Reduction of Linear Programming to Linear Approximation

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Now we recall relevent definitions.

An *affine function* of variables $x_1, \ldots, x_n$ is $b_0 + c_1x_1 + \cdots + c_nx_n$ where b_0, c_i are given numbers.

A *linear constraint* is any of the following constraints: $f \le g, f \ge g, f = g$, where f, g are affine functions.

A *linear program* is an optimization (maximization or minimization) of an affine function subject to a finite system of linear constraints.

An l^∞ linear approximation problem, also known as (discrete) *Chebyshev approximation problem* or finding the least-absolute-deviation fit, is the problem of minimization of the following function:

$$\max(|f_1|, \ldots, |f_m|) = \|(f_1, \ldots, f_m)\|_\infty,$$

where f_i are affine functions. This objective function is piece-wise linear and convex.

Given any Chebyshev approximation problem, here is a well-known reduction (Vaserstein, 2003) to a linear program with one additional variable t:

$$t \to \min, \text{ subject to } -t \le f_i \le t \text{ for } i = 1, \ldots, m.$$

This is a linear program with $n+1$ variables and $2m$ linear constraints.

Now we want to reduce an arbitrary linear program to a Chebyshev approximation problem. First of all, it is well known (Vaserstein, 2003) that every linear program can be reduced to solving a symmetric matrix game.

So we start with a matrix game, with the payoff matrix $M = -M^T$ of size N by N. Our problem is to find a column $x = (x_i)$ (an optimal strategy) such that

$$Mx \le 0, x \ge 0, \sum x_i = 1. \tag{1}$$

As usual, $x \geq 0$ means that every entry of the column x is ≥ 0. Later we write $y \leq t$ for a column y and a number t if every entry of y is $\leq t$. We go even further in abusing notation, denoting by $y - t$ the column obtaining from y by subtracting t from every entry. Similarly we denote by $M + c$ the matrix obtained from M by adding a number c to every entry.

This problem (1) (of finding an optimal strategy) is about finding a feasible solution for a system of linear constraints. It can be written as the following linear program with an additional variable t:

$$t \to \min, Mx \leq t, x \geq 0, \sum x_i = 1. \tag{2}$$

Now we find the largest entry c in the matrix M. If $c = 0$, then $M = 0$ and the problem (1) is trivial (every mixed strategy x is optimal). So we assume that $c > 0$.

Adding the number c to every entry of the matrix M, we obtain a matrix $M + c \geq 0$ (all entries ≥ 0). The linear program (2) is equivalent to

$$t \to \min, (M + c)x \leq t, x \geq 0, \sum x_i = 1 \tag{3}$$

in the sense that these two programs have the same feasible solutions and the same optimal solutions. The optimal value for (2) is 0 while the optimal value for (3) is c.

Now we can rewrite (3) as follows:

$$\|(M + c)x\|_\infty \to \min, x \geq 0, \sum x_i = 1 \tag{4}$$

which is a Chebyshev approximation problem with additional linear constraints. We used that $M + c \geq 0$, hence $(M + c)x \geq 0$ for every feasible solution x in (2). The optimal value is still c.

Now we rid off the constraints in(4) as follows:

$$\| \begin{pmatrix} (M + c)x \\ c - x \\ \sum x_i + c - 1 \\ -\sum x_i - c + 1 \end{pmatrix} \|_\infty \to \min . \tag{5}$$

Note that the optimization problems (4) and (5) have the same optimal value c and every optimal solution of (4) is optimal for (5). Conversely, for every x with a negative entry, the objective function in (5) is $> c$. Also, for every x with $\sum x_i \neq 1$, the objective function in (5) is $> c$. So every optimal solution for (5) is feasible and hence optimal for (4).

Thus, we have reduced solving any symmetric matrix game with $N \times N$ payoff matrix to a Chebyshev approximation problem (5) with $2N + 2$ affine functions in N variables.

Remark. It is well known that every l^1 linear approximation problem can be reduced to a linear program. Our result implies that every l^1 linear approximation problem can be reduced to a l^∞ linear approximation problem. I do not know whether the converse is true.

Note that our reduction of the l^1 linear approximation problem

$$\sum_{i=1}^{m} |f_i| \to \min \tag{6}$$

where f_i are affine functions in n variables, produces first the well-known linear program (Vaserstein, 2003)

$$\sum_{i=1}^{m} t_i \to \min, -t_i \leq f_i \leq t_i$$

with $m + n$ variables and $2m$ linear constraints, then a symmetric game with the payoff matrix of size $(3m + 2n + 1) \times (3m + 2n + 1)$, and finally a Chebyshev approximation problem with $6m + 4n + 4$ affine functions in $3m + 2n + 1$ variables.

By comparison, an obvious direct reduction produces

$$\max |f_1 \pm f_2 \pm \cdots \pm f_m| \to \min$$

which is a Chebyshev approximation problem with 2^{m-1} affine functions in n variables. So this reduction increases the size exponentially, while our reduction increases size linearly.

References

Vaserstein, L. N. (2003), *Introduction to Linear Programming,* Prentice Hall. (There is a Chinese translation by Mechanical Industry Publishing House ISBN: 7111173295.)