Integrals, partitions and MacMahon's Theorem

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Abstract

In two previous papers, the study of partitions with short sequences has been developed both for its intrinsic interest and for a variety of applications. The object of this paper is to extend that study in various ways. First, the relationship of partitions with no consecutive integers to a theorem of MacMahon and mock theta functions is explored independently. Secondly, we derive in a succinct manner a relevant definite integral related to the asymptotic enumeration of partitions with short sequences. Finally, we provide the generating function for partitions with no sequences of length K and part exceeding N.

1 Introduction

In his classic two volume work, Combinatory Analysis [5], P.A. MacMahon devotes Chapter IV of Volume 2 to "Partitions Without Sequences". His object in this chapter is to make a thorough study of partitions in which no consecutive integers (i.e. integers that differ by 1) occur. He concludes this chapter with what we will call MacMahon's Theorem.

Theorem 1.1. *The number of partitions of an integer N into parts $\not\equiv \pm 1$ (mod 6) equals the number of partitions of N with no consecutive integers as summands and no ones.*

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For example, for $n = 10$, the first set of partitions is 10, $8+2$, $6+4$, $6+2+2$, $4+4+2$, $4+3+3$, $4+2+2+2$, $3+3+2+2$, $2+2+2+2+2$; the second set is 10, $8+2$, $7+3$, $6+3$, $6+4$, $6+2+2$, $5+5$, $4+4+2$, $4+2+2+2$, $2+2+2+2+2$. The fact that each set of partitions has the same number of elements (in this case 9), is MacMahon's assertion.

In two previous papers [2, 4], MacMahon's ideas have been generalized to the consideration of partitions in which sequences of consecutive integers have been restricted to contain fewer than k terms (MacMahon only dealt with $k = 2$).

In Section 2 of this paper we shall explore in detail various aspects of MacMahon's work in [5; Vol. II, Ch. IV]. In Section 3 we discuss the generalization to partitions without k consecutive parts: First, we obtain a new and simplified proof of the Holroyd-Liggett-Romik definite integral that was used in [4] to obtain results on the asymptotic enumeration of these classes of partitions. Secondly, we strengthen the results of [2] by obtaining a double series representation of the generating function for partitions in which each part is $\leq N$ and sequences of consecutive integers have length less than k. Finally, Section 4 contains some remarks on a probabilistic interpretation of the mock theta function $\chi(q)$ studied by Ramanujan.

2 Investigation of MacMahon's theorem

We begin with some definitions.

Definition 2.1. *Let*

$$\begin{aligned} g_n &= \text{the number of partitions of } n \text{ with no two consecutive parts,} \\ h_n &= \text{the number of partitions of } n \text{ with no two consecutive parts} \\ &\quad\ \text{and no 1's,} \end{aligned}$$

$$G_2(q) = \sum_{n=0}^{\infty} g_n q^n, \tag{2.1}$$

$$H_2(q) = \sum_{n=0}^{\infty} h_n q^n, \tag{2.2}$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n}(1-q^j+q^{2j})}, \tag{2.3}$$

where $\chi(q)$ is one of the third-order mock theta function studied by Ramanujan [6]; p. 354.

2.1 A bijective proof of Theorem 1.1

Proof. By passing to the conjugate partition, the number of partitions of n with no 1's and no two consecutive parts is clearly seen to be equal to the number of partitions of n not containing any part *exactly once*. Here is a bijection between the set $\mathcal{C}_n$ of partitions of n not containing any part exactly once, and the set $\mathcal{B}_n$ of partitions of n into parts congruent to $0, 2, 3, 4 \bmod 6$: If $n = \sum_{k=1}^{\infty} k r_k$ is a partition in $\mathcal{C}_n$ (r_k is the multiplicity of k, or the number of parts equal to k in the partition), $r_k \in \{0, 2, 3, 4, \dots\}$, then each r_k can be written uniquely as $r_k = s_k + t_k$, where $s_k \in \{0, 3\}$ and $t_k \in \{0, 2, 4, 6, 8, \dots\}$. Define a partition $n = \sum_{j=1}^{\infty} j b_j$ by

$$\begin{aligned}
b_{6k+1} &= 0 \qquad\qquad\qquad\qquad (k = 0, 1, 2, 3, \dots) \\
b_{6k+5} &= 0 \\
b_{6k+2} &= \frac{1}{2} t_{3k+1} \\
b_{6k+4} &= \frac{1}{2} t_{3k+2} \\
b_{6k+3} &= \frac{1}{3} s_{2k+1} + t_{6k+3} \\
b_{6k+6} &= \frac{1}{3} s_{2k+2} + t_{6k+6}
\end{aligned}$$

This partition is in $\mathcal{B}_n$, and it is not difficult to check that any partition in $\mathcal{B}_n$ is obtained in this way from a unique partition in $\mathcal{C}_n$. □

2.2 A q-series for $G_2(q)$

We give a simplified proof of the following q-series representation for $G_2(q)$, which was stated in [2, eq. (4.2)]:

Theorem 2.2.

$$G_2(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n \prod_{j=1}^{n-1} (1 - q^j + q^{2j})}{\prod_{j=1}^{n} (1 - q^j)}. \tag{2.4}$$

Proof. Again by passing to the conjugate partition, we see that g_n is the number of partitions of n where all the parts except possibly the largest part do not appear exactly once.

Write (2.4) as

$$\begin{aligned} G_2(q) &= 1 + \sum_{n=1}^{\infty} \left[\frac{q^n}{1-q^n} \cdot \prod_{j=1}^{n-1} \left(\frac{1-q^j+q^{2j}}{1-q^j} \right) \right] \\ &= 1 + \sum_{n=1}^{\infty} \left[\frac{q^n}{1-q^n} \cdot \prod_{j=1}^{n-1} (1+q^{2j}+q^{3j}+q^{4j}+\dots) \right]. \end{aligned}$$

The coefficient of q^N in the n-th summand on the right-hand side is equal to the number of partitions of N with maximal part n, where no part except possibly the largest part appears exactly once. So the coefficient of q^N in the entire sum on the right-hand side is exactly g_N. $\square$

2.3 The MacMahon-Fine identity

In [2], it was shown that a combination of identities due to MacMahon [5]; Vol. II, p. 52, and Fine [3]; p. 57 show that

$$G_2(q) = H_2(q)\chi(q). \tag{2.5}$$

This identity can be given the following combinatorial interpretation:

Theorem 2.3. *For each integer $n \geq 1$ and $0 \leq k \leq \sqrt{n}$, let $f_{n,k}$ be the number of partitions of $n - k^2$ in which no part which is greater than k appears exactly once. Then for each $n \geq 1$,*

$$g_n = \sum_{k=0}^{\lfloor \sqrt{n} \rfloor} f_{n,k}. \tag{2.6}$$

Proof. From the remark at the beginning of the proof of Theorem 2.2, we can write

$$H_2(q) = \prod (1+q^{2j}+q^{3j}+q^{4j}+\dots) = \prod_{n=1}^{\infty} \frac{1-q^j+q^{2j}}{1-q^j}, \tag{2.7}$$

(this is an alternative way to prove Theorem 1.1). Now combining (2.5) and (2.7) and the definition of $\chi(q)$ gives

$$G_2(q) = \sum_{k=0}^{\infty} q^{k^2} \left(\prod_{j=1}^{k} \frac{1}{1-q^j} \right) \cdot \left(\prod_{j=k+1}^{\infty} \frac{1-q^j+q^{2j}}{1-q^j} \right).$$

The coefficients of q^n in the left- and right-hand side of this equation are clearly the left and right-hand sides of (2.6), respectively. $\square$

A natural question is whether Theorem 2.3 has a simple combinatorial explanation.

3 Partitions without k consecutive parts

3.1 The Holroyd-Liggett-Romik integral

In [4], the following result concerning the asymptotic enumeration of partitions without k consecutive parts was proved:

Theorem 3.1 (Holroyd, Liggett and Romik, [4]). *Let $p_k(n)$ denote the number of partitions of n not containing k consecutive parts. Then for each fixed $k > 1$, we have as $n \to \infty$*

$$p_k(n) = e^{(1+o(1))c_k\sqrt{n}},$$

where

$$c_k = \pi \sqrt{\frac{2}{3}\left(1 - \frac{2}{k(k+1)}\right)}.$$

The proof of this result relies on a special case of the following family of definite integrals, also proved in [4]: For every $0 < a < b$, a decreasing function $f_{a,b} : [0,1] \to [0,1]$ can be defined by $f_{a,b}(0) = 1, f_{a,b}(1) = 0$ and $f_{a,b}(x)^a - f_{a,b}(x)^b = x^a - x^b$ in between. In the simplest case $f_{1,2} - f_{1,2}^2 = x - x^2$, we have $f_{1,2}(x) = 1 - x$. Then we have:

Theorem 3.2 (Holroyd, Liggett and Romik, [4]).

$$\int_0^1 \frac{-\log f_{a,b}(x)}{x} dx = \frac{\pi^2}{3ab}.$$

We give here a new and shorter proof of this result. We remark that the proof given in [4], while considerably more complicated, seems to contain more interesting information, see [7].

Proof. The integral in the theorem can be interpreted as a double integral:

$$I_{a,b} := \int_0^1 \frac{-\log f_{a,b}(x)}{x} dx = \int_0^1 \frac{dx}{x} \int_{f_{a,b}(x)}^1 \frac{dy}{y} = \int\int_D \frac{dxdy}{xy},$$

where D is a symmetric domain bounded below by $y^a - y^b = x^a - x^b$, above by $y = 1$, and to the right by $x = 1$. Bisect it along its symmetry axis $y = x$ and substitute $y = xt, dy = xdt$ to get

$$I_{a,b} = 2 \int\int_{D'} \frac{dxdt}{xt},$$

where D' is bounded below by $x^{b-a} = (1 - t^a)/(1 - t^b)$, above by $t = 1$, and to the right by $x = 1$. Integrating x we get

$$I_{a,b} = \frac{2}{b-a} \int_0^1 \log\left(\frac{1-t^b}{1-t^a}\right) \frac{dt}{t}.$$

Finally, if we split the logarithm in two and substitute $x = t^b$ in the first integral and $x = t^a$ in the second, the desired result is obtained.

$$I_{a,b} = \frac{2}{b-a}\left(-\frac{1}{b} + \frac{1}{a}\right) \int_0^1 \frac{\log(1-x)}{x} dx = \frac{\pi^2}{3ab}.$$

□

3.2 The restricted generating function

We must now substantially extend the definitions that appear at the beginning of Section 2.

Let

$$\begin{aligned} g_{m,n}(k,N) &= \text{the number of partitions of } n \text{ into } m \text{ parts in which} \\ &\quad\ \text{each part is } \le N \text{ and there is no string of parts forming} \\ &\quad\ \text{a sequence of consecutive integers of length } k, \\ G_k(N;x,q) &= \sum_{m,n=0}^{\infty} g_{m,n}(k,N) x^m q^n. \end{aligned}$$

We note in passing that with regard to the definitions in Section 2,

$$g_n = \sum_{m\geq 0} g_{m,n}(2,\infty),$$

and

$$G_2(q) = G_2(\infty; 1, q).$$

In [2; eq. [2.5]], it was proven that

$$G_k(\infty; x, q) = \frac{1}{(xq;q)_\infty} \sum_{r,s\geq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}}}{(q^k;q^k)_s (q^{k+1};q^{k+1})_r}, \tag{3.1}$$

where

$$(A;q)_t = (1-A)(1-Aq)\dots(1-Aq^{t-1}), \qquad (A;q)_0 = 1.$$

Our object here is to prove the following result for $G_k(N;x,q)$ which reduces to (3.1) when $N\to\infty$.

Theorem 3.3.

$$\begin{aligned} G_k(N;x,q) &= \frac{1}{(xq;q)_N} \sum_{r,s\geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}} \\ &\quad \times \begin{bmatrix} N-kr-ks-r+1 \\ s \end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1}, \end{aligned} \tag{3.2}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_t = \begin{cases} 0 & \text{if } B<0 \text{ or } B>A, \\ \frac{(q^t;q^t)_A}{(q^t;q^t)_B (q^t;q^t)_{A-B}} & \text{for } 0\leq B\leq A. \end{cases}$$

Proof. We begin by noting that there is a defining recurrence for $G_k(N;x,q)$. Namely,

$$G_k(N;x,q) = \begin{cases} \frac{1}{(xq;q)_N}, \ \text{if } 0\leq N<k, \\ G_k(N-1;x,q)+ \\ \sum_{i=1}^{k-1} \frac{x^i q^{N+(N-1)+\dots+(N-i+1)} G_k(N-i-1;x,q)}{(1-xq^N)(1-xq^{N-1})\dots(1-xq^{N-i+1})} \end{cases} \tag{3.3}$$

This last assertion is easily verified as follows. If $N < k$, then there can be no sequences of k consecutive integers among the parts. Hence for $N < k$, all partitions with parts $\leq N$ must be included and the generating function in this case is

$$\frac{1}{(xq;q)_N}$$

as asserted.

To establish the bottom line of (3.3), we note that among the partitions generated by $G_k(N;x,q)$ there are some in which N does not appear as a part. These are generated by $G_k(N-1;x,q)$. If N does appear as a part, it then lies in a sequence of consecutive integers of maximal length i where $1 \leq i < k$. The portion of such partitions containing only parts in $[N-i+1, N]$ is generated by

$$\frac{x^i q^{N+(N-1)+\cdots+(N-i+1)}}{(1-xq^N)(1-xq^{N-1})\dots(1-xq^{N-i+1})},$$

and all other parts must be $< N-i$, and consequently are generated by $G_k(N-i-1;x,q)$. Hence the right-hand side of (3.3) generates precisely those partitions generated by $G_k(N;x,q)$ thus establishing the second line of (3.3).

We now define

$$S(k,N) = (xq;q)_N G_k(N;x,q). \tag{3.4}$$

Consequently $S(k,N)$ is uniquely determined by the recurrence

$$S(k,N) = \begin{cases} 1, & \text{if } 0 \leq N < k, \\ \sum_{i=1}^{k-1} x^i q^{N+(N-1)+\cdots+(N-i+1)}(1-xq^{N-i})S(k,N-i-1). \end{cases} \tag{3.5}$$

We now define

$$\begin{aligned} \sigma(k,N) &= \sum_{r,s\geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}} \\ &\quad \times \begin{bmatrix} N-kr-ks-r+1 \\ s \end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1}. \end{aligned} \tag{3.6}$$

We wish to show that $S(k,N) = \sigma(k,N)$ in order to complete the proof of this theorem. To do this we need only show that $\sigma(k,N)$ also satisfies the defining recurrence (3.5).

Immediately we see that if $N < k$, then the only non-vanishing term of the double sum in (3.6) occurs for $s = r = 0$. Hence

$$\sigma(k, N) = 1 \quad \text{if } 0 \leq N < k.$$

We shall prove the following equivariant recurrence for $\sigma(k, N)$ when $n \geq k$:

$$\begin{aligned}&\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}} \\ &\times (\sigma(k, N-i) - \sigma(k, N-i-1)) + x^k q^{kn-\binom{k}{2}} \sigma(k, N-k) = 0. \quad (3.7)\end{aligned}$$

We now simplify the left-hand side of (3.7).

$$
\begin{aligned}
&\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}} \left(\sigma(k,N-i)-\sigma(k,N-i-1)\right)\\
=\;&\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}} \sum_{r,s\geq 0}(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}}\\
&\times\left\{q^{k(N-i-kr-ks-r+1-s)}\begin{bmatrix}N-kr-ks-r-i\\ s-1\end{bmatrix}_k\begin{bmatrix}N-kr-ks-i\\ r\end{bmatrix}_{k+1}\right.\\
&\left.+q^{(k+1)(N-i-kr-ks-r)}\begin{bmatrix}N-kr-ks-r-i\\ s\end{bmatrix}_k\begin{bmatrix}N-kr-ks-1-i\\ r-1\end{bmatrix}_{k+1}\right\}\\
=\;&\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}} \sum_{r,s\geq 0}(-1)^{s+i} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-i+1}{2}}\\
&\times\left\{q^{k(N-i-k(r+s)-r+i+1-s-i)}\begin{bmatrix}N-kr-ks-r\\ s+i-1\end{bmatrix}_k\begin{bmatrix}N-kr-ks-i\\ r-i\end{bmatrix}_{k+1}\right.\\
&\left.+\,q^{(k+1)(N-i-k(r+s)-r+i)}\begin{bmatrix}N-kr-ks-r\\ s+i\end{bmatrix}_k\begin{bmatrix}N-kr-ks-i-1\\ r-i-1\end{bmatrix}_{k+1}\right\}\\
&\text{(having replaced } s \text{ by } s+i \text{ and } r \text{ by } r-i\text{)}\\
=\;&\sum_{i=0}^{k-1} q^{Ni-\binom{i}{2}} \sum_{r,s\geq 0}(-1)^{s+i} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-i+1}{2}}\\
&\times q^{k(N-i-k(r+s)-r-s+1)}\begin{bmatrix}N-kr-ks-r\\ s+i-1\end{bmatrix}_k\begin{bmatrix}N-i-kr-ks\\ r-i\end{bmatrix}_{k+1}\\
&+\sum_{i=1}^{k} q^{N(i-1)-\binom{i-1}{2}} \sum_{r,s\geq 0}(-1)^{s+i-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-i+2}{2}}\\
&\times q^{(k+1)(N-kr-ks-r)}\begin{bmatrix}N-kr-ks-r\\ s+i-1\end{bmatrix}_k\begin{bmatrix}N-i-kr-ks\\ r-i\end{bmatrix}_{k+1}\\
&\text{(having replaced } i \text{ by } i-1 \text{ in the second sum.)}
\end{aligned}
$$

Now examination of the exponents on x and q reveals that each term in the second sum for $1 \leq i \leq k-1$ is the negative of each term in the first sum. Hence all that remains after cancellation is the term $i=0$ in the first sum and the term $i=k$ in the second.

Hence

$$\begin{aligned}
&\sum_{i=0}^{k-1} x^i q^{Ni-\binom{i}{2}} \left(\sigma(k,N-i)-\sigma(k,N-i-1)\right)\\
= \quad & \sum_{r,s\geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}+k(N-(k+1)(r+s)+1)}\\
& \times \begin{bmatrix} N-kr-ks-r \\ s-1 \end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1}\\
& + q^{N(k-1)-\binom{k-1}{2}} \sum_{r,s\geq 0} (-1)^{s+k-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-k+2}{2}}\\
& \times q^{(k+1)(N-kr-ks-r)} \begin{bmatrix} N-kr-ks-r \\ s+k-1 \end{bmatrix}_k \begin{bmatrix} N-k-kr-ks \\ r-k \end{bmatrix}_{k+1}\\
:= \quad & S_1+S_2 \qquad (3.8)
\end{aligned}$$

Let us now define

$$\begin{aligned}
S_3 \quad := \quad & x^k q^{N+(N-1)+\cdots+(N-k+1)} \sigma(k,N-k) \qquad (3.9)\\
= \quad & x^k q^{kN-\binom{k}{2}} \sum_{r,s\geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}}\\
& \times \begin{bmatrix} N-k-kr-ks-r+1 \\ s \end{bmatrix}_k \begin{bmatrix} N-k-kr-ks \\ r \end{bmatrix}_{k+1}\\
= \quad & q^{kN-\binom{k}{2}} \sum_{r,s\geq 0} (-1)^{s-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s-1)^2+(k+1)\binom{r+1}{2}}\\
& \times \begin{bmatrix} N-kr-ks-r+1 \\ s-1 \end{bmatrix}_k \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1}\\
& \text{(where we have replaced } s \text{ by } s-1). \qquad (3.10)
\end{aligned}$$

In order to complete the proof of the recurrence (3.7) for $\sigma(k,n)$ we need only show that

$$S_1+S_2=-S_3.$$

Now

$$
\begin{aligned}
& S_1 + S_3 \\
= \ & \sum_{r,s\geq 0} (-1)^s x^{ks+(k+1)r} \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1} \\
& \left\{ q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}+k(N-(k+1)(r+s)+1)} \begin{bmatrix} N-kr-ks-r \\ s-1 \end{bmatrix}_k \right. \\
& \left. -q^{kN-\binom{k}{2}+\binom{k+1}{2}(r+s-1)^2+(k+1)\binom{r+1}{2}} \begin{bmatrix} N-kr-ks-r+1 \\ s-1 \end{bmatrix}_k \right\} \\
= \ & -\sum (-1)^s x^{ks+(k+1)r} \begin{bmatrix} N-kr-ks \\ r \end{bmatrix}_{k+1} q^{kN-\binom{k}{2}+\binom{k+1}{2}(r+s-1)^2+(k+1)\binom{r+1}{2}} \\
& \times q^{k(N-kr-ks-r-s+2)} \begin{bmatrix} N-kr-ks-r \\ s-2 \end{bmatrix}_k \\
& \text{(by [1; eq. (3.3.3), p. 35])} \\
& -\sum_{r,s\geq 0} (-1)^{s+k+1} x^{ks+(k+1)r} \begin{bmatrix} N-k-kr-ks \\ r-k \end{bmatrix}_{k+1} \\
& \times q^{kN-\binom{k}{2}+\binom{k+1}{2}(r+s)^2+(k+1)\binom{r-k+1}{2}} \\
& \times q^{k(N-k(r+s+1)-(r+s+1)+2)} \begin{bmatrix} N-kr-ks-r \\ s+k-1 \end{bmatrix}_k \\
= \ & -S_2.
\end{aligned}
$$

Thus $S_1 + S_2 = -S_3$; so the desired recurrence is established for $\sigma(k,n)$. Consequently $S(k,n) = \sigma(k,n)$ for all $k \geq 1, n \geq 0$ which is the result to be proved. □

4 Further remarks

4.1 A probabilistic interpretation of $\chi(q)$

The mock theta function $\chi(q)$ has an interpretation in terms of conditional probabilities in some probability space. Let $0 < q < 1$, and let $C_1, C_2, \ldots$ be a sequence of independent events with probabilities

$$P(C_n) = 1 - q^n, \quad n = 1, 2, 3 \ldots$$

Define events A and B by

$$\begin{aligned} A &= \bigcap_{n=1}^{\infty}(C_n \cup C_{n+1}), \\ B &= \bigcap_{n=2}^{\infty}(C_n \cup C_{n+1}). \end{aligned}$$

Theorem 4.1. *The following relations hold:*

$$\begin{aligned} \mathbf{P}(A|B) &= (1-q)\chi(q), \\ \mathbf{P}(C_1|A) &= 1/\chi(q). \end{aligned}$$

Proof. Let

$$F(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Holroyd, Liggett and Romik [4] proved that

$$\mathbf{P}(A) = \frac{G_2(q)}{F(q)},$$

and by a similar argument it follows that

$$\mathbf{P}(B) = \frac{H_2(q)}{(1-q)F(q)}.$$

Then, using (2.5):

$$\begin{aligned} \mathbf{P}(A|B) &= \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(1-q)G_2(q)}{H_2(q)} = (1-q)\chi(q), \\ \mathbf{P}(C_1|A) &= \frac{\mathbf{P}(C_1\cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(C_1\cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(C_1)\mathbf{P}(B)}{\mathbf{P}(A)} \\ &= \frac{(1-q)H_2(q)/(1-q)F(q)}{G_2(q)/F(q)} = 1/\chi(q). \end{aligned}$$

Incidentally, since probabilities are between 0 and 1, we get that for $0 < q < 1$,

$$\chi(q) < \frac{1}{1-q}.$$

□

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