Proof (cf. [9, Appendix to Chap. VII]). We may assume that the given isomorphism $\varphi:\mathfrak{M}(xE-A)\to M(A)$ is in the following form. We represent an arbitrary vector $\alpha(x) \in R[x]^{(m)}$ in the unique form $\alpha(x) = (xE - A)\beta(x) + \alpha_0$, where $\beta(x) \in R[x]^{(m)}$, $\alpha_0 \in R^{(m)}$. Then $\varphi(\alpha(x)+\mathfrak{C}(xE-A))=\alpha_0$.

For matrices of the form $xE - A$ over $R[x]$ nothing better can be said about the canonical form which would allow a simple criterion for equivalence. This follows from the results of [1, 2] mentioned in the introduction. However, some similarity of this form to the case of matrices over a field can be found.

THEOREM 4. Let $A \in R_m$. Then $xE - A$ is equivalent to some matrix $\mathcal{K}(x)$ of the form

$$\mathcal{K}(x)=(K_{ij}(x))_{m\times m}, \tag{4}$$

where

$$\Sigma \deg K_{ii}(x)=m, \tag{5}$$

and the polynomials $K_{ii}(x)$ are monic for $i = \overline{1, m}$;

$$\overline{K}_{ii}(x)\mid \overline{K}_{i+1,i+1}(x),\ i=\overline{1, m-1}, \tag{6}$$

and if $i \neq j$ then

$$K_{ij}(x)\in J(R)[x], \tag{7}$$

$$\deg K_{ij}(x)<\min\{\deg K_{ii}(x),\ \deg K_{jj}(x)\}. \tag{8}$$

Conversely, for every matrix (4) there exists a matrix $A \in R_m$ such that $(xE-A)\sim\mathcal{K}(x)$.

Proof. The existence of a matrix $\mathcal{K}(x)$ with properties (6), (7) which is equivalent to $xE - A$ follows clearly from the existence of a canonical matrix over $\overline{R}[x]$ which is equivalent to $xE - A$. According to Krull's theorem [11, Sec. 4], property (5) can be ensured by multiplying the rows of the matrix $\mathcal{K}(x)$ with suitable invertible polynomials of $R[x]$. Assume that $\mathcal{K}(x)$ has properties (5)-(7). Put $m_{ij} = \min\{\deg K_{ii}(x), \deg K_{jj}(x)\}$. Then there exists a natural number δ such that the coefficients of each polynomial $K_{ij}(x)$ for powers of x which are greater than or equal to m_{ij} belong to $J(R)^{\delta}$, i.e.,

$$\deg(K_{ij}(x)\bmod J^{\delta})<m_{ij}\quad \text{for all}\quad i\neq j. \tag{9}$$

Since there exists $n\in\mathbf{N}$ such that $J^n = 0$ it is clear that for the proof of the Theorem it suffices to find a method of constructing from $\mathcal{K}(x)$ an equivalent matrix $\mathcal{K}'(x)=(K'_{ij}(x))_{m\times m}$ with properties (5)-(7) such that

$$\deg(K'_{ij}(x)\bmod J^{\delta+1})<m_{ij}\quad \text{for all}\quad i\neq j. \tag{10}$$

To this end we proceed as follows. Divide with remainder each of the polynomials $K_{ij}(x)$ by $K_{ss}(x)$, where $s = \min\{i, j\}$: if $i < j$, then

$$K_{ij}(x)=Q_{ij}(x)K_{ii}(x)+L_{ij}(x),\ \deg L_{ij}<\deg K_{ii}; \tag{11}$$

if $i > j$, then

$$K_{ij}(x)=Q_{ij}(x)K_{jj}(x)+L_{ij}(x),\ \deg L_{ij}<\deg K_{jj}. \tag{12}$$

Note that in view of condition (9) and since the polynomials $K_{ii}(x)$ are monic, the coefficients of all polynomials $Q_{ij}(x)$ belong to J^{δ}. We introduce the following notation:

$$\mathcal{D}=\mathrm{Diag}\,(K_{11}(x),\ \ldots,\ K_{mm}(x)),\quad N=\mathcal{K}-\mathcal{D},$$

$$Q_{\ell}=\begin{pmatrix} 0 & \ldots & 0\\ Q_{21} & \ddots & \ldots\\ \ldots & & \\ Q_{m1} & \ldots\ Q_{mm-1} & 0\end{pmatrix},\quad Q_{r}=\begin{pmatrix} 0 & Q_{12} & \ldots & Q_{1m}\\ & & \ddots & \\ & & \ldots & Q_{m-1m}\\ 0 & & \ldots & 0\end{pmatrix}.$$

Then the matrices $E - Q_{\ell}$ and $E - Q_r$ are invertible and for the matrices $\mathcal{K}_1=(E-Q_{\ell})\mathcal{K}(E-Q_r)$ we have

$$\mathcal{K}_1\equiv\mathcal{D}+N-Q_{\ell}\mathcal{D}-\mathcal{D}Q_r\ (\mathrm{mod}\ J^{\delta+1}).$$

Thus, $\mathcal{K}_1\equiv\mathcal{D}+L\,(\mathrm{mod}\ J^{\delta+1})$, where

$$L=\begin{pmatrix} 0 & L_{12}(x) & \ldots & L_{1m}(x)\\ L_{21}(x) & 0 & \ldots & \ldots\\ \ldots & & & L_{m-1m}(x)\\ L_{m1}(x) & \ldots & L_{mm-1}(x) & 0\end{pmatrix},$$