Concave and Convex Compositions

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Dedicated to my good friends, Mourad Ismail and Dennis Stanton

Abstract

In a previous work, concave compositions were defined with the restriction that there be essentially the same number of parts on each side of the minimal part. In this work, we drop that restriction, and, in addition, we consider convex compositions as well. We shall link the related generating functions to combinations of classical, false, or mock theta functions and other Appell-Lerch sums.

Keywords: Compositions; mock theta functions; false theta functions.

1 Introduction

A *concave composition* of n is a sum of integers of the form

$$\sum_{i=1}^{R} a_i + c + \sum_{i=1}^{S} b_i = n \tag{1.1}$$

where $a_1 \geq a_2 \geq \cdots \geq a_R > c < b_1 \leq b_2 \leq \cdots \leq b_s$. c is called the central part and is ≥ 0. If all the "$\leq$" and "$\geq$" are replaced by "$<$" and "$>$", we refer to a strictly concave composition.

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We denote the number of concave compositions of n by $V(n)$ and the number of strictly concave compositions of n by $V_d(n)$. For example, $V(3) = 13$; the relevant concave compositions are $1\underline{0}2$, $1\underline{0}11$, $2\underline{0}1$, $11\underline{0}1$, $3\underline{0}\_$, $21\underline{0}\_$, $111\underline{0}\_$, $\_\underline{0}3$, $\_\underline{0}12$,$\_\underline{0}111$, $2\underline{1}\_$, $\_\underline{1}2$, $\_\underline{3}\_$ (where the underlined entry is the central part.)

On the other hand, $V_d(3) = 9$ because from the original thirteen concave compositions four must be excluded: $1\underline{0}11$, $11\underline{0}1$, $111\underline{0}\_$, $\_\underline{0}111$.

It should be noted that the concave compositions studied in this paper differ from those studied in [6] in that in the latter case $|R - S| \leq 1$.

Convex and *strictly convex compositions* are defined is the same way with all the inequality signs reversed. Thus there are twelve convex compositions of 5: $\_\underline{5}\_$, $\_\underline{4}1$, $1\underline{4}\_$, $1\underline{3}1$, $2\underline{3}\_$, $\_\underline{3}2$, $11\underline{3}\_$, $\_\underline{3}11$, $111\underline{2}\_$, $11\underline{2}1$, $1\underline{2}11$, $\_\underline{2}111$. Of these, six are strictly convex: $\_5\_$, $\_\underline{4}1$, $1\underline{4}\_$, $1\underline{3}1$, $2\underline{3}\_$, $\_\ \underline{3}2$. The related enumerating functions are $X(n)$ and $X_d(n)$. So $X(5) = 12$ and $X_d(5) = 6$.

Also, we can restrict the "strictly" convention to just one side of the central part. For definiteness we put the strict inequalities to the left of c. The related enumerating functions are $V_m(n)$ and $X_m(n)$. We refer to these as semi-strictly concave and semi-strictly convex compositions. $X_m(5) = 9$, the relevant compositions being $\_\underline{5}\_$, $\_\underline{4}1$, $1\underline{4}\_$, $1\underline{3}1$, $\_311$, $23\_$, $\_32$, $\_\underline{2}111$, $1\underline{2}11$. $V_m(3) = 11$, the relevant compositions being $3\underline{0}\_$, $\_\underline{0}3$, $\_\underline{0}12$, $21\underline{0}\_$, $\_\underline{0}111$, $1\underline{0}11$, $1\underline{0}2$, $2\underline{0}1$, $2\underline{1}\_$, $\_\underline{1}2$, $\_\ 3\ \_$.

We shall use lower case v and x for the related generating functions. Standard techniques [5; Ch. 1] allow us to observe:

$$v(q) := \sum_{n=0}^{\infty} V(n)q^n = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_\infty^2},$$

$$x(q) := \sum_{n=0}^{\infty} X(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q;q)_n^2},$$

$$v_d(q) := \sum_{n=0}^{\infty} V_d(n)q^n = \sum_{n=0}^{\infty} q^n \left(-q^{n+1};q\right)_\infty^2,$$

$$x_d(q) := \sum_{n=0}^{\infty} X_d(n)q^n = \sum_{n=0}^{\infty} q^{n+1}(-q;q)_n^2,$$

$$v_m(q) := \sum_{n=0}^{\infty} V_m(n)q^n = \sum_{n=0}^{\infty} \frac{q^n(-q^{n+1};q)_\infty}{(q^{n+1};q)_\infty},$$

$$x_m(q) := \sum_{n=0}^{\infty} X_m(n) q^n = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n+1}}{(q;q)_n}$$

(see (1.8) and (1.9) for definition of $(A;q)_n$).

Our immediate object is to relate these generating functions to classical, false, and mock theta functions. Additionally, $v(q)$ requires other Appell-Lerch type series.

Theorem 1.

$$\begin{aligned} v(q) = \frac{q^{-1}}{(q;q)_\infty^3} \Bigg\{ & \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+q^n) q^{n(3n+1)/2}}{(1-q^n)^2} \\ & - \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \qquad (1.2) \\ & + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}}{1-q^n} \Bigg\}, \end{aligned}$$

$$x(q) = \frac{q}{(q;q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}}, \tag{1.3}$$

$$v_d(q) = -\sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} + 2(-q;q)_\infty^2 \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}), \tag{1.4}$$

$$x_d(q) = \psi(-q) + 2(-q;q)_\infty^2 \alpha(-q), \tag{1.5}$$

$$v_m(q) = \frac{2}{(1+q)\phi(-q)} - \frac{1}{1+q}, \tag{1.6}$$

$$x_m(q) = \frac{q}{(1+q)\phi(-q)}, \tag{1.7}$$

where

$$(A;q)_\infty = \prod_{j=0}^{\infty} \left(1 - Aq^j\right), \tag{1.8}$$

$$(A;q)_n = \frac{(A;q)_\infty}{(Aq^n;q)_\infty}, \quad \left(= \begin{array}{c} (1-A)(1-Aq)\cdots(Aq^{n-1}) \\ \textit{where } n \textit{ is a positive integer} \end{array} \right) \tag{1.9}$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \tag{1.10}$$

($\psi(q)$ one of Ramanujan's third order mock theta functions [14; p. 62])

$$\alpha(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, \tag{1.11}$$

$$= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2}, \tag{1.12}$$

($\alpha(q)$ as given in (1.11) *and* (1.12) *occurs in Ramanujan's "Lost" Notebook [2; eqs. (3.25) and (3.27)] and has been termed a second order mock theta function by McIntosh [13], see also [12]).*

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (q^2;q^2)_\infty(-q;q^2)_\infty^2 = \frac{(-q;-q)_\infty}{(q;-q)_\infty}. \tag{1.13}$$

(by Jacobi's triple product [5; Cor. 29])

Part I
Unrestricted Parts

2 Proof of (1.2)

In light of (1.9), which defines the q-Pochhammer symbol for all integers n, we see that

$$\sum_{n=-\infty}^{\infty} \frac{(ab)^n q^{n^2}}{(aq;q)_n(bq;q)_n}$$
$$= \sum_{n=0}^{\infty} \frac{(ab)^n q^{n^2}}{(aq;q)_n(bq;q)_n} + (1-a)(1-b)\sum_{n=0}^{\infty} q^{n+1}(aq;q)_n(bq;q)_n. \tag{2.1}$$

Consequently

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{n+1}(aq;q)_n(bq;q)_n \\
&\qquad = \frac{1}{(1-a)(1-b)}\Bigg\{\sum_{n=-\infty}^{\infty}\frac{(ab)^n q^{n^2}}{(aq;q)_n(bq;q)_n} \\
&\qquad\qquad\qquad\qquad - \sum_{n=0}^{\infty}\frac{(ab)^n q^{n^2}}{(aq;q)_n(bq;q)_n}\Bigg\} \\
&\quad := \frac{1}{(1-a)(1-b)}\left\{S_1(a,b;q) - S_2(a,b;q)\right\}
\end{aligned} \tag{2.2}$$

Now Choi [9; p. 359] has proved that

$$S_1(a,b;q) = \frac{(q/a;q)_\infty}{(bq;q)_\infty(q;q)_\infty}\left(1 + (1-a)\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}} b^n}{1-aq^n}\right) \tag{2.3}$$

and by the q-analog of Whipple's theorem [10; p.242, eq. (III.18)]

$$\begin{aligned}
S_2(a,b;q) = \frac{1}{(abq;q)_\infty}\Bigg\{1 + (1-a)(1-b)\sum_{n=1}^{\infty}\frac{(abq;q)_{n-1}(1-abq^{2n})}{(q;q)_n} \\
\times \frac{(-1)^n q^{n(3n+1)/2}}{(1-aq^n)(1-bq^n)}\Bigg\}
\end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{n+1}(q;q)_n^2 \\
&\quad = \lim_{b\to 1}\frac{1}{1-b}\left(\lim_{a\to 1}\frac{1}{1-a}\left(S_1(a,b;q) - S_2(a,b;q)\right)\right) \\
&\quad = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(1+q^n)q^{n(3n+1)/2}}{(1-q^n)^2} \\
&\qquad + \lim_{b\to 1}\frac{1}{1-b}\left(\lim_{a\to 1}\frac{1}{1-a}\left\{-\frac{1}{(abq;q)_\infty} + \frac{(q/a;q)_\infty}{(bq;q)_\infty(q;q)_\infty}\right\}\right)
\end{aligned}$$

$$+\lim_{b\to 1}\frac{1}{1-b}\left(\frac{1}{(bq;q)_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n b^n q^{n(n+1)/2}}{1-q^n}\right)$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(1+q^n)q^{n(3n+1)/2}}{(1-q^n)^2}+S_3(q)+S_4(q) \tag{2.5}$$

Now by L'Hopital's rule,

$$\begin{aligned}
S_3(q) &= \lim_{b\to 1}\frac{1}{1-b}\left\{-\frac{1}{(bq;q)_\infty}\left(-\sum_{n=1}^{\infty}\frac{bq^n}{1-bq^n}+\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right)\right\}\\
&= \lim_{b\to 1}\frac{-1}{(bq;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)(1-bq^n)}\\
&= \frac{-1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2},
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
S_4(q) &= \lim_{b\to 1}\frac{1}{1-b}\frac{1}{(bq;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^n b^n q^{n(n+1)/2}}{1-q^n}+\sum_{n=1}^{\infty}\frac{(-1)^{-n}b^{-n}q^{n(n-1)/2}}{1-q^{-n}}\right)\\
&= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1-q^n}\lim_{b\to 1}\frac{b^n-b^{-n}}{1-b}\\
&= \frac{2}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{n(n+1)/2}}{1-q^n},
\end{aligned} \tag{2.7}$$

Substituting (2.7) and (2.6) into (2.5) and dividing by $q(q;q)_\infty^2$, we deduce (1.2). □

3 Proof of (1.3)

Once we examine the natural form of $x(q)$ the proof of (1.3) will rely on a well known specialization of the Heine transformation. Namely

$$x(q)=\sum_{n=0}^{\infty}\frac{q^{n+1}}{(q;q)_n^2}$$

$$= \frac{q}{(q;q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

This transformation (at least ones very close) can be found in [8; p. 683] and [1]. It easily follows from the Heine transformation [5; p. 19, Cor. 2.3] with $a \to 0, b \to 0, c = t = q$.

4 Proof of (1.4)

Here we begin by noting that

$$\begin{aligned} v_d(q) &= \sum_{n=0}^{\infty} q^n (-q^{n+1};q)_\infty^2 \\ &= (-q;q)_\infty^2 \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n^2} \\ &= (-q;q)_\infty^2 \left\{ -\frac{\sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2}}{(-q;q^2)_\infty^2} + 2 \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{(-q;q)_m} \right\} \\ &\qquad \text{(by [3; p. 144, eq. (3.6)] with } a = b = 1 \text{)} \\ &= -\sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2} + 2(-q;q)_\infty^2 \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1 - q^{2n+1}), \\ &\qquad \text{(by [5; p. 29, Ex. 10] with } x = 1.\text{)} \end{aligned}$$

which is (1.4). □

5 Proof of (1.5)

As noted in the introduction,

$$x_d(q) = \sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1}$$

Hence

$$
\begin{aligned}
q^{-1}x_d(q) &= \sum_{n=0}^{\infty}(-q;q)_n^2 q^n \\
&= \sum_{n=0}^{\infty}\frac{(-q;q)_n^2}{(q;q)_n} q^n (q^2;q^2)_n \\
&= (q^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{(-q;q)_n q^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{q^{m(2m+2)}}{(q^2;q^2)_m} \\
&\hphantom{=}\qquad\qquad\qquad\qquad \text{(by [5; p. 19, eq. (2.2.5)])} \\
&= (q^2;q^2)_\infty \sum_{m=0}^{\infty}\frac{q^{2n}}{(q^2;q^2)_m}\frac{(-q^{2m+2};q)_\infty}{(q^{2m+1};q)_\infty} \\
&= (-q;q)_\infty^2 \sum_{m=0}^{\infty}\frac{(q;q^2)_m q^{2m}}{(-q;q^2)_{m+1}(-q^2;q^2)_m}
\end{aligned}
$$

We now apply Theorem 1 from [3; p. 141] replacing q by q^2 and setting $A = 0, B = q, a = 1, b = q$.

Thus

$$
\begin{aligned}
q^{-1}x_d(q) &= \frac{(-q;q)_\infty^2}{1+q}\left\{\frac{-(q;q^2)_\infty}{(-q^3;q^2)_\infty(-q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2+2m}}{(-q;q^2)_{m+1}}\right. \\
&\qquad\qquad \left. + (1+q)\sum_{m=0}^{\infty}\frac{(-1;q^2)_{m+1}(-q)^m}{(-q;q^2)_{m+1}}\right\} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+1} q^{m^2+2m}}{(-q;q^2)_{m+1}} + (-q;q)_\infty^2\sum_{m=0}^{\infty}\frac{(-1;q^2)_{m+1}(-q)^m}{(-q;q^2)_{m+1}} \\
&= \psi(-q) + 2(-q;q)_\infty^2\alpha(-q),
\end{aligned}
$$

as desired. $\square$

6 Proofs of (1.6) and (1.7)

The mixed cases turn out to be straightforward. As in the previous sections,

$$\begin{aligned} v_m(q) &= \sum_{n=0}^{\infty} \frac{q^n(-q^{n+1};q)_\infty}{(q^{n+1};q)} \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(q;q)_n q^n}{(-q;q)_n} \\ &= \frac{1}{(1+q)\phi(-q)}(2-\phi(-q)) \\ &= \frac{2}{(1+q)(\phi(-q))} - \frac{1}{1+q}, \end{aligned}$$

where the penultimate line follows from the fact that

$$\sum_{j=0}^{N} \frac{(q;q)_j q^j}{(-q;q)_j} = \frac{2}{1+q} - \frac{(q;q)_{N+1}}{(1+q)(-q;q)_N} \tag{6.1}$$

which is easily proved by mathematical induction. Equation (6.1) is a special case of [3; eq. (2.15)].

As for $x_m(q)$, we see that

$$\begin{aligned} x_m(q) &= \sum_{n=0}^{\infty} \frac{q^{n+1}(-q)_n}{(q)_n} \\ &= \frac{q(-q^2;q)_\infty}{(q)_\infty} \\ &= \frac{q}{(1+q)\phi(-q)}, \qquad \text{by [5; p. 17, Th. 2.1].} \end{aligned}$$

Part II
Odd Parts

7 Statement of Theorem 2

As was mentioned in the introduction, this part will be devoted to concave (as given in (1.1)) and convex compositions with the added constraint that each of the a_i, b_i and c are positive odd integers. We have again six partition functions and their corresponding generating functions:

$$
\begin{aligned}
vo(q) &:= \sum_{n=0}^{\infty} VO(n)q^n = \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(q^{2n+1};q^2)_\infty^2},\\
xo(q) &:= \sum_{n=0}^{\infty} XO(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q^2)_n^2},\\
vo_d(q) &:= \sum_{n=0}^{\infty} VO_d(n)q^n = \sum_{n=1}^{\infty} q^{2n-1}(-q^{2n+1};q^2)_\infty^2\\
xo_d(q) &:= \sum_{n=0}^{\infty} XO_d(n)q^n = \sum_{n=0}^{\infty} q^{2n+1}(-q;q^2)_n^2,\\
vo_m(q) &:= \sum_{n=0}^{\infty} VO_m(n)q^n = \sum_{n=1}^{\infty} \frac{q^{2n-1}(-q^{2n+1};q^2)_\infty}{(q^{2n+1};q^2)_\infty},\\
xo_m(q) &:= \sum_{n=0}^{\infty} XO_m(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2n+1}(-q;q^2)_n}{(q;q^2)_n}.
\end{aligned}
$$

The fact that the parts are odd introduces a natural pairing in the statement of the analog of Theorem 1.

Theorem 2.

$$
\begin{aligned}
vo(q) &= \frac{-q^{-2}}{(q;q^2)_\infty^2}(xo_d(-q)+q)\\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{1-q^{2n+1}} - (-q;q)_\infty^2 \omega(q) \qquad (7.1)
\end{aligned}
$$

(a result due to Hikami [11; eq. (16)])

$$
\begin{aligned}
xo(q) &= q - \frac{q^2}{(q;q^2)_\infty^2} vo_d(-q)\\
&= q + \frac{q^2}{(q;q^2)_\infty^2} \sum_{m=0}^{\infty} (-1)^m q^{m^2+m} - \sum_{m=0}^{\infty} (-1)^m q^{3m^2+2m+2} \qquad (7.2)\\
xo_m(q) &= q - \frac{q^2(-q;q^2)_\infty}{(q;q^2)_\infty} vo_m(-q)\\
&= \frac{q-q^2}{1+q^2} + \frac{q^2(-q;q^2)_\infty}{(1+q^2)(q;q^2)_\infty}, \qquad (7.3)
\end{aligned}
$$

where

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2},$$

a third order mock theta function [14; p. 62].

8 Proofs of (7.1) and (7.2)

The first assertion in (7.1) is a simple exercise in comparing the infinite series on each side term by term. The second assertion in (7.1) is due to Hikami as noted in the statement of the theorem.

The first assertion in (7.2) is also a direct exercise in comparing the infinite series on each side term by term.

To prove the second assertion in (7.2), we specialize equation $(3.6)_{\mathrm{R}}$ [3; p. 144] by replacing q by q^2, then setting $a = b = q^{-1}$ and finally multiplying by q. This yields

$$xo(q) = \frac{q^2}{(q;q^2)_\infty^2} \sum_{m=0}^{\infty} (-1)^m q^{m^2+m} + (1-q) \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m+1}}{(q;q^2)_m}. \tag{8.1}$$

To conclude, we need only show that

$$(1-q) \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m+1}}{(q;q^2)_m} = q - \sum_{m=0}^{\infty} (-1)^m q^{3m^2+2m+2}(1+q^{2m+1}). \tag{8.2}$$

From Ramanujan's Lost Notebook [7; p. 237, entry 9.5.1] with $a = -1$, we know that

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m}}{(q;q^2)_{m+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n^2+2n}(1+q^{2n+1}). \tag{8.3}$$

Furthermore,

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m+1}}{(q;q^2)_m} - \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m+1}}{(q;q^2)_{m+1}} &= \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m+1}(1-q^{2m+1}-1)}{(q;q^2)_{m+1}} \\
&= \sum_{m-1}^{\infty} \frac{(-1)^m q^{m^2+m}}{(q;q^2)_m} \\
&= -1 + \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m}}{(q;q^2)_m},
\end{aligned}$$

so

$$
\begin{aligned}
(1-q)\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2+m+1}}{(q;q^2)_m} &= q - q^2\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2+m}}{(q;q^2)_{m+1}}\\
&= q - \sum_{m=0}^{\infty}(-1)^m q^{3m^2+2m+2}(1+q^{2m+1}),
\end{aligned}
$$

(by (8.3))

as desired. Thus (7.2) is proved.

9 Proof of (7.3)

As with (7.1) and (7.2), the first assertion in (7.3) is confirmed by comparing the infinite series on each side term by term. For the second assertion, we specialize identity (2.15) of [3; p. 140] with q replaced by $q^2, N = 0$, then $a = -q, b = q$ and both sides multiplied by q. Thus

$$
xo_m(q) = \frac{q^2(-q;q^2)_\infty}{(1+q^2)(q;q^2)_\infty} + \frac{q(1-q)}{1+q^2},
$$

which is (7.2).

Part III
Mixed Parity

10 Statement of Theorem 3

We return to the definition of concave (as given in (1.1)) and convex compositions now requiring that the a_i and c are odd while the b_i are even.

It should be remarked that this case is really the fully general mixed parity case owing to the symmetric roles played by the a_i's and b_i's. Requiring c to be even would only introduce a slight variation in the generating functions.

$$
\begin{aligned}
vp(q) = \sum_{n=0}^{\infty} VP(n)q^n &= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{2n+2};q^2)_\infty (q^{2n+3};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{2n+2};q)_\infty}, \\
xp(q) = \sum_{n=0}^{\infty} XP(n)q^n &= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q^2)_n (q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n}}, \\
vp_d(q) = \sum_{n=0}^{\infty} VP_d(n)q^n &= \sum_{n=0}^{\infty} q^{2n+1}(-q^{2n+3};q^2)_\infty(-q^{2n+2};q^2)_\infty \\
&= \sum_{n=0}^{\infty} q^{2n+1}(-q^{2n+2};q)_\infty, \\
xp_d(q) = \sum_{n=0}^{\infty} XP_d(n)q^n &= \sum_{n=0}^{\infty} q^{2n+1}(-q;q^2)_n(-q^2;q^2)_n \\
&= \sum_{n=0}^{\infty} q^{2n+1}(-q;q)_{2n}, \\
vp1_m(q) = \sum_{n=0}^{\infty} VP1_m(n)q^n &= \sum_{n=0}^{\infty} q^{2n+1}\frac{(-q^{2n+2};q^2)_\infty}{(q^{2n+3};q^2)_\infty}, \\
vp2_m(q) = \sum_{n=0}^{\infty} VP2_m(n)q^n &= \sum_{n=0}^{\infty} q^{2n+1}\frac{(-q^{2n+3};q^2)_\infty}{(q^{2n+2};q^2)_\infty}, \\
xp1_m(q) = \sum_{n=0}^{\infty} XP1_m(n)q^n &= \sum_{n=0}^{\infty} q^{2n+1}\frac{(-q^2;q^2)_n}{(q;q^2)_n}, \\
xp2_m(q) = \sum_{n=0}^{\infty} XP2_m(n)q^n &= \sum_{n=0}^{\infty} q^{2n+1}\frac{(-q;q^2)_n}{(q^2;q^2)_n}.
\end{aligned}
$$

The parity mixture produces a noticeable simplification in the analytic complexity of the generating functions as will by clear in the next result.

Theorem 3.
$$vp(q) = \frac{1}{2q}\left(\frac{R(q)}{(q;q)_\infty} - 1\right), \tag{10.1}$$
where $R(q)$ comes from Ramanujan's Lost Notebook [4; p. 157]
$$R(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(-q;q)_n}$$

$$xp(q) = \frac{q}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n^2} = \frac{q}{2}\left(\frac{1}{(q;q)_\infty} + \frac{1}{(-q;q)_\infty}\right) \tag{10.2}$$
$$vp_d(q) = \frac{1}{2}\left((-q;q)_\infty f(q) - 1\right), \tag{10.3}$$

where $f(q)$ is a third order mock theta function [14; p. 62].

$$xp_d(q) = \frac{1}{2}\left((-q;q)_\infty - \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1})\right) \tag{10.4}$$
$$vp1_m(q) = \frac{2q(-q^2;q^2)_\infty}{(q^3;q^2)_\infty(1+q^3)} - \frac{q}{1+q^3} \tag{10.5}$$
$$vp2_m(q) = \frac{1}{1+q}\left(\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} - 1\right) \tag{10.6}$$
$$xp1_m(q) = \frac{q}{1+q^3}\left(\frac{q(-q^2;q^2)_\infty}{(q;q^2)_\infty} + 1 - q\right) \tag{10.7}$$
$$xp2_m(q) = \frac{q(-q^3;q^2)_\infty}{(q^2;q^2)_\infty} \tag{10.8}$$

11 Proof of Theorem 3

The assertions in Theorem 3 are sufficiently closely tied to results in the literature that it seemed best to gather the proofs of the eight identities in one section. In several of the proofs of these eight identities we will require the $N = 0$ case of equation (2.15) of [3; p. 140]:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^n}{(b;q)_n} = \frac{q}{b-aq}\left(\frac{(a;q)_\infty}{(b;q)_\infty} - 1 + \frac{b}{q}\right) \tag{11.1}$$

For (10.1),

$$\begin{aligned}
vp(q) &= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{2n+2};q)_\infty} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{2n+1}(q;q)_{2n+1} \\
&= \frac{1}{2(q;q)_\infty} \sum_{n=0}^{\infty} q^{n}(q;q)_{n}(1-(-1)^n) \\
&= \frac{1}{2(q;q)_\infty} \left(q^{-1}(1-(q;q)_\infty) + q^{-1}(R(q)-1)\right)
\end{aligned}$$

(by (11.1), b=0,a=q and [4; p. 157, eq. $(1.6)_{\mathrm{R}}$])

$$= \frac{1}{2q}\left(\frac{R(q)}{(q;q)_\infty} - 1\right),$$

as desired.

For (10.2),

$$\begin{aligned}
xp(q) &= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n}} \\
&= \frac{q}{2}\sum_{n=0}^{\infty} \frac{q^{n}}{(q;q)_{n}}(1+(-1)^n) \\
&= \frac{q}{2}\left(\frac{1}{(q;q)_\infty} + \frac{1}{(-q;q)_\infty}\right)
\end{aligned}$$

(by [5; p. 19, eq. (2.2.5)])

$$= \frac{q}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n^2}$$

(by [5; p. 23, eq. (2.2.12)])

Next for (10.3),

$$
\begin{aligned}
vp_d(q) &= \sum_{n=0}^{\infty} q^{2n+1}(-q^{2n+2};q)_\infty \\
&= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(-q;q)_{2n+1}} \\
&= \frac{(-q;q)_\infty}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n}(1-(-1)^n) \\
&= \frac{(-q;q)_\infty}{2}\left(\frac{-1}{(-q;q)_\infty}+2\right) - \frac{(-q;q)_\infty}{2}(2-f(q)) \\
&\qquad\qquad\qquad\qquad \text{(by (11.1) and [11; p. 24, eq. (9)])} \\
&= \frac{(-q;q)_\infty}{2}\left(f(q)-\frac{1}{(-q;q)_\infty}\right) \\
&= \frac{1}{2}(-q;q)_\infty f(q) - \frac{1}{2},
\end{aligned}
$$

as desired.

As for (10.4),

$$
\begin{aligned}
xp_d(q) &= \sum_{n=0}^{\infty} q^{2n+1}(-q;q)_{2n} \\
&= \frac{q}{2}\sum_{n=0}^{\infty} q^n(-q;q)_n(1+(-1)^n) \\
&= \frac{1}{2}\left((-q;q)_\infty - 1\right) + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{n+1}(-q;q)_n \\
&\qquad\qquad\qquad\qquad \text{(by (11.1))} \\
&= \frac{1}{2}\left((-q;q)_\infty - \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1})\right) \\
&\qquad\qquad\qquad\qquad \text{(by [5; p. 29, } x=-1])
\end{aligned}
$$

,

and this completes (10.4).

As for (10.5)- (10.8), inspection reveals that each of these follows from a direct specialization of (11.1). We shall leave the details to the reader.

12 Conclusion

One surprising note is the mixture of second order and third order mock theta functions together in (1.5) and (7.1). It is surprising that $v(q)$ has the most intricate representation of all 20 generating functions. It would be nice if there were a simpler representation. Obviously some of the identities are sufficiently succinct that one might hope for bijective proofs.

Finally there are numerous conjectures possible concerning congruences modulo 2 and modulo 4. For example, from Theorem 1 it is immediate that $V_d(n)$ is odd precisely when n is triangular.

Problem 1 Prove combinatorially that $V_d(n)$ is odd if and only if n is triangular.

Problem 2 Prove combinatorially that

$$x(q)/qv_d(q)/(-q;q)_\infty^2 \pmod 4.$$

The analytic proof of this last congruence can be done easily term-by-term and is eventually seen to be a consequence of the fact that if $f(z)$ is a power series in z with integer coefficients, then

$$f(z)^2 \equiv f(-z)^2 \pmod 4$$

because

$$f(z)^2 - f(-z)^2 = \left(f(z) - f(-z)\right)\left(f(z) + f(-z)\right),$$

and the right-hand side is the product of twice the odd part of f times twice the even part of f.

More challenging is the following refinement of Problem 1:

Problem 3 Prove combinatorially that $V_d(n)$ is divisible by 4 except when n is a triangular number.

Analytically one can deduce this assertion from (1.4) by invoking Jacobi's identity for the cube of $(q;q)_\infty$ [7; p. 19, eq. (1.3.3)].

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