DEDEKIND'S ETA-FUNCTION AND ITS TRUNCATED PRODUCTS

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February 17, 2000.

1. Introduction and Statement of Results

Dedekind's eta function $\eta(z)$, defined by the infinite product

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{1}$$

where $q := e^{2\pi i z}$, is one of most fundamental modular forms. For example, Ramanujan's $\Delta(z)$ function, the unique normalized weight 12 cusp form on $SL_2(\mathbb{Z})$, is given by

$$\Delta(z) = \eta^{24}(z).$$

However, more is true. For instance, every modular form on $SL_2(\mathbb{Z})$ is a rational function in the eta-function since the classical Eisenstein series $E_4(z)$ and $E_6(z)$ satisfy

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n = \frac{\eta^{16}(z)}{\eta^8(2z)} + 256 \frac{\eta^{16}(2z)}{\eta^8(z)},$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sum_{d|n} d^5 q^n = \frac{\eta^{24}(z)}{\eta^{12}(2z)} - 480\eta^{12}(2z) - 16896 \frac{\eta^{12}(2z)\eta^8(4z)}{\eta^8(z)} + 8192 \frac{\eta^{24}(4z)}{\eta^{12}(2z)}.$$

Therefore, every modular form on $SL_2(\mathbb{Z})$ is essentially an algebraic expression in infinite products.

1991 *Mathematics Subject Classification.* Primary 11F20; Secondary 11P81.

Both authors thank the National Science Foundation for its generous research support. The second author thanks the Alfred P. Sloan Foundation and the David and Lucile Packard Foundation for their support.

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In a different direction, Borcherds [B] has shown that many suitable modular forms have Fourier expansions which are infinite products whose exponents are Fourier coefficients of certain corresponding modular forms. A special case of his work implies that

$$E_6(z) = 1 - 504q - 16632q^2 - \cdots = (1-q)^{504}(1-q^2)^{143388} \cdots = \prod_{n=1}^{\infty}(1-q^n)^{c(n^2)}$$

where the exponents $c(n^2)$ are coefficients of a specific modular form $f(z) = \sum c(n)q^n$.

In addition to their role in the theory of modular forms, such infinite products play a crucial role in q-series and the theory of partitions. Therefore, it is natural to investigate the modularity and the combinatorial properties of truncated products of such series. (Note. Empty products shall equal 1 throughout.) With this as motivation, we make a first step by examining the q-series obtained by summing the differences between $\eta(24z)$ and $1/\eta(24z)$ and all of their truncated products. Specifically, we study the two q-series

$$\sum_{n=0}^{\infty}\left(\frac{1}{\eta(24z)} - \frac{1}{q(1-q^{24})(1-q^{48})\cdots(1-q^{24n})}\right) = q^{23} + 3q^{47} + 6q^{71} + \cdots \tag{2}$$

$$\sum_{n=0}^{\infty}\left(\eta(24z) - q(1-q^{24})(1-q^{48})\cdots(1-q^{24n})\right) = -q^{25} - 2q^{49} - q^{73} - \cdots \tag{3}$$

(Note: A similar construction was studied by the first author in [A2] in connection with an identity appearing in Ramanujan's "Lost Notebook" involving a Mock theta type function.)

At first glance, there is no reason to suspect that the q-series in (2) and (3) are easily described in terms of modular forms. However, we show that these series are indeed modular forms up to a simple q-series. In addition, we give two interesting applications of these results to partitions.

Recall that a *partition* of a non-negative integer N is any nonincreasing sequence of positive integers whose sum is N. If $p(N)$ denotes the number of partitions of N, then we have

$$\frac{1}{\eta(24z)} = \sum_{N=0}^{\infty} p(N)q^{24N-1} = q^{-1}\prod_{n=1}^{\infty}\frac{1}{1-q^{24n}} = q^{-1} + q^{23} + 2q^{47} + 3q^{71} + 5q^{95} + \cdots . \tag{4}$$

Similarly, let $d_e(N)$ (resp. $d_o(N)$) denote the number of partitions of N into an even (resp. odd) number of distinct parts. Euler's Pentagonal Number Theorem asserts that the Fourier expansion of $\eta(24z) \in S_{1/2}(\Gamma_0(576), \chi)$ is

$$\eta(24z) = \sum_{N=0}^{\infty}\left(d_e(N) - d_o(N)\right)q^{24N+1} = q - q^{25} - q^{49} + \cdots = \sum_{n=1}^{\infty}\chi(n)q^{n^2} \tag{5}$$

where

$$\chi(n) := \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

To state our results, let $D(q)$ be the q-series defined by

$$D(q) := -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = -\frac{1}{2} + \sum_{n=1}^{\infty} d(n) q^n = -\frac{1}{2} + q + 2q^2 + 2q^3 + 3q^4 + \cdots, \tag{7}$$

where $d(n)$ denotes the number of positive divisors of a positive integer n.

Theorem 1. *The following q-series identity is true:*

$$\frac{1}{\eta(24z)} = 2\sum_{n=0}^{\infty}\left(\frac{1}{\eta(24z)} - \frac{1}{q(1-q^{24})(1-q^{48})\cdots(1-q^{24n})}\right) - 2D(q)\cdot\frac{1}{\eta(24z)}.$$

As a corollary, we obtain the following partition theoretic result.

Corollary 2. *If n is a positive integer, then let $a_n(N)$ denote the number of partitions of N into parts not exceeding n. For every positive integer N we have*

$$(N+1)p(N) = \sum_{n=1}^{N} p(N-n)d(n) + \sum_{n=1}^{N} a_n(N).$$

Corollary 2 is equivalent to the observation used by Erdös [E] in the beginning of his study of the asymptotics of $p(N)$ by elementary means.

Now we consider the q-series defined in (3).

Theorem 3. *If $R(z) \in S_{3/2}(\Gamma_0(576), \chi)$ is the cusp form with Fourier expansion*

$$R(z) := \sum_{n=1}^{\infty} \chi(n) n q^{n^2} = q - 5q^{25} - 7q^{49} + 11q^{121} + \cdots,$$

then

$$R(z) = 2\sum_{n=0}^{\infty}\big(\eta(24z) - q(1-q^{24})(1-q^{48})\cdots(1-q^{24n})\big) - 2D(q)\eta(24z).$$

As with Theorem 1, Theorem 3 has an interesting partition theoretic consequence which is similar in flavor to Euler's Pentagonal Number Theorem (5). We require some notation.

If π is a partition, then let m_π denote its largest part. Moreover, if N is an integer, then let $S_e(N)$ (resp. $S_o(N)$) denote the set of partitions of N into an even (resp. odd) number of distinct parts. Using this notation, define the two partition functions $A_e(N)$ and $A_o(N)$ by

$$A_e(N) := \sum_{\pi \in S_e(N)} m_\pi, \tag{8}$$

$$A_o(N) := \sum_{\pi \in S_o(N)} m_\pi \tag{9}$$

Corollary 4. *If N is a positive integer, then*

$$A_e(N) - A_o(N) =$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k d\left(N - (3k^2+k)/2\right) + \begin{cases} (-1)^k(3k) & \text{if } N = \frac{3k^2+k}{2} \text{ with } k \geq 0, \\ (-1)^k(3k-1) & \text{if } N = \frac{3k^2-k}{2} \text{ with } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In §2 we prove Theorem 1 and Corollary 2, and in §3 we prove Theorem 3 and Corollary 4. We conclude by raising two natural questions which arise from these results.

Question 1. Under what conditions does a modular form $f(z)$ given by an infinite product have the property that the sum of all the differences between $f(z)$ and its truncated products is a modular form modulo a function such as $2D(q)f(z)$?

Question 2. Is there a direct combinatorial proof of Corollary 4?

2. The proof of Theorem 1 and Corollary 2

We begin with an equivalent form of the conjectured identity which is simple to obtain.

Proposition 2.1. *Let $P(q)$ and $\mathcal{D}(q)$ be the q-series defined by*

$$P(q) := \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots,$$

$$\mathcal{D}(q) := \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} d(n)q^n = q + 2q^2 + 2q^3 + 3q^4 + \cdots.$$

The identity in Theorem 1 is true if and only if

$$\sum_{n=0}^{\infty} \left(P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \right) = P(q)\mathcal{D}(q).$$

Proof. This claim follows easily from Theorem 1 by first multiplying both sides of the conjectured identity in Theorem 1 by q, replacing q by $q^{1/24}$ and then by using the fact that

$$D(q) = -\frac{1}{2} + \mathcal{D}(q).$$

Q.E.D.

The next Lemma provides a convenient form of the left hand side of the conjectured identity in Proposition 2.1.

Lemma 2.2. *The following identity is true*

$$\sum_{n=0}^{\infty}\left(P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}\right) = \sum_{n=1}^{\infty}\frac{nq^n}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Proof. It is easy to see that

$$P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

is the generating function for the number of partitions of N with at least one part exceeding n. Therefore, each partition π of an positive integer N is counted with multiplicity m_π in the telescoping sum. On the other hand, the generating function for the number of partitions of π of N with largest part $m_\pi = n$ counted with multiplicity n is easily seen to be

$$\frac{nq^n}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

The proof follows immediately.

Q.E.D.

In view of Proposition 2.1 and Lemma 2.2, to prove Theorem 1 it suffices to prove that

$$\sum_{n=1}^{\infty}\frac{nq^n}{(1-q)(1-q^2)\cdots(1-q^n)} = P(q)\mathcal{D}(q). \tag{10}$$

For completeness we include a proof of (10). However, Uchimura [U] originally proved (10) and Dilcher [D] generalized it. To prove this identity, we require the following identity.

Proposition 2.3. *For $|z|, |q| < 1$ we have*

$$F(z) := 1 + \sum_{n=1}^{\infty} \frac{z^n q^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-zq^n)}.$$

Proof. This is a classical result of Euler. In particular, it follows by letting $t = zq$ in [Cor. 2.2, A1].

Q.E.D.

Proof of Theorem 1. Since the q-series $F'(z)$ is

$$F'(z) = \sum_{n=1}^{\infty} \frac{nz^{n-1} q^n}{(1-q)(1-q^2)\cdots(1-q^n)},$$

it follows that

$$F'(1) = \sum_{n=1}^{\infty} \frac{nq^n}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Therefore, in view of (10) it suffices to prove that

$$F'(1) = P(q)\mathcal{D}(q). \tag{11}$$

However, we may apply the product rule to $F(z)$ using its infinite product form from Proposition 2.3, and we find that

$$\begin{aligned} F'(z) &= \sum_{n=1}^{\infty} \frac{q^n}{(1-zq^n)\prod_{m=1}^{\infty}(1-zq^m)} \\ &= \frac{1}{\prod_{m=1}^{\infty}(1-zq^m)} \cdot \sum_{n=1}^{\infty} \frac{q^n}{1-zq^n}. \end{aligned}$$

Therefore, we have that

$$F'(1) = P(q)\mathcal{D}(q).$$

This is (11), and so the proof is complete.

Q.E.D.

Proof of Corollary 2. Here we use the identity of Theorem 1 in the form

$$\sum_{n=0}^{\infty} \left(P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \right) = P(q)\mathcal{D}(q). \tag{12}$$

If m is a non-negative integer, then the series

$$P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = q^{m+1} + \sum_{n=m+2}^{\infty} b_m(n)q^n \tag{13}$$

for some sequence of integers $b_m(n)$.

Now let N be a positive integer, then it is easy to see that the coefficient of N on the right hand side of (12) is

$$\sum_{n=1}^{N} p(N-n)d(n).$$

Therefore, to prove the corollary it suffices to show that the coefficient of q^N on the left hand side of of (12) is

$$(N+1)p(N) - \sum_{n=1}^{N} a_n(N). \tag{14}$$

By letting $m = N$ as in (13), it is easy to see that the coefficient of q^N on the left hand side of (12) is the coefficient of q^N in

$$\sum_{n=0}^{N} \left(P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \right).$$

Claim (14) follows from the obvious fact that

$$\sum_{N=0}^{\infty} a_n(N)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

This completes the proof.

Q.E.D.

3. The proof of Theorem 3 and Corollary 4

In this section we prove Theorem 3 and Corollary 4. The proof of Theorem 3 is more involved than the proof of Theorem 1. We begin with a reformulation of Theorem 3.

Proposition 3.1. *Let $E(q)$ be the q-series defined by*

$$E(q) := \prod_{n=1}^{\infty}(1-q^n) = 1 - q - q^2 + q^5 + \cdots .$$

The identity in Theorem 1 is true if and only if

$$\sum_{n=0}^{\infty}\big(E(q) - (1-q)(1-q^2)\cdots(1-q^n)\big) =$$
$$= E(q)\mathcal{D}(q) + \sum_{k=0}^{\infty}(-1)^k(3k)q^{(3k^2+k)/2} + \sum_{k=1}^{\infty}(-1)^k(3k-1)q^{(3k^2-k)/2}.$$

Proof. Begin by noticing the following identities:

$$\mathcal{D}(q) = -\frac{1}{2} + D(q),$$

and

$$\sum_{n=1}^{\infty}\chi(n)nq^{n^2} = q\sum_{k=1}^{\infty}(-1)^k(6k-1)q^{24k(3k-1)/2} + q\sum_{k=0}^{\infty}(-1)^k(6k+1)q^{24k(3k+1)/2}. \tag{15}$$

With these identities, one obtains the reformulation by dividing both sides of the identity in Theorem 3 by q and then replacing q by $q^{1/24}$.

Q.E.D.

As in §2, we reformulate the q-series appearing in the left hand side of the alleged identity in Proposition 3.1.

Lemma 3.2. *The following identity is true:*

$$\sum_{n=0}^{\infty}\big(E(q) - (1-q)(1-q^2)\cdots(1-q^n)\big) = -\sum_{n=1}^{\infty} nq^n(1-q)(1-q^2)\cdots(1-q^{n-1}).$$

Proof. As in (5), it is easy to see that

$$E(q) = \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=0}^{\infty}\left(d_e(n) - d_o(n)\right)q^n.$$

Since $E(q)$ is the generating function for the number of partitions into an even number of distinct parts minus the number into an odd number of distinct parts, it is easy to see that

$$E(q) - (1-q)(1-q^2)\cdots(1-q^n)$$

is the generating function whose coefficient of q^N is the number of partitions of N into an even number of distinct parts where at least one part exceeds n minus the number of partitions of N into an odd number of distinct parts where at least one part exceeds n. Consequently, each partition π into distinct parts is counted with multiplicity $(-1)^{n_\pi} m_\pi$ where n_π is the number of parts in π. This implies the claimed identity.

Q.E.D.

By Lemma 3.2, Theorem 3 is equivalent to the truth of the following identity:

$$\begin{aligned} -\sum_{n=1}^{\infty} nq^n(1-q)(1-q^2)\cdots(1-q^{n-1}) &= \\ &= E(q)\mathcal{D}(q) + \sum_{k=0}^{\infty}(-1)^k(3k)q^{(3k^2+k)/2} + \sum_{k=1}^{\infty}(-1)^k(3k-1)q^{(3k^2-k)/2}. \end{aligned} \tag{16}$$

The following Proposition is simple.

Proposition 3.3. *If $G(z)$ is the q-series defined by*

$$G(z) = 1 + \sum_{n=1}^{\infty}(1-q)(1-q^2)\cdots(1-q^n)z^n q^n,$$

then Theorem 3 is equivalent to the truth of the following identity

$$\begin{aligned} -qG(1) - qG'(1) &= \\ &= E(q)\mathcal{D}(q) + \sum_{k=0}^{\infty}(-1)^k(3k)q^{(3k^2+k)/2} + \sum_{k=1}^{\infty}(-1)^k(3k-1)q^{(3k^2-k)/2}. \end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned} &\sum_{n=1}^{\infty} nq^n(1-q)(1-q^2)\cdots(1-q^{n-1}) = \\ &= q\sum_{n=1}^{\infty} q^{n-1}(1-q)(1-q^2)\cdots(1-q^{n-1}) + q\sum_{n=1}^{\infty}(n-1)q^{n-1}(1-q)(1-q^2)\cdots(1-q^{n-1}) \\ &= q + q\sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^n) + q\sum_{n=1}^{\infty} nq^n(1-q)(1-q^2)\cdots(1-q^n) \\ &= qG(1) + qG'(1). \end{aligned}$$

The claim follows from (16).

Q.E.D.

Now we obtain a reformulation of the series $G(z)$.

Lemma 3.4. *The series $G(z)$ satisfies the following identity*

$$G(z) = E(q) \cdot \left(\frac{1}{1-zq} + \sum_{n=1}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-zq^{n+1})} \right).$$

Proof. This identity is a corollary of Heine's first transformation [Cor. 2.3, A1]. In particular, let $a = q, c = 0, b = q$ and $t = zq$ in the transformation.

Q.E.D.

Proposition 3.5. *The series $G(1)$ is given by*

$$G(1) = q^{-1} - q^{-1}E(q).$$

Proof. By letting $z = 1$ in Lemma 3.4, we find that

$$\begin{aligned} G(1) &= E(q) \left(\frac{1}{1-q} + \sum_{n=1}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^{n+1})} \right) \\ &= E(q) \left(\frac{1}{1-q} + q^{-1} \sum_{n=1}^{\infty} \frac{q^{n+1}}{(1-q)(1-q^2)\cdots(1-q^{n+1})} \right) \\ &= E(q) \left(\frac{1}{1-q} - \frac{1}{1-q} + q^{-1} \sum_{n=1}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} \right) \\ &= q^{-1}E(q) \left(\sum_{n=1}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} \right) \end{aligned} \tag{17}$$

By Proposition 2.3, we see that

$$1 + \sum_{n=1}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} = E^{-1}(q).$$

Therefore, (17) reduces to

$$G(1) = q^{-1}E(q) \left(-1 + E^{-1}(q)\right) = q^{-1} - q^{-1}E(q).$$

This completes the proof.

Q.E.D.

We record the following obvious Proposition for convenience.

Proposition 3.6. *The series $G'(1)$ satisfies the following identity*

$$G'(1) = \sum_{n=1}^{\infty} nq^n (1-q)(1-q^2)\cdots(1-q^n).$$

We now recall an identity due to L. J. Rogers (see [p. 29, Ex. 10, A1]).

Proposition 3.7. *If $H(z)$ is the series*

$$H(z) := z - \sum_{n=1}^{\infty} (1-z^2q)(1-z^2q^2)\cdots(1-z^2q^{n-1})z^{2n+3}q^n,$$

then

$$H(z) = z + \sum_{n=1}^{\infty} (-1)^n \left(q^{(3n^2-n)/2} z^{6n-1} + q^{(3n^2+n)/2} z^{6n+1} \right).$$

Proof. To obtain this identity, simply let $x = z^2$ in [p. 29, Ex. 10, A1] then multiply both sides of the identity by z.

Q.E.D.

Now we prove a crucial Lemma regarding $H(z)$.

Lemma 3.8. *The series $H'(1)$ satisfies the following identities:*

$$\begin{aligned} H'(1) &= 1 + \sum_{n=1}^{\infty} (-1)^n \left((6n-1) q^{(3n^2-n)/2} + (6n+1) q^{(3n^2+n)/2} \right) \\ &= -4 - 2qG'(1) + 5E(q) + 2 \sum_{n=1}^{\infty} q^n (1-q)(1-q^2)\cdots(1-q^{n-1}) \sum_{j=1}^{n-1} \frac{q^j}{1-q^j}. \end{aligned}$$

Proof. The first claim regarding $H'(1)$ follows trivially from Proposition 3.7.

Now we use the definition of $H(z)$ in Proposition 3.7 to give another formulation of $H'(1)$. We begin by computing $H'(z)$. We find by the product rule that

$$\begin{aligned} H'(z) = 1 &- \sum_{n=1}^{\infty} (1-z^2q)(1-z^2q^2)\cdots(1-z^2q^{n-1})(2n+3)z^{2n+2}q^n \\ &- \sum_{n=1}^{\infty} z^{2n+3}q^n (1-z^2q)(1-z^2q^2)\cdots(1-z^2q^{n-1}) \sum_{j=1}^{n-1} \frac{-2zq^j}{1-z^2q^j}. \end{aligned}$$

Therefore, we find that

$$\begin{aligned}
H'(1) = 1 &- \sum_{n=1}^{\infty}(1-q)(1-q^2)\cdots(1-q^{n-1})(2n+3)q^n \\
&- \sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{-2q^j}{1-q^j} \\
= 1 &- 2\sum_{n=1}^{\infty}(1-q)(1-q^2)\cdots(1-q^{n-1})(n-1)q^n - 5\sum_{n=1}^{\infty}(1-q)(1-q^2)\cdots(1-q^{n-1})q^n \\
&+ 2\sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{q^j}{1-q^j} \\
= 1 &- 2q\sum_{n=1}^{\infty}(1-q)(1-q^2)\cdots(1-q^{n})nq^n - 5q\sum_{n=0}^{\infty}(1-q)(1-q^2)\cdots(1-q^{n})q^n \\
&+ 2\sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{q^j}{1-q^j}.
\end{aligned}$$

However, by the definition of $G(1)$ and Proposition 3.6, we find that

$$\begin{aligned}
H'(1) = 1 - 2qG'(1) - 5qG(1) \\
+ 2\sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{q^j}{1-q^j}.
\end{aligned}$$

The claimed identity follows from Proposition 3.5.

Q.E.D.

Here is the last crucial Lemma that we require.

Lemma 3.9. *The following identity is true:*

$$\sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{q^j}{1-q^j} = 1 - E(q) - E(q)\mathcal{D}(q).$$

Proof. Let $I(z)$ be the series in the following identity

$$I(z) := 1 - \sum_{n=1}^{\infty} zq^n(1-zq)(1-zq^2)\cdots(1-zq^{n-1}) = \prod_{n=1}^{\infty}(1-zq^n).$$

It is easy to see, by differentiating both expressions above, that

$$
\begin{aligned}
I'(z) = -\sum_{n=1}^{\infty} q^n(1-zq)(1-zq^2)\cdots(1-zq^{n-1}) \\
-\sum_{n=1}^{\infty} zq^n(1-zq)(1-zq^2)\cdots(1-zq^{n-1})\sum_{j=1}^{n-1}\frac{(-q^j)}{1-zq^j} \\
=\prod_{n=1}^{\infty}(1-zq^n)\sum_{j=1}^{\infty}\frac{(-q^j)}{1-zq^j}.
\end{aligned}
$$

Therefore, we find that

$$
\begin{aligned}
&I'(1) = \\
&= -\sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1}) + \sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{q^j}{1-q^j} \\
&\quad = -E(q)\mathcal{D}(q).
\end{aligned}
$$

By the definition of $G(z)$, we now find that

$$
\sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{q^j}{1-q^j} = qG(1) - E(q)\mathcal{D}(q).
$$

Proposition 3.5 completes the proof.

Q.E.D.

Proof of Theorem 3. By Proposition 3.3 and Proposition 3.5, it suffices to show that

$$
\begin{aligned}
-1 + E(q) - qG'(1) - E(q)\mathcal{D}(q) = \\
= \sum_{k=0}^{\infty}(-1)^k(3k)q^{(3k^2+k)/2} + \sum_{k=1}^{\infty}(-1)^k(3k-1)q^{(3k^2-k)/2}.
\end{aligned}
$$

However, by Lemma 3.9 this is equivalent to

$$
\begin{aligned}
-2 + 2E(q) - qG'(1) + \sum_{n=1}^{\infty} q^n(1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{j=1}^{n-1}\frac{q^j}{1-q^j} \\
= \sum_{k=0}^{\infty}(-1)^k(3k)q^{(3k^2+k)/2} + \sum_{k=1}^{\infty}(-1)^k(3k-1)q^{(3k^2-k)/2}.
\end{aligned}
$$

However, by Lemma 3.8 we find that this is equivalent to

$$\frac{1}{2}H'(1) - \frac{1}{2}E(q) = \sum_{k=0}^{\infty}(-1)^k(3k)q^{(3k^2+k)/2} + \sum_{k=1}^{\infty}(-1)^k(3k-1)q^{(3k^2-k)/2}.$$

Lemma 3.8 and Euler's Pentagonal Number Theorem (5) completes the proof.

Q.E.D.

Proof of Corollary 4. The combinatorial interpretation of Lemma 3.2 together with Theorem 3 immediately implies the Corollary.

Q.E.D.

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