The elements on the main diagonal of $\mathscr{K}_1$ are in general not yet monic polynomials, they are monic only modulo $J^{\delta+1}$. But according to Krull's theorem they can be made monic by multiplying the rows of $\mathscr{K}_1$ with suitable invertible polynomials which are congruent to e modulo $J^{\delta+1}$. The resulting matrix will clearly be equivalent to $\mathscr{K}$, it will satisfy properties (5)-(7), and $\mathscr{K}' \equiv \mathscr{D} + L (\mathrm{mod}\, J^{\delta+1})$, i.e., by (11) and (12) it will have property (10). The first part of the Theorem is now established.

Remark. If one writes out the product $\mathscr{K}_1 = (E - Q_\ell)\mathscr{K}(E - Q_r)$ in detail it is easy to see that the element in the upper left corner of $\mathscr{K}_1$ remains the same as in $\mathscr{K}$. Consequently, it is possible to construct for matrices $\mathscr{K}$ satisfying (4)-(7) an equivalent matrix $\mathscr{K}'$ satisfying (4)-(8) and such that $K'_{11} = K_{11}$.

Now assume that we have a matrix (4) satisfying properties (5)-(8). On its main diagonal we collect the polynomials of degree zero and change the order of the entries to obtain

$$\mathscr{K} = \mathrm{Diag}\left(e, \ldots, e, \begin{pmatrix} K_{11} & \ldots & K_{1r} \\ \ldots & & \ldots \\ K_{r1} & \ldots & K_{rr} \end{pmatrix} \right),$$

where

$$\deg K_{ii}(x) = m_i > 0, \ \sum_{i=1}^{r} m_i = m, \ \deg K_{ij}(x) < m_{ij} = \min\{m_i, m_j\}.$$

Assume that

$$K_{ii}(x) = x^{m_i} - \sum_{s=0}^{m_i-1} K_{ii}^{(s)} x^s, \quad K_{ij}(x) = \sum_{s=0}^{m_{ij}-1} K_{ij}^{(s)} \cdot x^s.$$

Consider a matrix $A \in R_m$ of the form

$$A = \begin{pmatrix} A_{11} & \ldots & A_{1r} \\ \ldots & & \ldots \\ A_{r1} & \ldots & A_{rr} \end{pmatrix},$$

where

$$A_{ii} = \begin{pmatrix} 0 & e & 0 & \ldots & 0 \\ 0 & 0 & e & & \ldots \\ & & & \ddots & \\ \ldots & \ldots & & & 0 \\ 0 & 0 & \ldots & 0 & e \\ K_{ii}^{(0)} & & \ldots & & K_{ii}^{(m_i-1)} \end{pmatrix}, \qquad A_{ij} = \begin{pmatrix} 0 & & \ldots & & 0 \\ \ldots & & & & \ldots \\ 0 & & \ldots & & 0 \\ K_{ij}^{(0)} & \ldots & K_{ij}^{(m_{ij}-1)} & \ldots & 0 \end{pmatrix}.$$

We claim that $(xE - A) \sim \mathscr{K}(x)$. The chain of elementary operations which transform $xE - A$ into $\mathscr{K}(x)$ consists of successive reductions of each block $xE - A_{ii}$ to the form $\mathrm{Diag}(e, \ldots, e, K_{ii}(x))$ and subsequent obvious elementary operations. To fix our ideas we illustrate this in the case $r = 2$. Then

$$xE - A = \left(\begin{array}{ccccc|cccc} x & -e & 0 & \ldots & 0 & 0 & & \ldots & 0 \\ 0 & x & -e & \ldots & 0 & \ldots & & & \ldots \\ \ldots & \ddots & & \ddots & & & & & \\ 0 & & x & & -e & 0 & & \ldots & 0 \\ -K_{11}^{(0)} & \ldots & -K_{11}^{(m_1-2)} & & x - K_{11}^{(m_1-1)} & K_{12}^{(0)} & \ldots & K_{12}^{(m_{12}-1)} \ \ldots & 0 \\ \hline 0 & & \ldots & & 0 & x & -e & 0 \ \ldots & 0 \\ & & & & & 0 & x & -e & 0 \\ \ldots & & & & \ldots & \ldots & \ddots & \ddots & \ldots \\ 0 & & \ldots & & 0 & 0 & & x & -e \\ K_{21}^{(0)} & & \ldots & & K_{21}^{(m_{21}-1)} & -K_{22}^{(0)} & \ldots & -K_{22}^{(m_2-2)} & x - K_{22}^{(m_2-1)} \end{array}\right).$$

To column $m_1 - 1$ of the matrix $xE - A$ we add column m_1 multiplied by x, then add to column $m_1 - 2$ column $m_1 - 1$ multiplied by x and so on until finally one adds to column 1 column 2