Ramanujan for the Twenty First Century

by

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August 29, 2003

Abstract

This paper discusses Ramanujan as (1) an analyst (2) a formalist and (3) a computer. In each of these three categories we describe recent developments that owe much to Ramanujan's insight.

1 Introduction

In much of science, it is clear that decades old thought and ideas are passé. Fortunately mathematics does not suffer from this impermanence. Srinivasa Ramanujan (1887–1920) still has much to teach us. This is not surprising in light of G. H. Handy's evaluation [24, P. xxxv].

> "It is plainly impossible for me to attempt a reasoned estimate of Ramanujan's work. Some of it is very intimately connected with my own, and my verdict could not be impartial; there is much too that I am hardly competent to judge; and there is a mass of unpublished material, in part new and in part anticipated, in part proved and in part only conjectured, that still awaits analysis."

In this paper, we shall consider only three aspects of Ramanujan. In Section 2, we describe two identities from Ramanujan's Lost Notebook [24]

*Partially supported by National Science Foundation Grant DMS 9206993

whose recently discovered proof relies on recondite properties of entire functions. In Section 3, we look at two other identities from [24] that on their face are extremely perplexing; here we find that again there are underlying principles whose implications have yet to be fully explored. Section 4 examines a recent computer search algorithm that may well have been in Ramanujan's private arsenal. We conclude with some speculation about Ramanujan and the future.

2 Ramanujan as Analyst

This section concerns an aspect of Ramanujan's work that has only recently been verified [5], [6]. It concerns two identities that are so surprising to an insider that I avoided studying them for twenty five years out of fear. Here are these formidable identities. First from the middle of page 26 in Ramanujan's Lost Notebook [24]

$$\sum_{n=0}^{\infty} a^n q^{n^2} = \prod_{n=1}^{\infty} (1 + aq^{2n-1}(1 + y_1(n) + y_2(n) + \cdots)), \tag{2.1}$$

where

$$y_1(n) = \frac{\sum_{j=n}^{\infty}(-1)^j q^{j(j+1)}}{\sum_{j=0}^{\infty}(-1)^j (2j+1) q^{j(j+1)}} \tag{2.2}$$

and

$$y_2(n) = \frac{\left(\sum_{j=n}^{\infty}(j+1)(-1)^j q^{j(j+1)}\right)\left(\sum_{j=n}^{\infty}(-1)^j q^{j(j+1)}\right)}{\left(\sum_{j=0}^{\infty}(-1)^j (2j+1) q^{j(j+1)}\right)^2} \tag{2.3}$$

Our second result is the third identity on page 57 of [24]

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \left(1 + \frac{aq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \dots}\right), \tag{2.4}$$

where

$$y_1 = \frac{1}{(1-q)\psi^2(q)}, \tag{2.5}$$

$$y_2 = 0, \tag{2.6}$$

$$y_3 = \frac{q + q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3 \psi^6(q)}, \tag{2.7}$$

$$y_4 = y_1 y_3, \tag{2.8}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{2.9}$$

and where

$$(A;q)_n = (1-A)(1-Aq)(1-Aq^2)\cdots(1-Aq^{n-1}). \tag{2.10}$$

In both identities $0 < q < 1$.

It seems to have taken me forever to recognize that both these results lie in the realm of the theory of entire functions of the variable a.

To understand something of the depth of these discoveries it is necessary to provide at least an intuitive introduction to entire functions. First of all, an entire function is an analytic function of z that has no singularities in the finite portion of the z plane. Consequently, it has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that converges for all z.

We now turn to E. T. Copson [15, p. 158] who succinctly describes the analogy of entire functions and polynomials:

> "The most important property of a polynomial is that it can be expressed uniquely as a product of linear factors of the form
>
> $$Az^p\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\cdots\left(1-\frac{z}{z_n}\right),$$
>
> where A is a constant, p a positive integer or zero, and $z_1, z_2, \ldots, z_n$ the points, other than the origin, at which the polynomial vanishes, multiple zeros being repeated in the set according to their order. Conversely, if the zeros are given, the polynomial is determined apart from an arbitrary constant multiplier.
>
> Now a polynomial is an integral function [i.e. entire function] of a very simple type, its singularity at infinity being a pole. We naturally ask whether it is possible to exhibit in a similar manner the way in which any integral function depends on its zeros."

This prologue leads to the central fundamental theorm on entire functions, Hadamard's Factorization Theorem, [15, p.174]. The full theorem is not necessary for us. In fact, we need only the following special case.

Haramard's Factorization Theorem (weak case). Suppose $f(z)$ is an entire function with simple zeros at $z_1, z_2, z_3, \ldots$, $f(0) = 1$ and $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$, then

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

In each of (2.1) and (2.4), we note that each series defines an entire function of the complex variable a. In each identity, Ramanujan is presenting the Hadamard Factorization of the function in question and additionally is specifying explicit formulas, or at least approximations, for each of the zeros.

Once one understands that this is what is going on, generally we are still far from understanding why Ramanujan is able to make these assertions about the zeros of these functions.

The full details are presented in [5] and [6], but the main idea involved relies on polynomial approximations of these functions. For (2.4), the relevant polynomial sequence is

$$K_n(a) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j^2} a^j, \tag{2.11}$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{cases} 0 & \text{if } j < 0 \quad \text{or } j > n \\ 1 & \text{if } j = 0 \quad \text{or } n \\ \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-j+1})}{(1-q^j)(1-q^{j-1})\cdots(1-q)} & \text{otherwise.} \end{cases} \tag{2.12}$$

For (2.1), the relevant polynomial sequence is

$$p_n(a) = (q^2;q^2)_\infty (-aq;q^2)_n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 \frac{q^{2j}}{(-aq;q^2)_j}. \tag{2.13}$$

It turns out that

$$\lim_{n\to\infty} K_n(a) = \sum_{m=0}^{\infty} \frac{a^m q^{m^2}}{(q;q)_n}, \tag{2.14}$$

and

$$\lim_{n\to\infty} p_n(a) = \sum_{m=0}^{\infty} q^{m^2} a^m. \tag{2.15}$$

The sequence $K_n(a)$ are, in fact, the Stieltjes-Wigert polynomials. G. Szegö [25] had studied these at length as an interesting family of orthogonal polynomials. Indeed, his paper concludes by noting that the limiting function

$$\sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q;q)_n}$$

has (for $0 < q < 1$) real, negative simple zeros. This then is the obvious starting point for proving (2.4), and this is the tack taken in [5].

Proving (2.1) turns out to be much more problematic. Knowledge of q-hypergeometric series leads one inexorably to the sequence $p_n(x)$. However, they do NOT form a family of orthogonal polynomials. Consequently all of their important features must be deduced ex nihilo.

In light of the fact the (2.1) was first proved in March, 2003, it is not too surprising that very few of the implications of these formulas have been discerned. More important, it is clear that there are many more functions in the classical theory of q-series that are, in fact, entire functions of one of the variables. How many of these have expansions as intriguing as those in (2.1) and (2.4)? What are the combinatorial implications?

3 Ramanujan as Formalist

Here we turn again to G. H. Hardy for an assessment of Ramanujan [23; p.xxxv]:

> "It was his insight into algebraical formulae, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numericl examples; all of his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation, he combined a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that were often really startling, and made him, in his own peculiar field, without a rival in his day."

One might emphasize this by citing Hardy for specifics [23; p. xxxiv]

> "... it may be useful if I state, shortly and dogmatically, what seems to me Ramanujan's finest, most independent, and most characteristic work ... . It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities."

These are

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+3})(1-q^{5n+4})}. \tag{3.2}$$

Note that the series in (3.1) is the case $a = 1$ of the series in (2.4), and (3.2) is the case $a = q$ of the series in (2.4). Also it should be noted that the infinite product in (2.4) seems to be totally unrelated to the products in (3.1) and (3.2).

Hardy continues "... but here Ramanujan must take second place to Prof. Rogers; and if I had to select one formula from all of Ramanujan's work, I would agree with Major MacMahon in selecting ...

$$p(4) + p(9)x + p(14)x^2 + \cdots = \frac{5\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6},$$

where $p(n)$ is the number of partitions of n."

It is impossible to fault Hardy's judgement. Instead, I would like to add to the list some formulas from the Lost Notebook that stumped me for years and that eventually led to a number of surprising discoveries.

In the early part of the 18th century, L. Euler observed that for each integer n, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts. He provided a proof through generating functions. For subsequent purposes, we name the function in question $S(q)$. Euler observed

$$S(q) := \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{1-q^{2n+1}}. \tag{3.3}$$

Here is what Ramanujan does with $S(q)$ (taken from mid-page 14).

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n}$$

$$= 1 + \sum_{n=0}^{\infty} q^{n+1}(-1)^n (q;q)_n \tag{3.4}$$

$$= S(q) + 2\sum_{n=0}^{\infty} (S(q) - (-q;q)_n) - 2S(q)\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \tag{3.5}$$

$$= S(q) + 2\sum_{n=1}^{\infty} (S(q) - \frac{1}{(q;q^2)_n}) - 2S(q)\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \tag{3.6}$$

The last two lines here truly stunned me when I first gazed at them. The infinite series involving $S(q)$ are quite unlike anything I had seen before. In each of these series, Ramanujan is taking the difference between $S(q)$ and a partial product that converges to $S(q)$. Why in the world should this produce anything as orderly as these identities.

With a sense of great exhilaration I managed to prove these results in January of 1985. The proof appeared in 1986 [1].

Many years later, in 2000, Ken Ono rekindled my interest by noting that Don Zagier had proved a very similar result [26] and had applied it to an interesting L-function evaluation. Our subsequent explorations led us to the following very general result [8; p.403]:

Proposition 1 *Suppose that*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

is analytic for $|z| < 1$. If α is a complex number for which

(i) $\sum_{n=0}^{\infty} (\alpha - \alpha_n) < +\infty$,

(ii) $\lim_{n\to+\infty} n(\alpha - \alpha_n) = 0$,

then

$$\lim_{z\to 1^-} \frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty} (\alpha - \alpha_n).$$

From this general result, identities (3.4)–(3.6) are not that hard to prove.

In the ensuing time, Coogan, Lovejoy and Ono [13], [14], [19] have found many further applications of this result. Most recently, P. Freitas and I [7] have extended this result to the following theorem involving higher derivatives:

Proposition 2 *Let*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

be analytic for $|z| < 1$, and assume that for some positive integer p we have that

$$\sum_{n=0}^{\infty} \left[\prod_{j=1}^{p} (n+j) \right] (\alpha_{n+p} - \alpha_{n+p-1})$$

converges;

$$\sum_{n=0}^{\infty} \left[\prod_{j=1}^{p-1} (n+j) \right] (\alpha - \alpha_{n+p-1})$$

converges, where α is a fixed complex number such that

$$\lim_{n\to\infty} \left[\prod_{j=1}^{p} (n+j) \right] (\alpha_{n+p} - \alpha) = 0.$$

Then

$$\frac{1}{p} \lim_{z\to 1^-} \left\{ \frac{d^p}{dz^p} [(1-z)f(z)] \right\} = \sum_{n=0}^{\infty} \left[\prod_{j=1}^{p-1} (n+j) \right] (\alpha - \alpha_{n+p-1}).$$

We have applied this more general result to a wide variety of Ramanujan style identities. For example,

$$\sum_{n=0}^{\infty} \left[\frac{(t)_n}{(t)_\infty} - 1 \right] = \sum_{n=1}^{\infty} \frac{t^n}{(q)_n (1-q^n)},$$

$$\sum_{n=0}^{\infty} \left[\frac{(t)_n}{(q)_n} - \frac{(t)_\infty}{(q)_\infty} \right]^2 = \left[\frac{(t)_\infty}{(q)_\infty} \right]^2 \sum_{n=1}^{\infty} \frac{(q/t)_n}{(q)_n} \left[\frac{(q)_n}{(t)_n} - 1 \right] \frac{t^n}{1-q^n},$$

$$\sum_{n=0}^{\infty} \left[1 - \frac{(q)_\infty}{(q)_n} \right]^2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1-(q)_n}{1-q_n},$$

$$\sum_{n=0}^{\infty} \frac{(q)_\infty}{(q)_n} \left[1 - \frac{(q)_\infty}{(q)_n} \right] = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n(n+1)/2}}{1-q^n}.$$

4 Ramanujan and Computation

Anyone who studies Ramanujan's Lost Notebook comes away with some appreciation of the importance of computation in Ramanujan's work. Again G. H. Hardy provides a relevant comment [23; p. xxxv].

> "His memory, and his powers of calculation, were very unusual, but they could not reasonably be called "abnormal". If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or more accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper (15). This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general, slightly the quicker and more accurate of the two."

In a monograph on q series [2; p.87], I was led to speculate about Ramanujan and the age of computer algebra:

Sometimes when studying his work I have wondered how much Ramanujan could have done if he had had MACSYMA or SCRATCH-PAD or some other symbolic algebra package. More often I get the feeling that he was such a brilliant, clever, and intuitive computer himself that he really did not need them.

However much fun such speculation may be; it is more important that we examine the ways in which Ramanujan's computations and calculations have guided more recent discoveries.

Among the letters from Ramanujan that perhaps influenced Hardy's remark, we might mention the one on Ramanujan's empirical evidence for the Rogers-Ramanujan identities [24; pp. 360–361]. The study of the calculations led to a truncated version of the Bailey Chain method [3] which in turn led to positivity theorems about the differences of successive Gaussian polynomials [4].

However, there is perhaps a more revealing result in his early notebooks [12; p. 130]. In this identity, p_n denotes the nth prime. So $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$.

$$\prod_{n=1}^{\infty} \frac{1}{1-q^{p_n}} = 1 + \sum_{j=1}^{\infty} \frac{q^{p_1+p_2+\cdots+p_j}}{(1-q)(1-q^2)\ldots(1-q^j)}. \tag{4.1}$$

It should be noted that this assertion is FALSE, and also Ramanujan must have thought so because he drew a line through this formula. Here we face one of the few false results asserted by Ramanujan. The temptation is to forget (4.1) and move on to the countless valued discoveries by Ramanujan.

However, the question nags: Why would Ramanujan have written down (4.1) in the first place? In the late 1980's A. and J. Knopfmacher [18; Th. 1.4] proved the following theorem. Let $\mathcal{L} = \mathbf{C}((q))$ to the field of formal Laurent series over the complex numbers, $\mathbf{C}$. If

$$A = \sum_{n=\nu}^{\infty} \mathcal{C}_n q^n,$$

we call $\nu = \nu(A)$ the *order* of A and we define the *norm* of A to be

$$||A|| = 2^{-\nu(A)}.$$

In addition, we define the *integral part* of A by

$$[A] = \sum_{\nu \leqq n \leqq} \mathcal{C}_n \mathcal{Q}^n.$$

Theorem 1 *EXTENDED ENGEL EXPANSION THEOREM. Every $A \in \mathcal{L}$ has a finite or convergent (relative to the above norm) series expansion of the form*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n},$$

where $a_n \in \mathbf{C}[q^{-1}], a_0 = [A]$,

$$\nu(a_n) \leqq -n, \qquad \textit{and} \qquad \nu(a_{n+1}) \leqq \nu(a_n) - 1.$$

The series (1.4) is unique for A (up to constants in $\mathbf{C}$), and it is finite if and only if $A \in \mathbf{C}(q)$. In addition, if

$$a_0 + \sum_{j=1}^{n} \frac{1}{a_1 \cdots a_j} = \frac{p_n}{q_n}, \qquad \textit{where} \qquad q_n = a_1 a_2 \cdots a_n,$$

then

$$\left\| A - \frac{p_n}{q_n} \right\| \leqq \frac{1}{2^{n+1} \|q_n\|}$$

and

$$\nu\left(A - \frac{p_n}{q_n}\right) = -\nu(q_{n+1}) \geqq \frac{(n+1)(n+2)}{2}.$$

In fact, the a_n are given by

$$a_n = \left[\frac{1}{A_n}\right]$$

where $A_0 = A,\ a_0 = [A],$ and

$$A_{n+1} = a_n A_n - 1.$$

From this expansion theorem they observed empirically that the generalized Engel expansion for

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

is, in fact,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n},$$

i.e. they had found an empirical method that led to the First Rogers-Ramanujan identity (3.1). In subsequent work [9], it was shown that this observation can be rigorously established. In addition, while (3.1) is false, nonetheless, the generalized Engel expansion of the left side of (4.1) yields precisely the first four terms of the right side before chaos sets in in subsequent terms.

Thus while we cannot be certain how Ramanujan came upon either (3.1) or (3.2) or (4.1), we can note that our knowledge that "something's going on here" provides inspiration for the exploitation of results like the generalized Engel expansion.

As a final comment in this section, I hark back to Hardy's comparison of the computational skills of Ramanujan and P.A. MacMahon. Certainly at the beginning of the twentieth century MacMahon was the one other major researcher applying computational studies to problems in the theory of partitions.

MacMahon was perhaps most proud of his method of Partition Analysis [20; Sec. VIII]. The method fell into disuse because of the massive calculations necessary in its employment. However, with the advent of computer algebra system, this method has flourished. The most direct applications of

MacMahon's ideas are developed in [10]. Related systems have been developed by Stembridge [22] implementing the ideas of Stanley [21] and DeLoera [16] implementing Barvinok's work [1].

5 Conclusion

In this brief survey, I have touched a bit on the ways in which Ramanujan is influencing research more than 80 years after his death.

In a somber note [17; p. 14], G. H. Hardy remarked:

> "It is possible that the great days of formulae are finished, and that Ramanujan ought to have been born 100 years ago; but he was by far the greatest formalist of his time. There have been a good many more important, and I suppose one must say greater, mathematicians than Ramanujam during the last fifty years, but not one who could stand up to him on his own ground. Playing the game of which he knew the rules, he could give any mathematician in the world fifteen."

Owing to the development of computers and the concomitant interest in combinatorial and discrete mathematics, it is not unreasonable to conclude with the speculation that interest in and application of Ramanujan's work will continue to grow in the coming century. Indeed, Hardy's gloomy assessment is surely far off the mark.

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