10. Von Neumann regular rings. Class R3(n).

A ring A is called von Neumann *regular* if for every $a \in A$ there is an element $x \in A$ such that $axa = a$. Replacing x by xax, we can have both $axa = a$ and $xax = x$. In this section, we use "regular" for von Neumann regular.

It is clear from the definition that any ideal and any factor ring of a regular ring are regular and that the direct product of regular rings is regular.

Examples of a regular A are the matrix rings $\mathrm{M}_n D$ over division rings D. Indeed, we can write $a \in \mathrm{M}_n D$ (where D is a division ring) as $a = \alpha p \beta$ where $p = p^2$ is a diagonal matrix of zeros and ones and $\alpha, \beta \in \mathrm{GL}_n A$. Set $x = \beta^{-1}\alpha^{-1}$.

In this example, $\mathrm{sr}(A) = 1$, but there are regular A with $\mathrm{sr}(A) \neq 1$ [MM2]. For any commutative regular ring A, $\mathrm{sr}(A) = 1$. More generally, $\mathrm{sr}(A) = 1$. for any regular A which is abelian, i.e., every idenpotent in A is central [Go, Corollary 4.15].

Regular rings A with $\mathrm{sr}(A) = 1$ are known as *unit regular* rings. They can be defined as follows [Go, Proposition 4.12]: for every $a \in A$ there is $x \in \mathrm{GL}_1 A$ such that $axa = a$. They include $\mathrm{M}_n D$ with division rings D, and the direct product of unit regular rings is unit regular. Any ideal or factor ring of any unit regular ring is unit regular.

Here is a proof that any abelian regular ring A is unit regular. Let $a \in A$. By the regularity, $axa = a$ for some $x \in A$. Then $xaxa = xa$ is idenpotent. Since A is abelian, xa commutes with both a and x. Set $u := xax + 1 - xa \in 1 + A$. Then $aua = a$ and

$$u(axa + 1 - xa) = (axa + 1 - xa)u = 1$$

and $aua = a$.

For any artinian ring A (e.g., and finite-dimensional algebra over a field), the ring $A/\mathrm{rad}(A)$ is the finite direct product of matrix rings over division rings and hence regular. Also for any commutative semilocal ring A, the ring $A/\mathrm{rad}(A)$ is the finite direct product of fields and hence unit regular.

Another example of a regular ring is an arbitrary Boolean algebra, i.e., a ring A such that $a^2 = a$ for every $a \in A$. In this case, we can take $x = a$.

In probability theory, we use rings of (real or complex) measurable functions on probability spaces. Those rings are regular while the ring of continuous functions on the line is not regular.

Theorem 10.1. Let A be an associative ring with 1 and B an ideal of A such that the ring $A/\mathrm{rad}(A)$ is (von Neumann) regular. Then for any $n \geq 2$:

(a) $\mathrm{E}_n(A, B)$ contains all matrices of the form $1_n + vu$ where v is an n-column over A, u is an n-row over B, and $uv = 0$; in particular $\mathrm{E}_n(A, B)$ is normal in $\mathrm{GL}_n A$;

(b) $\mathrm{E}_n(A, B) \supset [\mathrm{E}_n A, \mathrm{G}_n(A, B)]$; in particular every subgroup H of $\mathrm{G}_n(A, B)$ containing $\mathrm{E}_n(A, B)$ is normalized by $\mathrm{E}_n A$;

(c) if B is regular, then $\mathrm{E}_n(A, B) = \mathrm{E}_n B$.

Proof. (a) We write $v = (v_i)$ and $u = (u_j)$.

Case 1: $1 + u_k v_k \in \mathrm{GL}_1 A$ for some k. In this case we are done by Proposition 8.3.

General case. We find $x \in B$ such that

$$u_n x u_n - u_n \in \mathrm{rad}(B).$$

Then $1_n + vu = \alpha\beta$ with $\alpha = 1_n + v(1 - u_n x)u$ and $\beta = 1_n + vu_n xu$.

By Case 1 with $k = n, \alpha \in \mathrm{E}_n(A, B)$. Concerning β, its $(1, n)$-entry is $\beta_{1,n} = v_1 u_n x u_n \equiv v_1 u_n \pmod{\mathrm{rad}(B)}$ and its $(1, 1)$-entry is $\beta_{1,1} = 1 + v_1 u_n x u_1$ so $\beta_{1,1} - \beta_{1,n} x u_1 \equiv 1 \pmod{\mathrm{rad}\ (B)}$ hence $\beta_{1,1} - \beta_{1,n} x u_1 \in \mathrm{GL}_1 B$.

By Case 1 with $n = 1$, the matrix $(xu_1)^{n,1}\beta(-xu_1)^{n,1}$ (which has the form $1_n + v'u'$ with $u'v' = 0$) belongs to $\mathrm{E}_n(A, B)$ hence $\beta \in \mathrm{E}_n(A, B)$.

(b) Let $a^{i,j}$ be an elementary matrix in $\mathrm{E}_n A$ and $\beta \in \mathrm{G}_n(A, B)$. We have to prove that $[a^{i,j}, \beta] \in \mathrm{E}_n(A, B)$. Since both $\mathrm{E}_n(A, B)$ and $\mathrm{G}_n(A, B)$ are normalized by all permutation matrices, we can assume that $(i, j) = (n, 1)$.

Then $[\ a^{n,1}, \beta] = a^{n,1}(1_n - vau)$, where v is the last column of β and u is the first row of β^{-1}. We write $u = (u_j) = (u', u_n)$ and $v = (v_i) = \begin{pmatrix} v' \\ v_n \end{pmatrix}$. Since $B/\mathrm{rad}(B)$ is regular, there are $x \in B$ such that $1 + u_n v_n a x_i \in \mathrm{GL}_1 A$ for all i and $\sum x_k = 1$.

We want to prove that

$$a^{n,1} = (a(1 - u_n x))^{n,1}(au_n x)^{n,1}$$

and β commute modulo the normal subgroup $\mathrm{E}_n(A, B)$ of $\mathrm{GL}_n A$. Since $a^{i,j} = \prod (ax_i)^{n,1}$, It suffices to show that both $(a(1-u_n x))^{n,1}$ and $(au_n x)^{n,1}$ commute with β modulo $\mathrm{E}_n(A, B)$.

Since $au_n x \in B$,

$$[(au_n x)^{n,1}, \beta] \in \mathrm{E}_n(A, B)$$

by (a).

Now we have to prove that

$$\alpha = [\beta, (-a(1 - u_n x))^{n,1}] = (1_n - va(1 - u_n x)u)(a(1 - u_n x))^{n,1} \in \mathrm{E}_n(A, B).$$

The (n, n)-entry of α is

$$d = 1 - v_n a(1 - u_n x)u_n \in \mathrm{GL}_1 B.$$

We set $w = (w', w_n) = a(1 - u_n x)u$.

By row addition operations over B we eliminate all non-diagonal entries of the last column and then of the last row of the matrix α using the invertible diagonal entry d. We obtain the following block-diagonal matrix α' (which belongs to $\mathrm{E}_n(A, B)$ if and only if α does): $\alpha' = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, where $d = \alpha_{n,n} = 1 + v_n w_n$ and

$$a = 1_{n-1} + v'w' - v'w_n d^{-1} v_n w' = 1_{n-1} + v'(1 - w_n d^{-1} v_n)w'$$

$$= 1_{n-1} + v'd'^{-1}w' = (1_{n-1} - v'w')^{-1}$$

with $d' = 1 + w_n v_n = 1 - w'v'$. By Lemma 3.5, $\alpha' \in \mathrm{E}_n(A, B)$

(c) Note that when B is regular, then $BB = B$ hence $\mathrm{E}_n(A, B) = \mathrm{E}_n(B)$ by Corollary 3.3 and the group $\mathrm{E}_n B$ is perfect provided that $n \geq 3$. For $n = 2$ we also have $\mathrm{E}_2(A, B) = \mathrm{E}_2 B$ but $\mathrm{E}_2 B$ need not to be perfect. QED.

Theorem 10.2. Let A be (von Neumann) regular, $n \geq 3$, and H a subgroup of $\mathrm{GL}_n A$. Then the following two conditions are equivalent:

(a) H is normalized by $\mathrm{E}_n A$;

(b) $\mathrm{E}_n(A, B) \subset H \subset \mathrm{G}_n(A, B)$ for an ideal B of A.

Proof. (a) $\Rightarrow$ (b). Let H is normalized by $\mathrm{E}_n A$. Set $B = \{b \in A : b^{1,2} \in H\}$. By (1.5), (1.11), B is an ideal of A and $\mathrm{E}_n(A, B) \subset H$. We have to prove that $H \subset \mathrm{G}_n(A, B)$.

Otherwise, the image H' of H in $\mathrm{GL}_n(A, B)$ is not central. Notice the H' is normalized by $\mathrm{E}_n A'$. We claim that H' contains $(b')^{1,2}$ with $0 \neq b' \in A/B$. To prove this we pick a non-central $\alpha' = (\alpha'_{i,j}) \in H'$.

Case 1: an off-diagonal entry of α' is zero. Then we can use Proposition 1.10.

Case 2: $\alpha'_{n,1} + \alpha'_{n,2} y' = 0$ for some $y' \in A'$. Then non-central matrix $\beta' = (\beta'_{i,j}) = (-y)^{2,1} \alpha' y'^{2,1} \in H'$ and $\beta'_{n,1} = \alpha'_{n,1)} + \alpha'_{n,2} y' = 0$, so we are reduced to Case 1.

Case 3: a diagonal entry of α' belongs to $\mathrm{GL}_1 A'$. Then we are done by Proposition 4.11.

General case. We find $x' \in A'$ such that

$$z = \alpha'_{n,2} x' \alpha'_{n,2} - \alpha'_{n,2} \in \mathrm{rad}(A').$$

We set $p = 1 - \alpha'_{n,2} x'$. If $p \alpha'_{n,1} = 0$, i.e., $(1 - \alpha'_{n,2} x') \alpha'_{n,1} = 0$ then we are done by Case 2.

Otherwise, the matrix

$$\beta' = (\beta'_{i,j}) = [\alpha', p^{1,n}] \in H'$$

is not central and

$$\beta'_{2,2})1 + \alpha'^{-1}_{2,1} p \alpha'_{n,2} = 1 - \alpha'^{-1}_{2,1} p z \in \mathrm{GL}_1 A'$$

so we are reduced to Case 3.

Thus, we have a non-central matrix in H'. Therefore $\alpha_1 = b^{1,3} \beta_1 \in H$ with $b \in A \setminus B$ and $\beta_1 \in \mathrm{GL}_n B$. Taking commutator of α_1 with $1^{3,2}$ and using that $[\beta_1, 1^{3,2}] \in \mathrm{E}_n(A, B)$ by Theorem 10.1 (b) , we obtain that $b^{1,2} \in H$ which contradicts the definition of B.

(b) $\Rightarrow$ (a). This follows from Theorem 10.1 (b) and is true for all $n \geq 2$. QED.

Denote by VNR the class of all von Neumann regular rings. By Theorem 10.1, VNR $\subset$ R1(n)$\cap$ R1(n) for every $n \geq 2$.

Now, for every $n \geq 2$, we introduce another class of rings, R3(n). Namely, we define R3(n) the class of all associative rings B such that

(10.3) for every associative ring A with 1 containing B as an ideal every noncentral subgroup H of $\mathrm{GL}_n B'$ which is normalized by $\mathrm{E}_n A$ either contains a nontrivial elementary matrix or is central in $\mathrm{GL}_n A$.

In the case when B has unity, the condition (10.3) simplifies to a the following condition:

(10.4) every noncentral subgroup H of $\mathrm{GL}_n B$ which is normalized by $\mathrm{E}_n B$ contains a nontrivial elementary matrix.

In general, it it suffices to require (10.3) for the rings $A = B_1$ obtained by adding 1 to B.

Theorem 10.2 implies that VNR $\subset$ R3(n) for every $n \geq 3$. Theorem 4.12 implies that R3(n) contains every ring B with $\mathrm{sr}(B) \leq n-1$ when $n \geq 3$. Proposition 8.6 implies that R3(n) contains all commutative rings B when $n \geq 3$. Theorem 9.8 implies that R3(n) contains every associative ring A with 1 satisfying the condition (9.4). In particular, R3(n) contains all special topological algebras with 1.

Proposition 10.5. Let A be an associative ring with 1, B a nonzero ideal of A, H a noncentral subgroup of $\mathrm{GL}_n A$ which is contained in $\mathrm{G}_n(A, B)$ and normalized by $\mathrm{E}_n A$. Assume that $\mathrm{E}_n A$ is perfect (e.g., $n \geq 3$). Then $H \cap \mathrm{GL}_n B$ is a noncetral subgroup of $\mathrm{GL}_n A$.

Proof. Let $\alpha \in H$ be a noncentral matrix. We claim that $[\alpha, \beta] \in H \cap \mathrm{GL}_n B$ is noncentral for some $\beta \in \mathrm{E}_n A$.

Otherwise, we have a homomorphism $\beta \to [\alpha, \beta]$ from a perfect group $\mathrm{E}_n A$ into a commutative group $\mathrm{G}_n(A, 0)$. Recall that the centralizer of $\mathrm{E}_n A$ in $\mathrm{GL}_n A$ is the center $\mathrm{G}_n(A, 0)$ of $\mathrm{GL}_n A$. QED.

Proposition 10.6. Let B be an associative ring, B' an ideal of B, and $n \geq 3$. Assume that both B' and B/B' belong to R3(n). Then $B \in$ R3(n).

Proof. Let $A = B_1$, H is a subgroup of $\mathrm{GL}_n B$ which is noramlized by $\mathrm{E}_n A$. Assume that H is not central in $\mathrm{GL}_n A$. we have to prove taht H contains a nontrivial elementary matrix.

Since $B/B' \in$ R3(n), reduction modulo B' shows that either $H \subset \mathrm{G}_n B'$ or H contains a matrix g whose reduction modulo B' is a nontrivial elementary matrix.

In the first case, by Proposition 10.5, $H \cap \mathrm{GL}_n B$ is a noncetral subgroup of $\mathrm{GL}_n A$. Since $B' \in$ R3(n), $H \cap \mathrm{GL}_n B$ contains a nontrivial elementary matrix.

In the second case, we can assume that the reduction g modulo B' is $b^{1,2}$ with $0 \neq b \in B/B'$. Then both matrices, $[1^{1,3}, \alpha], [1^{3,2}, \alpha] \in \mathrm{GL}_n B$. If one of them is not in the center $\mathrm{G}_n(A, 0)$ of $\mathrm{GL}_n A$, we are done. Otherwise, $[1^{1,2}, \alpha] = [[1^{1,3}, 1^{3,2}], \alpha] = 1_n$, hence the $(k, 1)$-entries of g are zero for $k \geq 2$ (as well as the $(2, k)$-entries), and we are done by Proposition 1.10. QED.

Proposition 10.7. Let $n \geq 3$ and A be an associative ring with 1. Assume that $A \in$ R1(n) and that every factor ring of A belongs to R3(n). Then a subgroup H of $\mathrm{GL}_n A$ is normalized by $\mathrm{E}_n A$ if and only if $\mathrm{E}_n(A, B) \subset H \subset \mathrm{G}_n(A, B)$ for an ideal B of A.

Proof. Since $n \geq 3$, the group $\mathrm{E}_n A$ is perfect. Since $A \in$ R1(n), the "if" part follows from Proposition 8.12.

To prove the "only if" part, consider any subgroup H of $\mathrm{GL}_n A$ is normalized by $\mathrm{E}_n A$ and set $B = \{a \in A : a^{1,2} \in H\}$. By (1.7), (1.11), B is an ideal of A. We claim that $H \subset \mathrm{G}_n(A, B)$.

Otherwise, passing to the factor ring A/B and using that condition $A/B \in$ R3(n), we conclude that the image of H in $\mathrm{GL}_n A/B$ contains a nontrivial elementary matrix. In other words, $a^{1,2}\alpha \in H$ with $a \in A \setminus B$ and $\alpha \in \mathrm{GL}_n B$. Since $A \in$ R1(n), $[1^{3,1}, \alpha] \in H$. So $[[1^{3,1}, a^{1,2}] = a^{3,1} \in H$, i.e., $a \in B$, which contradicts to the assumption $a \in A \setminus B$. QED.

Exercises.

1. Prove that every Boolean algebra A is commutative and $2A = 0$.

2. Prove that if A is (von Neumann) regular then the Jacobson radiacal, $\mathrm{rad}(A)$ is zero.

3. Suppose that $A \neq 0$ is regular and has no zero divisors. Prove that A is a division ring.

4. Prove that the following 5 conditions on a ring A are equivalent:
(a) A is regular,
(b) every principal left ideal is generated by an idempotent,
(c) every finitely generated left ideal is generated by an idempotent,
(d) every principal left ideal is a direct summand of the left A-module A,
(e) every finitely generated left ideal is a direct summand of the left A-module A.

5. Prove that the following 3 conditions on a commutative ring A are equivalent:
(a) A is regular,
(b) A has Krull dimension 0 and no nonzero nilpotent elements,
(c) the localization of A at every maximal ideal is a field.

6. Show that for any regular ring A and any integer $n \geq 1$, the matrix ring $\mathrm{M}_n A$ is regular.

7. Let A be the endomorphism ring of a right vector space over a division ring. Show that A is regular (even in the infinite-dimensional case),

8. Let A be a coommutative regular ring. Show that for every $a \in A$ there is exactly one $x \in A$ such that $axa = a$ and $xax = x$.

9. A ring A is called *strongly regular* if for any $a \in A$ there is $x \in A$ such that $a = a^2 x$. Show that A is strongly regular if and only if A is abelian regular.