MODULARITY OF THE CONCAVE COMPOSITION GENERATING FUNCTION

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Abstract. A composition of an integer constrained to have decreasing then increasing parts is called concave. We prove that the generating function for the number of concave compositions, denoted $v(q)$, is a mixed mock modular form in a more general sense than is typically used.

We relate $v(q)$ to generating functions studied in connection with "Moonshine of the Mathieu group" and the smallest parts of a partition. We demonstrate this connection in four different ways. We use the elliptic and modular properties of Appell sums as well as q-series manipulations and holomorphic projection.

As an application of the modularity results we give an asymptotic expansion for the number of concave compositions of n. For comparison, we give an asymptotic expansion for the number of concave compositions of n with strictly decreasing and increasing parts. The generating function of which is related to a false theta function rather than a mock theta function.

1. Introduction

A composition of an integer n is a sum of positive integers adding to n, in which order matters. The study of compositions has a long history dating back to MacMahon [25]. The book of Heubach and Mansour [22] contains more on the history of compositions. It is natural to impose restrictions on the ascents or descents of consecutive parts of a composition. For instance, compositions with no ascents correspond to integer partitions.

A concave composition of n is a sum of integers of the form

$$\sum_{i=1}^{L} a_i + c + \sum_{i=1}^{R} b_i = n$$

where $a_1 \geq \cdots \geq a_L > c < b_1 \leq \cdots \leq b_R$, where $c \geq 0$ is the central part of the composition. Let $V(n)$ be the number of concave compositions of n. For example, $V(3) = 13$ since $\{0,3\}$, $\{3,0\}$, $\{0,1,2\}$, $\{2,1,0\}$, $\{0,1,1,1\}$, $\{1,1,1,0\}$, $\{1,2\}$, $\{2,1\}$, $\{1,0,2\}$, $\{2,0,1\}$, $\{1,0,1,1\}$,$\{1,1,0,1\}$, and $\{3\}$ are all concave sequences. The first author [4] showed that the generating function for concave compositions with further restrictions is related to statistics for spiral self-avoiding random walks as well as other partition problems.

Standard techniques show that the generating function for the sequence $\{V(n)\}_{n=0}^{\infty}$ is given by

$$v(q) := \sum_{n=0}^{\infty} V(n)q^n = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_\infty^2}$$

where $(x)_n = (x;q)_n := \prod_{j=0}^{n-1}(1-xq^j)$ and $(x)_\infty = (x;q)_\infty := \prod_{j=0}^{\infty}(1-xq^j)$ (see [5]). We establish the modularity properties of $v(q)$ as a mixed mock modular form.

Theorem 1.1. *Let $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$. Define $f(\tau) = q(q)_\infty^3 v(q)$ and*

$$\widehat{f}(\tau) := f(\tau) - \frac{i}{2}\eta(\tau)^3 \int_{-\overline{\tau}}^{i\infty} \frac{\eta(z)^3}{(-i(z+\tau))^{\frac{3}{2}}}dz + \frac{\sqrt{3}}{2\pi i}\eta(\tau) \int_{-\overline{\tau}}^{i\infty} \frac{\eta(z)}{(-i(z+\tau))^{\frac{1}{2}}}dz$$

where the Dedekind η-function is given by $\eta(\tau) = q^{\frac{1}{24}}(q)_\infty$. The function $\widehat{f}$ transforms as a modular form of weight 2 for $SL_2(\mathbb{Z})$.

Following Zagier [35] a *mock theta function of weight* $k \in \{\frac{1}{2}, \frac{3}{2}\}$ is a q-series $H(q) = \sum_{n=0}^{\infty} a_n q^n$ such that there exists a rational number λ and a unary theta function of weight $2-k$, $g(\tau) = \sum_{n\in\mathbb{Q}^+} b_n q^n$, where $q = e^{2\pi i\tau} = e^{2\pi i(x+iy)}$, such that $h(\tau) = q^\lambda H(q) + g^*(\tau)$ is a non-holomorphic modular form of weight k, where

$$g^*(\tau) = (i/2)^{k-1} \sum_{n\in\mathbb{Q}^+} n^{k-1}\overline{b_n}\Gamma(1-k, 4\pi ny)\, q^{-n} \tag{1.1}$$

and $\Gamma(k,t) = \int_t^\infty u^{k-1}e^{-u}du$ is the incomplete gamma function. Such a non-holomorphic modular form is called a harmonic weak Maass form (see Section 2 for a definition and Ono's surveys [27, 28] for history). The theta function g is called the *shadow* of the mock theta function H.

The first author [5] established the following identity which is crucial in establishing Theorem 1.1. We have

$$v(q) = q^{-1}\left(v_1(q) + v_2(q) + v_3(q)\right) \tag{1.2}$$

where

$$v_1(q) := \frac{1}{(q)_\infty^3}\left(\sum_{n\neq 0} \frac{(-1)^{n+1}q^{\frac{3n(n+1)}{2}}}{(1-q^n)^2} - 3\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} + \frac{1}{12}\right) \tag{1.3}$$

$$v_2(q) := \frac{1}{(q)_\infty^3}\left(\sum_{n\neq 0} \frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n} - \frac{1}{4} - 2\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}\right) \tag{1.4}$$

$$v_3(q) := \frac{1}{(q)_\infty^3}\left(\frac{1}{6} + 2\sum_{n=1}^{\infty} q^n\left(\frac{1}{(1+q^n)^2} + \frac{1}{(1-q^n)^2}\right)\right). \tag{1.5}$$

Theorem 1.1 may be recast in the following terms.

Theorem 1.2. *With $q = e^{2\pi iz}$ and $z \in \mathbb{H}$ we have*

(1) $q^{-1/24}(q)_\infty^2 v_1(q)$ *is a mock theta function of weight* $3/2$ *with shadow proportional to* $\eta(z)$*. Consequentially,* $v_1(q)$ *is a mixed mock modular form.*
(2) $q^{-1/8}v_2(q)$ *is a mock theta function of weight* $1/2$ *with shadow proportional to* $\eta(z)^3$*.*
(3) $q^{-1/8}v_3(q)$ *is a modular form of weight* $1/2$*.*

Remark. Theorem 4.1 gives the level and shadow for each of the corresponding harmonic weak Maass form.

The mock theta function $v_2(q)$ has appeared in the recently work of Eguchi, Ooguri, and Tachikawa [17] and Cheng [13]. Their work is related to the character table of the Mathieu group M_{24} and "Moonshine of the Mathieu group". In their works $v_2(q)$ arises in a different form which is equivalent to the following identity. Moreover, the claim for $v_2(q)$ in Theorem 1.2 follows from the following theorem and the results of the third author's PhD thesis [36].

Theorem 1.3. *For* $|q| < 1$ *we have*

$$\widetilde{F}(q) := \frac{1}{(q)_\infty(-q)_\infty^2}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n(n+1)}{2}}}{1+q^n} = \frac{1}{(q)_\infty^3}\left(\frac{1}{2} + 4\sum_{n\geq 1}\frac{q^n}{(1+q^n)^2} - 2\sum_{n\neq 0}\frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n}\right).$$

We give three different ways of obtaining this identity. Each proof relies on different symmetries of Appell sums. The first uses an elliptic transformation property and Taylor expansions of the Jacobi theta function. The second proof uses the modular properties of the Appell sums. The third is via q-series manipulations. Finally, we sketch a connection with the holomorphic projection construction for mock modular forms.

Remark. Theorem 11.1 gives an analogous result for the Appell sum in the definition of $v_1(q)$. Moreover, it relates $v_1(q)$ to the smallest parts generating function studied by the first author [3].

There is a convenient graphical representation of a composition, where each part is represented by a vertical column of boxes (see Fig. 1).

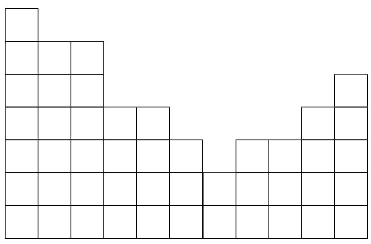

Fig. 1. This is a concave composition of 47.

Considering each composition of n, possibly from a restricted subset, it is natural to ask about the average limiting behavior of the graphical representation as $n \to \infty$. For example, there is a great deal of literature about the limiting shape of integer partitions (see, for instance, [19, 29, 34]).

Properties of the typical representation are often studied via probabilistic models. However, when the generating functions are modular forms very strong theorems can be proved for the statistics of interest. For instance, the modularity of the generating function for the number of partitions and the circle method yield the following asymptotic expansion for the number of partitions of n, denoted $p(n)$,

$$p(n) = \frac{2\sqrt{3}}{(24n-1)} \exp\left(\frac{\pi}{6}\sqrt{24n-1}\right)\left(1 - \frac{6}{\pi\sqrt{24n-1}} + O\left(\frac{1}{n^T}\right)\right)$$

for any $T \geq 0$.

As an application of the modular properties of $v(q)$ we give an asymptotic expansion for $V(n)$. Since the generating function is a mixed mock modular form, a version of the circle method developed by Bringmann and Mahlburg [10, 11] is used to establish the asymptotic.

Theorem 1.4. *For any $T \geq 1$ as $n \to \infty$ we have the following asymptotic expansion*

$$V(n) = \frac{2\sqrt{6}}{(12n+\frac{21}{2})^{\frac{3}{4}}} \exp\left(\frac{\pi}{3}\sqrt{12n+\frac{21}{2}}\right)\left(\sum_{t=1}^{T} \frac{(2t-1)!!3^t\alpha_t}{2^{2t}\pi^t}\frac{1}{(12n+\frac{21}{2})^{\frac{t}{2}}} + O\left(\frac{1}{n^{\frac{T+1}{2}}}\right)\right)$$

where α_t are defined by

$$\sum_{k=1}^{\infty} \alpha_k x^{2k} := \exp\left(\sqrt{1-2x^2} - 1 + x^2\right) x \frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh\left(\pi x\right)}.$$

In particular $\alpha_1 = \frac{2\pi}{3}$.

Remark. It is possible to obtain an asymptotic with a polynomial error term (see [10, 11, 31]).

Remark. A concave composition corresponds to a triple (λ, μ, c) where λ and μ are partitions, and $c \geq 0$ is an integer strictly smaller than the smallest parts in λ and μ, such that $n = |\lambda| + |\mu| + c$. Fristedt's [19] results imply that partitions of size n have a part of size 1 with probability roughly equal to $1 - \frac{\pi}{\sqrt{6n}}$. We expect at least one of the partitions μ or λ to have size not much smaller than $\frac{n}{2}$. That partition will almost surely contain a part of size 1. Thus we expect $c = 0$ for most triples. Therefore, we expect that $V(n)$ will agree to leading order with the asymptotic for the number of pairs of partitions (μ, λ) with $|\mu| + |\lambda| = n$. Standard circle method calculations show this number is

$$\frac{\sqrt{6}}{(12n-1)^{\frac{5}{4}}} \exp\left(\frac{\pi}{3}\sqrt{12n-1}\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

As expected this agrees with the leading order asymptotic of $V(n)$.

For comparison we introduce the notion of strongly concave compositions. A strongly concave composition of n is a sum of integers of the form

$$\sum_{i=1}^{L} a_i + c + \sum_{i=1}^{R} b_i = n$$

where $a_1 > \cdots > a_L > c < b_1 < \cdots < b_R$ and where $c \geq 0$. Let $V_d(n)$ be the number of strongly concave compositions of n. We have

$$\begin{aligned} v_d(q) &:= \sum_{n=0}^{\infty} V_d(n) q^n = \sum_{n=0}^{\infty} q^n (-q^{n+1})_\infty (-q^{n+1})_\infty \\ &= -\sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} + 2(-q)_\infty^2 \sum_{n \geq 0} \left(\frac{-12}{n}\right) q^{\frac{n^2-1}{24}}. \end{aligned} \tag{1.6}$$

where the second equality follows from standard techniques and the third equality is established by the first author in [5] and $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. The function $\sum_{n\geq 0} \left(\frac{-12}{n}\right) q^{\frac{n^2-1}{24}}$ is called a partial theta function. The asymptotic behavior of the partial theta function is discussed in Section 10 and used to derive the following theorem for $V_d(n)$.

Theorem 1.5. *For any $T \geq 0$ as $n \to \infty$ we have*

$$V_d(n) = \frac{\sqrt{3}}{(24n+2)^{\frac{3}{4}}} \exp\left(\frac{\pi}{6}\sqrt{24n+2}\right) \left(\sum_{m=0}^{T} \frac{(-1)^m}{2^{2m} m! (24n+2)^{\frac{m}{2}}} \gamma(m) + O\left(\frac{1}{n^{\frac{T+1}{2}}}\right)\right)$$

where

$$\gamma(m) = \sum_{a+b=m} \binom{m}{a} L(-2a) \left(\frac{\pi}{3}\right)^{a-b} p(b,a)$$

with $p(b,a) = \prod_{j=0}^{b} (4(a-1)^2 - j^2)$ and $L(-r) = -\frac{6^r}{r+1}\left(B_{r+1}\left(\frac{1}{6}\right) - B_{r+1}\left(\frac{5}{6}\right)\right)$, where $B_r(x)$ is the r-th Bernoulli polynomial.

Remark. Let $\mathbf{sm}(\lambda)$ be the smallest part in the partition λ. Strongly concave compositions are characterized by a pair of partitions into distinct parts (λ, μ) with $\mathbf{sm}(\lambda) \neq \mathbf{sm}(\mu)$. Let $\widetilde{u}(n)$ be the number of pairs of partitions into distinct parts with sizes summing to n. Therefore, we expect the asymptotic of

$$V_d(n) \sim (1 - \mathbf{Prob}\{\mathbf{sm}(\lambda) = \mathbf{sm}(\mu) : |\lambda| + |\mu| = n\})\, \widetilde{u}(n).$$

We have $V_d(n) \sim \frac{2}{3} \frac{\sqrt{3}}{(24n)^{\frac{3}{4}}} \exp\left(\frac{\pi}{6}\sqrt{24n}\right)$ and $\widetilde{u}(n) \sim \frac{\sqrt{3}}{(24n)^{\frac{3}{4}}} \exp\left(\frac{\pi}{6}\sqrt{24n}\right)$.

It follows from Fristedt [19] Theorem 9.1 that the smallest part of a partition into distinct parts has size j with probability roughly equal to $\frac{1}{2^j}$. Therefore, a pair of partitions into distinct parts will have the same smallest part with probability roughly equal to $\frac{1}{3}$, which agrees with the prediction.

It would be of interest to address some of the following questions as $n \to \infty$

(1) What is the distribution of the center part of a (strongly) concave composition?
(2) How many parts does a typical (strongly) concave composition of n have?
(3) What is the distribution of the number of parts to the left of the center part minus the number of parts to the right of the center part (see Section 9)?
(4) What is the distribution of the perimeter of the (strongly) concave composition of n?

Some of these questions can be answered by modelling a concave composition as the convolution of two random partitions (discussed above), while others can be treated via modular techniques of Bringmann, Mahlburg and the second author [12]. Section 9 contains some results addressing the third of these questions.

In Section 2 we recall some basic facts about holomorphic and non-holomorphic modular forms. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2. In Section 5 we give the first proof of Theorem 1.3, which proceeds via elliptic properties of Appell sums. In Section 6 we use q-series manipulations to prove Theorem 1.3. In Section 7 we prove Theorem 1.3 via "modular methods". Section 8 contains a calculation of the holomorphic projection operation which relates to Theorem 1.3. In Sections 9 and 10 we prove Theorems 1.4 and 1.5. Both use the circle method, modular properties of the generating functions of interest, and asymptotic analysis. Finally, in Section 11 we discuss the analog of Theorem 1.3 for the function $v_1(q)$.

Throughout the remainder of the text we let $q := e^{2\pi i\tau}$ with $\tau = x+iy$, $x, y \in \mathbb{R}$, and $y > 0$. Moreover, we let $z \in \mathbb{R}^+$, $h, k \in \mathbb{N}_0$, $0 \le h \le k$ with $(h,k) = 1$. Moreover we denote by $[a]_b$ the inverse of a modulo b.

2. Holomorphic Modular Forms and Harmonic Weak Maass Forms

In this section we define and give some basic properties of harmonic weak Maass forms. Before turning to non-holomorphic modular forms we describe the classic holomorphic modular forms of half integral weight.

We follow Shimura [32], see also [26], by setting

$$\left(\frac{a}{b}\right) = \eta \left(\frac{a}{|b|}\right)$$

where $\eta = -1$ if $a, b, < 0$ and $\eta = 1$ if $a > 0$ or $b > 0$. For an odd integer m, we put $\epsilon_m = 1$ if $m \equiv 3 \pmod 4$ and $\epsilon_m = i$ if $d \equiv 3 \pmod 4$. For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ let

$$j(\gamma, \tau) := \left(\frac{c}{d}\right) \epsilon_d^{-1} (c\tau + d)^{\frac{1}{2}}. \tag{2.1}$$

The Dedekind η-function is defined by $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$ and define $\chi(h, [-h]_k, k)$ to be the multiplier so that

$$\eta\left(\frac{1}{k}(h+iz)\right) = \sqrt{\frac{i}{z}}\chi\Big(h, [-h]_k, k\Big)\eta\left(\frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right). \tag{2.2}$$

By [26] Theorem 1.60, for instance,

$$\eta(8\tau)^3 = \sum_{n\in\mathbb{Z}} (-1)^n(2n+1)e^{2\pi i(2n+1)^2\tau} = \sum_{n=1}^{\infty}\left(\frac{-4}{n}\right) nq^{n^2}. \tag{2.3}$$

The following lemma follows from Theorem 1.44 of [26].

Lemma 2.1. *$\eta(8\tau)^3$ is a weight $3/2$ modular form on $\Gamma_0(64)$ with trivial Nebentypus.*

Before discussing harmonic weak Maass forms we introduce the quasimodular form

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \tag{2.4}$$

where $\sigma_1(n) := \sum_{d|n} d$. In particular it satisfies $E_2(\tau+1) = E_2(\tau)$ and

$$E_2(\tau) = \tau^{-2}E_2\left(-\frac{1}{\tau}\right) - \frac{6}{\pi i\tau}. \tag{2.5}$$

Therefore, the completed form of E_2 defined by

$$\widehat{E_2}(\tau) := E_2(\tau) - \frac{3}{\pi y} \tag{2.6}$$

transforms as a modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$.

We turn to harmonic weak Maass forms. Define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.7}$$

Definition 2.2. Suppose that $k \in \frac{1}{2}\mathbb{Z}$, N is a positive integer, and that ψ is a Dirichlet character with modulus $4N$. A *harmonic weak Maass* form of weight k on $\Gamma_0(4N)$ with Nebentypus character ψ is an smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$ and all $\tau \in \mathbb{H}$, we have
$$f(A\tau) = \psi(d)j(A,\tau)^{2k}f(\tau).$$
(2) We have that $\Delta_k f = 0$.
(3) The function f has at most linear exponential growth at all the cusps of $\mathbb{H}/\Gamma_0(N)$.

The third author [36] constructed a general class of harmonic weak Maass forms by "completing" certain Appell sums. The Appell sum is defined for $u, v \in \mathbb{C} \setminus (\mathbb{Z} + \mathbb{Z}\tau)$ by

$$\mu(u, v; \tau) := \frac{e^{\pi i u}}{\vartheta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi i n v} q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi i u} q^n}, \tag{2.8}$$

where

$$\vartheta(v; \tau) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu \left(v + \frac{1}{2}\right)}$$

is the Jacobi theta function. The Jacobi theta function satisfies the triple product identity

$$\vartheta(v; \tau) = -i q^{\frac{1}{8}} \zeta^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - \zeta q^{n-1})(1 - \zeta^{-1} q^n) \tag{2.9}$$

with $\zeta = e^{2\pi i v}$ and the following transformation

(2.10)

$$\vartheta\left(-ivz; \frac{1}{k}(h + iz)\right) = \chi^3(h, [-h]_k, k) \sqrt{\frac{i}{z}} e^{\pi k z v^2} \vartheta\left(v, \frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right).$$

The non-holomorphic correction term of the Appell sum requires the definition

(2.11)

$$R(u; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (-1)^{\nu - \frac{1}{2}} \left\{ \operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im}(u)}{\operatorname{Im}(\tau)}\right) \sqrt{2\operatorname{Im}(\tau)}\right) \right\} e^{-2\pi i \nu u} e^{-\pi i \nu^2 \tau}.$$

with $E(x)$ defined by

$$E(x) := 2 \int_0^x e^{-\pi u^2} du = \operatorname{sgn}(x) \left(1 - \beta\left(x^2\right)\right),$$

where for positive real x we let $\beta(x) := \int_x^\infty u^{-\frac{1}{2}} e^{-\pi u} du$. We have the following useful properties of R.

Proposition 2.3 (Propositions 1.9 and 1.10 of [36])**.** *If* $u \in \mathbb{C}$ *and* $\operatorname{Im}(\tau) > 0$*, then*

(1) $R(u + 1; \tau) = R(-u; \tau) = -R(u; \tau)$,
(2) $R(u; \tau + 1) = e^{-\frac{\pi i}{4}} R(u; \tau)$,
(3) $R(u; \tau) = -\sqrt{\frac{i}{\tau}} e^{\frac{i\pi u^2}{\tau}} \left(R\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) - H\left(\frac{u}{\tau}; -\frac{1}{\tau}\right)\right)$,

where the Mordell integral is defined by

$$H(u; z) := \int_{-\infty}^{\infty} \frac{e^{\pi i z x^2 - 2\pi x u}}{\cosh(\pi x)} dx.$$

Moreover, we need the following "dissection" property of R.

Proposition 2.4 (Proposition 2.3 of [9]). *For $n \in \mathbb{N}$, we have*

$$R\left(u; \frac{z}{n}\right) = \sum_{\ell=0}^{n-1} q^{-\frac{1}{2n}\left(\ell-\frac{n-1}{2}\right)^2} e^{-2\pi i\left(\ell-\frac{n-1}{2}\right)\left(u+\frac{1}{2}\right)} R\left(nu + \left(\ell - \frac{n-1}{2}\right) z + \frac{n-1}{2}; nz\right).$$

The completion of μ is defined by

$$\widehat{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau). \tag{2.12}$$

This function satisfies the following elliptic and modular transformation laws.

Theorem 2.5 (Theorem 1.11 of [36]). *Assume all of the notation and hypotheses from above. If $k, \ell, m, n \in \mathbb{Z}$, then we have*

(1) $\widehat{\mu}(u+k\tau+\ell, v+m\tau+n; \tau) = (-1)^{k+\ell+m+n} e^{\pi i\tau(k-m)^2+2\pi i(k-m)(u-v)} \widehat{\mu}(u, v; \tau)$,

(2) $\widehat{\mu}\left(-iuz, -ivz; \frac{1}{k}(h + iz)\right) = \chi^{-3}(h, [-h]_k, k)\sqrt{\frac{i}{z}} e^{-\pi kz(u-v)^2} \widehat{\mu}\left(u, v; \frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right).$

Finally, we have the following result which is useful in determining the shadow of a mock theta function.

Theorem 2.6 (Theorem 1.16 of [36]). *For $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $b \in \mathbb{R}$. we have*

$$\int_{-\overline{\tau}}^{i\infty} \frac{g_{a+\frac{1}{2}, b+\frac{1}{2}}(z)}{\sqrt{-i(\tau + z)}} dz = -e^{-\pi i a^2\tau + 2\pi i a\left(b+\frac{1}{2}\right)} R(a\tau - b; \tau)$$

where

$$g_{a,b}(z) := \sum_{\nu \in a+\mathbb{Z}} \nu e^{\pi i\nu^2 z + 2\pi i\nu b}.$$

3. Proof of Theorem 1.1

In this section we use the Jacobi properties of the Appell sums

(3.1)

$$A(u, v; \tau) := e^{\pi iu} \sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{2\pi inv} q^{\frac{n^2+n}{2}}}{1 - e^{2\pi iu} q^n} \quad \text{and} \quad A_3(u; \tau) := e^{\pi iu} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{3n^2+n}{2}}}{1 - e^{2\pi iu} q^n}$$

to deduce Theorem 1.1. The function $A_3(u; \tau)$ was studied by Bringmann [8].

Direct computation gives

$$f_2(\tau) := \sum_{n\neq 0} \frac{(-1)^{n+1} q^{\frac{3n(n+1)}{2}}}{(1 - q^n)^2} = \left(\frac{1}{2\pi i}\frac{\partial}{\partial u}\left(e^{-\pi iu} A_3(u, \tau) - \frac{1}{1 - e^{2\pi iu}}\right)\right)\Big|_{u=0}$$

$$f_1(\tau) := \sum_{n\neq 0} \frac{(-1)^{n+1} n q^{\frac{n(n+1)}{2}}}{1 - q^n} = -\frac{1}{2\pi i}\frac{\partial}{\partial v}\Big|_{u=v=0} A(u, v; \tau)$$

Thus the modular properties of the Appell sums will dictate the modular properties of our functions $v_1(q)$ and $v_2(q)$. In analogy to (2.12) the completed forms of these sums are

(3.2)
$$\widehat{A}(u,v;\tau) := A(u,v;\tau) + \frac{i}{2}\vartheta(v;\tau)R(u-v;\tau)$$
(3.3)
$$\widehat{A_3}(u;\tau) := A_3(u;\tau) + \frac{1}{2}\eta(\tau)q^{-\frac{1}{6}}\left(e^{2\pi iu}R(3u-\tau;3\tau) - e^{-2\pi iu}R(3u+\tau;3\tau)\right).$$

Using the transformation properties of μ found in Chapter 1 of [36] we have the following modular properties of $\widehat{A}$. For $\widehat{A_3}$ we will use Theorem 3.1 of [8].

Proposition 3.1. *For all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *we have*

$$\widehat{A}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)e^{\pi i \frac{c}{c\tau+d}(2uv-v^2)}\widehat{A}(u,v;\tau).$$

We then have the following theorem which together with the transformation properties of $\widehat{E_2}(\tau)$ and (1.2) yields Theorem 1.1.

Theorem 3.2. *With* $\tau = x+iy$, $x,y\in\mathbb{R}$ *and* $y>0$ *we let*

$$\begin{aligned}
\widehat{f_1}(\tau) =& f_1(\tau) - \frac{1}{4\pi y} - \frac{i}{2}\eta(\tau)^3\int_{-\overline{\tau}}^{i\infty}\frac{\eta(z)^3}{(-i(z+\tau))^{\frac{1}{2}}}dz\\
\widehat{f_2}(\tau) =& f_2(\tau) - \frac{1}{24} + \frac{3}{8\pi y} + \frac{\sqrt{3}}{2\pi i}\eta(\tau)\int_{-\overline{\tau}}^{i\infty}\frac{\eta(z)}{(-i(z+\tau))^{\frac{3}{2}}}dz.
\end{aligned}$$

Both $\widehat{f_1}$ *and* $\widehat{f_2}$ *transform as a weight 2 modular form for all of* $SL_2(\mathbb{Z})$ *with trivial Nebentypus.*

Proof. We establish the claim for f_1 first. We have

$$\begin{aligned}
\frac{1}{2\pi i}\frac{\partial}{\partial v}\Big|_{u=v=0}\widehat{A}(u,v;\tau) =& -f_1(\tau) + \frac{1}{4\pi}\vartheta'(0)R(0;\tau)\\
=& -f_1(\tau) + \frac{i}{2}\eta(\tau)^3\int_{-\overline{\tau}}^{i\infty}\frac{\eta(z)^3}{(-i(z+\tau))^{\frac{1}{2}}}dz\\
=& -\widehat{f_1}(\tau) - \frac{1}{4\pi y},
\end{aligned}$$

where we have used Theorem 2.6 in the second equality and the fact that $\vartheta(0) = 0$.

Additionally, from Proposition 3.1 we have (by taking $\frac{1}{2\pi i}\frac{\partial}{\partial v}\big|_{u=v=0}$ on both sides)

$$\begin{aligned}\frac{1}{(c\tau+d)}&\left(-\widehat{f}_1\left(\frac{a\tau+b}{c\tau+d}\right)-\frac{1}{4\pi}\cdot\frac{1}{\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)}\right)\\ &=(c\tau+d)\left(-\widehat{f}_1(\tau)-\frac{1}{4\pi y}\right)+c\lim_{u,v\to 0}ue^{\pi i\frac{c}{c\tau+d}(2uv-u)^2}\widehat{A}(u,v;\tau)\end{aligned}$$

Using $\lim_{u\to 0}u\widehat{A}(u,v;\tau)=-\frac{1}{2\pi i}$ or $\lim_{v\to 0}\widehat{A}(u,v;\tau)=-\frac{1}{2\pi i}\frac{\vartheta'(0)}{\vartheta(u)}$ we find that

$$\lim_{u,v\to 0}ue^{\pi i\frac{c}{c\tau+d}(2uv-u^2)}\widehat{A}(u,v;\tau)=-\frac{1}{2\pi i}.$$

Additionally, $\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)=\frac{y}{|c\tau+d|^2}$, thus

$$\begin{aligned}(c\tau+d)\widehat{f}_1(\tau)-\frac{1}{(c\tau+d)}\widehat{f}_1\left(\frac{a\tau+b}{c\tau+d}\right)=&\frac{1}{c\tau+d}\frac{1}{4\pi\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)}-\frac{c\tau+d}{4\pi y}-\frac{c}{2\pi i}\\ =&\frac{c\overline{\tau}+d}{4\pi y}-\frac{c\tau+d}{4\pi y}-\frac{c}{2\pi i}=0\end{aligned}$$

which gives the result for $\widehat{f}_1$.

We define

$$\widetilde{A_3}(u;\tau)=e^{-\pi iu}A_3(u,\tau)-\frac{1}{1-e^{2\pi iu}}.$$

By Theorem 3.1 of [8] we have

(3.4)
$$\begin{aligned}\widetilde{A_3}(u,\tau)=&\frac{1}{1-e^{2\pi iu}}-\frac{ie^{\frac{3\pi u^2}{z}-\pi iu-\frac{\pi u}{z}}}{z(1-e^{-\frac{2\pi u}{z}})}+\frac{i}{z}e^{\frac{3\pi u^2}{z}-\pi iu-\frac{\pi u}{z}}\widetilde{A_3}\left(\frac{iu}{z},-\frac{1}{\tau}\right)\\ &-(-i\tau)^{-\frac{1}{2}}\eta\left(-\frac{1}{\tau}\right)\int_{\mathbb{R}}e^{3\pi i\tau x^2}\left(\sum_{\pm}\frac{e^{\pm\frac{\pi i}{6}\mp\pi i\tau zx}}{1-e^{2\pi iu}e^{\pm\frac{\pi i}{3}\mp 2\pi i\tau zx}}\right)dx\end{aligned}$$

Remark. This may also be derived by using

$$e^{\frac{3\pi iu}{2}}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{\frac{3n(n+1)}{2}}e^{2\pi inv}}{1-e^{2\pi iu}q^n}=\sum_{k=0}^{2}e^{2\pi iku}\vartheta\left(v+k\tau+1;3\tau\right)\mu\left(3u,v+k\tau+1;3\tau\right)$$

and the results of Chapter 1 of [36].

Lemma 4.2 of [8] gives

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\Big|_{u=0}\int_{\mathbb{R}}e^{3\pi i\tau x^2}\left(\sum_{\pm}\frac{e^{\pm\frac{\pi i}{6}\mp\pi i\tau zx}}{1-e^{2\pi iu}e^{\pm\frac{\pi i}{3}\mp 2\pi i\tau zx}}\right)dx=\frac{\sqrt{3}(-i\tau)^2}{2\pi}\int_0^{\infty}\frac{\eta(iw)}{(-i(iw-i\tau))^{\frac{3}{2}}}dw.$$

Using the transformation $\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$ and applying $\frac{1}{2\pi i}\frac{\partial}{\partial u}\Big|_{u=0}$ to (3.4) we have

$$\begin{aligned}
f_2(\tau) =& \frac{1}{2\pi i}\frac{\partial}{\partial u}\left(\frac{1}{1-e^{2\pi i u}} + \frac{e^{\frac{3\pi i u^2}{\tau} - \pi i u - \frac{\pi i u}{\tau}}}{\tau(1-e^{-\frac{2\pi i u}{\tau}})} + \frac{1}{\tau}e^{\frac{3\pi i u^2}{\tau} - \pi i u - \frac{\pi i u}{\tau}}\widetilde{A_3}\left(\frac{u}{\tau}, -\frac{1}{\tau}\right)\right)\Big|_{u=0} \\
&+ \tau^2\eta(\tau)\frac{\sqrt{3}}{2\pi}\int_0^\infty \frac{\eta(iw)}{(-i(iw-i\tau))^{\frac{3}{2}}}dw \\
=& \frac{1}{2\pi i}\frac{\partial}{\partial u}\left(\frac{1}{1-e^{2\pi i u}} + \frac{e^{\frac{3\pi i u^2}{\tau} - \pi i u - \frac{\pi u}{\tau}}}{\tau(1-e^{-\frac{2\pi i u}{\tau}})}\right)\Big|_{u=0} + \tau^{-2}f_2\left(-\frac{1}{\tau}\right) \\
&+ \tau^2\eta(\tau)\frac{\sqrt{3}}{2\pi}\int_0^\infty \frac{\eta(iw)}{(-i(iw-i\tau))^{\frac{3}{2}}}dw
\end{aligned}$$

$$= \left(\frac{1}{24} - \frac{1}{24\tau^2} + \frac{3}{4\pi i\tau}\right) + \tau^{-2}f_2\left(-\frac{1}{\tau}\right) + \tau^2\eta(\tau)\frac{\sqrt{3}}{2\pi}\int_0^\infty \frac{\eta(iw)}{(-i(iw-i\tau))^{\frac{3}{2}}}dw \tag{3.5}$$

where we have used $\sum_{n\neq 0}\frac{(-1)^n q^{\frac{3n^2+n}{2}}}{1-q^n} = 0$.

Next define $\mathcal{S}(\tau) := \frac{\sqrt{3}}{2\pi i}\eta(\tau)\int_{-\overline{\tau}}^{i\infty}\frac{\eta(w)}{(-i(w+\tau))^{\frac{3}{2}}}dw$. Then the modular transformation of $\eta(\tau)$ implies that

$$\mathcal{S}(\tau) = \tau^2\mathcal{S}\left(-\frac{1}{\tau}\right) - \tau^2\frac{\sqrt{3}}{2\pi}\int_0^\infty \frac{\eta(iw)}{(-i(iw-i\tau))^{\frac{3}{2}}}dw.$$

Combining this with (3.5) gives $\widehat{f_2}(\tau) = \tau^{-2}\widehat{f_2}\left(-\frac{1}{\tau}\right)$. Using $\widehat{f_2}(\tau+1) = \widehat{f_2}(\tau)$ we obtain the result. □

Proof of Theorem 1.1. By (1.2) and

$$\sum_{n=1}^\infty \frac{q^n}{(1-q^n)^2} = \sum_{n=1}^\infty \frac{nq^n}{1-q^n} = \sum_{n\geq 1}\sigma_1(n)q^n = \frac{1-E_2(\tau)}{24} \tag{3.6}$$

we have $q(q)_\infty^3 v(q) = f_1(\tau) + f_2(\tau) + \frac{1}{24}\left(E_2(\tau)-1\right)$. The result now follows from Theorem 3.2 and (2.5). □

4. Proof of Theorem 1.2

In this section we give the following more precise version of Theorem 1.2.

Theorem 4.1. *With $q = e^{2\pi i\tau}$ we have*

$$q^{-1}(q^24;q^24)_\infty^2 v_1(q^{24}) + \frac{i}{4\sqrt{2}\pi}\int_{-\overline{\tau}}^{i\infty}\frac{\eta(24z)}{\left(-i\left(\tau+z\right)\right)^{\frac{3}{2}}}dz$$

is a harmonic weak Maass form of weight $\frac{3}{2}$ *on* $\Gamma_0(576)$ *with Nebentypus* $\left(\frac{12}{\cdot}\right)$. *Also,*

$$q^{-1}v_2(q^8) - i\sqrt{2}\int_{-\overline{\tau}}^{i\infty} \frac{\eta(8z)^3}{\sqrt{-i(z+\tau)}}dz$$

is a harmonic weak Maass form of weight $\frac{1}{2}$ *on* $\Gamma_0(64)$ *with trivial Nebentypus. Finally, let*

$$E_2^{odd}(\tau) := 1 + 24\sum_{n>0}\left(\sum_{d|n, d\ odd} d\right)q^n = 2E_2(2\tau) - E_2(\tau),$$

then

$$q^{-1}v_3(q^8) = \frac{E_2^{odd}(8\tau)}{6\eta(8\tau)^3}$$

is a weight $1/2$ *weakly holomorphic modular form on* $\Gamma_0(64)$ *with trivial Nebentypus.*

The modular properties of $v_1(q)$ and $v_3(q)$ are straightforward or follow from known results. Similar to (3.6) we have

$$\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}nq^n}{1-q^n} = -\sum_{n\geq 1}\left(\sum_{d|n}(-1)^d d\right)q^n \tag{4.1}$$

Hence

$$\sum_{n=1}^{\infty}q^n\left(\frac{1}{(1+q^n)^2}+\frac{1}{(1-q^n)^2}\right) = \frac{E_2^{odd}(\tau)-1}{12}. \tag{4.2}$$

The claim for $v_2(q)$ follows from Lemma 2.1 and the fact that $E^{odd}(\tau)$ is a holomorphic modular form of weight 2 on $\Gamma_0(4)$. The proof of the claims about $v_1(q)$ follows from (3.6) and the following theorem of Bringmann [8].

Theorem 4.2 (Theorem 1.1 of [8])**.** *Let* $q = e^{2\pi i\tau}$ *and* $\mathcal{R}(\tau) := \frac{q^{-1}}{(q^{24};q^{24})_\infty}\sum_{n\neq 0}\frac{(-1)^{n+1}q^{36n(n+1)}}{(1-q^{24n})^2}$. *Then*

$$\mathcal{M}(\tau) := \mathcal{R}(\tau) - \frac{i}{4\sqrt{2}\pi}\int_{-\overline{\tau}}^{i\infty}\frac{\eta(24z)}{(-i(\tau+z))^{\frac{3}{2}}}dz - \frac{1}{24\eta(24\tau)} + \frac{E_2(24\tau)}{8\eta(24\tau)}$$

is a harmonic weak Maass form of weight $\frac{3}{2}$ *on* $\Gamma_0(576)$ *with Nebentypus* $\left(\frac{12}{\cdot}\right)$.

The modularity of $v_2(q)$ follows easily from the identity of Theorem 1.3 and the results of Chapter 1 of [36]. Namely, the following theorem.

Theorem 4.3. *With* $q = e^{2\pi i\tau}$ *we have* $\widetilde{F}(q) = -2iq^{\frac{1}{8}}\mu\left(\frac{1}{2},\frac{1}{2};\tau\right)$ *is a mock theta function with shadow proportional to* $\eta^3(z)$. *More precisely*

$$q^{-1}\widetilde{F}(q^8) + 2i\sqrt{2}\int_{-\overline{\tau}}^{i\infty}\frac{\eta(8z)^3}{\sqrt{-i(z+\tau)}}dz$$

is a harmonic weak Maass form of weight $\frac{1}{2}$ *on* $\Gamma_0(64)$ *with trivial Nebentypus.*

Proof. Since this function is written in terms of μ the result follows from Chapter 1 of [36]. In particular, to compute the shadow we use Theorem 2.6 and Lemma 2.1. For additional details see [16] or [31]. □

We have used Theorem 1.3 to establish the modularity of $v_2(q)$. In the next three sections we will give different proofs of this theorem. Each proof uses different techniques and highlights different aspects of the Appell sums appearing in Theorem 1.3.

5. Elliptic Proof of Theorem 1.3

In this section we prove the identity of Theorem 1.3 via a transformation property of $\mu(u, v; \tau)$ with respect to the elliptic variables u and v.

Proposition 5.1 (Proposition 1.4 (7) of [36])**.** *For* $u, v, u+z, v+z \notin \mathbb{Z}\tau + \mathbb{Z}$ *we have*

$$\mu\left(u+z, v+z; \tau\right) - \mu\left(u, v; \tau\right) = \frac{1}{2\pi i} \frac{\vartheta'(0)\vartheta(u+v+z)\vartheta(z)}{\vartheta(u)\vartheta(v)\vartheta(u+z)\vartheta(v+z)}$$

where we write $\vartheta(u) = \vartheta(u;\tau)$ *when* τ *is understood.*

We will need the following lemma of [30]. Let

$$F_{2k}(\tau) := \sum_{n=1}^{\infty} \sum_{d|n} (-1)^d d^{2k-1} q^n \quad \text{and} \quad \Phi_{2k}(\tau) := \sum_{n=1}^{\infty} \sum_{d|n} d^{2k-1} q^n. \tag{5.1}$$

Lemma 5.2. *With* $Z = 2\pi i u$ *we have*

$$\vartheta(u;\tau) = -2\sin(\pi u)\eta^3(\tau)\exp\left(-2\sum_{\substack{\ell \text{ even} \\ \ell > 0}} \frac{Z^\ell}{\ell!}\Phi_\ell(\tau)\right)$$

and

$$\vartheta\left(u+\frac{1}{2};\tau\right) = -2\cos(\pi u)\frac{\eta(2\tau)^2}{\eta(\tau)}\exp\left(-2\sum_{\substack{\ell > 0 \\ \ell \text{ even}}} \frac{Z^\ell}{\ell!}F_\ell(\tau)\right).$$

With this lemma we turn to the proof of Theorem 1.3 using Proposition 5.1.

"Elliptic" Proof of Theorem 1.3. Throughout this proof let $x = e^{2\pi i u}$. From Proposition 5.1 and using $\vartheta'(0) = -2\pi\eta(\tau)^3$ we have

$$\vartheta(u)\vartheta(-u)\mu\left(u+\frac{1}{2}, -u+\frac{1}{2};\tau\right) - \vartheta(u)\vartheta(-u)\mu(u,-u;\tau) = \frac{i\eta^3(\tau)\vartheta\left(\frac{1}{2}\right)^2}{\vartheta\left(u+\frac{1}{2}\right)^2}. \tag{5.2}$$

Taylor expanding each of the three terms in this identity around $u = 0$ by applying Lemma 5.2 we have

(5.3)
$$\vartheta(u)\vartheta(-u)\mu\left(u+\frac{1}{2},-u+\frac{1}{2};\tau\right) = -4i\pi^2\left(\frac{\eta(\tau)^6}{\vartheta\left(\frac{1}{2}\right)}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n^2+n}{2}}}{1+q^n}\right)u^2+O(u^4)$$

$$\frac{i\eta^3(\tau)\vartheta\left(\frac{1}{2}\right)}{\vartheta\left(u+\frac{1}{2}\right)^2} = i\eta(\tau)^3+4i\pi^2\left(\frac{1}{4}-2F_2(\tau)\right)u^2+O(u^4) \tag{5.4}$$

The remaining term is more interesting. From the definition of $\mu(u,-u;\tau)$ we have

$$\vartheta(u)\vartheta(-u)\mu(u,-u;\tau) = \frac{i\vartheta(u)}{2\sin(\pi u)}+\vartheta(u;\tau)e^{\pi iu}\sum_{n\neq 0}\frac{(-1)^n x^{-n}q^{\frac{n^2+n}{2}}}{1-xq^n}.$$

Since $\sum_{n\neq 0}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{1-q^n} = 0$ we have

$$\sum_{n\neq 0}\frac{(-1)^n x^{-n}q^{\frac{n^2+n}{2}}}{1-xq^n} = 2\pi iu\left(\sum_{n\neq 0}\frac{(-1)^{n+1}nq^{\frac{n^2+n}{2}}}{1-q^n}+\sum_{n\neq 0}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{(1-q^n)^2}\right)+O(u^2).$$

Again applying Lemma 5.2 we have

$$-\vartheta(u)\vartheta(-u)\mu(u,-u;\tau) = i\eta^3(\tau)$$

(5.5)
$$-4i\pi^2\eta^3(\tau)\left(\sum_{n\neq 0}\frac{(-1)^n nq^{\frac{n^2+n}{2}}}{1-q^n}-\sum_{n\neq 0}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{(1-q^n)^2}-\Phi_1(\tau)\right)u^2+O(u^3).$$

Using (5.2), (5.3), (5.4), and (5.5) to compare the u^2 coefficient of the Taylor expansion around $u = 0$ of both sides of (5.2) we have

$$\frac{1}{\vartheta\left(\frac{1}{2}\right)}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n^2+n}{2}}}{1+q^n}-\frac{1}{\eta^3(\tau)}\left(\sum_{n\neq 0}\frac{(-1)^{n+1}nq^{\frac{n^2+n}{2}}}{1-q^n}+\sum_{n\neq 0}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{(1-q^n)^2}+\Phi_2(\tau)\right)$$
$$=\frac{1}{\eta^3(z)}\left(-\frac{1}{4}+2F_2(\tau)\right).$$

The identity follows from $\vartheta\left(\frac{1}{2}\right) = -2q^{\frac{1}{8}}(q)_\infty(-q)_\infty^2$, $\eta^3(\tau) = q^{\frac{1}{8}}(q)_\infty^3$, $\sum_{n\neq 0}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{(1-q^n)^2} = -\Phi_2(\tau)$, and $\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2} = -F_2(\tau)$ □

6. q-Series Proof of Theorem 1.3

In this section we will give a q-series proof of the identity in Theorem 1.3. We will divide the proof into several lemmas.

For negative values of n we define $(x)_n = (x)_\infty/(xq^n)_\infty$. For simplicity we write $(a,b,c,d;q)_n = (a;q)_n(b;q)_n(c;q)_n(d;q)_n$. We require the following facts from the literature

$$\vartheta_4 := \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = \frac{(q)_\infty}{(-q)_\infty} \tag{6.1}$$

$$\sum_{n\in\mathbb{Z}} \frac{(e)_n(f)_n\left(\frac{aq}{ef}\right)^n}{\left(\frac{aq}{c}\right)_n\left(\frac{aq}{d}\right)_n} = \frac{\left(\frac{q}{c},\frac{q}{d},\frac{aq}{e},\frac{aq}{f};q\right)_\infty}{\left(aq,a^{-1}q,\frac{aq}{cd},\frac{aq}{ef};q\right)_\infty}\sum_{n\in\mathbb{Z}}\frac{(1-aq^{2n})(c,d,e,f;q)_n\left(\frac{qa^3}{cdef}\right)^n q^{n^2}}{(1-a)\left(\frac{aq}{c},\frac{aq}{d},\frac{aq}{e},\frac{aq}{f};q\right)_n} \tag{6.2}$$

where the first is (2.2.12) of [2] and the second can be found in [7] or page 135 of [20].

Lemma 6.1.

$$2\sum_{n\in\mathbb{Z}}\frac{q^{\binom{n+1}{2}}}{1+q^n} = \frac{1}{\vartheta_4^2}\left(1+\sum_{n=1}^{\infty}\left(\frac{8q^{2n^2+n}}{(1+q^n)^2} - \frac{16nq^{2n^2}(1-q^n)}{1+q^n}\right)\right)$$

Proof. In (6.2) set $e=c=-1$, $a=1$, and let $d,f\to\infty$ to obtain

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n(n+1)}{2}}}{1+q^n} =& \frac{1}{\vartheta_4^2}\lim_{a\to1}\sum_{n\in\mathbb{Z}}\frac{(1-aq^2n)}{1-a}\frac{(-1)_n^2}{(-aq)_n^2}a^3nq^{2n^2}\\
=&\frac{1}{\vartheta_4^2}\left(1-\sum_{n=1}^{\infty}\frac{d}{da}\Big|_{a=1}\left(\frac{(1-aq^{2n})(-1)_n^2a^{3n}q^{2n^2}}{(-aq)_n^2}+\frac{(1-aq^{-2n})(-1)_{-n}^2a^{-3n}q^{2n^2}}{(-aq)_{-n}^2}\right)\right)\\
=&\frac{1}{\vartheta_4^2}\left(1-\frac{d}{da}\Big|_{a=1}\sum_{n=1}^{\infty}(1-aq^{2n})\left(\frac{(-1)_n}{(-aq)_n}\right)^2a^{3n}q^{2n^2}\right.\\
&\left.-\frac{d}{da}\Big|_{a=1}\sum_{n=1}^{\infty}(1-aq^{-2n})\left(\frac{(a+1)(a+q)\cdots(a+q^{n-1})}{(-q)_n^2}\right)^2a^{-3n}q^{2n^2+2n}\right)
\end{aligned}$$

where we use (6.1) and L'Hopital's rule in the second displayed equation. Continuing we have

$$
\begin{aligned}
2\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n(n+1)}{2}}}{1+q^n} =& \frac{1}{\vartheta_4^2}\Bigg(1+\sum_{n=1}^{\infty}\frac{4q^{2n^2+2n}}{(1+q^n)^2}-\sum_{n=1}^{\infty}(1-q^{2n})(-1)_n^2(-2(-q)_n^{-3})\sum_{j=1}^{n}\frac{q^j}{1+q^j}(-q)_nq^{2n^2}\\
&-\sum_{n=1}^{\infty}(1-q^{2n})\frac{12nq^{2n^2}}{(1+q^n)^2}+\sum_{n=1}^{\infty}\frac{4q^{2n^2}}{(1+q^n)^2}\\
&-\sum_{n=1}^{\infty}(1-q^{-2n})q^{2n^2+2n}\frac{2(-1)_n^2}{(-q)_n^2}\sum_{j=0}^{n-1}\frac{1}{1+q^j}+\sum_{n=1}^{\infty}(1-q^{-2n})q^{2n^2+2n}\frac{12n}{(1+q^n)^2}\Bigg)\\
=& \frac{1}{\vartheta_4^2}\Bigg(1+4\sum_{n=1}^{\infty}\frac{q^{2n^2+2n}}{(1+q^n)^2}+8\sum_{n=1}^{\infty}(1-q^{2n})\frac{q^{2n^2}}{(1+q^n)^2}\sum_{j=1}^{n}\frac{q^j}{1+q^j}\\
&-12\sum_{n=1}^{\infty}\frac{nq^{2n^2}(1-q^n)}{(1+q^n)}+4\sum_{n=1}^{\infty}\frac{q^{2n^2}}{(1+q^n)^2}\\
&+8\sum_{n=1}^{\infty}(1-q^{2n})\frac{q^{2n^2}}{(1+q^n)^2}\sum_{j=0}^{n-1}\frac{1}{1+q^j}-12\sum_{n=1}^{\infty}\frac{(1-q^n)nq^{2n^2}}{(1+q^n)}\Bigg)
\end{aligned}
$$

Noting that

$$
\sum_{j=1}^{n}\frac{q^j}{1+q^j}+\sum_{j=0}^{n-1}\frac{1}{1+q^j}=n+\frac{q^n}{1+q^n}-\frac{1}{2}
$$

we see that the right hand side equals

$$
\begin{aligned}
&\frac{1}{\vartheta_4^2}\left(1+4\sum_{n=1}^{\infty}\frac{(1+q^{2n})q^{2n^2}}{(1+q^n)^2}-16\sum_{n=1}^{\infty}\frac{n(1-q^n)q^{2n^2}}{(1+q^n)}+8\sum_{n=1}^{\infty}\frac{(1-q^n)q^{2n^2}}{(1+q^n)}\left(\frac{q^n}{1+q^n}-\frac{1}{2}\right)\right)\\
=&\frac{1}{\vartheta_4^2}\left(1+4\sum_{n=1}^{\infty}\frac{q^{2n^2+n}}{(1+q^n)^2}-16\sum_{n=1}^{\infty}\frac{n(1-q^n)q^{2n^2}}{1+q^n}\right)
\end{aligned}
$$

which gives the result. □

Lemma 6.2.

$$
\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{\frac{n(n+1)}{2}}}{1-q^n}=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}-2\sum_{n=1}^{\infty}\frac{n(1+q^n)q^{2n^2}}{1-q^n}+\sum_{n=1}^{\infty}\frac{q^{2n^2+n}}{(1-q^n)^2}
$$

Proof. Recall from the proof of (1.2) in [5] that

(6.3)

$$
\lim_{a,b\to 1}\frac{1}{(1-a)(1-b)}\left(S_1(a,b;q)-\frac{1}{(ab;q)_\infty}\right)=\frac{1}{(q)_\infty}\left(-\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{\frac{n(n+1)}{2}}}{1-q^n}\right)
$$

where

$$S_1(a,b;q) = \frac{(a^{-1}q;q)_\infty}{(bq,q;q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(1-a)(-b)^n q^{\frac{n(n+1)}{2}}}{1-aq^n}$$

We now apply (6.2) with $f, d \to \infty$ and setting a equal to ab. Then letting $e \to a$ we obtain

$$S_1(a,b;q) = \frac{\left(\frac{q}{a},\frac{q}{b};q\right)_\infty}{\left(abq,\frac{q}{ab},q;q\right)_\infty} \sum_{n\in\mathbb{Z}} \frac{(1-abq^{2n})}{(1-ab)} \frac{(a,b;q)_n}{(aq,bq;q)_n} (ab)^{2n} q^{2n^2}$$

Now

(6.4)

$$\lim_{a,b\to 1} \frac{1}{(1-a)(1-b)} \left(\frac{\left(\frac{q}{a},\frac{q}{b};q\right)_\infty}{\left(abq,\frac{q}{ab},q;q\right)_\infty} - \frac{1}{(abq)_\infty} \right)$$
$$= \lim_{a,b\to 1} \frac{1}{(1-a)(1-b)} \left(\frac{1}{(abq)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n \left(\frac{q}{ab}\right)^n}{(q)_n^2} - \frac{1}{(abq)_\infty} \right)$$
$$= \lim_{a,b\to 1} \frac{1}{(abq)_\infty} \sum_{n=1}^{\infty} \frac{(aq)_{n-1}(bq)_{n-1}\left(\frac{q}{ab}\right)^n}{(q)_n^2} = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}$$

where we use Corollary 2.3 of [2] in the second equality.

Next we have

(6.5)

$$\lim_{a,b\to 1} \frac{\left(\frac{q}{a},\frac{q}{b};q\right)_\infty}{(1-a)(1-b)\left(abq,\frac{q}{ab},q;q\right)_\infty} \sum_{n\neq 0} \frac{(1-abq^{2n})}{(1-ab)} \frac{(a,b;q)_n}{(bq,aq;q)_n} q^{2n^2}(ab)^{2n}$$
$$= \frac{1}{(q)_\infty} \lim_{a,b\to 1} \sum_{n\neq 0} \frac{(1-abq^{2n})q^{2n^2}(ab)^{2n}}{(1-ab)(1-aq^n)(1-bq^n)}$$
$$= \frac{1}{(q)_\infty} \lim_{b\to 1} \sum_{n=1}^{\infty} \left(\frac{(1-bq^{2n})b^{2n}q^{2n^2}}{(1-b)(1-q^n)(1-bq^n)} + \frac{(1-bq^{-2n})b^{-2n}q^{2n^2}}{(1-b)(1-q^{-n})(1-bq^{-n})} \right)$$
$$= -\frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{d}{db}\Big|_{b=1} \left(\frac{(1-bq^{2n})b^{2n}q^{2n^2}}{(1-b)(1-q^n)(1-bq^n)} + \frac{(q^{2n}-b)b^{-2n}q^{2n^2}}{(1-b)(1-q^n)(b-q^n)} \right)$$
$$= -\frac{4}{(q)_\infty} \sum_{n=1}^{\infty} \frac{n(1+q^n)q^{2n^2}}{1-q^n} + \frac{2}{(q)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n^2+n}}{1-q^n}$$

Comparing (6.3), (6.4), and (6.5) and multiplying by $(q)_\infty/2$ we obtain the result. $\square$

Lemma 6.3.

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}-\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2} = 4\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}$$
$$=\sum_{n=1}^{\infty}\frac{4n(1+q^{2n})q^{2n^2}}{1-q^{2n}}+4\sum_{n=1}^{\infty}\frac{q^{2n^2+2n}}{(1-q^{2n})^2}$$

Proof. The first equality is immediate and follows by combining the initial two sums term by term. Finally

$$\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}=\sum_{n,m=1}^{\infty}mq^{2nm}=\sum_{m=1}^{\infty}\left(\sum_{n=1}^{m}+\sum_{m+1}^{\infty}\right)mq^{2nm}$$
$$=\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}mq^{2nmn}+\sum_{m=1}^{\infty}\frac{mq^{2m^2+2m}}{1-q^2m}$$
$$=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(m+n)q^{2n(m+n)}+\sum_{m=1}^{\infty}\frac{mq^{2m^2+2m}}{1-q^{2m}}$$
$$=\sum_{n=1}^{\infty}\frac{q^{2n^2+2n}}{(1-q^{2n})^2}+\sum_{n=1}^{\infty}\frac{nq^{2n^2}}{1-q^{2n}}+\sum_{m=1}^{\infty}\frac{mq^{2m^2+2m}}{1-q^{2m}}$$
$$=\sum_{n=1}^{\infty}\frac{nq^{2n^2}(1+q^{2n})}{1-q^{2n}}+\sum_{n=1}^{\infty}\frac{q^{2n^2+2n}}{(1-q^{2n})^2}.$$

□

We are now ready to prove the main result.

q-Series Proof of Theorem 1.3. Comparing the statement of Theorem 1.3 with the assertion of Lemma 1, we see that the theorem is equivalent to the following

(6.6)
$$\sum_{n=1}^{\infty}\frac{q^{2n^2+n}}{(1+q^n)^2}-2\sum_{n=1}^{\infty}\frac{nq^{2n^2}(1-q^n)}{1+q^n}=\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n nq^{\frac{n(n+1)}{2}}}{1-q^n}$$

Now in (6.6) we replace the final sum on the right-hand side by the negative of the right-hand side of the identity given by Lemma 6.2. Hence Theorem 1.3 is equivalent to the assertion that

$$\sum_{n=1}^{\infty}\frac{q^{2n^2+n}}{(1+q^n)^2}-2\sum_{n=1}^{\infty}\frac{nq^{2n^2}(1-q^n)}{1+q^n}$$
$$=\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}-\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}+2\sum_{n=1}^{\infty}\frac{nq^{2n^2}(1+q^n)}{1-q^n}+\sum_{n=1}^{\infty}\frac{q^{2n^2+n}}{(1-q^n)^2}.$$

We now combine the first sum on the left-hand side with the final sum on the right-hand side and apply Lemma 6.3 to the first two sums on the right-hand side. Hence Theorem 1.3 is equivalent to the assertion that

$$-4\sum_{n=1}^{\infty}\frac{q^{2n^2+2n}}{(1-q^{2n})^2}-4\sum_{n=1}^{\infty}\frac{nq^{2n^2}(1-q^n)}{1+q^n}=\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}-\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2},$$

which is merely a restatement of Lemma 6.3. □

7. Modular Proof of Theorem 1.3

In this section we establish Theorem 1.3 by computing the modular transformations of the right hand side and demonstrate that they match those of the left hand side. Finally checking that enough of the Fourier coefficients are equal yields the result.

"Modular" proof of Theorem 1.3. We use Theorem 2.6 together with (2.12) to find

$$\begin{aligned}\widehat{\mu}(u,u;\tau)&=\mu(u,u;\tau)+\frac{i}{2}R(0;\tau)\\&=\mu(u,u;\tau)-\frac{i}{2}\int_{-\overline{\tau}}^{i\infty}\frac{g_{\frac{1}{2},\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}dz\\&=\mu(u,u;\tau)+\frac{1}{2}\int_{-\overline{\tau}}^{i\infty}\frac{\eta(z)^3}{\sqrt{-i(z+\tau)}}dz.\end{aligned}$$

Combining this with the definition of $\widehat{f_1}$ in Theorem 3.2 and the definition of $\widehat{E_2}$ in (2.6), we see that if we define

$$M(u;\tau):=f_1(\tau)+i\eta(\tau)^3\mu(u,u;\tau)-\frac{1}{12}E_2(\tau),$$

then this also equals

$$\widehat{f_1}(\tau)+i\eta(\tau)^3\widehat{\mu}(u,u;\tau)-\frac{1}{12}\widehat{E_2}(\tau).$$

Now using the transformation properties of $\widehat{f_1}$ from Theorem 3.2, $\widehat{\mu}$ from Theorem 2.5, and $\widehat{E_2}$ we get that M transforms as

$$M\left(\frac{u}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2M(u;\tau)$$

for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. Further, by the first part of Theorem 2.5 we have the elliptic transformation property

$$M(u+k\tau+l;\tau)=M(u;\tau),$$

for all $k,l\in\mathbb{Z}$. Using these we easily see that

$$m(\tau):=\begin{pmatrix}M\left(\frac{1}{2};\tau\right)\\M\left(\frac{\tau}{2};\tau\right)\\M\left(\frac{\tau+1}{2};\tau\right)\end{pmatrix}$$

transforms as a vectorvalued modular form of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$:

$$m(\tau+1)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}m(\tau)\qquad\text{and}\qquad m\left(-\frac{1}{\tau}\right)=\tau^2\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}m(\tau).$$

Fom this we see that $M(\frac{1}{2};\tau)$ transforms as a modular form of weight 2 on $\Gamma_0(2)$.

On the other hand,

$$g(\tau):=\begin{pmatrix}E_2(\tau)-2E_2(2\tau)\\E_2(\tau)-\frac{1}{2}E_2(\frac{\tau}{2})\\E_2(\tau)-\frac{1}{2}E_2(\frac{\tau+1}{2})\end{pmatrix}=\begin{pmatrix}\widehat{E}_2(\tau)-2\widehat{E}_2(2\tau)\\\widehat{E}_2(\tau)-\frac{1}{2}\widehat{E}_2(\frac{\tau}{2})\\\widehat{E}_2(\tau)-\frac{1}{2}\widehat{E}_2(\frac{\tau+1}{2})\end{pmatrix}.$$

satisfies the same modular transformation properties as m. Further we get directly from the definitions

$$g(\tau)=\begin{pmatrix}-1+O(q)\\\frac{1}{2}+O(q^{\frac{1}{2}})\\\frac{1}{2}+O(q^{\frac{1}{2}})\end{pmatrix},\qquad\text{and}\qquad m(\tau)=\begin{pmatrix}\frac{1}{6}+O(q)\\-\frac{1}{12}+O(q^{\frac{1}{2}})\\-\frac{1}{12}+O(q^{\frac{1}{2}})\end{pmatrix},$$

and so

$$m(\tau)+\frac{1}{6}g(\tau)=\begin{pmatrix}O(q)\\O(q^{\frac{1}{2}})\\O(q^{\frac{1}{2}})\end{pmatrix}.$$

If we take the product of the three components of $m+\frac{1}{6}g$, then we get a holomorphic cusp form on $\mathrm{SL}_2(\mathbb{Z})$ of weight 6, and hence this equals zero. This then implies that all three components are zero and so we get $m+\frac{1}{6}g=0$.

The first component equals

$$\frac{1}{6}\left(-1-24\sum_{n>0}\sigma_1^{odd}(n)q^n\right)-\frac{1}{12}E_2(\tau)+\sum_{n\neq 0}\frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n}+2\frac{(q)_\infty^2}{(-q)_\infty^2}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n(n+1)}{2}}}{1+q^n}$$

where we have used $\vartheta\left(\frac{1}{2};\tau\right)=-2q^{\frac{1}{8}}(q)_\infty(-q)_\infty^2$ and $\sigma_1^{odd}(n)=\sum_{\substack{d\mid n\\ d\text{ odd}}}d$. Using (4.1), (4.2), and (3.6) to rewrite the Eisenstein series terms gives the result. □

Remark. The calculations above show that $M(u;\tau)$ is a multiple of the Weierstrass $\wp$-function. Thus the conclusion follows from known facts about $\wp$ at half periods. We have included the details for completeness.

8. Holomorphic Projection

In this section we demonstrate that the function $B(\tau)=\frac{1}{\eta(\tau)^3}\sum_{n\neq 0}\frac{(-1)^n nq^{\frac{n(n+1)}{2}}}{1-q^n}$ from Section 7 arises from the holomorphic projection of $\eta^3(\tau)(\eta^3(\tau))^*$. The holomorphic projection operator is the unique linear map $\pi_{hol} = \pi_{hol,k}$ mapping the space of non-holomorphic modular forms of weight k to the space of cusp forms of weight k and level N satisfying

$$(h, \Phi) = (h, \pi(\Phi))$$

for all cusp forms h of weight k and level N and $(\cdot, \cdot)$ is the Petersson inner product.

If $\Phi(\tau) := \sum_{n\in\mathbb{Z}} a_n(y)q^n$ is a modular form of weight k, not necessarily holomorphic, on $\mathrm{SL}_2(\mathbb{Z})$ such that for $\tau \to i\infty$ there exists an $\epsilon > 0$ with

$$(\Phi \mid_k \gamma)(\tau) = O(y^{-\epsilon}), \tag{8.1}$$

then

$$\pi_{hol}(\Phi)(\tau) := \sum_{n=1}^{\infty} c_n(\Phi)q^n$$

with

$$c_n(\Phi) := \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \int_0^\infty a_n(y)e^{-4\pi n y} y^{k-2} dy. \tag{8.2}$$

See for instance [33] or Proposition 5.1 of [21].

Proposition 6.2 of [21] (see also Proposition 11 of [15]) suggest that $\frac{1}{\eta^3}\pi_{hol}\left(\eta^3(\eta^3)^*\right)$ is, after the addition of an Eisenstein series, the holomorphic part of a harmonic weak Maass form (see [27] for discussion of holomorphic parts of harmonic weak Maass forms). Precisely, we have the the following result.

Proposition 8.1. *With the notation from above*

$$\pi_{hol}\left(\eta^3(\eta^3)^*\right)(\tau) = \sum_{n\neq 0} \frac{(-1)^n n q^{\frac{n(n+1)}{2}}}{1-q^n}.$$

Proof. For simplicity let $g(\tau) = \eta^3(\tau)$. From (2.3) we have $g(\tau) = \sum_{n=1}^\infty \left(\frac{-4}{n}\right) n q^{\frac{n^2}{8}}$. Thus $(g(\tau))^* = \pi^{-\frac{1}{2}} \sum_{n=1}^\infty \left(\frac{-4}{n}\right) \Gamma\left(\frac{1}{2}; \frac{\pi n^2 y}{2}\right) q^{-\frac{n^2}{8}}$. So that

$$g(\tau)g^*(\tau) = \pi^{-\frac{1}{2}} \sum_{n,m>0} m \left(\frac{-4}{nm}\right) \Gamma\left(\frac{1}{2}; \frac{\pi n^2 y}{2}\right) q^{\frac{m^2-n^2}{8}}.$$

With $\ell = m^2 - n^2$ we need to compute

$$\begin{aligned}\int_0^\infty \Gamma\left(\frac{1}{2}; \frac{\pi n^2 y}{2}\right) e^{-\frac{\pi \ell y}{2}} dy &= \int_0^\infty \int_{\frac{\pi n^2 y}{2}}^\infty e^{-t} t^{\frac{1}{2}} e^{-\frac{\pi \ell y}{2}} \frac{dt}{t} dy = \int_0^\infty \int_1^\infty e^{-(wn^2+\ell)\frac{\pi}{2}y} (yw)^{\frac{1}{2}} \frac{dw}{w} dy \\ &= n\sqrt{\frac{\pi}{2}} \int_1^\infty w^{\frac{1}{2}} \frac{dw}{w} \int_0^\infty e^{-(wn^2+\ell)\frac{\pi}{2}y} y^{\frac{3}{2}} \frac{dy}{y} \\ &= \frac{n}{\sqrt{\pi}} \int_1^\infty \frac{1}{w^{\frac{1}{2}}(wn^2+\ell)^{\frac{3}{2}}} dw = \frac{n}{\sqrt{\pi}} \left(\frac{2\sqrt{w}}{\ell\sqrt{wn^2+\ell}} \Big|_1^\infty \right) \\ &= \frac{2n}{\sqrt{\pi}\ell} \left(\frac{1}{n} - \frac{1}{\sqrt{n^2+\ell}} \right) = \frac{2}{\sqrt{\pi} m(m+n)}.\end{aligned}$$

where we use $\int_0^\infty e^{-\alpha y} y^{\frac{1}{2}} \frac{dy}{y} = \alpha^{-\frac{3}{2}} \frac{\sqrt{\pi}}{2}$ in the third equality.

Inserting this into (8.2) gives

$$\pi_{hol}(gg^*)(\tau) = \sum_{m>n>0} \left(\frac{-4}{nm}\right)(m-n)q^{\frac{m^2-n^2}{8}}.$$

Now $\left(\frac{-4}{n}\right)\left(\frac{-4}{m}\right) = 0$ unless both n and m are odd. In which case we have $n = 2a+1$ and $m = 2b+1$. Thus $\left(\frac{-4}{nm}\right) = (-1)^{a+b}$ and we have

$$\pi_{hol}(gg^*)(\tau) = 2\sum_{a>b\geq 0} (-1)^{a+b}(a-b)q^{\frac{a^2+a-b^2-b}{2}} = 2\sum_{b\geq 0, h>0} (-1)^h hq^{\frac{h^2+h}{2}+bh} = 2\sum_{h>0} \frac{(-1)^h hq^{\frac{h(h+1)}{2}}}{1-q^h}$$

which yields the result. □

9. Proof of Theorem 1.4

In this section we compute an asymptotic expansion for $V(n)$. We follow a circle method argument used by Bringmann and Mahlburg [10] (see also [11]) to calculate an asymptotic for coefficients of mixed modular forms. Before turning to our calculation of the asymptotic of Theorem 1.4 it is convenient to define a two variable generating function for concave compositions that carries additional information about the shape of the composition.

9.1. **Where is the Position of the Central Part?** Let $R-L$ be the *tilt* of the concave composition given earlier. This quantity measures the position of the central part. Let $V(m,n)$ (resp. $V_d(m,n)$) be the number of concave composition (resp. strongly concave compositions) of n with tilt equal to m. Standard techniques give

$$v(x,q) := \sum_{n\geq 0, m\in\mathbb{Z}} V(m,n)x^m q^n = \sum_{n=0}^{\infty} \frac{q^n}{(xq^{n+1};q)_\infty (x^{-1}q^{n+1};q)_\infty}$$

$$v_d(x,q) := \sum_{n\geq 0, m\in\mathbb{Z}} V_d(m,n)x^m q^n = \sum_{n=0}^{\infty} q^n(-xq^{n+1};q)_\infty(-x^{-1}q^{n+1};q)_\infty.$$

The following identities are deduced in a similar manner to Theorem 1 of [5].

Theorem 9.1. *In the notation above*

$$v_d(x,q) = -\,x\sum_{n=0}^{\infty}(-1)^n x^{2n} q^{\frac{n(n+1)}{2}} + (x)_\infty (x^{-1}q)_\infty \sum_{n=0}^{\infty}\left(\frac{12}{n}\right) x^{\frac{n-1}{2}} q^{\frac{n^2-1}{24}}$$

$$qv(x,q) = -\frac{1}{(x)_\infty(x^{-1})_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq)_n(x^{-1}q)_n} + \frac{(1-x)}{(q)_\infty(x)_\infty(x^{-1})_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n^2+n}{2}}x^{-n}}{1-xq^n}.$$

Proof of Theorem 9.1. The identity for $v_d(x,q)$ follows from an application of (3.6) of [1] and then (13.3) and (6.1) of [18]. The second identity is a corollary of Theorem 4 of [14]. □

One should compare these generating functions to the rank of a partition studied in [12]. As is the case in of [12] the moments of the tilt statistic can be calculated precisely. We expect the tilt statistic to be asymptotically normally distributed.

9.2. **Circle Method and the Proof of Theorem 1.4.** By Cauchy's theorem we have

$$V(n-1)=\frac{1}{2\pi i}\int_C \frac{qv(q)}{q^{n+1}}dq,$$

where C is an arbitrary path inside the unit circle that loops around 0 in the counterclockwise direction. We choose the circle with radius $r=e^{-\pi/N^2}$ with $N:=\lfloor n^{\frac{1}{2}}\rfloor$, and use the parameterization $q=e^{-2\pi/N^2+2\pi it}$ with $0\le t\le 1$. As usual in the circle method, we define

$$\vartheta'_{h,k}:=\frac{1}{k(k_1+k)} \qquad \text{and} \qquad \vartheta''_{h,k}:=\frac{1}{k(k_2+k)},$$

where $\frac{h_1}{k_1}<\frac{h}{k}<\frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order $N:=\lfloor n^{\frac{1}{2}}\rfloor$. So $\frac{1}{k+k_j}\le\frac{1}{N+1}$ for $j=1,2$. Next, decompose the path of integration into paths along the Farey arcs $-\vartheta'_{h,k}\le\Phi\le\vartheta''_{h,k}$, where Φ is defined by $z=\frac{k}{N^2}-ki\Phi$ with $-\vartheta'_{h,k}\le\Phi\le\vartheta''_{h,k}$ and $0\le h\le k\le N$ with $(h,k)=1$. Hence

$$V(n-1)=\sum_{\substack{1\le k<N\\(h,k)=1}} e^{-\frac{2\pi ihn}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} v\left(e^{\frac{2\pi i}{k}(h+iz)}\right)e^{\frac{2\pi i}{k}(h+iz)}e^{\frac{2\pi nz}{k}}d\Phi. \tag{9.1}$$

To compute the asymptotic nature of $v\left(e^{\frac{2\pi i}{k}(h+iz)}\right)$ as $z\to 0$ the useful form of Theorem 9.1 is

$$qv(x,q)=q^{\frac{1}{12}}\frac{\eta(\tau)}{2\sin(\pi u)\vartheta(u;\tau)}R(u,\tau)-iq^{\frac{1}{8}}\mu\left(u,-u;\tau\right). \tag{9.2}$$

where we have set $q=e^{2\pi i\tau}$, $x=e^{2\pi iu}$, and

$$R(u,\tau)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq)_n(x^{-1}q)_n}$$

is the Rank generating function (see [27]). The idea is to compute the asymptotics as $z\to 0^+$ of the two variable version of the generating function and then set $u=0$. We are interested in exponential growth and will freely ignore terms once they are determined to have smaller growth.

The proof of Proposition 3.5 of [12] gives the following asymptotic evaluation for $R(u,\tau)$. Let $\widetilde{h}\in\{-1,0,1\}$ defined by $\widetilde{h}\equiv h \pmod 3$. Moreover, if $0\le\ell<k$, then we write

$$\alpha^{\pm}(\ell,k):=\frac{1}{k}\left(\pm\frac{1}{3}-\left(\ell-\frac{k-1}{2}\right)\right)$$

and $\widetilde{k} = \begin{cases} 3k & 3 \nmid k \\ \frac{k}{3} & 3 \mid k \end{cases}$ and

(9.3)

$$\widetilde{H}^{\pm}(u,h,\ell,k;z) := \pm e^{\frac{\widetilde{h}^2\pi}{3kz}} \xi_\ell\left(\frac{3h}{(3,k)}, \frac{k}{(3,k)}\right) H\left(\frac{(3,k)iu}{z} + \alpha^{\pm}\left(\ell, \frac{k}{(3,k)}\right) \mp \frac{\widetilde{h}i}{kz}; \frac{i}{\widetilde{k}z}\right).$$

where

$$\xi_\ell(h,k) := (-1)^{\ell+1} e^{-\frac{\pi i h}{4k}(2\ell+1)^2 \mp \frac{\pi i}{k}\left(\frac{\widetilde{h}-h}{3}\right)(2\ell+1) + \frac{2\pi i \widetilde{h}}{9k}}. \tag{9.4}$$

Proposition 9.2. *For $\tau = \frac{1}{k}(h+iz)$ with $0 \le h < k$ with $(h,k)=1$ set $q = e^{\frac{2\pi i}{k}(h+iz)}$ and $q_1 = e^{\frac{2\pi i}{k}\left([-h]_k + \frac{i}{z}\right)}$. With this notation we have*

$$R(u,\tau) = -i^{\frac{3}{2}} \chi^{-1}\Big(h,[-h]_k,k\Big)(qq_1^{-1})^{\frac{1}{24}} z^{-\frac{1}{2}} e^{\frac{3k\pi u^2}{z}} \frac{\sin(\pi u)}{\sinh\left(\frac{\pi u}{z}\right)} + \sum_{\ell=1}^{\infty} a_\ell(z) \frac{(2\pi i u)^\ell}{\ell!}$$
$$- \frac{i\sin(\pi u)}{\sqrt{\widetilde{k}z}} e^{-\frac{\pi z}{12k} - \frac{\pi i h}{4k} + \frac{3\pi k u^2}{z}} \sum_{\pm} \sum_{\ell=0}^{\frac{k}{(3,k)}-1} \widetilde{H}^{\pm}(u,h,\ell,k;z)$$

with $a_\ell(z) \ll_\ell |z|^{1-\ell} e^{-\frac{23\pi}{12k} Re\left(\frac{1}{z}\right)}$.

Additionally, we have

$$\frac{q^{\frac{1}{12}}\eta(\tau)}{\vartheta(u;\tau)} = q^{\frac{1}{12}}\chi^{-2}(h,[-h]_k,k)e^{\frac{k\pi u^2}{z}} \frac{\eta(\tau')}{\vartheta\left(\frac{iu}{z};\tau'\right)} = \frac{q^{\frac{1}{12}} q_1^{\frac{1}{24}}(1+O(q_1))e^{\frac{k\pi u^2}{z}}}{\chi^2(h,[-h]_k,k)\vartheta\left(\frac{iu}{z};\tau'\right)}.$$

Thus

(9.5)

$$\frac{q^{\frac{1}{12}}\eta(\tau)}{2\sin(\pi u)\vartheta(u;\tau)} R(u,\tau) = -i^{\frac{3}{2}}\chi^{-3}(h,[-h]_k,k)\, q^{\frac{1}{8}} \frac{z^{-\frac{1}{2}} e^{\frac{4\pi k u^2}{z}}}{2\sinh\left(\frac{\pi u}{z}\right)\vartheta\left(\frac{iu}{z};\tau'\right)} + O\left(e^{-\frac{\alpha}{k}\mathrm{Re}\left(\frac{1}{z}\right)}\right)$$
$$- i\chi^{-2}(h,[-h]_k,k) \frac{q^{\frac{1}{12}} e^{-\frac{\pi z}{12k} - \frac{\pi i h}{4k}} q_1^{\frac{1}{24}}(1+O(q_1))}{2\sqrt{\widetilde{k}z}\,\vartheta\left(\frac{iu}{z};\tau'\right)} \sum_{\pm} \sum_{\ell=0}^{\frac{k}{(3,k)}-1} \widetilde{H}^{\pm}(u,h,\ell,k;z)$$

for some $\alpha > 0$ independent of k as $z \to 0$ and $u \to 0$.

Turning to the other term we have the following proposition.

Proposition 9.3. *Let $\tau = \frac{1}{k}(h+iz)$ and $\tau' = \frac{1}{k}\left([-h]_k + \frac{i}{z}\right)$ then*

$$\mu(u,-u;\tau) = i^{\frac{1}{2}}\chi^{-3}(h,[-h]_k,k) z^{-\frac{1}{2}} e^{\frac{4\pi k u^2}{z}} \mu\left(\frac{iu}{z}, -\frac{iu}{z}; \tau'\right)$$
$$+ \frac{(-1)^{h+1}i}{2} e^{\frac{4\pi k u^2}{z}} \sum_{\ell=0}^{k-1} e^{-\frac{\pi i}{k}\left(\ell+\frac{1}{2}\right)^2} H\left(\frac{2iu}{z} - \frac{1}{k}\left(\ell - \frac{k-1}{2}\right); \frac{i}{kz}\right).$$

Proof. Theorem 2.5 implies that

$$\begin{aligned}(9.6)\quad \mu(u,-u;\tau) &= \chi^{-3}\Big(h,[-h]_k,k\Big)\sqrt{\frac{i}{z}}e^{\frac{4\pi k u^2}{z}}\mu\left(\frac{iu}{z},-\frac{iu}{z};\tau'\right)\\ &+\frac{i}{2}\chi^{-3}\Big(h,[-h]_k,k\Big)\sqrt{\frac{i}{z}}e^{\frac{4\pi k u^2}{z}}R\left(\frac{2iu}{z};\tau'\right)-\frac{i}{2}R\left(2u;\tau\right).\end{aligned}$$

Propositions 2.3 and 2.4 yield

$$\begin{aligned}R\left(2u;\tau\right) = & -\sum_{\ell=0}^{k-1} e^{-\frac{\pi i}{k}(h+iz)\left(\ell-\frac{k-1}{2}\right)^2-4\pi iu\left(\ell-\frac{k-1}{2}\right)}e^{-\pi i\left(\ell-\frac{k-1}{2}\right)-\frac{\pi ikh}{4}}\left(-1\right)^{\left(\ell-\frac{k-1}{2}\right)(h+1)}\\ \times & \frac{e^{\frac{\pi}{kz}\left(2uk+\left(\ell-\frac{k-1}{2}\right)iz\right)^2}}{\sqrt{kz}}\left(R\left(\frac{2iu}{z}-\frac{1}{k}\left(\ell-\frac{k-1}{2}\right);\frac{i}{kz}\right)-H\left(\frac{2iu}{z}-\frac{1}{k}\left(\ell-\frac{k-1}{2}\right);\frac{i}{kz}\right)\right).\end{aligned}$$

The non-holomorphic R-functions above will exactly cancel with the other term of (9.6); this can be shown as in [11] or [12]. Simplifying the factors multiplying the H-functions gives the result. □

We will evaluate the asymptotic nature of the two terms on the right hand side of Proposition 9.3. First consider the terms with the H-function. we have

$$H\left(\frac{2iu}{z}-\frac{1}{k}\left(\ell-\frac{k-1}{2}\right);\frac{i}{kz}\right)=O\left(\int_{-\infty}^{\infty}e^{-\frac{\pi x^2}{k}\mathrm{Re}\left(\frac{1}{z}\right)}dx\right)=O\left(\sqrt{k}\mathrm{Re}\left(\frac{1}{z}\right)^{-\frac{1}{2}}\right).$$

as $u\to 0$ and $z\to 0$. Next turning to the other term we have

$$\mu\left(\frac{iu}{z},-\frac{iu}{z};\tau'\right)=\frac{1}{2\sinh\left(\frac{\pi u}{z}\right)\vartheta\left(-\frac{iu}{z};\tau'\right)}+\frac{e^{-\frac{\pi u}{z}}}{\vartheta\left(-\frac{iu}{z};\tau'\right)}\sum_{n\neq 0}\frac{(-1)^n q_1^{\frac{n(n+1)}{2}}e^{\frac{2\pi nu}{z}}}{1-e^{-\frac{2\pi u}{z}}q_1^n}.$$

The summation is $O(uq_1)$, since it is 0 when $u=0$ by symmetry. Consequentially, using Proposition 5.2 we have $\vartheta\left(-\frac{iu}{z};\tau'\right)^{-1}=O\left(\frac{z}{u}q_1^{-\frac{1}{8}}\right)$, we see that the term with the summation is $O(q_1^{\frac{7}{8}})=O\left(e^{-\frac{7\pi}{4kz}}\right)$ after setting $u=0$. So it is of exponential decay and will not contribute to the asymptotic expansion of our generating function. Hence

(9.7)

$$\begin{aligned}-iq^{\frac{1}{8}}\mu\left(u,-u;\tau\right) &= -\,i^{\frac{3}{2}}\chi^{-3}(h,[-h]_k,k)q^{\frac{1}{8}}z^{-\frac{1}{2}}e^{\frac{4\pi ku^2}{z}}\mu\left(\frac{iu}{z},-\frac{iu}{z};\tau'\right)+O_k\left(1\right)\\ &=\frac{i^{\frac{3}{2}}q^{\frac{1}{8}}\chi^{-3}(h,[-h]_k,k)z^{-\frac{1}{2}}e^{\frac{4\pi ku^2}{z}}}{2\sinh\left(\frac{\pi u}{z}\right)\vartheta\left(\frac{iu}{z};\tau'\right)}+O\left(z^{-\frac{1}{2}}q_1^{-\frac{7}{8}}\right)+O_k\left(1\right)\end{aligned}$$

as $u\to 0$ when z is chosen as in the circle method calculations.

Using (9.2), (9.5), and (9.7) we see that as $u \to 0$ and $z \to 0$ we have the following asymptotic evaluation for $v(q)$ with $q = e^{\frac{2\pi i}{k}(h+iz)}$.

$$qv(q) \sim -i\chi^{-2}(h, [-h]_k, k) \frac{q^{\frac{1}{12}} e^{-\frac{\pi z}{12k} - \frac{\pi i h}{4k}} q_1^{\frac{1}{24}} (1 + O(q_1))}{2\sqrt{kz}\vartheta\left(\frac{iu}{z}; \tau'\right)} \sum_{\pm} \sum_{\ell=0}^{\frac{k}{(3,k)}-1} \widetilde{H}^{\pm}(u, h, \ell, k; z).$$

It is evident that the largest growth comes from the case when $k = 1$ and thus the largest contribution to (9.1) will come from the case $k = 1$. In this case we set

$$\begin{aligned}
\sum_{\pm} \widetilde{H}^{\pm}(u, 0, 0, 1; z) &= -H\left(\frac{iu}{z} + \frac{1}{3}; \frac{i}{3z}\right) + H\left(\frac{iu}{z} - \frac{1}{3}; \frac{i}{3z}\right) \\
&= -\int_{-\infty}^{\infty} \frac{e^{-\frac{\pi x^2}{3z}}}{\cosh(\pi x)} e^{-\frac{2\pi i x u}{z}} \left(e^{-\frac{2\pi x}{3}} - e^{\frac{2\pi x}{3}}\right) dx \\
&= -2i \int_{-\infty}^{\infty} e^{-\frac{\pi x^2}{3z}} \sin\left(\frac{2\pi x u}{z}\right) \frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} dx \\
&= -2i \cdot \frac{2\pi u}{z} \int_{-\infty}^{\infty} x e^{-\frac{\pi x^2}{3z}} \frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} dx + O\left(u^3\right)
\end{aligned}$$

Therefore, with $q = e^{-2\pi z}$ we have

$$\begin{aligned}
qv(q) &\sim \lim_{u \to 0} -e^{-\frac{\pi z}{4}} e^{-\frac{\pi}{12z}} \frac{1}{2\sqrt{3z}\vartheta\left(\frac{iu}{z}; \tau'\right)} \left(-\frac{4\pi i u}{z} \int_{-\infty}^{\infty} x e^{-\frac{\pi x^2}{3z}} \frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} dx\right) \\
&\sim \frac{e^{\frac{\pi}{6}\left(\frac{1}{z} - \frac{3}{2}z\right)}}{\sqrt{3z}} \int_{-\infty}^{\infty} x e^{-\frac{\pi x^2}{3z}} \frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cos(\pi x)} dx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
V(n-1) &\sim -\int_{-\vartheta'_{0,1}}^{\vartheta''_{0,1}} \frac{e^{\frac{\pi}{6}\left(\frac{1}{z} + (12n - \frac{3}{2})z\right)}}{\sqrt{3z}} \int_{-\infty}^{\infty} x e^{-\frac{\pi x^2}{3z}} \frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} dx d\Phi \\
&\sim -\frac{1}{\sqrt{3}} \int_{-\infty}^{\infty} x \frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} \int_{-\frac{1}{N}}^{\frac{1}{N}} z^{-\frac{1}{2}} e^{\frac{\pi}{6}\left((1-2x^2)\frac{1}{z} + (12n - \frac{3}{2})z\right)} d\Phi dx
\end{aligned}$$

where $z = \frac{1}{N^2} - i\Phi$ and we symmetrize the integral by using $\int_{-\vartheta'_{0,1}}^{\vartheta''_{0,1}} = \int_{-\frac{1}{N}}^{\frac{1}{N}} - \int_{-\frac{1}{N}}^{-\vartheta'_{0,1}} - \int_{\vartheta''_{0,1}}^{\frac{1}{N}}$ and $\sim$ represents the asymptotic expansion with respect to n. The final two sums contribute a polynomially bounded error with respect to N. We handle the resulting integral with respect to Φ exactly as in Proposition 3.2 of [10] (see Lemma 4.2 of [31] for an analogous calculation). This results in

$$\int_{-\frac{1}{N}}^{\frac{1}{N}} z^{-\frac{1}{2}} e^{\frac{\pi}{6}\left((1-2x^2)\frac{1}{z}+(12n-\frac{3}{2})z\right)} d\Phi = -2\pi \left(\frac{1-2x^2}{12n-\frac{3}{2}}\right)^{\frac{1}{4}} I_{\frac{1}{2}}\left(\frac{\pi}{3}\sqrt{(12n-\frac{3}{2})(1-2x^2)}\right)$$
$$= -2\sqrt{6}\left(12n-\frac{3}{2}\right)^{-\frac{1}{2}} \sinh\left(\frac{\pi}{3}\sqrt{\left(12n-\frac{3}{2}\right)(1-2x^2)}\right)$$

Therefore,

$$V(n-1) \sim \frac{2\sqrt{2}}{\left(12n-\frac{3}{2}\right)^{\frac{1}{2}}} \int_{-\infty}^{\infty} x\frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} \sinh\left(\frac{\pi}{3}\sqrt{\left(12n-\frac{3}{2}\right)(1-2x^2)}\right) dx$$
$$\sim \frac{2\sqrt{2}}{\left(12n-\frac{3}{2}\right)^{\frac{1}{2}}} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x\frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} \sinh\left(\frac{\pi}{3}\sqrt{\left(12n-\frac{3}{2}\right)(1-2x^2)}\right) dx$$
$$\sim \frac{2\sqrt{2}}{\left(12n-\frac{3}{2}\right)^{\frac{1}{2}}} \int_{0}^{\frac{1}{\sqrt{2}}} x\frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} \exp\left(\frac{\pi}{3}\sqrt{\left(12n-\frac{3}{2}\right)(1-2x^2)}\right) dx \tag{9.8}$$

where the second equality follows because $\sinh\left(\frac{\pi}{3}\sqrt{\left(12n-\frac{3}{2}\right)(1-2x^2)}\right) = O(1)$ for $|x| > \frac{1}{\sqrt{2}}$.

We have the following Lemma which gives the asymptotic expansion of the remaining integral.

Lemma 9.4. *Let α_k for $k \geq 1$ be defined by*

$$\sum_{k=1}^{\infty} \alpha_k x^{2k} = \exp\left(\sqrt{1-2x^2}-1+x^2\right) x\frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)}.$$

As $y \to \infty$ we have

$$\int_0^{\frac{1}{\sqrt{2}}} e^{y\sqrt{1-2x^2}} x\frac{\sinh\left(\frac{2\pi x}{3}\right)}{\cosh(\pi x)} \sim e^y \frac{\sqrt{\pi}}{2\sqrt{y}} \sum_{t=1}^{\infty} \frac{(2t-1)!!}{2^{2t}} \alpha_t \frac{1}{y^t}.$$

From (9.8) and Lemma 9.4 we have

$$V(n-1) \sim \frac{\sqrt{6}}{\left(12n-\frac{3}{2}\right)^{\frac{3}{4}}} \exp\left(\frac{\pi}{3}\sqrt{12n-\frac{3}{2}}\right) \sum_{t=1}^{\infty} \frac{(2t-1)!!3^t}{2^{2t}\pi^t} \alpha_t \left(12n-\frac{3}{2}\right)^{-\frac{t}{2}}$$

which gives Theorem 1.4.

Note we have

$$\sum_{k=1}^{\infty} \alpha_k x^{2k} = \frac{2\pi}{3}x^2 - \frac{23}{81}\pi^3 x^4 - \frac{4860\pi - 1681\pi^5}{14580}x^6 + \frac{1837080\pi + 257543\pi^7 - 782460\pi^3}{5511240}x^8 + \cdots$$

10. Proof of Theorem 1.5

In this section we compute an asymptotic for $V_d(n)$. We follow the standard circle method set-up as in Section 9. As above, we have

$$V_d(n) = \sum_{\substack{1\le k<N\\(h,k)=1}} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} v_d\left(e^{\frac{2\pi i}{k}(h+iz)}\right) e^{\frac{2\pi n z}{k}} d\Phi. \tag{10.1}$$

Lemma 10.1. *Let* $c_{h,k}(n) := \left(-\frac{12}{n}\right) e^{\frac{\pi i h}{12k}n^2}$*, then if* $h \neq 0$ *then* $c_{h,k}(n)$ *is* $24k$ *periodic and has mean value 0.*

Applying this lemma with the proposition in Section 3 of [24] we have the following.

Lemma 10.2. *Let* $q = e^{\frac{2\pi i}{k}(h+iz)}$ *then*

$$\sum_{n=0}^{\infty}\left(\frac{-12}{n}\right) q^{\frac{n^2}{24}} \sim \sum_{r=0}^{\infty} L(-2r,h,k)\left(\frac{-\pi}{12k}\right)^r \frac{z^r}{r!}$$

where

$$L(-r,h,k) := -\frac{(24k)^r}{r+1}\sum_{n=1}^{24k} c_{h,k}(n) B_{r+1}\left(\frac{n}{24k}\right)$$

where $B_r(x)$ *denotes the* r*th Bernoulli polynomial and* c_k *is defined in Lemma 10.1. If* $h=0$ *we have* $L(-r) := L(-r,0,1) = -\frac{6^r}{r+1}\left(B_{r+1}\left(\frac{1}{6}\right) - B_{r+1}\left(\frac{5}{6}\right)\right)$.

By (2.2) we have

$$(-q)_\infty^2 = \begin{cases} \frac{1}{2} e^{\frac{\pi}{12k}\left(\frac{1}{z}+2z\right)} i^{\frac{1-k}{2}} \omega_{h,k} e^{-\frac{\pi i h}{4k}+\frac{\pi i k h}{4}} \left(\frac{h}{k}\right)\left(1+O\left(e^{-\frac{\pi}{kz}}\right)\right) & 2\nmid k\\ O\left(e^{-\frac{\pi}{6kz}}\right) & 2\mid k\end{cases}.$$

So that we have the following results

Proposition 10.3. *Let* $q = e^{\frac{2\pi i}{k}(h+iz)}$ *with* $0\le h<k$ *and* $(h,k)=1$*. If* $2\mid k$ *then* $v_d(q) + \sum_{n\ge 0}(-1)^n q^{\frac{(n+1)(n+2)}{2}} = O\left(e^{-\frac{\pi}{6kz}}\right)$*. When* $2\nmid k$ *we have*

$$v_d(q) + \sum_{n\ge0}(-1)^n q^{\frac{(n+1)(n+2)}{2}} \sim \frac{1}{2} e^{\frac{\pi}{12k}\left(\frac{1}{z}+2z\right)} i^{\frac{1-k}{2}} \omega_{h,k} e^{-\frac{\pi i h}{4k}+\frac{\pi i k h}{4}} \left(\frac{h}{k}\right)\sum_{r=0}^{\infty} L(-2r,k)\left(\frac{-\pi}{12k}\right)^r\frac{z^r}{r!}$$

Using these asymptotics with (10.1) and the following integral evaluation (see [23] for details)

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{\pi}{12k}\left(mz+\frac{1}{z}\right)} z^{\frac{1}{2}-j} d\Phi = \frac{2\pi}{k} m^{\frac{j}{2}-\frac{3}{4}} I_{\frac{3}{2}-j}\left(\frac{\pi\sqrt{m}}{6k}\right) + O\left(N^{-\frac{1}{4}}\right) \tag{10.2}$$

we have

$$V_d(n) \sim (-1)^{n+1}\delta(n) + \pi \sum_{\substack{2\nmid k<N\\(h,k)=1}} \frac{\mathcal{A}_k(n)}{k}\sum_{r=0}^{\infty} L(-2r,k)\left(\frac{-\pi}{12k}\right)^r \frac{1}{r!}(24n+2)^{-\frac{r+1}{2}} I_{r-1}\left(\frac{\pi}{6k}\sqrt{24n+2}\right)$$

where

$$\mathcal{A}_k(n) = \sum_{(h,k)=1} i^{\frac{1-k}{2}} \left(\frac{h}{k}\right) \omega_{h,k} e^{-\frac{\pi i h}{4}\left(\frac{1}{k}+k\right) - \frac{2\pi i h n}{k}}$$

and $\delta(m) = \begin{cases} 1 & m = \frac{(n+1)(n+2)}{2} \text{ for some } n \\ 0 & \text{else} \end{cases}.$

The main term comes from $k = 1$ and so we have

$$\begin{aligned} V_d(n) \sim & (-1)^{n+1}\delta(n) + \pi \sum_{r=0}^{\infty} L(-2r) \left(\frac{-\pi}{12}\right)^r \frac{1}{r!} (24n+2)^{-\frac{r+1}{2}} I_{r-1}\left(\frac{\pi}{6}\sqrt{24n+2}\right) \\ \sim & (-1)^{n+1}\delta(n) + \frac{\sqrt{3} e^{\frac{\pi}{6}\sqrt{24n+2}}}{(24n+2)^{\frac{3}{4}}} \sum_{r=0}^{\infty} L(-2r) \left(\frac{\pi}{12}\right)^r \frac{(-1)^r}{r!} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \left(\frac{3}{4\pi}\right)^\ell p(\ell, r)(24n+2)^{-\frac{r+\ell}{2}} \\ \sim & (-1)^{n+1}\delta(n) + \frac{\sqrt{3} e^{\frac{\pi}{6}\sqrt{24n+2}}}{(24n+2)^{\frac{3}{4}}} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m! (24n+2)^{\frac{m}{2}}} \gamma(m). \end{aligned}$$

11. The q-series $v_1(q)$

We have used a variety of different methods to establish the modular properties of the q-series defined by $v_2(q)$ or $\sum_{n\neq 0} \frac{(-1)^{n+1} n q^{\frac{n^2+n}{2}}}{1-q^n}$. On the other hand, we quoted the results of Bringmann [8] for the modularity of $v_1(q)$. But this q-series is susceptible to similar methods of the ones discussed here. Analogous to Theorem 1.3 and Proposition 8.1 we have the following result for the Appell sum appearing in the definition in $v_1(q)$ and the Appell-like sum arising from the holomorphic projection operation.

Theorem 11.1. *In the notation above,*

$$\pi_{hol}\left(\eta\eta^*\right) = \sum_{n\neq 0} (-1)^n \left(\frac{-3}{n-1}\right) \frac{n q^{\frac{n(n+1)}{6}}}{1-q^n} = \sum_{n \neq 0} \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{(1-q^n)^2} + \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{n>0} \frac{(q;q)_{n-1} q^n}{(1-q^n)}.$$

Remark. It follows that $\frac{1}{(q)_\infty} \pi_{hol}(\eta^*\eta) = \sum_{n=1}^{\infty} \mathrm{spt}(n) q^n$ where $\mathrm{spt}(n)$ is the smallest parts function and is equal to the sum of the number of smallest parts in the partitions of n. See the first authors paper [3] for more on the spt-function.

Sketch of Proof of Theorem 11.1. The first equality is given in Zagier's Bourbaki lecture [35]. While the second equality can be proved via "modular" methods discussed above. It would be interesting to establish a q-series proof of the second equality.

Finally, the last equality is derived in [3]. It may also be derived from the recent results of the first author, Garvan, and Liang [6]. Namely, letting $z = 1$ in Theorem 2.4 of [6] gives

$$\sum_{n\geq 1} \frac{(q)_{n-1} q^n}{(1-q^n)} = \sum_{n\neq 0} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-q^n)^2} - \sum_{n\neq 0} \frac{(-1)^{n-1} q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2}.$$

The result follows from the well known identity $\sum_{n\neq 0} \frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}}{(1-q^n)^2} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}$. $\square$

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