5. Stabilization for $\mathbf{K}_1$

Our first goal is to prove Theorem 4.17. Under a stronger condition (which implies that $\mathrm{sr}(A) \le n$) the statement of the theorem is Bass's conjecture on p.514 of [B1]. The conjecture was partially proven in [B2] and [BMS]. A complete proof appeared in [V8] and [V9].

We assume that $n \ge \mathrm{sr}(A) + 1$ and construct a map

$$\delta : \mathrm{GL}_{n+1}B \to \mathrm{GL}_n A/\mathrm{E}_n(A, B)$$

which is trivial on $\mathrm{E}_{n+1}(A, B)$ and whose restriction on $\mathrm{GL}_n A$ is the canonical projection

$$\varepsilon : \mathrm{GL}_n B \to \mathrm{GL}_n B/\mathrm{E}_n(A, B).$$

This would imply that

(5.1) $$\mathrm{E}_{n+1}(A, B) \cap \mathrm{GL}_n A = \mathrm{E}_n(A, B).$$

Recall that $\mathrm{E}_n(A, B)$ is normal in $\mathrm{GL}_n A$ by Theorem 4.14(b). By Theorem 4.14(a), the inclusion $\mathrm{GL}_n B \subset \mathrm{GL}_{n+1}B$ gives a surjective homomorphism

(5.2) $$\mathrm{GL}_n B/\mathrm{E}_n(A, B) \to \mathrm{GL}_{n+1}B/\mathrm{E}_{n+1}(A, B).$$

Combining (5.1) and (5.2), we see that (5.2) is an isomorphism. As $n \to \infty$, we obtain the conclusion of Theorem 4.17.

A difficulty in constructing δ is that a matrix in $\mathrm{GL}_{n+1}B$ need not have any invertible entries (unless $B = A$ is local or $B \subset \mathrm{rad}(A)$). We define δ first on subsets X, X' of $\mathrm{GL}_{n+1}B$ and then on $X'X = \mathrm{GL}_{n+1}B$.

We start by defining δ on

$$X = \{(\alpha_{i,j}) \in \mathrm{GL}_{n+1}B : \alpha_{n+1,n+1} = 1\}$$

as follows:

$$\delta \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} = \varepsilon(a - bc) \in \mathrm{GL}_n B/\mathrm{E}_n(A, B).$$

Notice that $\mathrm{GL}_n B \subset X$ and that the restriction of δ on $\mathrm{GL}_n B$ is ε.

Next we define

$$\delta' : X' \to \mathrm{GL}_n B/\mathrm{E}_n(A, B)$$

on

$$X' = \{(\alpha_{i,j}) \in \mathrm{GL}_{n+1}B : \alpha_{1,1} = 1\}$$

by

$$\delta' \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = \varepsilon(d - cb) \in \mathrm{GL}_n B/\mathrm{E}_n(A, B).$$

Lemma 5.3. Let $\alpha \in X, \alpha' \in X', \beta \in \mathrm{GL}_n B, v$ an n-column over B, u an n-row over B. Then

$$\delta(\begin{pmatrix} \beta & v \\ 0 & 1 \end{pmatrix} \alpha) = \varepsilon(\beta)\delta(\alpha),$$

$$\delta(\alpha \begin{pmatrix} \beta & 0 \\ u & 1 \end{pmatrix}) = \delta(\alpha)\varepsilon(\beta),$$

$$\delta'(\begin{pmatrix} 1 & 0 \\ v & \beta \end{pmatrix} \alpha') = \varepsilon(\beta)\delta'(\alpha'),$$

$$\delta'(\alpha' \begin{pmatrix} 1 & u \\ 0 & \beta \end{pmatrix}) = \delta'(\alpha')\varepsilon(\beta).$$

Proof. This is an easy computation over an arbitrary ring A. Notice that the matrices on the left hand side belong to domains of δ, δ' respectively. QED.

Lemma 5.4. Let $\alpha = (\alpha_{i,j}) \in X, a \in B$ and $\alpha_{n,n+1} = 0$. Then

$$\delta(a^{n+1,n}\alpha(-a)^{n+1,n}) = \delta(\alpha).$$

Proof. Let $\alpha = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$ with $b_n = 0$ in $b = (b_i)$. Replacing α and $a^{n+1,n}\alpha(-a)^{n+1,n}$ by $\alpha \begin{pmatrix} 1_n & 0 \\ -c & 1 \end{pmatrix} (a - bc)^{-1}$ and $a^{n+1,n}\alpha \begin{pmatrix} 1_n & 0 \\ -c & 1 \end{pmatrix} (a - bc)^{-1}(-a)^{n+1,n}$ respectively which is OK by Lemma 5.3, we are reduced to the case when

$$\alpha = \begin{pmatrix} 1_n & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1_{n-1} & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \alpha' = \begin{pmatrix} 1_{n-1} & -ba & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this case, $\delta(\alpha) = \varepsilon(1_n) = 1$ while $\delta(a^{n+1,n}\alpha(-a)^{n+1,n}) = \varepsilon \begin{pmatrix} 1_{n-1} & -ba \\ 0 & 1 \end{pmatrix} = 1$. QED.

Corollary 5.5. $\delta(\alpha) = \delta'(\alpha)$ for all $\alpha \in X \cap X'$.

Proof. By addition operations which do not change $\delta(\alpha)$ or $\delta'(\alpha)$ (see Lemma 5.3) we can bring to the form

$$\alpha = \begin{pmatrix} 1 & 0 & z \\ v & d & 0 \\ z' & u & 1 \end{pmatrix}$$

where $z, z' \in B$, $v \in B^{n-1}$ is an $(n-1)$-column, u is an $(n-1)$-row over B, and $d \in 1_{n-1} + \mathrm{M}_n B$.

Since $\delta(\alpha) = \varepsilon \begin{pmatrix} 1 - zz' & -zu \\ v & d \end{pmatrix}$, it is clear that $\begin{pmatrix} 1 - zz' \\ v \end{pmatrix} \in \mathrm{Um}_n A$. On the other hand,

$$\delta'(\alpha) = \varepsilon \begin{pmatrix} d & -vz \\ u & 1 - z'z \end{pmatrix} = \varepsilon \begin{pmatrix} 1 - z'z & u \\ -vz & d \end{pmatrix}$$

since ε is invariant under conjugation by permutation matrices by Theorem 4.14(b).

By Lemma 4.13, there is an $(n-1)$-column $c \in A^{n-1}$ such that $v' = v + c(1 - zz') \in \mathrm{Un}_{n-1}A$. We find an $(n-1)$-row w over A such that $wv' = 1$.

Now

$$\begin{pmatrix} 1 & 0 \\ -c & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & zz'w \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 - zz' & -zu \\ v & d \end{pmatrix}$$
$$= \begin{pmatrix} 1 & zz'w(d - czu) - zu \\ -czz' & d - czz'w(d - czu) \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ cz & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & -z'w \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -cz & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 - z'z & u \\ -vz & d \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -z'w(d - czu)z + u \\ czz'z & d - czz'w(d - czu) \end{pmatrix}.$$

(When z is invertible the last two equality are obtained from each other by conjugation with a diagonal matrix.) We have

$$\varepsilon \begin{pmatrix} 1 & zz'w(d - czu) - zu \\ -czz' & d - czz'w(d - czu) \end{pmatrix} = \varepsilon \begin{pmatrix} 1 & -z'w(d - czu)z + u \\ czz'z & d - czz'w(d - czu) \end{pmatrix}$$

because both matrices can be brought to the same matrix $\begin{pmatrix} 1 & 0 \\ 0 & d' \end{pmatrix}$ with

$$d' = d - czz'w(d - czu) + czz'(zz'w(d - czu) - zu)$$

by two block-addition operations. Thus,

$$\delta(\alpha) = \varepsilon \begin{pmatrix} 1 - zz' & -zu \\ v & d \end{pmatrix} = \delta'(\alpha) = \varepsilon \begin{pmatrix} 1 - z'z & u \\ -vz & d \end{pmatrix}.$$

QED.

Corollary 5.6. Let $\alpha_1, \alpha_2 \in X, \alpha_3, \alpha_4 \in X'$, and $\alpha_3\alpha_1 = \alpha_4\alpha_2$. Then $\delta'(\alpha_3)\delta(\alpha_1) = \delta'(\alpha_4)\delta(\alpha_2)$.

Proof. Let $\alpha_3 = \begin{pmatrix} 1 & u \\ * & * \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} * & v \\ * & 1 \end{pmatrix}$. Replacing $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ by $\begin{pmatrix} 1 & u \\ 0 & 1_{n-1} \end{pmatrix} \alpha_1, \begin{pmatrix} 1_n & -v \\ 0 & 1 \end{pmatrix} \alpha_2, \alpha_3 \begin{pmatrix} 1 & -u \\ 0 & 1_n \end{pmatrix}, \alpha_4 \begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix}$ respectively, we can assume (see Lemma 5.3 with $\beta = 1_n$) that $\alpha_3 = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix}$.

Now $\alpha_3^{-1}\alpha_4 \in X'$ and $\delta'(\alpha_3^{-1}\alpha_4) = \delta'(\alpha_3)^{-1}\delta'(\alpha_4)$ by Lemma 5.3. Also $\alpha_1\alpha_2^{-1} \in X$ and $\delta(\alpha_1\alpha_2^{-1}) = \delta(\alpha_1)\delta'(\alpha_2)^{-1}$ by Lemma 5.3.

Since $\alpha_3^{-1}\alpha_4 = \alpha_1\alpha_2^{-1}$, we obtain our conclusion using Corollary 5.5. QED.

Thus, we have a well-defined map

$$\delta : X'X \to \mathrm{GL}_n B/\mathrm{E}_n(A,B)$$

given by $\delta(\alpha'\alpha) = \delta'(\alpha')\delta(\alpha)$ where $\alpha' \in X'$ and $\alpha \in X$. Clearly, this δ agrees with ε on $\mathrm{GL}_n B$. We want to prove that δ is defined on whole $\mathrm{GL}_{n+1} B$ and trivial on $\mathrm{E}_{n+1}(A,B)$.

Lemma 5.7. $X'X = \mathrm{GL}_{n+1}B$. Moreover every $\beta \in \mathrm{GL}_n B$ has the form $\beta = \alpha'\alpha$ with $\alpha' = (\alpha'_{i,j}) \in X', \alpha \in X$, and $\alpha'_{2,1} = 0$.

Proof. Consider the last column $\begin{pmatrix} b \\ v \end{pmatrix}$ of the matrix β where $b \in B$. Since $\mathrm{sr}(B) \le n-1$, by Proposition 4.1 there is an n-column $c = (c_i) \in B^n$ such that $v + cb \in \mathrm{Um}_n B$ and $c_1 = 0$. As in the proof of Theorem 4.4(a), $\mathrm{E}_n(A,B)$ acts transitively on $\mathrm{Um}_n B$. So the last entry of $\gamma(v+cb)$ is 1. for a matrix $\gamma \in \mathrm{E}_n(A,B)$.

Therefore $\alpha = \begin{pmatrix} 1 & 0 \\ \gamma & \gamma c \end{pmatrix} \beta \in X$ and $\alpha' = \begin{pmatrix} 1 & 0 \\ \gamma & \gamma c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \gamma^{-1} & -c \end{pmatrix} \in X'$. QED.

Lemma 5.8. Every $\beta \in \mathrm{GL}_n B$ has the form $\beta = \alpha'\alpha$ with $\alpha' =\in X', \alpha = (\alpha_{i,j}) \in X$, and $\alpha_{n+1,n} = 0$.

Proof. We proceed as in the previous proof, but use that the stable rank of the opposite ring B_0 is $\le n-1$ and the opposite analog of Proposition 4.1. So we consider the first row (u,b) of β and find a row $a = (a_i)$ with n entries in B such that $u + ba$ is unimodular (i.e.,$(u+ba)A^n = A$) and $a_n = 0$. The last row of α is going to be $(a,1)$. Then we find $\gamma \in \mathrm{E}_n(A,B)$ such that $(u+ba)\gamma = (1,0,\ldots,0)$. Then we set $\alpha = \begin{pmatrix} 1/\gamma & 0 \\ a & 1 \end{pmatrix}$. so $\alpha' = \beta\alpha^{-1} \in X'$.

Corollary 5.9. δ is invariant under conjugation by $\mathrm{E}_n A$.

Proof. Since $\mathrm{E}_n A$ is generated by elementary matrices $g = a^{i,j}$ with $a \in A$ and $|i-j| = 1$, it suffices to show that δ is invariant under conjugation by such g.

Case 1: $2 \le i,j \le n$ (this case is possible only if $n \ge 3$. Then g normalizes both X and X', and the conjugation by g does not change δ on X and δ' on X'.

Case 2: $(i,j) = (1,2)$. We write an arbitrary $\beta \in \mathrm{GL}_{n+1}B$ as $\beta = \alpha'\alpha$ with $\alpha' = (\alpha'_{i,j}) \in X', \alpha \in X$, and $\alpha'_{2,1} = 0$. Clearly, $g\alpha g^{-1} \in X$ and $\delta(g\alpha g^{-1}) = \delta(\alpha)$.

Since $\alpha'_{2,1} = 0$, $g\alpha' g^{-1} \in X'$ and $\delta(g\alpha' g^{-1}) = \delta'(g\alpha' g^{-1}) = \delta(\alpha)$ (cf. Lemma 5.4).

Thus, $\delta g\beta g^{-1}) = \delta(\beta)$.

Case 3: $(i,j) = (n, n+1)$. This case is similar to Case 2 but we use Lemma 5.8 instead of Lemma 5.7.

Case 4: $(i,j) = (1,2)$. We write an arbitrary $\beta \in \mathrm{GL}_{n+1}B$ as $\beta = \alpha'\alpha$ with $\alpha' = \begin{pmatrix} 1 & u \\ * & * \end{pmatrix} \in X', \alpha \in X$. Next we change α', α by

$$\alpha' \begin{pmatrix} 1 & -u \\ 0 & 1_n \end{pmatrix}, \begin{pmatrix} 1 & -u \\ 0 & 1_n \end{pmatrix} \alpha$$

without changing β. Now the conclusion is obvious.

Case 5: $(i,j) = (n+1,n)$. We write an arbitrary $\beta \in \mathrm{GL}_{n+1}B$ as $\beta = \alpha'\alpha$ with $\alpha' \in X', \alpha = \begin{pmatrix} * & v \\ * & 1 \end{pmatrix} \in X$. Next we change α', α by

$$\alpha' \begin{pmatrix} 1_n & v \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1_n & -v \\ 0 & 1 \end{pmatrix} \alpha$$

without changing β. Now the conclusion is obvious.

Thus, the corollary is proven. QED.

Now we can conclude our proof of Theorem 4.7. We used the map

$$\delta : \mathrm{GL}_{n+1}B = X'X \to \mathrm{GL}_n B/\mathrm{E}_n(A,B)$$

constructed above.

Let Y be the set of all $g \in \mathrm{GL}_{n+1}B$ such that $\delta(g\beta) = \delta(\beta)$ for all $\beta \in \mathrm{GL}_{n+1}B$ Clearly, Y is a subgroup of $\mathrm{GL}_{n+1}B$ By Lemma 5.3, $A^{2,1} \subset Y$. By Corollary 5.9, Y is normalized by $\mathrm{E}_n A$.

Therefore, Y contains $\mathrm{E}_{n+1}(A,B)$. So

$$\delta(\mathrm{E}_{n+1}(A,B)) = \delta(Y) = \delta(1_{n+1})$$

is trivial, hence we have proven (5.1).

For some rings A we may have

$$\mathrm{GL}_n A/\mathrm{E}_n(A,B) = \mathrm{K}_1(A,B)$$

with $n = \mathrm{sr}(B)$ or even smaller n. For example, this is the case when A is the ring of integers in a number field with infinite $\mathrm{GL}_1 A$, the ideal B is nonzero, and $n = 2 = \mathrm{sr}(A) = \mathrm{sr}(B)$ (Vaserstein [V19]).

Another example is when $A = B$ is the ring continuous functions $\mathbf{R}^d \to \mathbf{R}$ (so $\mathrm{sr}(A) = d+1$) and $n \geq 3$ (Thurston-Vaserstein [TV]). The following theorem gives more examples.

Theorem 5.10 (Suslin [Su1]). Let C be a field or the integers. Then

$$\mathrm{SL}_n(C[x_1,\ldots,x_k]) = \mathrm{E}_n(C[x_1,\ldots,x_k])$$

for all $n \geq 3$ and all k.

Another possible example of

$$\mathrm{GL}_n A/\mathrm{E}_n A = \mathrm{K}_1 A$$

with $n = \mathrm{sr}(A)$ is when A is a finitely generated regular algebra over a perfect C_1-field in which case it is known (R. A. Rao, van der Kallen [RK]) that $\mathrm{GL}_n A/\mathrm{E}_n A = \mathrm{K}_1 A$ for all $n \geq \dim(A)$ (the Krull dimension). However it is not known whether $\mathrm{sr}(A) = \dim(A)+1$ or $\mathrm{sr}(A) \leq \dim(A)$ except that the latter holds when the field is finite and $\dim(A) \geq 2$ (Vaserstein).

Bounded reduction

From the definition of the stable rank, it is easy to see that if $\mathrm{sr}(A) = m < \infty$, then every matrix in $\mathrm{GL}_{n+1}A$ with $n \geq m$ can be reduced to a matrix in $\mathrm{GL}_n A$ by $m+3n$ row addition operations. Now we will see that a bounded reduction for a fixed n has interesting consequences without any other assumptions on A.

First we observe that bounded reduction by row and column addition operations is equivalent to bounded reduction by row addition operations:

Proposition 5.11. Let A be an associative ring with 1 and $m \geq 1$. Assume that every matrix in $\mathrm{E}_{n+1}A$ can be reduced to $\mathrm{GL}_n A$ (resp., to $\mathrm{E}_n A$) by a bounded number of row and column addition operations. Then every matrix in $\mathrm{E}_{n+1}A$ can be reduced to $\mathrm{GL}_n A$ (resp., to $\mathrm{E}_n A$) by a bounded number of row addition operations.

Proof. Let $U(s)$ (resp., $L(s)$) be the subgroup of $\mathrm{E}_s A$ consiting of the matrices α of the form $\alpha = \begin{pmatrix} 1_{s-1} & * \\ 0 & 1 \end{pmatrix}$ (resp., $\alpha = \begin{pmatrix} 1_{s-1} & 0 \\ * & 1 \end{pmatrix}$).

Every elementary matrix in $\mathrm{E}_s A$ is a product of 4 matrices from $U(s) \cup L(s)$, so the condition of the theorem is equivalent to the following: every matrix in $\mathrm{E}_{n+1}A$ can be reduced to $\mathrm{GL}_n A$ (resp., to $\mathrm{E}_n A$) by a bounded number left and right multiplications by matrices from $U(n+1) \cup L(n+1)$. But $\mathrm{GL}_{n+1}A$ normalizes both $U(n+1)$ and $L(n+1)$, hence every matrix in $\mathrm{E}_{n+1}A$ can be reduced to $\mathrm{GL}_n A$ (resp., to $\mathrm{E}_n A$) by a bounded number of left multiplications by matrices from $U(n+1) \cup L(n+1)$.

Every matrix in $U(n+1) \cup L(n+1)$ is a product of n elementary matrices, hence every matrix $\mathrm{E}_{n+1}A$ can be reduced to $\mathrm{GL}_n A$ by a bounded number of row addition operations. QED.

Next we prove that bounded reduction of every matrix in $\mathrm{E}_{m+1}A$ to $\mathrm{GL}_m A$ implies bounded reduction of every matrix in $\mathrm{E}_n A$ to $\mathrm{GL}_m A$ for every $n > m$. To get a quantative form of this result, we use triangular matrices instead of elementary matrices. Every triangular matrix in $\mathrm{E}_n A$ we use is a product of $n(n-1)/2$ elementary matrices.

Let $U_s = U(2) \cdots U(s)$ (resp., $L_s = L(2) \cdots L(s)$) be the group of upper (resp., lower) diagonal matrices in $\mathrm{E}_s A$ with ones on the main diagonal. Let $T(s,t)$ be $(U_s L_s)^{t/2}$ when t is even and $T(s,t)$ be $(U_s L_s)^{(t-1)/2} U_s$ when t is odd. In other words, $T(s,t)$ consists the products $\alpha_1 \cdots \alpha_t$ of t matrices with $\alpha_{2i-1} \in U_s$ and $\alpha_{2i} \in L_s$.

Theorem 5.12. Let A be an associative ring with 1 and $m \geq 1$. Let H be a subgroup of $\mathrm{GL}_m A$. Assume that $\mathrm{E}_{m+1}A = T(m+1,t) \begin{pmatrix} H & 0 \\ 0 & 1 \end{pmatrix} = T(m+1,t) \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix}$ for some $t \geq 1$. Then for any $k, l \geq 0$ and $n = k+m+l > m$ we have

$$\mathrm{E}_n A = T(n,t) \begin{pmatrix} 1_k & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1_l \end{pmatrix}$$

Proof. We proceed by induction on n. When $n = m+1$ the conclusion is the condition, so we assume that $n \geq m+2$.

First we prove that

$$T(n,t) \begin{pmatrix} 1_k & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1_l \end{pmatrix} = T(n,t) \begin{pmatrix} H & 0 \\ 0 & 1_{n-m} \end{pmatrix} \tag{5.13}$$

is independent on k for $0 \leq k \leq n-m$ where $l = n-m-k$. We use that both $U(n)$ and $L(n)$ normalize both U_{n+1} and L_{n+1}.

When $l \geq 1$, we write

$$T(n,t) = \begin{cases} (U(n)L(n))^{t/2}T(n-1,t) & \text{if } t \text{ is even;} \\ (U(n)L(n))^{(t-1)/2}U(n)T(n-1,t) & \text{if } t \text{ is odd.} \end{cases}$$

By the induction hypothesis,

$$T(n-1,t)\begin{pmatrix} 1_k & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1_{l-1} \end{pmatrix} = T(n-1,t)\begin{pmatrix} H & 0 \\ 0 & 1_{n-1-m} \end{pmatrix}.$$

So $T(n,t)\begin{pmatrix} 1_k & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1_l \end{pmatrix} = T(n,t)\begin{pmatrix} H & 0 \\ 0 & 1_{n-m} \end{pmatrix}$.

It remains to consider the case when $l = 0$ and prove that $T(n,t)\begin{pmatrix} 1_{n-m} & 0 \\ 0 & H \end{pmatrix} = T(n,t)\begin{pmatrix} 1_{n-m-1} & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{pmatrix}$. But the case $k > 0$ is similar to the case $l > 0$. Namely, we write

$$T(n,t) = \begin{cases} (U'(n)L'(n))^{t/2}\begin{pmatrix} 1 & 0 \\ 0 & T(n-1,t) \end{pmatrix} & \text{if } t \text{ is even,} \\ (U'(n)L'(n))^{(t-1)/2}U'(n)\begin{pmatrix} 1 & 0 \\ 0 & T(n-1,t) \end{pmatrix} & \text{if } t \text{ is odd,} \end{cases}$$

where $U'(n) = \begin{pmatrix} 1 & * \\ 0 & 1_{n-1} \end{pmatrix}$ and $T'(n) = \begin{pmatrix} 1 & 0 \\ * & 1_{n-1} \end{pmatrix}$, etc.

When $\mathrm{E}_n A \subset H$ it is clear now that the set (5.13) is invariant under right multiplications by elementary matrices so it contains $\mathrm{E}_n A$ and we are done. Otherwise, having the equality (5.13) proven, we are only half way through our proof. Note that in the case of trivial H. our theorem is Lemma 7 of [V79] and the rest of our proof follows the proof of the lemma (due to K.Dennis).

It suffices to show that the set (5.13) is invariant under left multiplication by elementary matrices $a^{i,j}$ with $a \in A$ and $|i-j| = 1$ because such matrices generate the group $\mathrm{E}_n A$.

Case 1: $i < j,$. Then $a^{i,j} \in U_n$ so we can just include $a^{i,j}$ into the first matrix in $T(n,t) = U_n L_n \cdots$.

So we assume now that $i = j+1$.

Case 2: $i < n$. Then we rewrite the set (5.13) as

$$\begin{pmatrix} T(n-1,t)H & 0 \\ 0 & 1 \end{pmatrix} U(n)L(n)U(n)\ldots.$$

By the induction hypothesis, (5.13) is invariant under left multiplication by $a^{i,i-1}$.

Case 3: $i = n$. Then we rewrite the set (5.13) as

$$\begin{pmatrix} 1 & 0 \\ 0 & T(n-1,t)H \end{pmatrix} U'(n)L'(n)U'(n)\ldots.$$

By the induction hypothesis, (5.13) is invariant under left multiplication by $a^{n,n-1}$. QED.

Putting together Proposition 5.11 and Theorem 5.12, we obtain

Corollary 5.14. Assume that for some $m \geq 1$ every matrix in $\mathrm{E}_{m+1}A$ can be reduced to $\mathrm{GL}_m A$ by a bounded number of row and column addition operations. Then for any $n > m$ every matrix $\mathrm{E}_n A$ can be reduced to $\mathrm{GL}_m A$ by a bounded number of row addition operations.

Remark 5.15. When $\mathrm{sr}(A) = m < \infty$, the condition of Theorem 5.12 holds with $t = 4$.

Using the same trick as in the proof of Theorem 5.12, we obtain

Theorem 5.16. Let A be an associative ring with 1 and $m \geq 1$. Let D_s be the subgroup of diagonal matrices in $\mathrm{E}_s A$. Assume that $\mathrm{E}_m A = T(m,t)D_m$ for some $t \geq 1$. Then $\mathrm{E}_n A = T(n,t)D_n$ for every $n > m$.

Remark 5.17. When $\mathrm{st}(A) = 1$, the condition of Theorem 5.16 holds with $m = 2, t = 3$. Basicaly. it means that for every $n \geq 2$, every matrix in $\mathrm{GL}_n A$ is product $\lambda\rho\mu$ with $\lambda, \mu \in U_n$ and $\rho \in D_n L_n$ (a low triangular matrix). This is Theorem 1 of [V100].

Problems.

1. Let A be commutative, and assume that $\mathrm{sr}(A) < \infty$. Assume also that the group $\mathrm{SK}_1(A, B)$ is divisible for every ideal B of finite index in A. Prove that for $n \geq \mathrm{sr}(A) + 1$, every subgroup of finite index in $\mathrm{SL}_n A$ contains $\mathrm{SL}_n B$ for an ideal B of finite index in A.

2. For any Dedekind ring A of arithmetic type prove that every subgroup of finite index in $\mathrm{GL}_1 A$ contains $\mathrm{GL}_1 B$ with an ideal $B \neq 0$. (Hint: the group $\mathrm{GL}_1 A$ is finitely generated.)

3. (A converse to Problem 2) Let A be commutative, and assume that for all sufficiently large n every subgroup of finite index in $\mathrm{SL}_n A$ contains $\mathrm{SL}_n B$ for an ideal B of finite index in A. Show that all groups $\mathrm{SK}_1(A, B)$ with ideals B of finite index are divisible.

4. For an arbitrary group, prove that each subgroup of finite index contains a normal subgroup of finite index.

5. Let A be the algebraic integers. Show that the groups $\mathrm{SL}_n A$ and $\mathrm{GL}_n A$ have no proper subgroups of finite index.

6. Let A be the ring of all algebraic integers. Show that every number in A is a sum of two units. Moreover, for any integer $m \geq 2$, every number in A is a sum of m units.

7. In our proof of Theorem 4.7 we used the representation $\mathrm{GL}_{n+1} A = X'X$ which was used first by H. Bass who got a weaker version of the theorem. Show that $\mathrm{GL}_{n+1} A = X'X$ (or $\mathrm{E}_{n+1} A \subset X'X$) if and only if $\mathrm{sr}(A) \leq n$ and the group $\mathrm{GL}_n A$ acts transitively on $\mathrm{Um}_n A$.

8. In our proof of Theorem 4.7 we used also the representation $\mathrm{GL}_{n+1} A = X'_0 X$ where $X'_0 = \{(\alpha_{i,j}) \in X' : \alpha_{n+1,1} = 0\}$. Prove that $\mathrm{GL}_{n+1} A = X'_0 X$ (or $\mathrm{E}_{n+1} A \subset X'_0 X$) if and only if $\mathrm{sr}(A) \leq n - 1$.

9. Let A be an associative ring with 1, B an ideal of A such that $BB = 0$, and $n \geq 2$. Prove that $\mathrm{GL}_n B / \mathrm{E}_n(A, B)$ is the additive group of B modulo its subgroup generated by $ab - ba$ with $a \in A, b \in B$.

10. Let B be a ring with zero multiplication, i.e., $BB = 0$ (so the additive group of B is an arbitrary commutative group). Prove that $\mathrm{K}_1 B = B$ (more precisely, the additive group of B) but there is an associative ring A with 1 containing B as an ideal and such that $\mathrm{K}_1(A, B) = 0$.

11. (Open problem) Is the group $\mathrm{K}_1 A$ finitely generated for every finitely generated ring A?

12. (Open problem) Is every matrix in $\mathrm{E}_3 \mathbf{Z}[x]$ a product of a bounded number of elementary matrices? It is known that $\mathrm{E}_3 \mathbf{Z}[x] = \mathrm{SL}_3 \mathbf{Z}[x]$.

13. Let $A = \mathbf{R}^{\mathbf{R}}$ be the ring of real-valued continued functions on line. Show that for any $n \geq 1$ there is a matrix in $\mathrm{E}_2 A$ which is not a product of n elementary matrices.

14. Let A be an associative ring with 1 and $m = \mathrm{sr}(A) < \infty$. Show that for $n \geq m+1$, every commutator in $\mathrm{GL}_n A$ is a product of a bounded number of elementary matrices.

15. Let A be an associative ring with 1 and $n \geq 3$. Show that every upper triangular matrix in $\mathrm{E}_n A$ is a product of two commutators in $\mathrm{E}_n A$