Bibliography

[B1] Berkovitz, L. D., *Convexity and Optimization in R^n.* J. Wiley, New York, 2002.

[B2] Biegler, L. T., et al., *Large-Scale Optimization with Applications.* Springer-Verlag, New York, 1997.

[BL] Birge, J. R. and F. Louveaux, *Introduction to Stochastic Programming.* Springer Series in Operations Research. Springer-Verlag, New York, 1997.

[BV] Byrne C. and L. N. Vaserstein, An improved algorithm for finding saddlepoints of two-person zero-sum games, *Game Theory,* 20:2 (1991), 149–159.

[C1] Carlsson, C., *Fuzzy reasoning in decision making and optimization.* Physica-Verlag, Heidelberg/New York, 2002.

[C2] Chazelle, B., *The Discrepancy Method. Randomness and Complexity.* Cambridge University Press, Cambridge, U.K., 2000.

[CGT] Conn, A. R., N. I. M. Gould, and P. L. Toint, *Trust-Region methods.* SIAM, Philadelphia, 2000.

[CP1] Coope, I. D. and C. J. Price, On the convergence of grid-based methods for unconstrained optimization. *SIAM J. Optim.* 11 (2000), no. 4, 859–869

[CP2] Calude, C. S. and G. Paun, *Computing with Cells and Atoms: An Introduction to Quantum, DNA, and Membrane Computing.* Taylor & Francis, London/New York, 2001.

[CVL] Coello, C. C. A., D. A. Van Veldhuizen, and G. B. Lamont, *Evolutionary algorithms for solving multi-objective problems.* Kluwer Academic, New York, 2002.

[D] Dorn, W. S., Duality in quadratic programming, *Quarterly of Applied Mathematics* 18 (1960), no. 2, 155–162.

[DL] Dreyfus, S. E. and A. M. Law, *The Art and Theory of Dynamic Programming.* Mathematics in Science and Engineering, Vol. 130. Academic Press [Harcourt Brace Jovanovich, Publishers], New York/London, 1977.

[DPW] Du, D.-Z., P. M. Pardalos, and W. Wu (ed.), Mathematical Theory of Optimization. Kluwer Academic, Dordrecht/Boston, 2001.

[DZ1] Dor, D. and U. Zwick, Selecting the median, *SIAM J. Comput.* 28 (1999), no. 5, 1722–1758.

[DZ2] Dor, D. and U. Zwick, Median selection requires $(2+\epsilon)N$ comparisons. *SIAM J. Discrete Math.* 14 (2001), no. 3, 312–325.

[ES] Eiselt, H. A. and C.-L. Sandblom, *Integer Programming and Network Models.* With contributions by K. Spielberg, E. Richards, B. T. Smith, G. Laporte and B. T. Boffey. Springer-Verlag, Berlin, 2000.

[FMP] Ferris, M. C., O. L. Mangasarian, and J.-S. Pang (editors), *Complementarity: Applications, Algorithms, and Extensions.* Kluwer Academic, Boston, 2001.

[FV] Florenzano, M. and C. L. Van in cooperation with P. Gourdel, *Finite Dimensional Convexity and Optimization .* Springer-Verlag, Berlin/New York, 2001.

[FSS] Forgó, F., J. Szép, and F. Szidarovszky, *Introduction to the Theory of Games. Concepts, Methods, Applications.* Revised and expanded version of the 1985 original. Nonconvex Optimization and its Applications, 32. Kluwer Academic, Dordrecht/Boston, 1999.

[FW] Frank, M. and P. Wolfe, An algorithm for quadratic programming. *Naval Res. Logist. Quart.* 3 (1956), 95–110.

[G] Gramss, T. et al., Non-Standard Computation: Molecular Computation, Cellular Automata, Evolutionary Algorithms, Quantum Computers. Wiley-VCH, Weinheim/New York, 1998.

[GP] Gutin G. and A. P. Punnen (editors), *The Traveling Salesman Problem and its Variations* Kluwer Academic, Dordrecht/Boston, 2002.

[H1] Hačijan, L. G., A polynomial algorithm in linear programming. (Russian) *Dokl. Akad. Nauk SSSR* 244 (1979), no. 5, 1093–1096.

[H2] Hertog, D. den., *Interior Point Approach to Linear, Quadratic, and Convex Programming: Algorithms and Complexity.* Kluwer Academic, Dordrecht/Boston, 1994.

[HT] Horst, R. and H. Tuy, On the convergence of global methods in multiextremal optimization. *J. Optim. Theory Appl.* 54 (1987), no. 2, 253–271.

[K1] Karmarkar, N., A new polynomial-time algorithm for linear programming. *Combinatorica* 4 (1984), no. 4, 373–395.

[K2] Kelley, C. T., *Iterative Methods for Optimization* (Frontiers in Applied Mathematics, 18) SIAM, Philadelphia, 1999.

[K3] Knuth, D. E., *The art of computer programming. Volume 3. Sorting and searching.* Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley Publishing Co., Reading, Mass., 1973.

[K4] Klerk, E. de, *Aspects of Semidefinite Programming: Interior Point Algorithms and Selected Applications.* Kluwer Academic, Dordrecht/Boston, 2002.

[K5] Kalyanmoy D., *Multi-Objective Optimization Using Evolutionary Algorithms.* John Wiley & Sons, Chichester/ New York, 2001.

[KV] Korte, B. H. and J. Vygen, *Combinatorial Optimization: Theory and Algorithms,* 2nd ed. Springer-Verlag, New York, 2002.

[KW] Kall, P. and S. W. Wallace, *Stochastic programming.* Wiley-Interscience Series in Systems and Optimization. John Wiley & Sons, Ltd., Chichester, 1994.

[L] Luenberger, D. G., *Introduction to Linear and Nonlinear Programming.* Addison-Wesley, Reading, Mass., 1984.

[LP] Langdon, W. B. and R. Poli, *Foundations of Genetic Programming.* Springer-Verlag, New York, 2002.

[M] Morris, P., *Introduction to Game Theory.* Springer-Verlag, New York, 1994.

[MCC] Marathe, A., A. E. Condon, and R. M. Corn, On combinatorial DNA word design. *DNA Based Computers, V* (Cambridge, Mass, 1999), 75–89, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 54, Amer. Math. Soc., Providence, RI, 2000.

[NC] Nielsen, M. A. and I. L. Chuang, *Quantum Computation and Quantum Information.* Cambridge University Press, Cambridge, U. K., 2000.

[NN] Nesterov, Yu. and A. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM, Philadelphia, 1994.

[P] Porembski, M., Finitely convergent cutting planes for concave minimization, *Journal of Global Optimization,* 20, no. 2 (June 2000), 113–136.

[R] Renegar, J., A Mathematical View of Interior-Point Methods in Convex Optimization. SIAM/Mathematical Programming Society, Philadelphia, 2001.

[RI] Ramík, J. and M. Inuiguchi, *Fuzzy Mathematical Programming.* Papers from the session of the 7th Congress of the International Fuzzy Systems Association held in Prague, June 25–29, 1997. Edited by Jaroslav Ramk and Masahiro Inuiguchi. Fuzzy Sets and Systems 111 (2000), no. 1. North-Holland Publishing Co., Amsterdam, 2000.

[RV] Ramík, J. and M. Vlach, *Generalized Concavity in Fuzzy Optimization and Decision Analysis.* Kluwer Academic, Boston, 2002.

[S1] Schniederjans, M. J., *Goal Programming: Methodology and Applications.* Kluwer Academic, Dordrecht/Boston, 1995.

[S2] Sierksma, G., *Linear and Integer Programming: Theory and Practice.* 2nd ed. Marcel Dekker, New York, 2002.

[S3] Stefanov S. M., *Separable Programming: Theory and Methods.* Kluwer Academic, Dordrecht/Boston, 2001.

[S-M] Stancu-Minasian, I. M., *Fractional Programming: Theory, Methods and Applications.* Kluwer Academic, Dordrecht/Boston, 1997.

[SMY] Sarker R., M. Mohammadian, and X. Yao (editors), *Evolutionary Optimization* . Kluwer Academic Publishers, Boston, 2002.

[T] Tsurkov, V., *Large-Scale Optimization: Problems and Methods.* Applied Optimization, 51. Kluwer Academic, Dordrecht/Boston, 2001.

[VCS] Vaserstein, L. N., V. I. Chmil', and E. B. Sherman, A multiextremal problem for the growth and location of production with a concave objective function. (Russian), in *Mathematical Methods in Economic Research* (Russian), pp. 138–143. Izdat. Nauka, Moscow, 1974.

[V] Vaserstein, L. N., On the best choice of a damping sequence in iterative optimization methods, *Publ. Matem. Univ. Aut. Barcelona* 32 (1988), 275–287.

[W1] Wolkowicz, H. et al (editors), *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications.* International Series in Operations Research & Management Science, 27. Kluwer Academic, Dordrecht/Boston, 2000.

[W2] Wolsey, L. A. *Integer Programming.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, New York, 1998.

Answers to Selected Exercises

§1. What Is Linear Programming?

1. True

3. True

5. True. This is because for real numbers any square and any absolute value are nonnegative.

7. False. For $x = -1,\ 3(-1)^3 < 2(-1)^2$.

8. False (see Definition 1.4).

9. False (see Example 1.9 or 1.10).

11. False. For example, the linear program

$$\text{Minimize } x + y \text{ subject to } x + y = 1$$

has infinitely many optimal solutions.

13. True. It is a linear equation. A standard form is $4x = 8$ or $x = 2$.

15. No. This is not a linear form, but an affine function.

17. Yes if a and z do not depend on x, y.

18. No (see Definition 1.1).

19. No. But it is equivalent to a system of two linear constraints.

21. Yes. We can write $0 = 0 \cdot x$, which is a linear form.

22. True if y is independent of x and hence can be considered as a given number; see Definition 1.3.

23. Yes if a, b are given numbers. In fact, this is a linear equation.

25. No. We will see later that any system of linear constrains gives a convex set. But we can rewrite the constraint as follows $x \geq 1$ OR $x \leq -1$. Notice the difference between OR and AND.

27. See Problem 6.7.

29. $x = 3 - 2y$ with an arbitrary y.

31. $\min = 0$ at $x = y = 0, z = -1$. All optimal solutions are given as follows: $x = -y, y$ is arbitrary, $z = -1$.

33. $\max = 1$ at $x = 0$.

35. $\min = 0$ at $x = -y = 1/2, z = -1$.

37. No. This is a linear equation.

39. No

41. Yes

43. No
45. Yes
47. Yes
49. No
51. Yes
53. Yes. In fact, this is a linear equation.
55. No. This is not even equivalent to any linear constraint with rational coefficients.
57. No.
60. min $= 2^{-100} - 1$ at $x = 0, y = 0, z = 3\pi/2, u = -100, v = -100$. In every optimal solution, x, y, u, v are as before and $z = 3\pi/2 + 2n\pi$ with any integer n such that $-16 \leq n \leq 15$. So there are exactly 32 optimal solutions.

§2. Examples of Linear Programs

2. min $= 1.525$ at $a = 0, b = 0.75, c = 0, d = 0.25$

4. Let x be the number of quarters and y the number of dimes we pay. The program is

$$25x + 10y \to \min,$$
subject to
$$0 \leq x \leq 100,\ 0 \leq y \leq 90, 25x + 10y \geq C \text{ (in cents)},\ x, y \text{ integers}.$$

This program is not linear because the conditions that x, y are integers. For $C = 15$, an optimal solution is $x = 0, y = 2$. For $C = 102$, an optimal solution is $x = 3y = 3$ or $x = 1, y = 8$. For $C = 10000$, the optimization problem is infeasible.

5. Let x, y be the sides of the rectangle. Then the program is

$$xy \to \min,$$
subject to
$$x \geq 0,\ y \geq 0, 2x + 2y = 100.$$

Since $xy = x(50 - x) = 625 - (x - 25)^2 \leq 625$, max $= 625$ at $x = y = 25$.

7. We can compute the objective function at all 24 feasible solutions and find the following two optimal matchings: Ac, Ba, Cb, Dd and Ac, Bb, Ca, Dd with optimal value 7.

8. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9 + 9 + 7 + 9 + 9 = 43$ for the objective function. This bound cannot be achieved because of a conflict over c (the third column). So max ≤ 42. On the other hand, the matching Aa, Bb, Cc, De, Ed achieved 42, so this is an optimal matching.

9. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9+9+9+9+8+9+6 = 59$ for the objective function. However looking at B and C, we see that they cannot get $9 + 9 = 18$ because of the conflict over g. They cannot get more than $7 + 9 = 16$. Hence, we have the upper bound max ≤ 57. On the other hand, we achieve this bound 57 in the matching Ac, Bf, Cg, De, Eb, Fd, Ga.

11. Let c_i be given numbers. Let c_j be an unknown maximal number (with unknown j). The linear program is
$c_1x_1 + \cdots + c_nx_n \to \max$, all $x_i \geq 0$, $x_1 + \cdots + x_n = 1$.
Answer: $\max = c_j$ at $x_j = 1, x_i = 0$ for $i \neq j$.

§3. Graphical Method

1. Let SSN be 123456789. Then the program is
$-x \to \max, 7x \leq 5, 13x \geq -8, 11x \leq 10.$
Answer: $\max = 8/13$ at $x = -8/13$.

3. Let SSN be 123456789. Then the program is
$x + 2y \to \min, |12x + 4y| \leq 10, |5x + 15y| \leq 10, |x + y| \leq 24.$
Answer: $\min = -25/16$ at $x = -11/16, y = -7/16$.

5. $\min = -72$ at $x = 0, y = -9$
7. $\min = -1/4$ at $x = 1/2, y = -1/2$ or $x = -1/2, y = 1/2$.
6. The problem is unbounded ($\min = -\infty$).
9. $\max = 1$ at $x = y = 0$
11. $\max = 22$ at $x = 4, y = 2$
13. The program is unbounded.
15. $\max = 3$ at $x = y = 0, z = 1$. See the answer to Exercise 11 of §2.

§4. Logic

1. False.For $x = -1$, $|-1| = 1$.
3. False. For $x = -10$, $|-10| > 1$.
5. True. $1 \geq 0$.
7. True. $2 \geq 0$.
9. True. The same as Exercise 7.
11. False. $1 \geq 1$.
13. True. $5 \geq 0$.
15. False. For example, $x = 2$.
17. True. Obvious.
19. False. For example, $x = 1$.
21. True. $1 \geq 0$.

22. Yes, we can. 23. Yes. $10 \geq 0$.
25. No, it does not. $(-5)^2 > 10$. 27. True.
29. False. The first condition is stronger than the second one.
31. True. 33. (i) $\Rightarrow$ (ii), (iii), (iv)
35. (i) $\Rightarrow$ (iii) 37. (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii)
41. "only if".
42. This depends on the definition of it linear function.
43. No. $x \geq 1, x \leq 0$ are two feasible constraints, but the system is infeasible.
44. False. 45. False. Under our conditions, $|x| > |y|$.
49. No, it does not follow.
51. Yes, it does. Multiply the first equation by -2 and add to the second equation to obtain the third equation.

§5. Matrices

1. $[2, 1, -6, 6]$ 3. -14

5. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 3 \\ -2 & -4 & 0 & 6 \\ 4 & 8 & 0 & -12 \end{bmatrix}$

7. $(-14)^2 \cdot A^T B = \begin{bmatrix} 0 & -196 & -392 & 784 \\ 0 & -392 & -784 & 1568 \\ 0 & 0 & 0 & 0 \\ 0 & 588 & 1176 & -2352 \end{bmatrix}$ 9. No. $1 \neq 4$.

10. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 5 & 2 & 3 & -1 \\ 1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

15. $b = a - 1, c = -1/3, d = 7a - 4, a$ arbitrary. 19. $AB^T = 5$

21. $A^T B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 3 & 2 \\ 2 & -1 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 6 & 4 \end{bmatrix}$

23. $(A^T B)^2 = 5A^T B$, and see Answer to 21.

25. $(A^T B)^{1000} = A^T (BA^T)^{999} B = 5^{999} A^T B$, and see Answer to 21.

27. $AB^T = 4$.

29. $A^TB = \begin{bmatrix} -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ 1 & -1 & 0 & -3 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 6 & 6 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

31. $(A^TB)^2 = 4A^4B$, and see Answer to 29.

33. $(A^TB)^{1000} = 4^{999}A^4B$, and see Answer to 29.

35. $E_1C = \begin{bmatrix} 3 & 6 & 9 \\ -8 & -10 & -12 \end{bmatrix}$ $E_2C = \begin{bmatrix} 21 & 27 & 33 \\ 4 & 5 & 6 \end{bmatrix}$

$E_1^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}$ $E_2^n = \begin{bmatrix} 1 & 5n \\ 1 & 0 \end{bmatrix}$

37. $\begin{bmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix}$. 41. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. 43. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

§6. Systems of Linear Equations

1. $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ is invertible; $\det(A) = -4$

3. The matrix is invertible if and only if $abc \neq 0$; $\det(A) = abc$.

5. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is invertible; $\det(A) = 2$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 13/7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is invertible; $\det(A) = 13$

9. $0 = 1$ (no solutions)

11. $x = -z - 3b + 10,\ y = -z + 2b - 6$.

13. If $t \neq 6 + 2u$, then there are no solutions. Otherwise, $x = -2y + u + 3$, y arbitrary.

15. If $t = 1$, then $x = 1 - y$, y arbitrary.
If $t = -1$, there are no solutions.

If $t \neq \pm 1$, then $x = (t^2 + t + 1)/(t + 1), y = -1/(t + 1)$.

17. $y = 5b + x - 16, z = -3b - x + 10$

19. No. The half-sum of solutions is a solution.

21. $A^{-1} = \begin{bmatrix} 7/25 & 4/25 & -1/25 \\ 19/25 & -7/25 & 8/25 \\ -18/25 & 4/25 & -1/25 \end{bmatrix}$

23. $A^{-1} = \begin{bmatrix} -3/22 & -1/22 & -41/22 & 3/11 \\ -15/22 & -5/22 & -51/22 & 4/11 \\ 5/22 & 9/22 & 61/22 & -5/11 \\ 15/22 & 5/22 & 73/22 & -4/11 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$

27. This cannot be done. We have $0 = A_{1,1} = L_{1,1}U_{1,1} \neq 0$ since A is invertible, hence U, V are invertible.

29. $A = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & -2 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix}$

33. $x = -3(19 + 2d)/8, y = (15 + 2d)/8, z = -(3 + 2d)/8$

35. $x = (15u + 4v)/16, y = (11u + 4v)/16, z = -3u/4$

37. $x = y = 1, z = 0$

39. $x = y = 0, z = 100$

§7. Standard and Canonical Forms for Linear Programs

1. Set $u = y + 1 \geq 0$. Then $f = 2x + 3y = 2x + 3u - 3$ and $x + y = x + u - 1$. A canonical form is

$$-f = -2x - 3u + 3 \to \min, x + u \leq 6, u, x \geq 0.$$

A standard form is

$$-f = -2x - 3u + 3 \min, x + u + v = 6, u, v, x \geq 0$$

with a slack variable $v = 6 - x - u \geq 0$.

2. Excluding $y = x+1$ and using $y \geq 1$, we obtain the canonical form $-x \to \min,\ 2x \leq 8, x \geq 0$.

Introducing a slack variable $z = 8 - 2x$, we obtain the standard form $-x \to \min,\ 2x + z = 8, x \geq 0, z \geq 0$.

3. We solve the equation for x_3:

$x_3 = 3 - 2x_2 - 3x_4$
and exclude x_3 from the LP:

$$x_1 - 7x_2 + 3 \to \min,\ x_1 - x_2 + 3x_4 \geq 3, \text{ all } x_i \geq 0.$$

A canonical form is

$$x_1 - 7x_2 + 3 \to \min,\ -x_1 + x_2 - 3x_4 \leq -3,\ \text{ all } x_i \geq 0.$$

A standard form is

$$x_1 - 7x_2 + 3 \to \min,\ -x_1 + x_2 - 3x_4 + x_5 = -3,\ \text{ all } x_i \geq 0$$

with a slack variable $x_5 = x_1 - x_2 + 3x_4 - 3$.

5. Set $t = x+1 \geq 0, u = y-2 \geq 0, f = x = y+z = t+u+z+1$ (the objective function). Then a standard and canonical form for our problem is

$$x + u + z + 1 \to \min; t, u, z \geq 0.$$

7. Using standard tricks, a canonical form is

$$-x \to \min, x \leq 3, -x \leq -2, x \geq 0.$$

A standard form is

$$-x \to \min, x + u = 3, -x + v = -2; x, u, v \geq 0$$

with two slack variables.

9. One of the given equations reads
$-5 - x - z = 0,$
which is inconsistant with given constraints $x, z \geq 0$. So we can write very short canonical and standard forms:

$$0 \to \min, 0 \leq -1; x, y, z \geq 0 \text{ and } 0 \to \min, 0 = 1; x, y, z \geq 0.$$

11. Set $x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]^T$ and $c = [3, -1, 1, 3, 1, -5, 1, 3, 1]$. Using standard tricks, we obtain the canonical form
$cx \to \min, Ax \leq b, x \geq 0$
with

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -2 & 3 & 1 \\ 2 & -2 & -2 & 2 & 3 & -1 & -2 & 1 & 1 \\ -2 & 2 & 2 & -2 & -3 & 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 & 3 & -1 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

and $b = [-3, -1, 2, -2, 0, 0]^T$.

Excluding a couple of variables using the two given equations, we would get a canonical form with two variables and two constraints less. A standard form can be obtained from the canonical form by introducing a column u of slack variables:

$$cx \to \min, Ax + u = b, x \geq 0, u \geq 0.$$

§8. Pivoting Tableaux

1.
$$\begin{array}{cccccc|l} a & b & c & d & e & 1 & \\ \hline .3 & 1.2 & .7 & 3.5 & 5.5 & -50 & = u_1 \\ 73 & 96 & 20253 & 890 & 279 & 4000 & = u_2 \\ 9.6 & 7 & 19 & 57 & 22 & -100 & = u_3 \\ 10 & 15 & 5 & 60 & 8 & 0 & = C \to \min \end{array}$$

3. $A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ -1 & 0 & 0 & 2 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}$

11.
$$\begin{array}{cccc|l} z & a & 3 & x & \\ \hline 1 & 2 & b+3 & a+1 & = y \\ -1 & 2 & 3 & 1 & = 1 \end{array}$$

12.
$$\begin{array}{cccc|l} 1 & a & 3 & z & \\ \hline 1+a & -2a & b-3a & a & = y \\ 1 & -2 & -3 & 1 & = x \end{array}$$

13.
$$\begin{array}{c|l} 2 & \\ \hline 1/5 & = x \end{array}$$

§9. Standard Row Tableaux

2.
$$\begin{array}{ccc|l} x & y & 1 & \\ \hline -4 & -5 & 7 & = u \\ -2 & -3 & 0 & = -P \to \min \end{array}$$

with a slack variable $u = 7 - 4x - 5y$

3.
$$\begin{array}{ccccc|l} x' & x'' & y' & y'' & 1 & \\ \hline -1 & 1 & -1 & 1 & 1 & = u_1 \\ 1 & -1 & 1 & -1 & 1 & = u_2 \\ 1 & -1 & -1 & 1 & 1 & = u_3 \\ -1 & 1 & 1 & -1 & 1 & = u_4 \\ -1 & 1 & 0 & 0 & 0 & = -x \to \min \end{array}$$

with $x = x' - x'', y = y' - y''$ and slack variables u_i.

§10. Simplex Method, Phase 2

4. False. The converse is true.
5. True
6. True

13. If the row without the last entry is nonnegative, then the tableau is optimal; else the LP is unbounded.

§11. Simplex Method, Phase 1

1. The second row (v-row) is bad, so the LP is infeasible.

2. The tableau is optimal, so
$\min(w) = 0$ at $x = y = z = 0, u = 2, v = 0$.

3. This is a feasible tableau with a bad column (the z-column). So the LP is unbounded (z and hence w can be arbitrary large).

9. True

10. False

§12. Geometric Interpretation

1. The diamond can be given by four linear constraints $\pm x \pm y \leq 1$.

2. Any convex combination of convex combinations is a convex combination.

7. Both $x = 1$ and $x = -1$ belong to the feasible region, but $0 = x/2 + y/2$ does not.

8. $2tx + (1 - t^2)y \leq 1 + t^2$, where t ranges over all rational numbers.

9. A set S is called closed if it contains the limit points of all sequences in S. Any system of linear constraints gives a closed set, but the interval $0 < x < 1$ is not closed. Its complement is closed.

10. The rows of the identity matrix 1_6.

11. One

§13. Dual Problems

1.

$$\begin{array}{r} -x \\ -y \\ -z \\ 1 \end{array} \begin{bmatrix} 0 & -1 & -5 \\ 1 & -1 & -6 \\ 0 & 0 & -2 \\ 7 & -3 & 0 \end{bmatrix}$$
$$\begin{array}{ccc} \| & \| & \downarrow \\ v_1 & v_2 & \max \end{array}$$

5. Let $cx + d, cy + d$ be two feasible values, where x, y are two feasible solutions. We have to prove that

$\alpha(cx + d) + (1 - \alpha)(cy + d)$

is a feasible value for any α such that $0\alpha \leq 1$. But

$\alpha(cx + d) + (1 - \alpha)(cy + d) = c(\alpha x + (1 - \alpha)y) + d,$

where $\alpha x + (1 - \alpha)y$ is a feasible solution because the feasible region is convex.

§14. Sensitivity Analysis and Parametric Programming

3. $\min = 0$ at $d = e = 0, a \geq 0$ arbitrary

§15. More on Duality

1. No, it is not redundant.
2. Yes, it is 2$\cdot$ (first equation) + (second equation).
3. No, it is not redundant.
4. No, it is not redundant.
5. No, it is not redundant.
7. Yes, it is.

§16. Phase 1

1.

20	10	5		35
		5	15	20
20	10	10	15	

3.

30				35
90				90
11	91	9		111
		1	19	20
140	91	10	19	

§17. Phase 2

1.

	1	2	2	
0	1 175	2 25	3 (1)	200
0	1 (0)	2 100	2 200	300
	175	125	200	

§18. Job Assignment Problem

1. min = 7 at $x_{14} = x_{25} = x_{32} = x_{43} = x_{51} = 1$, all other $x_{ij} = 0$.

3. min = 7 at $x_{12} = x_{25} = x_{34} = x_{43} = x_{51} = x_{67} = x_{76} = 1$, all other $x_{ij} = 0$.

5. max = 14 at $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$, all other $x_{ij} = 0$.

7. max = 30 at $x_{15} = x_{26} = x_{33} = x_{41} = x_{54} = x_{62} = x_{77} = 1$, all other $x_{ij} = 0$.

§19. What are Matrix Games?

1. max min $= -1$. min max $= 0$. There are no saddle points.
$[1/3, 2/3, 0]^T$ gives at least $-2/3$ for the row player.
$[1/2, 0, 0, 0, 1/2]$ gives at least $1/2$ for the column player.
So $-2/3 \le$ the value of the game $\le -1/2$.

3. max min $= -1$. min max $= 2$. There are no saddle points.
(second row + 2·third row)/3 $\ge -2/3$.
(third column + sixth column)/2 ≤ 1.
So $-2/3 \le$ the value of the game ≤ 1.

5. We compute the max in each column (marked by *) and min in each row (marked by $^{\blacksquare}$).

$$\begin{array}{cc} & \begin{array}{ccccccccc} 4 & 2 & 3 & 5 & 4 & 3^{\blacksquare} & 7 & 6 & 3^{\blacksquare} \end{array} \\ \begin{array}{c} -4 \\ -2 \\ -4 \\ 0^* \\ -9 \end{array} & \begin{bmatrix} 4^* & -4^{\blacksquare} & 3^* & 0 & 0 & 0 & -1 & 1 & -2 \\ -1 & 0 & 2 & 1 & -2^{\blacksquare} & -2^{\blacksquare} & 1 & 0 & -2^{\blacksquare} \\ -4^{\blacksquare} & 0 & -2 & -2 & 1 & -1 & 1 & 6^* & 2 \\ 1 & 2^* & 2 & 5^* & 3 & 3^* & 7^* & 2 & 0^{\blacksquare} \\ -4 & -9^{\blacksquare} & -8 & 0 & 4^* & 2 & 2 & 0 & 3^* \end{bmatrix} \end{array}$$

Thus, max min $= 0$. min max $= 3$. There are no saddle points.

§20. Matrix Games and Linear Programming

1. The optimal strategy for the row player is $[2/3, 1/3, 0]^T$.
The optimal strategy for the column player is $[1/2, 1/2, 0]$.
The value of the game is 2.

3. The optimal strategy for the row player is $[0.2, 0, 0.8]^T$.
An optimal strategy for the column player is $[0, 0.5, 0.5, 0, 0, 0]$.
The value of the game is 1.

5. The optimal strategy for the row player is $[1/3, 2/3, 0]^T$.
The optimal strategy for the column player is $[2/3, 0, 0, 1/3]$.
The value of the game is $-2/3$.

7. The optimal strategy for the row player: $[1/8, 0, 7/8, 0]^T$.
The optimal strategy for the column player: $[0, 1/4, 0, 0, 0, 3/4]$.
The value of the game is -0.25.

§21. Other Methods

1. The first row and column are dominated. The optimal strategy for the row player is $[0, 0.5, 0.5]^T$. The optimal strategy for the column player is $[0, 0.25, 0.75]$. The value of the game is 2.5.

3. The optimal strategy for the row player is $[0, 0.4, 0, 0.6]^T$. The optimal strategy for the column player is $[0, 0.4, 0.6]$. The value of the game is 2.8.

5. The optimal strategy for the row player is $[1/3, 1/3, 1/3]^T$. The optimal strategy for the column player is $[0, 0, 2/7, 3/7, 2/7, 0]$. The value of the game is 0.

7. The value of the game is 0 because the game is symmetric.

9. The first two columns and the first row go by domination. The value of the game is 11/7.

11. 0 at a saddle point.

13. 0 at a saddle point.

§22. What is Linear Approximation?

1. The mean is $-2/5 = -0.4$. The median is 1. The midrange is $-5/2 = -2.5$.

3. The mean is $5/9$. The median is 0. The midrange is $1/2 = 0.5$.

5(b). 1, 2, 9.

5(d). Exercise 1.

5(f). Exercise 3.

§23. Linear Approximation and Linear Programming

1. min $= 0$ at $a = -15, b = 50$ for $w = a + bh$

and

min ≈ 19 at $a \approx 25.23$ for $w = ah^2$

2. $x + y + 0.3 = 0$

3. $a = 0.9, b \approx -0.23$

§24. More Examples

1. The model is $w = ah + b$, or $w - x_2 = a(h - 1988) + b'$ with $b = x_2 + -1988a$ and $x_2 = 37753/45 \approx 838.96$. Predicted production P in 1993 is $x_2 + 5a + b'$.

For $p = 1$, we have $a \approx 16.54, b' \approx 31, P \approx 953$.

For $p = 2$, we have $a \approx 0, b' \approx 32, P \approx 871$.

For $p = \infty$, we have $a \approx 17.59, b' \approx 32, x_5 \approx 959$.

So in this example l^∞-prediction is the best.

3. $a = \$4875, b = \$1500$

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