Matrix Games, Linear Programming, and Linear Approximation

LEONID N. VASERSTEIN,
Department of Mathematics, Penn State U., University Park, PA 16802
(e-mail: vstein@math.psu.edu)

Received Jan. 27, 2006

Abstract. The following four classes of computational problems are equivalent:
solving matrix games,
solving linear programs,
best l^∞ linear approximation,
best l^1 linear approximation.

Key words Matrix games,linear programming, linear approximation, least absolute deviations.

Definitions

First we recall relevent definitions.

An *affine function* of variables $x_1, \ldots, x_n$ is $b_0 + c_1 x_1 + \cdots + c_n x_n$ where b_0, c_i are given numbers.

An l^∞ *linear approximation problem,* also known as (discrete) *Chebyshev approximation problem* is the problem of minimization of the following function:

$$\max(|f_1|, \ldots, |f_m|) = \|(f_1, \ldots, f_m)\|_\infty, \tag{1}$$

where $f_1, \ldots, f_m$ are m affine functions of n variables. This objective function is piece-wise linear and convex.

An l^1 *linear approximation problem,* also known as finding the LAD (least-absolute-deviations) fit, is the problem of minimization of the following function:

$$\sum_{i=1}^{m} |f_i| = \|(f_1, \ldots, f_m)\|_1, \tag{2}$$

where $f_1, \ldots, f_m$ are m affine functions of n variables. This objective function is piece-wise linear and convex.

A *matrix game* is given by a (payoff) matrix A. To solve a matrix game is to find a row p (an optimal strategy for the row player), a column q (an optimal strategy for the column player), and a number v such that $p = (p_i) \geq 0, \sum p_i = 1, q = (q_j) \geq 0, \sum q_i = 1, pA \geq v \geq Aq$. The number v is known as the value of game. The pair (p, q) is known as an equilibrium for the matrix game.

As usual, $x \geq 0$ means that every entry of the vector x is ≥ 0. We write $y \leq t$ for a vector y and a number t if every entry of y is $\leq t$. We go even further in abusing notation, denoting by $y - t$ the vector obtaining from y by subtracting t from every entry. Similarly we denote by $M + c$ the matrix obtained from M by adding a number c to every entry.

A matrix game is called *symmetric* if the payoff matrix is skew-symmetric. Recall that the value of any symmetric game is 0, and the transposition gives a bijection between the optimal strategies of the players.

A *linear constraint* is any of the following constraints: $f \le g, f \ge g, f = g$, where f, g are affine functions. A *linear program* is an optimization (maximization or minimization) of an affine function subject to a finite system of linear constraints.

Statement of results

It is well known, that solving a matrix game can be reduced to solving a pair of linear programs, dual to each other. It is also known that solving any linear program can be reduced to finding an optimal strategy with positive last component for a symmetric matrix game. In both reductions, the size of data (in terms of the number of given numbers or the number of given bits) may increase at most two times.

A subtle point here is: how can we compute an optimal strategy (for a symmetric game) with a positive last entry or prove that no such strategy exists? An answer is that for any vertex in the set of optimal strategy with positive last entry is a solution of a system of linear equations whose coefficients are the entries of the payoff matrix or 0,1, so a positive lower bound α can be given for this entry (at least in the case when all given numbers are rational). Namely, let β be an upper bound for the absolute values of the numerators and denominators of the entries of the payoff matrix of size N by N. Then $\alpha = \beta^{-2N} N^{-N/2}$ will work. Notice that $0 < \alpha < 1$.

The mixed strategies for the column player with the last entry $\ge \alpha$ in the symmetric game are the mixed strategies for the column player for the modified game obtained by adding the $(\alpha/(1-\alpha))$-multiple of the last column to the other columns of the payoff matrix. The optimal strategies for a modified matrix game give optimal strategies with positive last entry for the original symmetric game provided that the value of the modified game stays 0 (otherwise, there are no optimal strategies with positive last entry for the original symmetric game hence the original linear program has no optimal solutions).

Given any l^∞ approximation problem with the objective function (1), here is a well-known reduction (Vaserstein, 2003) to a linear program with one additional variable t:

$$t \to \min, \text{ subject to } -t \le f_i \le t \text{ for } i = 1, \ldots, m.$$

This is a linear program with $n+1$ variables and $2m$ linear constraints. Since any linear program can be reduced to a matrix game (see above), we conclude that finding an Chebyshev fit can be reduced to solving a matrix game.

The converse reduction is a main goal of this paper:

Theorem 1. Solving any matrix game can be reduced to finding a Chebyshev fit. More precisely, when the game is given by an m by n matrix, we construct a Chebyshev approximation problem with $2m + 2n + 3$ affine functions of $m + n + 1$ variables as well as a bijection between the equilibria for the matrix game and the solutions for the approximation problem.

Given any l^1 approximation problem with the objective function (2), here is a well-known reduction (Vaserstein, 2003) to a linear program with m additional variables t_i:

$$\sum_{i=1}^{m} t_i \to \min, \text{ subject to } -t_i \le f_i \le t_i \text{ for } i = 1, \ldots, m.$$

This is a linear program with $n+m$ variables and $2m$ linear constraints. Since any linear program can be reduced to a matrix game (see above), we conclude that finding the best l^1-fit can be reduced to solving a matrix game.

The converse reduction is the second goal of this paper:

Theorem 2. Solving any matrix game can be reduced to solving an l^1 linear approximation problem. More precisely, when the game is given by an m by n matrix, we construct an l^1 approximation problem with $4m+4n+6$ affine functions of $m+n+1$ variables as well as a bijection between the equilibria for the matrix game and the solutions for the approximation problem.

Proof of Theorem 1

Consider any matrix game with the payoff matrix A with m rows and n columns. It can can be reduced to the symmetric game with the payoff matrix

$$M = \begin{pmatrix} 0 & A+C & -J \\ -A^T - C & 0 & J' \\ J^T & -J' & 0 \end{pmatrix},$$

where J (rest. J') is the column of m (resp., n) ones and the number C is such that $A+C>0$. The skew-symmetrix matrix $M=-M^T$ has size $(m+n+1)\times(m+n+1)$. (J. von Neumann suggested another reduction resulting in a skew-symmetric matrix of size $(mn)\times(mn)$ which is not so good from computational point of view.)

The bijection between the solutions (p,q,v) for the game with the matrix A and the optimal strategies for the row player in the symmetric game with the matrix M is given by

$$(p,q)\mapsto (p,q^T,v+C)/(2+v+C).$$

Note that the last entry of any optimal strategy for the symmetric game above is positive because $A+C>0$.

Now we start with any matrix game, with the payoff matrix $M=-M^T$ of size N by N. (In the situation above, $N=m+n+1$.) Our problem is to find a column $x=(x_i)$ (an optimal strategy) such that

$$Mx\le 0, x\ge 0, \sum x_i = 1. \tag{3}$$

This problem (3) (of finding an optimal strategy) is about finding a feasible solution for a system of linear constraints. It can be written as the following linear program with an additional variable t and the optimal value 0:

$$t\to \min, Mx\le t, x\ge 0, \sum x_i = 1. \tag{4}$$

Now we find the largest entry c in the matrix M. If $c=0$, then $M=0$ and the problem (1) is trivial (every mixed strategy x is optimal). So we assume that $c>0$.

Adding the number c to every entry of the matrix M, we obtain a matrix $M+c \geq 0$ (all entries ≥ 0). The linear program (4) is equivalent to

$$t \to \min, (M+c)x \leq t, x \geq 0, \sum x_i = 1 \tag{5}$$

in the sense that these two programs have the same feasible solutions and the same optimal solutions. The optimal value for (4) is 0 while the optimal value for (5) is c.

Now we can rewrite (5) as follows:

$$\|(M+c)x\|_\infty \to \min, x \geq 0, \sum x_i = 1 \tag{6}$$

which is a Chebyshev approximation problem with additional linear constraints. We used that $M+c \geq 0$, hence $(M+c)x \geq 0$ for every feasible solution x in (4). The optimal value is still c.

Now we rid off the constraints in (4) as follows:

$$\| \begin{pmatrix} (M+c)x \\ c-x \\ \sum x_i + c - 1 \\ -\sum x_i - c + 1 \end{pmatrix} \|_\infty \to \min\,. \tag{7}$$

Note that the optimization problems (6) and (7) have the same optimal value c and every optimal solution of (6) is optimal for (7). Conversely, for every x with a negative entry, the objective function in (7) is $> c$. Also, for every x with $\sum x_i \neq 1$, the objective function in (7) is $> c$. So every optimal solution for (5) is feasible and hence optimal for (6).

Thus, we have reduced solving any symmetric matrix game with $N \times N$ payoff matrix to a Chebyshev approximation problem (7) with $2N+2$ affine functions in N variables.

Proof of Theorem 2

As in the proof of Theorem 1, we first reduce our game to a symmetric N by N game where $N = m+n+1$ and set c to be largest entry in the matrix M. The case $c = 0$ is trivial, so let $c > 0$.

We want to find a column x such that

$$x \geq 0, \sum x_i = 1, Mx \leq 0.$$

Consider the l^1 approximation problem whose objective function is $f(x) =$

$$\| \begin{pmatrix} Mx \\ c+Mx \\ x \\ 1-x \\ -1+\sum x_i \\ 1-\sum x_i \end{pmatrix} \|_1 = \|Mx\|_1 + \|c+Mx\|_1 + \|x\|_1 + \|1-x\|_1 + \|-1+\sum x_i\|_1 + \|1-\sum x_i\|_1$$

with $4N + 2$ affine functions of N variables.

Note that $f(x) = Nc + N$ for every optimal strategy x and that $f(x) > Nc + N$ for every x which is not an optimal strategy. So solving this approximation problem is equivalent to solving the matrix game.

Remark. Our result implies that every l^1 linear approximation problem can be reduced to a l^∞ linear approximation problem and vice versa..

There is an obvious direct reduction of the l^1 approximation problem with the objective function (2) to

$$\max |f_1 \pm f_2 \pm \cdots \pm f_m| \to \min$$

which is a Chebyshev approximation problem with 2^{m-1} affine functions in n variables. This reduction increases the size exponentially, while our reductions increases the size linearly.

Remark. There are methods for solving l^1 approximation problems alternative to the simplex method [Bloomfield–Steiger 1983]. Our reductions allows us to use these methods for solving arbitrary linear programs and matrix games.

Remark. A preprint with Theorem 1 appeared at arXiv [Vaserstein 2006].

References

Bloomfield, Peter and Steiger, William L. *Least absolute deviations. Theory, applications, and algorithms.* Progress in Probability and Statistics, 6. Birkhuser Boston, Inc., Boston, MA, 1983. xiv+349 pp. ISBN 0-8176-3157-7.

Vaserstein, L. N. (2003), *Introduction to Linear Programming,* Prentice Hall. (There is a Chinese translation by Mechanical Industry Publishing House ISBN: 7111173295.)

Vaserstein L.N., Reduction of Linear Programming to Linear Approximation, arXiv.org, math.OC/0602339, 15 Feb 2006.