A REFINEMENT OF THE ALLADI-SCHUR THEOREM

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Abstract. K. Alladi first observed a variant of I. Schur's 1926 partition theore. Namely, the number of partitions of n in which all parts are odd and none apppears more than twice equals the number of partitions of n in which all parts differ by at least 3 and more than 3 if one of the parts is a multiple of 3. In this paper we refine this result to one that counts the number of parts in the relevant partitions.

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1. Introduction

In 1926, I. Schur [7] proved the following result:

Theorem 1. *Let $A(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$. Let $B(n)$ denote the number of partitions of n into distinct nonmultiples of 3. Let $D(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$ where $b_i - b_{i+1} \geq 3$ with strict inequality if $3|b_i$. Then*

$$A(n) = B(n) = D(n).$$

K. Alladi [1] has pointed out (cf. [4, p. 46, eq. (1.3)]) that if we define $C(n)$ to be the number of partitions of n into odd parts with none appearing more than twice, then also

$$C(n) = D(n).$$

This follows immediately from the fact that

$$\begin{aligned}
\sum_{n=0}^{\infty} C(n)q^n &= \prod_{n=1}^{\infty} \left(1+q^{2n-1}+q^{4n-2}\right) \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{6n-3})}{(1-q^{2n-1})} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{6n-3})}{(1-q^{6n-5})(1-q^{6n-3})(1-q^{6n-1})} \\
&= \prod_{n=1}^{\infty} \frac{1}{(1-q^{6n-5})(1-q^{6n-1})} \\
&= \sum_{n=0}^{\infty} A(n)q^n = \sum_{n=0}^{\infty} D(n)q^n.
\end{aligned}$$

Rather surprisingly the following refinement has been overlooked:

Theorem 2. *Let $C(m,n)$ denote the number of partitions of n into m parts, all odd and none appearing more than twice. Let $D(m,n)$ denote the number of partitions of n into parts of the type enumerated by $D(n)$ with the added condition that the total number of parts plus the number of even parts is m (i.e. m is the weighted count of parts where each even is counted twice).*

For example $C(4,16)=6$ with the relevant partitions being $11+3+1+1$, $9+5+1+1$, $9+3+3+1$, $7+7+1+1$, $7+5+3+1$, $5+5+3+3$ *while $D(4,16)=6$ with the relevant partitions being* $14+2$, $12+4$, $11+4+1$, $10+6$, $10+5+1$, $9+5+2$.

This theorem is analogous to W. Gleissberg's comparable refinement of the assertion that $B(n)=D(n)$ [5], and the proof is analogous to the proof of Gleissberg's theorem given in [2].

2. Proof of Theorem 2.

We define $d_N(x,q)=d_N(x)$ to be the generating function for partitions of the type enumerated by $D(m,n)$ with the added condition that all parts by $\leq N$.

We also define

$$\epsilon(n)=\begin{cases}1, & \text{if } n \text{ is odd,}\\ 2, & \text{if } n \text{ is even.}\end{cases} \tag{2.1}$$

Then for $n \geq 0$

$$d_{3n}(x)=d_{3n-1}(x)+x^{\epsilon(3n)}q^{3n}d_{3n-4}(x), \tag{2.2}$$

$$d_{3n+1}(x) = d_{3n}(x) + x^{\epsilon(3n+1)} q^{3n+1} d_{3n-2}(x), \tag{2.3}$$

$$d_{3n+2}(x) = d_{3n+1}(x) + x^{\epsilon(3n+2)} q^{3n+2} d_{3n-1}(x), \tag{2.4}$$

with the initial condition $d_{-1}(x) = d_{-2}(x) = 1$, $d_{-4}(x) = 0$.

In preparation for the essential functional equations needed to prove Theorem 2, we note that

$$d_{3n+1}(x) = d_{3n+2}(x) - x^{\epsilon(3n+2)} q^{3n+2} d_{3n-1}(x). \tag{2.5}$$

Thus substituting (2.2) and (2.5) into (2.3), we find

$$\begin{aligned} d_{3n+2}(x) = &\left(1 + x^{\epsilon(3n+1)} q^{3n+1} + x^{\epsilon(3n+2)} q^{3n+2}\right) d_{3n-1}(x) \\ &+ \left(x^{\epsilon(3n)} q^{3n} - x^{\epsilon(3n+1)+\epsilon(3n-1)} q^{6n}\right) d_{3n-4}(x). \end{aligned} \tag{2.6}$$

Consequently

$$\begin{aligned} d_{6n+2}(x) = &\left(1 + xq^{6n+1} + x^2 q^{6n+2}\right) d_{6n-1}(x) \\ &+ \left(x^2 q^{6n} - x^2 q^{12n}\right) d_{6n-4}(x), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} d_{6n-1}(x) = &\left(1 + x^2 q^{6n-2} + xq^{6n-1}\right) d_{6n-4}(x) \\ &+ \left(xq^{6n-3} - x^4 q^{12n-6}\right) d_{6n-7}(x). \end{aligned} \tag{2.8}$$

Lemma 3. *For $n \geq 1$,*

$$d_{6n+2}(x) = \left(1 + xq + x^2 q^2\right) d_{6n-1}(xq^2), \tag{2.9}$$

$$\begin{aligned} d_{6n-1}(x) = &\left(1 + xq + x^2 q^2\right) \Big\{ d_{6n-4}(xq^2) \\ &+ xq^{6n-1}(1 - qx) d_{6n-7}(xq^2) \Big\}, \end{aligned} \tag{2.10}$$

where $d_{-1}(x)$ is defined to by 1.

Proof. We define

$$F(n) = d_{6n+2}(x) - \left(1 + xq + x^2 q^2\right) d_{6n-1}(xq^2), \tag{2.11}$$

$$\begin{aligned} G(n) = d_{6n-1}(x) - &\left(1 + xq + x^2 q^2\right) \Big\{ d_{6n-4}(xq^2) \\ &+ xq^{6n-1}(1 - qx) d_{6n-7}(xq^2) \Big\}. \end{aligned} \tag{2.12}$$

To prove (2.9) and (2.10) we need only show that $F(n) = G(n) = 0$ for each $n \geq 1$.

In light of the fact that

$$d_2(x) = 1 + xq + x^2 q^2, \tag{2.13}$$

$$\begin{aligned} d_5(x) &= 1 + xq + x^2q^2 + xq^3 + x^2q^4 + x^3q^5 + x^3q^7 \\ &= (1 + xq + x^2q^2)d_2(xq^2) + xq^5(1 - xq), \end{aligned} \tag{2.14}$$

$$\begin{aligned} d_8(x) &= (1 + xq + x^2q^2)\big(1 + xq^3 + xq^5 + x^2q^6 + xq^7 \\ &\quad + x^2q^8 + x^2q^{10} + x^3q^{11} + x^3q^{13}\big) \\ &= \left(1 + xq + x^2q^2\right) d_5(xq^2), \end{aligned} \tag{2.15}$$

we see that

$$F(1) = G(1) = 0. \tag{2.16}$$

For simplicity in the remainder of the proof, we define

$$\lambda(x) = 1 + xq + x^2q^2. \tag{2.17}$$

We now replace x by xq^2 in (2.8) then multiply both sides of the resulting identity by $\lambda(x)$ and subtract from (2.7). The resulting identity simplifies to the following:

$$\begin{aligned} F(n) &= \left(1 + xq^{6n+1} + xq^{6n+2}\right) G(n) \\ &\quad + x^2q^{6n}\left(1 - q^{6n}\right) F(n-1). \end{aligned} \tag{2.18}$$

A second recurrence, now for $G(n)$, is somewhat more difficult. In (2.7) replace n by $n-1$, x by xq^2 and multiply the resulting identity by $\lambda(x)$; also in (2.8) replace n by $n-1$, x by xq^2 and multiply the resulting identity by $\lambda(x)xq^{6n-1}(1-qx)$. Now subtract both of these new identities from (2.8). The resulting identity simplifies to the following:

$$\begin{aligned} G(n) &= \left(1 + xq^{6n-1} + x^2q^{6n-2}\right) F(n-1) \\ &\quad + \left(-xq^{6n-3} + x^2q^{6n-2}\right) \lambda(x) d_{6n-7}(xq^2) \\ &\quad + (xq^{6n-3} - x^4q^{12n-6}) d_{6n-7}(x) \\ &\quad - \left(x^2q^{6n-2} - x^2q^{12n-8}\right) \lambda(x) d_{6n-10}(xq^2). \end{aligned} \tag{2.19}$$

Now in (2.19) replace the appearance of $d_{6n-7}(xq^2)$ with the right-hand side of (2.8) in which n has been replaced by $n-1$ and x replaced by xq^2. As a result, equation (2.19) is transformed after simplification into

$$\begin{aligned} G(n) &= \left(1 + xq^{6n-1} + x^2q^{6n-2}\right) F(n-1) \\ &\quad + \left(xq^{6n-3} - x^4q^{12n-6}\right) G(n-1). \end{aligned} \tag{2.20}$$

Finally the initial conditions $F(1) = G(1) = 0$ together with the recurrences (2.18) and (2.20) imply by mathematical induction that $F(n) = G(n) = 0$ for all $n \geq 1$, and this fact, as observed earlier, proves the lemma. $\square$

Lemma 4.

$$\lim_{n\to\infty} d_n(x) = \prod_{n=1}^{\infty}\left(1+xq^n+x^2q^{2n}\right). \tag{2.21}$$

Proof. By (2.6) we see directly that the above limit exists as a formal power series in q, and since $d_n(x)$ is dominated by the generating function for all partitions we see that if

$$A(x,q) = \lim_{n\to\infty} d_n(x),$$

then $A(x,q)$ is absolutely convergent provided $|q|<1$ and $|x|<\frac{1}{|q|}$.

Consequently

$$\begin{aligned} A(x,q) &= \lim_{n\to\infty} d_n(x) \\ &= \lim_{n\to\infty} d_{6n+2}(x) \\ &= \lim_{n\to\infty}\left(1+xq+x^2q^2\right) d_{6n-1}(xq^2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 3)} \\ &= \left(1+xq+x^2q^2\right) A(xq^2,q). \qquad (2.22) \end{aligned}$$

Iterating (2.21) we see that

$$\begin{aligned} A(x,q) &= A(0,q)\prod_{n=1}^{\infty}\left(1+xq^n+x^2q^{2n}\right) \\ &= \prod_{n=1}^{\infty}\left(1+xq^n+x^2q^{2n}\right), \end{aligned}$$

which is the desired result.

□

It is now an easy matter to deduce Theorem 2 from Lemma 3.

$$\begin{aligned} \sum_{n,m\geq 0} C(m,n)x^mq^n &= \prod_{n=1}^{\infty}\left(1+xq^n+x^2q^{2n}\right) \\ &= A(x,q) \\ &= \lim_{n\to\infty} d_n(x) \\ &= \sum_{n,m\geq 0} D(m,n)x^mq^n, \end{aligned} \tag{2.23}$$

and comparing coefficients in the extremes of (2.23) we establish the assertion in Theorem 2.

3. Conclusion

There are a couple of relevant observations. First, Alladi's addition to Schur's Theorem [1] given in Theorem 1 merits much closer study than it has received to date. Indeed, it would appear that it has been referred to in print subsequently only in [4].

Second, the conjectures of Kanade and Russell [6] suggest that the q-difference equation techniques, as initiated in [2], [3] need to be extended beyond partitions in which all parts are distinct. Part of the motivation for this paper was to show that such an extension is feasible.

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