PARTITIONS WITH DISTINCT EVENS

GEORGE E. ANDREWS

In honor of the 70th birthday of Georgy Egorychev

Abstract. Partitions with no repeated even parts (DE-partitions) are considered. A DE-rank for DE-partitions is defined to be the integer part of half the largest part minus the number of even parts. $\Delta(n)$ denotes the excess of the number of DE-partitions with even DE-rank over those with odd DE-rank. Surprisingly $\Delta(n)$ is (1) always non-negative, (2) almost always zero, and (3) assumes every positive integer value infinitely often.

1. Introduction

In [4] Ramanujan's series [16; p. 14]

$$R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty} S(n) q^n \tag{1.1}$$

was examined. Here

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{1.2}$$

It was shown [4] that $S(n)$ is almost always equal to zero and also assumes every integral value infinitely often. Combinatorially $S(n)$ is the excess of the number of partitions of n into distinct parts with even rank over those with odd rank. The *rank* of a partition is the largest part minus the number of parts [7], [5]. A similar theorem was proven [4; Sec. 5] for partitions into odd parts without gaps.

The results for $S(n)$ rely crucially on the identity [4; p. 392]

$$R(q) = \sum_{\substack{n \geqq 0 \\ |j|<n}} (-1)^{n+j} q^{n(3n+1)/2 - j^2}(1-q^{2n+1}). \tag{1.3}$$

It was noted at the end of [4] that there are numerous series similar in form to the right-hand expression in (1.3).

Indeed, results of this nature were given for Ramanujan's fifth order mock theta functions [2] (c.f. [16]), and such identities formed the basis for pathbreaking work by Zwegers [18] and Bringmann, Ono and Rhoades [6].

The object of this paper is to reveal a similar phenomenon connected to DE-partitions, i.e. partitions with no repeated even parts. Now DE-partitions have been examined previously. R. Honsberger [12] proved the following Euler-type theorem.

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Theorem 1. *Let $P_{DE}(n)$ denote the number of partitions of n with no repeated even parts. Let $P_{<4}(n)$ denote the number of partitions of n in which no part appears more than thrice. Let $P_{\nmid 4}(n)$ denote the number of partitions of n into parts not divisible by 4. Then*

$$P_{DE}(n) = P_{<4}(n) = P_{\nmid 4}(n)$$

for each $n \geqq 0$.

Honsberger's proof is immediate from the following identification of the related generating functions

$$\begin{aligned}\sum_{n\geqq 0} P_{DE}(n)q^n &= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty} \\ &= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \sum_{n\geqq 0} P_{\nmid 4}(n)q^n \\ &= \prod_{n=1}^{\infty}(1+q^n+q^{2n}+q^{3n}) = \sum_{n\geqq 0} P_{<4}(n)q^n.\end{aligned}$$

The fact that $P_{<4}(n) = P_{\nmid 4}(n)$ is due to J. W. L. Glaisher [10], and the asymptotics of these partition functions has been completely examined by P. Hagis [11].

From here on, our focus will be on the DE-rank of DE-partitions which is defined to be the integer part of half the largest part minus the number of even parts.

We let $\delta(m,n)$ denote the number of DE-partitions of n with DE-rank m.

Theorem 2.

$$\sum_{m,n\geqq 0}\delta(m,n)z^m q^n = 1 + \sum_{j\geqq 0}\frac{(-z^{-1}q^2;q^2)_j z^j q^{2j+1}(1+q)}{(q;q^2)_{j+1}}. \tag{1.4}$$

Next we write

$$\Delta(n) = \sum_{m\geqq 0}(-1)^m\delta(m,n). \tag{1.5}$$

Theorem 3.

$$\sum_{n\geqq 0}\Delta(n)(-q)^n = \sum_{n\geqq 0}\frac{(-1)^n q^{n(n+1)/2}(q;q)_n}{(-q)_n}. \tag{1.6}$$

This leads to a result analogous to (1.3) which allows us to obtain all the desired arithmetic properties of $\Delta(n)$.

Theorem 4.

$$\sum_{n\geqq 0}\Delta(n)q^n = \sum_{n=0}^{\infty}\left(q^{\binom{2n+1}{2}} + q^{\binom{2n+2}{2}}\right)\sum_{j=-n}^{n} q^{-j^2}. \tag{1.7}$$

From here we can relate $\Delta(n)$ to the arithmetic of $Q(\sqrt{2})$.

Theorem 5. *$\Delta(n)$ is the number of inequivalent elements of the ring of integers of $Q(\sqrt{2})$ with norm $8n+1$.*

Corollary 6. *$\Delta(n)$ is always non-negative.*

Corollary 7. *$\Delta(n)$ is almost always equal to zero.*

Corollary 8. $\Delta(n)$ *is equal to any given positive integer infinitely often.*

I thank Dean Hickerson for an extensive set of comments on this paper. In particular he has noted that $\Delta(n)$ is also the number of divisors of $8n+1$ which are congruent to ± 1 modulo 8 minus the number which are congruent to 3 or 5 modulo 8. Consequently, $\Delta(n)$ is the coefficient of $8n+1$ in

$$\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\left(\frac{2}{n}\right) q^n}{1-q^n},$$

where $\left(\frac{2}{n}\right)$ is the Legendre symbol. Hickerson also suggests that a similar investigation with $8n+1$ replaced by $8n+7$ seems to be of interest.

Finally I note that A. Patkowski [15] has recently found two related theorems for DE-partitions. His theorems provide other lacunary series arising from DE-partition statistics other than the rank.

2. Proof Theorem 2

For those DE-partitions with largest part $2j+1$, the DE-rank generating function is

$$\frac{(1+z^{-1}q^2)(1+z^{-1}q^4)\cdots(1+z^{-1}q^{2j})z^j q^{2j+1}}{(1-q)(1-q^3)\cdots(1-q^{2j+1})}.$$

For those DE-partitions with largest part $2j+2$, the DE-rank generating function is

$$\frac{(1+z^{-1}q^2)(1+z^{-1}q^4)\cdots(1+z^{-1}q^{2j})z^j q^{2j+2}z^{-1}}{(1-q)(1-q^3)\cdots(1-q^{2j+1})}.$$

We take the empty partition to have DE-rank 0, and so adding together the empty case, the odd case and the even case we find

$$\sum_{m,n \geqq 0} \delta(m,n) z^m q^n = 1 + \sum_{j=0}^{\infty} \frac{(-z^{-1}q^2;q^2)_j z^j (q^{2j+1}+q^{2j+2})}{(q;q^2)_{j+1}},$$

which is equivalent to Theorem 2. $\square$

3. Proof of Theorem 3

By Theorem 2 with z replaced by -1 and q replaced by $-q$, we see that

$$\begin{aligned}\sum_{n \geqq 0} \Delta(n)(-q)^n &= \sum_{m,n \geqq 0} \delta(m,n)(-1)^{m+n} q^n \\ &= 1 + \sum_{j \geqq 0} \frac{(q^2;q^2)_j (-1)^{j-1} q^{2j+1}(1-q)}{(-q;q^2)_{j+1}} \\ &= 1 - \frac{q(1-q)}{1+q} \sum_{j \geqq 0} \frac{(q^2;q^2)_j (q^2;q^2)_j (-q^2)^j}{(q^2;q^2)_j (-q^3;q^2)_j} \\ &= 1 + \sum_{j=0}^{\infty} \frac{(q;q^2)_{j+1}(-q)^{j+1}}{(-q^2;q^2)_{j+1}}\end{aligned}$$

(by [9; eq. (III.2), p. 241 with $q \to q^2$, then $a = b = q^2$, $z = -q^2$, $c = -q^3$])

$$= \sum_{j=0}^{\infty} \frac{(q;q^2)_j(-q)^j}{(-q^2;q^2)_j}$$

$$= \sum_{j\geqq 0} \frac{(q;q^2)_j(q^2;q^2)_j}{(-q^2;q^2)_j(-q;q^2)_j}(q^3)^j q^{2j^2-2j}(1+q^{4j+2})$$

(by [3; eq. (9.1.1), p. 223, $q \to q^2$, then $\alpha = q$, $\beta = -q^2$, $\tau = -q$])

$$= \sum_{j\geqq 0} \frac{(q;q)_{2j}}{(-q;q)_{2j+1}} q^{2j^2+j}\big(1+q^{2j+1}-q^{2j+1}(1-q^{2j+1})\big)$$

$$= \sum_{j=0}^{\infty} \frac{(q;q)_{2j}q^{\binom{2j+1}{2}}}{(-q;q)_{2j}} - \sum_{j=0}^{\infty} \frac{(q;q)_{2j+1}q^{\binom{2j+2}{2}}}{(-q;q)_{2j+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(q;q)_n}{(-q;q)_n}. \qquad \square$$

4. Proof of Theorem 4

We begin with the limiting case of Bailey's Lemma [1; p. 270, $p = q$, $\rho_2, N \to \infty$]. If

$$\beta_n = \sum_{r=0}^{\infty} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{4.1}$$

then

$$(1-a)\sum_{n\geqq 0} \frac{(-1)^n(q;q)_n a^n q^{n(n-1)/2}\alpha_n}{(a;q)_n} = \sum_{n\geqq 0}(q;q)_n(-1)^n a^n q^{n(n-1)/2}\beta_n. \tag{4.2}$$

Now if $a = q$, and $\beta_n = 1/(-q;q)_n$, then by [2; eq. (4.8) and eq. (5.7)]

$$\alpha_n = \frac{q^{n^2+\binom{n}{2}}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n}(-1)^j q^{-j^2}. \tag{4.3}$$

Substituting into (4.2), we see that

$$\begin{aligned} \sum_{n\geqq 0}(-1)^n q^{2n^2+n}(1-q^{2n+1}) \sum_{j=-n}^{n}(-1)^j q^{j^2} &= \sum_{n\geqq 0} \frac{(-1)^n(q;q)_n q^{n(n+1)/2}}{(-q;q)_n} \\ &= \sum_{n\geqq 0} \Delta(n)(-q)^n, \end{aligned} \tag{4.4}$$

by Theorem 3. If we now replace q by $-q$ in (4.4), we have Theorem 4.

5. Proof of Theorem 5 and Its Corollaries

Let us recall the relevant information concerning the ring of integers in $Q(\sqrt{2})$. There is a complete account occupying Chapter VII of [17], but numerous books on algebraic number theory provide the same material.

The ring of integers in $Q(\sqrt{2})$ is $\mathbb{Z}[\sqrt{2}]$. The group of units in $\mathbb{Z}[\sqrt{2}]$ consists of

$$\pm(1+\sqrt{2})^n, \quad -\infty < n < \infty.$$

The norm of $a+b\sqrt{2}$ is

$$N(a+b\sqrt{2}) = a^2 - 2b^2 = (a+b\sqrt{2})(a-b\sqrt{2}).$$

The conjugate of $a+b\sqrt{2}$ is $a-b\sqrt{2}$ and is often written $\overline{a+b\sqrt{2}}$.

$\mathbb{Z}[\sqrt{2}]$ is a unique factorization domain. Two elements of $\mathbb{Z}[\sqrt{2}]$ are said to be equivalent if each is equal to the other multiplied by a unit.

The primes in $\mathbb{Z}[\sqrt{2}]$ all arise from the factorization of primes in $\mathbb{Z}$ as follows:

(i) $\sqrt{2}$ is prime.

(ii) All primes of the form $8N \pm 1$ in $\mathbb{Z}$ split into two primes π_1 and $\overline{\pi_1}$ in $\mathbb{Z}[\sqrt{2}]$.

(iii) All primes of the form $8N \pm 3$ in $\mathbb{Z}$ remain prime in $\mathbb{Z}[\sqrt{2}]$.

Arguments exactly analogous to those used for the Gaussian integers [13; p. 126] show that each integer

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

is the norm of an element of $\mathbb{Z}[\sqrt{2}]$ if and only if α_i is even for each prime $\equiv \pm 3 \pmod 8$.

Next we note that the proof of Lemma 3 in [3; p. 396] is adequate to show that among the equivalent solutions of

$$u^2 - 2v^2 = (u-\sqrt{2}v)(u+\sqrt{2}v) = m > 0$$

there is exactly one (u_1, v_1) with

$$-\tfrac{1}{2}u_1 < v_1 \leqq \tfrac{1}{2}u_1, \quad u_1 > 0.$$

Proof of Theorem 5. By Theorem 4, $\Delta(m)$ is the number of solutions of

$$m = \binom{n+1}{2} - j^2$$

where $|j| \leqq \frac{1}{2}n$ and $n > 0$. Thus

$$8m+1 = (2n+1)^2 - 2(2j)^2.$$

Now any solution of

$$8m+1 = u^2 - 2v^2 = N(u+\sqrt{2}v)$$

must necessarily have u odd and v even. Furthermore the solution $u = 2n+1$, $v = 2j$ has

$$-\tfrac{1}{2}(2n+1) < 2j \leqq \tfrac{1}{2}(2n+1)$$

which is equivalent to $|j| \leqq \frac{1}{2}n$.

Consequently $\Delta(m)$ is the number of inequivalent elements of $\mathbb{Z}[\sqrt{2}]$ with norm $8m+1$. $\square$

Proof of Corollary 6 follows directly from Theorem 5.

Proof of Corollary 7 follows exactly in the manner of the proof of Theorem 4 with the alteration

$$\sum_{p\equiv\pm3 \pmod 8} \frac{1}{p}$$

diverges owing to the strong version of Dirichlet's theorem [14; bottom of p. 217].

Proof of Corollary 8 follows directly from the fact that there are infinitely many primes $p \equiv \pm1 \pmod 8$, and for each of these

$$\Delta\left(\frac{p^\alpha - 1}{8}\right) = (\alpha + 1).$$

Thus every integer > 1 appears infinitely often.

On the other hand, there are infinitely many primes $p \equiv \pm3 \pmod 8$, and

$$\Delta\left(\frac{p^2 - 1}{8}\right) = 1.$$

6. Conclusion

There are a number of natural questions that arise from this study. First, a combinatorial proof of Theorem 4 might be possible and is much to be desired. The same is true for Theorem 3, and this may well be the easier project.

In addition, the ordinary rank of Dyson has led both to explications of the Ramanujan congruence for $p(n)$ [5], [7] and to surprising and appealing combinatorial theorems (cf. [8; eqs. (2.3.91), (2.4.6)]. These aspects of the DE-rank are completely unexplored.

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Department of Mathematics, The Pennsylvania State University, University Park, PA 16802

E-mail address: andrews@math.psu.edu