Section 2, we analyze two theorems that were originally found by F. H. Jackson and are listed as identities (38) and (39) in Slater [14]. In this case the exchange of the roles of odds and evens yields two of the mock theta functions listed in [6].

In Section 3, we begin with Slater's identity (119) [14, p. 165]. In this case, the reversed roles of odds and evens leads to a result equivalent to Slater's (81) [14, p. 160].

In Section 4, events take a surprising turn. We begin with Slater's (44) and (46) [14, p. 156]. Each of these makes condition (2) of the Sylvester constraints rather cumbersome. So the terms of the series in (44) and (46) are slightly altered to streamline condition (2). The result is new mock theta functions, and the odd-even reversal yields further mock theta functions.

Finally in Section 5, we start with Slater's (53). This requires us to move from odd-even (or modulus 2) conditions to modulus 4 conditions. In this case, the role reversal takes us from Slater's(53) to Slater's (55).

Section 6 is the conclusion where we discuss a variety of potential projects foreshadowed by this paper.

2. Identities of modulus 8

Of course, there are two famous modulus 8, Rogers-Ramanujan identities. They are due to Lucy Slater [14, eqs. (36), (34)]:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{\substack{n=1 \\ n\equiv 1,4,7 \pmod 8}}^{\infty} \frac{1}{1-q^n}, \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \prod_{\substack{n=1 \\ n\equiv 3,4,5 \pmod 8}}^{\infty} \frac{1}{1-q^n}. \tag{2.2}$$

Although Slater first obtained these results in her Ph.D. thesis in the late 1940's, they have become known as the Göllnitz-Gordon identities because in the early 1960's both H. Göllnitz [9] and B. Gordon [10] discovered their partition theoretic interpretation.

Interestingly F. H. Jackson [11] partitially found, and Slater [14, eqs. (39), (38)] re-found closely related results which we now consider in slightly altered form:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^{2n+1};q^2)_\infty}{(q^2;q^2)_n} = \prod_{\substack{n=1 \\ n\equiv 1,4,7 \pmod 8}}^{\infty} \frac{1}{1-q^n}, \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n} = \prod_{\substack{n=1 \\ n\equiv 3,4,5 \pmod 8}}^{\infty} \frac{1}{1-q^n}. \tag{2.4}$$

Let us rewrite these series in a form where the partition theoretic interpretation is obvious.

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{q^{2+2+4+4+\cdots+(2n-2)+(2n-2)+2n}(1+q^{2n+1})(1+q^{2n+3})(1+q^{2n+5})\cdots}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} \\ &= \prod_{\substack{n=1 \\ n\equiv 1,4,7 \pmod 8}}^{\infty} \frac{1}{1-q^n},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{q^{2+2+4+4+\cdots+2n+2n}(1+q^{2n+3})(1+q^{2n+5})(1+q^{2n+7})\cdots}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} \\ &= \prod_{\substack{n=1 \\ n\equiv 3,4,5 \pmod 8}}^{\infty} \frac{1}{1-q^n}.\end{aligned} \tag{2.6}$$

The standard methods for generating partitions from q-series and products [2, Ch. 1] allows us to interpret (2.5) and (2.6) as follows.

Theorem 4. *Let $G_1(n)$ denote the number of partitions of n into parts $\equiv 1, 4$ or 7 (mod 8). Let $R_1(n)$ denote the number of partitions of n in which, (i) odd parts are distinct and each is larger than any even part, and (ii) all even integers less than the largest even part appears at least twice. Then for each $n \geq 0$,*

$$G_1(n) = R_1(n).$$

For example, the 12 partitions enumerated by $G_1(15)$ are 15, $12+1+1+1$, $9+4+1+1$, $9+1+1+\cdots+1$, $7+7+1$, $7+4+4$, $7+4+1+1+1+1$, $7+1+1+\cdots+1$, $4+4+4+1+1+1$, $4+4+1+1+\cdots+1$, $4+1+1+\cdots+1$, $1+1+\cdots+1$, and the 12 partitiions enumerated by $R_1(15)$ are 15, $11+3+1$, $9+5+1$, $7+5+3$, $13+2$, $11+2+2$, $9+2+2+2$, $7+2+2+2+2$, $5+2+2+2+2+2$, $3+2+2+\cdots+2$, $7+4+2+2$, $5+4+2+2+2$.

Theorem 5. *Let $G_2(n)$ denote the number of partitions of n into parts $\equiv 3$, 4, or 5 (mod 8). Let $R_2(n)$ denote the number of partitions of n in which, (i) odd parts are distinct, greater than 1, and each is larger than the largest even$+2$, and (ii) all even integers up to and including the largest even part appear at least twice. The for each $n \geq 0$*

$$G_2(n) = R_2(n).$$

For example, the 7 partitions enumerate by $G_2(16)$ are $13+3$, $12+4$, $11+5$, $5+5+3+3$, $5+4+4+3$, $4+4+4+4$, $4+3+3+3$, and the 7 partitions enumerated by $R_2(16)$ are $13+3$, $11+5$, $9+7$, $7+5+2+2$, $4+4+2+2+2+2$, $4+4+4+2+2$, $2+2+\cdots+2$.

Now let us reverse the roles played by the evens and odds. The resulting counterpart of (2.5) is

$$\begin{aligned}\sum_{n\geq 1} \frac{q^{1+1+3+3+\cdots+(2n-3)+(2n-3)+(2n-1)}(-q^{2n};q^2)_\infty}{(q;q^2)_n} &= q\sum_{n\geq 0} \frac{q^{2n^2+2n}(-q^{2n+2};q^2)_\infty}{(q;q^2)_{n+1}} \\ &= q(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;-q)_{2n+1}} \\ &:= q(-q^2;q^2)_\infty \mathcal{G}_1(q), \end{aligned} \tag{2.7}$$

where [6]

$$\begin{aligned}\mathcal{G}_1(-q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(-q;q)_{2n+1}} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2-3n}(q^{14n+7}-1) \sum_{j=-n}^{n} (-1)^j q^{-j^2}.\end{aligned}$$

The latter is the now familiar form of a Hecke-type series for a mock theta function. This particular identity is noted in [6].

The resulting counterpart of (2.6) is

$$\begin{aligned}\sum_{n\geq 1}\frac{q^{1+1+3+3+\cdots+(2n-1)+(2n-1)}(-q^{2n+2};q^2)_\infty}{(q;q^2)_n} &= \sum_{n\geq 0}\frac{q^{2n^2}(-q^{2n+2};q^2)_\infty}{(q;q^2)_n}\\ &= (-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;-q)_{2n}}\\ &= (-q^2;q^2)_\infty\mathcal{G}_2(q),\end{aligned} \tag{2.8}$$

where [6]

$$\begin{aligned}\mathcal{G}_2(-q) &= \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-q;q)_{2n}}\\ &= \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{4n^2+n}(1-q^{6n+3})\sum_{j=-n}^{n}(-1)^j q^{-j^2}.\end{aligned}$$

Thus, as was mentioned in the Introduction, the even-odd reversal transformed the related generating functions from classical theta functions into mock theta functions.

3. Identities of Modulus 28

Suppose now we allow some mixing of odds and evens in our Sylvester constraints. Let us turn to identity (119) in Slater's [13, p. 165] which we write as follows:

$$\sum_{n=0}^{\infty}\frac{q^{1+3+\cdots+(2n-1)}(-q^{2n+2};q^2)_\infty}{(q;q)_{2n+1}} = q\prod_{\substack{n=1\\ n\not\equiv 0,\pm4,\pm5,\pm9,14 \pmod{28}}}^{\infty}\frac{1}{1-q^n}. \tag{3.1}$$

We directly deduce from this the following partition identity.

Theorem 6. *Let $H_1(n)$ denote the number of partitions of n into parts $\not\equiv 0,\pm4,\pm5,$ $\pm9,14 \pmod{28}$. Let $S_1(n)$ denote the number of partitions of n in which odd parts do appear and without gaps while the evens larger than the largest odd part are distinct. Then for $n\geq 1$*

$$H_1(n-1) = S_1(n).$$

For example, the 18 partitions enumerated by $H_1(9)$ are $8+1$, $7+2$, $7+1+1$, $6+3$, $6+2+1$, $6+1+1+1$, $3+3+3$, $3+3+2+1$, $3+3+1+1+1$, $3+2+2+2$, $3+2+2+1+1$, $3+2+1+1+1+1$, $3+1+\cdots+1$, $2+2+2+2+1$, $2+2+1+1+1$, $2+2+1+1+1+1+1$, $2+1+1+\cdots+1$, $1+1+\cdots+1$, and the 18 partitions enumerated by $S_1(10)$ are $8+1+1$, $6+3+1$, $6+2+1+1$, $6+1+1+1+1$, $5+3+1+1$, $4+3+2+1$, $4+3+1+1+1$, $4+2+1+1+1+1$, $3+3+3+1$, $4+1+1+\cdots+1$, $3+3+2+1+1$, $3+3+1+1+1+1$, $3+2+2+2+1$, $3+2+2+1+1+1$, $3+2+1+1+\cdots+1$, $3+1+1+\cdots+1$, $2+1+1+\cdots+1$, $1+1+\cdots+1$.

When we now reverse the roles of evens and odds, we find that, instead of a mock theta function arising, we obtain another identity of Slater's [14]. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{2+4+\cdots+2n}(-q^{2n+1};q^2)_\infty}{(q;q)_{2n}} &= (-q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}(-q;q^2)_n} \\ &= (-q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_n} \\ &= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 2,\pm 10,\pm 12,14 \pmod{28}}}^{\infty} \frac{1}{1-q^n},\end{aligned}$$

by [14, p. 160, eq. (81)].

This result is then directly interpretable in the following theorem.

Theorem 7. *Let $H_2(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm 2, \pm 10, \pm 12, 14 \pmod{28}$. Let $S_2(n)$ denote the number of partitions of n in which even parts appear without gaps and the odd parts larger than the largest even part are distinct. Then*

$$H_2(n) = S_2(n).$$

For example, the 15 partitions enumerated by $H_2(9)$ are 9, $8+1$, $7+1+1$, $6+3$, $6+1+1+1$, $5+4$, $5+3+1$, $5+1+1+1+1$, $4+4+1$, $4+3+1+1$, $4+1+1+\cdots+1$, $3+3+3$, $3+3+1+1+1$, $3+1+1+\cdots+1$, $1+1+\cdots+1$, and the 15 partitions enumerated by $S_2(9)$ are 9, $7+2$, $5+3+1$, $5+2+2$, $5+2+1+1$, $4+3+2$, $4+2+1+1+1$, $4+2+2+1$, $3+2+2+2$, $3+2+2+1+1$, $3+2+1+1+1+1$, $2+2+2+2+1$, $2+2+2+1+1+1$, $2+2+1+1+\cdots+1$, $2+1+1+\cdots+1$.

4. Identities Stemming From Modulus 20

As is apparent by now, each section of this paper is devoted to some different outcome when extending Sylvester's three conditions to the interpretation of Slater's identities. In this section we begin with two of Slater's formulas that, upon inspection, suggest rather cumbersome partition identities. The modifications necessary to reduce the awkwardness again lead us to mock theta functions.

The identities in question are Slater's (44) and (46) [14, p. 156] slightly rewritten:

$$\sum_{n\geq 0} \frac{q^{1+1+2+3+3+\cdots+(2n-1)+(2n-1)+2n+(2n+1)}(-q^{2n+3};q^2)_\infty}{(q)_{2n+1}} = q\prod_{\substack{n=1 \\ n\not\equiv 0,\pm 2,\pm 4,\pm 6,10 \pmod{20}}}^{\infty} \frac{1}{1-q^n}. \tag{4.1}$$

and

$$\sum_{n\geq 0} \frac{q^{1+1+2+3+3+\cdots+(2n-3)+(2n-3)+(2n-2)+(2n-1)+2n}(-q^{2n+1};q^2)_\infty}{(q)_{2n}} = q\prod_{\substack{n=1 \\ n\not\equiv 0,\pm 2,\pm 6,\pm 8,10 \pmod{20}}}^{\infty} \frac{1}{1-q^n}. \tag{4.2}$$

One can interpret (4.1) and (4.2) in the Sylvester manner, but, in doing so, condition (2) in the Sylvester constraints becomes quite complicated.

So instead we consider closely related series where the interpretations are more natural. Let

$$\sum_{n\geq 0} J_1(n)q^n := \sum_{n\geq 0} \frac{q^{1+1+2+3+3+4+\cdots+(2n-1)+(2n-1)+2n}(-q^{2n+1};q^2)_\infty}{(q)_{2n}} \tag{4.3}$$
$$= \sum_{n\geq 0} \frac{q^{3n^2+n}(-q^{2n+1};q^2)_\infty}{(q)_{2n}}.$$

and

$$\sum_{n\geq 0} J_2(n)q^n := \sum_{n\geq 0} \frac{q^{1+1+2+3+3+4+\cdots+2n+(2n+1)+(2n+1)}(-q^{2n+3};q^2)_\infty}{(q)_{2n+1}} \tag{4.4}$$
$$= \sum_{n\geq 0} \frac{q^{3n^2+5n+2}(-q^{2n+3};q^2)_\infty}{(q)_{2n+1}}.$$

Now $J_1(n)$ and $J_2(n)$ may be viewed as enumerating partitions that mix "partitions with initial 2-representations" with "partitions without gaps."

Namely, $J_1(n)$ is the number of partitions of n in which (1) all odd integers smaller than the largest even part appear at least twice, (2) even parts appear without gaps, and (3) odd parts larger than the largest even part are distinct.

The formulation of $J_2(n)$ is even more straightforward. $J_2(n)$ is the number of partitions of n in which (1) each odd integer smaller than a repeated odd part is a repeated odd part and (2) every even integer smaller than the largest repreated odd part is a part, and (3) there are no other even parts.

Theorem 8.

$$\sum_{n\geq 0} J_1(n)q^n = \frac{1}{\psi(-q)} \sum_{n=0}^{\infty} q^{4n^2+2n}(1-q^{4n+2}) \sum_{j=-\lfloor \frac{n}{2}\rfloor}^{\lfloor \frac{n}{2}\rfloor} (-1)^j q^{-6j^2+2j} \tag{4.5}$$

and

$$\sum_{n\geq 0} J_2(n)q^n = \frac{q^2}{\psi(-q)} \sum_{n=0}^{\infty} q^{4n^2+6n}(1-q^{4n+4}) \sum_{j=-\lfloor \frac{n}{2}\rfloor}^{\lfloor \frac{n+1}{2}\rfloor} (-1)^j q^{-6j^2+2j} \tag{4.6}$$

where

$$\psi(q) := \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}. \tag{4.7}$$

Proof. Using representations (4.3) and (4.4) we see that (4.5) and (4.6) are equivalent to the following assertions.

(4.8)
$$\sum_{n=0}^{\infty} \frac{q^{3n^2+n}}{(q^2;q^2)_n(q^2;q^4)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2+2n}(1-q^{4n+2}) \sum_{|2j|\leq n} (-1)^j q^{-6j^2+2j}$$

and
(4.9)
$$\sum_{n=0}^{\infty}\frac{q^{3n^2+5n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{4n^2+6n}(1-q^{4n+4})\sum_{-n\le 2j\le n+1}(-1)^j q^{-6j^2+2j}.$$

Identities (4.8) and (4.9) may be reduced to Bailey pair identities following the use of the strong form of Bailey's Lemma [3, p. 270]. In the case of (4.8) we replace q by q^2 in Bailey's Lemma and set $a=q^2$. In the case of (4.9) we replace q by q^2 in Bailey's Lemma and set $a=1$. If we then invoke the weak form of Baileys Lemma [4, p. 27, eq. (3.33)] we see that (4.8) and (4.9) are equivalent to the assertions (4.18) and (4.19) below.

Let

$$a_1(n,q)=\sum_{j=0}^{n}\frac{(q^{-n};q)_j(q^{n+1};q)_j q^{\binom{j+1}{2}}}{(q;q)_j(q;q^2)_j}, \tag{4.10}$$

$$a_2(n,q)=\sum_{j=1}^{n}\frac{(q^{-n};q)_j(q^{n};q)_j q^{\binom{j+1}{2}}}{(q;q)_{j-1}(q;q^2)_j}, \tag{4.11}$$

$$a_3(n,q)=\sum_{j=0}^{n}\frac{(q^{-n};q)_j(q^{n};q)_j q^{\binom{j+1}{2}}}{(q;q)_j(q;q^2)_j}, \tag{4.12}$$

Our proof relies on proving the following three identities. This in the spirit of the method developed at length in [6].

$$a_1(n,q)+q^n a_1(n-1,q)=(1+q^n)a_3(n,q), \tag{4.13}$$

$$q^n a_2(n,q)-(1-q^n)a_1(n,q)=-(1-q^n)a_3(n,q), \tag{4.14}$$

$$a_3(n,q)=\begin{cases}0 & \text{if } n \text{ is odd}\\ (-1)^\nu q^{-\nu^2} & \text{if } n=2\nu\end{cases}. \tag{4.15}$$

First we prove (4.13).

$$\begin{aligned}a_1(n,q)+q^n a_1(n-1,q)&=\sum_{j=0}^{n}\frac{(q^{-n+1};q)_{j-1}(q^{n+1};q)_{j-1}q^{\binom{j+1}{2}}}{(q;q)_j(q;q^2)_j}\\&\qquad\times\left\{(1-q^{-n})(1-q^{n+j})+q^n(1-q^{-n+j})(1-q^n)\right\}\\&=(1+q^n)\sum_{j=0}^{n}\frac{(q^{-n};q)_j(q^{n};q)_j q^{\binom{j+1}{2}}}{(q;q)_j(q;q^2)_j}\\&=(1+q^n)a_3(n,q).\end{aligned}$$

Next we treat (4.14).

$$\begin{aligned}
a_2(n,q)-(1-q^n)a_1(n,q) &= \sum_{j\geq 0}\frac{(q^{-n};q)_j(q^n;q)_j q^{\binom{j+1}{2}}}{(q;q)_j(q;q^2)_j}\left((1-q^j)-(1-q^{n+j}\right)\\
&= -(1-q^n)\sum_{j\geq 0}\frac{(q^{-n};q)_j(q^n;q)_j q^{\binom{j+1}{2}+j}}{(q;q)_j(q;q^2)_j}\\
&= -(1-q^n)\sum_{j\geq 0}\frac{(q^{-n};q)_j(q^n;q)_j q^{\binom{j+1}{2}}\left(1-(1-q^j)\right)}{(q;q)_j(q;q^2)_j}\\
&= -(1-q^n)a_3(n,q)+(1-q^n)a_2(n,q),
\end{aligned}$$

which is equivalent to (4.14).

Finally we move to (4.15) using the notation of [8, p. 4] and invoking [8, p. 242, eq. III.13].

$$\begin{aligned}
a_3(n,q) &= \lim_{\tau\to 0} {}_3\phi_2\left(\begin{matrix} q^{-n}, q^n, -\frac{q}{\tau}; q, \tau \\ q^{\frac12}, -q^{\frac12}\end{matrix}\right)\\
&= \frac{1}{(-q^{\frac12};q)_n}\lim_{\tau\to 0} {}_3\phi_2\left(\begin{matrix} q^{-n}, -\frac{q}{\tau}, q^{\frac12-n}; q, q \\ q^{\frac12}, \frac{q^{\frac32-n}}{\tau}\end{matrix}\right)\\
&= \frac{1}{(-q^{\frac12};q)_n}{}_2\phi_1\left(\begin{matrix} q^{-n}, q^{\frac12-n}; q, -q^{\frac12+n} \\ q^{\frac12}\end{matrix}\right)\\
&= \begin{cases} 0 & \text{if } n \text{ is odd}\\ (-1)^\nu q^{-\nu^2} & \text{if } n=2\nu,\end{cases}
\end{aligned}$$

where the final line follows from the q-analog of Kummer's theorem [8, p. 236, eq. (II.9)].

From (4.13)–(4.15) it is clear that each of $a_1(n,q)$, $a_2(n,q)$ and $a_3(n,q)$ is recursively defined as a Laurent polynomial in q. It is then a straightforward matter to show via mathematical induction that

$$a_1(n,q)=\begin{cases} -q^n a_1(n-1,q) & \text{if } n \text{ odd}\\ q^{\binom{n+1}{2}}\sum_{-\nu}^{\nu}(-1)^j q^{-j(3j+1)} & \text{if } n=2\nu.\end{cases} \tag{4.16}$$

$$a_2(n,q)=(1-q^n)(-1)^n q^{\binom{n}{2}}\sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^j q^{-j(3j+1)}. \tag{4.17}$$

Equating (4.10) and (4.16) are equivalent to the assertion that

$$\begin{cases} \alpha_n = \frac{(-1)^n q^{n^2-n}(1-q^{4n+2})}{(1-q^2)}a_1(n,q^2)\\ \beta_n = \frac{q^{n^2-n}}{(q^2;q^2)_n(q^2;q^4)_n}\end{cases} \tag{4.18}$$

are a Bailey pair (where $q\to q^2$ and $a=q^2$) (see [3] especially Bailey's Lemma on page 270 and eq. (4.1) on page 278). We may now insert this Bailey pair into the weak form of Bailey's Lemma [4, p. 27, eq. (3.33)] with $q\to q^2$, $a=q^2$], and then (4.16) and simplification yields (4.8).

Equations (4.11) and (4.17) are equivalent to the assertion that

$$\begin{cases} \overline{\alpha}_n = (-1)^n q^{n^2-n}(1+q^{2n})a_2(n,q) \\ \overline{\beta}_n = \frac{q^{n^2-n}(1-q^{2n})}{(q^2;q^2)_n(q^2;q^4)_n} \end{cases} \tag{4.19}$$

are a Bailey pair (with $q \to q^2$, $a = 1$) [3, pp. 270 and 278]. We may now insert this Bailey pair into the weak form of Bailey's Lemma [4, p. 27, eq. (3.33) with $q \to q^2$, $a = 1$]; then (4.17) and simplification yields (4.9). $\square$

Notice that our starting position in this section, namely (4.3) and (4.4) (inspired by (4.1) and (4.2)) landed us in the world of mock theta functions immediately. So what will happen when we reverse the roles of evens and odds? We define

$$\begin{aligned} \sum_{n\geq 0} K_1(n)q^n &:= \sum_{n\geq 0} \frac{q^{1+2+2+3+4+4+\cdots+2n+2n+(2n+1)}(-q^{2n+2};q^2)_\infty}{(q)_{2n+1}} \\ &= \sum_{n\geq 0} \frac{q^{3n^2+4n+1}(-q^{2n+2};q^3)_\infty}{(q)_{2n+1}}, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} \sum_{n\geq 0} K_2(n)q^n &:= \sum_{n\geq 0} \frac{q^{1+2+2+3+\cdots+2n+2n}(-q^{2n+2};q^2)_\infty}{(q)_{2n}} \\ &= \sum_{n\geq 0} \frac{q^{3n^2+2n}(-q^{2n+2};q^2)_\infty}{(q)_{2n}}. \end{aligned} \tag{4.21}$$

We shall not formally provide the partition-theoretic interpretations of $K_1(n)$ and $K_2(n)$ because they are identical with those of $J_1(n)$ and $J_2(n)$ respectively where the roles of odds and evens have been exchanged.

Theorem 9.

$$\sum_{n\geq 0} K_1(n)(-q)^n = \frac{1}{(-q;q)_\infty(q;q^5)_\infty(q^4;q^5)_\infty} - \sum_{n=0}^{\infty} K_2(n)(-q)^n, \tag{4.22}$$

and

$$\sum_{n\geq 0} K_2(n)q^n = \frac{1}{\phi(-q^2)} \sum_{n\geq 0} q^{4n^2+2n}(1-q^{4n+2}) \sum_{j=-n}^{n} (-1)^j(-q)^{-j(3j-1)/2}, \tag{4.23}$$

with $\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$.

Proof. Using representations (4.20) and (4.21) we see that (4.22) and (4.23) are equivalent to the following assertions.

$$\begin{aligned} &\sum_{n\geq 0} \frac{q^{3n^2+4n+1}}{(q;q)_{2n+1}(-q^2;q^2)_n} \\ &\quad = \frac{1}{(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n(-q)^{n(5n+3)/2}\right) - \sum_{n=0}^{\infty} \frac{q^{3n^2+2n}}{(q;q)_{2n}(-q^2;q^2)_n}. \end{aligned} \tag{4.24}$$

$$\sum_{n\geq 0} \frac{q^{3n^2+2n}}{(q;q)_{2n}(-q^2;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2+2n}(1-q^{4n+2}) \sum_{j=-n}^{n} (-1)^j (-q)^{j(3j-1)/2}. \tag{4.25}$$

Identities (4.24) and (4.25) may be reduced to Bailey pair identities following the use of the strong form of Bailey's Lemma [3, p. 270]. For both (4.24) and (4.25) we replace q by q^2 in Bailey's Lemma and set $a = q^2$. If we then invoke the weak form of Bailey's Lemma [4, p. 27, eq. (3.33)] we see (4.24) and (4.25) are equivalent to the assertions (4.36) and (4.37) below.

Let

$$A_1(n,q) = \sum_{j=0}^{n} \frac{(q^{-2n};q^2)_j (q^{2n+2};q^2)_j q^{j^2+4j+1}}{(q;q)_{2j+1}(-q^2;q^2)_j}, \tag{4.26}$$

$$A_2(n,q) = \sum_{j=0}^{n} \frac{(q^{-2n};q^2)_j (q^{2n+2};q^2)_j q^{j^2+2j}}{(q;q)_{2j}(-q^2;q^2)_j}, \tag{4.27}$$

$$A_3(n,q) = \sum_{j=0}^{n} \frac{(q^{-2n};q^2)_j (q^{2n+2};q^2)_j q^{j^2+2j}}{(q;q)_{2j+1}(-q^2;q^2)_j}, \tag{4.28}$$

$$A_4(n,q) = \sum_{j=0}^{n} \frac{(q^{-2n};q^2)_j (q^{2n};q^2)_j q^{j^2+2j}}{(q;q)_{2j}(-q^2;q^2)_j}. \tag{4.29}$$

Our proof requires the following identities.

$$A_3(n,q) - A_1(n,q) = A_2(n,q), \tag{4.30}$$

$$A_2(n,q) + q^{2n} A_2(n-1,q) = (1+q^{2n}) A_4(n,q), \tag{4.31}$$

$$A_3(n,q) = \frac{(-q)^{-\binom{n}{2}}}{1-q^{2n+1}}, \tag{4.32}$$

$$A_4(n,q) = \frac{(-q)^{-\binom{n}{2}}\,(1+(-q)^n)}{1+q^{2n}}. \tag{4.33}$$

First we prove (4.30).

$$\begin{aligned} A_3(n,q) - A_1(n,q) &= \sum_{j=0}^{n} \frac{(q^{-2n};q^2)_j (q^{2n+2};q^2)_j q^{j^2+2j}(1-q^{2j+1})}{(q;q)_{2j+1}(-q^2;q^2)_j} \\ &= \sum_{j=0}^{n} \frac{(q^{-2n};q^2)_j (q^{2n+2};q^2)_j q^{j^2+2j}}{(q;q)_{2j}(-q^2;q^2)_j} = A_2(n,q). \end{aligned}$$

Next comes (4.31).

$$\begin{aligned}
&A_2(n,q)+q^{2n}A_2(n-1,q)\\
&\qquad=\sum_{j\geq 0}\frac{(q^{-2n+2};q^2)_{j-1}(q^{2n+2};q^2)_{j-1}q^{j^2+2j}}{(q;q)_{2j}(-q^2;q^2)_j}\\
&\qquad\quad\times\left\{(1-q^{2n})(1-q^{2n+2j})+q^{2n}(1-q^{-2n+2j})(1-q^{2n})\right\}\\
&\qquad=(1+q^{2n})\sum_{j\geq 0}\frac{(q^{-2n};q^2)_j(q^{2n};q^2)_jq^{j^2+2j}}{(q;q)_{2j}(-q^2;q^2)_j}
\end{aligned}$$

Now we treat (4.32) using the notation of [8, p. 4].

$$\begin{aligned}
A_3(n,q)&=\frac{1}{1-q}\lim_{\tau\to 0}{}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n+2},-\frac{q}{\tau};q^2,q^2\tau\\ q^3,-q^2\end{matrix}\right)\\
&=\frac{1}{(q;q^2)_{n+1}}\lim_{\tau\to 0}{}_3\phi_2\left(\begin{matrix}q^{-2n},-\frac{q}{\tau},-q^{-2n};q^2,q^2\\ -q^2,-\frac{q^{2n}}{\tau}\end{matrix}\right)\\
&\hspace{18em}\text{by [8, p. 242, eq. (III.13)]}\\
&=\frac{1}{(q;q^2)_{n+1}}{}_2\phi_1\left(\begin{matrix}q^{-2n},-q^{2n};q^2,q^{2n+3}\\ -q^2\end{matrix}\right)\\
&=\frac{1}{(q;q^2)_{n+1}}\sum_{j=0}^{n}\frac{(q^{-4n};q^4)_jq^{(2n+3)j}}{(q^4;q^4)_j}\\
&=\frac{(q^{3-2n};q^4)_n}{(q;q^2)_{n+1}}=\frac{(-q)^{-\binom{n}{2}}}{1-q^{2n+1}},
\end{aligned}$$

where the penultimate assertion follows from [8, p. 236, eq. (II.7)].

Finally we treat the fourth identity (4.33).

$$\begin{aligned}
A_4(n,q)&=\lim_{\tau\to 0}{}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n},-\frac{q}{\tau};q^2,q^2\tau\\ -q^2,q\end{matrix}\right)\\
&=\frac{1}{(q;q^2)_n}{}_2\phi_1\left(\begin{matrix}q^{-2n},-q^{2-2n};q^2,q^{1+2n}\\ -q^2\end{matrix}\right)\\
&\hspace{18em}\text{by [8, p. 241, eq. (III.9)]}\\
&=\frac{1}{(q;q^2)_n}\sum_{j=0}^{n}\frac{(q^{4-4n};q^4)_{j-1}(1-q^{-2n})(1+q^{-2n+2j})q^{j(1+2n)}}{(q^4;q^4)_j}\\
&=\frac{1}{(q;q^2)_n(1+q^{-2n})}\sum_{j=0}^{n}\frac{(q^{-4n};q^4)_j}{(q^4;q^4)_j}\left(q^{j(1+2n)}+q^{-2n+j(3+2n)}\right)\\
&=\frac{q^{2n}}{(q;q^2)_n(1+q^{2n})}\left((q^{1-2n};q^4)+(q^{3-2n};q^4)_n\right)\\
&=\frac{(-q)^{-\binom{n}{2}}(-q)^n}{1+q^{2n}}+\frac{(-q)^{-\binom{n}{2}}}{1+q^{2n}}\\
&=(-q)^{-\binom{n}{2}}\frac{(1+(-q)^n)}{1+q^{2n}},
\end{aligned}$$

as desired.

From (4.30)–(4.33), it follows by mathematical induction that

$$A_1(n,q) = -q^{n^2+n}(-1)^n \sum_{j=-n}^{n} (-1)^j (-q)^{(-j(3j-1))/2} + \frac{(-q)^{-\frac{n(n-1)}{2}}}{1-q^{2n+1}}, \tag{4.34}$$

$$A_2(n,q) = (-1)^n q^{n^2+n} \sum_{j=-n}^{n} (-1)^j (-q)^{-j(3j-1)/2}. \tag{4.35}$$

Let us treat (4.23) or rather its equivalent formulation (4.25) first. Identity (4.35) is equivalent to the assertion that

$$\begin{cases} \alpha'_n = \frac{q^{2n^2}(1-q^{4n+2})}{(1-q^2)}) \sum_{j=-n}^{n} (-1)^j (-q)^{-j(3j-1)/2} \\ \beta'_n = \frac{q^{n^2}}{(q;q)_{2n}(-q^2;q^2)_n} \end{cases} \tag{4.36}$$

are a Bailey pair (where $q \to q^2$ and $a = q^2$). Inserting this Bailey pair into the weak form of Bailey's Lemma, we obtain (4.25) by invoking (4.35) and simplifying.

As for (4.22), or rather its equivalent formulation (4.24), we see from (4.34) and (4.35) that

$$\begin{cases} \alpha''_n = -\alpha'_n + \frac{(-1)^n(-q)^{\binom{n}{2}}(1+q^{2n+1})}{(1-q^2)} \\ \beta''_n = \frac{q^{n^2+2n+1}}{(q;q)_{2n+1}(-q^2;q^2)_n} \end{cases} \tag{4.37}$$

form a Bailey pair. Furthermore

$$\begin{aligned} \sum_{n\geq 0} K_1(n)q^n &= \sum_{n=0}^{\infty} q^{2n^2+2n}\beta''_n \\ &= \frac{1}{(q^4;q^2))\infty} \sum_{n=0}^{\infty} q^{2n^2+2n}\alpha''_n \\ &= \frac{1}{(q^4;q^2)_\infty} \sum_{n=0}^{\infty} q^{2n^2+2n} \left(-\alpha'_n + \frac{(-1)^n(-q)^{\binom{n}{2}}(1+q^{2n+1})}{1-q^2} \right) \\ &= -\sum_{n\geq 0} K_2(n)q^n + \frac{1}{(q^2;q^2)_\infty} \sum_{n--\infty}^{\infty} (-1)^n(-q)^{\frac{5n^2}{2}+\frac{3n}{2}}, \end{aligned}$$

and invoking Jacobi's triple product identity [2, Th. 2.8, p. 21], we see that (4.24) is established. □

5. Identities of Modulus 12

As is obvious by now, we are choosing a variety of examples from Slater's compendium to illustrate the variety that arises when we mix parity with the Sylvester constraints. We close our presentation with a move beyond parity to conditions modulo 4.

Recall that *evenly even* numbers are numbers divisible by 4 while *oddly even* numbers are numbers congruent to 2 modulo 4.

We shall examine Slater's (53) and (55) [14, p. 157].

$$\prod_{\substack{n=1\\ n\equiv \pm1,\pm3,\pm4 \pmod{12}}} \frac{1}{1-q^n} = \sum_{n\geq 0} \frac{q^{4n^2}}{(q^4;q^4)_{2n}(q^{4n+1};q^2)_\infty} \tag{5.1}$$
$$= \frac{1}{(q;q^2)_\infty} + \frac{q^{2+2}}{(1-q^{2+2})(1-q^{4+4})(q^5;q^2)_\infty} + \frac{q^{2+2+6+6}}{(1-q^{2+2})(1-q^{4+4})(1-q^{6+6})(1-q^{8+8})(q^9;q^2)_\infty} + \cdots$$

and

$$\prod_{\substack{n=1\\ n\equiv \pm3,\pm4,\pm5 \pmod{12}}} \frac{1}{1-q^n} \tag{5.2}$$
$$= \sum_{n\geq 0} \frac{q^{4n^2+4n}}{(q^4;q^4)_{2n+1}(q^{4n+3};q^2)_\infty}$$
$$= \frac{1}{(1-q^{2+2})(q^3;q^2)_\infty} + \frac{q^{4+4}}{(1-q^{2+2})(1-q^{4+4})(1-q^{6+6})(q^7;q^2)_\infty} + \frac{q^{4+4+8+8}}{(1-q^{2+2})(1-q^{4+4})(1-q^{6+6})(1-q^{8+8})(1-q^{10+10})(q^{11};q^2)_\infty} + \cdots.$$

In both (5.1) and (5.2), the extended final forms are given so that the following theorems are immediately interpreted from these forms.

Theorem 10. *Let $L_1(n)$ denote the number of partitions of n into parts that are $\equiv \pm1,\pm3,\pm4 \pmod{12}$. Let $T_1(n)$ denote the number of partitions of n in which (1) all even parts must appear an even number of times, (2) each oddly even integer not exceeding the largest even part must appear, (3) each odd part is at least 3 greater than each oddly even part. Then for $n \geq 0$,*

$$L_1(n) = T_1(n).$$

For example, the 20 partitions enumerated by $T_1(13)$ are 13, $11+1+1$, $9+3+1$, $9+2+2$, $9+1+1+1+1$, $7+5+1$, $7+3+3$, $7+3+1+1+1$, $7+1+1+\cdots+1$, $5+5+3$, $5+5+1+1+1$, $5+3+3+1+1$, $5+3+1+\cdots+1$, $5+2+2+\cdots+2$, $5+1+1+\cdots+1$, $3+3+3+3+1$, $3+3+3+1+1+1+1$, $3+3+1+1+\cdots+1$, $3+1+1+\cdots+1$, $1+1+\cdots+1$, and the 20 partitions enumerated by $L_1(13)$ are 13, $11+1+1$, $9+4$, $9+3+1$, $9+1+1+1+1$, $8+4+1$, $8+3+1+1$, $8+1+1+\cdots+1$, $4+4+4+1$, $4+4+3+1+1$, $4+4+1+1+\cdots+1$, $4+3+3+3$, $4+3+3+1+1+1$, $4+3+1+1+\cdots+1$, $4+1+1+\cdots+1$, $3+3+3+3+1$, $3+3+3+1+1+1+1$, $3+3+1+1+\cdots+1$, $3+1+1+\cdots+1$, $1+1+\cdots+1$,

Theorem 11. *Let $L_2(n)$ denote the number of partitions of n into parts that are $\equiv \pm3,\pm4,\pm5 \pmod{12}$. Let $T_2(n)$ denote the number of partitions of n in which (1) all even parts must appear an even number of times, (2) each evenly even integer not exceeding the largest even part must appear as a part, (3) each odd part is larger than 1 and at least 3 larger than the largest evenly even part. Then for $n \geq 0$,*

$$L_2(n) = T_2(n).$$

For example the 10 partitions enumerated by $L_2(15)$ are 15, $9+3+3$, $8+7$, $8+4+3$, $7+5+3$, $7+4+4$, $5+5+5$, $5+4+3+3$, $4+4+4+3$, $3+3+3+3+3$, and the 10 partitions enumerated by $T_2(15)$ are 15, $11+2+2$, $9+3+3$, $7+5+3$, $7+4+4$, $7+2+2+2+2$, $5+5+5$, $5+3+3+2+2$, $3+3+3+3+3$, $3+2+2+\cdots+2$.

6. Conclusion

This paper is in no way meant to be exhaustive. Indeed we have chosen a handful of Slaters identities for consideration. The examples were chosen to illustrate the variety of possible outcomes.

There are many further formulas in Slater's paper [14] that can be interpreted using the approach we have developed. Indeed this can be done for the original Rogers-Ramanujan identities [14, p. 133–134 (14)–(18)] and also for variants on the Rogers-Ramanujan identities (c.f. Slater's (15), (16), (19), (20) and (25)). Others like the modulus 6 results (Slater's (22)–(30)) are either quite classical (e.g. (23) is effectively due to Euler) or seem to require some alternative analysis. The identities with modulus 27 (Slater's (88)–(93)) seem quite distant from these developments as do those identities like (97), or (101)–(112), or (125)–(130) that apparently are not reducible to a single product.

It would certainly be interesting to determine if there is an alternative to Sylvester's constraints that leads to explanations of further Slater identities that could not be treated here.

It is interesting to note that in each case where a Slater identity was modified to fit the Sylvester paradigm, the resulting infinite product was always of the nicest form imaginable, namely

$$\prod_{n=1}^{\infty}{}' \frac{1}{1-q^n}$$

where the $'$ indicates only that the n are restricted to a specified set of arithmetic progressions.

Finally the relation of (4.24) to the original Roger's Ramanujan function is striking. Indeed one can provide an alternative proof of (4.24) by adding together the left-hand sides of (4.24) and (4.25) and proving (slightly non-trivially) that the result is, in fact, Slater's (15) [14, p. 153] with q replaced by $-q$.

In fact, it is possible to prove that, instead of (4.24),

$$\sum_{n=0}^{\infty} \frac{q^{3n^2+4n+1}}{(q;q)_{2n+1}(-q^2;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2-2n}(1-q^{12n+6}) \sum_{j=-n}^{n} (-1)^j (-q)^{-j(3j-1)/2}. \tag{6.1}$$

In addition

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q)_{2n}(-q^2;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2}(1-q^{8n+4}) \sum_{j=-n}^{n} (-1)^j (-q)^{-j(3j-1)/2}. \tag{6.2}$$

If we denote the left-hand side of (6.2) by $T(q)$, then Slater's (19) [14, p. 154] asserts

$$T(-q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q;q^5)_\infty}{(q^2;q^2)_\infty} \tag{6.3}$$

Identities of this nature combined with the results in Section 4 suggest a variety of new mock theta type results related to the Rogers-Ramanujan identities.

References

[1] G. E. Andrews, On a partition problem of J. J. Sylvester, J. London Math. Soc.(2), 2 (1970), 571–576.

[2] G. E. Andrews, The Theory of Partitions, Encycl. Math and Its Appl., Addison-Wesley, Reading, 1976. Reissued: Cambridge Univ. Press, 1998.

[3] G. E. Andrews, Multiple series Rogers-Ramaujan type identities, Pac. J. Math., 114 (1984), 267–283.

[4] G. E. Andrews, q-Series: Their Develpment..., C.B.M.S. Regional Conf. Series in Math., No. 66, Amer. Math. Soc., Providence, 1986.

[5] G. E. Andrews, Partitions with initial repetitions, Acta Math. Sinica, English Series, 25 (2009), 1437–1442.

[6] G. E. Andrews, q-Othogonal polynomials, Rogers-Ramanujan identities, and mock theta functions, (to appear).

[7] N. J. Fine, Basic Hypergeometric Series and Applications, Amer. Math. Soc., Providence, 1988.

[8] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encycl. Math. and Its Appl., Vol. 35, 1990, Cambrige University Press, Cambridge.

[9] H. Göllnitz, Partitionen mit Differenzenbedingungen, J. Reine u. Angew. Math., 225 (1967), 154–190.

[10] B. Gordon, Some continued fractions of the Rogers-Ramanujan type, Duke Math. J., 32 (1965), 741–748.

[11] F. H. Jackson, Examples of a generalization of Euler's transformation for power series, Messenger of Math., 57 (1928), 169–187.

[12] S. Ramanujan, Collected Papers, Cambridge University Press, Cambridge, 1927. (Reprinted: Chelsea, New York, 1962).

[13] A. Sills, Finite Rogers-Ramanujan type identities, Elec. J. Comb., 10 (2003), #R13, 122 pp.

[14] L. J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc.(2), 54 (1952), 147–167.

[15] J. J. Sylvester, "Unsolved questions", Mathematical Questions and Solutions from the Educational Times, 45 (1886), 125–145.

[16] G. N. Watson, The final problem: an account of the mock theta functions, J. London Math. Soc., 11 (1936), 55–80.

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