

# The Discovery of Ramanujan's Lost Notebook

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IN HONOR OF THE 125th BIRTHDAY OF RAMANUJAN

## Abstract

The Lost Notebook has now become the starting point for much current mathematical research. In this paper, the unlikely chain of events leading to its discovery is chronicled.

Key Words: Ramanujan, Lost Notebook, mock theta functions

AMS Classification: Primary 01A60, Secondary 33D15.

## 1 Introduction

The story of the discovery of Ramanujan's Lost Notebook has been recounted several times. I have written several paragraphs on the discovery in each of [5], [6], and [10]. In Part I of Ramanujan's Lost Notebook [8], Bruce Berndt and I provide a plausible account of the travels of the Lost Notebook as it slowly made its way from Ramanujan's home to G. N. Watson's study.

I have not previously provided a fully documented account of the unlikely and seemingly unrelated events in my own life that preceded the discovery.

## 2 Rademacher and Mock Theta Functions

In 1961, I became a graduate student at the University of Pennsylvania. The Penn Mathematics Department was devoting a year to honoring Hans Rademacher at age 70. During the year, Rademacher taught a course on analytic number theory, and I was hooked. When I asked him if I could be his student, he agreed and suggested that I consider improving the work of

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Leila Dragonette [9] on Ramanujan's mock theta functions. This eventually led to my Ph.D. thesis [3]. At this time, Nathan Fine offered a course on Basic Hypergeometric Functions. During Fine's course, he asked me to report on G. N. Watson's paper on the fifth order mock theta functions [11]. Thus when I graduated from Penn, I probably knew more than anyone at that time about the mock theta functions especially those of fifth order. In the late 1960's, I published three papers [1], [2], [4] which built upon Watson's paper.

### 3 The First Letter from Lucy Slater

In 1970, I was to have my first sabbatical from Penn State. I wrote to Lucy Slater asking if I could visit her. Her response was truly intriguing:

SIDGWICK AVENUE  
CAMBRIDGE

Feb. 23rd. 70

Dear Dr. Andrews,

Thank you for your letter of Feb. 14th. I have a great collection of papers inherited from Prof. G.N. Watson, Prof. W.N. Bailey, and Rev. Dr. F.H. Jackson, in fact, complete collections of all the published papers of all three, whom I knew very well. There are also some letters from L.J. Rogers on R-R identities, and one MSS headed  $p(n)$  and  $t(n)$  functions. This was a copy, I think of the last paper written by Ramanujan, just before his death. Watson said it provided him with the starting ideas for his long series of papers on modular functions. Mrs. Watson, his widow, sent most of his papers to the library of Trinity College here, so any unpublished papers would be there. F.H. Jackson's papers went to his sister, who had my help in sorting them out,

and there were no letters from Ram. there. W.N. Bailey's papers were also sorted by me with the help of his widow, and there were no unpublished Ram. mss or letters there. The only other source would be Hardy's papers, which also, I think, went to his college. I have just heard from the Trinity librarian Mr. A. Halcrow, who says he has several boxes of uncatalogued papers from Watson's estate. The Trinity library is at present being extended so when you come across, you can write to him, asking for the papers to be brought out for you. This will take about a week. Hope you will be able to fix up about your finances.

Yours sincerely,

Dr. L.J. Slater Sc.D.

Unfortunately, I was unable to fund a trip to England in 1970, and the birth of our daughter Amy constrained travel in any event.

## 4 Winter in Wisconsin in 1976

During the early 1970's, Richard Askey and I corresponded extensively about  $q$ -series and related topics. His interest in them arose from his work on orthogonal polynomials and special functions while I came at them from the world of integer partitions.

This led to Askey inviting me to spend the year 1975-76 at the Mathematics Research Center in Madison.

In the spring of that year, I was invited by Dominique Foata to speak at a week long combinatorics conference in Strasbourg. Now an interesting quirk in airline pricing policy opened an opportunity for me. If one spent three weeks in Europe, the airfare was  $\epsilon$ , and if one stayed less than three weeks the fare was  $1/\epsilon$ . Could I justify to the MRC a three week visit to Europe? I recalled Lucy Slater's letter from February 1970. Perhaps I could build something around a visit to her. So I wrote the following to her:

February 23, 1976

Dr. L.J. Slater  
Department of Applied Economics  
University of Cambridge  
Sidgwick Avenue  
Cambridge, U.K.

Dear Dr. Slater,

At the end of April, I shall attend a mathematics conference in Strasbourg. I hope at that time to be able to spend a few weeks in France and England. I am very much hoping that you will be available in the first couple of weeks in May, for I would very much like to have the opportunity to meet you and to discuss basic hypergeometric functions.

I had hoped to get to England in 1970; however unfortunately my plans fell through at that time. You mentioned at that time in a letter to me in February 1970 that you possess a number of papers, manuscripts etc. by Bailey, Jackson, Watson and others that sound extremely interesting to me.

I do hope that this time my plans do not fall through. Best wishes!

Sincerely yours,

George E. Andrews

P.S. I thought you might enjoy the enclosed reprint.

Recalling her 1970 letter, I also wrote to the Trinity College librarian:

February 25, 1976

Librarian  
Trinity College  
Cambridge, England

Dear Sir:

During the first two weeks of May of this year I plan to visit Cambridge. Dr. Lucy J. Slater of the Applied Economics Department of the University of Cambridge informed me sometime back that there are several boxes of uncatalogued papers from the estate of G.N. Watson in your possession. I would like very much to make arrangements with you to see those papers when I visit. If there are any official requests that I should file before I come, please let me know.

Sincerely yours,

George E. Andrews  
Visiting Professor of Mathematics

The response from the librarian has been lost, but I recall that I was assured there would be no difficulty in accessing the Watson boxes.

I do have Lucy Slater's response:

SIDGWICK AVENUE  
CAMBRIDGE  
ENGLAND  
CB3 9DD

March 1st, 1976

Dear Dr. Andrews,

Thank you for your letter of Feb. 23rd. I confirm that I shall be in England and indeed in Cambridge during the month of May next.

Perhaps, when you have finalised your plans you would drop a note to The Secretary, Faculty of Pure Mathematics, Mill Lane, Cambridge, giving the dates you will be here, and saying that you will be visiting me. You will then probably be offered full rights as a formal visitor, with desk, and library rights, and the use of the University Centre. She may also be able to help you with accommodation, if you are alone. If your wife is with you, I can offer the use of a double room in my own house, as my guest, but you would have to look after yourselves, as I have no domestic help. Looking forward to seeing you

Yours sincerely,

L.J. Slater  
Head of Computing

P.S. Please contact also Harold Exton, who is more active in the field than I am. He may be able to pay a short visit to Cambridge at the same time. His address is 27 Hollinhurst Ave, Penwortham, Preston, Lancs, England.

He has just finished a book on multiple hypergeometric functions.

Now it was up to me to convince the MRC director, Ben Noble, that my three weeks in Europe were justified:

March 9, 1976

Dear Professor Noble:

I am requesting permission to visit France and England from April 22 to May 14, 1976. The first week of my trip will be spent in Strasbourg, France attending the conference "Combinatorics and the Representation Theory of the Symmetric Group". The second week will be spent in Paris, Southhampton, and London; in these cities I have arranged to confer with several mathematicians (Paris: Malliavin, Comtet; Southhampton: Lloyd, King; London: Hughes) whose research activities are related to my own and who I feel will be able to contribute significantly to my own research work. Also in London I hope to investigate some materials in the archives of several institutions (such as the Royal Society); these materials are also related to my current research. The third week will be spent in Cambridge. Part of the time I will be conferring with L.J. Slater (H. Exton, et al.) concerning the hypergeometric series work that I am doing with Dick Askey, and part will be spent investigating the invaluable, unpublished materials of the late G.N. Watson that are kept in the Trinity College Library (see attached letter).

I hope that you will find it possible to grant the above requested administrative leave.

Sincerely yours,

George E. Andrews  
Visiting Professor

Ben Noble's response was just what I had hoped for:

March 18, 1976

Professor George E. Andrews  
Mathematics Research Center

Dear George:

You can certainly have administrative leave from April 22 to May 14, 1976 to continue your research activities in Europe.

When you are at Trinity College, Cambridge, you might ask if they have the notebooks and manuscripts of Clark Maxwell, of electro-magnetic theory fame. There is one result in his commentary on the collected papers of Cavendish that requires elucidation and it might be possible to track this down in his manuscript. I will try to get some information about this to you before you leave.

Sincerely,

Ben Noble  
Director

Prior to the actual trip to France and on to England, there followed one further letter from Lucy Slater:

SIDGWICK AVENUE  
CAMBRIDGE  
ENGLAND  
CB3 9DD

March 22nd, 1976



Dear Dr. Andrews,

Thank you for your letter of March 11th. The point about stopping at my house is two fold. First, you may have considerable difficulty in finding any accomodation at all in Cambridge, as May is the peak time for tourists, and full term for the students. Secondly, I have a largeish house, and since the death of my mother last year I live there alone, so welcome all visitors.

I have taught the infant school and your daughters will be as welcome as you and your wife. The third point is that, if no one else is stopping in the house, I can hardly let Harold Exton come for the night to meet you, and it is too far for him to drive over from Preston for a day trip. So please, come and stop with me. You will then have all my collection of papers on Hyp. Geo. on the spot, and can read them at your leisure.

Yours sincerely,

Dr. L.J. Slater Sc.D.

So everything was set. All the promised activities looked interesting, but it is worth noting that there is no suggestion in any of this correspondence that Ramanujan's Lost Notebook might be lurking in the wings.

## 5 May, 1976

The actual day of discovery is recounted as follows in [6, p. 5-6]:

When the librarian presented me with a list of the items from Watson's estate, I noticed one in particular: "A 139 page manuscript by S. Ramanujan on  $q$ -series." Upon examination, I found enough development of the mock theta functions to become convinced that I held in my hands the mathematical work that had been done by Ramanujan during the last year of his life

when he had returned to India and was quite ill. The clue to this understanding lay in the single letter written to Hardy by Ramanujan after his return to India. The letter described Ramanujan's new discovery: mock theta functions. He included a few results to tantalize Hardy, but the theory was quite incomplete and no details were given. Since this manuscript contained all of Ramanujan's statements about the mock theta functions, it had to contain a substantial amount of work out of his last year. I realized most of the above shortly after starting to look at the document because I had written my Ph.D. thesis on mock theta functions and had studied them intermittently ever since.

I was now quite excited and a bit worried. I *had* to get a copy of this document since I was to leave Cambridge in a couple of days. Would they Xerox it? If I had not been so desperate, I wouldn't have had the nerve to ask. The Wren Library seems more like a shrine than the Instructional Media Center of American experience. However I gathered up my courage, tiptoed up to the desk, and stammered out my request. "Of course we can Xerox it," replied the friendly woman behind the desk. "We should have it for you in a couple of weeks." I explained that I was leaving England in a few days so could she airmail it to me in the U.S. This seemed somewhat extravagant to her; however if I wished to pay the additional postage of more than £7, it was okay. Since I was prepared to take a second mortgage on my house to get hold of this document, I happily shelled out the extra £7.

The next day I returned to examine some other materials in the Wren Library. Upon entering I noticed a large manila envelope at the desk that was addressed to me and ready to be taken to the post office. Apparently British Xeroxing was somewhat more efficient than I had been led to believe. With some difficulty I persuaded the new person behind the desk that since the package was addressed to me, it would be all right if I just took it. Thus I flew out of England in May 1976 with a complete copy of Ramanujan's "Lost" Notebook.

## 6 Epilogue

There are a few minor details to add as a denouement.

I was so pleased and thrilled by the discovery of the Lost Notebook that I took my wife and daughters (Amy, then age 5, and Katy, age 3) punting on the Cam. A punt is a long boat propelled by a person standing at the very end of the boat who directs it with a long pole. I had successfully punted on the Cam in 1961 when my wife and I lived in Cambridge. However, my punting skill had apparently atrophied, and a stuck pole pulled me into the river. The taxi ride back to Lucy's produced a very angry cab driver when he found that my wet clothes soaked the seat of the cab.

Finally, on this same trip I had the privilege of studying Lucy Slater's collection of letters. The background of this story was recounted as follows in [7, p. 975]. The most interesting items in the collection were two letters from L.J. Rogers to F.H. Jackson [7, p. 977-991] describing Rogers' response to the hubbub surrounding the Rogers-Ramanujan identities:

I discovered this letter during my visit to Cambridge in 1976, when I also discovered Ramanujan's "Lost" Notebook. My wife, Joy, and I, and our small daughters, Amy and Katy, were house guests of Lucy Slater. One evening, Lucy announced: "When a mathematician visits Cambridge, his host takes him off to Hall (a formal hall dinner with the college Fellows), and his wife is left to babysit. Tonight I will be taking your wife to Hall and you will be babysitting." I would note as explanation that the gender segregation of the Cambridge colleges in 1976 meant that Lucy could not invite me to Hall at Lady Lucy Cavendish College. At the time, I am sure Lucy thought that my wife was getting the best deal. My reaction, however, was much like that of Brer Rabbit, upon being tossed in the briar patch. After Amy and Katy were in bed, I got to go through Lucy's extensive collection of letters and reprints. By far the most appealing discovery of the evening was this illuminating letter of Rogers.

Lucy Slater's collection of letters subsequently was given to Harold Exton who then gave them to me in the 1990's. Here are the complete Rogers letters:

The University of  
Leeds

Feb. 13th

Dear Sir:

It is with a certain amount of amusement that a theorem which I proved nearly 24 years ago should have remained in obscurity so long, and recently brought (sic) into prominence as a conjecture.

MacMahon wrote to me on the publication of his work regretting that he had overlooked my work before it was too late. Since then I have found out other ways of proving both the identities in a more direct way, which have been published in the London Math. Soc.. My extra copies are at my home and I will send you one.

I have another proof besides which I sent to MacMahon is case he brings out a new edition which is as simple as one can make it I think. I enclose two old papers of mine. They have lain by many years and Leeds dirt penetrates everywhere, so I must apologize for their condition.

They contain too a disgraceful number of errata, which I have listed in my recent paper.

I have tried to extend the idea but without success viz. seeing that

$$\begin{aligned} 1 - \frac{q}{1-q} + \frac{q^3}{(1-q)(1-q^2)} - \frac{q^6}{(1-q)(1-q^2)(1-q^3)} + \cdots \\ = (1-q)(1-q^2)(1-q^3) \cdots \\ = 1 - q - q^2 + q^5 + q^7 - \cdots \end{aligned}$$

a  $\theta$ -function of 3/2th order.

Also

$$\left. \begin{array}{l} 1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \dots \\ 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \dots \end{array} \right\} \text{depend upon } \theta\text{-functions of order } 5/2.$$

Will

$$\left. \begin{array}{l} 1 + \frac{q^3}{1-q} + \frac{q^9}{(1-q)(1-q^2)} + \dots \\ 1 + \frac{q^2}{1-q} + \frac{q^7}{(1-q)(1-q^2)} + \dots \\ 1 + \frac{q}{1-q} + \frac{q^5}{(1-q)(1-q^2)} + \dots \end{array} \right\} \text{depend upon } \theta\text{-functions of order } 7/2?$$

I do not get at anything, but as you will see from my last paper certain simple basic series are connected with  $\theta$ -functions of order  $3/2, 3, 5/2, 5, 7/2, 15/2, 21/2$ , but I got no further.

I cannot just now lay my hands on a copy of my old “first memoir” which dealt with the operator

$$\eta f(x) = \frac{f(x) - f(xq)}{x}$$

and its affects on  $(1 - \lambda x)(1 - \lambda xq)(1 - \lambda xq^2) \dots$  to  $\infty$ , but you will find it in the London M.S. of about 1893.

With many thanks for your paper and any other if you have them.

Yours truly,

L.J. Rogers

6 Hollin Lane, Leeds

Feb. 14th

Dear Sir:

The proof of the identities which I referred to in my last letter to you has not been published, but is as follows: Let  $C_r$  denote  $(1-x)(1-xq)\cdots(1-xq^{r-1})/(1-q)(1-q^2)\cdots(1-q^r)$ . Take  $V_m$  to denote

$$(1 - a_0x^m) - a_0a_1x^nq^{n+1-m}C_1(1 - a_2x^mq^{2m}) \\ + a_0a_1a_2a_3x^{2n}q^{4n+3-2m}C_2(1 - a_4x^mq^{4m}) - \cdots$$

The general term being

$$+(-1)^ra_0a_1\cdots a_{2r-1}x^{rn}q^{r^2n+r(r+1)/2-rm}C_r(1 - a_{2r}x^mq^{2rm}) \\ \vdots$$

Then

$$V_m - V_{m-1} = a_0x^{m-1}(1-x) \\ - a_0a_1x^nq^{n+1-m}C_1(1-q+a_2x^{m-1}q^{2m-1}(1-xq)) + \cdots$$

so that

$$\frac{V_m - V_{m-1}}{1-x} = a_0x^{m-1}((1 - a_1x^{n-m+1}q^{n+1-m}) \\ - a_1a_2x^nq^{n+m}(\eta C_1)(1 - a_3x^{n+1-m}q^{3(n+1-m)}) + \cdots)$$

where

$$\eta f(x) = f(xq).$$

Then

$$\frac{V_m - V_{m-1}}{1-x} = a_0x^{m-1}\eta V_{n-m+1} \quad (2.1)$$

with the suffixes of the  $a$ 's increased by 1. This is unnecessarily general as we only want the case where every  $a = 1$ .

If

$$V_m = \nu_m(1-x)(1-xq)\cdots \text{ to } \infty$$

we have

$$\nu_m - \nu_{m-1} = x^{m-1}\eta\nu_{n-m+1}.$$

Let  $n = 2$ , then since evidently  $\nu_0 = 0$

$$\nu_1 = \eta\nu_2 \quad \text{when} \quad m = 1$$

$$\nu_2 - \nu_1 = x\eta\nu_1 \quad \text{when} \quad m = 2$$

$$\therefore \quad \nu_1 = \eta\nu_1 + xq\eta^2\nu_1.$$

Let

$$\nu_1 = 1 + c_1x + c_2x^2 + \cdots$$

then

$$\begin{aligned} 1 + c_1x + c_2x^2 + \cdots &= 1 + c_1qx + c_2q^2x^2 + \cdots \\ &+ xq(1 + c_1q^2x + c_2q^4x^2 + \cdots) \end{aligned}$$

$$v_1 = 1 + \frac{xq}{1-q} + \frac{x^2q^4}{(1-q)(1-q^2)} + \cdots = \frac{1}{\prod_{n=1}^{\infty}(1-xq^n)}V_1.$$

Let  $x = q$ , then all the  $C's = 1$  and

$$V_1 = 1 - q - q^4 + \cdots$$

the  $\theta$ -series. The other identity comes from  $\nu_2$ , when  $x = q$ . You will see that the idea of  $V_m$  is very widely general and when all the  $a's$  are unity leads to extensions for higher values of  $n$ .

It may be shown that  $V_m$  actually vanishes for all the factors  $1-x, 1-xq, 1-xq^2, \dots$  equated to zero, but this property is not necessary for the above proof.

When all  $a's = 1$ , or all  $= -1$ , we also get

$$\frac{V_m - xV_{m-1}}{1-x} = \eta V_{n+m} \quad \text{and} \quad V_{-m} = -a_0x^{-m}V_m$$

which accounts for all suffixes (positive, negative, or zero).

When all the  $a's = -1$  and  $x = q$ ,  $V_m$  becomes a false  $\theta$ -series like those in §9 p. 332 of the enclosed paper, which seem to me very remarkable. I have not considered them from the point of view of the  $V$ -method.

Yours truly,

L.J. Rogers

N.B. I find we have friends in common in the Gotts our present Lord Mayor and Lady Mayoress. I do not see them as much as I could wish as they nearly always go to Newby Wiske for weekends and other weekdays are most devoted to committees. Otherwise my common interests with Mr and Mrs Gott in in alpinism.

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