The Theory of Compositions, IV: Multicompositions

by

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Abstract

The theory of ordered partitions, or compositions, originated with P. A. MacMahon. In this paper, we explore compositions wherein several copies of the integers are used as summands.

1 Introduction

Compositions are ordered partitions of integers. For example, there are eight compositions of 4: 4, $3+1$, $1+3$, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, $1+1+1+1$. In his initial study of compositions [4, p. 151], P. A. MacMahon noted that there are 2^{n-1} compositions of n.

Our object in this paper is to consider compositions wherein we use k copies of the positive integers as summands: $\{1_1, 1_2, \ldots, 1_k, 2_1, 2_2, \ldots, 2_k, 3_1, 3_2, \ldots, 3_k, ...\}$. We shall refer to these new compositions as multicompositions (or k-compositions when using exactly k copies of the integers). We shall add one restriction on the summands, namely, **THE LAST SUBSCRIPT IN THE COMPOSITION MUST BE 1**. If we do not throw this in then we will have k sets of k-compositions which will be identical

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except for the last subscript. For example, in the case of bi-partitions of 3, our restriction allows

$$3_1, 2_1+1_1, 2_2+1_1, 1_1+2_1, 1_2+2_1, 1_1+1_1+1_1, 1_2+1_1+1_1, 1_1+1_2+1_1, 1_2+1_2+1_1$$

Whereas allowing any final subscript would add nine additional bi-compositions of 3, namely

$$3_2, 2_1+1_2, 2_2+1_2, 1_1+2_2, 1_2+2_2, 1_1+1_1+1_2, 1_2+1_1+1_2, 1_1+1_2+1_2, 1_2+1_2+1_2.$$

In the interest of keeping the hand calculation of the relevant sums within reason, we have added this restriction.

In 1964, H. Gould [3, p. 251] studied compositions of n with relatively prime summands. We shall call this number the Gould function, $g_1(n)$. Thus $g_1(4) = 6$ because the compositions enumerated are $3+1$, $1+3$, $2+1+1$, $1+2+1$, $1+1+2$ and $1+1+1+1$. Clearly as is implied in Gould's paper

$$2^{n-1} = \sum_{d|n} g_1(d). \tag{1.1}$$

Recently E. Deutsch, in a problem in the American Mathematical Monthly [2], noted that $3|g_1(n)$ if $n \geq 3$, which means that since $g(1) = g(2) = 1$ the only time $g(n)$ is prime is when $n = 3$. Actually $g_1(n)$ is highly composite as the following table reveals.

n	$g_1(n)$	$g_1(n)$ factored	n	$g_1(n)$	$g_1(n)$ factored
1	1	1	11	1023	$3 \cdot 11 \cdot 31$
2	1	1	12	2010	$2 \cdot 3 \cdot 5 \cdot 67$
3	3	3	13	4095	$3^2 \cdot 5 \cdot 7 \cdot 13$
4	6	$2 \cdot 3$	14	8127	$3^3 \cdot 7 \cdot 43$
5	15	$3 \cdot 5$	15	16365	$3 \cdot 5 \cdot 1091$
6	27	3^3	16	32640	$2^7 \cdot 3 \cdot 5 \cdot 17$
7	63	$3^2 7$	17	65535	$3 \cdot 5 \cdot 17 \cdot 257$
8	120	$2^3 35$	18	130788	$2^2 \cdot 3^3 \cdot 7 \cdot 173$
9	252	$2^2 3^2 7$	19	262143	$3^3 \cdot 7 \cdot 19 \cdot 73$
10	495	$3^2 511$	20	523770	$2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 19$

Obviously there is much more divisibility going on here than just divisibility by 3. Is there a more general factorization theorem for $g_1(n)$ than that given by Deutsch's problem?

Of course, we may naturally generalize Gould's function to $g_k(n)$ the number of k-compositions of n wherein the summands are relatively prime (here we ignore subscripts, so the summands in $4_2 + 2_1 + 2_2$ are viewed as having 2 as a common divisor).

Here is a table of the first twenty values of $g_2(n)$:

n	$g_2(n)$	$g_2(n)$ factored	n	$g_2(n)$	$g_2(n)$ factored
1	1	1	11	59048	$2^3 \cdot 11^2 61$
2	2	2	12	176880	$2^4 \cdot 3 \cdot 5 \cdot 11 \cdot 67$
3	8	2^3	13	531440	$2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 73$
4	24	$2^3 \cdot 3$	14	1593592	$2^3 \cdot 7 \cdot 11 \cdot 13 \cdot 199$
5	80	$2^4 \cdot 5$	15	4782880	$2^5 \cdot 5 \cdot 167 \cdot 179$
6	232	$2^3 \cdot 29$	16	14346720	$2^5 \cdot 3^7 \cdot 5 \cdot 41$
7	728	$2^3 \cdot 7 \cdot 13$	17	43046720	$2^6 \cdot 5 \cdot 17 \cdot 41 \cdot 193$
8	2160	$2^4 \cdot 3^3 \cdot 5$	18	129133368	$2^3 \cdot 3^2 \cdot 7 \cdot 13 \cdot 19709$
9	6552	$2^3 \cdot 3^2 \cdot 7 \cdot 13$	19	387420488	$2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 757$
10	19600	$2^4 \cdot 5^2 \cdot 7^2$	20	1162241760	$2^5 \cdot 3 \cdot 5 \cdot 41 \cdot 73 \cdot 809$

The object of this paper will be to prove the following result. We use the notation $\phi(n)$ to denote the number of positive integers $\leqq n$ and relatively prime to n.

Theorem 1. *If the prime factorization of n is $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, then the number*

$$\ell.c.m \left(k(k+2)_3 (k+1)^{\phi(p_1^{\alpha_1})} - 1, \quad (k+1)^{\phi(p_2^{\alpha_2})} - 1, \dots, (k+1)^{\phi(p_r^{\alpha_r})} - 1 \right)$$

divides $g_k(n)$ provided $n \geqq 3$.

For example, if $n = 18 = 3^2 2$, $k = 2$, then

$$\operatorname{lcm}\left(2 \cdot 4, 3^{\phi(2)} - 1, 3^{\phi(3)} - 1\right) = \operatorname{lcm}(8, 2, 728) = 728 = 2^3 \cdot 7 \cdot 13.$$

The next section will be devoted to a proof of this theorem. We will conclude with some observations and open problems.

2 Proof of Theorem 1

In order to prove Theorem 1, it is sufficient to prove that each of the entries in the $\ell.c.m$ (= least common multiple) expression divides $g_k(n)$. First we require some preliminary results.

Lemma 2. *Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of integers,*

$$\sum_{n=m}^{\infty} b_n q^n = \sum_{n=m}^{\infty} \frac{a_n q^n}{1-q^n}, \tag{2.1}$$

and $j|b_n$ for $n \geqq m$. Then $j|a_n$ for $n \geqq m$.

Proof. We proceed by induction. We note that

$$\sum_{n=m}^{\infty} \frac{a_n q^n}{1-q^n} = \sum_{n=m}^{\infty} \sum_{s=1}^{\infty} a_n q^{ns}. \tag{2.2}$$

Hence the coefficient of q^m on the right-hand side is a_m; therefore $a_m = b_m$, so $j|a_m$.

Now assume that $j|a_n$ for $m \leqq n < N$. So

$$\begin{aligned} \sum_{n=N}^{\infty} \frac{a_n q^n}{1-q^n} &= \sum_{n=m}^{\infty} b_n q^n - \sum_{n=m}^{N-1} \frac{a_n q^n}{1-q^n} \\ &= \sum_{n=m}^{\infty} b_n q^n - \sum_{n=m}^{N-1} \sum_{s=1}^{\infty} a_n q^{ns}. \end{aligned} \tag{2.3}$$

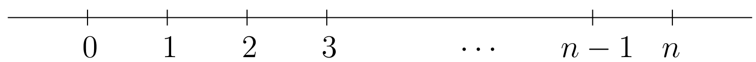
All the coefficients on the right-hand side of (2.3) are divisible by j, and in particular the coefficient of q^N is so divisible. But on the left-hand side we see that the coefficient of q^N is just a_N. Hence $j|a_N$, and the result follows by mathematical induction. $\square$

Lemma 3. *The total number of k-compositions of n is $(k+1)^{n-1}$.*

Proof. We modify MacMahon's proof of the case $k = 1$. Namely, we can geometrically represent the k-compositions as follows: choose n_1 numbers among $\{1, 2, \ldots, n-1\}$ to be labelled "1"; choose n_2 numbers among those remaining to be labelled "2", etc. up through k. The number of possible choices is

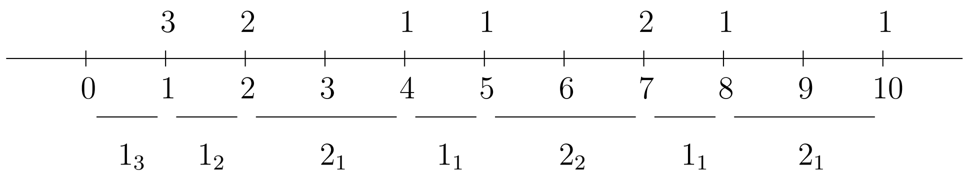
$$\begin{aligned} &\binom{n-1}{n_1, n_2, \ldots, n_k, n-1-n_1-n_2-\cdots-n_k} \\ &= \frac{(n-1)!}{n_1! n_2! \cdots n_k! (n-1-n_1-n_2-\cdots-n_k)!} \end{aligned}$$

These labelled points define integer length segments on the unit line,



and these segments make up the k-composition with the stipulation that the label of the right end determines the subscript of the part (n is labelled “1”).

For example. If $k = 3$, $n = 10$, $n_1 = 3$, $n_2 = 2$, $n_3 = 1$, then one possible choice is



yielding the 3-composition of 10: $1_3 + 1_2 + 2_1 + 1_1 + 2_2 + 1_1 + 2_1$. Thus the number of k-compositions of n with n_1 parts with subscript 1 (excluding the final part), n_2 parts with subscript 2, etc. is

$$\binom{n-1}{n_1, n_2, \ldots, n_k, n-1-n_1-n_2-\cdots-n_k}$$

and so the total number of k-compositions of n is

$$\begin{aligned} &\sum_{n_1,n_2,\ldots,n_k \geqq 0} \binom{n-1}{n_1, n_2, \ldots, n_k, n-1-n_1-n_2-\cdots-n_k} \\ &= (1 + \underbrace{1+1+1+\cdots+1}_{k \text{ terms}})^{n-1} \\ &= (k+1)^{n-1}. \end{aligned}$$

□

We are now in a position to prove Theorem 1. First we shall prove that $k(k+2)$ divides $g_k(n)$ for $n \geqq 3$.

We see immediately that

$$(k+1)^{n-1} = \sum_{d|n} g_k(d),$$

by classifying the k-compositions of n according to the greatest common divisor of their parts.

We may translate this into generating function form as follows:

$$\begin{aligned}
\frac{q}{1-(k+1)q} &= \sum_{n=1}^{\infty}(k+1)^{n-1}q^n \\
&= \sum_{n=1}^{\infty}\sum_{d\cdot e=n} g_k(d)q^n \\
&= \sum_{d=1}^{\infty}\sum_{e=1}^{\infty} g_k(d)q^{ed} \\
&= \sum_{d=1}^{\infty}\frac{g_k(d)q^d}{1-q^d}\,.
\end{aligned}$$

Now $g_k(1) = 1$ because $g_k(1)$ only counts 1_1, and $g_k(2) = k$ because $g_k(2)$ counts $1_1 + 1_1$, $1_2 + 1_1, \ldots, 1_k + 1_1$. Hence

$$\begin{aligned}
\sum_{d=3}^{\infty}\frac{g_k(d)q^d}{1-q^d)} &= \frac{q}{1-(k+1)q} - \frac{q}{1-q} - \frac{kq^2}{1-q^2} \\
&= \frac{k(k+2)q^3}{(1-q)(1-q^2)(1-(k+1)q)}
\end{aligned}$$

Therefore by Lemma 1, $k(k+2)$ divides $g_k(n)$ for $n \geqq 3$.

Next we must consider each of the primes p_i which occur in the prime factorizations of $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$.

So let

$$n = p^{\alpha}m,$$

where $(p,m) = 1$.

Then by Möbius inversion [1, p. 87]

$$\begin{aligned}
g_k(n) &= \sum_{d|n}\mu(d)(k+1)^{\frac{n}{d}-1} \\
&= \sum_{d|m}\mu(d)(k+1)^{p^{\alpha}\frac{m}{d}-1} \\
&\quad + \sum_{d|m}\mu(pd)(k+1)^{p^{a-1}\frac{m}{d}-1} \\
&\qquad\qquad \text{(because } \mu(d) = 0 \text{ if } p^2|d)
\end{aligned}$$

$$= \sum_{d|m} \mu(d)(k+1)^{p^{\alpha-1}\frac{m}{d}-1}\left((k+1)^{\frac{m}{d}(p^\alpha - p^{\alpha-1})} - 1\right).$$

Now each term in this last expression is clearly divisible by

$$(k+1)^{p^\alpha - p^{\alpha-1}} - 1 = (k+1)^{\phi(p^\alpha)} - 1$$

This concludes the proof of Theorem 1 because we have now shown that each of the factors in the ℓ.c.m. divides $g_k(n)$. □

Corollary 4. *If all the prime factors of n are relatively prime to $k+1$, then $n|g_k(n)$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Since $(p_i, k+1) = 1$, we know by Euler's Theorem [1, p. 62] that

$$(k+1)^{\phi(p_i^{\alpha_i})} \equiv 1 \pmod{p_i^{\alpha_i}}.$$

Hence for each i,

$$p_i^{\alpha_i} \left| \left\{ (k+1)^{\phi(p_i^{\alpha_i})} - 1 \right\} \right| g_k(n);$$

therefore,

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \Big| g_k(n).$$

□

Corollary 5. *If n is odd, then $n|g_1(n)$.*

Proof. This follows from Corollary 4 with $k = 1$. □

3 Conclusion

While Theorem 1 explains a lot about why $g_k(n)$ has many small prime factors, it clearly doesn't explain everything. For example,

$$g_3(36) = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 67 \cdot 241 \cdot 1{,}025{,}663{,}893.$$

whereas

$$\begin{aligned}\ell cm(3\cdot 5, 4^{\phi(9)}-1,\, 4^{\phi(4)}-1) &= \ell cm(3\cdot 5,\, 4^6-1,\, 4^2-1)\\ &= 3^2\cdot 5\cdot 7\cdot 13.\end{aligned}$$

This leaves $2^{10}\cdot 17$ unexplained.

So a natural project is the following:

Find other theorems like Theorem 1 that account for the other small prime factors of $g_k(n)$.

Also one may view $g_k(n)$ as a polynomial in k. In that case, Theorem 1 is still valid and asserts the divisibility of polynomials. So when $n = 36$

$$\begin{aligned}g_k(36) = k(k+1)^5&(k+2)(k^2+k+1)(k^2+2k+2)\\ &\times (k^2+3k+3)(k^4+4k^3+5k^2+2k+1)\\ &\times (k^{18}+18k^{17}+\cdots+48620\,k^9+\cdots+24k+1).\end{aligned}$$

On the other hand

$$\begin{aligned}&\ell.c.m.(k(k+2), (k+1)^{\phi(2^2)}-1,\, (k+1)^{\phi(3^2)}-1)\\ &= k(k+2)(k^2+k+1)(k^2+3k+3),\end{aligned}$$

and this leaves unexplained the factors

$$\begin{aligned}&(k+1)^5(k^2+2k+2)(k^4+4k^3+5k^2+2k+1)\\ &= (k+1)^5((k+1)^2+1)((k+1)^4-(k+1)^2+1).\end{aligned}$$

We note that each of the irreducible polynomial factors $p(k)$ of our ℓ.c.m. is such that $p(k-1)$ is a cyclotomic polynomial. So our final project is to account for all the other factors or $g_k(n)$ that are instances of cyclotomic polynomials evaluated at $k+1$.

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