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## Answers to Selected Exercises

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§1. What Is Linear Programming?

1. True
3. True
5. True. This is because for real numbers any square and any absolute value are nonnegative.
7. False. For  $x = -1$ ,  $3(-1)^3 < 2(-1)^2$ .
8. False (see Definition 1.4).
9. False (see Example 1.9 or 1.10).
11. False. For example, the linear program

Minimize  $x + y$  subject to  $x + y = 1$

has infinitely many optimal solutions.

13. True. It is a linear equation. A standard form is  $4x = 8$  or  $x = 2$ .
15. No. This is not a linear form, but an affine function.
17. Yes if  $a$  and  $z$  do not depend on  $x, y$ .
18. No (see Definition 1.1).
19. No. But it is equivalent to a system of two linear constraints.
21. Yes. We can write  $0 = 0 \cdot x$ , which is a linear form.
22. True if  $y$  is independent of  $x$  and hence can be considered as a given number; see Definition 1.3.
23. Yes if  $a, b$  are given numbers. In fact, this is a linear equation.
25. No. We will see later that any system of linear constraints gives a convex set. But we can rewrite the constraint as follows  $x \geq 1$  OR  $x \leq -1$ . Notice the difference between OR and AND.
27. See Problem 6.7.
29.  $x = 3 - 2y$  with an arbitrary  $y$ .
31.  $\min = 0$  at  $x = y = 0, z = -1$ . All optimal solutions are given as follows:  $x = -y, y$  is arbitrary,  $z = -1$ .
33.  $\max = 1$  at  $x = 0$ .
35.  $\min = 0$  at  $x = -y = 1/2, z = -1$ .
37. No. This is a linear equation.
39. No
41. Yes

43. No

45. Yes

47. Yes

49. No

51. Yes

53. Yes. In fact, this is a linear equation.

55. No. This is not even equivalent to any linear constraint with rational coefficients.

57. No.

60.  $\min = 2^{-100} - 1$  at  $x = 0, y = 0, z = 3\pi/2, u = -100, v = -100$ . In every optimal solution,  $x, y, u, v$  are as before and  $z = 3\pi/2 + 2n\pi$  with any integer  $n$  such that  $-16 \leq n \leq 15$ . So there are exactly 32 optimal solutions.

## §2. Examples of Linear Programs

2.  $\min = 1.525$  at  $a = 0, b = 0.75, c = 0, d = 0.25$

4. Let  $x$  be the number of quarters and  $y$  the number of dimes we pay. The program is

$$25x + 10y \rightarrow \min,$$

subject to

$$0 \leq x \leq 100, 0 \leq y \leq 90, 25x + 10y \geq C \text{ (in cents), } x, y \text{ integers.}$$

This program is not linear because the conditions that  $x, y$  are integers. For  $C = 15$ , an optimal solution is  $x = 0, y = 2$ . For  $C = 102$ , an optimal solution is  $x = 3, y = 3$  or  $x = 1, y = 8$ . For  $C = 10000$ , the optimization problem is infeasible.

5. Let  $x, y$  be the sides of the rectangle. Then the program is

$$xy \rightarrow \min,$$

subject to

$$x \geq 0, y \geq 0, 2x + 2y = 100.$$

Since  $xy = x(50 - x) = 625 - (x - 25)^2 \leq 625$ ,  $\max = 625$  at  $x = y = 25$ .

7. We can compute the objective function at all 24 feasible solutions and find the following two optimal matchings: Ac, Ba, Cb, Dd and Ac, Bb, Ca, Dd with optimal value 7.

8. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound  $9 + 9 + 7 + 9 + 9 = 43$  for the objective function. This bound cannot be achieved because of a conflict over  $c$  (the third column). So  $\max \leq 42$ . On the other hand, the matching Aa, Bb, Cc, De, Ed achieved 42, so this is an optimal matching.

9. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound  $9 + 9 + 9 + 9 + 8 + 9 + 6 = 59$  for the objective function. However looking at B and C, we see that they cannot get  $9 + 9 = 18$  because of the conflict over g. They cannot get more than  $7 + 9 = 16$ . Hence, we have the upper bound  $\max \leq 57$ . On the other hand, we achieve this bound 57 in the matching Ac, Bf, Cg, De, Eb, Fd, Ga.

11. Let  $c_i$  be given numbers. Let  $c_j$  be an unknown maximal number (with unknown  $j$ ). The linear program is

$$c_1x_1 + \cdots + c_nx_n \rightarrow \max, \text{ all } x_i \geq 0, x_1 + \cdots + x_n = 1.$$

Answer:  $\max = c_j$  at  $x_j = 1, x_i = 0$  for  $i \neq j$ .

### §3. Graphical Method

1. Let SSN be 123456789. Then the program is

$$-x \rightarrow \max, 7x \leq 5, 13x \geq -8, 11x \leq 10.$$

Answer:  $\max = 8/13$  at  $x = -8/13$ .

3. Let SSN be 123456789. Then the program is

$$x + 2y \rightarrow \min, |12x + 4y| \leq 10, |5x + 15y| \leq 10, |x + y| \leq 24.$$

Answer:  $\min = -25/16$  at  $x = -11/16, y = -7/16$ .

5.  $\min = -72$  at  $x = 0, y = -9$

7.  $\min = -1/4$  at  $x = 1/2, y = -1/2$  or  $x = -1/2, y = 1/2$ .

6. The problem is unbounded ( $\min = -\infty$ ).

9.  $\max = 1$  at  $x = y = 0$

11.  $\max = 22$  at  $x = 4, y = 2$

13. The program is unbounded.

15.  $\max = 3$  at  $x = y = 0, z = 1$ . See the answer to Exercise 11 of §2.

### §4. Logic

1. False. For  $x = -1, |-1| = 1$ .

3. False. For  $x = -10, |-10| > 1$ .

5. True.  $1 \geq 0$ .

7. True.  $2 \geq 0$ .

9. True. The same as Exercise 7.

11. False.  $1 \geq 1$ .

13. True.  $5 \geq 0$ .

15. False. For example,  $x = 2$ .

17. True. Obvious.

19. False. For example,  $x = 1$ .

21. True.  $1 \geq 0$ .

22. Yes, we can. 23. Yes.  $10 \geq 0$ .  
 25. No, it does not.  $(-5)^2 > 10$ . 27. True.  
 29. False. The first condition is stronger than the second one.  
 31. True. 33. (i)  $\Rightarrow$  (ii), (iii), (iv)  
 35. (i)  $\Rightarrow$  (iii) 37. (i)  $\Leftrightarrow$  (ii)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii)  
 41. "only if".  
 42. This depends on the definition of it linear function.  
 43. No.  $x \geq 1, x \leq 0$  are two feasible constraints, but the system is infeasible.  
 44. False. 45. False. Under our conditions,  $|x| > |y|$ .  
 49. No, it does not follow.  
 51. Yes, it does. Multiply the first equation by -2 and add to the second equation to obtain the third equation.

## §5. Matrices

1.  $[2, 1, -6, 6]$  3. -14  
 5.  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 3 \\ -2 & -4 & 0 & 6 \\ 4 & 8 & 0 & -12 \end{bmatrix}$   
 7.  $(-14)^2 \cdot A^T B = \begin{bmatrix} 0 & -196 & -392 & 784 \\ 0 & -392 & -784 & 1568 \\ 0 & 0 & 0 & 0 \\ 0 & 588 & 1176 & -2352 \end{bmatrix}$  9. No.  $1 \neq 4$ .  
 10.  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  11.  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$   
 12.  $\begin{bmatrix} 5 & 2 & 3 & -1 \\ 1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$   
 15.  $b = a - 1, c = -1/3, d = 7a - 4, a$  arbitrary. 19.  $AB^T = 5$   
 21.  $A^T B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 3 & 2 \\ 2 & -1 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 6 & 4 \end{bmatrix}$   
 23.  $(A^T B)^2 = 5A^T B$ , and see Answer to 21.  
 25.  $(A^T B)^{1000} = A^T (BA^T)^{999} B = 5^{999} A^T B$ , and see Answer to 21.



$$27. AB^T = 4.$$

$$29. A^T B = \begin{bmatrix} -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ 1 & -1 & 0 & -3 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 6 & 6 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$31. (A^T B)^2 = 4A^4 B, \text{ and see Answer to 29.}$$

$$33. (A^T B)^{1000} = 4^{999} A^4 B, \text{ and see Answer to 29.}$$

$$35. E_1 C = \begin{bmatrix} 3 & 6 & 9 \\ -8 & -10 & -12 \end{bmatrix} E_2 C = \begin{bmatrix} 21 & 27 & 33 \\ 4 & 5 & 6 \end{bmatrix}$$

$$E_1^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix} E_2^n = \begin{bmatrix} 1 & 5n \\ 1 & 0 \end{bmatrix}$$

$$37. \begin{bmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix} \cdot 41. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot 43. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

## §6. Systems of Linear Equations

$$1. \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \text{ is invertible; } \det(A) = -4$$

$$3. \text{ The matrix is invertible if and only if } abc \neq 0; \det(A) = abc.$$

$$5. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \text{ is invertible; } \det(A) = 2$$

$$7. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 13/7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ is invertible; } \det(A) = 13$$

$$9. 0 = 1 \text{ (no solutions)}$$

$$11. x = -z - 3b + 10, y = -z + 2b - 6.$$

$$13. \text{ If } t \neq 6 + 2u, \text{ then there are no solutions. Otherwise, } x = -2y + u + 3, y \text{ arbitrary.}$$

$$15. \text{ If } t = 1, \text{ then } x = 1 - y, y \text{ arbitrary.}$$

$$\text{If } t = -1, \text{ there are no solutions.}$$

If  $t \neq \pm 1$ , then  $x = (t^2 + t + 1)/(t + 1)$ ,  $y = -1/(t + 1)$ .

17.  $y = 5b + x - 16$ ,  $z = -3b - x + 10$

19. No. The half-sum of solutions is a solution.

21.  $A^{-1} = \begin{bmatrix} 7/25 & 4/25 & -1/25 \\ 19/25 & -7/25 & 8/25 \\ -18/25 & 4/25 & -1/25 \end{bmatrix}$

23.  $A^{-1} = \begin{bmatrix} -3/22 & -1/22 & -41/22 & 3/11 \\ -15/22 & -5/22 & -51/22 & 4/11 \\ 5/22 & 9/22 & 61/22 & -5/11 \\ 15/22 & 5/22 & 73/22 & -4/11 \end{bmatrix}$

24.  $A = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$

27. This cannot be done. We have  $0 = A_{1,1} = L_{1,1}U_{1,1} \neq 0$  since  $A$  is invertible, hence  $U, V$  are invertible.

29.  $A = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix}$

30.  $A = \begin{bmatrix} 1 & -2 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix}$

33.  $x = -3(19 + 2d)/8$ ,  $y = (15 + 2d)/8$ ,  $z = -(3 + 2d)/8$

35.  $x = (15u + 4v)/16$ ,  $y = (11u + 4v)/16$ ,  $z = -3u/4$

37.  $x = y = 1$ ,  $z = 0$

39.  $x = y = 0$ ,  $z = 100$

## §7. Standard and Canonical Forms for Linear Programs

1. Set  $u = y + 1 \geq 0$ . Then  $f = 2x + 3y = 2x + 3u - 3$  and  $x + y = x + u - 1$ . A canonical form is

$$-f = -2x - 3u + 3 \rightarrow \min, x + u \leq 6, u, x \geq 0.$$

A standard form is

$$-f = -2x - 3u + 3 \min, x + u + v = 6, u, v, x \geq 0$$

with a slack variable  $v = 6 - x - u \geq 0$ .

2. Excluding  $y = x + 1$  and using  $y \geq 1$ , we obtain the canonical form  $-x \rightarrow \min, 2x \leq 8, x \geq 0$ .

Introducing a slack variable  $z = 8 - 2x$ , we obtain the standard form  $-x \rightarrow \min, 2x + z = 8, x \geq 0, z \geq 0$ .

3. We solve the equation for  $x_3$ :

$$x_3 = 3 - 2x_2 - 3x_4$$

and exclude  $x_3$  from the LP:

$$x_1 - 7x_2 + 3 \rightarrow \min, \quad x_1 - x_2 + 3x_4 \geq 3, \quad \text{all } x_i \geq 0.$$

A canonical form is

$$x_1 - 7x_2 + 3 \rightarrow \min, \quad -x_1 + x_2 - 3x_4 \leq -3, \quad \text{all } x_i \geq 0.$$

A standard form is

$$x_1 - 7x_2 + 3 \rightarrow \min, \quad -x_1 + x_2 - 3x_4 + x_5 = -3, \quad \text{all } x_i \geq 0$$

with a slack variable  $x_5 = x_1 - x_2 + 3x_4 - 3$ .

5. Set  $t = x + 1 \geq 0, u = y - 2 \geq 0, f = x = y + z = t + u + z + 1$  (the objective function). Then a standard and canonical form for our problem is

$$x + u + z + 1 \rightarrow \min; t, u, z \geq 0.$$

7. Using standard tricks, a canonical form is

$$-x \rightarrow \min, x \leq 3, -x \leq -2, x \geq 0.$$

A standard form is

$$-x \rightarrow \min, x + u = 3, -x + v = -2; x, u, v \geq 0$$

with two slack variables.

9. One of the given equations reads

$$-5 - x - z = 0,$$

which is inconsistent with given constraints  $x, z \geq 0$ . So we can write very short canonical and standard forms:

$$0 \rightarrow \min, 0 \leq -1; x, y, z \geq 0 \text{ and } 0 \rightarrow \min, 0 = 1; x, y, z \geq 0.$$

11. Set  $x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]^T$  and  $c = [3, -1, 1, 3, 1, -5, 1, 3, 1]$ . Using standard tricks, we obtain the canonical form

$$cx \rightarrow \min, Ax \leq b, x \geq 0$$

with

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -2 & 3 & 1 \\ 2 & -2 & -2 & 2 & 3 & -1 & -2 & 1 & 1 \\ -2 & 2 & 2 & -2 & -3 & 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 & 3 & -1 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

and  $b = [-3, -1, 2, -2, 0, 0]^T$ .

Excluding a couple of variables using the two given equations, we would get a canonical form with two variables and two constraints less. A standard form can be obtained from the canonical form by introducing a column  $u$  of slack variables:

$$cx \rightarrow \min, Ax + u = b, x \geq 0, u \geq 0.$$

§8. Pivoting Tableaux

$$1. \quad \begin{array}{cccccc} & a & b & c & d & e & 1 \\ \left[ \begin{array}{cccccc} .3 & 1.2 & .7 & 3.5 & 5.5 & -50 \\ 73 & 96 & 20253 & 890 & 279 & 4000 \\ 9.6 & 7 & 19 & 57 & 22 & -100 \\ 10 & 15 & 5 & 60 & 8 & 0 \end{array} \right] & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = C \rightarrow \min \end{array} \end{array}$$

$$3. \quad A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ -1 & 0 & 0 & 2 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$$11. \quad \begin{array}{cccc} z & a & 3 & x \\ \left[ \begin{array}{cccc} 1 & 2 & b+3 & a+1 \\ -1 & 2 & 3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = 1 \end{array} \end{array}$$

$$12. \quad \begin{array}{cccc} 1 & a & 3 & z \\ \left[ \begin{array}{cccc} 1+a & -2a & b-3a & a \\ 1 & -2 & -3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = x \end{array} \end{array}$$

$$13. \quad \begin{array}{c} 2 \\ [1/5] \end{array} = x$$

§9. Standard Row Tableaux

$$2. \quad \begin{array}{ccc} x & y & 1 \\ \left[ \begin{array}{ccc} -4 & -5 & 7 \\ -2 & -3 & 0 \end{array} \right] & \begin{array}{l} = u \\ = -P \rightarrow \min \end{array} \end{array}$$

with a slack variable  $u = 7 - 4x - 5y$

$$3. \quad \begin{array}{ccccc} x' & x'' & y' & y'' & 1 \\ \left[ \begin{array}{ccccc} -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \end{array} \right] & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = -x \rightarrow \min \end{array} \end{array}$$

with  $x = x' - x'', y = y' - y''$  and slack variables  $u_i$ .

§10. Simplex Method, Phase 2

4. False. The converse is true.
5. True
6. True

13. If the row without the last entry is nonnegative, then the tableau is optimal; else the LP is unbounded.

§11. Simplex Method, Phase 1

1. The second row ( $v$ -row) is bad, so the LP is infeasible.
2. The tableau is optimal, so  
 $\min(w) = 0$  at  $x = y = z = 0, u = 2, v = 0$ .
3. This is a feasible tableau with a bad column (the  $z$ -column).

So the LP is unbounded ( $z$  and hence  $w$  can be arbitrary large).

9. True
10. False

§12. Geometric Interpretation

1. The diamond can be given by four linear constraints  $\pm x \pm y \leq 1$ .
2. Any convex combination of convex combinations is a convex combination.
7. Both  $x = 1$  and  $x = -1$  belong to the feasible region, but  $0 = x/2 + y/2$  does not.
8.  $2tx + (1 - t^2)y \leq 1 + t^2$ , where  $t$  ranges over all rational numbers.
9. A set  $S$  is called closed if it contains the limit points of all sequences in  $S$ . Any system of linear constraints gives a closed set, but the interval  $0 < x < 1$  is not closed. Its complement is closed.
10. The rows of the identity matrix  $1_6$ .
11. One

§13. Dual Problems

1.

$$\begin{array}{rcccl}
 -x & \left[ \begin{array}{ccc} 0 & -1 & -5 \\ 1 & -1 & -6 \\ 0 & 0 & -2 \\ 7 & -3 & 0 \end{array} \right] \\
 -y & & & & \\
 -z & & & & \\
 1 & & & & \\
 & \parallel & \parallel & \downarrow & \\
 & v_1 & v_2 & \max &
 \end{array}$$

5. Let  $cx + d, cy + d$  be two feasible values, where  $x, y$  are two feasible solutions. We have to prove that

$$\alpha(cx + d) + (1 - \alpha)(cy + d)$$

is a feasible value for any  $\alpha$  such that  $0 \leq \alpha \leq 1$ . But

$$\alpha(cx + d) + (1 - \alpha)(cy + d) = c(\alpha x + (1 - \alpha)y) + d,$$

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where  $\alpha x + (1 - \alpha)y$  is a feasible solution because the feasible region is convex.

#### §14. Sensitivity Analysis and Parametric Programming

3.  $\min = 0$  at  $d = e = 0, a \geq 0$  arbitrary

#### §15. More on Duality

1. No, it is not redundant.
2. Yes, it is  $2 \cdot (\text{first equation}) + (\text{second equation})$ .
3. No, it is not redundant.
4. No, it is not redundant.
5. No, it is not redundant.
7. Yes, it is.

#### §16. Phase 1

- 1.

20	10	5		35
		5	15	20
20	10	10	15	

- 3.

30				35
90				90
11	91	9		111
		1	19	20
140	91	10	19	

§17. Phase 2

1.

	1	2	2	
0	1 175	2 25	3 (1)	200
0	1 (0)	2 100	2 200	300
	175	125	200	

§18. Job Assignment Problem

1.  $\min = 7$  at  $x_{14} = x_{25} = x_{32} = x_{43} = x_{51} = 1$ , all other  $x_{ij} = 0$ .

3.  $\min = 7$  at  $x_{12} = x_{25} = x_{34} = x_{43} = x_{51} = x_{67} = x_{76} = 1$ , all other  $x_{ij} = 0$ .

5.  $\max = 14$  at  $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$ , all other  $x_{ij} = 0$ .

7.  $\max = 30$  at  $x_{15} = x_{26} = x_{33} = x_{41} = x_{54} = x_{62} = x_{77} = 1$ , all other  $x_{ij} = 0$ .

§19. What are Matrix Games?

1.  $\max \min = -1$ .  $\min \max = 0$ . There are no saddle points.

$[1/3, 2/3, 0]^T$  gives at least  $-2/3$  for the row player.

$[1/2, 0, 0, 0, 1/2]$  gives at least  $1/2$  for the column player.

So  $-2/3 \leq \text{the value of the game} \leq -1/2$ .

3.  $\max \min = -1$ .  $\min \max = 2$ . There are no saddle points.

$(\text{second row} + 2 \cdot \text{third row})/3 \geq -2/3$ .

$(\text{third column} + \text{sixth column})/2 \leq 1$ .

So  $-2/3 \leq \text{the value of the game} \leq 1$ .

5. We compute the max in each column (marked by \*) and min in each row (marked by ■).

$$\begin{array}{r}
 \begin{array}{cccccccccc}
 & 4 & 2 & 3 & 5 & 4 & 3^\blacksquare & 7 & 6 & 3^\blacksquare \\
 -4 & \left[ \begin{array}{cccccccccc}
 4^* & -4^\blacksquare & 3^* & 0 & 0 & 0 & -1 & 1 & -2 \\
 -1 & 0 & 2 & 1 & -2^\blacksquare & -2^\blacksquare & 1 & 0 & -2^\blacksquare \\
 -4 & -4^\blacksquare & 0 & -2 & -2 & 1 & -1 & 1 & 6^* & 2 \\
 0^* & 1 & 2^* & 2 & 5^* & 3 & 3^* & 7^* & 2 & 0^\blacksquare \\
 -9 & -4 & -9^\blacksquare & -8 & 0 & 4^* & 2 & 2 & 0 & 3^*
 \end{array} \right]
 \end{array}
 \end{array}$$

Thus,  $\max \min = 0$ .  $\min \max = 3$ . There are no saddle points.

## §20. Matrix Games and Linear Programming

1. The optimal strategy for the row player is  $[2/3, 1/3, 0]^T$ .  
The optimal strategy for the column player is  $[1/2, 1/2, 0]$ .  
The value of the game is 2.
3. The optimal strategy for the row player is  $[0.2, 0, 0.8]^T$ .  
An optimal strategy for the column player is  $[0, 0.5, 0.5, 0, 0, 0]$ .  
The value of the game is 1.
5. The optimal strategy for the row player is  $[1/3, 2/3, 0]^T$ .  
The optimal strategy for the column player is  $[2/3, 0, 0, 1/3]$ .  
The value of the game is  $-2/3$ .
7. The optimal strategy for the row player:  $[1/8, 0, 7/8, 0]^T$ .  
The optimal strategy for the column player:  $[0, 1/4, 0, 0, 0, 3/4]$ .  
The value of the game is  $-0.25$ .

## §21. Other Methods

1. The first row and column are dominated. The optimal strategy for the row player is  $[0, 0.5, 0.5]^T$ . The optimal strategy for the column player is  $[0, 0.25, 0.75]$ . The value of the game is 2.5.
3. The optimal strategy for the row player is  $[0, 0.4, 0, 0.6]^T$ .  
The optimal strategy for the column player is  $[0, 0.4, 0.6]$ . The value of the game is 2.8.
5. The optimal strategy for the row player is  $[1/3, 1/3, 1/3]^T$ .  
The optimal strategy for the column player is  $[0, 0, 2/7, 3/7, 2/7, 0]$ .  
The value of the game is 0.
7. The value of the game is 0 because the game is symmetric.
9. The first two columns and the first row go by domination.  
The value of the game is  $11/7$ .
11. 0 at a saddle point.
13. 0 at a saddle point.

## §22. What is Linear Approximation?

1. The mean is  $-2/5 = -0.4$ . The median is 1. The midrange is  $-5/2 = -2.5$ .
3. The mean is  $5/9$ . The median is 0. The midrange is  $1/2 = 0.5$ .
- 5(b). 1, 2, 9.
- 5(d). Exercise 1.
- 5(f). Exercise 3.



§23. Linear Approximation and Linear Programming

1.  $\min = 0$  at  $a = -15, b = 50$  for  $w = a + bh$

and

$\min \approx 19$  at  $a \approx 25.23$  for  $w = ah^2$

2.  $x + y + 0.3 = 0$

3.  $a = 0.9, b \approx -0.23$

§24. More Examples

1. The model is  $w = ah + b$ , or  $w - x_2 = a(h - 1988) + b'$  with  $b = x_2 + -1988a$  and  $x_2 = 37753/45 \approx 838.96$ . Predicted production  $P$  in 1993 is  $x_2 + 5a + b'$ .

For  $p = 1$ , we have  $a \approx 16.54, b' \approx 31, P \approx 953$ .

For  $p = 2$ , we have  $a \approx 0, b' \approx 32, P \approx 871$ .

For  $p = \infty$ , we have  $a \approx 17.59, b' \approx 32, x_5 \approx 959$ .

So in this example  $l^\infty$ -prediction is the best.

3.  $a = \$4875, b = \$1500$

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