DIFFERENCES OF PARTITION FUNCTIONS — THE ANTI-TELESCOPING METHOD

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Dedicated to the memory of the great Leon Ehrenpreis.

Abstract. The late Leon Ehrenpreis originally posed the problem of showing that the difference of the two Rogers-Ramanujan products had positive coefficients without invoking the Rogers-Ramanujan identities. We first solve the problem generalized to the partial products and subsequently solve several related problems. The object is to introduce the Anti-Telescoping Method which is capable of wide generalization.

1. Introduction

At the 1987 A.M.S. Institute on Theta Functions, Leon Ehrenpreis aksed if one could prove that

$$\prod_{j=1}^{\infty} \frac{1}{(1-q^{5j-4})(1-q^{5j-1})} - \prod_{j=1}^{\infty} \frac{1}{(1-q^{5j-3})(1-q^{5j-2})}$$

has non-negative coefficients in its power series expansion without resorting to the Rogers-Ramanujan identites.

In [1], Rodney Baxter and I answered this question "sort of." Actually the point of our paper was to show that if one begins trying to solve Ehrenpreis's problem, then there is a natural path to the solution which has the Rogers-Ramanujan identities as a corollary. Indeed, as we say there [1, p. 408]: "It may well be objected that we presented a somewhat stilted motivation. Indeed if [the Rogers-Ramanujan identities] were not in the back of our minds, we would never have thought to construct [the path to the solution of Ehrenpreis's problem]." Subsequently in 1999, K. W. J. Kadell [9] constructed an injection of the partitions of n whose parts are $\equiv \pm 2 \pmod 5$ into partitions of n whose parts are $\equiv \pm 1 \pmod 5$. Finally in 2005, Berkovich and Garvan [6, Sec. 5] improved upon Kadell's work by providing ingenious, injective proofs for an infinite family of partition function inequalities related to finite products (including Theorem 1 below).

In this paper, we introduce a new method which mixes analytic and injective arguments. We illustrate the method on the most famous problem, Theorem 1. We note that Theorem 2 is also a direct corollary of [6, Sec. 5].

Theorem 1 (the Finite Ehrenpreis Problem, cf. [6])**.** *For $n \geq 1$, the power series expansion of*

$$\prod_{j=1}^{n} \frac{1}{(1-q^{5j-4})(1-q^{5j-1})} - \prod_{j=1}^{n} \frac{1}{(1-q^{5j-3})(1-q^{5j-2})}$$

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has non-negative coefficients.

We should note that the original question can be answered trivially if one invokes the Rogers-Ramanujan identities [5, p. 82] because

(1.1)
$$\begin{aligned}
&\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})} - \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})} \\
&= \left(1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}\right)\left(1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}\right) \\
&= q + \sum_{n=2}^{\infty} \frac{q^{n^2}}{(1-q^2)(1-q^3)\cdots(1-q^n)},
\end{aligned}$$

which clearly has non-negative coefficients.

However, there is no possibility of proving Theorem 1 in this manner because there are no known refinements of the Rogers-Ramanujan identities fitting these finite products. A new method is required.

Our method of proof might be called "Anti-Telescoping." Namely we want to write the first line of (1.1) as

$$\sum_{j=1}^{n} (P_j - P_{j-1}) \tag{1.2}$$

where each P_i is a finite product with

$$P_0 = \prod_{j=1}^{n} \frac{1}{(1-q^{5j-3})(1-q^{5j-2})}$$

and

$$P_n = \prod_{j=1}^{n} \frac{1}{(1-q^{5j-4})(1-q^{5j-1})}.$$

We construct the P_i so that they gradually change from P_1 to P_n. The proof then follows from an intricate, term-by-term analysis of (1.2).

In Section 2, we construct (1.2) and provide some analysis of the terms. In Section 3, we provide an injective map of partitions to show that each term of the constructed (1.2) has at most one negative coefficient. From there the proof of Theorem 1 is given quickly in Section 4.

We wish to emphasize that Anti-Telescoping is applicable to many problems of this nature. To make this point, we provide three further examples.

Theorem 2 (finite Göllnitz-Gordon)**.** *For $n \geq 1$, the power series expansion of*

$$\prod_{j=1}^{n} \frac{1}{(1-q^{8j-7})(1-q^{8j-4})(1-q^{8j-1})} - \prod_{j=1}^{n} \frac{1}{(1-q^{8j-5})(1-q^{8j-4})(1-q^{8j-3})}$$

has non-negative coefficients.

This theorem falls to the Anti-Telescoping method much more easily than the finite Ehrenpreis problem (Theorem 1).

Theorem 3 (finite little Göllnitz)**.** *For $n \geq 1$, the power series expansion of*

$$\prod_{j=1}^{n} \frac{1}{(1-q^{8j-7})(1-q^{8j-3})(1-q^{8j-2})} - \prod_{j=1}^{n} \frac{1}{(1-q^{8j-6})(1-q^{8j-5})(1-q^{8j-1})}$$

has non-negative coefficients.

This theorem requires a rather intricate application of Anti-Telescoping. We have chosen it to illustrate the breadth of this method.

We note that the partial products in Theorem 2 are from the Göllnitz-Gordon identities [2, (1.7) and (1.8) pp. 945–946], and the partial products in Theorem 3 are from identities termed by K. Alladi, The Little Göllnitz identities, [6, Sätze 2.3 and 2.4, pp. 166-167] (cf. [3, pp. 449–452]).

We conclude our applications of Anti-Telescoping by proving a finite version of differences between partition functions from the Rogers-Ramanujan-Gordon theorem ([8], cf. [1]). Again the proof goes without difficulty; however, a few cases must be excluded including the result in Theorem 1.

Theorem 4 (finite Rogers-Ramanujan-Gordon)**.** *For $\frac{k}{2} > s > r \geq 1$ and $n \geq 1$, the power series expansion of*

$$\prod_{\substack{j=1 \\ j \not\equiv 0, \pm s \ (\mathrm{mod}\ k)}}^{kn} \frac{1}{1-q^j} \quad - \prod_{\substack{j=1 \\ j \not\equiv 0, \pm r \ (\mathrm{mod}\ k)}}^{kn} \frac{1}{1-q^j}$$

has non-negative coefficients except *possibly in the case s prime and $s = r+1$ and $k = 3r+2$.*

The final section of the paper provides a number of open problems.

2. Anti-Telescoping

In this short section, we construct the telescoping sum (1.2). Namely

$$P_j = \frac{1}{(q, q^4; q^5)_j (q^{5j+2}, q^{5j+3}; q^5)_{n-j}} \tag{2.1}$$

where

$$(a; q)_s = (1-a)(1-aq)\cdots(1-aq^{s-1}),$$

and

$$(a_1, a_2, \ldots, a_r; q)_s = \prod_{i=1}^{r} (a_i; q)_s.$$

Clearly

$$P_n = \frac{1}{(q, q^4; q^5)_n}$$

and

$$P_0 = \frac{1}{(q^2, q^3; q^5)_n}.$$

So

$$\frac{1}{(q, q^4; q^5)_n} - \frac{1}{(q^2, q^3; q^5)_n} = \sum_{j=1}^{n} (P_j - P_{j-1}). \tag{2.2}$$

We let

$$T(n, j) := P_j - P_{j-1}.$$

So for $1 \le j \le n$,

$$T(n,j) = \frac{(1-q^{5j-2})(1-q^{5j-3})-(1-q^{5j-4})(1-q^{5j-1})}{(q,q^4;q^5)_j(q^{5j-3},q^{5j-2};q^5)_{n+1-j}} \tag{2.3}$$
$$= \frac{q^{5j-4}(1-q)(1-q^2)}{(q,q^4;q^5)_j(q^{5j-3},q^{5j-2};q^5)_{n+1-j}}$$
$$= \frac{q^{5j-4}(1-q^2)}{(q^6;q^5)_{j-1}(q^4;q^5)_j(q^{5j-3},q^{5j-2};q^5)_{n+1-j}}$$

and for $2 \le j \le n$

$$T(n,j) = q^{5j-8}\left(\frac{q^4}{1-q^4}-\frac{q^6}{1-q^6}\right)\frac{1}{(q^{11};q^5)_{j-2}(q^9;q^5)_{j-1}(q^{5j-3},q^{5j-2};q^5)_{n+1-j}}. \tag{2.4}$$

So for $n \ge 1$

$$T(n,1) = \frac{q}{(1-q^3)(1-q^4)(q^7,q^8;q^5)_{n-1}}, \tag{2.5}$$

for $n \ge 2$

$$T(n,1)+T(n,2) = \frac{q+q^4+q^5+q^6+q^9}{(1-q^6)(1-q^7)(1-q^8)(1-q^9)(q^{12},q^{13};q^5)_{n-2}}, \tag{2.6}$$

for $n \ge 3$,

$$\begin{aligned} &T(n,1)+T(n,2)+T(n,3) \\ &= \frac{q+q^{11}+q^{21}}{(1-q^8)(1-q^9)(1-q^{11})(1-q^{14})(q^{12},q^{13};q^5)_{n-2}} \\ &\qquad + \frac{q^4+q^{11}}{(1-q)(1-q^9)(1-q^{11})(1-q^{14})(q^{12},q^{13};q^5)_{n-2}}, \end{aligned} \tag{2.7}$$

and for $n \ge 4$

$$\begin{aligned} &T(n,1)+T(n,2)+T(n,3)+T(n,4) \\ &= \frac{q+q^{12}}{(1-q^3)(q^{12};q)_3(q^{16};q)_4(q^{22},q^{23};q^5)_{n-4}} \\ &\quad + \frac{(2q^{11}+q^{21})(1+q^3+q^6+q^9+q^{12}+q^{15}+q^{18})}{(q^{11};q)_4(q^{16};q)_4(q^{22},q^{23};q^5)_{n-4}} \\ &\quad + \frac{q^5+q^{13}}{(1-q^3)(q^{11};q)_4(1-q^{16})(q^{18};q)_2(q^{22},q^{23};q^5)_{n-4}} \\ &\quad + \frac{q^6+q^9+q^{10}+q^{15}+q^{16}+q^{19}+q^{20}}{(1-q^3)(q^{11};q)_4(q^{16};q)_2(1-q^{19})(q^{22},q^{23};q^5)_n}. \end{aligned} \tag{2.8}$$

Lemma 5. *For $n \ge 4$, the first terms of the power series expansion are given by*

$$T(n,1)+T(n,2)+T(n,3)+T(n,4) = q+q^4+q^5+q^6+q^7+q^8+2q^9+\cdots$$

and the remaining coefficients are all ≥ 2.

Proof. By direct computation we may establish that the assertion of Lemma 5 is valid through the first seventeen terms.

Next we note that the coefficients in question must all be at least as large as those of

$$\frac{q+q^{12}}{(1-q^3)}+\frac{q^5+q^6+q^9+q^{10}+q^{15}+q^{19}+q^{20}}{(1-q^3)}$$
$$=q+q^4+q^5+q^6+q^7+q^8+2q^9+2q^{10}+q^{11}+3q^{12}+2q^{13}+q^{14}$$
$$+4q^{15}+2q^{16}+q^{17}+\frac{2q^{18}}{1-q}+\frac{q^{18}(2+q)}{1-q^3},$$

and the coefficients in this last expression are all ≥ 2 beyond q^{17} owing to $\frac{2q^{18}}{(1-q)}$. $\square$

3. The Injection

Our first goal is to interpret $T(n,j)$ as given in (2.4) as the difference between two partition generating functions.

First, we define a set of integers for $n \geq j \geq 5$

$$S(n,j) := \{9,11,14,16,19,\ldots,5j-4,5j-1\}$$
$$\cup\,\{5j-3,5j-2,5j+2,5j+3,\ldots,5n-3,5n-2\}$$

We say that 4-*partitions* are partitions whose parts lie in $\{4\}\cup S(n,j)$ with the condition that at least one 4 is a part.

We say 6-*partitions* are partitions whose parts lie in $\{6\}\cup S(n,j)$ with the condition that at least one 6 is a part.

We let $p4_{n,j}(m)$ (resp. $p6_{n,j}(m)$) denote the number of 4-partitions (resp. 6-partitions) of m.

Thus by (2.4) and the standard construction of product generating functions [2, p. 45], we see that for $n \geq j \geq 4$

$$T(n,j)=q^{5j-8}\sum_{m\geq 0}\big(p4_{n,j}(m)-p6_{n,j}(m)\big)q^m \tag{3.1}$$

Lemma 6. *For $m \geq 0$, $n \geq j \geq 5$,*

$$p4_{n,j}(m)-p6_{n,j}(m)=\begin{cases}-1, & \text{if } m=6\\ \geq 0 & \text{if } m\neq 6.\end{cases}$$

Proof. Clearly for $m \leq 6$, $p4_{n,j}(m)=0$ except for $m=4$ when it is 1, and $p6_{n,j}(m)=0$ except for $m=6$ when it is 1. Hence Lemma 3 is proved for $m\leq 6$. From here on we assume $m>6$.

We now construct an injection of the 6-partitions of m into the 4-partitions of m to conclude our proof.

Case 1. There are $2k$ 6's in a given 6-partition. Replace these by $3k$ 4's.

Case 2. There are $(2k+1)$ 6's in a given 6-partition (with $k>0$). Replace these by $(3k-2)$ 4's and one 14.

Case 3. The given 6 partition has exactly *one* 6. Since $m>6$, there must be a smallest summand coming from $S(n,j)$. Call this summand the *second summand*. We must replace the unique 6, the second summand, and perhaps one or two other summands by some fours and some elements of $S(n,j)$ that are (except in the instances indicated with $(*)$) no larger than the second summand with the added proviso that either

(i) the number of 4's is $\equiv 2 \pmod 3$

or

(ii) the number of 4's is $\equiv 1 \pmod 3$ and no 14 occurs in the image.

The table below provides the replacement required in each case. The first column describes the pre-image partition; the single 6 and the second summand are always given explicitly as the first two summands. After the few sumands that are to be altered are listed there is a parenthesis such as (≥ 11, no 14's) which means that the remaining summands are taken from $S(n, j)$, all are ≥ 11 and there are no 14's. The second column describes the image partition. The parts indicated parenthetically are unaltered in the mapping.

	pre-image partition	$\longrightarrow$	image partition
$(*)$	$6 + 9 + (\geq 11$, no 14's$)$	$\longrightarrow$	$4 + 11 + (\geq 11$, no 14's$)$
$(*)$	$6 + 9 + 9 + (\geq 11$, no 14's$)$	$\longrightarrow$	$4 + 9 + 11 + (\geq 11$, no 14's$)$
	$6 + 9 + 9 + 9 + 11 + (\geq 9)$	$\longrightarrow$	eleven 4's $+(\geq 11)$
	$6 + 9 + 9 + 9 + 16 + (9$'s or $\geq 16)$	$\longrightarrow$	ten 4's $+9 + (9$'s or $\geq 16)$
	$6 + 9 + 9 + 9 + (9$'s or $\geq 16)$	$\longrightarrow$	$4 + 9 + 9 + 11 + (9$'s or $\geq 16)$
	$6 + 9 + 14 + (\geq 9)$	$\longrightarrow$	$4 + 4 + 4 + 4 + 4 + 9 + (\geq 9)$
	$6 + 11 + (\geq 11)$	$\longrightarrow$	$4 + 4 + 9 + (\geq 11)$
$(*)$	$6 + 14 + (\geq 16)$	$\longrightarrow$	$4 + 16 + (\geq 16)$
	$6 + 14 + 14 + (\geq 14)$	$\longrightarrow$	$4 + 4 + 4 + 4 + 4 + 14 + (\geq 14)$
	$6 + 16 + (\geq 16)$	$\longrightarrow$	$4 + 4 + 14 + (\geq 16)$
	$6 + 19 + (\geq 19)$	$\longrightarrow$	$4 + 4 + 4 + 4 + 9 + (\geq 19)$
	$6 + 21 + (\geq 21)$	$\longrightarrow$	$4 + 4 + 19 + (\geq 21)$
	$6 + 24 + (\geq 24)$	$\longrightarrow$	$4 + 4 + 11 + 11 + (\geq 24)$
	$6 + 26 + (\geq 26)$	$\longrightarrow$	eight 4's $+(\geq 26)$
	$6 + 29 + (\geq 29)$	$\longrightarrow$	$4 + 4 + 4 + 4 + 19 + (\geq 29)$

Now for $j \geq i > 6$

	$6 + (5i - 4) + (\geq 5i - 4)$	$\longrightarrow$	$4 + 4 + (5i - 6) + (\geq 5i - 4)$
	$6 + (5i - 1) + (\geq 5i - 1)$	$\longrightarrow$	$4 + 4 + 11 + (5i - 14) + (\geq 5i - 1)$

(remember that $j \geq 5$)

$(*)$	$6 + (5j - 3) + (\geq 5j - 3)$	$\longrightarrow$	$4 + (5j - 1) + (\geq 5j - 3)$
	$6 + (5j - 2) + (\geq 5j - 2)$	$\longrightarrow$	$4 + 4 + (5j - 4) + (\geq 5j - 2)$
	$6 + (5j + 2) + (\geq 5j + 2)$		
	if $j = 5$	$\longrightarrow$	$4 + 4 + 11 + 14 + (\geq 5j + 2)$
	if $j = 6$	$\longrightarrow$	$4 + 9 + 9 + 16 + (\geq 5j + 2)$
	if $j > 6$	$\longrightarrow$	$4 + 4 + 16 + (5j - 16) + (\geq 5j + 2)$
	$6 + (5j + 7) + (\geq 5j + 7)$	$\longrightarrow$	$4 + 4 + 4 + 4 + (5j - 3) + (\geq 5j + 2)$

and for $i \geq j + 3$

	$6 + (5i - 3) + (\geq 5i - 3)$	$\longrightarrow$	five 4's $+(5i - 17) + (\geq 5i - 3)$
	$6 + (5j + 3) + (\geq 5j + 3)$		
	if $j = 5$	$\longrightarrow$	$4 + 4 + 4 + 4 + 9 + 9 + (\geq 5j + 3)$
	if $j \geq 6$	$\longrightarrow$	five 4's $+(5j - 11) + (\geq 5j + 3)$
	$6 + (5i + 8) + (\geq 5j + 8)$	$\longrightarrow$	$4 + 4 + 4 + 4 + (5j - 2) + (\geq 5j + 8)$
	$6 + (5j + 13) + (\geq 5j + 13)$	$\longrightarrow$	five 4's $+(5j - 1) + (\geq 5j + 13)$

and for $i \geq j + 4$

	$6 + (5i - 2) + (\geq 5i - 2)$	$\longrightarrow$	$4 + 4 + 9 + 9 + (5i - 22) + (\geq 5i - 2)$

The important points to keep in mind in checking for the injection are: (A) every possible pre-image is accounted for and (B) there is no overlap among the images.

Point (A) follows from direct inspection of the construction of the first column where each line accounts for every possible second summand.

Point (B) requires serious scrutiny. We note that if two partitions have a different number of 4's, then they can't be the same partition. There are single lines where the image has 11, 10, 8, 7 fours; so these are unique. There are five lines with five 4's, and inspection of these reveals they are all different. There are five lines with four 4's, and they all are clearly different in the explicitly given parts. There are ten lines with two 4's, and inspection of these reveals only two lines of possible concern, namely

$$6+(5j-2)+(\geq 5j-2) \quad \longrightarrow \quad 4+4+(5j-4)+(\geq 5i-2)$$

and at $j>6$

$$6+(5j+2)+(\geq 5j+2) \quad \longrightarrow \quad 4+4+16+(5j-16)+(\geq 5j+2)$$

Here the upper line if j were 4 would be $4+4+16+(\geq 18)$ while the bottom line is $4+4+16+(5j-16)+(\geq 5j-2)$, and so we would have a possible identity of images if j were 4. Fortunately, j is specified to be ≥ 5. There are seven lines with a single 4. These seven can be displayed with their smalest parts in evidence

$$\begin{aligned}
&4+11+\cdots\\
&4+9+11+\cdots\\
&4+16+\cdots\\
&4+(5j-1)+\cdots\\
&4+9+9+16+\cdots\\
&4+9+16+\cdots;
\end{aligned}$$

so clearly all of these lines are distinct. Thus we have constructed the required injection. □

4. Proof of Theorem 1

For $n=1$,

$$\frac{1}{(1-q)(1-q^4)}-\frac{1}{(1-q^2)(1-q^3)}=\frac{q}{(1-q^3)(1-q^4)}. \tag{4.1}$$

For $n=2$,

$$\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)}-\frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)} \\ =\frac{q+q^4+q^5+q^6+q^9}{(1-q^6)(1-q^7)(1-q^8)(1-q^9)}. \tag{4.2}$$

For $n=3$,

$$\frac{1}{(q,q^4;q^5)_3}-\frac{1}{(q^2,q^3;q^5)_3}=T(3,1)+T(3,2)+T(3,3). \tag{4.3}$$

For $n=4$,

$$\frac{1}{(q,q^4;q^5)_4}-\frac{1}{(q^2,q^3;q^5)_4}=T(4,1)+T(4,2)+T(4,3)+T(4,4). \tag{4.4}$$

The non-negativity of the power series coefficients in (4.1) and (4.2) is obvious by inspection. The non-negativity for (4.3) follows directly from (2.7), and that in (4.4) follows directly from (2.8).

So for the remainder of the proof we can assume $n \geq 5$. Hence

$$\begin{aligned}\frac{1}{(q,q^4;q^5)_n} - \frac{1}{(q^2,q^3;q^5)_n} &= T(n,1)+T(n,2)+T(n,3)+T(n,4)+\sum_{j=5}^{n} T(n,j)\\ &= T(n,1)+T(n,2)+T(n,3)+T(n,4)\\ &\qquad + \sum_{j=5}^{n} q^{5j-8}\sum_{m=0}^{\infty}\big(p4_{n,j}(m)-p6_{n,j}(m)\big)q^m.\end{aligned}$$

With the $T(n,1)+T(n,2)+T(n,3)+T(n,4)$ term we know by Lemma 5 that all coefficients are ≥ 2 from q^9 onward. The j-th term in the sum has exactly one negative coefficient which is -1 and occurs as the coefficient of q^{5j-2} $(5 \leq j \leq n)$ and these single subtractions of 1 occur against terms in $T(n,1)+T(n,2)+T(n,3)+T(n,4)$ where the corresponding coefficient is ≥ 2. Hence all terms have non-negative coefficients. $\square$

Corollary 7. *In the power series expansion of*

$$\frac{1}{(q,q^4;q^5)_n} - \frac{1}{(q^2,q^3;q^5)_n}$$

the coefficient of q^m is positive except for the cases $n = 1$ with $m = 0$, 2, 3, and 6 and $n \geq 2$ with $m = 0, 2, 3$.

Proof. For $n = 1$

$$\frac{1}{(1-q)(1-q^4)} - \frac{1}{(1-q^2)(1-q^3)} = q + q^4 + q^5 + \frac{q^7}{1-q} + \frac{q^{13}}{(1-q^3)(1-q^4)},$$

and clearly the only zero coefficients occur for q^0, q^2, q^3 and q^6.

For $n = 2$

$$\begin{aligned}&\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)} - \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)}\\ &= q + \frac{q^4}{1-q} + \frac{q^9+q^{10}+q^{13}+q^{14}+q^{15}}{(1-q^6)(1-q^7)(1-q^8)(1-q^9)} + \frac{q^{11}}{(1-q)(1-q^8)(1-q^9)}\\ &\qquad \frac{q^{12}(1+q^5+q^7+q^{12})}{(1-q^6)(1-q^8)(1-q^9)} + \frac{q^{13}}{(1-q)(1-q^8)(1-q^9)},\end{aligned}$$

and now the only zero coefficients occur for q^0, q^2, and q^3.

For $n = 3$, the assertion follows from (2.7).

For $n \geq 4$ we see that the proof of Theorem 1 shows that all the coefficients are positive for q^m with $m \geq 9$, and Lemma 5 together with the proof of Theorem 1 proves the result for $m < 9$. $\square$

5. Proof of Theorem 2

We define

$$g_n = (q,q^4,q^7;q^8)_n$$

and

$$h_n = (q^3,q^4,q^5;q^8)_n.$$

Then Theorem 2 is the assertion that

$$\frac{1}{g_n}-\frac{1}{h_n}$$

has non-negative coefficients. So

$$\begin{aligned}\frac{1}{g_n}-\frac{1}{h_n}&=\frac{1}{h_n}\left(\frac{h_n}{g_n}-1\right)\\&=\frac{1}{h_n}\sum_{j=1}^{n}\left(\frac{h_j}{g_j}-\frac{h_{j-1}}{g_{j-1}}\right)\\&:=\sum_{j=1}^{n}U(n,j)\end{aligned}$$

where

$$\begin{aligned}U(n,j)&=\frac{h_{j-1}}{g_jh_n}\Big((1-q^{8j-5})(1-q^{8j-4})(1-q^{8j-3})\\&\qquad\qquad-(1-q^{8j-7})(1-q^{8j-4})(1-q^{8j-1}\Big)\\&=\frac{q^{8j-7}(1+q)}{(q^9,q^{12};q^8)_{j-1}(q^7;q^8)_j(q^{8j-5},q^{8j-3};q^8)_{n+1-j}(q^{8j+4};q^8)_{n-j}}\end{aligned}$$

and $U(n,j)$ clearly has non-negative coefficients. □

6. Proof of Theorem 3

We have chosen this third theorem to illustrate some of the problems that can arise using the Anti-Telescoping method and to show how to surmount arising difficulties.

If we were to follow exactly the steps in the proof of Theorem 2, we would repace g_n with $(q,q^5q^6;q^8)_n$ and h_n with $(q^2,q^3,q^7;q^8)_n$. The resulting $U(n,j)$ is frought with difficulties. $U(n,1)$ has no negative coefficients, but for $j>1$ $U(n,j)$ has scads of negative coefficients many of which are not just -1 or -2. Thus the smooth ride of Section 5 or the “6's$\longrightarrow$ 4's” injection of Section 3 seems to become a nightmare.

The secret is to adjust the anti-telescoping. Namely, we let

$$G_n=(q^6,q^9,q^{13};q^8)_n, \tag{6.1}$$

and

$$H_n=(q^7,q^{10},q^{11};q^8)_n, \tag{6.2}$$

with
(6.3)

$$W(n,j)=\begin{cases}\frac{1}{H_{n-1}(1-q)(1-q^5)(1-q^{8n-2})}\left(\frac{H_j}{G_j}-\frac{H_{j-1}}{G_{j-1}}\right), & 1\le j<n\\ \frac{1}{H_{n-1}}\left(\frac{1}{(1-q)(1-q^5)(1-q^{8n-2})}-\frac{1}{(1-q^2)(1-q^3)(1-q^{8n-1})}\right) & \text{if } j=0.\end{cases}$$

Then

(6.4)

$$\begin{aligned}\sum_{j=0}^{n-1} W(n,j) &= \frac{1}{(1-q)(1-q^5)(1-q^{8n-2})G_{n-1}} - \frac{1}{H_{n-1}(1-q)(1-q^3)(1-q^{8n-1})} \\ &\quad + \frac{1}{H_{n-1}}\left(\frac{1}{(1-q)(1-q^5)(1-q^{8n-2})} - \frac{1}{(1-q^2)(1-q^3)(1-q^{8n-1})}\right) \\ &= \frac{1}{(q,q^5,q^6;q^8)_n} - \frac{1}{(q^2,q^3,q^7;q^8)_n}\end{aligned}$$

The advantage of this altered anti-telescoping is that the denominator factors $(1-q)$ and $(1-q^5)$ help reduce the terms with negative coefficients to at most one for the $W(n,j)$.

Indeed,

$$W(1,0) = \frac{q(1+q^4)}{(1-q^5)(1-q^6)(1-q^7)}, \tag{6.5}$$

$$W(2,0) = \frac{1}{H_1}\left(\frac{q(1+q^4)}{(1-q^{14})(1-q^{15})} + \frac{q^7(1+q^4)}{(1-q^3)(1-q^{14})(1-q^{15})}\right), \tag{6.6}$$

and for $n \geq 3$

$$\begin{aligned} W(n,0) = \frac{1}{H_{n-1}}\Bigg(&\frac{q(1+q^4)(1+q^{8n-3}) + q^7(1+q^3)}{(1-q^{10})(1-q^{8n-2})(1-q^{8n-1})} \\ &+ \frac{q^{13}(1-q^{8n-14})}{(1-q^3)(1-q^{10})(1-q^{8n-1})(1-q^{8n-2})}\Bigg)\end{aligned} \tag{6.7}$$

and the numberator factor $(1-q^{8n-14})$ cancels with th same factor in H_{n-1}. Hence the non-negativity of the coefficients of $W(n,0)$ is clear upon inspection.

Next for $n \geq 2$

$$W(n,1) = \frac{q^6(1+q^3-q^5-q^6-q^9-q^{10}+q^{12}+q^{15}}{H_{n-1}(1-q^5)(1-q^6)(1-q^9)(1-q^{13})(1-q^{8n-2})} \tag{6.8}$$

and

$$\begin{aligned} &1+q^3-q^5-q^6-q^9-q^{10}+q^{12}+q^{15} \\ &\quad = \frac{(1-q^5)(1-q^6)(1-q^9)+q^3(1-q^7)(1-q^{13})+q^{12}(1-q^5)(1-q^9)}{1+q^{11}}.\end{aligned} \tag{6.9}$$

So

(6.10)

$$W(n,1) = \frac{q^6\left\{(1-q^5)(1-q^6)(1-q^9)+q^3(1-q^7)(1-q^{13})+q^{12}(1-q^5)(1-q^9)\right\}}{H_{n-1}(1-q^5)(1-q^6)(1-q^9)(1-q^{13})(1-q^{8n-2})(1+q^{11})}$$

and in light of the facts that: (1) each factor $1-q^i$ in the numerator also appears either explicitly in the denominator or in H_{n-1}, and (2) the factor of $1-q^{11}$ from H_{n-1} combines with $1+q^{11}$ to leave $1-q^{22}$ in the denominator, we see that $W(n,1)$ has non-negative coefficients for $n \geq 2$.

Now for $n > j \geq 2$.

$$
\begin{aligned}
W(n,j) &= \frac{H_{j-1}q^{8j-2}\left\{(1-q^5)(1-q^{8j+2})+q^3(1-q^3)(1-q^{8j-2})\right\}}{H_{n-1}(1-q^5)G_j(1-q^{8n-2})}\\
&= \frac{q^{8j-2}}{(q^{8j-1},q^{8j+3};q^8)_{n-j}(q^{8j+10};q^8)_{n-j-1}G_j(1-q^{8n-2})}\\
&\quad + q^{8j-5}\left(\frac{q^6}{1-q^6}-\frac{q^9}{1-q^9}\right)\frac{1}{(q^{8j-1},q^{8j+2},q^{8j+3};q^8)_{n-j}}\\
&\qquad\qquad \times \frac{1}{(1-q^5)(q^{14};q^8)_{j-2}(q^{17};q^8)_{j-1}(q^{13};q^8)_j}\\
&:= W_1(n,j)+W_2(n,j) \qquad (6.11)
\end{aligned}
$$

Now it is immediate that $W_1(n,j)$ has non-negative coefficients. Also because $1-q^6$ is in the denominator, we see that the coefficient of q^{8j+4} is ≥ 1. In addition, because the only factors in the denominator with exponents ≤ 11 are $(1-q^6)$ and $(1-q^9)$ we see that the coefficient of q^{8j+9} in $W_1(n,j)$ is zero.

We are now in a position to show via an injection involving $W_1(n,j)$ that $W(n,j)$ has only one negative coefficient which is -1 and occurs for q^{8j+9}. This requires an analysis analogous to that in Section 3.

We define for $n > j \geq 2$

$$
\begin{aligned}
\Sigma(n,j) := \{&5,13,14,17,21,\ldots,8j-10,8j-7,8j-3,8j-1,8j+1,8j+2,\\
&8j+3,8j+5,\ldots,8n-9,8n-6,8n-5,8n-2\}
\end{aligned}
$$

In other words, the elements of $\Sigma(n,j)$ are the numbers that appear as exponents in the factors $1-q^x$ making up the denominator of $W_1(n,j)$ (excluding 6 and 9).

We shall say that *6-partitions* (*a new definition from that in Section 3*) are partitions whose parts lie in $\{6\}\cup\Sigma(n,j)$ with the condition that at least one 6 is a part.

We shall say that *9-partitions* are partitions whose parts lie in $\{9\}\cup\Sigma(n,j)$ with the condition that at least one 9 is a part.

We let $P6_{n,j}(m)$ (resp. $P9_{n,j}(m)$) denote the number of 6-partitions (resp. 9-partitions) of m. We use capital "P" so that this $P6$ will not be confused with the $p6$ of Section 3. Thus by (6.11) and the standard construction of product generating functions [2, p. 45], we see that for $n > j \geq 2$

$$
W_2(n,j) = q^{8j-5}\sum_{m\geq 0}\left(P6_{n,j}(m)-P9_{n,j}(m)\right)q^m. \qquad (6.12)
$$

Lemma 8. *For $m \geq 0$, $n > j \geq 2$*

$$
P6_{n,j}(m)-P9_{n,j}(m) = \begin{cases} -1 & \text{if } m = 9 \text{ or } 14\\ \geq 0 & \text{if } m \neq 9 \text{ or } 14\end{cases}
$$

Proof. Clearly for $m \geq 14$, $P6_{n,j}(m) = 0$ except at $m = 6$ and $m = 11$ $(= 6+5)$ when it is 1, and $P9_{n,j}(m) = 0$ except for 9 and 14 $(= 9+5)$. Thus Lemma 8 is proved for $m \leq 14$. From here on we assume $m > 14$.

We now construct an injection of the 9-partitions into the 6-partitoins of m to conclude our proof.

Case 1. There are $2k$ 9's in a given 9-partition, replace these by $3k$ 6's.

Case 2. There are $2k+1$ 9's in a given 9-partition (with $k>0$). Replace these with $(3k-2)$ 6's and one 21. (Note that there is 21 present in $\Sigma(n,j)$ because $j \geq 2$).

Case 3. The given 9-partition has exactly *one* 9. Since $m > 14$, there must either be at least two 5's in the partition or else a *second summand* (i.e. the least summand other than the one 9) coming from $\Sigma(n,j)$.

As in Section 3, we must replace the unique 9, the second summand (or the 5's), and perhaps one or two other summands by some 6's and some elements of $\Sigma(n,j)$ that are (except in the instances indicated with $(*)$) no larger than the orignal second summand with the added proviso that either

(i) the number of 6's is $\equiv 2 \pmod 3$

and

(ii) the number of 6's is $\equiv 1 \pmod 3$ and no 21 occurs in the image.

The table below provides the replacement in each case. As in Section 3, the first column describes the preimage partition; the single 9 and the second summand are given explicitly as the first two summands. After the few summands that are to be altered are listed, there is a parenthesis such as (≥ 14, no 21) which means that the remaining summands are taken from $\Sigma(n,j)$, all are ≥ 14 and 21 does not appear. The second column describes the image partition. The parts indicated parenthetically are unaltered by the mapping.

$$\begin{array}{lll}
 & \text{pre-image partition} & \longrightarrow \quad \text{image partition} \\
 & 9+5+5+(\geq 5, \text{ no } 21) & \longrightarrow \quad 6+13+(\geq 5, \text{ no } 21) \\
 & 9+5+5+21+(\geq 5) & \longrightarrow \quad 6+6+6+6+6+5+5+(\geq 5) \\
(*) & 9+14+(\geq 14, \text{ no } 21) & \longrightarrow \quad 6+17+(\geq 14, \text{ no } 21) \\
 & 9+14+21+(\geq 14) & \longrightarrow \quad 6+6+6+6+6+14+(\geq 14) \\
 & 9+17+(> 21) & \longrightarrow \quad 6+5+5+5+5+(> 21) \\
 & 9+17+21+(\geq 21) & \longrightarrow \quad 6+6+6+6+6+17+(\geq 21) \\
 & 9+21+(\geq 21) & \longrightarrow \quad 6+6+6+6+6+(\geq 21)
\end{array}$$

now for $i \geq 4$

$$\begin{array}{lll}
 & 9+(8i-7)+(\geq 8i-7) & \longrightarrow \quad 6+6+(8i-10)+(\geq 8i-7) \\
 & 9+(8i-3)+(\geq 8i-3) & \longrightarrow \quad 6+6+5+(8i-11)+(\geq 8i-3) \\
(*) & 9+(8i-2)+(\geq 8i-2) & \longrightarrow \quad 6+(8i+1)+(\geq 8i-2) \\
 & 9+(8j-1)+(\geq 8j-1) & \longrightarrow \quad 6+5+(8j-3)+(\geq 8j-1) \\
 & 9+(8j+2)+(\geq 8j+2) & \longrightarrow \quad 6+6+(8j-1)+(\geq 8j-2) \\
 & 9+(8j+3)+(\geq 8j+3) & \longrightarrow \quad 6+5+(8j+1)+(\geq 8j+3)
\end{array}$$

now for $i > j$

$$\begin{array}{ll}
9+(8i-1)+(\geq 8i-1) & \longrightarrow \quad 6+6+5+(8i-9)+(\geq 8i-1) \\
9+(8i+2)+(\geq 8i+2) & \longrightarrow \quad 6+5+5+(8i-5)+(\geq 8i+2) \\
9+(8i+3)+(\geq 8i+3) & \longrightarrow \quad 6+6+5+(8i-5)+(\geq 8i+3)
\end{array}$$

finally

$$9+(8n-2)+(\text{more } 8n-2\text{'s}) \quad \longrightarrow \quad 6+6+(8n-5)+(\text{more } 8n-2\text{'s})$$

The comments that followed the table in Section 3 are again relevant here. However, the task here is simpler. The subtle aspect treated in Section 3 was the concern with overlapping images. The two lines marked $(*)$ clearly do not coincide with each other nor with the other five lines that have a unique 6 in the image. This concludes the proof of Lemma 8. ☐

We are now positioned to conclude the proof of Theorem 3.

The case $n = 1$ follows directly from (6.4) and (6.5). The case $n = 2$ follows from (6.4), (6.6) and (6.10).

Now suppose $n > 2$. Then by (6.4)

$$\begin{aligned}\frac{1}{(q, q^5, q^6; q^8)_n} - \frac{1}{(q^2, q^3, q^7; q^8)_n} &= \sum_{j=0}^{n-1} W(n, j)\\ &= W(n, 0) + W(n, 1) + \sum_{j=2}^{n-1} W(n, j)\\ &= W(n, 0) + W(n, 1) + \sum_{j=2}^{n-1} \big(W_1(n, j) + W_2(n, j)\big). \qquad (6.13)\end{aligned}$$

By examining (6.7), we see that the coefficients of $W(n, 0)$ (for $n > 2$) are at least as large as those of

$$\frac{q^{13}}{(1-q^3)(1-q^{10})} = q^{13} + q^{16} + q^{19} + q^{22} + q^{23} + q^{25} + q^{26} + q^{28} + q^{29} + \frac{q^{31}}{1-q} + \frac{q^{43}}{(1-q^3)(1-q^{10})}. \qquad (6.14)$$

In particular, this means that the coefficient of q^{25} in $W(n, 0)$ is positive and all coefficients of q^{31} and higher powers are positive.

In addition, we know that the coefficients of $W(n, 1)$ are non-negative. We have also established that the coefficient of q^{8j+4} in $W_1(n, j)$ is at least 1. Lemma 6 establishes $W_2(n, j)$ has its only negative coefficients at q^{8j+4} and q^{8j+9} and that these negative coefficients are both -1. Thus $W(n, j)$ $(= W_1(n, j) + W_2(n, j))$ has at most one negative coefficient which occurs at q^{8j+9} and is, at worst, -1. These occur for $j \geq 2$, i.e., the sum in (6.13) has possibly -1's as a coefficient of q^{25}, q^{33}, q^{41},.... However the comments following (6.14) show that these -1's are all cancelled out by positive terms in $W(n, 0)$.

Hence there are no negative coefficients on the right-hand side of (6.13). Therefore Theorem 3 is proved.

7. Proof of Theorem 4

We proceed as in Section 5 where injections were unnecessary. We define

$$J_n := \prod_{\substack{j=1\\ j\not\equiv 0,\pm s \pmod{k}}}^{kn} \frac{1}{1-q^j} = \frac{(q^s, q^{k-s}, q^k; q^k)_n}{(q;q)_{kn}}$$

and

$$K_n := \prod_{\substack{j=1\\ j\not\equiv 0,\pm r \pmod{k}}}^{kn} \frac{1}{1-q^j} = \frac{(q^r, q^{k-r}, q^k; q^k)_n}{(q;q)_{kn}},$$

where $\frac{k}{2} > s > r \geq 1$, *and* we exclude the case where s is prime and $s = r + 1$ and $k = 3r + 2$ hold.

The object is to prove that $J_n - K_n$ has non-negative power series coefficients. Thus

$$
\begin{aligned}
&J_n - K_n \\
&= K_n \left(\frac{J_n}{K_n} - 1 \right) \\
&= K_n \sum_{j=1}^{n} \left(\frac{J_j}{K_j} - \frac{J_{j-1}}{K_{j-1}} \right) \\
&= K_n \sum_{j=1}^{n} \frac{J_{j-1}(1-q^{jk})}{K_j} \Big((1-q^{kj-k+5})(1-q^{kj-5}) - (1-q^{kj-k+5})(1-q^{kj-5}) \Big) \\
&= K_n \sum_{j=1}^{n} \frac{J_{j-1}(1-q^{jk})}{K_j} q^{jk-k+r}(1-q^{s-r})(1-q^{k-s-r}) \\
&= \frac{1}{(q)_{kn}} \sum_{j=1}^{n} (q^s, q^{k-s}; q^k)_{j-1} (q^{jk+r}, q^{jk+k-r}, q^k)_{n-j} q^{jk-k+r} (1-q^{s-r})(1-q^{k-s-r}).
\end{aligned}
$$

There are $2n$ binary factors of the form $1 - q^i$ in the numerator where $1 \leq i < kn$. If all of the numerator factors are distinct for each j, they will cancel with the corresponding terms in the denominator and the non-negativity of the coefficients will follow.

For every j we see that all the factors of

$$(q^s, q^{k-s}; q^k)_{j-1} (q^{jk+r}, q^{jk+k-r}; q^k)_{n-j}$$

are distinct. So our only worry is whether $(1 - q^{s-r})$ and $(1 - q^{k-s-r})$ can overlap with other terms.

We note that $s - r \neq k - s - r$ because $\frac{k}{2} > s$.

If $j = 1$, then $(1 - q^{s-r})$ and $(1 - q^{k-s-r})$ are the only terms with exponents $< k$ and so all coefficients in the $j = 1$ term are positive.

Next we note that

$$s - r < s < k - s,$$

and for $j > 1$ the terms with exponents less than k are

$$(1 - q^{s-r}),\ (1 - q^s),\ (1 - q^{k-s}),\ \text{and}\ (1 - q^{k-s-r}).$$

The only possible equality here occurs when $s = k - s - r$. So if $k \neq 2s + r$, then we have distinct factors in the numerator and the j^{th} term has non-negative coefficients.

Suppose that $k = 2s + r$ so that there are now two factors $(1 - q^s)$ in the numerator. Noting

$$\frac{(1-q^s)^2}{(1-q)(1-q^s)} = 1 + q + \cdots + q^{s-1},$$

we see that if the $(1 - q)$ has not been cancelled from the numerator, then the coefficients are again non-negative.

Thus the only way that we are in danger of having negative coefficients in any term is if $k = 2s + r$ and $(1 - q)$ is cancelled from the denominator by $(1 - q^{s-r})$.

I.e. the cases that can't be handled occur when both $k = 2s + r$ and $s - r = 1$, or $k = 3r + 2$ and $s = r + 1$.

This latter case can be handled if s is composite. Because then there is a $t \mid s$ with $1 < t < s$ so that

$$\frac{1 - q^s}{1 - q^t} = 1 + q^t + \cdots q^{s-t}.$$

Thus cancellation can still be managed in the $s = r + 1$, $k = 2s + r$ case if s is composite. Hence the only situation not accounted for is where s is prime, and $s = r + 1$ and $k = 2s + r = 3r + 2$. □

8. Conclusion

The method of anti-telescoping should be applicable in a variety of further problems. The obvious first extension would be the Gordon generalization of Rogers-Ramanujan [3]:

Conjecture 1. *For each $n \geq 1$ and $1 \leq j < i < \frac{k}{2}$,*

$$\frac{(q^i, q^{k-i}, q^k; q^k)_n}{(q;q)_{kn}} - \frac{(q^j, q^{k-j}, q^k; q^k)_n}{(q;q)_{kn}}$$

has non-negative power series coefficients.

The case $k = 5$, $i = 2$, $j = 1$ is Theorem 1. Theorem 4 takes care of most cases. The only open cases are for i prime and $i = j + 1$ with $k = 3j + 2$.

To make the method more easily applicable to results like Theorems 1 and 3, it would be of value to explore the following question:

Suppose that S is a set of positive integers and i and j are not in S. Let

$$T_1 = \{i\} \cup S$$
$$T_2 = \{j\} \cup S$$

with $i < j$. Let $p(S, n)$ denote the number of partitions of n whose parts are in S. Under what conditions can we assert that $p(T_1, n) \geq p(T_2, n)$ except for an explicitly given finite set of values for n.

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