Restricted Bipartitions.

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ABSTRACT: We consider bipartitions subject to certain restrictions and show that $b_e(m,n) \geq b_o(m,n)$, where $b_e(m,n)$ (respectively, $b_o(m,n)$) denotes the number of these partitions with an even (respectively, odd) number of even parts. Our principal tool is a lemma concerning the non-negativity of the coefficients of a certain rational function. As another corollary of this lemma, we deduce an inequality between the rank-counting numbers, $N(r,m,n)$.

§1 Introduction.

A *bipartite* number is a pair $(m,n) \in \mathbb{N} \times \mathbb{N}$ and a bipartite number (m,n) is *even* (respectively, *odd)* if $m+n$ is even (respectively, odd). A *bipartite partition* or, simply, *bipartition* of a bipartite number (m,n) is a sequence of bipartite numbers (ordered lexicographically) whose (vector) sum is (m,n). We shall be considering bipartitions satisfying two conditions:

(a) the parts (i,j) each satisfy $|i-3j| \leq 3$,
(b) the even parts are distinct.

Let $b(m,n)$ denote the number of bipartitions of (m,n) that satisfy (a) and (b) and let $b_e(m,n)$ (respectively, $b_o(m,n)$) denote the number of such bipartitions wherein the number of even parts is even (respectively, odd). We shall prove

Theorem. *$b_e(m,n) \geq b_o(m,n)$ holds for all $(m,n) \in \mathbb{N} \times \mathbb{N}$ and $b_e(m,n) = b_o(m,n)$, if $m \equiv -1 \bmod 3$.*

As an example, $b_e(5,1)$ counts the seven bipartitions: $(4,1)+(1,0)$; $(3,1)+(2,0)$; $(3,0)+(2,1)$; $(3,0)+(0,1)+(1,0)+(1,0)$; $(2,1)+(1,0)+(1,0)+(1,0)$; $(2,0)+(1,1)+(1,0)+(1,0)$; $(0,1)+(1,0)+(1,0)+(1,0)+(1,0)+(1,0)$ and $b_o(5,1)$ counts the seven bipartitions: $(5,1)$; $(3,1)+(1,0)+(1,0)$; $(3,0)+(2,0)+(0,1)$; $(3,0)+(1,0)+(1,1)$; $(2,1)+(2,0)+(1,0)$; $(2,0)+(1,0)+(1,0)+(1,0)+(0,1)$; $(1,1)+(1,0)+(1,0)+(1,0)+(1,0)$.

Our main tool is the lemma proved in §2 below.

We will use the familiar notation

$$(z;q)_\infty := \prod_{n\in\mathbb{N}} (1 - zq^n)$$

(for $|q| < 1$) and we also write

$$[z;q]_\infty := (z;q)_\infty (z^{-1}q;q)_\infty$$

(for $|q| < 1, z \neq 0$).

§2 The proof.

Define

$$R(x,y) = \frac{[xy;x^3y]_\infty (x^3y;x^3y)_\infty}{[x;x^3y]_\infty [y;x^3y]_\infty}. \tag{2.1}$$

Lemma. *The coefficients of all $x^n y^m$ in $R(x,y)$ are nonnegative and are 0, if $n \equiv -1$ modulo 3.*

Proof. Taking $b = aq$ in Ramanujan's ${}_1\Psi_1$ identity [1; (C.1), p.115], viz.

$$\sum_{n\in\mathbb{Z}} \frac{(a;q)_n t^n}{(b;q)_n} = \frac{(b/a;q)_\infty (at;q)_\infty (q/at;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b/at;q)_\infty (b;q)_\infty (t;q)_\infty},$$

(for $|b/a| < |t| < 1$) gives

$$\sum_{n\in\mathbb{Z}} \frac{t^n}{1 - aq^n} = \frac{[at;q]_\infty (q;q)_\infty^2}{[a;q]_\infty [t;q]_\infty} \tag{2.2}$$

for $|q| < |t| < 1$.

Note that

$$\begin{aligned} R(x,y) &= \frac{[xy;x^3y]_\infty (x^3y;x^3y)_\infty}{[x;x^3y]_\infty [y;x^3y]_\infty} \\ &= \frac{1}{(x^3y;x^3y)_\infty} \sum_{n\in\mathbb{Z}} \frac{x^n}{1 - y(x^3y)^n} \end{aligned} \tag{2.3}$$

(by (2.2), with $q = x^3y, a = y, t = x$). Splitting this last sum according to the residue classes of n mod 3, we have

$$R(x,y) = R_0(x,y) + R_1(x,y) + R_{-1}(x,y) \tag{2.4}$$

where

$$R_i(x,y) = \frac{1}{(x^3y;x^3y)_\infty} \sum_{n\in\mathbb{Z}} \frac{x^{3n+i}}{1-y(x^3y)^{3n+i}} \tag{2.5}$$

Note that, in $R_i(x,y)$, all powers of x are congruent to i mod 3.

Now

$$\begin{aligned} R_{-1}(x,y) &= \frac{x^{-1}}{(x^3y;x^3y)_\infty} \sum_{n\in\mathbb{Z}} \frac{x^{3n}}{1-x^{-3}(x^3y)^{3n}} \\ &= \frac{x^{-1}[1;x^9y^3]_\infty (x^9y^3;x^9y^3)_\infty^2}{(x^3y;x^3y)_\infty [x^{-3};x^9y^3]_\infty [x^3;x^9y^3]_\infty}, \end{aligned} \tag{2.6}$$

(by (2.2)),

$$= 0.$$

Next,

$$\begin{aligned} R_0(x,y) &= \frac{1}{(x^3y;x^3y)_\infty} \sum_{n\in\mathbb{Z}} \frac{x^{3n}}{1-y(x^3y)^{3n}} \\ &= \frac{1}{(x^3y;x^3y)_\infty} \frac{[x^3y;x^9y^3]_\infty (x^9y^3;x^9y^3)_\infty^2}{[x^3;x^9y^3]_\infty [y;x^9y^3]_\infty} \\ &= \frac{(x^9y^3;x^9y^3)_\infty}{[x^3;x^9y^3]_\infty [y;x^9y^3]_\infty} \\ &= \prod_{n\in\mathbb{N}} \frac{(1-(x^9y^3)^{2n+1})}{(1-y(x^9y^3)^n)(1-x^9y^2(x^9y^3)^n)} \\ &\quad \times \frac{1}{1-x^3} \prod_{n\in\mathbb{N}} \frac{(1-(x^9y^3)^{2n+2})}{(1-x^{12}y^3(x^9y^3)^n)(1-x^6y^3(x^9y^3)^n)} \\ &= \frac{1}{1-x^3} \prod_{n\in\mathbb{N}} \frac{(1-x^{18n+9}y^{6n+3})}{(1-x^{9n}y^{3n+1})(1-x^{9n+9}y^{3n+2})} \\ &\quad \times \prod_{n\in\mathbb{N}} \frac{(1-x^{18n+18}y^{6n+6})}{(1-x^{9n+12}y^{3n+3})(1-x^{9n+6}y^{3n+3})} \end{aligned} \tag{2.7}$$

It follows that the coefficients of $x^n y^m$ in $R_0(x,y)$ are all non-negative, because each multiplicand in the final expression for $R_0(x,y)$ has the form

$$\frac{1-AB}{(1-A)(1-B)} = \frac{1}{1-A} + \frac{B}{1-B} = 1 + \sum_{n=1}^{\infty}(A^n + B^n). \tag{2.8}$$

Finally

$$\begin{aligned}
R_1(x,y) &= \frac{x}{(x^3y;x^3y)_\infty} \sum_{n\in\mathbb{Z}} \frac{x^{3n}}{1-x^3y^2(x^3y)^{3n}} \\
&= \frac{x}{(x^3y;x^3y)_\infty} \frac{[x^3y;x^9y^3]_\infty (x^9y^3;x^9y^3)_\infty^2}{[x^3;x^9y^3]_\infty [x^3y^2;x^9y^3]_\infty} \\
&= \frac{x(x^9y^3;x^9y^3)_\infty}{[x^3;x^9y^3]_\infty [x^3y^2;x^9y^3]_\infty} \\
&= x \prod_{n\in\mathbb{N}} \frac{(1-(x^9y^3)^{2n+1})}{(1-x^3(x^9y^3)^n)(1-x^6y^3(x^9y^3)^n)} \\
&\quad \times \frac{1}{1-x^6y} \prod_{n\in\mathbb{N}} \frac{(1-(x^9y^3)^{2n+2})}{(1-x^3y^2(x^9y^3)^n)(1-x^{15}y^4(x^9y^3)^n)} \\
&= \frac{x}{1-x^6y} \prod_{n\in\mathbb{N}} \frac{(1-x^{18n+9}y^{6n+3})}{(1-x^{9n+3}y^{3n})(1-x^{9n+6}y^{3n+3})} \\
&\quad \times \prod_{n\in\mathbb{N}} \frac{(1-x^{18n+18}y^{6n+6})}{(1-x^{9n+3}y^{3n+2})(1-x^{9n+15}y^{3n+4})}
\end{aligned}$$

Once again, all the multiplicands in $R_1(x,y)$ are of the form (2.8), and the coefficients of all $x^n y^m$ in $R_1(x,y)$ are non-negative. The decomposition (2.1) proves the lemma. $\square$

Now, a little thought shows that

$$\sum_{m,n\in\mathbb{N}\times\mathbb{N}}^{\infty} \Big(b_e(m,n) - b_o(m,n)\Big) x^m y^n = R(x,y)$$

and the Theorem follows from this Lemma.

Let $N(r,m,n)$ denote the number of partitions of n having rank [2] congruent to r modulo m. Our original motivation in introducing the function $R(x,y)$ was to give a proof of

Corollary. *For all $n \in \mathbb{N}$,*

$$N(1,9,3n+1) \leq N(0,9,3n+1),$$

and the inequality is strict for $n > 1$.

Proof. It is shown in [3; p.116] that

$$\sum_{n\in\mathbb{N}} \Big(N(0,9,3n+1) - N(1,9,3n+1)\Big) q^n = \frac{[q^4;q^9]_\infty (q^9;q^9)_\infty}{[q^2;q^9]_\infty^2 [q^3;q^9]_\infty} = \frac{1}{[q^2;q^9]_\infty} R(q^2,q^3) \tag{2.9}$$

By the Lemma, $R(q^2,q^3)$ has nonnegative coefficients and it is plain that $[q^2;q^9]_\infty^{-1}$ has positive coefficients from q^7 onwards. So the same is true of the series (2.9). But this series begins $1 + 2q^2 + q^3 + 2q^4 + q^5 + 4q^6 + \cdots$. $\square$

It is pointed out in [3; p.87] that three further inequalities follow easily from the identities

$$\sum_{n\in\mathbb{N}} \Big(N(1,9,3n+2) - N(0,9,3n+2)\Big) q^n = \frac{(q^9;q^9)_\infty}{[q^2;q^9]_\infty [q^3;q^9]_\infty}$$

$$\sum_{n\in\mathbb{N}} \Big(N(0,9,3n+1) - N(2,9,3n+1)\Big) q^n = \frac{(q^9;q^9)_\infty}{[q;q^9]_\infty [q^3;q^9]_\infty}$$

$$\sum_{n\in\mathbb{N}} \Big(N(2,9,3n) - N(3,9,3n)\Big) q^n = \frac{(q^9;q^9)_\infty}{[q^4;q^9]_\infty [q^3;q^9]_\infty}$$

[3; pp.116,117] (and without the help of our Lemma), namely

$$N(1,9,3n+2) > N(0,9,3n+2), \qquad (n \geq 2),$$

$$N(0,9,3n+1) > N(2,9,3n+1), \qquad (n \geq 0),$$

$$N(2,9,3n) > N(3,9,3n), \qquad (n \geq 3).$$

References.

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