An Algebraic Identity of F.H. Jackson and its Implications for Partitions.

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ABSTRACT: *An identity of F.H. Jackson is used to derive new partition generating functions and related identities.*

§1 **Introduction**

In the study of the theory of partitions, we often consider infinite product generating functions. This is perhaps epitomised in the study of partition ideals of order 1 (cf. [2, pp.124-128]). In every previous instance, the factors in the infinite product arise from either the finite geometric series

$$1 + q^n + \ldots + q^{(r-1)n} = \frac{1 - q^{rn}}{1 - q^n} \tag{1.1}$$

or the infinite geometric series

$$1 + q^n + q^{2n} + \ldots = \frac{1}{1 - q^n}. \tag{1.2}$$

Thus in every case, we see the familiar factors $1 - q^{an}$ arising in the numerator and denominator. Multiplying instances of these two identities together yields every generating function for partitions in which different parts appear independently although the number of appearances of each individual part may be restricted (i.e. partition ideals of order 1).

In this way we are led quickly to the products of Euler [2; pp. 4,5]:

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n; \tag{1.3}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{O}(n) q^n &= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} \\ &= \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n} \\ &= \prod_{n=1}^{\infty} (1+q^n) = \sum_{n=0}^{\infty} \mathcal{D}(n) q^n, \end{aligned} \tag{1.4}$$

where $p(n)$ is the total number of partitions of n, $\mathcal{O}(n)$ is the number of partitions of n into odd parts and $\mathcal{D}(n)$ is the number of partitions of n into distinct parts.

In the same vein, we have the infinite product portions of the Rogers-Ramanujan identities [2; §§7.1.6, 7.1.7]

$$G(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})}, \tag{1.5}$$

$$H(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}. \tag{1.6}$$

In this note, we bring to light identities more recondite than (1.1) and (1.2) which give rise to interesting and useful results in the theory of partitions.

We begin with a surprising but little known identity due to F.H. Jackson [5]:

$$1 - \frac{a(1-b)(1-c)(1-d)(1-a^2bcd)}{(1-ab)(1-ac)(1-ad)(1-abcd)} = \frac{(1-a)(1-abc)(1-abd)(1-acd)}{(1-ab)(1-ac)(1-ad)(1-abcd)}. \tag{1.7}$$

In fact, as Jackson pointed out, this identity is the instance $n = q = 1$ of his $q-$analogue of Dougall's theorem [5].

If we differentiate both sides of (1.7) w.r.t. a and then set $a = 1$ we find

$$1 + \frac{b}{1-b} + \frac{c}{1-c} + \frac{d}{1-d} - \frac{bcd}{1-bcd} = \frac{(1-bc)(1-bd)(1-cd)}{(1-b)(1-c)(1-d)(1-bcd)}. \tag{1.8}$$

Setting $d = 0$ in (1.8) gives the simple

$$1+\sum_{n=1}^{\infty}(b^n+c^n) = 1+\frac{b}{1-b}+\frac{c}{1-c} = \frac{1-bc}{(1-b)(1-c)} \tag{1.9}$$

and, setting $d = bc$ in (1.8), we find

$$1+\sum_{n=1}^{\infty}(b^n+c^n+(bc)^{2n-1}) = 1+\frac{b}{1-b}+\frac{c}{1-c}+\frac{bc}{1-b^2c^2} = \frac{(1-b^2c)(1-bc^2)}{(1-b)(1-c)(1-b^2c^2)}. \tag{1.10}$$

In §2, we will use (1.9) and (1.10) to obtain a number of infinite product generating functions for partitions. Subsequently, in §3, we shall exhibit some identities for these newly defined partition functions.

§2 Generating Functions.

We shall be considering partitions whose parts lie in prescribed arithmetic progressions modulo k. In each example, we shall ask that the parts differ by a certain amount.

Definition 1. Suppose $0 < a < b < k$. Let $W_1(a,b;k;n)$ denote the number of partitions of n in which the parts are congruent to a or b modulo k and such that, for any j, $kj+a$ and $kj+b$ are not both parts.

Definition 2. Suppose $0 < 2a < 2b < k$. Let $W_2(a,b;k;n)$ denote the number of partitions of n in which the parts are congruent to $\pm a$ or $\pm b$ modulo k and such that any two parts each congruent to $\pm a$ modulo k do not differ by $2a$ and any two parts congruent to $\pm b$ modulo k do not differ by $k-2b$.

Definition 3. Supose $0 < a < b < a+b \leq k$. Let $W_3(a,b;k;n)$ denote the number of partitions of n in which the parts are congruent to a or b modulo k and in which, if two parts p_1 and p_2 are incongruent modulo k and differ by $b-a$, then the total number of appearances of p_1 is odd and equals the total number of appearances of p_2.

Definition 4. For $i = 1, 2, 3$,

$$\mathcal{W}_i(a,b;k;q) := \sum_{n=0}^{\infty} W_i(a,b;k;n)q^n. \tag{2.1}$$

We have, by (1.9),

$$(2.2)\ \mathcal{W}_1(a,b;k;q) = \prod_{n=0}^{\infty}\left(1+\sum_{m=1}^{\infty} q^{(kn+a)m} + \sum_{k=1}^{\infty} q^{(kn+b)m}\right)$$
$$= \prod_{n=0}^{\infty} \frac{(1-q^{2kn+a+b})}{(1-q^{kn+a})(1-q^{kn+b})}$$

and (1.9) also gives

$$(2.3)\ \mathcal{W}_2(a,b;k;q) = \sum_{n=0}^{\infty} q^{an} \prod_{n=0}^{\infty}\left(1+\sum_{m=1}^{\infty} q^{(kn+k+a)m} + \sum_{m=1}^{\infty} q^{(kn+k-a)m}\right)$$
$$\times \prod_{n=0}^{\infty}\left(1+\sum_{m=1}^{\infty} q^{(kn+b)m} + \sum_{m=1}^{\infty} q^{(kn+k-b)m}\right)$$
$$= \frac{1}{1-q^a} \prod_{n=0}^{\infty} \frac{(1-q^{2kn+2k})}{(1-q^{kn+k+a})(1-q^{kn+k-a})}$$
$$\times \prod_{n=0}^{\infty} \frac{(1-q^{2kn+k})}{(1-q^{kn+b})(1-q^{kn+k-b})}$$
$$= \prod_{n=0}^{\infty} \frac{1-q^{kn+k}}{(1-q^{kn+a})(1-q^{kn+k-a})(1-q^{kn+b})(1-q^{kn+k-b})}.$$

Finally, (1.10) gives

$$(2.4)\ \mathcal{W}_3(a,b;k;q) = \prod_{n=0}^{\infty}\left(1+\sum_{m=1}^{\infty} q^{(kn+a)m} + \sum_{m=1}^{\infty} q^{(kn+b)m} + \sum_{m=1}^{\infty} q^{(2kn+a+b)(2m-1)}\right)$$
$$= \prod_{n=0}^{\infty} \frac{(1-q^{3kn+2a+b})(1-q^{3kn+a+2b})}{(1-q^{kn+a})(1-q^{kn+b})(1-q^{4kn+2a+2b})}.$$

Note that the infinite products appearing in (2,2), (2.3) and (2.4) are not a priori power series with nonnegative coefficients. However, since each is a partition generating function, this must in fact be the case.

§3 Identities.

Theorem 1. *The number of partitions of n into non-multiples of 3 in which no two parts differ by 1 equals the number of partitions of n in which no part appears more than twice and no two parts differ by 1.*

Proof.. The first class of partitions is enumerated by $W_1(1,2;3;n)$. The second class of partitions is enumerated by the partition function $B_{1,1}(n)$ from [1] and we know from this reference that

$$\sum_{n=0}^{\infty} B_{1,1}(n)q^n = \prod_{n=0}^{\infty} \frac{(1-q^{6n+3})^2(1-q^{6n+6})}{(1-q^{n+1})}. \tag{3.1}$$

Now by (2.2)

$$\begin{aligned} \sum_{n=0}^{\infty} W_1(1,2;3;n)q^n = \mathcal{W}_1(1,2;3;q) &= \prod_{n=0}^{\infty} \frac{(1-q^{6n+3})}{(1-q^{3n+1})(1-q^{3n+2})} \\ &= \prod_{n=0}^{\infty} \frac{(1-q^{6n+3})(1-q^{3n+3})}{(1-q^{n+1})} \\ &= \prod_{n=0}^{\infty} \frac{(1-q^{6n+3})^2(1-q^{6n+6})}{(1-q^{n+1})} \end{aligned} \tag{3.2}$$

and, comparing (3.1) and (3.2) we see that

$$W_1(1,2;3;n) = B_{1,1}(n)$$

for every $n \in \mathbb{N}$ and so we have proved Theorem 1. $\square$

To illustrate Theorem 1, we note that the five partitions enumerated by $W(1,2;3;7)$ are $7,\ 5+2,\ 5+1+1,\ 4+1+1+1$ and $1+1+1+1+1$ while the five partitions enumerated by $B_{1,1}(7)$ are $7,\ 6+1,\ 5+2,\ 5+1+1$ and $3+3+1$.

Corollary. *The number of partitions of n into non-multiples of 3 in which at least two parts differ by 1 equals the number of partitions of n in which no part appears more than twice and at least two parts differ by 1.*

Proof.

This follows from Theorem 1 and Glaisher's Theorem [2; p.6], a special case of which asserts that the number of partitions of n into non-multiples of 3 equals the number of partitions of n in which no part appears more than twice. $\square$

Theorem 2. *Suppose $0 < a < b < k = a + b$. Then $W_3(a, b; k; n)$ equals the number of partitions into parts congruent to $a, 3a + 2b, b$ or $3b + 2a$ modulo $3k$ or congruent to $2k$ modulo $4k$.*

Proof. Bearing in mind that $a + b = k$, we see that, by (2.4),

$$\begin{aligned}\sum_{n=0}^{\infty} W_3(a, b; k; n)q^n &= \prod_{n=0}^{\infty} \frac{(1 - q^{3kn+2a+b})(1 - q^{3kn+a+2b})}{(1 - q^{kn+a})(1 - q^{kn+b})(1 - q^{4kn+2a+2b})} \\ &= \prod_{n=0}^{\infty} \frac{(1 - q^{3kn+2a+b})}{(1 - q^{3kn+a})(1 - q^{3kn+2a+b})(1 - q^{3kn+3a+2b})} \\ &\quad \times \prod_{n=0}^{\infty} \frac{(1 - q^{3kn+a+2b})}{(1 - q^{3kn+b})(1 - q^{3kn+a+2b})(1 - q^{3kn+2a+3b})} \times \prod_{n=0}^{\infty} \frac{1}{(1 - q^{4kn+2k})} \\ &= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{3kn+a})(1 - q^{3kn+3a+2b})(1 - q^{3kn+b})(1 - q^{3kn+2a+3b})(1 - q^{4kn+2k})},\end{aligned}$$

and this last infinite product is the generating function for the partitions of the type described in the theorem. □

Theorem 3. *$W_1(1, 3; 4; n)$ equals the number of partitions of n into odd parts in which no part appears more than three times.*

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} W_1(1, 3; 4; n)q^n &= \prod_{n=0}^{\infty} \frac{(1 - q^{8n+4})}{(1 - q^{4n+1})(1 - q^{4n+3})} \\ &= \prod_{n=0}^{\infty} \frac{(1 - q^{8n+4})}{(1 - q^{2n+1})} \\ &= \prod_{n=0}^{\infty} (1 + q^{2n+1} + q^{2(2n+1)} + q^{3(2n+1)}),\end{aligned}$$

and this last infinite product is the generating function for the partitions described in the theorem. □

We end this section with a result of a different type.

Theorem 4. *For* $0 < a < b < a+b < k$,

$$\mathcal{W}_2(a,b;k;q) = \frac{\sum_{n=-\infty}^{\infty} \frac{q^{an}}{1-q^{kn+b}}}{\sum_{n=-\infty}^{\infty} (-1)^n q^{kn(n+1)/2-(a+b)n}}$$

Proof. By (2.3)

$$\begin{aligned}\mathcal{W}_2(a,b;k;n) &= \frac{1}{\prod_{n=0}^{\infty}(1-q^{kn+a+b})(1-q^{kn+k-a-b})(1-q^{kn+k})} \\ &\quad \times \prod_{n=0}^{\infty} \frac{(1-q^{kn+a+b})(1-q^{kn+k-a-b})(1-q^{kn+k})^2}{(1-q^{kn+a})(1-q^{kn+k-a})(1-q^{kn+b})(1-q^{kn+k-b})} \\ &= \frac{\sum_{n=-\infty}^{\infty} \frac{q^{an}}{1-q^{kn+b}}}{\sum_{n=-\infty}^{\infty} (-1)^n q^{kn(n+1)/2-(a+b)n}}\end{aligned}$$

which follows by applying Jacobi's Triple Product identity [2;2.2.10] to the first factor and Ramanujan's ${}_1\Psi_1$ summation ([3;(C.1), p.115] , with q^a for t, q^k for q, q^{b+k} for b and q^b for a) to the second. $\square$

§4 Conclusion.

Our object has not been to provide an exhaustive account of these new partition functions. Indeed our main goal has been to reveal that infinite product generating functions can arise in subtler ways than had previously been encountered.

It would be of great interest if bijective proofs could be found for Theorems 1, 2 and 3. Theorems 2 and 3 are the most likely to yield to this approach, since their proofs involve only the manipulation of infinite products; a bijective proof of Theorem 1 would probably require a bijective proof of the expression given for $\sum_{n=0}^{\infty} B_{1,1}(n)q^n$.

We remark that the classes of partitions related to Definitions 1 and 3 can be extended to a very general setting. Namely, suppose $S_1 = \{a_n\}_{n=1}^{\infty}$ and $S_2 = \{b_n\}_{n=1}^{\infty}$ are disjoint sets of integers. Define $W_1(S_1, S_2; n)$ to be the number of partitions of N into elements of $S_1 \cup S_2$ which do not contain as parts both a_j and b_j, for any j and define $W_3(S_1, S_2; n)$

to be the number of partitions of n into elements of $S_1 \cup S_2$ in which, if a_j and b_j do both appear as parts, they appear the same odd number of times. Then

$$\sum_{n=0}^{\infty} W_1(S_1, S_2; n) q^n = \prod_{j=1}^{\infty} \frac{(1 - q^{a_j + b_j})}{(1 - q^{a_j})(1 - q^{b_j})}$$

and

$$\sum_{n=0}^{\infty} W_3(S_1, S_2; n) q^n = \prod_{j=1}^{\infty} \frac{(1 - q^{2a_j + b_j})(1 - q^{2b_j + a_j})}{(1 - q^{a_j})(1 - q^{b_j})(1 - q^{2a_j + 2b_j})}$$

We chose not to begin with these more general partition functions because all the results of interest that we found related to arithmetic progrssions.

In [4], some of the principles stated here are used to establish certain inequalities between the rank-counting numbers $N(r, 9, n)$.

References.

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