Partitions with Initial Repetitions

by

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Abstract

A variety of interesting connections with modular forms, mock theta functions and Rogers-Ramanujan type identities arise in consideration of partitions in which the smaller integers are repeated as summands more often than the larger summands. In particular, this concept leads to new interpretations of the Rogers-Selberg identities and Bailey's modulus 9 identities.

1   Introduction

Apparently, N. J. Fine was the first one to consider partitions without gaps. In [6, p. 57], he notes that the mock theta function

$$
\begin{aligned}
\psi(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})} \\
&= \sum_{n=1}^{\infty} \beta(n)q^n,
\end{aligned}
\tag{1.1}
$$

where $\beta(n)$ is the number of partitions of n into odd parts without gaps, i.e. partitions of n of the form

$$
n = n;1 + n;3 + \cdots + n_{2k-1}(2k-1), \qquad n_j > 0.
$$

In addition, Fine noted in lectures based on [6], that the partitions conjugate to partitions into distinct parts and partitions without gaps. Here we list the six partitions of 8 into distinct parts opposite the six partitions of 8 without gaps:

$$
\begin{array}{cc}
8 & 1+1+1+1+1+1+1+1 \\
7+1 & 2+1+1+1+1+1+1 \\
6+2 & 2+2+1+1+1+1 \\
5+3 & 2+2+2+1+1 \\
5+2+1 & 3+2+1+1+1 \\
4+3+1 & 3+2+2+1
\end{array}
$$

Partitions without gaps turn out to be the very simplest case of the types of partitions to be explored in this paper.

Definition. A partition with initial k-repetitions is a partition in which if any j appears at least k times as a part then each positive integer less than j appears at least k times as a part.

Clearly partitions with initial 1-repetitions are Fine's partitions without gaps.

In Section 2, we relate partitions with initial k-repetitions to Glaisher's classic generalization of Euler's theorem [2, Cor.1.3, p. 6].

Theorem 1. *The number of partitions of n with initial k-repetitions equals the number of partitions of n into parts not divisible by $2k$ and also equals the number of partitions of n in which no part is repeated more than $2k - 1$ times.*

This elementary result will be proved in Section 2. In Section 3, we refine our considerations by adding a classification according to the number of different parts in each partition. Sections 4 and 5 build on Fine's ideas by restricting the initially repeated parts to a specified arithmetic progression.

2   Glaisher Revisited

For brevity we require the following standard notation

$$(A; q)_n = (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}).$$

Clearly the generating function for partitions with initial k-repetitions is given by

$$\sum_{n=0}^{\infty} \frac{q^{k\cdot 1 + k\cdot 2 + \cdots + k\cdot n}}{(q;q)_n} \prod_{j=n+1}^{\infty} (1 + q^j + q^{2j} + \cdots + q^{(k-1)j})$$

$$= \sum_{n=0}^{\infty} \frac{q^{kn(n+1)/2}}{(q;q)_n} \prod_{j=n+1}^{\infty} \frac{(1 - q^{kj})}{(1 - q^j)}$$

$$= \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{kn(n+1)/2}}{(q^k;q^k)_n}$$

$$= \frac{(q^k;q^k)_\infty (-q^k;q^k)_\infty}{(q;q)_\infty}$$

$$= \frac{(q^{2k};q^{2k})_\infty}{(q;q)_\infty}$$

$$= \prod_{j=1}^{\infty} (1 + q^j + q^{2j} + \cdots + q^{(2k-1)j})$$

The last two expressions are respectively the generating functions for the number of partitions into parts not divisible by $2k$ and the number of partitions in which no part appears more than $2k - 1$ times. Thus Theorem 1 is proved.

3   The Number of Different Parts

A distinction must be made between "the number of different parts" and "the number of distinct parts". In the partition

$$7 + 7 + 6 + 4 + 4 + 4 + 2 + 1$$

there are five different parts namely $7, 6, 4, 2$ and 1, while there are three distinct parts $6, 2$ and 1.

Definition. Let $\mathcal{D}_e(m, n)$ (resp. $\mathcal{D}_o(m, n)$) denote the number of partitions of n with initial 2-repetitions, with m different parts and an even (resp. odd) number of distinct parts.

Theorem 2.

$$\mathcal{D}_e(m, n) - \mathcal{D}_o(m, n) = \begin{cases} (-1)^j & \text{if} \quad m = j, n = j(j+1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$\sum_{m,n \geq 0} (\mathcal{D}_e(m, n) - \mathcal{D}_o(m, n)) z^m q^n$$

$$= \sum_{n=0}^{\infty} \frac{z^n q^{2\cdot1+2\cdot2+\cdots+2\cdot n}}{(q; q)_n} \prod_{j=n+1}^{\infty} (1 - zq^j)$$

$$= (zq; q)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q; q)_n (zq; q)_n}. \tag{3.1}$$

In [7, eq. (III.1), p. 359], we let $a, b \to 0$, set $z = q$ and then set $c = zq$. It follows that

$$\sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n (zq; q)_n} = \frac{1}{(q; q)_\infty (zq; q)_\infty} \sum_{j=0}^{\infty} (-1)^j z^j q^{j(j+1)/2}. \tag{3.2}$$

In [7, eq. (III.3), p. 359], we also let $a, b \to 0$, set $z = q$ and then set $c = zq$. Now we find that

$$\sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n (zq; q)_n} = \frac{1}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+n} z^n}{(q; q)_n (zq; q)_n}. \tag{3.3}$$

Identifying the right-hand sides of (3.2) and (3.3) and comparing with (3.1), we see that

$$\sum_{m,n \geq 0} (\mathcal{D}_e(m, n) - \mathcal{D}_o(m, n)) z^m q^n = \sum_{j=0}^{\infty} (-1)^j z^j q^{j(j+1)/2},$$

and comparing coefficients of $z^m q^n$ on both sides, we obtain the assertion of Theorem 2. $\qquad \square$

It should be noted F. C. Auluck [4, p. 681, eq. (11)] had this analytic identity in the case $z = q$ (cf. [1, p. 575, eq. (A1)]). In unpublished work, A. Berkovich has found elegant finite series versions of these results.

4 Partitions With Initial Repetitions of Given Parity

In light of Fine's observations on (1.1), it is natural to consider what happens when we require that the repeating initial parts have the same parity.

Definition. Let $F_e(n)$ denote the number of partitions of n in which no odd parts are repeated and if an even part $2j$ is repeated then each even positive integer smaller than $2j$ appears in the partition as a repeated part and no odd integers smaller than $2j$ appear.

Theorem 3. $F_e(n)$ *equals the number of partitions of* n *into parts* $\not\equiv 0, \pm 2$ (mod 7).

Proof. We note that

$$
\begin{aligned}
\sum_{n=0}^{\infty} F_e(n) q^n &= \sum_{n=0}^{\infty} \frac{q^{2+2+4+4+\cdots+2n+2n}}{(q^2; q^2)_n} \prod_{j=2n+1}^{\infty} (1 + q^j) \\
&= (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2; q^2)_n (-q; q)_{2n}} \\
&= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 2 \ (\text{mod } 7)}}^{\infty} \frac{1}{1 - q^n}
\end{aligned}
$$

by [9, eq. (3.2), p. 155 (one of the Rogers-Selberg identities)]. In light of the fact that this infinite product is the generating function for partitions into parts $\not\equiv, \pm 2$ (mod 7), we see that Theorem 3 is established. $\qquad \square$

Definition. Let $F_o(n)$ denote the number of partitions of n in which no even parts are repeated and no even part is smaller than a repeated odd part, and if an odd part $2j - 1$ is repeated then each odd positive integer smaller than $2j - 1$ appears in the partition as a repeated part.

Theorem 4. $\sum_{n=1}^{\infty} F_o(n) q^n = (-q; q)_{\infty} f(q^2)$, *where* $f(q)$ *is one of Ramanujan's seventh order mock theta functions, namely [8, p. 355]*

$$
f(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n; q)_n}
$$

5

Proof. As in Theorem 3,

$$\sum_{n=1}^{\infty} F_o(n)q^n = \sum_{n=1}^{\infty} \frac{q^{1+1+3+3+\cdots+(2n-1)+(2n-1)}}{(q;q^2)_n} \prod_{j=2n}^{\infty}(1+q^j)$$

$$= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n(-q;q)_{2n-1}}$$

$$= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n^2}(q^2;q^2)_{n-1}}{(q;q^2)_n(-q;q^2)_n(-q^2;q^2)_{n-1}(q^2;q^2)_{n-1}}$$

$$= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n^2}(q^2;q^2)_{n-1}}{(q^2;q^2)_{2n-1}}$$

$$= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(q^{2n};q^2)_n}$$

$$= (-q;q)_\infty f(q^2).$$

□

5 Partitions With Initial Repetitions Divisible By 3 and Bailey's Modulus 9 Identities

The appearance of a modulus 7 Rogers-Ramanujan type identity in Section 4 suggests that there should be a similar result for Bailey's modulus 9 identities [5, p. 422].

Definition. $B(n)$ denotes the number of partitions of n wherein: (1) all parts are > 1, (2) if a multiple of 3, say $3j$, appears more than once, then each positive multiple of 3 smaller than $3j$ appears more than once as a part, no non-multiples of 3 smaller than $3j + 2$ appear, (3) no two parts that are non-multiples of 3 differ by exactly 1.

Theorem 5. $B(n)$ *equals the number of partitions of n into parts $\not\equiv 0, \pm 1$* (mod 9).

Proof.

$$\sum_{n=0}^{\infty} B(n)q^n$$

$$= \sum_{n=0}^{\infty} \frac{q^{3+3+6+6+\cdots+3n+3n}}{(q^3;q^3)_n(1-q^{3n+2})} \prod_{j=n+1}^{\infty} (1+q^{3j}) \prod_{h=n+1}^{\infty} \left(1 + \frac{q^{3h+1}}{1-q^{3h+1}} + \frac{q^{3h+2}}{1-q^{3h+2}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{q^{3n^2+3n}(-q^{3n+3};q^3)_\infty}{(q^3;q^3)_n(1-q^{3n+2})} \prod_{h=n+1}^{\infty} \frac{(1-q^{6h+3})}{(1-q^{3h+1})(1-q^{3h+2})}$$

$$= \sum_{n=0}^{\infty} \frac{q^{3n^2+3n}(-q^{3n+3};q^3)_\infty(q^{6n+9};q^6)_\infty}{(q^3;q^3)_n(q^{3n+2};q^3)_\infty(q^{3n+4};q^3)_\infty}$$

$$= \frac{(q^3;q^3)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{3n^2+3n}(q^2;q^3)_n(q;q^3)_{n+1}}{(q^3;q^3)_n(-q^3;q^3)_n(q^3;q^6)_{n+1}}$$

$$= \frac{(q^3;q^3)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{3n^2+3n}(q;q)_{3n+1}}{(q^3;q^3)_n^2(-q^3;q^3)_n(q^3;q^6)_{n+1}}$$

$$= \frac{(q^3;q^3)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{3n^2+3n}(q;q)_{3n+1}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}}$$

$$= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 1 \pmod 9}}^{\infty} \frac{1}{1-q^n} \qquad \text{(by [5, eq. (1.7), p. 422])}$$

This last infinite product is the generating function for partitions into parts $\not\equiv 0,\pm 1 \pmod 9$, and comparing coefficients of q^n in the extremes of the above string of inequalities, we see that our theorem is proved. $\qquad \square$

6   The Combinatorial Questions

6.1   A New Conjugacy

Recall our observation in Section 1 that the partitions without gaps (i.e. partitions with initial 1-repetitions) are the conjugates of the partitions with distinct parts. However, when $k > 1$, the partitions with k-repetitions are not mapped via conjugation onto partitions in which no part appears more than $2k-1$ times.

Question 1. What is an analog of the conjugacy map that will provide a bijection between partitions with initial k repetition and partitions in which no part appears more than $2k - 1$ times?

William Keith has both answered this question and generalized the new map.

6.2 An Analog of the Franklin Map

It is possible to rephrase Euler's Pentagonal Number Theorem [1; p. 10] in the language of this paper.

Theorem. $d_e(n) - d_o(n) = \begin{cases} (-1)^j & \text{if } n = j(3j \pm 1)/2 \\ 0 & \text{otherwise} \end{cases}$,

where $d_e(n)$ (resp. $d_o(n)$) is the number of partitions of n into partitions with initial 1-repetitions and largest part even (resp. odd).

Among the most famous proofs of this result is F. Franklin's "almost" bijection [2, pp. 10–11].

Question. Is there a proof of Theorem 2 that is an analog of the Franklin "almost" bijection?

Mike Rowell has discovered a beautiful bijection answer to this question.

6.3 The Borwein Conjecture

In [3], Peter Borwein's conjecture is studied in detail.

The Borwein Conjecture. The polynomial $A_n(q), B_n(q)$ and $C_n(q)$ given by

$$\prod_{j=1}^{n} (1 - q^{3j-2})(1 - q^{3j-1}) = A_n(q^3) - q B_n(q^3) - q^2 C_n(q^3)$$

all have non-negative coefficients.

By (3.1) and (4.9) of [3], we see immediately that it is only necessary to prove that $C_n(q)$ has non-negative coefficients.

Furthermore it is established [3, eq. (4.7)] that

$$C_n(q) = \sum_{0 \leq 3j \leq n-1} \frac{(q^3; q^3)_{n-j-1}(1 - q^{3j+1} - q^{n+3j+2} + q^n)(q; q)_{3j} q^{3j^2+3j}}{(q)_{n-3j-1}(q^3; q^3)_{2j+1}(q^3; q^3)_j}.$$

Letting $n \to \infty$ yields the Bailey identity which was central to the proof of Theorem 5.

Question. Is it possible to find some partition statistic associated with the partitions enumerated by $B(n)$ so that $C_N(q)$ is the generating function for these partitions subject to constraints on this new statistic?

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