

INSTRUCTOR'S SOLUTIONS MANUAL

Introduction to Linear Programming

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Preface

This manual gives solutions or answers to all exercises in the textbook. It extends and corrects

Answers to Selected Exercises
(pages 305–317 of the first print of textbook). Note that the exercises may have many correct solutions and even several correct answers.

See Students Manual
<http://www.math.psu.edu/vstein/LPbook/sol2.pdf>
for corrections to the textbook, additional references, and tips. See
<http://www.math.psu.edu/vstein/LPbook/corrections.txt>
for corrections to the textbook in plain text.

Chapter 1. Introduction

§1. What Is Linear Programming?

1. True.
2. True.
3. True.
4. True. $-35 \geq -36$.
5. True. This is because for real numbers any square and any absolute value are nonnegative.
6. False. Take $x = -1$.
7. False. For $x = -1$, $3(-1)^3 < 2(-1)^2$.
8. False (see Definition 1.5).
9. False (see Example 1.9 or 1.10).
10. False.
11. False. For example, the linear program
$$\text{Minimize } x + y \text{ subject to } x + y = 1$$
has infinitely many optimal solutions.
12. False.
13. True. It is a linear equation. A standard form is $4x = 8$ or $x = 2$.
14. Yes.
15. No. This is not a linear form, but an affine function.
16. Yes, if z is independent of x, y .
17. Yes if a and z do not depend on x, y .
18. No (see Definition 1.1).
19. No. But it is equivalent to a system of two linear constraints.
20. Yes, this is a linear equation.
21. Yes. We can write $0 = 0 \cdot x$, which is a linear form.
22. True if y is independent of x and hence can be considered as a given number; see Definition 1.3.
23. Yes if a, b are given numbers. In fact, this is a linear equation.
24. Yes, it is equivalent to the system of two linear constraints $-1 \leq x \leq 1$.
25. No. We will see later that any system of linear constraints gives a convex set. But we can rewrite the constraint as follows $x \geq 1$ OR $x \leq -1$. Notice the difference between OR and AND.
26. Yes, it is equivalent to $x = 0$.
27. See Problem 6.12.

2 §1. What Is Linear Programming?

28. We multiply the first equation by 5 and subtract the result from the second equation:

$$\begin{cases} x + 2y = 3 \\ -y = -11. \end{cases}$$

Multiplying the second equation by -1, we solve it for y . Substituting this into the first equation, we find x . The answer is

$$\begin{cases} x = -19 \\ y = 11. \end{cases}$$

29. $x = 3 - 2y$ with an arbitrary y .

30. Multiplying the first equation by 3 and subtract the result from the second equation we obtain $0 = -9$ which can be simplified to $0 = 1$. So the system has no solutions.

31. $\min = 0$ at $x = y = 0, z = -1$. All optimal solutions are given as follows: $x = -y, y$ is arbitrary, $z = -1$.

32. $\min = 1$ when $-2 \leq x \leq -2.5$.

33. $\max = 1$ at $x = 0$.

34. The problem is unbounded.

35. $\min = 0$ at $x = -y = 1/2, z = -1$.

36. Yes, it is.

37. No. This is a linear equation.

38. No. Suppose $x + y^2 = ax + by$ with a, b independent of x, y . Setting $x = 0, y = 1$ we find that $b = 1$. Setting $x = 0, y = -1$ we find that $b = -1$.

39. No.

40. Yes, $y = 0 \cdot x + 1 \cdot y$.

41. Yes.

42. Yes, $(x + 1)^2 + 2y - x^2 - 1 = 2x + 2y$.

43. No.

44. No. Suppose that xy is an affine function $ax + by + c$ of x, y . Setting $x = y = 0$, we find that $c = 0$. Setting $x = 0, y = 1$, we find that $b = 0$. Setting $x = 1, y = 0$, we find that $a = 0$. Setting $x = y = 1$, we find that $1 = 0$.

45. Yes.

46. No.

47. Yes.

48. No.

49. No.

50. No, this is a linear form.

51. Yes.

52. Yes, this is a linear equation because $0 = 0 \cdot x + 0 \cdot y$ is a linear form.

53. Yes. In fact, this is a linear equation.

54. Yes.

55. No. This is not even equivalent to any linear constraint with rational coefficients.

56. Yes.

57. No, see Exercise 44.

57 (58 in the next print). Let $f(x, y) = cx + dy$ be a linear form. Then $f(ax, ay) = cax + day = a(cx + dy) = af(x, y)$ for all a, x, y and $f(x_1 + x_2, y_1 + y_2) = c(x_1 + x_2) + d(y_1 + y_2) = cx_1 + dy_1 + cx_2 + dy_2 = f(x_1, y_1) + f(x_2, y_2)$ for all x_1, x_2, y_1, y_2 .

58 (59 in the next print). We set $c = f(1, 0)$ and $d = f(0, 1)$. Then $f(x, y) = f(x + 0, 0 + y) = f(x, 0) + f(0, y) = f(1, 0)x + f(0, 1)y = cx + dy$ for all x, y .

59 (60 in the next print). $\min = 2^{-100} - 1$ at $x = 0, y = 0, z = 3\pi/2, u = -100, v = -100$. In every optimal solution, x, y, u, v are as before and $z = 3\pi/2 + 2n\pi$ with any integer n such that $-16 \leq n \leq 15$. So there are exactly 32 optimal solutions.

§2. Examples of Linear Programs

1. Date: Nov 19, 2002. Store: <http://www.peapod.com>.

	Cereal by General Mills	Box Size	Price per oz	Price per box
A	APPLE CINNAMON			
	CHEERIOS	15 OZ	\$.27	\$3.99
B	BASIC 4	16.2 OZ	\$.26	\$4.29
C	CHEERIOS	15 OZ	\$.25	\$3.79
F	FIBER ONE	16 OZ	\$.25	\$3.99
G	GOLDEN GRAHAMS	13 OZ	\$.31	\$3.99
H	HARMONY	16.7 OZ	\$.25	\$4.19
K	KIX	12.7 OZ	\$.28	\$3.59
L	LUCKY CHARMS	14 OZ	\$.30	\$4.19
M	Multi-Bran CHEX	15.6 OZ	\$.26	\$3.99
T	TOTAL Corn Flakes	10 OZ	\$.40	\$3.99

4 §2. Examples of Linear Programs

Food composition and Dietary Reference Intakes (DRIs) are taken from <http://www.nal.usda.gov/fnic/etext/000020.html> (Food and Nutrition Information Center, US Dept of Agriculture). The DRIs are actually a set of four reference values: Estimated Average Requirements (EAR), Recommended Dietary Allowances (RDA), Adequate Intakes (AI), and Tolerable Upper Intake Levels, (UL) that have replaced the 1989 Recommended Dietary Allowances (RDAs). DRIs for the vitamins are for a male of age 21. The RDA for protein is taken for 62.5 kg of body weight. Vitamin B_2 is also known as thiamin. Data are per serving unit. The DRIs represents daily requirements.

	serving size	Protein g	A IU	B_1 mg	C mg	B_6 mg	B_{12} mcg
A	30 g	1.8	500.1	0.375	6	0.501	1.5
B	55 g	4.3	392.7	0.297	0	0.39	1.155
C	30 g	3.3	500.1	0.375	6	0.501	1.5
F	30 g	2.4	0	0.375	6	0.501	1.5
G	30 g	1.5	500.1	0.375	6	0.501	1.5
H	100 g	11	909	2.73	55	1.82	7.6
K	30 g	1.8	529.2	0.375	6.3	0.501	1.5
L	30 g	2.1	500.1	0.375	6	0.501	1.5
M	49 g	3.43	445.41	0.333	5.39	0.446	1.323
T	30 g	1.815	428.7	1.5	60	2.001	6
DRI /	day	50	900	1.2	90	1.3	2.4

Our variables are amounts of foods. Since most of data are per serving size, we measure all variables in serving sizes. Now we compute prices per serving size. We use prices per box rather than less precise prices per ounce. We use the US standard: 1 pound = 453.59237 grams = 16 ounces, hence 1 oz = 28.3495 g. We round off prices to 4 digits after the decimal point. For example, the price in dollars of one serving of APPLE CINNAMON CHEERIOS (A) is

$$30 \cdot 3.99 / (15 \cdot 28.3495) = 0.281486 \approx 0.2815.$$

serial serial	variable (servings)	price per serving
A	a	\$ 0.2815
B	b	\$ 0.5138
C	c	\$ 0.2674
F	f	\$ 0.2640
G	g	\$ 0.3448
H	h	\$ 0.8850
K	k	\$ 0.2991
L	l	\$ 0.3167
M	m	\$ 0.4421
T	t	\$ 0.4222

Now we can write our objective function, the total cost (in dollars per day):

$$0.2815a + 0.5138b + 0.2674c + 0.2640f + 0.3448g + 0.8850h \\ + 0.2991k + 0.3167l + 0.4421m + 0.4222t \rightarrow \min.$$

The constraints are

$$a, b, c, f, g, h, k, l, m, t \geq 0,$$

$$\begin{bmatrix} 1.8 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 4.3 & 392.7 & 0.297 & 0 & 0.39 & 1.155 \\ 3.3 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 2.4 & 0 & 0.375 & 6 & 0.501 & 1.5 \\ 1.5 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 11 & 909 & 2.73 & 55 & 1.82 & 7.6 \\ 1.8 & 529.2 & 0.375 & 6.3 & 0.501 & 1.5 \\ 2.1 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 3.43 & 445.41 & 0.333 & 5.39 & 0.446 & 1.323 \\ 1.815 & 428.7 & 1.5 & 60 & 2.001 & 6 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \\ f \\ g \\ h \\ k \\ l \\ m \\ t \end{bmatrix} \geq \begin{bmatrix} 50 \\ 900 \\ 1.2 \\ 90 \\ 1.3 \\ 2.4 \end{bmatrix}.$$

$$2. \min = 1.525 \text{ at } a = 0, b = 0.75, c = 0, d = 0.25$$

$$3. \max(P) = 100 \text{ at } x_1 = 20, x_2 = 20,$$

4. Let x be the number of quarters and y the number of dimes we pay. The program is

$$25x + 10y \rightarrow \min,$$

subject to

$$0 \leq x \leq 100, 0 \leq y \leq 90, 25x + 10y \geq C \text{ (in cents), } x, y \text{ integers.}$$

This program is not linear because the conditions that x, y are integers. For $C = 15$, an optimal solution is $x = 0, y = 2$. For

6 §2. Examples of Linear Programs

$C = 102$, an optimal solution is $x = 3, y = 3$ or $x = 1, y = 8$. For $C = 10000$, the optimization problem is infeasible.

5. Let x, y be the sides of the rectangle. Then the program is

$$\begin{aligned} xy &\rightarrow \max, \\ \text{subject to} \\ x &\geq 0, y \geq 0, 2x + 2y = 100. \end{aligned}$$

Since $xy = x(50 - x) = 625 - (x - 25)^2 \leq 625$, $\max = 625$ at $x = y = 25$.

6. If we choose the best worker for each work, we obtain the upper bound

$$\max \leq 20 + 70 + 90 + 67$$

for the objective function. However, this is not a matching, because B has two jobs, “a” and “d”. If “a” is done by somebody else, the objective function drops by at least 5. If “d” is done by somebody else, the objective function drops by at least 12. So Ab, Bd, Cc, Da is the optimal matching with $\max = 15 + 70 + 90 + 67 = 242$.

7. We can compute the objective function at all 24 feasible solutions and find the following two optimal matchings: Ac, Ba, Cb, Dd and Ac, Bb, Ca, Dd with optimal value 7.

8. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9 + 9 + 7 + 9 + 9 = 43$ for the objective function. This bound cannot be achieved because of a conflict over c (the third column). So $\max \leq 42$. On the other hand, the matching aa, Bb, Cc, De, Ed achieved 42, so this is an optimal matching.

9. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9 + 9 + 9 + 9 + 8 + 9 + 6 = 59$ for the objective function. However looking at B and C, we see that they cannot get $9 + 9 = 18$ because of the conflict over g. They cannot get more than $7 + 9 = 16$. Hence, we have the upper bound $\max \leq 57$. On the other hand, we achieve this bound 57 in the matching Ac, Bf, Cg, De, Eb, Fd, Ga.

10. We find the best match for A, then the best remaining match for B, and so on. We obtain matching Aa, Bb, Cg, Df, Ec, Fd, Ge with $8 + 9 + 9 + 5 + 1 + 9 + 1 = 42$ (years) total. A similar procedure for columns yield the matching Aa, Bb, Cc, Fd, De, Ef, Gg with $8 + 9 + 7 + 9 + 1 + 5 + 4 = 43$ total, which is a better matching.

11. Let c_i be given numbers. Let c_j be an unknown maximal number (with unknown j). The linear program is

$$c_1x_1 + \cdots + c_nx_n \rightarrow \max, \text{ all } x_i \geq 0, x_1 + \cdots + x_n = 1.$$

Answer: $\max = c_j$ at $x_j = 1, x_i = 0$ for $i \neq j$.

12. $|a - x| + |b - x| + |c - x| \rightarrow \min$.

§3. Graphical Method

1. Let SSN be 123456789. Then the program is

$-x \rightarrow \max, 7x \leq 5, 13x \geq -8, 11x \leq 10$.

Answer: $\max = 8/13$ at $x = -8/13$.

2. Let SSN be 123456789. Then the program is $f = x - 3y \rightarrow \min, |6x + 4y| \leq 14, |5x + 7y| \leq 8, |x + y| \leq 17$.

Answer: $\min = -22$ at $x = -65/11, y = 59/11$.

3. Let SSN be 123456789. Then the program is

$x + 2y \rightarrow \min, |12x + 4y| \leq 10, |5x + 15y| \leq 10, |x + y| \leq 24$.

Answer: $\min = -25/16$ at $x = -11/16, y = -7/16$.

4. The first constraint is equivalent to 2 linear constraints $-7 \leq x \leq 3$. The feasible region for the second constraint is also an interval, $-8 \leq x \leq 2$. The feasible region for the linear program is the interval $-7 \leq x \leq 2$. In Case (i), the objective function is an increasing function of x and reaches its maximum 14 at the right endpoint $x = 2$. In Case (ii), the objective function is a decreasing function of x and reaches its maximum 63 at the left endpoint $x = -7$. In Case (ii),

$$\max = \begin{cases} 2b \text{ at } x = 2 & \text{if } b > 0, \\ 0 \text{ when } -7 \leq x \leq 2 & \text{if } b = 0, \\ -7b \text{ at } x = -7 & \text{if } b < 0. \end{cases}$$

5. $\min = -72$ at $x = 0, y = -9$

6. The objective function is not defined when $y = 0$. When $y = -1$ and $x \rightarrow \infty$, we have $x/y \rightarrow -\infty$. So this minimization problem is unbounded, $\min = -\infty$.

7. $\min = -1/4$ at $x = 1/2, y = -1/2$ or $x = -1/2, y = 1/2$.

8. There is only one feasible solution. Answer: $\max = 3$ at $x = 1, y = 1, z = 1$.

9. $\max = 1$ at $x = y = 0$

10. This is a linear equation for b . If $x \neq 0$, then the unique solution is $b = c/x$. If $x = 0 = c$, then every b is a solution. If $x = 0, c \neq 0$, then there are no solutions.

11. $\max = 22$ at $x = 4, y = 2$

12. Our variables, a and b are amounts of the foods in grams (including refuse). The constraints are

8 §3. Graphical Method

$$a \geq 0, b \geq 0,$$

$$0.4 \cdot 5.78a + 0.98 \cdot 0.56b \geq 2000 \text{ (energy in kcal),}$$

$$0.4 \cdot 0.0024a + 0.98 \cdot 0.0005b \geq 1.1 (B_1 \text{ in mg}),$$

$$0.4 \cdot 0.0081a + 0.98 \cdot 0.0005b \geq 1.1 (B_2 \text{ in mg}).$$

The objective function in dollars to be minimized is

$$2a/454 + 3b/454.$$

We took 1 pound ≈ 454 g (the exact value is 453.59237). The program can be solved graphically. However, it is clear that almond are both cheaper and have more contents for all 3 ingredients, so $b = 0$ in any optimal solution. Answer:

$$\min \approx \$5.05 \text{ at } a \approx 1146 \text{ g} \approx 2.5 \text{ lb}, b = 0.$$

13. The program is unbounded.

14. The feasible region can be given by 4 linear constraints: $-5 \leq x \leq 0, 2 \leq y \leq 3$. It is a rectangle with 4 corners $[x, y] = [0, 3], [-5, 3], [-5, -2], [0, -2]$. The objective function is not affine. Its level $|x| + y^2 = c$ is empty when $c < 0$, is a point when $c = 0$, and is made of 2 parabola pieces when $c > 0$. It is clear that $\max = 14$ at $x = -5, y = 3$. The optimal solution is unique.

15. $\max = 3$ at $x = y = 9, z = 1$. See the answer to Exercise 11 of §2.

Chapter 2. Background

§4. Logic

1. False. For $x = -1$, $|-1| = 1$.
2. False. Take $y = 1 \neq 0, x = 0$.
3. False. For $x = -10$, $|-10| > 1$.
4. True.
5. True. $1 \geq 0$.
6. False. Take $x = -2$.
7. True. $2 \geq 0$.
8. True.
9. True. The same as Exercise 7.
10. True.
11. False. $1 \geq 1$.
12. False. Take $x = -1$.
13. True. $5 \geq 0$.
14. True.
15. False. For example, $x = 2$.
16. True.
17. True. Obvious.
18. True.
19. False. For example, $x = 1$.
20. True. The product of two nonzero numbers is nonzero.
21. True. $1 \geq 0$.
22. Yes, we can.
23. Yes. $10 \geq 0$.
24. True.
25. No, it does not. $(-5)^2 > 10$.
26. True.
27. True.
28. True.
29. False. The first condition is stronger than the second one.
30. False. The converse is true.
31. True.
32. $(i) \Rightarrow (ii), (iii), (iv)$.
33. $(i) \Rightarrow (ii), (iii), (iv)$.
34. $(i) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (ii)$.
35. $(i) \Rightarrow (iii)$.

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36. (i) \Rightarrow (ii) (add the constraints in (i) to obtain (ii)); (iv) \Rightarrow (iii).
37. (i) \Leftrightarrow (ii) \Rightarrow (iv) \Rightarrow (iii)
38. (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).
39. (i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (iii).
40. given that, assuming that, supposing that, in the case when, granted that.
41. “only if”
42. This depends on the definition of *linear function*.
43. No. $x \geq 1, x \leq 0$ are two feasible constraints, but the system is infeasible.
44. False.
45. False. Under our conditions, $|x| > |y|$.
46. Correct.
47. Correct (add the two constraints in the system and the constraint $0 \leq 1$).
48. Correct (add the two constraints in the system).
49. No, it does not follow.
50. Yes, it does. It is the sum of the first two equations.
51. Yes, it does. Multiply the first equation by -2 and add to the second equation to obtain the third equation.
52. Yes, it does. It is the difference of the first two equations.

§5. Matrices

1. $[2, 1, -6, 6]$
2. -14
3. -14
4.
$$\begin{bmatrix} 0 & -1 & -2 & 4 \\ 0 & -2 & -4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & -12 \end{bmatrix}$$
5.
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 3 \\ -2 & -4 & 0 & 6 \\ 4 & 8 & 0 & -12 \end{bmatrix}$$
6. $-14A^TB = \begin{bmatrix} 0 & 14 & 28 & -56 \\ 0 & 28 & 56 & -116 \\ 0 & 0 & 0 & 0 \\ 0 & -42 & -84 & 168 \end{bmatrix}$

$$7. (-14)^2 \cdot A^T B = \begin{bmatrix} 0 & -196 & -392 & 784 \\ 0 & -392 & -784 & 1568 \\ 0 & 0 & 0 & 0 \\ 0 & 588 & 1176 & -2352 \end{bmatrix}$$

$$8. -14^{999} A^T B = \begin{bmatrix} 0 & 14^{999} & 2 \cdot 14^{999} & -4 \cdot 14^{999} \\ 0 & 2 \cdot 14^{999} & 4 \cdot 14^{999} & -8 \cdot 14^{999} \\ 0 & 0 & 0 & 0 \\ 0 & -3 \cdot 14^{999} & -6 \cdot 14^{999} & 12 \cdot 14^{999} \end{bmatrix}$$

$$9. \text{No. } 1 \neq 4.$$

$$10. A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$11. A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$12. \begin{bmatrix} 5 & 2 & 3 & -1 \\ 1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$13. \begin{bmatrix} 0 & 2 & 3 & 0 & -1 \\ 1 & -1 & -3 & 0 & -2 \\ 0 & 0 & -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$14. \begin{bmatrix} 0 & 2 & 3 & 0 & 1 & -1 \\ 1 & -1 & -3 & -1 & 0 & -2 \\ 5 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$15. b = a - 1, c = -1/3, d = 7a - 4, a \text{ arbitrary}$$

$$16. \text{We permute the columns of the coefficient matrix:}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & -2 & -1 & -3 \\ 1 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ a \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Next we subtract the first row from the last row, and then add the third row to the first and the second rows with coefficients 1 and 2:

12 §5. Matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & -5 & -15 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \begin{bmatrix} x \\ a \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$$

Now we write the answer: $x = 3c - 1, a = 5b + 15c - 4, y = 2b + 6c - 2$, where b, c are arbitrary.

17. We will solve the system for x, d, a . So we rewrite the system (see the solution to Exercise 14 above):

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & -1 & 1 & -1 & -3 & -2 \\ 1 & 0 & 5 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now we subtract the first row from the last one and multiply the second row by -1 :

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 5 & -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Then we multiply the last row by $1/5$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 1 & -2/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2/5 \end{bmatrix}$$

Finally, we add the last row to the second one:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 3/5 & 12/5 & 2 \\ 0 & 0 & 1 & -2/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2/5 \\ -2/5 \end{bmatrix}$$

So our answer is

$$x = -2b - 3c + y + 1,$$

$$d = -0.6b - 2.4c - 2y - 0.4,$$

$$a = 0.4b + 0.6c - 0.4$$

with arbitrary b, c, y .

$$18. 2A + 3B = [-6, 5, -2, 0, 9, 10].$$

$$19. AB^T = 5$$

$$20. BA^T = 5$$

$$21. A^T B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 3 & 2 \\ 2 & -1 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 6 & 4 \end{bmatrix}$$

$$22. B^T A = \begin{bmatrix} 0 & -2 & 2 & 0 & 0 & -4 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 6 \\ 0 & 2 & -2 & 0 & 0 & 4 \end{bmatrix}$$

$$23. (A^T B)^2 = 5A^T B, \text{ and see Answer to 21.}$$

$$24. (A^T B)^3 = 125$$

$$25. (A^T B)^{1000} = A^T (BA^T)^{999} B = 5^{999} A^T B, \text{ and see Answer to 21.}$$

$$26. 2A + 3B = [-1, 5, -2, 9, 9, 10, -3].$$

$$27. AB^T = 4$$

$$28. BA^T = 4$$

$$29. A^T B = \begin{bmatrix} -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ 1 & -1 & 0 & -3 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 6 & 6 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$30. B^T A = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & 0 \\ -1 & -1 & 1 & 0 & 0 & -2 & 0 \end{bmatrix}$$

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$$31. (A^T B)^2 = 4A^T B = \begin{bmatrix} -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ 4 & -4 & 0 & -12 & -12 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 8 & 0 & 24 & 24 & 16 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$32. (A^T B)^3 = 16A^T B \\ = \begin{bmatrix} -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ 16 & -16 & 0 & -48 & -48 & -32 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 32 & 0 & 96 & 96 & 64 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$33. 4^{999} A^T B$$

$$34. AB^T = \begin{bmatrix} 89/4 & 23 \\ 341/8 & 107/2 \end{bmatrix}, BA^T = \begin{bmatrix} 89/4 & 341/8 \\ 23 & 107/2 \end{bmatrix},$$

$$A^T B = \begin{bmatrix} 9/2 & 21/4 & 23 \\ 1/2 & 9/4 & -3 \\ 27/2 & 63/4 & 69 \end{bmatrix}, B^T A = \begin{bmatrix} 9/2 & 1/2 & 27/2 \\ 21/4 & 9/4 & 63/4 \\ 23 & -3 & 69 \end{bmatrix},$$

$$(A^T B)^2 = \begin{bmatrix} 2667/8 & 6363/16 & 6699/4 \\ -297/8 & -633/16 & -809/4 \\ 8001/8 & 19089/16 & 20097/4 \end{bmatrix},$$

$$(A^T B)^3 = \begin{bmatrix} 777861/32 & 1857429/64 & 1952517/16 \\ -93351/32 & -222039/64 & -235047/16 \\ 2333583/32 & 5572287/64 & 5857551/16 \end{bmatrix}.$$

$$35. E_1 C = \begin{bmatrix} 3 & 6 & 9 \\ -8 & -10 & -12 \end{bmatrix}, E_2 C = \begin{bmatrix} 21 & 27 & 33 \\ 4 & 5 & 6 \end{bmatrix},$$

$$(E_1)^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}, (E_2)^n = \begin{bmatrix} 1 & 5n \\ 1 & 0 \end{bmatrix}.$$

$$36. CE_1 = \begin{bmatrix} 2 & 6 & 12 \\ 8 & 15 & 24 \end{bmatrix}, CE_2 = \begin{bmatrix} -8 & 2 & 3 \\ -14 & 5 & 6 \end{bmatrix},$$

$$DE_1 = \begin{bmatrix} 18 & 24 & 28 \\ 12 & 15 & 16 \\ 6 & 6 & 4 \end{bmatrix}, DE_2 = \begin{bmatrix} -12 & 8 & 7 \\ -6 & 5 & 4 \\ 0 & 2 & 1 \end{bmatrix},$$

$$E_1 E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -12 & 0 & 4 \end{bmatrix}, E_2 E_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -6 & 0 & 4 \end{bmatrix},$$

$$(E_1)^n = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}, (E_2)^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3n & 0 & 1 \end{bmatrix}.$$

$$37. \begin{bmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix}$$

38. For $n \times n$ diagonal matrices

$$A = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}, B = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix},$$

we have

$$A + B = \begin{bmatrix} a_1 + b_1 & 0 & \dots & 0 \\ 0 & a_2 + b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n + b_n \end{bmatrix} \text{ and}$$

$$AB = BA = \begin{bmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n b_n \end{bmatrix}.$$

For $m < n$, diagonal $m \times n$ matrices have the form $[A, 0]$ and $[B, 0]$ with $m \times m$ diagonal matrices A, B , and $[A, 0] + [B, 0] = [A + B, 0]$, where 0 is the zero $m \times (n - m)$ matrix. Similarly, sum of diagonal $m \times n$ matrices is diagonal in the case $m > n$.

Any nondiagonal entry of the product of diagonal matrices is the dot product of two rows, each having at most one nonzero entry, and these entries are located at different positions. So the product is a diagonal matrix.

39. Let $A = [a_{ij}], B = [b_{ij}]$ be upper triangular, i.e., $a_{ij} = 0 = b_{ij}$ whenever $i > j$. Then $(A + B)_{ij} = a_{ij} + b_{ij} = 0$ whenever $i > j$, so $A + B$ is upper triangular. For $i > j$, the entry $(AB)_{ij}$ is the product of a row whose first $i - 1 > j$ entries are zero and a column whose entries are zero with possible exception of the first j entries. So this $(AB)_{ij} = 0$. Thus, AB is upper triangular.

Take upper triangular matrices $A = [1, 2], B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB \neq BA$ (they have different sizes). Here is an example with square matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

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Now $AB = B \neq A = BA$.

40. Solution is similar to that of Exercise 39, and these Exercises can be reduced to each other by matrix transposition.

41.
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

42. Adding the first column to the third column we obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & -2 \\ 2 & 4 & 7 \\ 8 & -2 & 7 \end{bmatrix}.$$

Adding the second column multiplied by 2 to the third column we obtain a lower matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 2 & 4 & 15 \\ 8 & -2 & 3 \end{bmatrix}.$$

Now we kill nondiagonal entries in the second, third, and fourth rows using multiples of previous rows. Seven row addition operations bring our matrix to the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix}.$$

43.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

44. We kill the entries 5, 5 in the first column by two row addition operations:

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}.$$

Adding a multiple of the second row to the last row, we obtain the upper triangular matrix

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix}.$$

Since the diagonal entries are nonzero, we can bring this matrix to its diagonal part

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & -73/11 & 0 \end{bmatrix}$$

by five column addition operations.

§6. Systems of Linear Equations

1. $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ is invertible; $\det(A) = -4$

2. If $a \neq 0$, then by a row and a column addition operations we can reduced A to $\begin{bmatrix} a & 0 \\ 0 & d - bc/a \end{bmatrix}$. Similarly if $d \neq 0$, by two addition operations we can get $\begin{bmatrix} a - bc/d & 0 \\ 0 & d \end{bmatrix}$. If $a = d = 0, b \neq 0$, then

adding the second column to the first column we obtain $\begin{bmatrix} b & b \\ c & 0 \end{bmatrix}$.

Two more addition operations produce $\begin{bmatrix} b & 0 \\ 0 & -c \end{bmatrix}$. In all three cases, the product of diagonal entries in the diagonal form is $ad - bc$, so matrix is invertible if and only if $ad - bc \neq 0$. Finally, when $a = b = 0$, then the matrix is not invertible because it has a zero row, and $ad - bc = 0$. So in all possible cases the matrix is invertible if and only if $\det(A) = ad - bc \neq 0$.

3. The matrix is invertible if and only if $abc \neq 0$; $\det(A) = abc$.

4. $1 \begin{matrix} \nearrow \\ \uparrow \\ \nwarrow \end{matrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mapsto -1 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mapsto$

$1 \begin{matrix} \nearrow \\ \uparrow \\ \nwarrow \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The matrix is invertible,

$\det(A) = -1$.

5. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is invertible; $\det(A) = 2$

6. $-2 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} -5 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} \mapsto -4 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 4 & 7 \end{bmatrix} \mapsto$

$\nearrow \xrightarrow{1} \searrow$

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$$\begin{array}{c} \nearrow \quad \searrow \\ -8 \\ \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{array} \right] \mapsto \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{array} \right]. \end{array}$$

The matrix is invertible, $\det(A) = -25$.

7. $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 13/7 & 0 \\ 0 & 0 & 7 \end{array} \right]$ is invertible; $\det(A) = 13$

$$8. \quad \begin{array}{c} \nearrow \quad \xrightarrow{1} \quad \searrow \\ \nearrow \quad \xrightarrow{2} \quad \searrow \end{array}$$

$$\left[\begin{array}{cccc} 1 & -2 & -1 & 0 \\ 5 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 6 & 0 & 7 \end{array} \right] \mapsto \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 5 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 16 & 5 & 7 \end{array} \right]. \text{ By two row addition}$$

operations, we obtain $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{array} \right]$. Now we perform three more

row addition operations:

$$\begin{array}{c} -10 \nearrow \\ -16 \searrow \end{array} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{array} \right] \mapsto -1 \nwarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & -9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -9 \end{array} \right] \mapsto$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & -9 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 5 & -9 \end{array} \right]. \text{ By two column addition operations, we clear}$$

the second row and obtain $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 5 & -9 \end{array} \right]$. Now we work on the

submatrix $\begin{bmatrix} -8 & 9 \\ 5 & -9 \end{bmatrix}$. Adding the last column to the previous

one, we obtain $\begin{bmatrix} 1 & 9 \\ -4 & -9 \end{bmatrix}$. Now we use the entry 1 to clear the

first row and the first column. By two addition operations we obtain $\begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix}$. Thus, by addition operations with integral coefficients we

reduced the matrix A to the diagonal matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 27 \end{bmatrix}$. So

the matrix A is invertible, and $\det(A) = 27$.

9. $0 = 1$ (no solutions)

10. We do one row addition operation with the augmented matrix and then drop the zero row:

$-2 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & 2 & | & 3 \\ 2 & 4 & | & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix} \mapsto [1, 2 | 3]$. Answer: $x = 3 - 3y$, y being arbitrary.

11. $x = -z - 3b + 10$, $y = -z + 2b - 6$.

12. We perform two addition operations on the augmented matrix:

$$-1 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & 4 & | & 1 \\ 1 & 5 & | & -8 \end{bmatrix} \mapsto -4 \begin{array}{c} \nearrow \\ \swarrow \end{array} \begin{bmatrix} 1 & 4 & | & 1 \\ 0 & 1 & | & -9 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & | & 37 \\ 0 & 1 & | & -9 \end{bmatrix}.$$

Answer: $x = 37$, $y = -9$.

13. If $t \neq 6 + 2u$, then there are no solutions. Otherwise, $x = -2y + u + 3$, y arbitrary.

14. We do a row addition operation on the augmented matrix:

$$-2 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & t & | & 3 \\ 2 & 4 & | & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & t & | & 3 \\ 0 & 4 - 2t & | & -5 \end{bmatrix}.$$

If $t = 2$, the second equation shows that the system has no solutions. Otherwise we perform a multiplication and an addition operation:

$$\begin{aligned} 1/(4 - 2t) \begin{bmatrix} 1 & t & | & 3 \\ 0 & 4 - 2t & | & -5 \end{bmatrix} &\mapsto -t \begin{array}{c} \nearrow \\ \swarrow \end{array} \begin{bmatrix} 1 & t & | & 3 \\ 0 & 1 & | & 5/(2t - 4) \end{bmatrix} \\ &\mapsto \begin{bmatrix} 1 & 0 & | & 3 - 5t/(2t - 4) \\ 0 & 1 & | & 5/(2t - 4) \end{bmatrix}. \end{aligned}$$

Answer: $0 = 1$ when $t = 2$; $x = (t - 12)/(2t - 4)$, $y = 5/(2t - 4)$ when $t \neq 2$.

15. If $t = 1$, then $x = 1 - y$, y arbitrary.

If $t = -1$, there are no solutions.

If $t \neq \pm 1$, then $x = (t^2 + t + 1)/(t + 1)$, $y = -1/(t + 1)$.

16. We do a row addition operation on the augmented matrix:

$$-2 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & t & | & t^2 \\ 2 & u & | & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & t & | & t^2 \\ 0 & u - 2t & | & 1 - 2t^2 \end{bmatrix}.$$

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If $u - 2t = 0 = 1 - 2t^2$, then the second row is zero. We drop this row and the answer is $x = t^2 - ty$, y arbitrary. If $u - 2t = 0 \neq 1 - 2t^2$, there are no solutions.

If $u - 2t \neq 0$, we perform a multiplication and an addition operation:

$$\begin{aligned} & 1/(u - 2t) \left[\begin{array}{cc|c} 1 & t & t^2 \\ 0 & u - 2t & 1 - 2t^2 \end{array} \right] \mapsto \\ & \quad -t \begin{array}{l} \nearrow \\ \nwarrow \end{array} \left[\begin{array}{cc|c} 1 & t & t^2 \\ 0 & 1 & (1 - 2t^2)/(u - 2t) \end{array} \right] \\ & \mapsto \left[\begin{array}{cc|c} 1 & 0 & t^2 - t(1 - 2t^2)/(u - 2t) \\ 0 & 1 & (1 - 2t^2)/(u - 2t) \end{array} \right]. \end{aligned}$$

Answer: $0 = 1$ when $u = 2t \neq \sqrt{2}$; $x = t^2 - ty$, y arbitrary when $u = 2t = \sqrt{2}$; $x = t^2 - t(1 - 2t^2)/(u - 2t)$, $y = (1 - 2t^2)/(u - 2t)$ when $u \neq 2t$.

17. It is convenient to write the augmented matrix corresponding to the variables y, z, x (rather than x, y, z). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$\begin{aligned} & -5/3 \begin{array}{l} \nwarrow \\ \searrow \end{array} \left[\begin{array}{ccc|c} y & z & x & \\ 3 & 5 & 2 & 2 \\ 5 & 8 & 3 & b \end{array} \right] \mapsto \\ & 1/3 \cdot \left[\begin{array}{ccc|c} 3 & 5 & 2 & 2 \\ 0 & -1/3 & -1/3 & b - 10/3 \end{array} \right] \mapsto \\ & -5/3 \begin{array}{l} \nearrow \\ \nwarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5/3 & 2/3 & 2/3 \\ 0 & 1 & 1 & -3b + 10 \end{array} \right] \mapsto \\ & \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5b - 16 \\ 0 & 1 & 1 & -3b + 10 \end{array} \right]. \end{aligned}$$

Answer: $y = x + 5b - 16$, $z = -x - 3b + 10$, x is arbitrary.

18. It is convenient to write the augmented matrix corresponding to the variables x, z, y (rather than x, y, z). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$\begin{aligned} & -3/2 \begin{array}{l} \nwarrow \\ \searrow \end{array} \left[\begin{array}{ccc|c} x & z & y & \\ 2 & 5 & 3 & 2 \\ 3 & 8 & 5 & b \end{array} \right] \mapsto \\ & 1/2 \cdot \left[\begin{array}{ccc|c} 2 & 5 & 3 & 2 \\ 0 & 1/2 & 1/2 & b - 3 \end{array} \right] \mapsto \end{aligned}$$

$$-5/2 \begin{array}{l} \nearrow \\ \nwarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5/2 & 3/2 & 1 \\ 0 & 1 & 1 & 2b-6 \end{array} \right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & -1 & -5b+16 \\ 0 & 1 & 1 & 2b-6 \end{array} \right].$$

Answer: $x = y - 5b + 16$, $z = -y + 2b - 6$, y is arbitrary.

19. No. The halfsum of solutions is a solution.

20. We augment A by the identity matrix 1_4 and then make it 1_4 by elementary row operations. First we permute rows:

$$\begin{bmatrix} 0 & 0 & -1 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & | & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & | & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 1 & | & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 & 1 & 2 & 1 & | & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & | & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now we perform two addition, four multiplication operations, and four addition operations: $[A|1_4] =$

$$1 \mid \begin{array}{l} \nwarrow \\ \nearrow \end{array} 3 \begin{array}{l} \nwarrow \\ \nearrow \end{array} \left[\begin{array}{cccc|cccc} -1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 & 0 & 0 \end{array} \right] \mapsto$$

$$\begin{array}{l} -1 \cdot \\ -1 \cdot \\ -1 \cdot \\ 1/2 \cdot \end{array} \left[\begin{array}{cccc|cccc} -1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 1 & 1 & 0 \end{array} \right] \mapsto$$

$$2 \mid \begin{array}{l} \nearrow \\ \nwarrow \end{array} 1 \begin{array}{l} \nearrow \\ \nwarrow \end{array} \left[\begin{array}{cccc|cccc} 1 & -1 & -2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \end{array} \right] \mapsto$$

$$2 \downarrow \begin{array}{l} \nearrow \\ \nwarrow \end{array} 1 \downarrow \begin{array}{l} \nearrow \\ \nwarrow \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \end{array} \right] \mapsto$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \end{array} \right] = [1_4, A^{-1}]$$

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$$21. A^{-1} = \begin{bmatrix} 7/25 & 4/25 & -1/25 \\ 19/25 & -7/25 & 8/25 \\ -18/25 & 4/25 & -1/25 \end{bmatrix}$$

$$22. [A, 1_3] =$$

$$\begin{array}{l} \begin{array}{c} \swarrow \quad \swarrow \\ -2 \downarrow \quad -5 \searrow \\ \quad \searrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 5 & 1 & 3 & | & 0 & 1 & 0 \\ 2 & 4 & 5 & | & 0 & 0 & 1 \end{bmatrix} \mapsto \\ \\ -8/11 \begin{array}{c} \swarrow \quad \swarrow \\ \quad \searrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 11 & 8 & | & -5 & 1 & 0 \\ 0 & 8 & 7 & | & -2 & 0 & 1 \end{bmatrix} \mapsto \\ \\ 1/11 \cdot \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 11 & 8 & | & -5 & 1 & 0 \\ 0 & 0 & 13/11 & | & 18/11 & -8/11 & 1 \end{bmatrix} \mapsto \\ 11/13 \cdot \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 11 & 8 & | & -5 & 1 & 0 \\ 0 & 0 & 13/11 & | & 18/11 & -8/11 & 1 \end{bmatrix} \mapsto \\ \\ -8/11 \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 8/11 & | & -5/11 & 1/11 & 0 \\ 0 & 0 & 1 & | & 18/13 & -8/13 & 11/13 \end{bmatrix} \mapsto \\ \\ 1 \uparrow \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array} 2 \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -19/13 & 7/13 & -8/13 \\ 0 & 0 & 1 & | & 18/13 & -8/13 & 11/13 \end{bmatrix} \mapsto \\ \\ \begin{bmatrix} 1 & 0 & 0 & | & -7/13 & 6/13 & -5/13 \\ 0 & 1 & 0 & | & -19/13 & 7/13 & -8/13 \\ 0 & 0 & 1 & | & 18/13 & -8/13 & 11/13 \end{bmatrix} = [1_3, A^{-1}]. \end{array}$$

$$23. A^{-1} = \begin{bmatrix} -3/22 & -1/22 & -41/22 & 3/11 \\ -15/22 & -5/22 & -51/22 & 4/11 \\ 5/22 & 9/22 & 61/22 & -5/11 \\ 15/22 & 5/22 & 73/22 & -4/11 \end{bmatrix}$$

$$24. [A, 1_2] =$$

$$-6 \begin{array}{c} \swarrow \quad \swarrow \\ \quad \searrow \end{array} \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 6 & 8 & | & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -4 & | & -6 & 1 \end{bmatrix} = [U, L^{-1}]$$

$$\text{with } A = LU. \text{ So } A = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$$

25. If $a \neq 0$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d - bc/a \end{bmatrix}.$$

If the first column of A is zero, then $A = 1_2 A = LU$. If the first row of A is zero, then $A = A 1_2 = LU$. Finally, if $a = 0 \neq bc$, then $A \neq LU$.

26. The matrix $U = A$ is upper triangular, and $A = 1_3 A = LU$.

27. This cannot be done. Suppose $A = LU$. At the position (1,1), we have $0 = A_{11} = L_{11}U_{11}$. Since the first row of A is nonzero, we conclude that $L_{11} \neq 0$. Since the first column of A is nonzero, we conclude that $U_{11} \neq 0$. Thus, $0 = A_{11} = L_{11}U_{11} \neq 0$.

28. This cannot be done, because A is invertible (see the solution of Exercise 5) and $A_{11} = 0$. See the solution of Exercise 27.

$$29. A = LU = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix}.$$

$$UL = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -1 \\ 21 & 33 & 8 \\ -50 & -100 & -25 \end{bmatrix}.$$

$$30. A = LU$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix}.$$

$$UL = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -30/11 & -1 \\ 71 & 185/11 & 8 \\ 26/11 & 104/121 & 13/11 \end{bmatrix}.$$

31. We multiply A by a lower triangular matrix E_1^{-1} on left and obtain $E_1^{-1}A$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 5 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 6 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}.$$

Then we multiply by a lower triangular matrix E_2^{-1} on left and obtain $E_2^{-1}E_1^{-1}A$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/11 & 1 & 0 \\ 0 & 16/11 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -8/11 & 10/11 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix}.$$

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Then we multiply by a lower triangular matrix E_3^{-1} on left and obtain an upper triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 73/8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -8/11 & 10/11 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix} \\ = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -8/11 & 10/11 \\ 0 & 0 & 0 & -11/4 \end{bmatrix} = R.$$

$$\text{So } A = LR \text{ with } L = E_1 E_2 E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 1/11 & 1 & 0 \\ 5 & 16/11 & 73/8 & 1 \end{bmatrix}.$$

32. Answer:

$$x = a + b^2 + c^3 - d,$$

$$y = a + b^2 - 3c^3 + 2d,$$

$$z = -a - b^2 + 2c^3 - d.$$

$$33. x = -3(19 + 2d)/8, y = (15 + 2d)/8, z = -(3 + 2d)/8$$

34. Answer:

$$x = u_1 + u_2 - u_3,$$

$$y = u_1 - 3u_2 + 2u_3,$$

$$z = -u_1 + 2u_2 - u_3.$$

$$35. x = (15u + 4v)/16, y = (11u + 4v)/16, z = -3u/4$$

$$36. x = y = 0, z = 5.$$

$$37. x = y = 1, z = 1$$

$$38. x = y = 1, z = 0.$$

$$39. x = y = 0, z = 100$$

40. It is clear that any nonzero column with at least two entries can be reduced to the first column of the identity matrix by row addition operations. By induction on the number of columns, it follows that any $m \times n$ matrix with linearly independent columns can be reduced by row addition operations to the matrix of the first n columns of 1_m provided that $m > n$. Therefore any invertible $m \times m$ matrix can be reduced by by row addition operations to the diagonal matrix with the first $m - 1$ diagonal entries being ones, and the last entry being the determinant. One row multiplication operation applied to this matrix gives 1_m .

Therefore multiplication by an invertible matrix on the left is equivalent to performing row addition operations and a row multiplication operation.

If $A = 0$, then $b = 0$, $A' = 0$ and $b' = 0$, so there is nothing to prove. Similarly, the case $A' = 0$ is trivial. Assume now that $A \neq 0$ and $A' \neq 0$.

Let B be the submatrix in $[A, b]$ such that the rows of B form a basis for the row space of A , and let B' be a similar matrix for $[A', b']$. By Theorem 6.11, $B' = DB$ and $B = D'B'$ for some matrices D, D' . We have $B = D'DB$ and $B' = DD'B'$. Since the rows of B are linearly independent, $D'D$ is the identity matrix. Since the rows of B' are linearly independent, DD' is the identity matrix. So D is invertible, hence B, B' have the same size. So row operations on A allows us to change the rows of B to the rows of B' . Now by row addition operations we can make the other rows of $[A, b]$ (if any) equal to remaining rows of $[A', b']$ (if any). A row permutation operation finishes the job.

41. We use the parts of the previous solution. In particular, it is clear that the rank of the matrices of $[A, b]$ and $[A', b']$ are the same. By row addition operations we can make the last $m - m'$ rows of $[A, b]$ to be zeros. Then, as shown in the previous solution, we can manipulate the first m' rows to be the rows of $[A', b']$ by row addition operations and a row multiplication operation.

The only thing remaining to show is how to replace a row multiplication operation by row addition operations in the presence of a zero row. Here is how this can be done:

$$1 \begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} \mapsto (d-1) \begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix} \begin{bmatrix} r \\ r \end{bmatrix} \mapsto -1/d \begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix} \begin{bmatrix} dr \\ r \end{bmatrix} \mapsto \begin{bmatrix} dr \\ 0 \end{bmatrix},$$

where multiplication of a row r by a nonzero number d is accomplished by three addition operations.

Chapter 3. Tableaux and Pivoting

§7. Standard and Canonical Forms for Linear Programs

1. Set $u = y + 1 \geq 0$. Then $f = 2x + 3y = 2x + 3v - 3$ and $x + y = x + u - 1$. A canonical form is

$$-f = -2x - 3v + 3 \rightarrow \min, x + u \leq 6, u, x \geq 0.$$

A standard form is

$$-f = -2x - 3v + 3 \rightarrow \min, x + u + v = 6, u, v, x \geq 0$$

with a slack variable $v = 6 - x - u \geq 0$.

2. Excluding $y = x + 1$ and using $y \geq 1$, we obtain the canonical form

$$-x \rightarrow \min, 2x \leq 8, x \geq 0.$$

Introducing a slack variable $z = 8 - 2x$, we obtain the standard form

$$-x \rightarrow \min, 2x + z = 8, x \geq 0, z \geq 0.$$

3. We solve the equation for x_3 :

$$x_3 = 3 - 2x_2 - 3x_4$$

and exclude x_3 from the LP:

$$x_1 - 7x_2 + 3 \rightarrow \min, x_1 - x_2 + 3x_4 \geq 3, \text{ all } x_i \geq 0.$$

A canonical form is

$$x_1 - 7x_2 + 3 \rightarrow \min, -x_1 + x_2 - 3x_4 \leq -3, \text{ all } x_i \geq 0.$$

A standard form is

$$x_1 - 7x_2 + 3 \rightarrow \min, -x_1 + x_2 - 3x_4 + x_5 = -3, \text{ all } x_i \geq 0$$

with a slack variable $x_5 = x_1 - x_2 + 3x_4 - 3$.

4. We can solve the given equation for x_4 :

$$x_4 = 1 - 2x_2/3 - x_3/3.$$

Then exclude x_4 from our LP. The objective function f becomes $x_1 - 7x_2 + 3$. The constraint $x_1 - x_2 + 3x_4 \geq 3$ becomes $x_1 - 3x_2 - x_3 \geq 0$. So a canonical form for our LP is

$$-f = -x_1 + 7x_2 - 3 \rightarrow \min, -x_1 + 3x_2 + x_3 \leq 0; x_1, x_2, x_3 \geq 0.$$

To get a standard form we introduce a slack variable x_5 :

$$-f = -x_1 + 7x_2 - 3 \rightarrow \min, -x_1 + 3x_2 + x_3 + x_5 = 0; x_1, x_2, x_3, x_5 \geq 0.$$

Note that we cannot have the variable x_4 in a normal or standard form unless we know that it takes only nonnegative values.

5. Set $t = x + 1 \geq 0, u = y - 2 \geq 0$. The objective function is $x + y + z = t + u + z + 1$. Then a standard and a canonical form for our problem is

$$x + u + z + 1 \rightarrow \min; t, u, z \geq 0.$$

6. This mathematical program has exactly two optimal solution, but the set of optimal solutions of any LP is convex and hence cannot consist of exactly two optimal solutions (cf. Exercise 19 in §6.). Each of two optimal solution can be the optimality region for a linear program. For example, $\min = -26$ at $x = 1, y = -3, z = 0$ is the only answer for the linear program $-26 \rightarrow \min, x = 1, y = -3, z = 0$.

7. Using standard tricks, a canonical form is

$$-x \rightarrow \min, x \leq 3, -x \leq -2, x \geq 0.$$

A standard form is

$$-x \rightarrow \min, x + u = 3, -x + v = -2; x, u, v \geq 0$$

with two slack variables.

8. Excluding $y = 1 - x$ from the LP, we obtain

$$f = -x + z + 2 \rightarrow \max, z \geq 0.$$

Writing $x = u - v$ and replacing f by $-f$, we obtain a normal and standard form:

$$-f = u - v - z - 2 \rightarrow \min; u, v, z \geq 0.$$

It is clear that the program is unbounded.

9. One of the given equations reads

$$-5 - x - z = 0,$$

which is inconsistent with given constraints $x, z \geq 0$. So we can write very short canonical and standard forms:

$$0 \rightarrow \min, 0 \leq -1; x, y, z \geq 0 \text{ and } 0 \rightarrow \min, 0 = 1; x, y, z \geq 0.$$

10. The first matrix product is not defined.

11. Set $x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}^T]$ and $c = [3, -1, 1, 3, 1, -5, 1, 3, 1]$. Using standard tricks, we obtain the canonical form

$$cx \rightarrow \min, Ax \leq b, x \geq 0$$

with

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -2 & 3 & 1 \\ 2 & -2 & -2 & 2 & 3 & -1 & -2 & 1 & 1 \\ -2 & 2 & 2 & -2 & -3 & 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 & 3 & -1 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

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and $b = [-3, -1, 2, -2, 0, 0]^T$.

Excluding a couple of variables using the two given equations, we would get a canonical form with two variables and two constraints less. A standard form can be obtained from the canonical form by introducing a column u of slack variables:

$$cx \rightarrow \min, Ax + u = b, x \geq 0, u \geq 0.$$

§8. Pivoting Tableaux

$$1. \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \wedge & \wedge & \wedge & \wedge & \wedge \\ a & b & c & d & e & 1 \\ \left[\begin{array}{ccccc} .3 & 1.2 & .7 & 3.5 & 5.5 & -50 \\ 73 & 96 & 20253 & 890 & 279 & -4000 \\ 9.6 & 7 & 19 & 57 & 22 & -1000 \\ 10 & 15 & 5 & 60 & 8 & 0 \end{array} \right] \begin{array}{l} = u_1 \geq 0 \\ = u_2 \geq 0 \\ = u_3 \geq 0 \\ = C \rightarrow \min \end{array} \end{array}$$

$$2. \begin{array}{ccccc} a & b & c & d \\ \left[\begin{array}{cccc} .9 & .8 & .7 & .6 \\ 1 & 1 & 1 & 1 \\ 1.2 & 1.4 & 1.7 & 1.9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} = .75 \\ = 1 \\ = C \rightarrow \min \\ \geq 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \end{array} \end{array}$$

$$3. A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ -1 & 0 & 0 & 2 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

4. Canonical form:

$$z + 2 \rightarrow \min,$$

$$-3x - 2y - z - 3 \leq 0,$$

$$-x + 3y \leq 3,$$

$$x - 3y \leq -3,$$

$$-x - 2y + 2z \leq 0;$$

$$x, y, z \geq 0.$$

Standard form:

$$z + 2 \rightarrow \min,$$

$$-3x - 2y - z - 3 + u = 0,$$

$$-x + 3y = 3,$$

$$-x - 2y + 2z + v = 0;$$

$$x, y, z, u, v \geq 0.$$

5. Canonical form:

$$y - 5z + 2 \rightarrow \min,$$

$$-3x - y + 5z \leq 3,$$

$$-x - y \leq -10,$$

$$x + y \leq 10,$$

$$-2y + 10z \leq -7;$$

$$x, y, z \geq 0.$$

Standard form:

$$y - 5z + 2 \rightarrow \min,$$

$$-3x - y + 5z + u = 3,$$

$$x + y = 10,$$

$$-2y + 10z + v = -7;$$

$$x, y, z, u, v \geq 0.$$

6. Canonical form: $-z + 6 \rightarrow \min$,

$$-2y + z \leq 0, 2y - z \leq 0, x + 3y \leq -3, 2y + 2z \leq 0; x, y, z \geq 0.$$

Standard form: $-z + 6 \rightarrow \min$,

$$-2y + z = 0, x + 3y + u = -3, 2y + 2z + v = 0; x, y, z, u, v \geq 0.$$

It is clear that the program is infeasible (the constraints $x \geq 0, y \geq 0, x + 3y \leq -3$ are inconsistent).

7. Canonical form: $-x + z + 2 \rightarrow \min$,

$$-3x - 2y - z \leq 2, x - 3y \leq 1, 2y - 2z \leq 0; x, y, z \geq 0.$$

Standard form: $-x + z + 2 \rightarrow \min$,

$$-3x - 2y - z + u = 2, x - 3y + v = 1, 2y - 2z + w = 0;$$

$$x, y, z, u, v, w \geq 0.$$

8.

$$(a) \quad \begin{array}{cccc|l} x' & x'' & u & 1 & \\ \hline -3 & -2 & 1 & -b & = y \\ -1 & 3 & 0 & 1 & = a \\ -6 & -6 & 2 & -2b & = x' \\ -3 & -2 & 1 & 2 - b & = z \end{array}$$

(b)

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$$\begin{array}{cccc} x' & x'' & z & 1 \\ \left[\begin{array}{cccc} 3 & 2 & 1 & b-2 \\ -1 & 3 & 0 & 1 \\ 0 & -2 & 2 & -4 \\ 0 & 0 & 1 & -2 \end{array} \right] & \begin{array}{l} = u \\ = a \\ = x' \\ = y \end{array} \end{array}$$

9. The matrix is not square.

$$10. \begin{array}{cccc} 1 & a & 3 & x \\ \left[\begin{array}{cccc} 1^* & 0 & b & a \\ -1 & 2 & 3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = z \end{array} & \mapsto \end{array}$$

$$\begin{array}{cccc} y & a & 3 & x \\ \left[\begin{array}{cccc} 1 & 0 & -b & -a \\ -1 & 2 & 3+b & 1+a \end{array} \right] & \begin{array}{l} = 1 \\ = z \end{array} \end{array}$$

$$11. \begin{array}{cccc} z & a & 3 & x \\ \left[\begin{array}{cccc} 1 & 2 & b+3 & a+1 \\ -1 & 2 & 3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = 1 \end{array} \end{array}$$

$$12. \begin{array}{cccc} 1 & a & 3 & z \\ \left[\begin{array}{cccc} 1+a & -2a & b-3a & a \\ 1 & -2 & -3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = x \end{array} \end{array}$$

$$13. \begin{array}{c} 2 \\ \left[1/5 \right] \end{array} = x$$

$$14. \begin{array}{cccc} 1 & a & 0 & x & x \\ \left[\begin{array}{ccccc} 1 & 0 & b & a & -3 \\ -1 & 2^* & 3 & 1 & 0 \end{array} \right] & \begin{array}{l} = y \\ = z \end{array} & \mapsto \end{array}$$

$$\begin{array}{ccccc} 1 & z & 0 & x & x \\ \left[\begin{array}{ccccc} 1 & 0 & b & a & -3 \\ 1/2 & 1/2 & -3/2 & -1/2 & 0 \end{array} \right] & \begin{array}{l} = y \\ = a \end{array} \end{array}$$

$$15. \begin{array}{ccccc} 1 & u & 0 & x & 1 \\ \left[\begin{array}{ccccc} 1 & 0 & b & a & -3 \\ 0 & 1 & 0 & 0 & -1 \\ 1/2 & 1/2 & -3/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1 \end{array} \right] & \begin{array}{l} = y \\ = z \\ = 0 \\ = v \end{array} \end{array}$$

$$\begin{array}{l}
16. \quad \begin{array}{ccccc} x_1 & x_2 & x_8 & x_4 & 1 \\ \left[\begin{array}{ccccc} 4/3 & -2/3 & 1/3 & 2/3 & -3 \\ -1/3 & -4/3 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1/3 & -2/3 & 1/3 & -1/3 & 0 \\ -1 & 1 & 0 & 1 & 1 \end{array} \right] & \begin{array}{l} = x_5 \\ = x_6 \\ = x_7 \\ = x_3 \\ = v \end{array} \end{array} \\
\\
17. \quad \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_7 & x_6 & 1 \\ \left[\begin{array}{ccccccc} 1/3 & 0 & 1/3 & 1/3 & -1/3 & 1/3 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -2/3 & 2 & 10/3 & 4/3 & -1/3 & 1/3 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -2/3 & 1 & 1/3 & 4/3 & -1/3 & 7/3 & 3 \end{array} \right] & \begin{array}{l} = x_5 \\ = x_8 \\ = x_9 \\ = x_{10} \\ = v \end{array} \end{array}
\end{array}$$

18. Let us show that every column b of A equals to the corresponding column b' of A' . We set the corresponding variable on the top to be 0, and the other variables on the top to be zeros. Then the variables on the side take certain values, namely, $y = b = b'$.

§9. Standard Row Tableaux

1. Passing from the standard row tableau on page 95 to the canonical form (i.e., dropping the slack variables), we obtain a Linear program with one variable: $y/2 - 15/2 \rightarrow \min$,

$$16y - 26 \geq 0, -3y/2 + 15/2 \geq 0, 3y/2 - 15/2 \geq 0, y \geq 0.$$

We rewrite our constraints: $y \geq 0, 13/8, 5; y \leq 5$, so $y = 5$. In terms of the standard tableau, our answer is

$$\min(-z) = -5 \text{ at } y = 5, w_1 = 54, w_2 = w_3 = 0.$$

In terms of the original variables, our answer is

$$\max(z) = 5 \text{ at } u = -3, v = -4, x = -1, y = 5.$$

$$2. \quad \begin{array}{ccc} x & y & 1 \\ \left[\begin{array}{ccc} -4 & -5 & 7 \\ -2 & -3 & 0 \end{array} \right] & \begin{array}{l} = u \\ = -P \rightarrow \min \end{array} \end{array}$$

with a slack variable $u = 7 - 4x - 5y$

$$3. \quad \begin{array}{cccc} x' & x'' & y' & y'' & 1 \\ \left[\begin{array}{cccc} -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \end{array} \right] & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = -x \rightarrow \min \end{array} \end{array}$$

with $x = x' - x'', y = y' - y''$ and slack variables u_i .

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4. We introduce a slack variable u_1 for the first constraint and slack variables u_2, u_3 for the third constraint (replaced by two inequalities). Here is the resulting standard tableau:

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & -1 \\ \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -3 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{array} \right] & \begin{array}{l} = u_1 \\ = x_8 \\ = u_2 \\ = u_3 \\ = v \rightarrow \min \end{array} \end{array}$$

$x_9, x_{10} \geq 0.$

To get a smaller standard tableau, we can solve the equation for x_5 and exclude x_5 from the first and fourth rows. In other words we pivot on third entry 1 in the x_5 -column and then drop the column with 0 on top. This gives the standard tableau

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_6 & -1 \\ \left[\begin{array}{cccccc|c} -2 & 6 & 10 & 4 & 1 & 0 \\ -1 & 0 & 2 & 1 & 1 & -2 \\ 1 & -2 & -3 & -1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 2 & 3 \end{array} \right] & \begin{array}{l} = u_1 \\ = x_8 \\ = x_5 \\ = v \rightarrow \min \end{array} \end{array}$$

$x_9, x_{10} \geq 0.$

5. We multiply the last row by -1 and remove the the second and third rows from the tableau

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\ \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & -1 & -2 & -3 \end{array} \right] & \begin{array}{l} = x_7 \\ = x_2 \\ = -v \rightarrow \min \end{array} \end{array}$$

$$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 3x_3 + x_4 + x_5.$$

The tableau is not standard because x_2 occurs twice. We pivot on 1 in the x_2 -row and x_6 -column:

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\ \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1^* & 1 \\ 1 & -1 & 0 & -1 & -1 & -2 & -3 \end{array} \right] & \begin{array}{l} = x_7 \\ = x_2 \\ = -v \rightarrow \min \end{array} \end{array} \mapsto$$

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_2 & 1 \\ \left[\begin{array}{cccccc|c} 1 & -2 & -2 & 0 & -3 & 1 & -1 \\ 0 & -2 & -3 & -1 & 0 & 1 & -1 \\ 1 & 3 & 6 & 1 & -1 & -2 & -1 \end{array} \right] & \begin{array}{l} = x_7 \\ = x_6 \\ = -v \rightarrow \min \end{array} \end{array}.$$

Now we combine two x_2 from the top and obtain the standard tableau

$$\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & 1 \\
\left[\begin{array}{cccccc} 1 & -1 & -2 & 0 & -3 & -1 \\ 0 & -1 & -3 & -1 & 0 & -1 \\ 1 & 1 & 6 & 1 & -1 & -1 \end{array} \right] & & & & & \\
& & & & & & = x_7 \\
& & & & & & = x_6 \\
& & & & & & = -v \rightarrow \min
\end{array}$$

The equations

$$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 5x_3 + x_4 + x_5$$

relate this LP with the original LP. After we solve the program without x_8 and x_9 , we complete the answer with the values for x_8 and x_9 . By the way, looking at the equation for x_6 (the second row of the standard tableau), we see the the program is infeasible.

6. We combine the first and the third column, subtract the first row from the second one, and multiply the third row by -1:

$$\begin{array}{cccccc}
x_1 & x_2 & x_4 & x_5 & x_6 & 1 \\
\left[\begin{array}{cccccc} -1 & 0 & 1 & -3 & 1 & 0 \\ -4 & 0 & 0 & 3 & 0 & -2 \\ 9 & -2 & -1 & -1 & 0 & 0 \\ -11 & 2 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 & 2 & 3 \end{array} \right] & & & & & \\
& & & & & & = x_7 \\
& & & & & & = 0 \\
& & & & & & \geq 0 \\
& & & & & & = x_3 \\
& & & & & & = v \rightarrow \min
\end{array}$$

We introduce a slack variable u for the third row, pivot on -4 , and drop the first column, and obtain the standard tableau

$$\begin{array}{cccccc}
x_2 & x_4 & x_5 & x_6 & 1 \\
\left[\begin{array}{cccccc} 0 & 1 & -15/4 & 1 & 1/2 \\ 0 & 0 & 3/4 & 0 & -1/2 \\ -2 & -1 & 23/4 & 0 & -9/2 \\ 2 & 1 & -33/4 & 1 & 13/2 \\ 1 & 1 & 1/4 & 2 & 7/2 \end{array} \right] & & & & & \\
& & & & & & = x_7 \\
& & & & & & = x_1 \\
& & & & & & = u \\
& & & & & & = x_3 \\
& & & & & & = v \rightarrow \min
\end{array}$$

7. We pivot on the first 1 in the first row and then on 3 in the second row:

$$\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\
\left[\begin{array}{cccccc} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \end{array} \right] & & & & & & \\
& & & & & & = x_2 \\
& & & & & & = x_4 \\
& & & & & & = v \rightarrow \min
\end{array} \mapsto$$

$$\begin{array}{cccccc}
x_2 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\
\left[\begin{array}{cccccc} 1 & 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & 3^* & 2 & -3 & 2 & -2 \\ -1 & 2 & 4 & 2 & -2 & 1 & 0 \end{array} \right] & & & & & & \\
& & & & & & = x_1 \\
& & & & & & = x_4 \\
& & & & & & = v \rightarrow \min
\end{array} \mapsto$$

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$$\begin{array}{ccccccc} x_2 & x_2 & x_4 & x_4 & x_5 & x_6 & 1 \\ \left[\begin{array}{ccccccc} 2/3 & 0 & -1/3 & -1/3 & 2 & -1/3 & -2/3 \\ 1/3 & 0 & 1/3 & -2/3 & 1 & -2/3 & 2/3 \\ 1/3 & 2 & 4/3 & -2/3 & 2 & -5/3 & 8/3 \end{array} \right] & \begin{array}{l} = x_1 \\ = x_3 \\ = v \rightarrow \min \end{array} \end{array}.$$

Now we combine two x_2 -columns and two x_4 -columns and obtain the standard tableau

$$\begin{array}{ccccccc} x_2 & x_4 & x_5 & x_6 & 1 \\ \left[\begin{array}{ccccccc} 2/3 & -2/3 & 2 & -1/3 & -2/3 \\ 1/3 & -1/3 & 1 & -2/3 & 2/3 \\ 7/3 & 2/3 & 2 & -5/3 & 8/3 \end{array} \right] & \begin{array}{l} = x_1 \\ = x_3 \\ = v \rightarrow \min \end{array} \end{array}.$$

Do not forget the constraints $x_7, x_8, x_9, x_{10} \geq 0$ outside the tableaux.

8. We make three pivot steps:

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & -4 \\ \left[\begin{array}{ccccccc} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1^* & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{array} \right] & \begin{array}{l} \geq 0 \\ = 1 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 & -4 \\ \left[\begin{array}{ccccccc} 2 & 0 & -1 & 0 & -3 & 1 & 2 \\ 1 & 0 & -2 & -1 & 0 & 1 & 2 \\ -1 & 2 & 3 & 1 & 1^* & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 & 3 \\ 1 & 1 & -4 & -1 & 1 & 2 & 7 \end{array} \right] & \begin{array}{l} \geq 0 \\ = x_6 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & 0 & 1 & -4 \\ \left[\begin{array}{ccccccc} -1 & 6 & 8 & 3 & -3 & 1 & 2 \\ 1 & 0 & -2 & -1 & 0 & 1 & 2 \\ 1 & -2 & -3 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1^* & 0 & 0 & 1 & 3 \\ 2 & -1 & -7 & -2 & 1 & 2 & 7 \end{array} \right] & \begin{array}{l} \geq 0 \\ = x_6 \\ = x_5 \\ = x_1 \\ = v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{ccccccc} x_1 & x_2 & x_1 & x_4 & 0 & 1 & -4 \\ \left[\begin{array}{ccccccc} -1 & -10 & 8 & 3 & -3 & -7 & -22 \\ 1 & 4 & -2 & -1 & 0 & 3 & 8 \\ 1 & 4 & -3 & -1 & 1 & 3 & 9 \\ 0 & -2 & 1 & 0 & 0 & -1 & -3 \\ 2 & 13 & -7 & -2 & 1 & 9 & 28 \end{array} \right] & \begin{array}{l} \geq 0 \\ = x_6 \\ = x_5 \\ = x_3 \\ = v \rightarrow \min. \end{array} \end{array}$$

Now we introduce a slack variable u for the first row, combine some columns, and obtain the following standard tableau:

$$\begin{array}{cccc|l} x_1 & x_2 & x_4 & 1 & \\ \hline \begin{bmatrix} 7 & -10 & 3 & -29 \\ -1 & 4 & -1 & 11 \\ -2 & 4 & -1 & 12 \\ 1 & -2 & 0 & -4 \\ -5 & 13 & -2 & 37 \end{bmatrix} & & & & \begin{array}{l} = u \\ = x_6 \\ = x_5 \\ = x_3 \\ = v \rightarrow \min. \end{array} \end{array}$$

9. We pivot the three zeros from the right margin to the top and drop the corresponding columns:

$$\begin{array}{cccccc|l} x_7 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 & \\ \hline \begin{bmatrix} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} & & & & \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \end{array} \mapsto$$

$$\begin{array}{cccccc|l} x_2 & x_3 & x_4 & x_5 & x_6 & 1 & \\ \hline \begin{bmatrix} 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 3 & 2 & -3 & 2 & -2 \\ 2 & 4 & 2 & -2 & 1^* & 0 \\ 2 & 4 & 2 & -3 & 2 & 1 \\ 1 & 1 & 2 & -2 & 3 & 3 \end{bmatrix} & & & & \begin{array}{l} = x_7 \\ = 0 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \end{array} \mapsto$$

$$\begin{array}{cccc|l} x_2 & x_3 & x_4 & x_5 & 1 & \\ \hline \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ -4 & -5 & -2 & 1^* & -2 \\ -2 & -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 1 & 1 \\ -5 & -11 & -4 & 4 & 3 \end{bmatrix} & & & & \begin{array}{l} = x_7 \\ = 0 \\ = x_6 \\ = x_1 \\ = v \rightarrow \min \end{array} \end{array} \mapsto$$

$$\begin{array}{cccc|l} x_2 & x_3 & x_4 & 1 & \\ \hline \begin{bmatrix} 6 & 8 & 3 & 2 \\ 4 & 5 & 2 & 2 \\ 6 & 6 & 2 & 4 \\ 2 & 1 & 0 & 3 \\ 11 & 9 & 4 & 11 \end{bmatrix} & & & & \begin{array}{l} = x_7 \\ = x_5 \\ = x_6 \\ = x_1 \\ = v \rightarrow \min. \end{array} \end{array}$$

Now we take the first equation

$$x_7 = 6x_2 + 8x_3 + 3x_4 + 2$$

outside the table (since x_7 is not required to be ≥ 0) and obtain the standard tableau

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$$\begin{array}{cccc} x_2 & x_3 & x_4 & 1 \\ \left[\begin{array}{cccc} 4 & 5 & 2 & 2 \\ 6 & 6 & 2 & 4 \\ 2 & 1 & 0 & 3 \\ 11 & 9 & 4 & 11 \end{array} \right] & \begin{array}{l} = x_5 \\ = x_6 \\ = x_1 \\ = v \rightarrow \min. \end{array} \end{array}$$

10. We take the row

$$x_7 = -x_1 + 2x_2 + 3x_3 + x_4 + x_5$$

out the tableau, multiply the last row and column by -1, and add the first row to the second row with coefficient -2. We obtain

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ -3 & 0 & 0 & -1 & 6 & -2 & 2 \\ -1 & 2 & 3 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & -1 & -2 & 3 \end{array} \right] & \begin{array}{l} = x_1 \\ = 0 \\ = x_2 \\ = -v \rightarrow \min. \end{array} \end{array}$$

Now we take three pivot steps

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\ \left[\begin{array}{cccccc} 1 & 0 & 1^* & 1 & -3 & 1 & 0 \\ -3 & 0 & 0 & -1 & 6 & -2 & 2 \\ -1 & 2 & 3 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & -1 & -2 & 3 \end{array} \right] & \begin{array}{l} = x_1 \\ = 0 \\ = x_2 \\ = -v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_1 & x_4 & x_5 & x_6 & 1 \\ \left[\begin{array}{cccccc} -1 & 0 & 1 & -1 & 3 & -1 & 0 \\ -3 & 0 & 0 & -1^* & 6 & -2 & 2 \\ -4 & 2 & 3 & -2 & 9 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & 3 \end{array} \right] & \begin{array}{l} = x_3 \\ = 0 \\ = x_2 \\ = -v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_1 & 0 & x_5 & x_6 & 1 \\ \left[\begin{array}{cccccc} 2 & 0 & 1 & 1 & -3 & 1 & -2 \\ -3 & 0 & 0 & -1 & 6 & -2 & 2 \\ 2 & 2 & 3 & 2 & -3 & 2^* & -5 \\ -2 & -1 & 0 & -1 & 5 & -4 & 5 \end{array} \right] & \begin{array}{l} = x_3 \\ = x_4 \\ = x_2 \\ = -v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_1 & 0 & x_5 & x_2 & 1 \\ \left[\begin{array}{cccccc} 1 & -1 & -1/2 & 0 & -3/2 & 1/2 & 1/2 \\ -1 & 2 & 3 & 1 & 3 & -1 & -3 \\ -1 & -1 & -3/2 & -1 & 3/2 & 1/2 & 5/2 \\ 2 & 3 & 6 & 3 & -1 & -2 & -5 \end{array} \right] & \begin{array}{l} = x_3 \\ = x_4 \\ = x_6 \\ = -v \rightarrow \min. \end{array} \end{array}$$

Now we drop the column with 0 on top, combine two columns with x_1 on top, combine two columns with x_2 on top, which gives the following standard tableau:

$$\begin{array}{cccc}
x_1 & x_2 & x_5 & 1 \\
\left[\begin{array}{cccc}
-1/2 & -1/2 & -3/2 & 1/2 \\
2 & 1 & 3 & -3 \\
-5/2 & -1/2 & 3/2 & 5/2 \\
8 & 1 & -1 & -5
\end{array} \right] & & & \\
& & & = x_3 \\
& & & = x_4 \\
& & & = x_6 \\
& & & = -v \rightarrow \min .
\end{array}$$

Chapter 4. Simplex Method

§10. Simplex Method, Phase 2

1. The tableau is optimal, so the basic solution is optimal:

$\min = 0$ at $a = b = c = d = 0$,

$y_1 = 0.4, y_2 = 0.4, y_3 = 0, y_4 = 0.5, y_5 = 1, y_6 = 0.1$.

2. The y_2 -row is bad. The program is infeasible.

3. The first column is bad. However since the tableau is not feasible, this is not sufficient to conclude that the program is unbounded. Still we set $z_2 = z_3 = z_4 = 0$, and see what happens as $z_1 \rightarrow \infty$. We have $y_1 = 0.4 \geq 0, y_2 = 3z_1 + 0.4 \geq 0, y_3 = 0.6z_1 \geq 0, y_4 = 0.6z_1 + 0.5 \geq 0, y_5 = 0.1z_1 - 0.1 \geq 0, y_6 = 0.1 \geq 0$ for $z_1 \geq 1$, and the objective function $-11z_1 \rightarrow -\infty$. So $\min = -\infty$.

4. False. The converse is true.

5. True

6. True

7. First we write the program in a standard tableau and then we apply the simplex method (Phase 2):

$$\begin{array}{cccc|cl}
 x_1 & x_2 & x_3 & 1 & & \\
 \left[\begin{array}{cccc}
 -40^* & -20 & -60 & 1200 \\
 -4 & -1 & -6 & 300 \\
 -0.2 & -0.7 & -2 & 40 \\
 -100 & -100 & -800 & 8000 \\
 -0.1 & -0.3 & -0.8 & 8 \\
 -2 & -3 & -7 & 0
 \end{array} \right] & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \rightarrow \min
 \end{array} & \mapsto \\
 \\
 u_1 & x_2 & x_3 & 1 & & \\
 \left[\begin{array}{cccc}
 -0.025 & -0.5 & -1.5 & 30 \\
 0.1 & 1 & 0 & 180 \\
 0.005 & -0.6 & -1.7 & 34 \\
 2.5 & -50 & -650 & 5000 \\
 0.0025 & -0.25^* & -0.65 & 5 \\
 0.05 & -2 & -4 & -60
 \end{array} \right] & \begin{array}{l} = x_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \rightarrow \min
 \end{array} & \mapsto
 \end{array}$$

$$\begin{array}{ccccccc}
u_1 & u_5 & x_3 & 1 & & & \\
\left[\begin{array}{cccc} -0.03 & 2 & -0.2 & 20 \\ 0.11 & -4 & -2.6 & 200 \\ 0.001 & 2.4 & -0.14 & 22 \\ 2 & 200 & -520 & 4000 \\ 0.01 & -4 & -2.6 & 20 \\ 0.03 & 8 & 1.2 & -100 \end{array} \right] & & & & & \\
& & & & = x_1 & & \\
& & & & = u_2 & & \\
& & & & = u_3 & & \\
& & & & = u_4 & & \\
& & & & = x_2 & & \\
& & & & = -P \rightarrow \min. & &
\end{array}$$

This tableau is optimal, so

$$\max(P) = 100 \text{ at } x_1 = 20, x_2 = 20, x_3 = 0.$$

The zero values for the nonbasic slack variables u_1 and u_5 indicate that the corresponding resource limits are completely used (no slack there). The other resources are not completely used; some reserves left.

8. $\min = -\infty$. For example, we set $x = y = 0, v = -1$. Then the objective function $z \rightarrow -\infty$ as $z \rightarrow -\infty$.

9. First we solve the system of linear equations for a, b and the objective function f and hence obtain the standard tableau

$$\begin{array}{ccc}
c & d & 1 \\
\left[\begin{array}{ccc} 1 & 2 & -0.5 \\ -2 & -3 & 1.5 \\ 0.1 & 0.1 & 1.5 \end{array} \right] & & \\
& & = a \\
& & = b \\
& & = f \rightarrow \min.
\end{array}$$

The tableau is not row feasible so we cannot apply Phase 2. Until we learn Phase 1, we can pivot at random:

$$\begin{array}{ccc}
c & d & 1 \\
\left[\begin{array}{ccc} 1^* & 2 & -0.5 \\ -2 & -3 & 1.5 \\ 0.1 & 0.1 & 1.5 \end{array} \right] & & \\
& & = a \\
& & = b \\
& & = f \rightarrow \min
\end{array} \mapsto$$

$$\begin{array}{ccc}
a & d & 1 \\
\left[\begin{array}{ccc} 1 & -2 & 0.5 \\ -2 & 1 & 0.5 \\ 0.1 & -0.1 & 1.55 \end{array} \right] & & \\
& & = c \\
& & = b \\
& & = f \rightarrow \min.
\end{array}$$

Now the tableau is feasible, and we can use Phase 2:

$$\begin{array}{ccc}
a & d & 1 \\
\left[\begin{array}{ccc} 1 & -2^* & 0.5 \\ -2 & 1 & 0.5 \\ 0.05 & -0.1 & 1.55 \end{array} \right] & & \\
& & = c \\
& & = b \\
& & = f \rightarrow \min
\end{array} \mapsto$$

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$$\begin{array}{ccc|c} a & c & 1 & \\ \hline 0.5 & -0.5 & 0.25 & = d \\ -1.5 & -0.5 & 0.75 & = b \\ 0.15 & 0.05 & 1.525 & = f \rightarrow \min. \end{array}$$

The tableau is optimal, so

$\min = 1.525$ at $a = 0, b = 0.75, c = 0, d = 0.25$.

10. First we write the program in the standard tableau

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & 1 & \\ \hline 1 & -1 & 0 & 1 & 3 & = x_5 \\ 1 & -1 & 2 & 1 & 1 & = x_6 \\ 0 & 1 & 2 & 0 & -2 & \rightarrow \min \end{array}$$

The tableau is optimal, so $\min = -2$ at $x_1 = x_2 = x_3 = x_4 = 0, x_5 = 3, x_6 = 1$.

11. Set $f = x_2 + 2x_3 - 2$ (the objective function). We write the program in the standard tableau

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & 1 & \\ \hline 1 & 1 & 0 & 1 & 3 & = x_5 \\ 1 & 1 & 2 & 1 & 1 & = x_6 \\ 0 & -1 & -2 & 0 & 2 & = -f \rightarrow \min \end{array}$$

The tableau is feasible, and two columns are bad (namely, the x_2 -column and x_3 -column), so the program is unbounded ($\max(f) = \infty$).

12. Set $f = x_2 + 2x_3 + 2$ (the objective function). We write the program in the standard tableau

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & 1 & \\ \hline 1 & 1 & 0 & 1 & -3 & = x_5 \\ 1 & 1 & 2 & 1 & -1 & = x_6 \\ 0 & -1 & 2 & 0 & -2 & = -f \rightarrow \min \end{array}$$

We set $x_1 = 3$ and obtain a feasible tableau

$$\begin{array}{cccc|c} x_2 & x_3 & x_4 & 1 & \\ \hline 1 & 0 & 1 & 0 & \\ 1 & 2 & 1 & 2 & \\ -1 & 2 & 0 & -2 & = -f \rightarrow \min \end{array}$$

with the first column bad. So the program is unbounded.

13. If the row without the last entry is nonnegative, then the tableau is optimal; else the LP is unbounded.

14. Either the tableau is optimal, or the row is bad, or one pivot step according to the simplex method makes the tableau optimal.

§11. Simplex Method, Phase 1

1. The second row (v -row) is bad, so the LP is infeasible.
2. The tableau is optimal, so the basic solution is optimal:
 $\min = 0$ at $x = y = z = 0, u = 2, v = 0$.
 This is the only optimal solution.

3. This is a feasible tableau with a bad column (the z -column).
 So the LP is unbounded (z and hence w can be arbitrarily large).

4. First we make the tableau standard by scaling the second and fourth columns:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1 & 0 & -1 & -4 \\ 2 & 1 & 0 & 6 \\ 0 & -2 & -1 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array}$$

According to the simplex method, we pivot on 1 in the first row:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1^* & 0 & -1 & -4 \\ 2 & 1 & 0 & 6 \\ 0 & -2 & -1 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} a & b & d & 1 \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 4 \\ 2 & 1 & 2 & 14 \\ 0 & -2 & -1 & 0 \end{array} \right] & = c \\ & = e \\ & \rightarrow \min \end{array}.$$

Now we have a feasible tableau with two bad columns (b - and d -columns), so the program is unbounded ($\min = -\infty$).

5. The tableau is standard. According to the simplex method, we pivot on 1 in the first row:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1^* & 2 & 3 & -1 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 0 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} d & b & c & 1 \\ \left[\begin{array}{cccc} 1 & -2 & -3 & 1 \\ 2 & -4 & -5 & 5 \\ -1 & 3 & 3 & -1 \end{array} \right] & = a \\ & = e \\ & \rightarrow \min \end{array}$$

The tableau is feasible, and the d -column is bad, so the program is unbounded ($\min = -\infty$).

6. We scale the last column to get the standard tableau

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$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 3 \\ 2 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array} .$$

This is a feasible tablea with a bad column (the fist column), so $\min = -\infty$.

7. We scale the last column and then pivot on the first 1 in the first row to get both c on the top:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1^* & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 \end{array} \right] & = c \\ & = d \\ & = f \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} c & b & c & 1 \\ \left[\begin{array}{cccc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & -1 \end{array} \right] & = a \\ & = d \\ & = f \rightarrow \min . \end{array}$$

Now we combine two c -columns and obtain the standard tableau

$$\begin{array}{ccc} b & c & 1 \\ \left[\begin{array}{ccc} 0 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{array} \right] & = a \\ & = d \\ & = f \rightarrow \min . \end{array}$$

The first row is bad, so the program is infeasible. In fact, the first constraint in the original tableau is inconsistent with the constraint $a \geq 0$.

8. We scale the last column and obtain the standard tableau

$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 \end{array} \right] & = e \\ & = d \\ & = f \rightarrow \min . \end{array}$$

Then we pivot on the first 1 in the second row:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1^* & 1 & -1 \\ 1 & 2 & 3 & 0 \end{array} \right] & = e \\ & = d \\ & = f \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} a & d & c & 1 \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 1 & 2 \end{array} \right] & = e \\ & = b \\ & = f \rightarrow \min . \end{array}$$

The tableau is optimal, so the basic solution is optimal:

$\min = 2$ at $a = d = c = 0, e = 1, b = 1$.

9. True

10. False

11. We use the simplex method:

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & 1 & \\ 1^* & 0 & -2 & -3 & -1 & = x_5 \\ -1 & 1 & 1 & 1 & 1 & = x_6 \\ 2 & -1 & 0 & 1 & 3 & = x_7 \\ 1 & -1 & 1 & 0 & 2 & \rightarrow \min \end{array} \right] \mapsto$$

$$\left[\begin{array}{ccccc|c} x_5 & x_2 & x_3 & x_4 & 1 & \\ 1 & 0 & 2 & 3 & 1 & = x_1 \\ -1 & 1 & -1 & -2 & 0 & = x_6 \\ 2 & -1^* & 4 & 7 & 5 & = x_7 \\ 1 & -1 & 3 & 3 & 3 & \rightarrow \min \end{array} \right] \mapsto$$

$$\left[\begin{array}{ccccc|c} x_5 & x_7 & x_3 & x_4 & 1 & \\ 1 & 0 & 2 & 3 & 1 & = x_1 \\ 1 & -1 & 3 & 5 & 5 & = x_6 \\ 2 & -1 & 4 & 7 & 5 & = x_2 \\ -1 & 1 & -1 & -4 & -2 & \rightarrow \min. \end{array} \right]$$

Phase 1 was done in one pivot step, and Phase 2 also was done in one pivot step, because we obtain a feasible tableau with a bad column (x_5 -column). The program is unbounded.

12. We use the simplex method:

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & 1 & \\ 1^* & 0 & -1 & -1 & = x_4 \\ -1 & 3 & 1 & 0 & = x_5 \\ 3 & -1 & 2 & 1 & = x_6 \\ 1 & -1 & 1 & 0 & = x_7 \\ 1 & -1 & -1 & 0 & \rightarrow \min \end{array} \right] \mapsto$$

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$$\left[\begin{array}{cccc} x_4 & x_2 & x_3 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 3^* & 0 & -1 \\ 3 & -1 & 5 & 4 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} = x_1 \\ = x_5 \\ = x_6 \\ = x_7 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[\begin{array}{cccc} x_4 & x_5 & x_3 & 1 \\ 1 & 0 & 1 & 1 \\ 1/3 & 1/3 & 0 & 1/3 \\ 8/3 & -1/3 & 5 & 11/3 \\ 2/3 & -1/3^* & 2 & 2/3 \\ 2/3 & -1/3 & 0 & 2/3 \end{array} \right] \begin{array}{l} = x_1 \\ = x_2 \\ = x_6 \\ = x_7 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[\begin{array}{cccc} x_4 & x_7 & x_3 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 6/3 & 1 & 3 & 9/3 \\ 2 & -3 & 6 & 2 \\ 0 & 1 & -2 & 0 \end{array} \right] \begin{array}{l} = x_1 \\ = x_2 \\ = x_6 \\ = x_5 \\ \rightarrow \min. \end{array}$$

The x_3 -column is bad, so the program is unbounded.

13. We use the simplex method:

$$\left[\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 3 & 1^* & 0 & -2 & -1 \\ 3 & -1 & 2 & 1 & 2 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & -1 & 2 \end{array} \right] \begin{array}{l} = x_6 \\ = x_7 \\ = x_8 \\ = x_9 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[\begin{array}{cccccc} x_1 & x_2 & x_7 & x_4 & x_5 & 1 \\ 0 & 3 & -1 & -1 & -2 & 0 \\ 1 & -3 & 1^* & 0 & 2 & 1 \\ 5 & -7 & 2 & 1 & 6 & 3 \\ 2 & -4 & 1 & 0 & 1 & 2 \\ -1 & 2 & -1 & 0 & -3 & 1 \end{array} \right] \begin{array}{l} = x_6 \\ = x_3 \\ = x_8 \\ = x_9 \\ \rightarrow \min. \end{array}$$

The first column in this feasible tableau is bad, so the program is unbounded.

§12. Geometric Interpretation

1. The diamond can be given by four linear constraints $\pm x \pm y \leq 1$.
2. Any convex combination of convex combinations is a convex combination.
3. We have to prove that if $u = [x_1, y_1], v = [x_2, y_2]$ are feasible, i.e.,

$$x_1^4 + y_1^4 \leq 1 \text{ and } x_2^4 + y_2^4 \leq 1,$$

then the point $au + (1 - a)v$ is also feasible for $0 \leq a \leq 1$, i.e.,

$$(ax_1 + (1 - a)x_2)^4 + (ay_1 + (1 - a)y_2)^4 \leq 1.$$

Clearly, it suffices to do this in the case when $x_1, x_2, y_1, y_2 \geq 0$. In other words, it suffices to prove that the region

$$x^4 + y^4 \leq 1, x \geq 0, y \geq 0$$

is convex. The function $y = (1 - x^4)^{1/4}$ is smooth on the interval $0 < x < 1$, so it suffices to show that its slope decreases. At the point $[x, y] = [x, (1 - x^4)^{1/4}]$ the slope is $-x^3/y^3$ so it decreases.

Similarly, we can prove that the region $|x|^p + |y|^p \leq 1$ is convex for any $p \geq 1$. In the case $p = 1$ the slope is -1 , a constant function.

4. The halfsum of the top point of the diamond and the top point of the triangle does not belong to the union.

5. The points $[x, y] = [3, 1], [3, -1]$ belong to the circle but the halfsum $[3, 0]$ (the center of the circle) does not.

6. The halfsum of the lowest points of the intervals does not belong to the union.

7. Both $x = 1$ and $x = -1$ belong to the feasible region, but $0 = x/2 + y/2$ does not.

8. The tangent to the disc at the point $[x, y] = [2t/(1 + t^2), (1 - t^2)/(1 + t^2)]$ is $2tx + (1 - t^2)y = 1 + t^2$. The family of linear constraints $2tx + (1 - t^2)y \leq 1 + t^2$, where t ranges over all rational numbers, gives the disc.

9. A set S is called closed if it contains the limit points of all sequences in S . Any system of linear constraints gives a closed set, but the interval $0 < x < 1$ is not closed. Its complement is closed.

10. The rows of the identity matrix 1_6 . If the vectors are written as columns, take the columns of 1_6 .

11. One.

12. In terms of two variables on the top, we have $m + 1$ linear constraints (including two sign restrictions). So the feasible region in plane has at most $m + 1$ corners (vertices). Therefore Phase 2

terminates in at most m pivot steps. In the case of m steps, the initial and the terminal vertex are on one side (edge), so one pivot step would be sufficient. Thus, a better choice for the first pivot step results in at most $m - 1$ pivot steps.

13. Since x_i are affine, (a) \Rightarrow (b). It is also clear that (b) \Rightarrow (c) and (d) \Rightarrow (e) \Rightarrow (a). So it remains to prove that (c) \Rightarrow (d). The last implication follows from the well-known inequality

$$(|x_1| + \cdots + |x_n|)/n \leq ((x_1^2 + \cdots + x_n^2)/n)^{1/2}.$$

14. Let x be in S is not a vertex. We find distinct y, z in S such that $x = (y + z)/2$. The linear constraints giving S restricted on the line $ay + (1 - a)z$ give linear constraints on a . The interval $0 \leq a \leq 1$ is a part of the feasible set. Any of the constraints is either tight on the whole interval, or is tight only at an end point. So the tight constraints are the same for all points $ay + (1 - a)z$ with $0 < a < 1$.

The “only if” part proven, consider now the “if” part. Here is a counter example with an infinite system of constraints: The linear constraints are $x \geq c$ where c runs over all negative numbers. The feasible set S is the ray $x \geq 0$. No defining constraint is tight for any feasible x but $x = 0$ is the only vertex.

So we assume the S is given by a *finite* system of linear constraints. Let x, y be in S and x is a vertex and y has the same tight constraints. If $y \neq x$ then the same constraints are tight for every point on the line $ax + (1 - a)y$. For a number $a_0 > 1$ sufficiently closed to 1, all other constraints are also satisfied (here we use the finiteness). We pick such a number $a_0 < 2$. Then

$$x = ((a_0x + (1 - a_0)y) + ((2 - a_0)x + (a_0 - 1)y))/2$$

is not a vertex because $a_0x + (1 - a_0)y$ and $(2 - a_0)x + (a_0 - 1)y$ are distinct points in S .

15. Suppose that x is optimal, y, z are in the convex set S , and $x = (y + z)/2$. We have to prove that $y = z$.

Since f is affine, $f(x) = (f(y) + f(z))/2$. Since x is optimal, so are y and z . By uniqueness of optimal solution, $y = x = z$.

16. Every vertex is the basic solution for a feasible tableau. Then we can change the last row of this tableau, so its entries (except the last entry whose value does not matter) are positive. For the LP given by such an optimal tableau the optimal solution is unique.

Let S be the open diamond $|x| + |y| \leq 1$ with all convex combinations of the points $[1/3, 2/3], [2/3, 1/3]$ added. Then these two points are vertices. A linear form $ax + by$ is minimized at $[1/3, 2/3]$

if and only if $a = b \leq 0$. But in this case it is also minimized at every convex combination of $[1/3, 2/3], [2/3, 1/3]$.

17. Our set S is a subset of R^n . Let x be a point of S such that it is the limit of a sequence $y^{(1)}, y^{(2)}, \dots$ of points outside S (in other words, x belongs to the boundary of S). For example, x could be a vertex of S .

For each t , we find the point z in the closure of S closest to $y^{(t)}$. (We use the Euclidean distance $((y - z) \cdot (y - z))^{1/2}$ between points y, z in R^n .) We consider the linear constraint

$$(y^{(t)} - z^{(t)}) \cdot X \leq (y^{(t)} - z^{(t)})(y^{(t)} + y^{(t)})/2.$$

All points in S satisfy this constraint, while the point $y^{(t)}$ does not. Now we scale this constraint so it takes the form $c^{(t)}X \leq b^{(t)}$ with $c^{(t)} \cdot c^{(t)} = 1$. Then we take a limit constraint $c \cdot X \leq b$. Then all points in S satisfy the latter constraint and $c \cdot x = b$. Thus, x is maximizer of the linear form $f = c \cdot X \neq 0$ over S .

When the set S is closed, and x is a vertex, we can arrange x to be an unique maximizer.

18. Suppose that x is not a vertex in S' . Then $x = (y + z)/2$ with distinct y, z in S' . Since y, z are in S , x is not a vertex in S .

Chapter 5. Duality

§13. Dual Problems

1. Let $f = 5x - 6y + 2z$ be the objective function. Here is a standard column tableau:

$$\begin{array}{r} -x \\ -y \\ -z \\ 1 \end{array} \left[\begin{array}{ccc} 0 & -1 & 5 \\ 1 & -1 & -6 \\ 0 & 0 & 2 \\ 7 & -3 & 0 \end{array} \right]$$

$$\begin{array}{ccc} \parallel & \parallel & \downarrow \\ v_1 & v_2 & -f \end{array} \rightarrow \max$$

2. Let f be the objective function. Here is a standard column tableau:

$$\begin{array}{r} -a \\ -b \\ -c \\ -d \\ -e \\ -f \\ -g \\ -h \\ -i \\ -j \\ 1 \end{array} \left[\begin{array}{cccccc} -15 & -25 & -1 & -1 & -1 & -3 & .39 \\ -15 & -25 & -25 & 0 & -25 & -2 & .11 \\ -15 & -25 & -25 & -25 & -25 & -1 & .18 \\ -15 & 0 & -25 & -25 & -25 & -2 & .21 \\ -15 & 0 & 0 & -30 & -30 & -4 & .35 \\ -15 & -6 & -30 & -25 & -25 & -2 & .44 \\ -10 & -10 & -25 & -25 & -25 & -1 & .25 \\ -15 & -25 & -25 & -25 & -25 & -2 & .25 \\ -15 & 0 & -25 & -25 & -25 & -3 & .23 \\ -10 & -10 & -25 & -25 & -25 & -3 & .24 \\ -100 & -100 & -100 & -100 & -100 & -70 & 0 \end{array} \right]$$

$$\begin{array}{cccccc} \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \downarrow \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & -f \end{array} \rightarrow \max$$

$$3. \quad \begin{array}{ccc} x & y & z & 1 \\ \left[\begin{array}{ccc} 0 & -1 & 0 & 7 \\ 1 & 1 & 0 & -3 \\ 5 & -6 & 2 & 0 \end{array} \right] & \begin{array}{l} = v_1 \\ = v_2 \\ = f \end{array} & \rightarrow \min \end{array}$$

The matrix in Exercise 2 is so big that the transposed matrix may not fit on the page. So we reduce it as follows. The fifth constraint follows from the fourth constraint because $b \geq 0$ so we drop the redundant constraint.

Given any feasible solution, we can replace g, h by $0, g + h$ and obtain a feasible solution with the same value for the objective function f . So setting $g = 0$ we do not change the optimal value.

$$\begin{array}{cccccccccc}
 a & b & c & d & e & f & h & i & j & 1 \\
 \left[\begin{array}{cccccccccc}
 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 10 & -100 \\
 25 & 25 & 25 & 0 & 6 & 25 & 25 & 0 & 10 & -100 \\
 1 & 25 & 25 & 25 & 0 & 30 & 25 & 25 & 25 & -100 \\
 1 & 0 & 25 & 25 & 30 & 25 & 25 & 25 & 25 & -100 \\
 3 & 2 & 1 & 2 & 4 & 2 & 2 & 3 & 3 & -70 \\
 .39 & .11 & .18 & .21 & .35 & .44 & .25 & .23 & .24 & 0
 \end{array} \right] & \begin{array}{l} = v_1 \\ = v_2 \\ = v_3 \\ = v_4 \\ = v_6 \\ \rightarrow \min
 \end{array}
 \end{array}$$

4. $\min = -\infty$. Namely, $z = -0.5x_4 \rightarrow -\infty$ as $x_1 = x_2 = x_5 = 0, x_3 = 1.5x_4, x_6 = -10 + 2.5x_4, x_7 = -20 + .5x_4, x_8 = -2 + 0.5x_4, x_4 \rightarrow \infty$.

5. Let $cx + d, cy + d$ be two feasible values, where x, y are two feasible solutions. We have to prove that

$$\alpha(cx + d) + (1 - \alpha)(cy + d)$$

is a feasible value for any α such that $0\alpha \leq 1$. But

$$\alpha(cx + d) + (1 - \alpha)(cy + d) = c(\alpha x + (1 - \alpha)y) + d,$$

where $\alpha x + (1 - \alpha)y$ is a feasible solution because the feasible region is convex.

6. In the basic solution, $x_7 = -4$, so it is not feasible.

7. The first equation does not hold, so this is not a solution.

8. First we check that this $X = [x_i]$ is a feasible solution (i.e., satisfies all constraints) with $z = 2$. We introduce the dual variables y_i corresponding to x_i , write the dual problem as the column problem, and set $y_i = 0$ whenever $x_i \neq 0$ (assuming that X is optimal, cf. Problem 13.10 and its solution).

$$\begin{array}{ccccccc}
 -y_6 & \left[\begin{array}{cccccc}
 7 & -2 & -6 & 6 & -1 & 15 \\
 -1 & 0 & 1 & 0 & -1 & -2 \\
 4 & 0 & -3 & 5 & 3 & 4
 \end{array} \right] \\
 -y_8 & & & & & & \\
 1 & & & & & & \\
 & = 0 & = y_2 & = 0 & = y_4 & = 0 & = w & \rightarrow \max.
 \end{array}$$

We have a system of three linear equations for y_6, y_8 , and the system has no solutions. So X is not optimal.

9. Proceeding as in the solutions of Problem 13.10 and Exercise 8, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1] \begin{bmatrix} 7 & -2 & -6 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix} = 0.$$

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The system has the unique solution $y_6 = 1, y_7 = 2, y_8 = 1$. Moreover, this solution is feasible (the basic y_i are nonnegative). Since we have feasible solutions for the primal and dual problems and $x_i y_i = 0$ for all i , both solutions are optimal.

10. Proceeding as in the solutions of Problem 13.10 and Exercises 8,9, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1] \begin{bmatrix} -2 & -6 & 6 & -1 \\ 1 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 5 & 3 \end{bmatrix} = 0.$$

The system has no solutions, so the answer is: This is not optimal.

§14. Sensitivity Analysis and Parametric Programming

1. The tableau is not standard, so we treat the row and column programs separately. We pivot the row program on -1 in the b -column:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 2 & -1^* & 0 & -3 \\ 0 & 2 & 1 & 0 \end{array} \right] & \begin{array}{l} = d \\ = c \\ = w \end{array} & \rightarrow \min & \mapsto \end{array}$$

$$\begin{array}{cccc} a & c & c & 1 \\ \left[\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & -3 \\ 4 & -2 & 1 & -6 \end{array} \right] & \begin{array}{l} = d \\ = b \\ = w \end{array} & \rightarrow \min. & \end{array}$$

Then we combine the two c -columns:

$$\begin{array}{ccc} a & c & 1 \\ \left[\begin{array}{ccc} 1 & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{array} \right] & \begin{array}{l} = d \\ = b \\ = w \end{array} & \rightarrow \min. \end{array}$$

This tableau is standard. We use the simplex method:

$$\begin{array}{ccc} a & c & 1 \\ \left[\begin{array}{ccc} 1^* & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{array} \right] & \begin{array}{l} = d \\ = b \\ = w \end{array} & \rightarrow \min \quad \mapsto \\ \\ d & c & 1 \\ \left[\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 4 & 3 & 2 \end{array} \right] & \begin{array}{l} = a \\ = b \\ = w \end{array} & \rightarrow \min. \end{array}$$

This is an optimal tableau, so $\min = 2$ at $a = 2, b = 1, c = d = 0$.

Now we rewrite the column program in a standard column tableau:

$$\begin{array}{r} -g \\ -h \\ 1 \end{array} \left[\begin{array}{cccc} -1 & 0 & 1 & 2 \\ -2 & 1 & 0 & 3 \\ 0 & -2 & -1 & 0 \end{array} \right] \\ \begin{array}{cccc} = i & = j & = k & = -u \end{array} \rightarrow \max.$$

The k -column is bad, so this program is infeasible.

2. The tableau is not standard, so we treat the row and column programs separately. The second constraint in the row program reads $2a + c + 3 = c$, hence $a = -3/2$. So the row program is infeasible.

Now we rewrite the column program in a standard column tableau:

$$\begin{array}{r} -g \\ -h \\ 1 \end{array} \left[\begin{array}{cccc} -1 & -2 & -3 & -1 \\ -2 & 0 & -1 & 3 \\ 1 & -1 & 0 & 0 \end{array} \right] \\ \begin{array}{cccc} = i & = j & = k & = -u \end{array} \rightarrow \max.$$

The first row is bad, so the column program is either infeasible or unbounded. It is clear without pivoting that the program is unbounded: as $g \rightarrow \infty, g \geq 1/2, h = 0$, we have

$$-w = g \rightarrow \infty, i = g + 1 \geq 0, j = 2g - 1 \geq 0, k = 3g \geq 0.$$

3. $\min = 0$ at $d = e = 0, a \geq 0$ arbitrary

4. We rewrite the program in a standard row tableau:

$$\begin{array}{ccc} a & b & 1 \\ \left[\begin{array}{ccc} 1 & 0 & -1 - \varepsilon \\ 0 & 1 & -1 \\ -1 & -2 & 0 \end{array} \right] & = c \\ & = d \\ & = -w \rightarrow \min. \end{array}$$

Since we have a bad column (in fact, both columns are bad), our program is either infeasible or unbounded. Without pivoting, it is easy to see that the program is unbounded. Namely, let both $a, b \rightarrow \infty$. Then $w \rightarrow \infty$ while $c \geq 0$ when $a \geq 1 - \varepsilon$, and $d \geq 0$ when $b \geq 1$.

§15. More on Duality

1. No, it is not redundant.
2. Yes, it is $2 \cdot (\text{first equation}) + (\text{second equation})$.
3. No, it is not redundant.

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4. No, it is not redundant.
5. Yes, it is Adding the first two equations, we obtain

$$6x + 8y + 10z = 12$$

which implies the last constraint because $12 \geq 0$.

6. It is redundant because it can be obtained as follows: multiply the first constraint by 2 and add it to the second constraint.

7. Yes, it is.

8. We solve the system

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 6$$

$$6x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 = 1$$

$$x_1 - x_2 + x_3 - x_4 + x_5 = f$$

for x_1, x_2, f :

$$x_1 = -4 + x_3 + 2x_4 + 3x_5$$

$$x_2 = 5 - 2x_3 - 3x_4 - 4x_5$$

$$f = -9 + 4x_3 + 4x_4 + 8x_5.$$

Setting $x_3 = x_4 = x_5 = 0, x_1 = -4, x_2 = 5$, we satisfy the equations but $f = -9 < 0$, so the constraint $f \geq 0$ is not redundant.

9. For the dual problem, the first column reads $-1 \geq 0$, hence the dual program is infeasible. So the row program (the row program with a bad column) is either infeasible or unbounded. Pivoting on -5 in the x_3 -column yield a row-feasible tableau (and x_1 -column stays bad), so the row program is unbounded.

10. Let u be the nonbasic dual variable (corresponding to the first row). The dual constraints are: $-6u + 1 \geq 0, -8u + 2 \geq 0, -5u + 5 \geq 0, -6u + 1 \geq 0, -7u + 1 \geq 0, -8u + 1 \geq 0, -3u + 2 \geq 0, -5u \geq 0, -4u + 5 \geq 0$ and, of course, $u \geq 0$. The only feasible solution is $u = 0$, which is optimal (for any objective function). By complementary slackness, $x_i = 0$ for $i \neq 8$ for any optimal solution for the primal problem. The first row reads $x_8 \geq 1/5$. So $\min = 0$ when $x_8 \geq 1/5$ and all other $x_i = 0$.

11. Let y_i be the dual variable corresponding to x_i . The first 7 columns of the tableau give 7 linear constraints for y_8, y_9 . Two additional constraints are $y_8, y_9 \geq 0$. We can plot the feasible region (given by these 9 constraints) in the (y_8, y_9) -plane. (The constraints corresponding to x_5, x_6, x_7 are redundant.) The answer is $\max = 2.5$ at $y_8 = 0, y_9 = 1.25$. By complementary slackness, for any optimal solution $[x_i]$ of the primal problem, we have $x_i = 0$ for $i \neq 1, 8$. For such a solution, we have $3x_1 - 1 = x_8 \geq 0, 4x_1 - 2 = x_9 = 0$. Thus,

$\min = 2.5$ at $x_1 = 0.5$, all other $x_i = 0$.

12. Let y_i be the dual variable corresponding to x_i . The first 8 columns of the tableau give 8 linear constraints for y_9, y_{10} . Two additional constraints are $y_9, y_{10} \geq 0$. We can plot the feasible region (given by these 10 constraints) in the (y_9, y_{10}) -plane. The answer is $\max = 90/41$ at $y_9 = 10/41, y_{10} = 40/41$. By complementary slackness, for any optimal solution $[x_i]$ of the primal problem, we have $x_i = 0$ for $i \neq 1, 5$. For such a solution, we have $6x_1 + 5x_5 - 1 = x_9 = 0, 19x_1 + 9x_5 - 2 = x_{10} = 0$. Thus,

$\min = 90/41$ at $x_1 = 1/41, x_5 = 7/41$ all other $x_i = 0$.

13. Let y_i be the dual variable corresponding to x_i , and let u, v be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for u, v . Two additional constraints are $u, v \geq 0$. We can plot the feasible region (given by these 11 constraints) in the (u, v) -plane. The answer is $\max = 75/34$ at $u = 15/34, v = 15/17$. By complementary slackness, for any optimal solution $[x_i]$ of the primal problem, we have $x_i = 0$ for $i \neq 4, 6$. For such a solution, we have $6x_4 + 8x_6 - 1 = 0, 14x_4 + 13x_6 - 2 = 0$. Thus,

$\min = 75/34$ at $x_4 = 3/34, x_6 = 1/17$ all other $x_i = 0$.

14. Let y_i be the dual variable corresponding to x_i , and let u, v be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for u, v . Two additional constraints are $u, v \geq 0$. The constraint corresponding to the x_8 column reads $-5u - 8v + 0 \geq 0$, hence $u = v = 0$. On the other hand, then the constraint corresponding to the x_1 column reads $-1 \geq 0$. So the column problem is infeasible.

On the other hand, it is easy to find a feasible solution for the row program, for example, $x_9 = 1$ and $x_i = 0$ for all other i . By the theorem on four alternatives, the row problem is unbounded.

Chapter 6. Transportation Problems

§16. Phase 1

1.

20	10	5		35
		5	15	20
20	10	10	15	

2. By the northwest method,

35				35
5	4			9
	6	5		11
		5	15	20
40	10	10	15	

3. The total supply is 256, while the total demand is 260. So the problem is infeasible.

4. The balance condition $50 = 50$ holds. Each time, we pick a position with the minimal cost: $x_{2,1} = 2, x_{24} = 5, x_{33} = 4$ at zero cost, $x_{17} = 3, x_{25} = 1, x_{29} = 12, x_{32} = 3, x_{38} = 5, x_{49} = 4$ at unit cost 1, and $x_{45} = 2, x_{18} = 7, x_{48} = 2$ at unit cost 2. The total number of selected positions is 12, which equals $4 + 9 - 1$. Total cost is $0 \cdot 11 + 1 \cdot 28 + 2 \cdot 11 = 50$.

5. The balance condition $50 = 50$ holds. Each time, we pick a position with the minimal cost: $x_{21} = 2, x_{24} = 13, x_{33} = 4$ at zero cost, $x_{14} = 2, x_{17} = 3, x_{49} = 1, x_{42} = 7, x_{32} = 1, x_{38} = 4, x_{36} = 2$ at unit cost 1, and $x_{15} = 10, x_{35} = 1$ at unit cost 2. The total number of selected positions is 12, which equals $4 + 9 - 1$. Total cost is $0 \cdot 19 + 1 \cdot 20 + 2 \cdot 11 = 42$.

6. The balance condition $50 = 50$ holds. Each time, we pick a position with the minimal cost: $x_{21} = 5, x_{33} = 12$ at zero cost, $x_{14} = 5, x_{17} = 3, x_{49} = 1, x_{42} = 8, x_{41} = 7, x_{43} = 2$ at unit cost 1, $x_{15} = 1, x_{18} = 4, x_{46} = 0$ at unit cost 2, and $x_{16} = 2$ at unit cost 3. The total number of selected positions is 12, which equals $4 + 9 - 1$. Total cost is $0 \cdot 17 + 1 \cdot 26 + 2 \cdot 5 + 3 \cdot 2 = 42$.

7. The balance condition $130 = 130$ holds. Each time, we pick a position with the minimal cost: $x_{16} = 30$ at zero cost, $x_{12} = 10, x_{35} = 15$ at unit cost 30, $x_{21} = 25, x_{34} = 30$ at unit cost 35, $x_{13} = 10$, at unit cost 40, $x_{33} = 5$, at unit cost 95, and $x_{23} = 5$, at unit cost 100. The total number of selected positions is 8, which equals $3 + 6 - 1$. Total cost is $0 \cdot 30 + 30 \cdot 25 + 35 \cdot 55 + 40 \cdot 10 + 95 \cdot 5 + 100 \cdot 5 = 4050$.

8. Suppose our solution $x = (y + z)/2$ where y, z are feasible solution. We compare the values of x, y, z at the first selected position. Since the values for x was maximal possible, the values of y, z are the same. Then we consider the second selected position and conclude that x, y, z agree there. Continuing the process, we conclude that x, y, z agree at all selected positions. Then $x = y = z$. thus, x is a vertex.

§17. Phase 2

1.

	1	2	2	
0	$\begin{smallmatrix} 1 \\ 175 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 25 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ (1) \end{smallmatrix}$	200
0	$\begin{smallmatrix} 1 \\ (0) \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 100 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 200 \end{smallmatrix}$	300
	175	125	200	

This is an optimal table, and the corresponding solutions are optimal. The minimal cost for the transportation problem is $1 \cdot 175 + 2 \cdot 25 + 2 \cdot 100 + 2 \cdot 200 = 825$. The maximal profit for the dual problem is $1 \cdot 175 + 2 \cdot 125 + 2 \cdot 200 - 0 \cdot 200 - 0 \cdot 300 = 825$.

2.

	1	2	3	
0	$\begin{smallmatrix} 1 \\ (0) \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ 200 \end{smallmatrix}$	200
0	$\begin{smallmatrix} 1 \\ 175 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 125 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ (-1) \end{smallmatrix}$	300
	175	125	200	

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The table is not optimal. Adjusting the flow along a cycle may improve the total cost:

	1	2	3	
0	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ 0 + \varepsilon \end{matrix}$	$\begin{matrix} 3 \\ 200 - \varepsilon \end{matrix}$	200
0	$\begin{matrix} 1 \\ 175 \end{matrix}$	$\begin{matrix} 2 \\ 125 - \varepsilon \end{matrix}$	$\begin{matrix} 2 \\ \varepsilon (-1) \end{matrix}$	300
	175	125	200	

Namely, taking $\varepsilon = 125$ we decrease the total cost by 125.

3. We start with the basic feasible solution found in the solution of Exercise 4, §16 and compute the corresponding dual basic solution:

	0	2	1	0	1	2	1	2	1	
0	$\begin{matrix} 1 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ (0) \end{matrix}$	$\begin{matrix} 3 \\ (2) \end{matrix}$	$\begin{matrix} 1 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 3 \\ (1) \end{matrix}$	$\begin{matrix} 1 \\ 3 \end{matrix}$	$\begin{matrix} 2 \\ 7 \end{matrix}$	$\begin{matrix} 3 \\ (2) \end{matrix}$	10
0	$\begin{matrix} 0 \\ 2 \end{matrix}$	$\begin{matrix} 3 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 0 \\ 5 \end{matrix}$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 2 \\ (0) \end{matrix}$	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ (0) \end{matrix}$	$\begin{matrix} 1 \\ 12 \end{matrix}$	20
1	$\begin{matrix} 2 \\ (3) \end{matrix}$	$\begin{matrix} 1 \\ 3 \end{matrix}$	$\begin{matrix} 0 \\ 4 \end{matrix}$	$\begin{matrix} 1 \\ (2) \end{matrix}$	$\begin{matrix} 2 \\ (2) \end{matrix}$	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ (2) \end{matrix}$	$\begin{matrix} 1 \\ 5 \end{matrix}$	$\begin{matrix} 1 \\ (1) \end{matrix}$	12
0	$\begin{matrix} 1 \\ (1) \end{matrix}$	$\begin{matrix} 1 \\ (-1) \end{matrix}$	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ (2) \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ 2 \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ 2 \end{matrix}$	$\begin{matrix} 1 \\ 4 \end{matrix}$	8
	2	3	4	5	1	2	3	14	16	

There is only one negative $w_{42} = -1$. We select this position and get the loop $(4, 2), (3, 2), (3, 8), (4, 8)$. The maximal $\varepsilon = 2$, and we deselect the position $(4, 8)$. The total cost decreases by 2. Here is the new basic feasible solution and the corresponding dual basis

solution:

	1	2	1	1	2	3	1	2	2	
0	1 (0)	2 (0)	3 (2)	1 (0)	2 (0)	3 (0)	1 3	2 7	3 (1)	10
1	0 2	3 (2)	2 (2)	0 5	1 1	2 (0)	1 (1)	2 (1)	1 12	20
1	2 (2)	1 1	0 4	1 (1)	2 (1)	1 (-1)	2 (2)	1 7	1 (0)	12
1	1 (1)	1 2	1 (1)	2 (2)	2 (1)	2 2	2 (2)	2 (1)	1 4	8
	2	3	4	5	1	2	3	14	16	

Again we have a negative $w_{36} = -1$. The loop is (3, 6), (3, 2), (4, 2), (4,6). The total cost decreases by $-w_{36} \cdot \varepsilon = 1$. Here is the new basic feasible solution and the corresponding dual basis solution:

	0	1	1	0	1	2	1	2	1	
0	1 (1)	2 (1)	3 (2)	1 (1)	2 (1)	3 (1)	1 3	2 7	3 (2)	10
0	0 2	3 (2)	2 (1)	0 5	1 1	2 (0)	1 (0)	2 (0)	1 12	20
1	2 (3)	1 (1)	0 4	1 (2)	2 (2)	1 1	2 (2)	1 7	1 (1)	12
0	1 (1)	1 3	1 (0)	2 (2)	2 (1)	2 1	2 (1)	2 (0)	1 4	8
	2	3	4	5	1	2	3	14	16	

This table is optimal with total cost at $\min = 47$.

4. We start with the basic feasible solution found in the solution

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of Exercise 5, §16 and compute the corresponding dual basic solution:

	1	1	0	1	2	1	1	1	1	
0	1 (0)	2 (1)	3 (3)	1 2	2 10	3 (2)	1 3	2 (1)	3 (2)	15
1	0 2	3 (3)	2 (3)	0 13	1 (0)	2 (2)	1 (1)	2 (2)	1 (1)	15
0	2 (1)	1 1	0 4	1 (0)	2 1	1 2	2 (1)	1 4	1 (0)	12
0	1 (0)	1 7	1 (1)	2 (1)	2 (0)	2 (1)	2 (1)	2 (1)	1 1	8
	2	8	4	15	11	2	3	4	1	

The table is optimal, $\min = 42$.

5. We start with the basic feasible solution found in the solution of Exercise 6, §16 and compute the corresponding dual basic solution:

	2	2	2	1	2	3	1	2	2	
0	1 (-1)	2 (0)	3 (1)	1 5	2 1	3 2	1 3	2 4	3 (1)	15
2	0 5	3 (3)	2 (2)	0 (1)	1 (1)	2 (1)	1 (2)	2 (2)	1 (1)	5
2	2 (2)	1 (1)	0 12	1 (2)	2 (2)	1 (0)	2 (3)	1 (1)	1 (1)	12
1	1 7	1 8	1 2	2 (2)	2 (1)	2 0	2 (2)	2 (1)	1 1	18
	12	8	14	5	1	2	3	4	1	

There is only one negative $w_{11} = -1$. The loop is (1,1), (1, 6), (4, 6), (4, 1). The decrease in the total cost is 2. Here is the new basic

solution and the corresponding dual basic solution:

	1	1	1	1	2	2	1	2	1	
0	1 2	2 (1)	3 (2)	1 5	2 1	3 (1)	1 3	2 4	3 (2)	15
1	0 5	3 (3)	2 (2)	0 (0)	1 (0)	2 (1)	1 (1)	2 (1)	1 (1)	5
1	2 (2)	1 (1)	0 12	1 (1)	2 (1)	1 (0)	2 (2)	1 (0)	1 (1)	12
0	1 5	1 8	1 2	2 (1)	2 (0)	2 2	2 (1)	2 (0)	1 1	18
	12	8	14	5	1	2	3	4	1	

This table is optimal, and $\min = 40$.

6. We start with the optimal solution with $t = 0$ given in Table 17.22:

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	100 (60)	45 (10)	60 (30)	0 5	30
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

Now we replace 30 by $30 + t$ at the right margin and at the bottom margin (remember that the last column correspond to the fictitious store). We also replace the corresponding entry 5 in the table by $5 + t$.

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	100 (60)	45 (10)	60 (30)	0 $5+t$	$30+t$
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	$30+t$	

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When $t \geq -5$ (e.g., $|t| \leq 1$), this is an optimal table, The total cost is at $\min = 3475$ (independent of t).

7. When $t \geq 25$, see the previous solution. When $t < 0$, the total supply is less than the total demand, so the program is infeasible. So it remains to consider the case $0 \leq t \leq 25$.

8. In the optimal tableau of Table 17.22, we replace c_{23} by t and w_{23} by $t - 40$:

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	t ($t - 40$)	45 (10)	60 (30)	0 5	30
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

When $t \geq 40$, this table is optimal, with $\min = 3475$ (independent of t).

Assume now that $t < 40$. We select the position (2, 3) and get the loop (2, 3), (1,3), (1,6), (2,6). The corresponding $\varepsilon = 5$. The decrease in the total cost is $5(40 - t)$. Here is the new table:

	$75 - t$	30	40	35	30	0	
0	55 ($t - 20$)	30 10	40 15	50 (15)	40 (10)	0 25	50
40 $-t$	35 25	30 ($40 - t$)	t 5	45 ($50 - t$)	60 ($70 - t$)	0 ($40 - t$)	30
0	40 ($t - 35$)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

This table is optimal when $35 \leq t \leq 40$.

Assume now that $t < 35$. We select the position (3, 1) and get the loop (3, 1), (3, 6), (1,6), (1, 3), (2,3), (2,1). The corresponding $\varepsilon = 5$. The decrease in the total cost is $5(35 - t)$. Here is the new

table:

	$75 - t$	30	40	$70 - t$	$65 - t$	0	
0	55 ($t - 20$)	30 10	40 10	50 ($t - 20$)	40 ($t - 25$)	0 30	
40 $-t$	35 20	30 ($40 - t$)	t 10	45 (15)	60 (35)	0 ($40 - t$)	
35 $-t$	40 5	60 ($65 - t$)	95 ($90 - t$)	35 30	30 15	0 ($35 - t$)	
	25	10	20	30	15	30	

This table is optimal when $25 \leq t \leq 35$.

Assume now that $t < 25$. We select the position (1, 5) and obtain the loop (1, 5), (3, 5), (3, 1), (2, 1), (2, 3), (1, 3). The corresponding $\varepsilon = 10$. The decrease in the total cost is $10(25 - t)$. Here is the new table:

	50	30	$t + 15$	45	40	0	
0	55 (5)	30 10	40 ($25 - t$)	50 (5)	40 10	0 30	50
15	35 10	30 (15)	t 20	45 (15)	60 (35)	0 $40 - t$	30
10	40 15	60 (40)	95 ($90 - t$)	35 30	30 5	0 (10)	50
	25	10	20	30	15	30	

This table is optimal when $t \leq 25$.

Thus, we solve the program for all t . The minimal cost is

$$\begin{cases} 2900 + 20t & \text{for } t \leq 25 \\ 3400 + 5(t - 25) & \text{for } 25 \leq t \leq 35 \\ 3475 & \text{for } t \geq 40. \end{cases}$$

§18. Job Assignment Problem

1. $\min = 7$ at $x_{14} = x_{25} = x_{32} = x_{43} = x_{51} = 1$, all other $x_{ij} = 0$.

2. Subtract 1 from the rows 1,2,5 and the column 4. Subtract 2 from the row 4. Now we can place the flow at positions with zero

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cost, namely, positions (1, 4), (2, 3), (3,2), (4, 1), (5, 6), (6, 5), So $\min = 6$.

3. $\min = 7$ at $x_{12} = x_{25} = x_{34} = x_{43} = x_{51} = x_{67} = x_{76} = 1$, all other $x_{ij} = 0$.

4. We subtract: 1 from the rows 2,3,4,5,8; 2 from the row 7; 1 from the column 8:

0	2	2	4	0	1	5	0	4
1	2	0	2	3	2	1	2	0
1	0	3	2	3	2	0	0	3
3	1	3	1	1	1	0	3	1
1	2	4	0	1	3	1	2	3
5	2	2	4	0	1	1	0	4
0	1	2	1	2	1	0	1	3
4	1	3	2	1	0	1	2	0
1	2	1	2	0	1	2	3	5

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Hungarian method (Remark 17.25). Let us try to place the flow at positions with zero cost. In each of the following six lines there is only one zero: r4 (row 4), r5, r9, c2 (column 2), c3, c6 (we pass c4 because the conflict with r5: the position (5,4) is already selected in r5). We select the positions with these zeros and add the six lines c7, c4, c5, r3, r2, r8 to our list L of covering lines. The remaining matrix is

	$c1$	$c8$	$c9$
$r1$	0	0	4
$r6$	5	0	4
$r3$	0	1	3

We cannot place all flow at positions with zero cost, but we can cover all zeros by $2 < 3$ lines, namely, $c1$ and $c8$. The complete list L consists of 8 lines $c7, c4, c5, r3, r2, r8, c1, c8$. The least uncovered number is $m = 1$. We subtract 1 from all uncovered entries and add

1 to all twice-covered entries:

0	1	1	4	0	0	5	0	3
2	2	0	3	4	2	2	3	0
2	0	3	3	4	2	1	1	3
3	0	2	1	1	0	0	3	0
1	1	3	0	1	2	1	2	2
5	1	1	4	0	0	1	0	3
0	0	1	1	2	0	0	1	2
5	1	3	3	2	0	2	3	0
1	1	0	2	0	0	2	3	4

Now we can place the flow at the positions with zero costs:

						*		
		*						
	*						*	
			*					
							*	
*								
				*				
							*	

For this program, $\min = 0$. However we changed the objective function (without changing the optimal solutions). For the original problem, $\min = 1 + 1 + 1 + 1 + 1 + 1 + 2 + 1 + 0 = 9$.

5. $\max = 14$ at $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$, all other $x_{ij} = 0$.

6. First we convert the maximization problem to a minimization problem by subtracting each entry from the maximal entry in its row:

0	2	2	2	3	2
2	1	3	2	0	0
2	4	3	3	3	0
2	1	0	1	0	1
1	2	3	2	0	3
1	0	1	0	2	1

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Hungarian

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method (Remark 17.25). The following five lines cover all zeros: rows 1,4,6 and columns 5,6. Since $t = 5 < n = 6$, we cannot place all flow at positions with zero cost. The least uncovered number is $m = 1$. We subtract 1 from all uncovered entries and add 1 to all twice-covered entries:

0	2	2	2	4	3
1	0	2	1	0	0
1	3	2	2	3	0
2	1	0	1	1	2
0	1	2	1	0	3
1	0	1	0	3	2

Now we can place all flow at positions with zero cost:

*					
	*				
					*
		*			
			*		
			*		

For this program, $\min = 0$. However we changed the objective function (without changing the optimal solutions). For the original problem, $\max = 4 + 3 + 4 + 4 + 4 + 2 = 21$.

Note that the sum of maximal entries in rows is 22, and we cannot get this much because the conflict in the last column. This proves the the solution is optimal.

7. $\max = 30$ at $x_{15} = x_{26} = x_{33} = x_{41} = x_{54} = x_{62} = x_{77} = 1$, all other $x_{ij} = 0$.

8. To convert the maximization problem to a minimization problem, we subtract every entry from the maximal number in its row;

5	3	3	1	5	4	0	4	1
2	1	3	1	0	1	2	0	3
2	3	0	1	0	1	3	2	0
1	3	1	3	3	3	4	0	3
3	2	0	4	3	1	3	1	1
0	3	3	1	5	4	4	4	1
3	2	1	2	1	2	3	1	0
0	3	1	2	3	4	3	1	4
4	3	4	3	5	4	3	1	0

Now we subtract 1 from columns 2,4, and 6 to create a zero in each column:

5	2	3	0	5	3	0	4	1
2	0	3	0	0	0	2	0	3
2	2	0	0	0	0	3	2	0
1	2	1	2	3	2	4	0	3
3	1	0	3	3	0	3	1	1
0	2	3	0	5	3	4	4	1
3	1	1	1	1	1	3	1	0
0	2	1	1	3	3	3	1	4
4	2	4	2	5	3	3	1	0

We are ready to apply the Hungarian method (Remark 17.25). We cannot place the flow at the positions with zero cost, so we can cover all zeros by < 9 lines. Our 8 lines are the rows 1–5 and the columns 1, 3, 9. Now we subtract 1 from uncovered numbers and add 1 to twice-covered numbers (this correspond to decreasing by 1 the potentials for the covering columns and uncovering rows):

6	2	3	1	5	3	0	4	2
3	0	3	1	0	0	2	0	4
3	2	0	1	0	0	3	2	1
2	2	1	3	3	2	4	0	4
4	1	0	4	3	0	3	1	2
0	1	2	0	4	2	3	3	1
3	0	0	1	0	0	2	0	0
0	1	0	1	2	2	2	0	4
4	1	3	2	4	2	2	0	0

Now we can place the flow at the positions with zero cost:

							*	
*								
	*							
								*
				*			*	
		*						
			*					
*								
								*

The maximal efficiency for the original problem is
 $5 + 3 + 4 + 5 + 4 + 4 + 4 + 5 + 5 = 39$.

Chapter 7. Matrix Games

§19. What are Matrix Games?

1. $\max \min = -1$. $\min \max = 0$. There are no saddle points.

$[1/3, 2/3, 0]^T$ gives at least $-2/3$ for the row player.

$[1/2, 0, 0, 0, 1/2]$ gives at least $1/2$ for the column player.

So $-2/3 \leq \text{the value of the game} \leq -1/2$.

2. Computing $\max \min$ and $\min \max$, we find that the value v of game belongs to the interval $1 \leq v \leq 2$, the best pure for the row player (giving him at least 1) is the second row, and the best pure strategies for the column player (when she pays at most 2 are columns 1, 2, 5. there are no saddle points.

The columns 3 and 4 are dominated by column 2, so we obtain

a smaller game:
$$\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 2 \\ -1 & 2 & 0 \\ -4 & -8 & 2 \end{bmatrix}.$$

Now the first and the last rows are dominated by the second row, so we obtain a smaller game:
$$\begin{bmatrix} 2 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}.$$

Now the last column is dominated by the first column, so we obtain a smaller game:
$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

This game can be easily solved using slopes: optimal strategies are $[3/4, 1/4]^T$, $[1/4, 3/4]$, and $v = 5/4$. For the original big game, optimal strategies are $[0, 3/4, 1/4, 0]^T$, $[1/4, 3/4, 0, 0]$, and $v = 5/4$.

3. $\max \min = -1$. $\min \max = 2$. There are no saddle points.

$(\text{second row} + 2 \cdot \text{third row})/3 \geq -2/3$.

$(\text{third column} + \text{sixth column})/2 \leq 1$.

So $-2/3 \leq \text{the value of the game} \leq 1$.

4. We compute $\max \min = -1$ and $\min \max = 0$ in pure strategies. So $-1 \leq v \leq 0$ for the value v of game. There are no saddle points.

Choosing the row 4, he (the row player) gets at least -1 (i.e., he pays at most 1), but using the mixed strategy $[0, 1/3, 1/3, 1/3]^T$ he gets at least $-1/3$. Thus, $-1/3 \leq v \leq 0$.

5. We compute the max in each column (marked by *) and min in each row (marked by ■).

$$\begin{array}{c} \begin{array}{cccccccccc} 4 & 2^{\blacksquare} & 3 & 5 & 4 & 3^{\blacksquare} & 7 & 6 & 3 \\ -4 & 4^* & -4^{\blacksquare} & 3^* & 0 & 0 & 0 & -1 & 1 & -2 \\ -2 & -1 & 0 & 2 & 1 & -2^{\blacksquare} & -2^{\blacksquare} & 1 & 0 & -2^{\blacksquare} \\ -4 & -4^{\blacksquare} & 0 & -2 & -2 & 1 & -1 & 1 & 6^* & 2 \\ 0^* & 1 & 2^* & 2 & 5^* & 3 & 3^* & 7^* & 2 & 0^{\blacksquare} \\ -9 & -4 & -9^{\blacksquare} & -8 & 0 & 4^* & 2 & 2 & 0 & 3^* \end{array} \end{array}$$

Thus, $\max \min = 0$, $\min \max = 2$. There are no saddle points.

6. We compute $\max \min = -1$ and $\min \max = 0$ in pure strategies. So $-1 \leq v \leq 0$ for the value v of game. There are no saddle points.

Choosing the row 4, he (the row player) gets at least -1 (i.e., he pays at most 1), but using the mixed strategy $[0, 1/3, 0, 2/3, 0]^T$ he gets at least $-2/3$. Thus, $-2/3 \leq v \leq 0$. Choosing column 2 or 10, she pays him at most 0, but choosing the halfsum of this columns, she pays him at most $-1/2$ (i.e., she gets at least $1/2$). Thus, $-2/3 \leq v \leq -1/2$.

7. Optimal strategies are

$$[0, 1/3, 0, 0, 7/15, 1/5]^T, [0, 2/3, 0, 0, 0, 0, 0, 0, 0, 1/3],$$

and the value of game is $-2/3$.

8. Optimal strategies are

$$[0, 0, 0, 19/121, 94/121, 8/121]^T, \\ [11/121, 25/121, 0, 0, 0, 0, 0, 0, 0, 85/121],$$

and the value of game is $-14/121$.

9. Optimal strategies are

$$[55/137, 0, 0, 44/137, 0, 0, 38/137]^T, \\ [0, 0, 20/137, 0, 65/137, 0, 0, 0, 0, 0, 52/137],$$

and the value of game is $-54/137$.

10. Optimal strategies are

$$[0, 1/4, 0, 0, 0, 0, 0, 3/4, 0]^T, \\ [0, 1/2, 0, 0, 0, 0, 0, 0, 0, 1/2],$$

and the value of game is $-1/2$.

11. We have seen that $a_{i,j} = a_{i',j} = a_{i,j'} = a_{i',j'}$ because $a_{i,j} \geq a_{i',j} \geq a_{i',j'} \geq a_{i,j'} \geq a_{i,j}$. Since $a_{i,j}$ and $a_{i',j}$ are maximal in their columns j and j' , so are $a_{i',j}$ and $a_{i,j'}$. Since $a_{i,j}$ and $a_{i',j}$ are minimal in their columns i and i' , so are $a_{i',j}$ and $a_{i,j'}$. Thus, (i, j') and (i', j) are saddle points.

12. We compute all payoffs corresponding to the mixed strategies

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$$p = [9/68, 23/68, 28/68, 8/68]^T, q = [5/12, 0, 2/12, 0, 5/12] :$$

$$\begin{array}{c|cccccc} A \setminus B & (4,0) & (3,1) & (2,2) & (1,3) & (0,4) & q \\ \hline (3,0) & -3 & 0 & 4 & 3 & 2 & 1/4 \\ (2,1) & 0 & -2 & -1 & 2 & 1 & 1/4 \\ (1,2) & 1 & 2 & -1 & -2 & 0 & 1/4 \\ (0,3) & 2 & 3 & 4 & 0 & -3 & 1/4 \\ p & 1/4 & 1/2 & 1/4 & 1/4 & 1/4 & 1/4^{*'} \end{array} .$$

Since the last $1/4$ is both maximal in its column and minimal in its row, we got a saddle point. there are other optimal strategies, e.g., $[5/12, 0, 1/6, 0, 5/12]$.

§20. Matrix Games and Linear Programming

1. The optimal strategy for the row player is $[2/3, 1/3, 0]^T$.

The optimal strategy for the column player is $[1/2, 1/2, 0]$.

The value of the game is 2.

2. Let r_1, r_2, r_3 (resp., c_1, c_2, c_3) denote the strategies of the row (resp., column) player. Since r_2 is dominated by $(2r_1 + 3r_3)/5$, r_2 can be dropped. Since c_1 is dominated by $(c_2 + c_3)/2$, c_1 can be dropped. The remaining 2×2 matrix game can be easily solved using slopes. The optimal strategies are $[2/9, 0, 7/9]^T$ and $[0, 4/9, 5/9]$, and the value of the game is $8/9$.

3. The optimal strategy for the row player is $[0.2, 0, 0.8]^T$.

An optimal strategy for the column player is $[0, 0.5, 0.5, 0, 0, 0]$.

The value of the game is 1.

4. By domination, we can remove repeated rows and columns and obtain the matrix

$$\begin{bmatrix} 1 & -3 & 5 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} .$$

Using domination by mixed strategies as in the solution of Exercise 2 above, we can eliminate the second row and the first column. The remaining 2×2 matrix game with the payoff matrix $\begin{bmatrix} -3 & 5 \\ 2 & 0 \end{bmatrix}$ can be easily solved using slopes: the optimal strategies are $[0.2, 0.8]^T$, $[0.5, 0.5]$, and the value of game is 1. For the original game, the answer is: optimal strategies are

$$[0.2, 0, 0, 0, 0.8]^T, [0, 0, 0, 0.5, 0.5],$$

and the value of game is 1.

5. The optimal strategy for the row player is $[1/3, 2/3, 0]^T$.
 The optimal strategy for the column player is $[2/3, 0, 0, 0, 1/3]$.
 The value of the game is $-2/3$.
6. Using slopes, the optimal strategy for the column player is $[7/12, 5/12]$ and the value of game is $15/4$.
7. The optimal strategy for the row player: $[1/8, 0, 7/8, 0]^T$.
 The optimal strategy for the column player: $[0, 1/4, 0, 0, 0, 3/4]$.
 The value of the game is -0.25 .
8. Optimal strategies are
 $[0, 1/2, 1/2, 0, 0]^T, [0, 1/2, 1/4, 0, 0, 0, 1/4]$,
 and the value of game is 0.
9. Optimal strategies are
 $[0, 0, 0, 7/8, 1/8]^T, [3/8, 0, 0, 0, 0, 0, 0, 0, 5/8]$,
 and the value of game is $3/8$.
10. Optimal strategies are
 $[0, 0, 1/3, 2/3, 0]^T, [0, 2/3, 0, 0, 0, 0, 0, 0, 1/3]$,
 and the value of game is $-2/3$.

11. Note that the second constraint is redundant, because it follows from the first one. We solve the equation for x_6 and exclude this from our LP. We obtain an equivalent LP with all $x_i \geq 0$:

$$10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5 - 9 \rightarrow \min,$$

$$3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4,$$

$$(x_6 + 3 =) 3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 3.$$

Again, the second constraint is redundant.

Now we take advantage of the fact that in this problem all coefficients in the objective function and all right-hand parts of constraints are positive. We set $v = 1/(10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5) > 0$ on the feasible region and

$$p_1 = 10x_1v, p_2 = 5x_2v, p_3 = 4x_3v, p_4 = 7x_4v, p_5 = 4x_5v.$$

All $p_i \geq 0$ and $p_1 + p_2 + p_3 + p_4 + p_5 = 1$. The minimization of $1/v - 9$ is equivalent to the maximization of v .

The constraint $3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4$ take the form

$$(3p_1/10 + p_2/5 + p_3/4 + 2p_4/7 + p_5/4)/4 \geq v.$$

This is the row player program for the matrix game with the

payoff matrix $\begin{bmatrix} 3/40 \\ 1/20 \\ 1/16 \\ 1/14 \\ 1/16 \end{bmatrix}$. Our effort to get a smaller game pays, be-

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cause with can solve this game: the value of game is $v = 1/14$ and the optimal strategy is $[p_1, p_2, p_3, p_4, p_5]^T = [0, 0, 0, 1, 0]^T$.

This translates to $\max(1/v - 9) = 5$ at $x_1 = x_2 = x_3 = x_5 = 0, x_4 = 2, x_6 = 2x_4 - 3 = 1$.

12. We write our LP in a canonical form $-Ax \leq b, x \geq 0, cx \rightarrow \min$ corresponding to their standard row tableau (13.4) with

$$x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T, \\ c = [-1, -2, -4, -1, -1, -1, 0], b = [-5, -6, 7]^T,$$

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & -2 & 1 \\ 3 & 1 & 1 & 2 & 1 & 1 & -1 \\ -3 & -1 & -1 & -2 & -1 & 1 & 0 \end{bmatrix}$$

Then

$$M = \begin{bmatrix} 0 & -A & -b \\ A^T & 0 & -c^T \\ b^T & c & 0 \end{bmatrix}$$

is our payoff matrix (see page 219 of the textbook).

13. We take advantage of the fact that all coefficients in the objective function and all right-hand parts of constraints are positive. We set $v = 1/(x_1 + 2x_2 + x_3 + x_4 + x_5 + 3x_6 + x_7 + x_8) > 0$ on the feasible region (because the point where all $x_i = 0$ is not a feasible solution) and $p_1 = x_1v, p_2 = 2x_2v, p_3 = x_3v, p_4 = x_4v, p_5 = x_5v, p_6 = 3x_6v, p_7 = x_7v, p_8 = x_8v$. Our constraints take the form

$$3p_1 + p_2/2 + p_3 + 2p_4 + p_5 + p_6/3 + p_7 - 3p_8 \geq v, \\ 3p_1/5 + p_2/10 + p_3/5 + 2p_4/5 - p_5/5 + p_6/15 + p_7/5 + 3p_8/5 \geq v, \\ 3p_1 + p_2/2 + p_3 + 2p_4 + p_5 - p_6/3 + p_7 + p_8 \geq v.$$

Other constraints are: $p_i \geq 0$ for all i and $\sum_{i=1}^8 p_i = 1$. The minimization of $1/v$ is equivalent to the maximization of v .

Thus, we obtain the LP for the row player, with the payoff

$$\text{matrix} \begin{bmatrix} 3 & 3/5 & 3 \\ 1/2 & 1/10 & 1/2 \\ 1 & 1/5 & 1 \\ 1 & -1/5 & 1 \\ 1/3 & 1/15 & -1/3 \\ 1 & 1/5 & 1 \\ -3 & 3/5 & 1 \end{bmatrix}.$$

At the position (1,2), we have a saddle point, so the value of game is 0.6 and an optimal strategy for the row player is

$$[p_1, p_2, p_3, p_4, p_5, p_6]^T = [1, 0, 0, 0, 0, 0]^T.$$

This translates to $\min = 5/3$ at $x_1 = 5/3$, the other $x_i = 0$.

§21. Other Methods

1. The first row and column are dominated. The optimal strategy for the row player is $[0, 0.5, 0.5]^T$. The optimal strategy for the column player is $[0, 0.25, 0.75]$. The value of the game is 2.5.

2. The second row is dominated by the first row, and the second column is dominated by the third column. So we obtain a 2×2 game $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ which can be easily solved using slopes: optimal strategies are $[0.4, 0.6]^T$, $[0.4, 0.6]$, and the value of game is 1.2. For the original game, the answer is: optimal strategies are $[0.4, 0, 0.6]^T$, $[0.4, 0, 0.6]$, and the value of game is 1.2.

3. The optimal strategy for the row player is $[0, 0.4, 0, 0.6]^T$. The optimal strategy for the column player is $[0, 0.4, 0.6]$. The value of the game is 2.8.

4. By domination, we can reduce our matrix to $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$. this

game can be solved easily by graphical method. Answer: optimal strategies are $[0, 0, 0, 1, 0, 0]^T$ (a pure strategy corresponding to the fifth row) and $[0, 0.5, 0.5]$, and the value of game is 2.

5. An optimal strategy for the row player is $[1/3, 1/3, 1/3]^T$. An optimal strategy for the column player is $[0, 0, 2/7, 3/7, 2/7, 0]$. The value of the game is 0.

6. Using slopes, the value of game is $15/4$.

7. The value of the game is 0 because the game is symmetric.

8. There is a saddle point at the position (1, 3). The value of game is 1, and optimal strategies are $[1, 0, 0, 0, 0]^T$, $[0, 0, 1, 0, 0]$.

9. The first two columns and the first row go by domination. It is easy to solve the remaining 2×2 matrix game. The value of game is $11/7$, and optimal strategies are $[0, 5/14, 9/14]^T$, $[0, 0, 5/7, 2/7]$.

10. The value of game is 5, and optimal strategies are $[0, 1, 0, 0, 0]^T$ (the second row) and $[0.5, 0, 0.5, 0]$.

11. There is a saddle point at the position (1, 1). So the value of game is 0.

The last column is dominated by any other column. After dropping this column, we get a symmetric game. This is another way to see that the value of game is 0.

12. The value of game is $1/2$, and optimal strategies are

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$[0, 1/6, 1/2, 1/3, 0]^T$ and $[1/4, 0, 3/4, 0, 0]$.

13. 0 at a saddle point (at position (1, 1)). The row player has also other optimal strategies, e.g., $[1/3, 1/3, 0, 0, 1/3]^T$

14. The value of game is 1, and optimal strategies are $[21/59, 33/295, 32/295, 14/59, 11/59, 0]^T$ and $[0, 0, 0, 0, 0, 1, 0]$ (the seventh column).

Chapter 8. Linear Approximation

§22. What is Linear Approximation?

1. The mean is $-2/5 = -0.4$. The median is 1. The midrange is $-5/2 = -2.5$.

2. The mean is $148/7 \approx 21.1429$. The median is 8. The midrange is $61/2 = 20.5$.

3. The mean x_2 is $5/9$. The median x_1 is 0. The midrange x_∞ is $1/2 = 0.5$.

4. $x_2 = 3.1; 3 \leq x_1 \leq 5; x_\infty = 1$.

5. (a) 0, 0, 2, 2, 5.

(b) 1, 2, 9.

(c) 0, 0, 2, 2, 3.

(d). Exercise 1.

(e) 1, 2, 2.2, 3, 3.

(f). Exercise 3.

6. Consider the model $w = ch^2$ with $h = [1, 1.5, 2]$ and $w = [45, 50, 200]$. The median of $[w_1/h_1^2, w_2/h_2^2, w_3/h_3^2] = [45, 200/9, 50]$ is 45, while the optimal solution to

$$|w_1 - ch_1^2| + |w_2 - ch_2^2| + |w_3 - ch_3^2| \rightarrow \min$$

is $c = 50$. This is because the slope of the objective function is

$$\begin{cases} 7.25 & \text{for } c > 50 \\ -0.75 & \text{for } 45 < c < 50. \end{cases}$$

7. The program

$$|65 - 1.6c| + |60 - 1.5c| + |70 - 1.7c| \rightarrow \min$$

can be easily solved by computing slopes. E.g., the slope of the objective function on the interval $60/1.5 = 40 < c < 65/1.6 = 40.625$ is $1.5 - 1.6 - 1.7 = -1.8$ while the slope on the interval $65/1.6 = 40.625 < c < 70/1.7 \approx 42.2$ is $1.6 + 1.5 - 1.7 = 1.4$. Thus, $\min = 1.875$ at $c = 40.625$.

Since $1.875 < 6.25$ the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^1 -metric.

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The program

$$(65 - 1.6c)^2 + (60 - 1.5c)^2 + (70 - 1.7c)^2 \rightarrow \min$$

can be easily solved by differentiation. We obtain

$$1.6(65 - 1.6c) + 1.5(60 - 1.5c) + 1.7(70 - 1.7c) = 0,$$

hence $c \approx 40.6494$, $\min \approx 1.75325$. So the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^2 -metric.

The program

$$\max(|65 - 1.6c|, |60 - 1.5c|, |70 - 1.7c|) \rightarrow \min$$

can be easily solved by computing slopes. Near $c = 40.625$, the objective function is $\max(70 - 1.7c, 1.5c - 60)$. So $\min = 0.9375$ at $c = 40.625$. So the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^∞ -metric.

8. The program

$$|65 - 1.6^3a| + |60 - 1.5^3a| + |70 - 1.7^3a| \rightarrow \min$$

can be easily solved by computing slopes. E.g., the slope of the objective function near the point $a = 65/1.6^3 \approx 15.8691$ are $-1.6^3 + 1.7^3 - 1.5^3 < 0$ on the left and $1.6^3 + 1.7^3 - 1.5^3 > 0$ on the right. Thus, $\min = 14.4067$ at $a = 65/1.6^3 \approx 15.8691$.

The model $w = ch^2$ is better than the model $w = ch^3$ in Example 22.7 when we use l^1 -metric.

The program

$$(65 - 1.6^3a)^2 + (60 - 1.5^3a)^2 + (70 - 1.7^3a)^2 \rightarrow \min$$

can be easily solved by differentiation. We obtain

$$1.6^3(65 - 1.6^3a) + 1.5^3(60 - 1.5^3a) + 1.7^3(70 - 1.7^3a) = 0,$$

hence $a \approx 15.5366$, $\min \approx 99.1546$. So the model $w = ch^2$ is better than the model $w = ch^3$ in Example 22.7 when we use l^2 -metric.

The solve the program

$$\max(|65 - 1.6^3a|, |60 - 1.5^3a|, |70 - 1.7^3a|) \rightarrow \min$$

with *Mathematica*. The command is

```
FindMinimum[Max[Abs[65 - 1.6^ 3a], Abs[60 - 1.5^ 3a],
Abs[70 - 1.7^ 3a]], {a,1,2}]
```

The answer is $\min \approx 7.06202$ at $a \approx 15.6853$. So the model $w = ch^3$ gives a worse l^1 -fit than the model $w = ch^2$ in Example 22.7.

9. We enter the data $h = [1.6, 1.5, 1.7, 1.8]$, $w = [65, 60, 70, 80]$ to *Mathematica* as

```
h = { 1.6, 1.5, 1.7, 1.8}
w = {65, 60, 70, 80}
```

For $p = 1$, the objective function (to be minimized) is

```
f=Apply[Plus,Abs[w-a*h^ 2]]
```

An optimization command is

```
FindMinimum[f,{a,1,2}]
```

The answer is $\min \approx 7.59259$ at $a \approx 24.6914$. For comparison, the model $w = b$ gives $\min = 25$ when $65 \leq b \leq 70$ (medians).

For $p = 2$, we enter

```
f=Apply[Plus,(w-a*h^ 2)^ 2]; FindMinimum[f,{a,1,2}]
```

and obtain $\min \approx 21.0733$ at $a \approx 25.0412$. For comparison, the model $w = b$ gives $\min = 218.75$ when $b = 68.75$ (the mean).

For $p = \infty$, we enter

```
f=Max[Abs[w-a*h^ 2]]; FindMinimum[f,{a,1,2}]
```

and obtain $\min \approx 3.09339$ at $a \approx 25.2918$. For comparison, the model $w = b$ gives $\min = 10$ when $b = 70$ (the midrange).

So the model $w = ah^2$ gives better l^p -fits for our data than the model $w = b$ for $p = 1, 2, \infty$.

10. We enter data as in the previous exercise, and then we enter:

```
f=Apply[Plus,Abs[w-a*h]]; FindMinimum[f,{a,1,2}]
```

with the answer $\min \approx 3.63636$ at $a \approx 42.4242$ for $p = 1$;

```
f=Apply[Plus,(w-a*h)^ 2]; FindMinimum[f,{a,1,2}]
```

with the answer $\min \approx 34.5978$ at $a \approx 41.77332$ for $p = 2$;

```
f=Max[Abs[w-a*h]]; FindMinimum[f,{a,1,2}]
```

with the answer $\min \approx 3.63636$ at $a \approx 42.4242$ for $p = \infty$.

11. We enter data as in the previous exercise, and then we enter

```
f=Apply[Plus,Abs[w-a*h^ 3]]; FindMinimum[f,{a,1,2}]
```

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with the answer $\min \approx 21.6477$ at $a \approx 14.2479$ for $p = 1$;

$f = \text{Apply}[\text{Plus}, (w - a * h^3)^2]; \text{FindMinimum}[f, \{a, 1, 2\}]$

with the answer $\min \approx 167.365$ at $a \approx 14.8198$ for $p = 2$;

$f = \text{Max}[\text{Abs}[w - a * h^3]]; \text{FindMinimum}[f, \{a, 1, 2\}]$

with the answer $\min \approx 8.68035$, at $a \approx 15.2058$ for $p = \infty$.

12. For the model $p_n = cn$,

the best l^1 -fit is for $c \approx 4.98611$ (with $\min \approx 1968$),

the best l^2 -fit is for $c \approx 4.96398$ (with $\min \approx 47428$),

the best l^∞ -fit is for $c \approx 5.03077$ (with $\min \approx 38$).

For the model $p_n = cn \log(n)$,

the best l^1 -fit is for $c \approx 1.16302$ (with $\min \approx 251$),

the best l^2 -fit is for $c \approx 1.16038$ (with $\min \approx 1056$),

the best l^∞ -fit is for $c \approx 1.15849$ (with $\min \approx 8.5$).

We can see that $cn \log(n)$ is a better l^p -fit to p_n than cn for all three values of $p = 1, 2, \infty$. The model $p_n = cn \log(n + 1)$ gives slightly better fits.

13. These are not linear approximations. Taking log of both sides, we obtain a model $\log_2(F_t) = ct$. For this model,

the best l^1 -fit is for $c \approx 0.680907$ (with $\min \approx 9.5$),

the best l^2 -fit is for $c \approx 0.680096$ (with $\min \approx 2.8$),

the best l^∞ -fit is for $c \approx 0.671023$ (with $\min \approx 0.67$).

The limit value for c when we take morw and more terms of the sequence is $\log_2(\sqrt{5} + 1) - 1 \approx 0.694242$.

§23. Linear Approximation and Linear Programming

1. $\min = 0$ at $a = -15, b = 50$ for $w = a + bh$

and

$\min \approx 19$ at $a \approx 25.23$ for $w = ah^2$

2. $x + y + 0.3 = 0$

3. $a = 0.9, b \approx -0.23$

4. We set

$w = [85, 60, 70, 80], h = [1.6, 1.5, 1.7, 1.8]; c = [1, 1, 1, 1]$.

We want to find a least squares solution to $[c^T, h^T][a, b]^T = w^T$,

i.e., a solution to $\begin{bmatrix} c \cdot c & c \cdot h \\ c \cdot h & h \cdot h \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \cdot w \\ h \cdot w \end{bmatrix}$.

The solution is $a \approx 37.2348, b \approx 13.3511$.

5. We enter the data in *Mathematica*:

$h = \{1.5, 1.6, 1.7, 1.7, 1.8\}; w = \{60, 65, 70, 75, 80\}$

Then we enter

$\text{FindMinimum}[\text{Apply}[\text{Plus}, \text{Abs}[w - a * h^2]], \{a, 1, 2\}]$

The answer is $\min \approx 10.1367$ at $a \approx 25.3906$.

6. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Apply[Plus,Abs[w-a*h^ 3]],{a,1,2}]
```

The answer is $\min \approx 24.9797$ at $a \approx 15.2656$.

7. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Apply[Plus,Abs[w-a*h-b]],{a,1,2},{b,1,2}]
```

The answer is $\min \approx 13.7647$ at $a \approx 40.5882, b \approx 1$.

8. It makes sense to compare the best l^p -fits for various models with the same p . For instance, for data in Problem 23.2 Exercises 5–7 show that the model $w = ah + b$ gives a better fit than the model $w = ah^2$, and the model $w = ah^2$ is worst (all in l^1 -sense). It is harder to compare the best l^2 -fit and the best l^1 -fit of the form $w = ah^2$ —either is best in its own sense. We can compare the l^1 -average error $\approx 13.7647/5 \approx 2.75294$ in Exercise 7 with the l^2 -average error $\approx ((\sum_{i=1}^5 (w_i + 41.7 - 67.3h_i)^2)/5)^{1/2} \approx (67.3077/5)^{0.5} \approx 3.669$. So in some sense l^1 -fit is better than l^2 -fit for our data. However one should recall that l^p -fit of positive numbers cannot be worse than l^q -fit of the same numbers when $1 \leq p < q$.

9. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Max[Abs[w-a*h^ 2]],{a,1,2}]
```

The answer is $\min \approx 3.09339$ at $a \approx 25.2918$.

10. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Max[Abs[w-a*h^ 3]],{a,1,2}]
```

The answer is $\min \approx 8.68035$ at $a \approx 15.2058$.

11. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Max[Abs[w-a*h-b]],{a,1,2},{b,1,2}]
```

The answer is $\min \approx 3.72727$ at $a \approx 41.8182, b \approx 1$.

12. We get the least squares fit to p_n by $an + b$ when $a \approx 5.53069, b \approx -37.9697$ with $\min \approx 11923$ being the minimal value for

$$\sum_{i=1}^{100} (p_n - an - b)^2,$$

while

$$\sum_{i=1}^{100} (p_n - n \log(n))^2 \approx 161206.$$

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13. This is not a linear approximation. Taking log of the both sides, we get a linear model $\log_2(F_n) = \log_2(a) + bn$. The least squares fit is

$\min \approx 0.269904$ at $b \approx 0.693535$, $\log_2(a) \approx -0.443517$,
so $a \approx 0.74$.

14. We introduce three new variables a, b, c , and here is our linear program with 9 variables and 15 linear constraints:

$$a + b + c \rightarrow \min$$

subject to

$$e_1 = 2x_1 + 3x_2 - 1, -a \leq e_1 \leq a,$$

$$e_2 = x_1 - 2x_2 - 2, -b \leq e_2 \leq b,$$

$$e_3 = -x_1 + x_2 + 3, -c \leq e_3 \leq c,$$

$$e_4 = x_1 - x_2 - 4 - c \leq e_4 \leq c,$$

$$-e_3 \leq e_1 \leq e_3 \leq 4.$$

The variables e_1, e_2, e_3, e_4 can be easily excluded.

Answer;

$\min \approx 2.42857$ at $x_1 \approx 1.14286, x_2 \approx -0.428571$.

§24. More Examples

1. The model is $w = ah + b$, or $w - x_2 = a(h - 1988) + b'$ with $b = x_2 + 1988a$ and $x_2 = 37753/45 \approx 838.96$. Predicted production P in 1993 is $x_2 + 5a + b'$.

For $p = 1$, we have $a \approx 16.54, b' \approx 31, P \approx 953$.

For $p = 2$, we have $a \approx 0, b' \approx 32, P \approx 871$.

For $p = \infty$, we have $a \approx 17.59, b' \approx 32, x_5 \approx 959$.

So in this example l^∞ -prediction is the best.

2. For our data, the equation $w = ar + b$ is the system of linear equations $AX = w$, where

$$w = [5, 13, 29, 45, 81, 113, 149, 197, 253]^T,$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T, \text{ and } X = \begin{bmatrix} a \\ b \end{bmatrix}.$$

The least-squares solutions are solutions to $A^TAX = A^Tw$, i.e.,

$$\begin{bmatrix} 285 & 45 \\ 45 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6277 \\ 885 \end{bmatrix}.$$

The solution is $a = 463/15, b = -56$. The error

$$\sum_{i=1}^9 (w_i - 463i/15 + 56)^2 = 48284/15 \approx 3218.93.$$

For $w = ar^2$, the system is $Ba = w$, where w is as above and $B = [1^2, 3^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2]^T$. The equation $B^T Ba = B^T w$ is $415398a = 47598$, hence $a = 23799/7699 \approx 3.09118$. The error

$$\sum_{i=1}^9 (w_i - 23799i^2/7699)^2 = 2117089/7699 \approx 274.982.$$

So the model $w = ar^2$ gives a much better least-squares fit than the model $w = ar + b$ although it has one parameter instead of two.

The model $w = \pi r^2$ without parameters gives the error

$$\sum_{i=1}^9 (w_i - \pi i)^2 = 147409 - 95196\pi + 15398\pi^2 \approx 314.114.$$

It is known that $w_i/h_i^2 \rightarrow \pi$ as $i \rightarrow \infty$. Moreover, $|w_i - \pi r_i^2|/r_i$ is bounded as $i \rightarrow \infty$. It follows that $a \rightarrow \pi$ for the the best L_p -fit $w = ar^2$ for any $p \geq 1$ as we use data for $r = 1, 2, \dots, n$ with $n \rightarrow \infty$.

3. $a = \$4875, b = \1500

4. We enter data for 1900-1998 in *Mathematica* with t replaced by $t - 1990$:

$t = \{0, 1, 2, 3, 4, 5, 6, 7, 8\};$

$x = \{421, 429, 445, 449, 457, 460, 481, 503, 514\};$

$y = \{628, 646, 764, 683, 824, 843, 957, 1072, 1126\}$

To get the best l^1 -fit, we enter

`FindMinimum[Apply[Plus, Abs[y-a*t-b*x-c]],
{a,1,2}, {b,1,2}, {c,1,2}]`

with response

`{280.599, {a -> 47.7205, b -> 1.39226, c -> 1.}}`

Then we plug these values for a, b, c to $1070 - 9a - 523b - c$ (to get the difference between actual value 1070 and the prediction) and obtain ≈ -89 .

To get the best l^2 -fit, we enter

`FindMinimum[Apply[Plus, (y-a*t-b*x-c)^2],
{a,1,2}, {b,1,2}, {c,1,2}]`

with response

`{8275.95, {a -> -0.0169939, b -> 5.63814, c -> -1767.27}}`

hence $1070 - 9a - 523b - c \approx -111$.

To get the best l^∞ -fit, we enter

`FindMinimum[Max[Abs[y-a*t-b*x-c]],
{a,1,2}, {b,1,2}, {c,1,2}]`

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with response

$$\{8275.95, \{a \rightarrow 90.521, b \rightarrow 1.09929, c \rightarrow 1.\}\}$$

hence $1070 - 9a - 523b - c \approx -321$.

So the l^1 -fit gave the best prediction.