The Number of Smallest Parts in the Partitions of n

by

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June 20, 2007

Abstract

We denote by $\mathrm{spt}(n)$ the total number of appearances of the smallest part in each integer partition of n. We shall relate $\mathrm{spt}(n)$ to the Atkin–Garvan moments of ranks, and we shall prove that $5|\mathrm{spt}(5n+4)$, $7|\mathrm{spt}(7n+5)$ and $13|\mathrm{spt}(13n+6)$.

1 Introduction

In the study of the partitions of integers, it has often been of interest to consider weighted counts of partitions. In a series of three papers [1], [2], [3], K. Alladi really initiated the study of weighted counts of partitions including [2, Sec. 3] attaching certain simple weights to the Rogers–Ramanujan partitions (i. e. partitions with parts differing by at least 2) so that the weighted count was actually $p(n)$, the number of partitions of n.

In [10], Fokkink, Fokkink and Wang proved that a weighted sum over partitions of n into distinct parts equals the number of divisors of n.

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*Partially supported by National Science Foundation Grant DMS 0457003.
†Key Words: partitions, ranks, moments of ranks.
‡AMS Classification numbers: 05A17, 05A19, 11P83

Theorem 1. [10] *If $\mathcal{D}_n$ denotes the set of partitions π of n into distinct parts, then*

$$-\sum_{\pi\in\mathcal{D}_n}(-1)^{\#(\pi)}\sigma(\pi)=d(n), \tag{1.1}$$

where $\#(\pi)$ is the number of parts of π, $\sigma(\pi)$ is the smallest part of π and $d(n)$ is the number of divisors of n.

Their proof is a lovely combination of combinatorial analysis applied to a remarkable sequence of polynomials. However, it is instructive to note that the result is effectively a corollary of the differentiation of the q-analog of Gauss's theorem [4, p. 20, Cor. 2.4]. Namely, if $\mathrm{FFW}(n)$ denotes the left-hand side of (1.1), then standard partition-theoretic arguments (cf. [4, Ch. 1] show that

$$\begin{aligned}
\sum_{n\geqq 0}\mathrm{FFW}(n)q^n &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{(q;q)_n(1-q^n)} \\
&\qquad\qquad\text{(by consideration of conjugate partitions)}\\
&= \left[\frac{d}{dz}\sum_{n\geqq 0}\frac{(z;q)_n(-1)^nq^{n(n+1)/2}}{(q;q)_n(zq;q)_n}\right]_{z=1}\\
&= \frac{d}{dz}\left[\frac{(q;q)_\infty}{(zq;q)_\infty}\right]_{z=1}\ \big(\text{by [4, p.20, Cor.2.4 with } a=z, b=q/\tau, c=zq, \tau\to 0]\big)\\
&= \left[\frac{(q;q)_\infty}{(zq;q)_\infty}\sum_{j=1}^{\infty}\frac{q^j}{1-zq^j}\right]_{z=1}\\
&= \sum_{j=1}^{\infty}\frac{q^j}{1-q^j}=\sum_{n=1}^{\infty}d(n)q^n.
\end{aligned}$$

We have used the standard notation $(A;q)_n=(1-A)(1-Aq)\dots(1-Aq^{n-1})$.

Our purpose in presenting this argument in the introduction is to highlight the role of differentiation in proving theorems on weighted counts of partitions.

As mentioned in the abstract, this paper is devoted to the study of $\mathrm{spt}(n)$, the total number of appearances of the smallest parts in all of the partitions of n. In light of the nice theorem about $\mathrm{FFW}(n)$, it is not unreasonable that $\mathrm{spt}(n)$ be interesting as well.

As is well-known [4, Ch. 10], $p(n)$ has a number of interesting congruence properties, namely

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.3}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.4}$$

Similar results hold for $\mathrm{spt}(n)$.

Theorem 2.

$$spt(5n+4) \equiv 0 \pmod{5}, \tag{1.5}$$

$$spt(7n+5) \equiv 0 \pmod{7}, \tag{1.6}$$

$$spt(13n+6) \equiv 0 \pmod{13}. \tag{1.7}$$

The appearance of 13 in this result is completely unexpected.

In Section 2, we shall collect the necessary background results. Section 3 will be devoted to proving

Theorem 3.

$$spt(n) = np(n) - \frac{1}{2}N_2(n),$$

where $N_2(n)$ is the second Atkin-Garvan moment (see eq. (2.13) for the definition).

Section 4 is devoted to a proof of Theorem 2.

I must give special thanks to Frank Garvan who not only pointed out O'Brien's thesis [12] to me but also supplied me with a copy.

2 Background

In this section, we collect necessary facts from analysis, q-series and the theory of partitions.

We begin with three minor observations concerning any function $f(z)$ that is at least twice differentiable at $z = 1$.

$$-\frac{1}{2}\left[\frac{d^2}{dz^2}(1-z)(1-z^{-1})f(z)\right]_{z=1} = f(1). \tag{2.1}$$

This follows immediately once one computes the second derivative of the expression in question.

We also require Watson's celebrated q-analog of Whipple's theorem [11, p. 43, eq. (2.5.1)]

$$
\begin{aligned}
&{}_8\phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N}; q, \frac{a^2q^{2+N}}{bcde} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{N+1} \end{matrix}\right) \\
&= \quad \frac{(aq;q)_N(\frac{aq}{de};q)_N}{(\frac{aq}{d};q)_N(\frac{aq}{e};q)_N}\,{}_4\phi_3\left(\begin{matrix} \frac{aq}{bc}, d, e, q^{-N}; q, q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{de\,q^{-N}}{a} \end{matrix}\right),
\end{aligned} \tag{2.2}
$$

where

$$
{}_{r+1}\phi_r\left(\begin{matrix} a_0, a_1, \ldots, a_r; q, t \\ b_1, \ldots, b_r \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_0,q)_n(a_1;q)_n\cdots(a_r;q)_n t^n}{(q;q)_n(b_1;q)_n\cdots(b_r;q)_n}. \tag{2.3}
$$

Next we have another differentiation identity:

$$
-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} = (q;q)_\infty^2 \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \tag{2.4}
$$

This must be well-known, but we were unable to find a reference. The proof of (2.4) depends on the Jacobi triple product identity [4, p. 21, Th. 2.8]

$$
\begin{aligned}
&-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\frac{\sum_{n=-\infty}^{\infty}(-z)^n q^{n(n-1)/2}}{1-z}\right]_{z=1} \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=0}^{\infty}(-z)^{-n}q^{n(n+1)/2}\left(\frac{1-z^{2n+1}}{1-z}\right)\right]_{z=1} \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sum_{j=0}^{2n} z^{-n+j}\right]_{z=1} \\
&= -\frac{1}{2(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sum_{j=0}^{2n}(-n+j)(-n+j-1) \\
&= -\frac{1}{2(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\,\frac{1}{3}\,n(n+1)(2n+1)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{q}{3(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{n(n+1)}{2}q^{n(n+1)/2-1} \\
&= -\frac{q}{3(q;q)_\infty}\frac{d}{dq}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \\
&= -\frac{q}{3(q;q)_\infty}\frac{d}{dq}(q;q)_\infty^3 \\
&\qquad\qquad \text{(by [4, p. 176, ex. 7])} \\
&= (q;q)_\infty^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},
\end{aligned}
$$

where the last line follows by logarithmic differentiation.

To conclude this section we collect some results concerning the ranks of partitions, a topic begun by Dyson with his celebrated conjectures. We shall follow closely the notation as developed in [7], [8] and [12].

The rank of a partition is defined to be the largest part minus the number of parts. Thus the partition of 18 given by $5+4+4+2+1+1+1$ has rank $5-7=-2$.

We define $N(m,n)$ to be the number of partitions of n with rank m, and $N(m,Q,n)$ to be the number of partitions of n with rank $\equiv m \pmod{Q}$. Next

$$
r_{a,b}(d) = \sum_{n=0}^{\infty}(N(a,Q,Qn+d)-N(b,Q,Qn+d))q^{Qn}. \tag{2.5}
$$

The only case we shall be considering for $r_{a,b}(d)$ in detail is where $Q=13$. In this instance, we additionally define

$$
\begin{aligned}
S_1(d) &= r_{0,1}(d) - 6r_{5,6}(d), && (2.6)\\
S_2(d) &= r_{1,2}(d) - 5r_{5,6}(d), && (2.7)\\
S_3(d) &= r_{2,3}(d) - 4r_{5,6}(d), && (2.8)\\
S_4(d) &= r_{3,4}(d) - 3r_{5,6}(d), && (2.9)\\
S_5(d) &= r_{4,5}(d) - 2r_{5,6}(d), && (2.10)
\end{aligned}
$$

With these definitions (remembering $Q=13$), we have [12, p. 44]

$$
S_1(6) + 2\,S_2(6) - 5\,S_5(6) \equiv 0 \pmod{13}, \tag{2.11}
$$

and

$$
S_2(6) + 5\,S_3(6) + 3\,S_4(6) + 3\,S_5(6) \equiv 0 \pmod{13}. \tag{2.12}
$$

In [6], Atkin and Garvan introduce the moments of ranks

$$N_j(n) = \sum_{m=-\infty}^{\infty} m^j N(m,n). \tag{2.13}$$

Conjugation of partitions maps a partition with rank m to one of n with rank $-m$. Consequently,

$$N_j(n) = 0 \qquad \text{if } j \text{ is odd}. \tag{2.14}$$

In [5, eqs. (1.3) and (1.10)], it is shown that

$$\begin{aligned} R_1(z;q) :&= 1 + \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} N(m,n) z^m q^n & (2.15)\\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n} & (2.16)\\ &= \frac{(1-z)}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n}\,. & (2.17) \end{aligned}$$

3 The Two Main Theorems

Theorem 4.

$$\sum_{n \geqq 1} spt(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}\,. \tag{3.1}$$

We restate Theorem 3 from the introduction.

Theorem 3.

$$\mathrm{spt}(n) = np(n) - \frac{1}{2} N_2(n). \tag{3.2}$$

From these results we will deduce the congruences presented in the introduction.

Proof of Theorem 4.

$$
\begin{aligned}
&\sum_{n\geqq 1} \mathrm{spt}(n)q^n \\
&= \sum_{n=1}^{\infty}(q^n + 2q^{2n} + 3q^{3n} + \cdots) \prod_{j=n+1}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots) \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty} \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=0}^{\infty}\frac{(z;q)_n(z^{-1};q)_n q^n}{(q;q)_n}\right]_{z=1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (2.1))} \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(q;q)_\infty^2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)(z;q)_n(z^{-1};q)_n}{(\frac{q}{z};q)_n(zq;q)_n}\right)\right]_{z=1} \\
&\qquad\qquad\qquad \text{(by (2.2) with } a\to 1, d=e^{-1}=z,\ b,c,N\to\infty) \\
&= -\frac{1}{2(q;q)_\infty^3}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} + \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (2.1))} \\
&= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2} \qquad \text{(by (2.4)).}
\end{aligned}
$$

Proof of Theorem 3.

The object here is to show that the first expression on the right-hand side of (3.1) is the generating function for $n\,p(n)$ and the second is the generating function for $\frac{1}{2}N_2(n)$.

Recalling [4, Ch. 1]

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}\,,$$

we see that

$$\sum_{n=0}^{\infty} n\,p(n)q^n = q\,\frac{d}{dq}\frac{1}{(q;q)_\infty} = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{n\,q^n}{1-q^n}, \tag{3.3}$$

by logarithmic differentiation.

Next

$$
\begin{aligned}
&\sum_{n\geqq 0}\frac{1}{2}N_2(n)q^n\\
&=\sum_{n=0}^{\infty}\frac{N_2(n)-N_1(n)}{2}\,q^n\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(because } N_1(n)=0 \text{ by (2.14))}\\
&=\frac{1}{2}\left[\frac{d^2}{dz^2}R(z;q)\right]_{z=1} \qquad\qquad \text{(by (2.15))}\\
&=\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1-z)}{1-zq^n}\right]_{z=1} \qquad\qquad \text{(by (2.17))}\\
&=\frac{-1}{2(q;q)_\infty}\left[\frac{d}{dz}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1-q^n)}{(1-zq^n)^2}\right]_{z=1}\\
&=\frac{-1}{(q;q)_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n q^{n(3n+1)/2+n}}{(1-q^n)^2}\\
&=\frac{-1}{(q;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{-n}q^{n(3n-1)/2-n}}{(1-q^{-n})^2}+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2+n}}{(1-q^n)^2}\right)\\
&=\frac{-1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2} \qquad\qquad (3.4)
\end{aligned}
$$

Finally substituting (3.3) and (3.4) into (3.1) and comparing coefficients of q^n, we derive (3.2).

4 The Congruences

The object in this section is to prove Theorem 2 which contains three congruences. The first two are fairly direct consequences of the results of Atkin and Swinnerton-Dyer [8].

$$
\begin{aligned}
&\mathrm{spt}(5n+4) \\
&= (5n+4)p(5n+4) - \frac{1}{2}\, N_2(5n+4) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Th. 3)} \\
&\equiv -\frac{1}{2}\, N_2(5n+4) \pmod 5 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (1.2))} \\
&= -\frac{1}{2} \sum_{m=-\infty}^{\infty} m^2 N(m, 5n+4) \\
&\equiv -\frac{1}{2} \Big(N(1,5,5n+4) - N(2,5,5n+4) \\
&\qquad\qquad\qquad - N(3,5,5n+4) + N(4,5,5n+4) \Big) \pmod 5 \\
&\equiv 0 \pmod 5 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by [8, p. 85, eq. (2.4)])} \\
&\mathrm{spt}(7n+5) \\
&= (7n+5)p(7n+5) - \frac{1}{2}\, N_2(7n+5) \\
&\equiv -\frac{1}{2}\, N_2(7n+5) \pmod 7 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (1.3))} \\
&= -\frac{1}{2} \sum_{m=-\infty}^{\infty} m^2 N(m, 7n+5) \\
&\equiv -\Big(N(1,7,7n+5) + 4N(2,7,7n+5) + 2N(3,7,7n+5) \Big) \\
&\equiv 0 \pmod 7 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by [8, p. 85, eq. (2.10)])}
\end{aligned}
$$

The final congruence is a bit trickier and requires J. N. O'Brien's deep achievements concerning the modulus 13.

$$
\begin{aligned}
&\operatorname{spt}(13n+6)\\
&= (13n+6)p(13n+6) - \frac{1}{2}\,N_2(13n+6)\\
&\equiv 6p(13n+6) - \frac{1}{2}\sum_{j=0}^{12}(12+j^2-12)N(j,13,13n+6) \pmod{13}\\
&\equiv \sum_{j=0}^{12} 6(j^2-12)N(j,13,13n+6) \pmod{13}\\
&\equiv 6N(0,13,13n+6) + 12N(1,13,13n+6)\\
&\quad + 4N(2,13,13n+6) + 8N(3,13,13n+6)\\
&\quad + 11N(4,13,13n+6) + 0\cdot N(5,13,13n+6)\\
&\quad + N(6,13,13n+6) + N(7,13,13n+6)\\
&\quad + 0\cdot N(8,13,13n+6) + 11N(9,13,13n+6)\\
&\quad + 8N(10,13,13n+6) + 4N(11,13,13n+6)\\
&\quad + 12N(12,13,13n+6) \qquad\qquad \pmod{13}\\
&\equiv 2\Big(3N(0,13,13n+6) - N(1,13,13n+6)\\
&\quad +4N(2,13,13n+6) - 5N(3,13,13n+6)\\
&\quad -2N(4,13,13n+6) + N(6,13,13n+6)\Big) \pmod{13}\\
&\qquad\qquad (\text{since } N(m,13,13+6) = N(13-m,13,13n+6))
\end{aligned}
$$

This last congruence can be cast equivalently in terms of generating functions:

$$3r_{0,1}(6) + 2r_{1,2}(6) + 6r_{2,3}(6) + r_{3,4}(6) - r_{4,5}(6) - r_{5,6}(6) \equiv 0 \pmod{13}$$

Using (2.6)–(2.10), we may rewrite this last congruence using the $S_i(d)$, namely

$$
\begin{aligned}
&3(S_1(6) + 6r_{5,6}(6)) + 2(S_2(6) + 5r_{5,6}(6))\\
&+ 6(S_3(6) + 4r_{5,6}(6)) + S_4(6) + 3r_{5,6}(6)\\
&- S_5(6) - 2r_{5,6}(6) - r_{5,6}(6) \equiv 0 \pmod{13}.
\end{aligned}
$$

In this last congruence we see that the coefficient of $r_{5,6}(6)$ is $52 = 4 \cdot 13$; so all we need to prove is

$$3S_1(6) + 2S_2(6) + 6S_3(6) + S_4(6) - S_5(6) \equiv 0 \pmod{13} \tag{4.1}$$

But

$$\begin{aligned}
&3S_1(6) + 2S_2(6) + 6S_3(6) + S_4(6) - S_5(6)\\
&\equiv 3(S_1(6) + 2S_2(6) - 5S_5(6))\\
&\quad - 4(S_2(6) + 5S_3(6) + 3S_4(6) + 3S_5(6))\\
&\equiv 0 \qquad\qquad \pmod{13},
\end{aligned}$$

by (2.11) and (2.12).

This concludes the proof of Theorem 3. $\square$

5 Conclusion

The connection between spt (n) and the second order moment of Atkin and Garvan [6] in Theorem 3 suggests that there should be much more arithmetic information about spt (n) still to be unearthed.

In addition the connection of $N_2(n)/2$ to the enumeration of 2-marked Durfee symbols in [5] suggests the fact that there are also serious problems concerning combinatorial mappings that should be investigated.

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