An identity relating a theta function to a sum of Lambert series

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Abstract

We derive an identity connecting a theta function and a sum of Lambert series. As a consequence of this identity, we deduce a number of results of Jacobi, Dirichlet, Lorenz, Ramanujan and Rademacher.

1 Introduction

Suppose throughout that q is a complex number of modulus < 1. We will use the familiar notation

$$(z;q)_\infty := \prod_{n=0}^{\infty}(1-zq^n)$$

and we also set

$$[z;q]_\infty := (z;q)_\infty(z^{-1}q;q)_\infty.$$

and

$$[a,b,\ldots,y,z;q]_\infty := [a;q]_\infty[b;q]_\infty\cdots[y;q]_\infty[z;q]_\infty.$$

It is easy to see that

$$[z^{-1};q]_\infty = [zq;q]_\infty = -z^{-1}[z;q]_\infty. \tag{1}$$

Note that, as a function of z, $[z;q]_\infty$ has an essential singularity at $z=0$, no other singularities and simple zeros at $z=q^n$ for each $n\in\mathbb{Z}$.

Our main purpose is to prove

Theorem 1 *Suppose $a, b, c \neq q^n$ (for any $n \in \mathbb{Z}$) are non-zero complex numbers with $abc \neq q^n$. Then*

$$\begin{aligned}\frac{[bc, ca, ab; q]_\infty (q;q)_\infty^2}{[a,b,c,abc;q]_\infty} &= 1 + \sum_{n=0}^{\infty} \frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty} \frac{q^n/a}{1-q^n/a} \\ &+ \sum_{n=0}^{\infty} \frac{bq^n}{1-bq^n} - \sum_{n=1}^{\infty} \frac{q^n/b}{1-q^n/b} \\ &+ \sum_{n=0}^{\infty} \frac{cq^n}{1-cq^n} - \sum_{n=1}^{\infty} \frac{q^n/c}{1-q^n/c} \\ &- \sum_{n=0}^{\infty} \frac{abcq^n}{1-abcq^n} + \sum_{n=1}^{\infty} \frac{q^n/abc}{1-q^n/abc}. \end{aligned} \tag{2}$$

We give two proofs of Theorem 1 in §2. In §3, we use this theorem to give proofs of the two theorems which follow:

Theorem 2

$$\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} = 1 + 4\sum_{n=0}^{\infty} \frac{q^{4n+1}}{1-q^{4n+1}} - 4\sum_{n=0}^{\infty} \frac{q^{4n+3}}{1-q^{4n+3}}, \tag{3}$$

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} q^{m^2+2n^2} &= 1 + 2\left(\sum_{n=0}^{\infty} \frac{q^{8n+1}}{1-q^{8n+1}} + \sum_{n=0}^{\infty} \frac{q^{8n+3}}{1-q^{8n+3}}\right) \\ &\quad - 2\left(\sum_{n=0}^{\infty} \frac{q^{8n+5}}{1-q^{8n+5}} + \sum_{n=0}^{\infty} \frac{q^{8n+7}}{1-q^{8n+7}}\right), \end{aligned} \tag{4}$$

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} q^{m^2+3n^2} &= 1 + 2\left(\sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{3n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{3n+2}}\right) \\ &\quad + 4\left(\sum_{n=0}^{\infty} \frac{q^{12n+4}}{1-q^{12n+4}} - \sum_{n=0}^{\infty} \frac{q^{12n+8}}{1-q^{12n+8}}\right), \end{aligned} \tag{5}$$

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} q^{m^2+7n^2} &= 1 + 2\Bigg(\sum_{n=0}^{\infty} \frac{q^{7n+1}}{1-q^{7n+1}} + \sum_{n=0}^{\infty} \frac{q^{7n+2}}{1-q^{7n+2}} + \sum_{n=0}^{\infty} \frac{q^{7n+4}}{1-q^{7n+4}} \\ &\quad - \sum_{n=0}^{\infty} \frac{q^{7n+3}}{1-q^{7n+3}} - \sum_{n=0}^{\infty} \frac{q^{7n+5}}{1-q^{7n+5}} - \sum_{n=0}^{\infty} \frac{q^{7n+6}}{1-q^{7n+6}}\Bigg) \\ &\quad - 4\Bigg(\sum_{n=0}^{\infty} \frac{q^{28n+2}}{1-q^{28n+2}} + \sum_{n=0}^{\infty} \frac{q^{28n+18}}{1-q^{28n+18}} + \sum_{n=0}^{\infty} \frac{q^{28n+22}}{1-q^{28n+22}} \\ &\quad - \sum_{n=0}^{\infty} \frac{q^{28n+6}}{1-q^{28n+6}} - \sum_{n=0}^{\infty} \frac{q^{28n+10}}{1-q^{28n+10}} - \sum_{n=0}^{\infty} \frac{q^{28n+26}}{1-q^{28n+26}}\Bigg). \end{aligned} \tag{6}$$

(3) is due to Jacobi (see [10]) and (4) and (5) are due to Dirichlet and Lorenz, respectively (see [9]). (6) was found by Ramanujan; it is Entry 17(ii) in chapter 19 of his second Notebook [5, p.304]. The proof given in [5] uses a modular equation of the seventh order given in Entry 19(i); we believe that our proof of (6) is somewhat more transparent.

Theorem 3

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} - 8\sum_{n=1}^{\infty}(-1)^n \frac{nq^{2n}}{1-q^{2n}}, \tag{7}$$

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^6 = 1 + 16\sum_{n=1}^{\infty} \frac{n^2q^n}{1+q^{2n}} + 4\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)^2q^{2n-1}}{1-q^{2n-1}}, \tag{8}$$

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^8 = 1 + 16\sum_{n=1}^{\infty}(-1)^n \frac{n^3q^{2n}}{1-q^{2n}} + 16\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^{2n}}. \tag{9}$$

(7), (8) and (9) are all due to Jacobi (see [11; §90.2-90.4]).

Not one of (3)-(9) is new. What we believe are new are the uniform proofs of these identities provided by Theorem 1.

2 Two proofs of Theorem 1

Following [3], we say that points $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ are *equivalent* if $z_2 = q^n z_1$, for some $n \in \mathbb{Z}$. We say that a function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is *q-elliptic* if

- the only singularities of f are (isolated) poles,
- $f(qz) = q^{-1}f(z)$.

Then it may be shown that q-elliptic functions have the useful property ([8, Lemma 2], [6, Lemma]):

$$\sum_{\pi\in\mathcal{P}} \mathrm{res}(f;\pi) = 0, \tag{10}$$

where $\mathcal{P}$ is a complete set of inequivalent poles of f.

Remark: (10) is, essentially, the familiar fact that the sum of the residues of an (ordinary) elliptic function at its poles in a period parallelogram is zero.

Suppose now that $a, b, c \in \mathbb{C}\backslash\{0\}$ and that $a, b, c, abc \neq q^n$, for any $n \in \mathbb{Z}$. For $z \in \mathbb{C}\backslash\{0\}$, define

$$f(z) := \frac{[z/a;q]_\infty [z/b;q]_\infty [z/c;q]_\infty}{z[z;q]_\infty^2 [z/abc;q]_\infty} \tag{11}$$

It follows from (1) that f is q-elliptic, so (10) holds. We may take $\mathcal{P} = \{1, abc\}$, where $z = abc$ is a simple pole of f and $z = 1$ is a pole of order 2. Thus (10) gives

$$\operatorname{res}(f;1) + \operatorname{res}(f;abc) = 0. \tag{12}$$

Now

$$\begin{aligned}\operatorname{res}(f;abc) &= \lim_{z\to abc}(z-abc)f(z)\\ &= \frac{[bc;q]_\infty[ca;q]_\infty[ab;q]_\infty}{abc[abc;q]_\infty^2}\lim_{z\to abc}\frac{z-abc}{[z/abc;q]_\infty}\\ &= -\frac{[bc;q]_\infty[ca;q]_\infty[ab;q]_\infty}{[abc;q]_\infty^2(q;q)_\infty^2}\end{aligned} \tag{13}$$

To calculate the residue of f at 1 we need to be a little more cunning. Set

$$F(z) := (z-1)^2 f(z) = \frac{[z/a;q]_\infty[z/b;q]_\infty[z/c;q]_\infty}{z(zq;q)_\infty^2(q/z;q)_\infty^2[z/abc;q]_\infty}$$

(since $[z;q]_\infty = (1-z)(zq;q)_\infty(q/z;q)_\infty$). Then

$$\operatorname{res}(f;1) = \lim_{z\to 1}\frac{d}{dz}F(z).$$

But

$$\begin{aligned}\frac{d}{dz}F(z) &= F(z)\frac{d}{dz}\ln F(z)\\ &= F(z)\frac{d}{dz}\Bigg(\ln[z/a;q]_\infty + \ln[z/b;q]_\infty + \ln[z/c;q]_\infty\\ &\qquad - \ln z - 2\ln(zq;q)_\infty - 2\ln(q/z;q)_\infty - \ln[z/abc;q]_\infty\Bigg)\\ &= F(z)\Bigg\{-\sum_{n=0}^\infty\frac{q^n/a}{1-zq^n/a} + \sum_{n=1}^\infty\frac{aq^n/z^2}{1-aq^n/z}\\ &\qquad - \sum_{n=0}^\infty\frac{q^n/b}{1-zq^n/b} + \sum_{n=1}^\infty\frac{bq^n/z^2}{1-bq^n/z}\\ &\qquad - \sum_{n=0}^\infty\frac{q^n/c}{1-zq^n/c} + \sum_{n=1}^\infty\frac{cq^n/z^2}{1-cq^n/z}\\ &\qquad + \sum_{n=0}^\infty\frac{q^n/abc}{1-zq^n/abc} - \sum_{n=1}^\infty\frac{abcq^n/z^2}{1-abcq^n/z}\\ &\qquad - \frac{1}{z} + 2\sum_{n=1}^\infty\frac{q^n}{1-zq^n} - 2\sum_{n=1}^\infty\frac{q^n/z^2}{1-q^n/z}\Bigg\}.\end{aligned} \tag{14}$$

Now, when $z=1$, the bracketed sum in (14) is

$$1+\sum_{n=0}^{\infty}\frac{aq^n}{1-aq^n}-\sum_{n=1}^{\infty}\frac{q^n/a}{1-q^n/a}+\sum_{n=0}^{\infty}\frac{bq^n}{1-bq^n}-\sum_{n=1}^{\infty}\frac{q^n/b}{1-q^n/b}$$
$$+\sum_{n=0}^{\infty}\frac{cq^n}{1-cq^n}-\sum_{n=1}^{\infty}\frac{q^n/c}{1-q^n/c}-\sum_{n=0}^{\infty}\frac{abcq^n}{1-abcq^n}+\sum_{n=1}^{\infty}\frac{q^n/abc}{1-q^n/abc} \tag{15}$$

and we have, by (1) ,

$$\lim_{z\to 1}F(z)=\frac{[1/a;q]_\infty[1/b;q]_\infty[1/c;q]_\infty}{(q;q)_\infty^4[1/abc;q]_\infty}=\frac{[a;q]_\infty[b;q]_\infty[c;q]_\infty}{(q;q)_\infty^4[abc;q]_\infty}. \tag{16}$$

Now (12), (13), (15) and (16) together prove (2). $\boxed{\text{qed}}$

(By considering instead the function

$$f(z):=\frac{[z/a;q]_\infty[z/b;q]_\infty[z/c;q]_\infty[z/d;q]_\infty}{z[z;q]_\infty^2[z/ab;q]_\infty[z/cd;q]_\infty}$$

(where $a,b,c,d,ab,cd\neq q^n$), which is q-elliptic, we could have given an identity more general than (2). However (2) is all we need and, in its generalisation, seems to lose much of its elegance.)

This was how (2) was found, though we later came up with a neater proof, based on Bailey's ${}_6\Psi_6$ summation [4]. We now give this alternative proof, which uses the elementary identity (see [2]):

$$1+\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{abc}{1-abc}=\frac{(1-bc)(1-ac)(1-ab)}{(1-a)(1-b)(1-c)(1-abc)}. \tag{17}$$

We have

$$\begin{aligned}
&+\textstyle\sum_{n=0}^{\infty}\frac{cq^n}{1-cq^n}-\sum_{n=1}^{\infty}\frac{q^n/c}{1-q^n/c}-\sum_{n=0}^{\infty}\frac{abcq^n}{1-abcq^n}+\sum_{n=1}^{\infty}\frac{q^n/abc}{1-q^n/abc}\\
&=1+\tfrac{a}{1-a}+\tfrac{b}{1-b}+\tfrac{c}{1-c}-\tfrac{abc}{1-abc}\\
&+\textstyle\sum_{n=1}^{\infty}\left(\frac{aq^n}{1-aq^n}+\frac{bq^n}{1-bq^n}-\frac{c^{-1}q^n}{1-c^{-1}q^n}-\frac{abcq^n}{1-abcq^n}\right)\\
&+\textstyle\sum_{n=1}^{\infty}\left(-\frac{q^n/a}{1-q^n/a}-\frac{q^n/b}{1-q^n/b}+\frac{cq^n}{1-cq^n}-\frac{q^n/abc}{1-q^n/abc}\right)\\
&=-c^{-1}\textstyle\sum_{n=-\infty}^{\infty}\frac{(1-bc)(1-ac)(1-abq^{2n})q^n}{(1-aq^n)(1-bq^n)(1-c^{-1}q^n)(1-abcq^n)}
\end{aligned}$$

by (17)

$$\begin{aligned}
&=\frac{(1-bc)(1-ac)(1-ab)}{(1-a)(1-b)(1-c)(1-abc)}\times{}_6\Psi_6\left[\begin{matrix}q\sqrt{ab},-q\sqrt{ab},a,b,1/c,abc\\ \sqrt{ab},-\sqrt{ab},bq,aq,abcq,q/c\end{matrix};q,q\right]\\
&=\frac{[bc,ca,ab;q]_\infty(q;q)_\infty^2}{[a,b,c,abc;q]_\infty},
\end{aligned}$$

by Bailey's summation. $\boxed{\text{qed}}$

3 Proofs of Theorems 2 and 3.

We now apply Theorem 1 to give uniform proofs of Theorems 2 and 3. We first note that it follows from Jacobi's Triple Product Identity [1, Theorem 2.8] which, in our notation, states

$$[z;q]_\infty (q;q)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{n(n-1)/2} \tag{18}$$

that

$$\sum_{n=-\infty}^{\infty} q^{n^2} = [-q;q^2]_\infty (q^2;q^2)_\infty \tag{19}$$

(under $q \mapsto q^2,\ z \mapsto -q$). Then

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty} \tag{20}$$

follows from (19).

The proof of Theorem 2.

We now establish (3), (4), (5) and (6). For (3), take q^4 for q in (2) and then take $a,b,c = q,q,q$. The RHS of (2) is the RHS of (3) and the LHS of (2) becomes

$$\begin{aligned}\frac{[q^2,q^2,q^2;q^4]_\infty (q^4;q^4)_\infty^2}{[q,q,q,q^3;q^4]_\infty} &= \frac{[q^2;q^4]_\infty^2 (q^2;q^2)_\infty^2}{[q;q^4]_\infty^4} \\ &= \frac{[q^2;q^4]_\infty^2 (q^2;q^2)_\infty^2}{[q;q^2]_\infty^2} \\ &= [-q;q^2]_\infty^2 (q^2;q^2)_\infty^2\end{aligned}$$

which, by (19),

$$= \sum_{m,n=-\infty}^{\infty} q^{m^2+n^2}. \qquad \boxed{\text{qed}}$$

To prove (4), we replace q by q^8 in (2) and take for parameters $a, b, c = q, q, q^3$. The RHS of (2) is the RHS of (4) and the LHS is

$$\begin{aligned}
\frac{[q^4, q^4, q^2; q^8]_\infty (q^8; q^8)_\infty^2}{[q, q, q^3, q^5; q^8]_\infty} &= \frac{[q^2, q^4; q^8]_\infty (q^4; q^4)_\infty^2}{[q; q^2]_\infty} \\
&= \frac{[q^2, q^4; q^8]_\infty (q^2; q^2)_\infty (q^4; q^4)_\infty}{[q; q^2]_\infty (q^2; q^4)_\infty} \\
&= \frac{[q^2; q^4]_\infty [q^4; q^8]_\infty (q^2; q^2)_\infty (q^4; q^4)_\infty}{[q; q^2]_\infty [q^2; q^4]_\infty} \\
&= [-q; q^2]_\infty (q^2; q^2)_\infty [-q^2; q^4]_\infty (q^4; q^4)_\infty \\
&= \sum_{m,n=-\infty}^{\infty} q^{m^2+2n^2}. \qquad \boxed{\text{qed}}
\end{aligned}$$

Now we prove (5). First take q^6 for q in (2) and set $a, b, c = -q, -q^2, -q^3$. We get

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+3n^2} &= \frac{(q; q)_\infty (q^3; q^3)_\infty}{(-q; q)_\infty (-q^3; q^3)_\infty} \qquad \text{(by (20))} \\
&= \frac{2[q, q^2, q^3; q^6]_\infty (q^6; q^6)_\infty}{[-q, -q^2, -q^3, -q^6; q^6]_\infty} \\
= 2\Bigg(1 - \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1+q^{6n+1}} &+ \sum_{n=0}^{\infty} \frac{q^{6n+5}}{1+q^{6n+5}} - \sum_{n=0}^{\infty} \frac{q^{6n+2}}{1+q^{6n+2}} + \sum_{n=0}^{\infty} \frac{q^{6n+4}}{1+q^{6n+4}} \\
- \sum_{n=0}^{\infty} \frac{q^{6n+3}}{1+q^{6n+3}} &+ \sum_{n=0}^{\infty} \frac{q^{6n+3}}{1+q^{6n+3}} + \sum_{n=0}^{\infty} \frac{q^{6n+6}}{1+q^{6n+6}} - \sum_{n=0}^{\infty} \frac{q^{6n}}{1+q^{6n}}\Bigg) \\
= 1 - 2\Bigg(\sum_{n=0}^{\infty} \frac{q^{6n+1}}{1+q^{6n+1}} &- \sum_{n=0}^{\infty} \frac{q^{6n+5}}{1+q^{6n+5}} + \sum_{n=0}^{\infty} \frac{q^{6n+2}}{1+q^{6n+2}} - \sum_{n=0}^{\infty} \frac{q^{6n+4}}{1+q^{6n+4}}\Bigg).
\end{aligned} \tag{21}$$

Replacing q by $-q$ in (21) yields (5). $\boxed{\text{qed}}$

To prove (6), take q^7 for q and set $a, b, c = -q, -q^2, -q^4$ in (2). We get

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{7n^2} &= \frac{(q; q)_\infty}{(-q; q)_\infty} \frac{(q^7; q^7)_\infty}{(-q^7; q^7)_\infty} \\
&= 2 \frac{[q, q^2, q^3; q^7]_\infty (q^7; q^7)_\infty^2}{[-1, -q, -q^2, -q^3; q^7]_\infty}
\end{aligned} \tag{22}$$

Replacing q by q^7 and setting $a := -1, b := -q, c := -q^2$ in (2), we obtain

$$\frac{[q, q^2, q^3; q^7]_\infty (q^7; q^7)^2_\infty}{[-1, -q, -q^2, -q^3; q^7]_\infty} =$$
$$1 - \sum_{n=0}^{\infty} \frac{q^{7n}}{1+q^{7n}} + \sum_{n=1}^{\infty} \frac{q^{7n}}{1+q^{7n}} - \sum_{n=0}^{\infty} \frac{q^{7n+1}}{1+q^{7n+1}} + \sum_{n=0}^{\infty} \frac{q^{7n+6}}{1+q^{7n+6}}$$
$$- \sum_{n=0}^{\infty} \frac{q^{7n+2}}{1+q^{7n+2}} + \sum_{n=0}^{\infty} \frac{q^{7n+5}}{1+q^{7n+5}} + \sum_{n=0}^{\infty} \frac{q^{7n+3}}{1+q^{7n+3}} - \sum_{n=0}^{\infty} \frac{q^{7n+4}}{1+q^{7n+4}}. \tag{23}$$

Amalgamating (22) and (23) and replacing q by $-q$ gives (6). $\boxed{\text{qed}}$

For our proof of Theorem 3, we will need the following two lemmas. The first of these, Lemma 6, was given by Bailey [4]. We derive this result from Theorem 1.

Lemma 4

$$\begin{aligned}\frac{b[a/b, ab; q]_\infty (q;q)^4_\infty}{[a, b; q]^2_\infty} &= \sum_{n=1}^{\infty} \frac{bq^n}{(1-bq^n)^2} + \sum_{n=0}^{\infty} \frac{b^{-1}q^n}{(1-b^{-1}q^n)^2} \\ &\quad - \sum_{n=1}^{\infty} \frac{aq^n}{(1-aq^n)^2} - \sum_{n=0}^{\infty} \frac{a^{-1}q^n}{(1-a^{-1}q^n)^2} \\ &= \sum_{n=-\infty}^{\infty} \frac{bq^n}{(1-bq^n)^2} - \sum_{n=-\infty}^{\infty} \frac{aq^n}{(1-aq^n)^2}. \end{aligned} \tag{24}$$

Proof Divide each side of (2) by $1 - ca$ and then let $c \to a^{-1}$. We obtain (24). $\boxed{\text{qed}}$

Noting that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{aq^n}{(1-aq^n)^2} &= \frac{a}{(1-a)^2} + \left\{ \sum_{n=1}^{\infty} \frac{aq^n}{(1-aq^n)^2} + \frac{q^n/a}{(1-q^n/a)^2} \right\} \\ &= \frac{a}{(1-a)^2} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(a^m + a^{-m}) q^{mn} \\ &= \frac{a}{(1-a)^2} + \sum_{m=1}^{\infty} \frac{m(a^m + a^{-m})q^m}{1-q^m}\end{aligned}$$

we see that the identity (24) can be written as

$$\frac{b}{(1-b)^2} - \frac{a}{(1-a)^2} + \sum_{n=1}^{\infty} \frac{n(b^n + b^{-n} - a^n - a^{-n})q^n}{1-q^n} = \frac{b[a/b, ab; q]_\infty (q;q)^4_\infty}{[a, b; q]^2_\infty} \tag{25}$$

If we divide both sides of (25) by $b-a$ and then let $b \to a$, we get

$$\frac{a(1+a)}{(1-a)^3} + \sum_{n=1}^{\infty}(a^n - a^{-n})\frac{n^2q^n}{1-q^n} = \frac{a[a^2;q]_\infty (q;q)_\infty^6}{[a;q]_\infty^4} \tag{26}$$

In the proof of (8) we also use

Lemma 5

$$[-q;q^2]_\infty^4 = [q;q^2]_\infty^4 + q[-1;q^2]_\infty^4. \tag{27}$$

(A more familiar form of (27) is the identity: $\theta_3^4 = \theta_2^4 + \theta_4^4$).

Proof A proof of (27) is given in [6]. It is also a consequence of [8, Lemma 2, cor.] on taking $w = q^2$ and $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, -q, -q^2; q, q, q)$. $\boxed{\text{qed}}$

Proof of Theorem 3. We first dispose of (7). Put $a = -i, b = -1$ in (25) and we get

$$\begin{aligned}\left(\sum_{n=0}^{\infty}(-1)^n q^{n^2}\right)^4 &= \frac{(q;q)_\infty^4}{(-q;q)_\infty^4} \\ &= 1 + 8\sum_{n=1}^{\infty}(-1)^n \frac{nq^n}{1+q^n}\end{aligned} \tag{28}$$

and changing q to $-q$ in (28) yields

$$\left(\sum_{n=0}^{\infty} q^{n^2}\right)^4 = 1 + 8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n} = 1 + 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} - 8\sum_{n=1}^{\infty}(-1)^n\frac{nq^{2n}}{1-q^{2n}} \tag{29}$$

which is (7). $\boxed{\text{qed}}$

Now divide each side of (26) by $1+a$ and then set $a = -1$. We get

$$1 + 16\sum_{n=1}^{\infty}\frac{(-1)^n n^3 q^n}{1-q^n} = \frac{(q;q)_\infty^8}{(-q;q)_\infty^8} = \left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^8$$

and, changing q to $-q$, we have (9). $\boxed{\text{qed}}$

Finally we establish (8). First we take $a = i = \sqrt{-1}$ in (26), obtaining

$$1 + 4\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)^2q^{2n-1}}{1-q^{2n-1}} = \frac{(q;q)_\infty^6(-q;q)_\infty^2}{(-q^2;q^2)_\infty} = (q^2;q^2)_\infty^6[-q;q^2]_\infty^2[q;q^2]_\infty^4. \tag{30}$$

On the other hand, using the elementary identity

$$\frac{a(1+a)}{(1-a)^3} = \sum_{n=1}^{\infty} n^2 a^n,$$

we can rewrite (26) as

$$\sum_{n=1}^{\infty} \frac{n^2(a^n - a^{-n}q^n)}{1-q^n} = \frac{a[a^2;q]_\infty (q;q)_\infty^6}{[a;q]_\infty^4}. \tag{31}$$

Now change q to q^4 in (31) and replace a by q and we get

$$16\sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^{2n}} = 16\frac{q[q^2;q^4]_\infty (q^4;q^4)_\infty^6}{[q;q^4]_\infty^4} = q(q^2;q^2)_\infty^6 [-q;q^2]_\infty^2 [-q^2;q^2]_\infty^4. \tag{32}$$

Adding (30) and (32) we have

$$\begin{aligned} 1 + 4\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)^2 q^{2n-1}}{1-q^{2n-1}} &+ 16\sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^{2n}} \\ &= (q^2;q^2)_\infty^6 [-q;q^2]_\infty^2 \left([q;q^2]_\infty^4 + q[-q^2;q^2]_\infty^4 \right) \\ &= [-q;q^2]_\infty^6 (q^2;q^2)_\infty^6 = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^6 \end{aligned}$$

by (27) and (19). This is (8). $\boxed{\text{qed}}$

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