It is clear that conversely any monic polynomial $F(x) \in R[x]$ satisfying condition (24) is a minimal polynomial for A.

THEOREM 7. Assume that (24) holds and that $G(x)$ is a monic polynomial in $R[x]$ such that $\bar{G}(x)$ is a minimal polynomial of the matrix $\bar{A}$. Then the following conditions are equivalent

(a) The ring $R[x]$ contains a unique minimal polynomial for the matrix A.

(b) $\deg F(x) = \deg G(x)$.

(c) Ann(A) is a principal ideal.

Proof. It is clear that $\bar{G}(x)|\bar{F}(x)$. Therefore, if (a) holds but (b) is false, then $\deg G(x) < \deg F(x)$. Therefore, it we choose a nonzero element $\pi \in J$ with $\pi \cdot J = 0$ we obtain that $\pi G(A) = 0$ and $F(x) + \pi G(x)$ is a minimal polynomial for A different from $F(x)$.

Assume that condition (b) holds. It is clear by (24) that (c) is equivalent to the condition $\mathrm{Ann}(A) = (F(x))$. Assume that (c) does not hold. Then there exists a polynomial $L(x) \in \mathrm{Ann}(A) \cap J[x]$ such that $\deg L(x) < \deg G(x)$. If necessary, we may of course multiply the polynomial $L(x)$ by a suitable element $\pi \in J$, and thus we may assume that $L(x)$ is a nonzero polynomial with coefficients from the ideal $(0:J)$. This ideal can be considered as a finite-dimensional space over the field $\bar{R}$. Let $\pi_1, \ldots, \pi_t$ be a basis of this space. Then the polynomial $L(x)$ can be represented in the form $L(x) = L_1(x)\pi_1 + \ldots + L_t(x)\pi_t$, where $\deg L_i(x) < \deg G(x)$ for $i = \overline{1, t}$ and for at least one $i \le t$ we have $\bar{L}_i(x) \neq 0$. Assume that $\bar{L}_1(x) \neq \bar{0}$. Then it follows from $L(A) = 0$ that $\bar{L}_1(\bar{A}) = \bar{0}$, contrary to the definition of $G(x)$. Thus (b) implies (c).

If (c) holds we have obviously $\mathrm{Ann}(A) = (F(x))$ and thus (a). This concludes the proof of Theorem 7.

Remark. The equivalence of conditions (a) and (c) of Theorem 7 in the case of R being a principal ideal ring was noted in [8, Theorem II.4]. However, in the proof of this result use is made twice of the following erroneous assertion: if $J(R) = \pi R$ and there exists $U(x) \in R[x]$ such that $U(A) \in \pi^s R_m$ then there exists $V(x) \in R[x]$ such that $U(A) = \pi^s V(A)$. This does not hold for example if $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \in (\mathbf{Z}/4)_3$ and $U(x) = x^2$.

THEOREM 8. A matrix $A \in R_m$ is polynomially determined if and only if the following conditions hold:

(a) Ann(A) is a principal ideal;

(b) if two elementary divisors of the matrix $x\bar{E} - \bar{A}$ are not relatively prime, they are equal.

If these conditions hold, all Fitting invariants of $xE - A$ are principal ideals, and if $\mathrm{Ann}(A) = (F(x))$ and the decomposition of $F(x)$ into primary pairwise relatively prime factors is of the form $F(x) = F_1(x) \cdot \ldots \cdot F_t(x)$, then

$$A \approx \mathrm{Diag}(S(F_1), \ldots, S(F_1), S(F_2), \ldots, S(F_t)).$$

Proof. By Theorem 2 it is clear that it suffices to consider the case that $\chi_A(x)$ is a primary polynomial; this will be assumed in what follows. Assume that (a) and (b) hold and that $F(x)$ is a minimal polynomial for A. Then we have by Theorem 7 that $\mathrm{Ann}(A) = (F(x))$, $\bar{F}(x)$ is the minimal polynomial of $\bar{A}$, and by (b) all elementary divisors of the matrix $x\bar{E} - \bar{A}$ equal $\bar{F}(x)$. If the decomposition of the module $M(\bar{A})$ into a direct sum of cyclic modules is of the form

$$M(\bar{A}) = (\bar{\alpha}_1) \dotplus \ldots \dotplus (\bar{\alpha}_k), \tag{25}$$

then if $\deg \bar{F}(x) = r$ we have $\dim_{\bar{R}}(\bar{\alpha}_i) = r$ and $\bar{\alpha}_i, \bar{A}\bar{\alpha}_i, \ldots, \bar{A}^{r-1}\bar{\alpha}_i$ is a basis of the vector space $(\bar{\alpha}_i)$ over $\bar{R}$ for $i = \overline{1, r}$. Let α_i be an inverse image of $\bar{\alpha}_i$ in $R^{(m)}$. The system of vectors $\alpha_i, A\alpha_i, \ldots, A^{r-1}\alpha_i$ is then clearly free over R, and since $F(A)\alpha_i = 0$ the vector $A^r\alpha_i$ is a linear combination of these vectors. Therefore, the cyclic submodule (α_i), generated by α_i in $M(A)$ is a free R-module of dimension r, $(\alpha_i) = R\alpha_i + RA\alpha_i + \ldots + RA^{r-1}\alpha_i$ and by (25) we have $M(A) = (\alpha_1) \dotplus \ldots \dotplus (\alpha_k)$ and $A \approx \mathrm{Diag}(S(F), \ldots, S(F))$. Since these considerations are obviously valid for any matrix $B \in R_m$ for which $\mathrm{Ann}(B) = (F(x))$ and $\bar{B} \approx \bar{A}$, we find $B \approx A$ and therefore A is polynomially determined.