Chapter 6

Transportation Problems

§16. Phase 1

Have you ever had the opportunity to read magazines from years gone by? The advertisements in these magazines are fascinating. In one old magazine, blue jeans, touted as an essential piece of clothing apparel for agricultural workers, were advertised for $2.99 a pair. However, in small print, was the following phrase:

Prices may be higher west of the Rockies.

Why did the East Coast manufacturers of these blue jeans feel compelled to put that disclaimer in their advertisement? Obviously, they were faced with the daunting task, and equally daunting expense, of transporting their product across the Rocky Mountains to the consumers in the West who wanted blue jeans. Now, with manufacturing plants located throughout the United States, instead of primarily in the East, and with improved means of shipping, it might not be necessary to put such disclaimers in advertisements. Nonetheless, transportation costs continue to be an important consideration for business and industry.

In Example 2.4 of Chapter 1, we discussed a particular case of transportation problems and solved this problem by using a trial-and-error method. Since transportation problems provide particularly simple and nice examples of linear programs, we will revisit these examples, setting them up as linear programs and then solving them using the methods we have just learned. Historically speaking, these problems were investigated in detail many years before the general linear program was. Interestingly, the concept of duality, which we considered in Chapter 5, first appeared in transportation problems.

In this chapter we will see that Phase 1 of solving linear programs —finding a feasible solution—can be done by hand. Also, Phase 2—finding an optimal solution—is easier in transportation problems than for general linear programs.

To introduce graphs and tables, we consider again Example 15.7 and begin with a graphical representation (Fig. 16.1).

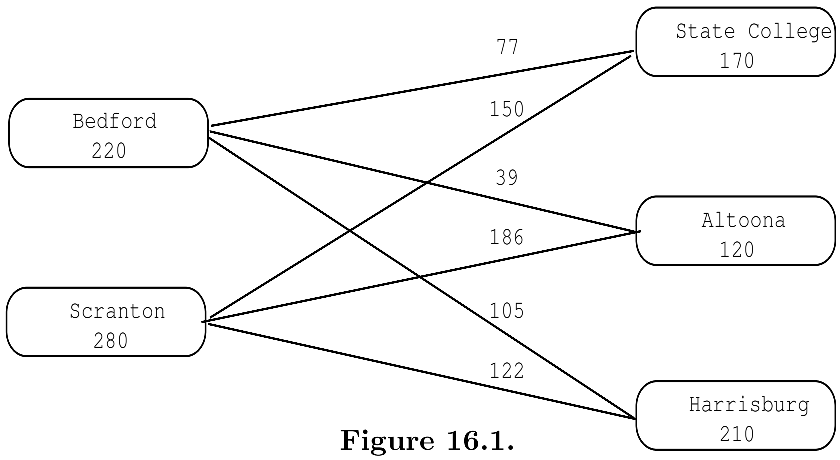

Figure 16.1.
Graphical representation of Example 15.7

Also, the numerical data can be written in a table:

77	39	105	220
150	186	122	280
170	120	210	

Table 16.2. Example 15.7 tabled

We left room in the table for values of the unknowns x_{ij} which are sometimes referred to as a $flow$. Any transportation problem can be treated similarly. Note that in Example 15.7 the *balance condition* holds: The total supply (the sum of the numbers in the right margin) equals the total demand (the sum of the numbers across the bottom).

Example 2.4 presented an open transportation problem, that is, a problem without the balance condition. That problem can be written as a closed (balanced) transportation problem, with the balance condition satisfied, by introducing a fictitious store, *dummy demand point,* where any surplus product goes at no cost. If we set the demand at the fictitious store to be exactly the total surplus supply, then the balance condition is restored.

A position in the table corresponds to an arrow in the graphical representation, and entering a number in the i, j position in the table corresponds to allocating stock from warehouse i to store j. Here is a way to find a feasible solution. First check the balance condition, which is satisfied in this problem so we can proceed. Pick a position in the table and write the maximum possible number, namely the available stock at the corresponding warehouse.

For example, in Example 15.7 we choose the first position in the first row and write there 170.

170			~~220~~ 50
			280
~~170~~	120	210	

We suppressed the objective function because we do not use it yet. Note that the first column of the table, corresponding to the store in State College, is done. We indicate this by crossing out the demand 170. The second entry in the first column must now be zero, because the demand is satisfied. We suppress entering this zero. We keep track of unallocated warehouse stock by entering a new number 50 instead of 220 at the right margin of the table.

Thus, we entered one number, crossed out one number, and adjusted another number. The first column is done (we found the flow in it), and we proceed with a smaller, 2-by-2 table.

We pick a second position in the table and again write the maximum possible number 50 = min(50, 120):

170	50		~~220 50~~
			280
~~170~~	~~120~~ 70	210	

Now both the first row and the first column are done. We repeat this procedure for remaining 1-by-2 matrix, as long as any choice remains:

170	50		~~220~~ ~~50~~
	70		~~280~~ 210
~~170~~ 0	~~120~~ ~~70~~	210	

At the last step the allocation will have been determined by the previous choices:

170	50		~~220~~ ~~50~~
	70	210	~~280~~ ~~210~~
~~170~~	~~120~~ ~~70~~	~~210~~	

The corresponding total cost is

$$77 \cdot 170 + 39 \cdot 50 + 186 \cdot 70 + 122 \cdot 210.$$

Here is the resulting feasible solution, with two zeros suppressed:

170	50		220
	70	210	280
170	120	210	

This method of finding a feasible solution works for any transportation problem under the balance condition. Now it is time to state an arbitrary transportation problem.

The General Transportation Problem.

A manufacturer has m warehouses and n retail stores. The warehouse $\# i$ has a_i units of product available and the store $\# j$ needs b_j units.

It is assumed that the following *balance condition* holds:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

For instance, in Example 15.7 the balance condition is satisfied. This is not the case in Example 2.4, as we observed. The cost of shipping one unit from warehouse $\# i$ to store $\# j$ is denoted by c_{ij} and the number of units shipped from warehouse $\# i$ to store $\# j$ is denoted by x_{ij}. The linear program can be stated as follows:

$$\begin{cases} \text{minimize} \quad & C(x_{11}, \ldots, x_{mn}) = \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij} \\ \text{subject to} \quad & \displaystyle\sum_{i=1}^{n} x_{ij} \geq b_j, \quad j = 1, \ldots, m \\ & \displaystyle\sum_{j=1}^{m} x_{ij} \leq a_i, \quad i = 1, \ldots, n \\ & x_{ij} \geq 0,\ i = 1, \ldots, n,\ j = 1, \ldots, m, \end{cases}$$

where C is the total cost function to be minimized, the constraints

$$\sum_{i=1}^{n} x_{ij} = b_j, \quad j = 1, \ldots, m$$

represent the number of units that store $\# j$ receives from warehouse $\# i$, and the remaining constraints represent the number of units that warehouse $\# i$ can supply to store $\# j$, in addition to the usual non-negativity constraints.

In this chapter we will show how the simplex method solves this problem. For now, here are some questions you might want to consider:

(a) We have been looking at ways to minimize shipping costs. However, does it ever make sense to maximize shipping costs rather than minimize them? Consider this situation: You are working for an interstate mover and get paid for shipping. In this situation, would you try to minimize or maximize the shipping cost?

(b) Is the general transportation problem *general* enough? Is Example 2.4 a particular case of the general transportation problem? In other words, can the constraints in Example 2.4 be written as equations rather than inequalities? Note that the total supply in that example is 130, which is larger than the total demand of 100. So if we just replace the inequalities by equations, then we would not have any feasible solutions.

Remark 16.3. Historically, transportation problems were the first linear programs that were explicitly stated, studied theoretically, solved, and used in industry. ■

Now we observe that the method of finding a feasible solution works for any transportation problem. We select any position and write there the maximal possible entry – namely, the minimum of the number in the right margin and the number in the bottom margin. Then we cross out one of these two numbers, which equals the written entry, and adjust the other one. At the last step, when a 1×1 table is left, we cross out both numbers. Thus, the total number of steps is $m + n - 1$, where m is the number of rows and n is the number of columns. Our method gives a feasible solution with at most $m+n-1$ nonzero values for mn variables.

To proceed with finding an optimal solution, it is convenient to make this number of selected positions exactly $m+n-1$ by crossing out only one margin entry (and making the other 0) in the case when both are the same (unless this is the last step) and allowing the value 0 to be entered at a selected position if either of the numbers at the margin is 0. By this method, we cross out one row or one column at each step, except the last. We should not cross out the last remaining row or column unless it is the last step, step $m + n - 1$.

The $m + n - 1$ positions we obtain this way correspond to the basic variables (those at the right margin) in a feasible standard row tableau. We write in the table the values of the basic variables and we do not write the zero values for nonbasic variables to leave room for the values of dual variables.

Thus, a feasible solution can be found easily by hand for any problem small enough so that the data can be written down by hand. But in fact, what we are doing is Phase 1 of the simplex method. We use smaller tables rather than standard tableaux. Initially, all variables x_{ij} are on the top, with slack variables at the right margin. The selected positions (i, j) correspond to variables x_{ij} which we pivot to the right margin. The pivot entries are 1 or -1.

Example 16.4. Find a basic feasible solution for the transportation problem

			2
			3
2	2	1	

We did not give an objective function, because we do not need it to find a feasible solution. We use the *northwest* method. According to this method, we choose the position in the upper-left corner. Note that we cross out 2 under the first column and adjust 2 at the right margin to 0:

2			~~2~~ 0
			3
~~2~~	2	1	

Now the first column is done, and the northwest method tells us to choose the second position in the first row, and the maximal number we can put there is 0:

2	0		~~2~~ ~~0~~
			3
~~2~~	2	1	

Now both the first row and first column are done. The next position we choose is in the second row and the second column:

2	0		~~2~~ ~~0~~
	2		~~3~~ 1
~~2~~	~~2~~	1	

Now only the last row and column remain. We write the last entry 1 and cross out the last two numbers at the margin:

2	0		~~2~~ ~~0~~
	2	1	~~3~~ ~~1~~
~~2~~	~~2~~	~~1~~	

In this example, $m = 2, n = 3$, and we wrote $m + n - 1 = 4$ numbers in the table, one of them being 0. ■

In graphical terms, the chosen positions are $m + n - 1$ edges (arcs) that connect all vertices (nodes), forming a graph (network) called a *tree*. This tree has exactly $m + n - 1$ arrows (or edges), is connected, and has no loops (cycles).

As mentioned previously, if the problem is expressed as a standard row tableau, the chosen positions correspond to the basic variables at the right margin of a row feasible tableau. The other positions correspond to the nonbasic variables across the top of a row feasible tableau. The method just described in fact implements Phase 1 of the simplex method. It is not necessary to work with the entire tableau for transportation problems, because the A block of the tableau always consists of 0s, 1s, and -1s. The b block of the tableau consists of the values in the chosen positions of the table. Construction of the c block will be discussed later.

We always obtain a feasible solution if the balance condition holds, and if it does not we declare the problem infeasible and stop. This is why we check the balance condition before beginning to choose entries. If the supply and demand data consist of integers, then the feasible solution determined by this method is likewise integral. It is always a vertex of the feasible region. The proof is left as an exercise.

There are many ways for choosing the table positions in our method. The northwest method ignores the objective function. But other methods for choosing positions may result in a better feasible solution, depending on the objective function. For example, in the minimum cost method, we go for a position with minimal cost.

To proceed with the second stage of the simplex method, we must recover the c-part of the tableau for the problem from the b-part, that is, from the table entries chosen to construct a feasible solution. The method for doing this is best described in terms of *potentials*. We introduced the potentials in Example 15.7.

In the notation of the general transportation problem, the dual problem is written

$$\text{maximize} - \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$$w_{ij} = c_{ij} + u_i - v_j \geq 0 \ \forall \ i, j$$

$$u \geq 0, v \geq 0.$$

In Example 15.7,

$$[a_1, a_2] = [220, 280], [b_1, b_2, b_3] = [170, 120, 210],$$

$$[c_{i,j}] = \begin{bmatrix} 77 & 39 & 105 \\ 150 & 186 & 122 \end{bmatrix}.$$

While the potentials correspond to the constraints on each retail store and each warehouse (or to the corresponding slack variables), there are other variables w_{ij} in the dual problem that correspond to the decision variable x_{ij} of the primal problem. They are the slack variables for the dual problem written in terms of potentials. We will call the variables w_{ij} *discrepancies*.

At start, all x_{ij} are at the right margin of a standard tableau (see Example 15.7) and all w_{ij} are at the left margin. After a few pivot steps, the variables x_{ij} corresponding to the selected positions (i, j) are on the top of a feasible tableau, and the corresponding w_{ij} are at the bottom margin. Their current values are in the last row of the tableau, its "c-part."

To proceed with the second stage of the simplex method, we have to recover the c-part of a row feasible tableau from the b-part. We know that if x_{ij} is a basic variable for the row problem—that is, it appears in the right margin of the tableau—then the corresponding dual variable w_{ij} appears in the left margin and w_{ij} takes the value 0 in the basic solution of the column problem (involving the potentials).

Thus, $w_{ij} = c_{ij} + u_i - v_j = 0$ for all chosen positions (i, j). It turns out that these equations determine potentials uniquely up to an additive constant. This follows from the fact that the chosen positions determine a tree in the graphical representation (Figure 16.5).

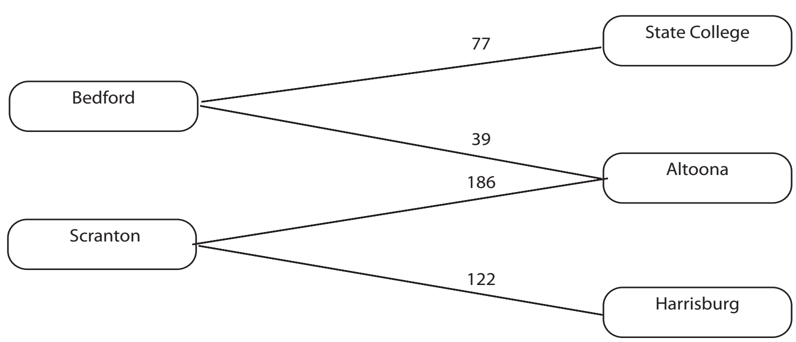

Figure 16.5. Tree
for the feasible solution of Example 15.7

We continue now with Example 15.7. We marked with a * the positions that were chosen as we found a feasible solution (Table 16.6).

77 *	39 *	105	220
150	186 *	122 *	280
170	120	210	

Table 16.6. Basic positions for the feasible solution of Example 15.7

The edges correspond to the selected positions, and the cost per unit is written on them. Because we seek a solution up to an additive constant, to solve for the potentials we can start by setting $u_1 = 0$. Then the equation $c_{11} = v_1 - u_1$ forces $v_1 = 77$. Similarly, the equation $c_{12} = v_2 - u_1$ forces $v_2 = 39$. Using this value of v_2 in the equation for c_{22} forces $u_2 = -147$, which, when substituted into the equation for c_{23}, forces $v_3 = -25$. Thus, we have found all the potentials based on the choice of $u_1 = 0$. All other solutions to the system of equations just solved are clearly obtained by adding the same constant to all potentials. We represent the potentials in Figure 16.7 and Table 16.8.

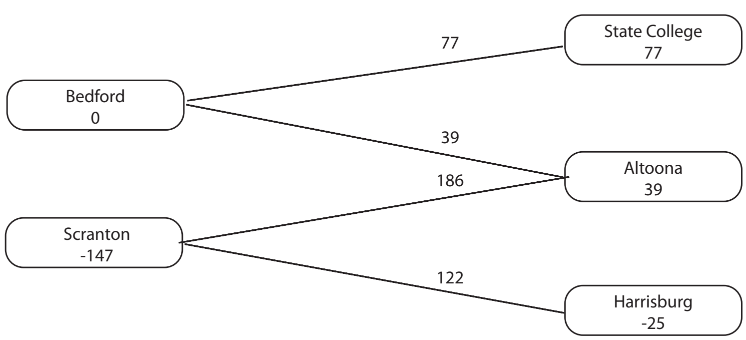

Figure 16.7. Tree representation of the potentials in Example 15.7

	77	39	−25	
0	77 *	39 *	105	220
−147	150	186 *	122 *	280
	170	120	210	

Table 16.8. Potentials
for the feasible solution of Example 15.7

Now we can compute the values $w_{ij} = c_{ij} + u_i - v_j$ for the other (nonbasic) positions. We put them in Table 16.8 in parentheses (Table 16.9). Because one of w_{ij} in this example is negative, our tableau is not optimal. We could show the basic solution for the primal problem by replacing each * representing a chosen position with the value assigned to x_{ij} at the time of that choice.

	77	39	−25	
0	77 *	39 *	105 (130)	220
−147	150 (−74)	186 *	122 *	280
	170	120	210	

Table 16.9. The basic solution
for the dual problem to Example 15.7

As mentioned previously, the computed values of w_{ij} correspond to the entries of the last row of the tableau which we have in mind without writing it down, except for the last entry in the tableau which is the current value of the objective function. Now we are ready for a pivot step.

Exercises

1–3. Find a feasible solution for each transportation problem.

1.

				35
				20
20	10	10	15	

2.

				35
				9
				11
				20
40	10	10	15	

3.

				35
				90
				111
				20
140	91	10	19	

4–7. Find a feasible solution and the corresponding total transportation cost.

4.

1	2	3	1	2	3	1	2	3	10
0	3	2	0	1	2	1	2	1	20
2	1	0	1	2	1	2	1	1	12
1	1	1	2	2	2	2	2	1	8
2	3	4	5	1	2	3	14	16	

5.

1	2	3	1	2	3	1	2	3	15
0	3	2	0	1	2	1	2	1	15
2	1	0	1	2	1	2	1	1	12
1	1	1	2	2	2	2	2	1	8
2	8	4	15	11	2	3	4	1	

6.

1	2	3	1	2	3	1	2	3	15
0	3	2	0	1	2	1	2	1	5
2	1	0	1	2	1	2	1	1	12
1	1	1	2	2	2	2	2	1	18
12	8	14	5	1	2	3	4	1	

7. Example 2.4.

55	30	40	50	40	0	50
35	30	100	45	60	0	30
40	60	95	35	30	0	50
25	10	20	30	15	30	

8. Prove that any feasible solution derived by our method is a vertex of the feasible region.

§17. Phase 2

Suppose that after computing potentials, some w_{ij} is negative. Then according to the simplex method, Phase 2, we have to exclude w_{ij} from the basis and include another variable instead. In terms of the row problem, we must include x_{ij} into the basis instead of another variable. When we do this, the objective function should improve or at least keep the same value. Graphically, this change of basis corresponds to adding a new edge to the tree and removing one edge from the resulting loop to form a new tree. We will use Example 2.4 to illustrate this process. We begin by using the northwest method to find a basic feasible solution (Table 17.1).

55 25	30 10	40 15	50	40	0	50
35	30	100 5	45 25	60	0	30
40	60	95	35 5	30 15	0 30	50
25	10	20	30	15	30	

Table 17.1. First feasible solution of Example 2.4

The corresponding transportation cost is $25 \cdot 55 + 10 \cdot 30 + 15 \cdot 40 + 5 \cdot 100 + 25 \cdot 45 + 5 \cdot 35 + 15 \cdot 30 + 30 \cdot 0 = 1375 + 300 + 600 + 500 + 1125 + 175 + 450 + 0 = 4525$. We will see that ignoring the objective function will result in this case in many pivot steps on the way to an optimal table. Usually, it pays to spend some time at Phase 1 of solving a transportation problem so we need to spend less time at Phase 2. As before, we proceed to find the potentials (Table 17.2).

	55	30	40	−15	−20	−50	
0	55 25	30 10	40 15	50	40	0	50
−60	35	30	100 5	45 25	60	0	30
−50	40	60	95	35 5	30 15	0 30	50
	25	10	20	30	15	30	

Table 17.2. Potentials for Table 17.1

We took the first potential 0 for the first row arbitrarily. The transportation cost can be also computed using potentials instead of the flow: $55\cdot25+30\cdot10+40\cdot20-15\cdot30-20\cdot15-50\cdot30-(0\cdot50-60\cdot30-50\cdot50)=4525$. As in the previous example, the feasible solution determines a tree in the graphical representation (Figure 17.3).

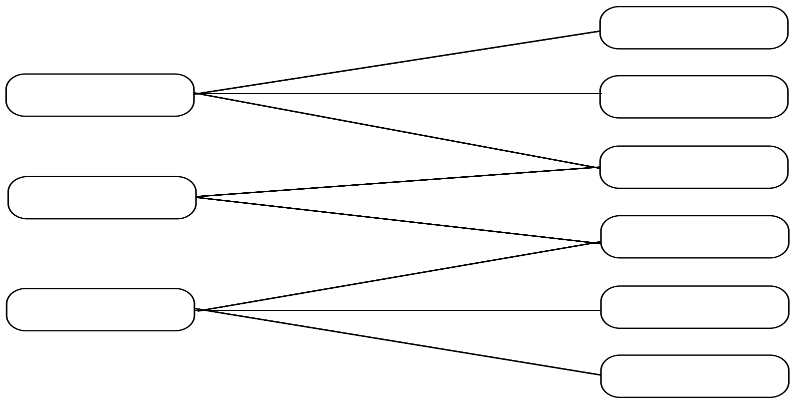

Figure 17.3. Tree for Table 17.1

Next we compute the values for the dual slack variables (discrepancies) w_{ij}. In tableau form, each nonbasic w_{ij} appears at the left margin with a basic x_{ij} at the right margin, and its value at the basic solution is 0. The basic w_{ij} corresponds to nonbasic x_{ij} and appears at the bottom of the tableau, so their values belong to the last row. Discrepancies w_{ij} are determined by the selected positions in Table 17.1. Recall that the discrepancies are zeros at the selected positions and are not written there but are written in parentheses elsewhere. In other words, the discrepancies, like the flow, are determined by the tree (see Figure 17.3).

	55	30	40	-15	-20	-50	
0	55 25	30 10	40 15	50 (65)	40 (60)	0 (50)	
-60	35 (-80)	30 (-60)	100 5	45 25	60 (20)	0 (-10)	30
-50	40 (-65)	60 (-20)	95 (5)	35 5	30 15	0 30	50
	25	10	20	30	15	30	

Table 17.4. Discrepancies w_{ij} for Table 17.2

Some of the basic w_{ij} are negative, so the solution is not optimal. We must pick a negative basic w_{ij} and switch it with a nonbasic $w_{i'j'}$, which is accomplished in the table by switching a nonbasic x_{ij} with a basic $x_{i'j'}$. We pick $w_{21} = -80$ and decide to switch x_{21} with a basic variable. In graphical terms, we are adding an edge from warehouse 2 to store 1, which destroys the tree structure of the graph by creating the loop $(2,1),(2,3),(1,3),(1,1)$ (Figure 17.5).

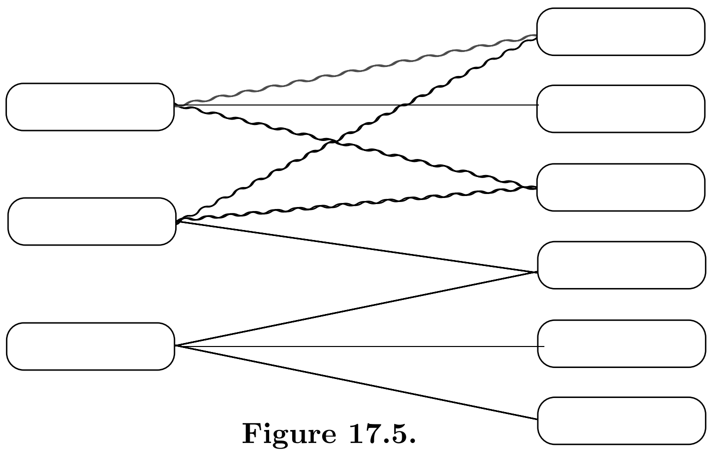

Figure 17.5.
Cycle arising from adding x_{21} to the basis

We start a flow of $\varepsilon \geq 0$ on the new chosen edge, from warehouse 2 to store 1; that is, we set $x_{21} = \varepsilon$. We must adjust the flow around the loop to satisfy the balance constraint. Therefore we subtract ε from x_{23} so as not to exceed the supply at warehouse 2. Then we add ε to x_{13} to meet the demand of store 3. On the last edge of the loop we subtract ε from x_{11}, which avoids exceeding the supply of warehouse 1 and the demand at store 1 (Table 17.6).

	55	30	40	-15	-20	-50	
0	55 $25-\varepsilon$	30 10	40 $15+\varepsilon$	50 (65)	40 (60)	0 (50)	50
-60	35 (-80) ε	30 (-60)	100 $5-\varepsilon$	45 25	60 (20)	0 (-10)	30
-50	40 (-65)	60 (-20)	95 (5)	35 5	30 15	0 30	50
	25	10	20	30	15	30	

Table 17.6.
Adjusting by ε the flow around the loop

The change in the transportation cost is $(35-100+40-55)\varepsilon = -80\varepsilon$. Note that the coefficient -80 is exactly the value of w_{ij}! To get the best improvement in the objective function, we give ε the maximum possible value—namely 5. This determines the x_{ij} to deselect, or, equivalently, the edge to remove from the graph to restore it to a tree. We move x_{21} into the basis and x_{23} out of the basis to arrive at a new basic solution (Table 17.7).

55 20	30 10	40 20	50	40	0	50
35 5	30	100	45 25	60	0	30
40	60	95	35 5	30 15	0 30	50
25	10	20	30	15	30	

Table 17.7.
Second feasible solution to Example 2.4

The transportation cost is now $20 \cdot 55 + 10 \cdot 30 + 20 \cdot 40 + 5 \cdot 35 + 25 \cdot 45 + 5 \cdot 35 + 15 \cdot 30 + 30 \cdot 0 = 4125 = 4525 - 80 \cdot 5$. We proceed to find the new values for the dual variables (Table 17.8).

	55	30	40	65	60	30	
0	55 20	30 10	40 20	50 (-15)	40 (-20)	0 (-30)	50
20	35 5	30 (20)	100 (80)	45 25	60 (20)	0 (-10)	30
30	40 (15)	60 (60)	95 (85)	35 5	30 15	0 30	50
	25	10	20	30	15	30	

Table 17.8.
Potentials and discrepancies w_{ij} for Table 17.7

We have executed one pivot step, replacing x_{23} with x_{21} as a basic variable, resulting in a new tree (Figure 17.9).

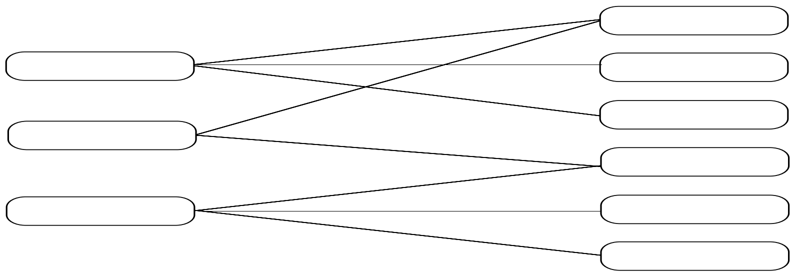

Figure 17.9. Tree resulting from removal of x_{23} from the basis

Some w_{ij} are still negative, however, so the table is not optimal. We have three negative entries in the table from which to choose. Choosing $w_{14} = -15$ corresponds to adding an edge to the graph from warehouse 1 to store 4, creating the loop

$$(1,4), (1,1), (2,1), (2,4).$$

Choosing $w_{15} = -20$ corresponds to adding an edge to the graph from warehouse 1 to store 5, creating the loop

$$(1,5), (1,1), (2,1), (2,4), (3,4), (3,5) \quad \text{(Figure 17.10)}.$$

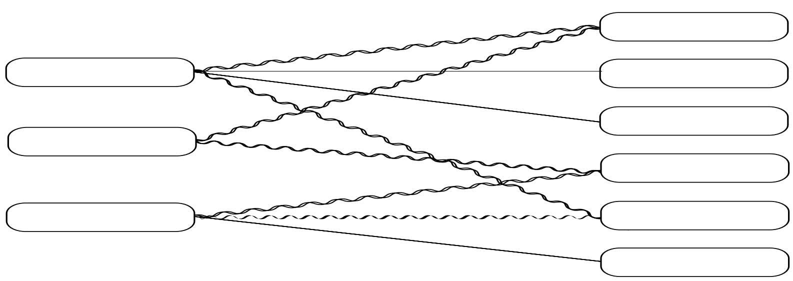

Figure 17.10. Loop resulting from adding x_{15} to the basis

To illustrate the process of finding a new basic feasible solution with a longer loop, we proceed with the latter choice. We start a flow of ε on the edge from warehouse 1 to store 5; that is, we set $x_{ij} = \varepsilon$. Then we subtract ε from x_{11} because we cannot exceed the supply of warehouse 1. Then we add ε to x_{21} to meet the demand at store 1.

We continue in this manner around the loop, subtracting ε from x_{24}, adding ε to x_{34}, and, finally, subtracting ε from x_{35} (see Table 17.11).

	55	30	40	65	60	30	
0	55 $20-\varepsilon$	30 10	40 20	50 (-15)	40 (-20) ε	0 (-30)	50
20	35 $5+\varepsilon$	30 (20)	100 (80)	45 $25-\varepsilon$	60 (20)	0 (-10)	30
30	40 (15)	60 (60)	95 (85)	35 $5+\varepsilon$	30 $15-\varepsilon$	0 30	50
	25	10	20	30	15	30	

Table 17.11.
Adjusting by ε the flow around the new loop

The change in the value of the objective function is
$\varepsilon(c_{15} - c_{35} + c_{34} - c_{23} + c_{21} - c_{11})$
$= \varepsilon(40 - 30 + 35 - 45 + 35 - 55) = -20\varepsilon = w_{15}\varepsilon.$

In this case, the maximum value that ε can take is 15, limited by the demand at store 5. Adjusting the x_{ij} as prescribed by the flow with $\varepsilon = 15$ results in a new feasible solution that decreases the objective function to $4125 - 20 \cdot 15 = 3825$ (Table 17.12). In the new solution, x_{15} replaces x_{35}, restoring our graphical representation to a tree (Figure 17.13).

55 5	30 10	40 20	50	40 15	0	50
35 20	30	100	45 10	60	0	30
40	60	95	35 20	30	0 30	50
25	10	20	30	15	30	

Table 17.12. Third feasible solution to Example 2.4

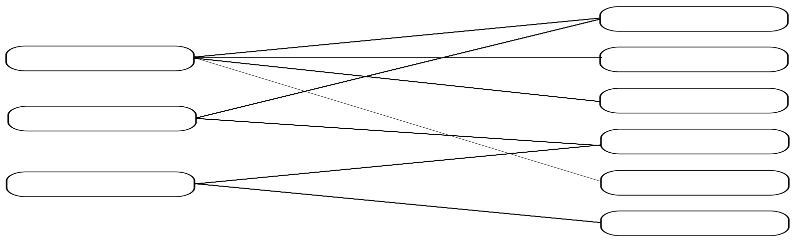

Figure 17.13. Tree resulting from deleting x_{35} from the basis

We proceed as before to find the new potentials, the new w_{ij}, a negative $w_{14} = -15$, creating the loop $(1,4), (1,1), (2,1), (2,4)$, and the adjusted flow around the loop (Table 17.14).

	55	30	40	65	40	30	
0	55 $5-\varepsilon$ ■	30 10	40 20	50 (−15) ε ■	40 15	0 (−30)	50
20	35 $20+\varepsilon$ ■	30 (20)	100 (80)	45 $10-\varepsilon$ ■	60 (40)	0 (−10)	30
30	40 (15)	60 (60)	95 (85)	35 20	30 (20)	0 30	50
	25	10	20	30	15	30	

Table 17.14. Dual variables and adjusted flow for Table 17.13

We set $\varepsilon = 5$, the maximum as constrained by the supply at warehouse 1, to obtain a new feasible solution which reduces the objective function by $w_{14}\varepsilon = 75$ (Table 17.15).

55	30 10	40 20	50 5	40 15	0	50
35 25	30	100	45 5	60	0	30
40	60	95	35 20	30	0 30	50
25	10	20	30	15	30	

Table 17.15. Fourth feasible solution to Example 2.4

We compute the potentials, the new w_{ij}, a new negative $w_{16} = -15$ and the new flow of ε around the loop

$$(1,6),(1,4),(3,4),(3,6)$$

(Table 17.16).

	40	30	40	50	40	15	
0	55 (15)	30 10	40 20	50 $5-\varepsilon$	40 15	0 (-15) ε	50
5	35 25	30 (5)	100 (65)	45 5	60 (25)	0 (-10)	30
15	40 (15)	60 (45)	95 (70)	35 $20+\varepsilon$	30 (5)	0 $30-\varepsilon$	50
	25	10	20	30	15	30	

Table 17.16. Dual variables and adjusted flow for Table 17.15

Again ε is constrained to 5 by the supply at warehouse 1, and setting $\varepsilon = 5$ gives a new feasible solution that improves the objective function by 75 (Table 17.17).

55	30 10	40 20	50	40 15	0 5	50
35 25	30	100	45 5	60	0	30
40	60	95	35 25	30	0 25	50
25	10	20	30	15	30	

Table 17.17. Fifth feasible solution to Problem 2.4

We compute again the new potentials and the new w_{ij}, find a negative $w_{35} = -10$, create the loop $(3,5),(3,6),(1,6),(1,5)$, and set $x_{35} = \varepsilon$, $x_{36} = 25 - \varepsilon$, $x_{16} = 5 + \varepsilon$, and $x_{15} = 15 - \varepsilon$ (Table 17.18).

	25	30	40	35	40	0	
0	55 (30)	30 10	40 20	50 (15)	40 $15-\varepsilon$	0 $5+\varepsilon$	50
−10	35 25	30 (−10)	100 (50)	45 5	60 (10)	0 (−10)	30
0	40 (15)	60 (30)	95 (55)	35 25	30 (−10) ε	0 $25-\varepsilon$	50
	25	10	20	30	15	30	

Table 17.18. Dual variables and adjusted flow for Table 17.17

Setting ε to the maximum value of 15, constrained by the demand at store 5, yields a new feasible solution that reduces the transportation cost by 150 (Table 17.19).

55	30 10	40 20	50	40	0 20	50
35 25	30	100	45 5	60	0	30
40	60	95	35 25	30 15	0 10	50
25	10	20	30	15	30	

Table 17.19. Sixth feasible solution to Example 2.4

Then we compute the new potentials u_i and v_j and the new dual slack variables w_{ij}. This time we choose position $(2,6)$ where $w_{26} = -10$. We adjust the flow around the loop by setting $x_{26} = \varepsilon$, $x_{24} = 5 - \varepsilon$, $x_{34} = 25 + \varepsilon$, and $x_{36} = 10 - \varepsilon$.

Here is the new table:

	25	30	40	35	30	0	
0	55 (30)	30 10	40 20	50 (15)	40 (10)	0 20	50
-10	35 25	30 (-10)	100 (50)	45 $5-\varepsilon$	60 (20)	0 (-10) ε	30
0	40 (15)	60 (30)	95 (55)	35 $25+\varepsilon$	30 15	0 $10-\varepsilon$	50
	25	10	20	30	15	30	

Table 17.20. Dual variables and adjusted flow for Table 17.19

We set ε to the maximum of 5, constrained by the supply at store 2, to get a feasible solution that improves the objective function by 50 (Table 17.21).

55	30 10	40 20	50	40	0 20	50
35 25	30	100	45	60	0 5	30
40	60	95	35 30	30 15	0 5	50
25	10	20	30	15	30	

Table 17.21. Seventh feasible solution to Example 2.4

We compute again the potentials and the dual slack variables—that is, the c-part of the corresponding tableau (Table 17.22).

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	100 (60)	45 (10)	60 (30)	0 5	30
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

Table 17.22. Potentials and w_{ij} for Table 17.21

Finally all the basic w_{ij}, which correspond to the c-part of a standard tableau, are nonnegative. So the seventh feasible solution is optimal. In fact this is the solution found by an educated guess in Example 2.4, although educated guesses would be difficult in larger problems. The transportation cost is $10 \cdot 30 + 20 \cdot 40 + 20 \cdot 0 + 25 \cdot 35 + 5 \cdot 0 + 30 \cdot 35 + 15 \cdot 30 + 5 \cdot 0 = 3475$. Figure 17.23 shows a graphical representation of our optimal solution.

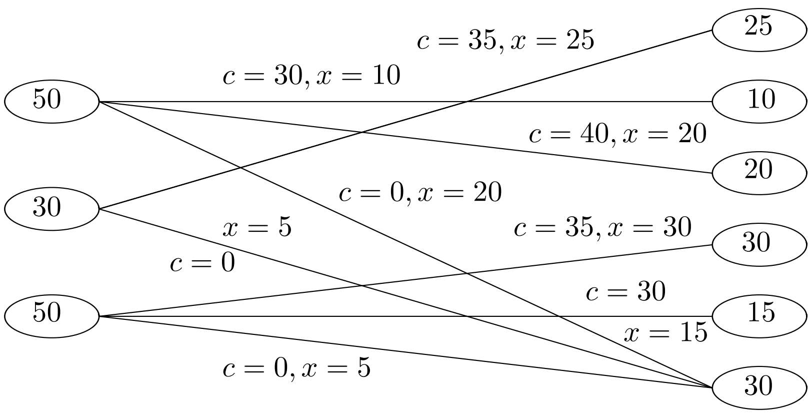

Figure 17.23. Optimal solution corresponding to Table 17.22

Since $w_{22} = 0$, x_{22} can be increased without hurting the objective function. Other optimal solutions may arise if this x_{ij} can be increased without violating any constraints. We can investigate this possibility by considering adding an edge to our graph from warehouse 2 to store 2, creating the loop $(2,2),(2,6),(1,6),(1,2)$. It is evident that the per-unit cost of shipping from warehouse 1 to store 2 equals the cost from warehouse 2 to store 2, and the cost of leaving units in stock—that is, shipping to store 6 is free for all warehouses. Thus we conclude that supplying some of the demand for store 2 from warehouse 2 instead of warehouse 1 and leaving fewer units in stock at warehouse 2 and more in stock at warehouse 1 would not change the transportation cost. We can only adjust the allocation in this way up to 5 units, because warehouse 2 only has 5 units to spare. All optimal solutions are obtained in this way, because all other w_{ij} are strictly positive.

By putting some extra numbers in our graph, we can represent the full solution (Figure 17.24). Each edge in the graph is marked by the number x_{ij} of units to be shipped from warehouse i to store j. Remember that "store 6" is fictitious.

So for each i, x_{i6} is actually the number of units that remain in stock at warehouse i. We can mark each warehouse with total supply and each store with total demand, which at store 6 equals the total excess supply across all warehouses. We can include all the optimal solutions by adding the edge from warehouse 2 to store 2 and making x_{12} and x_{22} functions of ε (Figure 17.24).

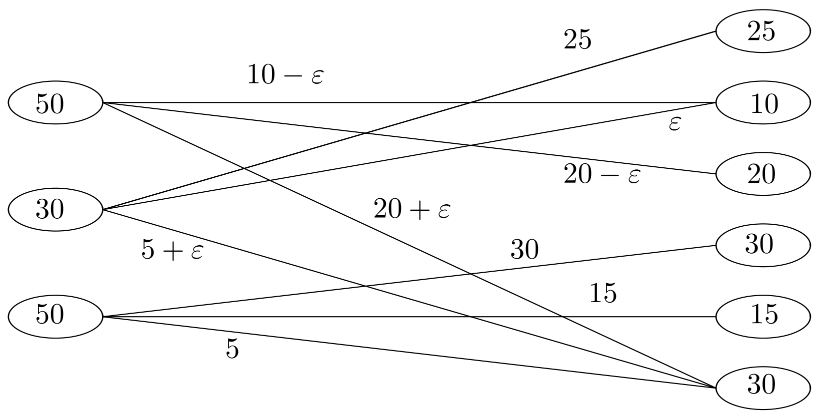

Figure 17.24. All optimal solutions, $0 \le \varepsilon \le 5$

The possibility of cycling in transportation problems is very remote, and Bland's method prevents it. If the given supply and demand data are integral, then the optimal solution is also integral. Moreover, the size of integers in feasible solutions does not exceed the size of integers in the supply and demand.

It is possible to use old potentials to compute new ones. This decreases the amount of computations but increases the complexity of the method and creates new opportunities for errors.

Remark 17.25. The dual simplex method for transportation problems is known as the Hungarian method. It is based on the observation that adding a constant to a row or a column of the cost matrix does not change the optimality region because it changes the objective function by a constant. If we manage to obtain a cost matrix with all entries ≥ 0 and enough zero entries to place the whole flow at the positions with zero costs, then we obtain an optimal solution (with zero adjusted total cost). The dual method is particularly advisable for transportation problems that come from job assignment problems (see the next section) where degeneracy slows down the primal method. In those problems, $m = n$ and $a_i = b_j = 1$ for all i, j.

Here we outline the Hungarian method for these problems. The method works with nonnegative matrices having a zero in every row and column. To get an inital $n \times n$ matrix, we subtract the least element of each row from that row of the cost matrix $[c_{i,j}]$. Then we do likewise for each column.

1. Given any matrix C with nonnegative entries, we draw the minimum number t of lines through the rows and columns to cover all zeros in C. If $t = n$, then it is very easy to find a basic feasible solution as in §16 selecting positions with zero costs and crossing out these n lines one after another. Otherwise, we proceed to Step 2.

2. Now $t < n$. We compute the minimum m of all uncovered entries. Then we subtract m from each uncovered entry and add it to each twice-covered entry (i.e., covered by both a horizontal and a vertical line). Return to Step 1.

Exercises.

1–2. Compute the feasible solution and the corresponding dual basic solution to the transportation problem, where the stars (*) mark the basic variables for the row problem. Put both basic solutions into the same table. Are those solutions optimal?

1.

1 *	2 *	3	200
1	2 *	2 *	300
175	125	200	

2.

1	2 *	3 *	200
1 *	2 *	2	300
175	125	200	

3–5. Solve the transportation problems in Exercises 4–6 of §16.

6–8. Solve the transportation problem in Example 2.4 with

6. The supply 30 replaced by $30 + t$ where t is a parameter and $|t| \le 1$

7. The supply 30 replaced by a parameter t

8. The unit cost $c_{23} = 100$ is replaced by a parameter t

Hint: Use Table 17.22 to start.

§18. Job Assignment Problem

We introduced this class of problems in Chapter 1 (see Example 2.5 and Exercises 6–9 in §2.) In general, in terms of variables and constraints, the problem can be stated as follows. We set

$$x_{ij} = 0 \text{ or } 1 \tag{18.1}$$

depending on whether the person i is assigned to do the job j. The total time is

$$\sum_i \sum_j c_{ij} x_{ij} (\text{to be minimized}). \tag{18.2}$$

The condition that every person i is assigned to exactly one job is

$$\sum_j x_{ij} = 1 \quad \text{for all } i. \tag{18.3}$$

The condition that exactly one person is assigned to every job j is

$$\sum_i x_{ij} = 1 \quad \text{for all } j. \tag{18.4}$$

Thus, the problem is stated as an optimization problem (18.1–18.4). The objective function (18.2) is linear, as are the constraints (18.3) and (18.4). However constraints (18.1) are not linear.

In view of (18.3) and (18.4), constraints (18.1) can be replaced by the following constraints: $x_{ij} \geq 0, x_{ij}$ are integers. This is an example of an integer programming problem.

If we drop the condition that x_{ij} are integers (keeping the conditions $x_{ij} \geq 0$), then mathematically our problem looks like a particular case of the transportation problem with all given supplies and demands equal to 1 and the balance condition meaning that the number of persons equals the number of jobs.

Let us forget about the integrability condition and solve the resulting linear problem by the simplex method. So we regard each worker as a "warehouse" supplying work and each job as a "store" with a demand for work. We set both the supply of each worker and the demand of each job equal to 1. Total time becomes the total transportation cost.

Because integral data guarantee an integral solution, our optimal solution has only integral values for all variables. So we obtain an optimal solution for the job assignment problem.

Thus, we have reduced any job assignment problem to a transportation problem. A variation of the assignment problem involves maximizing the sum S of ratings instead of minimizing the total cost (instead of time) of completing a set of jobs. Using a standard trick, we can reduce maximization to minimization. The resulting transportation problem may have negative coefficients c_{ij} which we did not see in previous examples. However, the simplex method does work the same way no matter whether coefficients are positive, negative, or zeros, or any mixture of those. So this version can be also solved as previously.

Problem 18.5. Solve the job assignment problem

2	2	2	1	2
2	3	1	2	4
2	0	1	1	1
2	3	4	3	3
3	2	1	2	1

where the table data are interpreted as cost per person per job and the goal is to minimize total cost.

Solution. Unless you like pivoting, it pays to spend more time on Phase 1 rather than use the northwest method. We observe the minimal cost per person, that is, the minimal cost in each row: 1, 1, 0, 2, 1. So the minimal total cost is at least 5 (i.e., min ≥ 5). Let us try to get this cost by choosing a position with minimal cost in each row. This works nicely and uniquely until we come to the last row, where there are two positions with cost 1. But one of them collides with the selected position in the second row, so we select the last position in the last row. Therefore, we get an optimal solution and it is unique:

			*	
		*		
	*			
*				
				*

Note that there are $5! = 120$ feasible solutions, and trying all of them would take more time than our common-sense approach.

Problem 18.6. Solve the job assignment problem with data as in Problem 18.5, but now the table data are interpreted as the rating of each worker on each job and the goal is to maximize efficiency.

Solution. Again before indulging in pivoting we try to use common sense. Let us try first to choose the best job for each worker (this worked so well in the previous problem). Here is the pattern of maximal entries in rows:

*	*	*		*
				*
*				
		*		
*				

We get the max $\leq 2+4+2+4+3 = 15$, but there is no way to do job 4 if we choose the best number in each row. Let us then try to choose a maximal entry in each column:

	*			*
	*	*	*	
*				

Now the first and third workers have nothing to do, and our upper bound for the maximum is max $\leq 3+3+4+3+4 = 17$, which is weaker than the previous bound.

In fact since choosing the maximal number in each row does not work, we conclude that max < 15. Since the maximum is an integer, we obtain that max ≤ 14. But now looking again at the data and the pattern of maximal entries in each row, we easily see a feasible solution with total rating 14:

	*			
				*
*				
		*		
			*	

where the workers 1–4 take the best jobs for each and the worker 5 takes the second best job 4.

Problem 18.7. Minimize total grief in the matching problem.

1	1	2	1
2	3	2	3
4	3	3	4
4	4	5	6

Solution. Taking a minimal entry

$$\boxed{\begin{array}{cccc} * & * & * & * \\ & & * & \\ & & & \\ & & & \end{array}}$$

in each column does not produce any feasible solutions, and we get the bound min > 5. Taking a minimal entry

$$\boxed{\begin{array}{cccc} * & * & & * \\ * & & * & \\ & * & * & \\ * & * & & \end{array}}$$

in each row does produce two optimal solutions. However, for a change, we ignore the optimal solution and use the simplex method. The point of this exercise is that although common sense and special tricks give shortcuts for some problems, the simplex method works for every linear program.

So we treat this problem as a transportation problem. As an initial basic solution we take the one found by a student in 2000:

$$\begin{array}{|cccc|cc} \hline {}^{1}1 & {}^{1} & {}^{2}0 & {}^{1} & \cancel{1} & \cancel{0} \\ {}^{2} & {}^{3} & {}^{2} & {}^{3}1 & \cancel{1} & \\ {}^{4} & {}^{3}1 & {}^{3} & {}^{4} & \cancel{1} & \\ {}^{4} & {}^{4}0 & {}^{5}1 & {}^{6}0 & \cancel{1} & \cancel{0} \\ \hline \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} & & \\ & \cancel{0} & & \cancel{0} & & \end{array}$$

Note that the total cost at this solution is $1 + 3 + 3 + 5 = 12$ which is better than $1+3+3+6 = 13$ given by the northwest method.

Then we find potentials and w_{ij}:

$$\begin{array}{c|cccc|}
 & 5 & 5 & 6 & 7 \\ \hline
4 & {}^{1}1 & {}^{1}(0) & {}^{2}0 & {}^{1}(-2) \\
4 & {}^{2}(1) & {}^{3}(2) & {}^{2}(0) & {}^{3}1 \\
2 & {}^{4}(1) & {}^{3}1 & {}^{3}(-1) & {}^{4}(-1) \\
1 & {}^{4}(0) & {}^{4}0 & {}^{5}1 & {}^{6}0 \\ \hline
\end{array}$$

The first potential 5 on top was chosen arbitrarily. Now there are three negative w_{ij}. We select the position (3, 3) with negative $w_{33} = -1$ (there were two other possible choices). This leads to a loop of length 4. (The choice of $w_{14} = -2$ would lead to a loop of length 4 and a degenerate pivot step.) Here is an adjustment of the flow along the loop:

$$\begin{array}{|cccc|} \hline
{}^{1}1 & {}^{1}(0) & {}^{2}0 & {}^{1}(-2) \\
{}^{2}(1) & {}^{3}(2) & {}^{2}(0) & {}^{3}1 \\
{}^{4}(1) & {}^{3}1-\varepsilon & {}^{3}\varepsilon & {}^{4}(-1) \\
{}^{4}(0) & {}^{4}0+\varepsilon & {}^{5}1-\varepsilon & {}^{6}0 \\ \hline
\end{array}$$

We take $\varepsilon = 1$ and deselect the position (4, 3) [another possible choice was the position (3, 2)]. The objective function improved by 1. Here are our new basic feasible solution, new potentials, and new adjustment along a loop of length 6:

$$\begin{array}{c|cccc|}
 & 5 & 6 & 6 & 8 \\ \hline
4 & {}^{1}1 & {}^{1}(-1) & {}^{2}0-\varepsilon & {}^{1}(-3)\ \varepsilon \\
5 & {}^{2}(2) & {}^{3}(2) & {}^{2}(1) & {}^{3}1 \\
3 & {}^{4}(2) & {}^{3}0-\varepsilon & {}^{3}1+\varepsilon & {}^{4}(-1) \\
2 & {}^{4}(1) & {}^{4}1+\varepsilon & {}^{5}(1) & {}^{6}0-\varepsilon \\ \hline
\end{array}$$

We have selected the position (1, 4) as a new basic position (there were two other choices). We must take $\varepsilon = 0$, so it is going to be a degenerate pivot step that does not change the feasible solution (but it changes the basis, i.e., the set of basic variables). Now we deselect the position (4, 4) [the other possible choices were (1, 3) and (3, 2)], compute again the new potentials and w_{ij}, and find a negative w_{ij} (once we find one we do not need to compute other w_{ij}) and the corresponding loop.

Here is our next table:

$$
\begin{array}{c|cccc|}
 & 5 & 6 & 6 & 5 \\
\hline
4 & {}^{1}1-\varepsilon & {}^{1}(-1) & {}^{2}0 & {}^{1}0+\varepsilon \\
2 & {}^{2}(-1)\varepsilon & {}^{3}(-1) & {}^{2}(-2) & {}^{3}1-\varepsilon \\
3 & {}^{4}(2) & {}^{3}0 & {}^{3}1 & {}^{4}(2) \\
2 & {}^{4}(1) & {}^{4}1 & {}^{5}(1) & {}^{6}(3) \\
\hline
\end{array}
$$

We have selected the position (2, 1), and $\varepsilon = 1$. We deselect the position (2, 4). Next we compute new potentials and w_{ij}:

$$
\begin{array}{c|cccc|}
 & 5 & 5 & 5 & 5 \\
\hline
4 & {}^{1}0 & {}^{1}(0) & {}^{2}(1) & {}^{1}1 \\
3 & {}^{2}1 & {}^{3}(1) & {}^{2}0 & {}^{3}(1) \\
2 & {}^{4}(1) & {}^{3}0 & {}^{3}1 & {}^{4}(1) \\
1 & {}^{4}(0) & {}^{4}1 & {}^{5}(1) & {}^{6}(2) \\
\hline
\end{array}
$$

Now all $w_{ij} \geq 0$, so the table is optimal. The optimal assignment it gives is

$$
\begin{array}{|cccc|}
\hline
 & & & * \\
* & & & \\
 & & * & \\
 & * & & \\
\hline
\end{array}
$$

The optimal value is min = 10.

Remark 18.8. Any job assignment problem with n workers and jobs has $n!$ feasible solutions. After it is written as a transportation problem, these solutions become the vertices of the feasible region which is a simplex of dimension $n! - 1$ (in particular, an interval when $n = 2$). For each vertex, there are 2^{n-1} choices for the basis. There are $(n^2)!$ ways to place the basic variables at the left margin of the standard tableau, and $(n^2)!$ ways to place the nonbasic variables at the top margin. So we can imagine $(n^2)!^2 \cdot 2^{n-1} \cdot n!$ standard tableaux for the problem. Fortunately, the simplex method does not need to go through all these tableaux. In fact, the simplex method for transportation problems proved to be very efficient in practice.

Exercises

1–4. Solve the job assignment problem where the table data are interpreted as cost per job per person and the goal is to minimize total cost.

1.

1	2	3	1	2
3	1	2	1	0
0	1	3	2	1
2	3	4	3	3
1	3	3	1	2

2.

4	2	2	2	1	2
2	3	1	2	4	4
2	0	1	1	1	4
2	3	4	3	4	3
3	2	1	2	4	1
1	2	1	2	0	1

3.

2	1	4	3	4	3	1
4	2	2	2	1	5	2
2	3	5	1	2	4	4
5	2	0	1	1	1	4
2	3	4	3	4	3	5
5	3	2	1	2	4	1
1	2	1	2	0	1	5

4.

0	2	2	4	0	1	5	1	4
2	3	1	3	4	3	2	4	1
2	1	4	3	4	3	1	2	4
4	2	4	2	2	2	1	5	2
2	3	5	1	2	4	2	4	4
5	2	2	4	0	1	1	1	4
2	3	4	3	4	3	2	4	5
5	2	4	3	2	1	2	4	1
1	2	1	2	0	1	2	4	5

5–8. Solve the job assignment problem 1–4 with the table data interpreted as the rating of each worker on each job and the goal is to maximize efficiency.