7. Functors K_0, K_2, and algebraic K-theory

Functor K_1

In Section 3, we made a commutative group $\mathrm{K}_1 A$ from invertible matrices over A. Namely, two matrices $\alpha \in \mathrm{GL}_m A$ and $\beta \in \mathrm{GL}_n A$ are equivalent if the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1_{n+k} \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & 1_{m+k} \end{pmatrix} \in \mathrm{GL}_{m+n+k} A$$

can be reduced to each other by row addition operations for some k. The group $\mathrm{K}_1 A$ consists of the equivalence classes.

Following the crowd and contrary to what we did before we will use the additive notation for the group operation on $\mathrm{K}_1 A$. Thus,

$$\mathrm{wh}(\alpha\beta) = \mathrm{wh}(\alpha) + \mathrm{wh}(\beta) = \mathrm{wh}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

for $\alpha\,\beta \in \mathrm{GL}_n A$. The second equality follows from the Whitehead lemma (see (1.5), and it shows that the group $\mathrm{K}_1 A$ is commutative.

In particular, $\mathrm{K}_1 F$ is the multiplicative group of F written additively. If you do not like that now $\mathrm{wh}(1) = 0$, make additive shift on A by 1, which makes the group operation on the multiplicative group look like $(a, b) \mapsto a \times b = a + b + ab$; after the shift, $\mathrm{wh}(\alpha \times \beta) = \mathrm{wh}(\alpha) + \mathrm{wh}(\beta)$ and $\mathrm{wh}(0) = 0$ (as a bonus, $\mathrm{GL}_n B$ becomes a subset of $\mathrm{M}_n B$ for any ideal B of A which makes it clear that $\mathrm{GL}_n B$ depends only on the ring B).

For any morphism $f : A \to A'$ of rings with 1 and any n, we have the induced morphisms $\mathrm{M}_n A \to \mathrm{M}_n A'$, $\mathrm{M}A \to \mathrm{M}A'$ of rings and the induced morphisms $\mathrm{GL}_n A \to \mathrm{GL}_n A'$, $\mathrm{E}_n A \to \mathrm{E}_n A'$ of groups, hence also group morphisms $\mathrm{GL}A \to \mathrm{GL}A'$, $\mathrm{E}A \to \mathrm{E}A'$, and $\mathrm{K}_1 A \to \mathrm{K}_1 A'$.

All these functors as well as any functor F from the the category of associative ring with 1 (with morphisms taking 1 to 1) to the category of groups can be extended to the category of all associative rings as follows: $\mathrm{F}(A) = \ker(\mathrm{F}(A_1) \to \mathrm{F}\mathbf{Z})$ where A_1 is the ring obtained by adjointing 1 formally to the ring A.

The relative group $\mathrm{F}(A, B)$ where A is a ring with 1 and B an ideal of A can be defined as $\ker(\mathrm{F}A' \to \mathrm{F}A)$ where A' is the double of A along B.

Functor K_0

Now we will make another commutative group, $\mathrm{K}_0 A$ from idempotent matrices $p = p^2$ over A. Namely, two matrices $p = p^2 \in \mathrm{M}_m A$ and $q = q^2 \in \mathrm{M}_n A$ are called equivalent if $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_{m+n+k} A$ and $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_{m+n+k} A$ are similar for some $k \geq 0$. The equivalence classes $\mathrm{P}A$ can be added using $p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$. So $\mathrm{P}A$ is a commutative semigroup.

For example, if A is a field then for any n every matrix $p = p^2$ in $\mathrm{M}_n\mathrm{A}$ is similar to $\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}$ where $r \leq n$ is the rank of p. So $\mathrm{P}A$ can be identified with the semigroup $\mathbf{N}$ of all non-negative integers. The same is true when A is any division ring or the integers $\mathbf{Z}$.

The column space pA^n of any idempotent matrix $p = p^2$ is a finitely generated projective module (that is, a module which is a direct summand of a free module A^n). In fact,

$pA^n \oplus (1-p)A^n = A^n$, the n-columns over A. Also every finitely generated projective module is the column space of an idempotent matrix.

So $\mathrm{P}A$ is the set of isomorphism classes of finitely generated projective A-modules with the addition induced by direct sum.

If A is the ring of continuous real (resp., complex) valued functions on a compact space X, then $\mathrm{P}A$ consists of isomorphism classes of real (resp., complex) vector bundles over X (Swan [Sw]). For an arbitrary topological space X, the category $\mathrm{P}A$ is equivalent to the category of all vector bundles of finite type over X, see [V61].

A similar result for algebraic varieties is due to Grothendieck (and published by Borel-Serre [BS]).

Now we can make a commutative group, K_0A from the commutative semigroup $\mathrm{P}A$ like we make the integers from non-negative integers $\mathbf{N}$. Namely, $\tilde{p}-\tilde{q} = \tilde{p}'-\tilde{q}'$ if $p+q'\oplus 1_k$ and $p'+q\oplus 1_k$ are equivalent for some integer $k \geq 0$. In the case of $\mathbf{N}$, we can restrict p or q to any infinite set, and we do not need the term 1_k. In general, we can restrict p or q to any infinite set of 1_k's, and it may happen that different elements in $\mathrm{P}A$ becomes the same in K_0A. Namely, two A-modules P and Q have the same image in K_0A if and only if they are *stably isomorphic*, i.e., $P \oplus A^k$ and $Q \oplus A^k$ are isomorphic for some k.

If P is "big" in one sense or another, stable isomorphism implies isomorphism. In other words, A^k can be cancelled in the isomorphism $P \oplus A^k \cong Q \oplus A^k$. Here is an easy example of a cancellation theorem.

Theorem 7.1. Let A be an associative ring with 1, $n \leq \mathrm{sr}(A) < \infty$, and P is a right A-module which contains A^{n-1} as a direct summand. Assume also that GL_nA acts transitively on Um_nA (e.g., $n=1$ or A is commutative and $n=2$). If Q is a right A-module and $P \oplus A^k \cong Q \oplus A^k$ for some k then $P \cong Q$.

Proof. Proceeding by induction on k we see that it suffice to handle the case $k=1$. Applying the isomorphism $P \oplus A^1 \cong Q \oplus A^1$ to a generator of A^1 in $Q \oplus A^1$ we obtain an unimodular element $p + a = \begin{pmatrix} p \\ a \end{pmatrix} \in P \oplus A^1$ with $p \in P$ and $a \in A^1 = A$ such that $(P \oplus A^1)/(p+a)A \cong Q$.

The unimodularity of $p+a$ means that there is an A-linear homomorphism

$$f : P \oplus A^1 \to A$$

such that $f(p+a) = 1$. Set $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in P \oplus A^1$ where $0 \in P$ and $1 \in A^1 = A$. This element is also unimodular and $(P \oplus A^1)/vA = P$. We will show that an authomorphism of the A-module $P \oplus A^1$ takes $p+a$ to v.

By the condition of the theorem, we can write $P \oplus A^1 = P' \oplus A^n$ and $p + a = p' + b$ with $p' \in P'$ and $b \in A^n$. The element $v \in P' \oplus A^n$ looks like $\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ where $e_n \in A^n$ is the last column of 1_n. The column $\begin{pmatrix} b \\ fp' \end{pmatrix} \in A^{n+1}$ is unimodular, so by the condition of the theorem there is $q \in A^n$ such that $b + qfp' \in \mathrm{Um}_nA$. Also there is an isomorphism of A^n which takes $b + qfp'$ to e_n.

Now we can define an automorphism of $P \oplus A^1 = A^n \oplus P'$ which takes $p + a = \begin{pmatrix} b \\ p' \end{pmatrix}$ to $v = \begin{pmatrix} e_n \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} b \\ p' \end{pmatrix} \mapsto \begin{pmatrix} b + qfp' \\ p' \end{pmatrix} \mapsto \begin{pmatrix} e_n \\ p' \end{pmatrix} \mapsto \begin{pmatrix} e_n \\ 0 \end{pmatrix}.$$

This gives an isomorphism between

$$(P \oplus A^1)/(p+a)A \cong Q$$

and

$$(P \oplus A^1)/vA = (P \oplus A^1)/A^1 \cong P.$$

QED.

Remark. We did not require P or Q in the theorem to be projective or finitely generated. When $\mathrm{sr}(A) = 1$, we have cancellation without any conditions on P, Q. The condition on transitivity in the theorem can be replaced by the condition that P contains A^n as a direct summand.

Remark. When A is commutative, the tensor product gives a commutative ring structure on the additive group $\mathrm{K}_0 A$. Moreover, symmetric and exterior powers of modules give an additional structure on $\mathrm{K}_0 A$ turning it into a λ-ring.

Example. Now we consider the case when A is the ring of integers in a number field F (so F is a finitely dimensional over the rational numbers). If the class number of F is 1, i.e., every ideal of A is principal, the situation is the same as for the integers, so $\mathrm{P}A = \mathbf{N}$ and $\mathrm{K}_0 A = \mathbf{Z}$.

In general, every element in $\mathrm{P}A$ is represented by $A^k \oplus B$ with some $k \geq 0$ and a nonzero ideal B of A. Two ideals B and B' are isomorphic as A-modules if and only if they differ by a nonzero factor in F. Note that $B \oplus B'$ is isomorphic to $A \oplus BB'$. Thus, $\mathrm{K}_0 A$ can be identified with $\mathbf{Z} \oplus \mathrm{CL}(A)$ where where $\mathrm{Cl}(A)$ is the ideal class group (the group of all nonzero fractional ideals modulo the subgroup $\mathrm{GL}_1 F$ of principal nonzero fractional ideals).

The same description of $\mathrm{K}_0 A$ holds in the more general case when A is any Dedekind domain, i.e., A is a commutative domain such that for any pair of ideals $C \subset B$ of A there is an ideal D of A such that $B = CD$. See [M, §1] for details. While the group $\mathrm{Cl}(A)$ is finite for any Dedekind ring of arithmetic type (Hasse domain, i.e., when A is the S-integers in a global field), it could be infinite for some other Dedekind domains.

Functor K_2

Now we define our next commutative group $\mathrm{K}_2 A$. Let $\mathrm{St}A$ be the group generated by symbols $a^{i,j}$ where $a \in A$, $i \neq j$, subject to the relations (1.7), (1.11) and (3.2).

Since the relations hold for elementary matrices, we have a homomorphism $\mathrm{St}A \to \mathrm{E}A$. The group $\mathrm{K}_2 A$ is defined as the kernel of this homomorphism. An analog of Whitehead lemma is that $\mathrm{K}_1 A$ is the center of the group $\mathrm{St}A$.

In homological terms, $\mathrm{K}_1 A = \mathrm{H}_1(\mathrm{GL}A)$ and $\mathrm{K}_2 A = \mathrm{H}_2(\mathrm{E}A)$. In other words, $\mathrm{K}_2 A$ is the kernel of an universal central extension $\mathrm{St}A \to \mathrm{E}A$ of $\mathrm{E}A$. Shur introduced universal central extensions for finite groups G so $\mathrm{H}_2(G)$ is often called the Shur multiplier of G for any group G.

Informally speaking, $\mathrm{K}_2 A$ describes all nontrivial relations among the elementary matrices. In the case when $A = F$ is a field, all relations come from relations between diagonal matrices arising as products of 4 elementary matrices as in (1.5). So the group $\mathrm{K}_2 F$ in this case is generated by the symbols $x \cdot y$ with $x, y \in \mathrm{GL}_1 F = F \setminus 0$ which satisfy the relations

(7.1) $(x-y)\cdot z = x\cdot z - y\cdot z,\ x\cdot y = -y\cdot x,$ and $u\cdot(1-u) = 0$ for all $x, y, z, u \in \mathrm{GL}_1 F,\ u \neq 1.$

(In $\mathrm{K}_1 F$ we use now $x - y$ instead of x/y but then how we write $1 - u$?)

Matsumoto proved that for any field F the group $\mathrm{K}_2 F$ is generated by the symbols $x\cdot y$ subject to the defining relations (7.1). Keune [Ke1] showed that this is equivalent easily to Theorem 6.7 with $n = 2, A = F[t], B = A(t^2 - t)$ in which case $\mathrm{K}_2 F = \mathrm{SK}_1(A, B)$. For this Dedekind ring A, the special case of the theorem proved by Bass-Milnor-Serre is sufficient. The theorem describes $\mathrm{K}_2 F$ in terms of Mennicke symbols which gives the Matsumoto theorem. Later Keune [Ke2] gave (a more complicated) presentation of $\mathrm{K}_2 A$ for any ring A with $\mathrm{sr}(A) = 1$.

The multiplication $\mathrm{K}_1 A \times \mathrm{K}_1 A \to \mathrm{K}_2 A$ which generalizes the above multiplication for fields, can be defined [M] for any commutative ring A but in general $K_2 A$ is not generated by the image and other additional relations may appear.

For a finite field $\mathbf{F}_q$, it is known that $\mathrm{K}_2 \mathbf{F}_q = 0$,

For the integers $\mathbf{Z}$, the group $\mathrm{K}_2 \mathbf{Z}$ has order 2, and the only non-trivial element is (-1)$\cdot$(-1) where -1 is the only non-trivial element of $\mathrm{K}_1 \mathbf{Z}$. The group $\mathrm{K}_2 \mathbf{Z}$ survives in $\mathrm{K}_2 \mathbf{R}$ and gives a generator of the fundamental group $\pi_1 \mathrm{SL}_n \mathbf{R}, n \geq 3$, where $\mathrm{SL}_n \mathbf{R}$ is considered as a topological space with usual Hausdorff topology.

The group $\mathrm{K}_2 F$ for global fields F is connected with reciprocity laws so it is of great interest in number theory. A complete answer is still unknown. The following exact sequence is known (as Moore's reciprocity law):

$$\mathrm{K}_2 F \to \oplus \mu(F_v) \to \mu(F) \to 0$$

where $\mu(F')$ is the group of roots of 1 in F', F_v a completion of F at a place v, the direct sum is taken over all finite and real palaces v of F, the first homomorphism is defined using the norm-residue symbols (generalizing the Hilbert symbol), and the homomorphism $\mu(F_v) \to \mu(F)$ is raising to $[\mu(F_v) : \mu(F)]$-th power. The kernel of the first homomorphism is shown to be trivial for some F including $F = \mathbf{Q}$ and $F = \mathbf{F}_q(t)$. Thus, for those F, the group $\mathrm{K}_2 F$ is computed. The groups $\mathrm{K}_2 F_v$ for the local fields (including $\mathbf{C}$) are uncountable, $\mu(F_v$ representing their "continuous" parts (the "continuous" part of $\mathrm{K}_2 \mathbf{C}$ being trivial).

Exact sequence

The group $\mathrm{K}_0, \mathrm{K}_1, \mathrm{K}_2$ are related by an exact sequence. For example for an ideal B of A we have an exact sequence

$$\mathrm{K}_2 A \to \mathrm{K}_2(A/B) \to \mathrm{K}_1(A, B) \to \mathrm{K}_1 A \to \mathrm{K}_1(A/B) \to \mathrm{K}_0(A, B) \to \mathrm{K}_0 A \to \mathrm{K}_0(A/B).$$

Here $\mathrm{K}_1(A, B)$ is a familiar group. It also can be defined as the kernel of the homomorphism $\mathrm{K}_1 A' \to \mathrm{K}_1(A/B)$ induced by the first (or second) projection, where $A' = \{(a, a') \in A \times A : a - a' \in B\}$ is the double of A along the ideal B.

The group $\mathrm{K}_0(A, B)$ in the exact sequence can be defined as the kernel of the homomorphism $\mathrm{K}_0 A' \to \mathrm{K}_0(A/B)$ induced by the first (or second) projection $\mathrm{K}_0(A, B)$. This group depends only on the ring B (while $\mathrm{K}_1(A, B)$ depends also on A).

The homomorphism $\mathrm{K}_2(A/B) \to \mathrm{K}_1(A, B)$ is easy to describe. Given any relation

$$(a'_1)^{i(1),j(1)} (a'_2)^{i(2),j(2)} \cdots (a'_m)^{i(m),j(m)} = 1$$

between elementary matrices $(a'_k)^{i(k),j(k)}$ over A/B, we represent $a'_k = a_k + B \in A/B$ by $a_k \in A$ and obtain

$$\alpha = (a_1)^{i(1),j(1)} (a_2)^{i(2),j(2)} \cdots (a_m)^{i(m),j(m)} \in \mathrm{GL}B.$$

If we change representatives, the matrix α does not change modulo $\mathrm{E}(A, B)$. The same is true when we change the relation using the trivial relations (1.7), (1.11) and (3.2). So we obtain a well-defined $\mathrm{wh}(\alpha) \in \mathrm{K}_1(A, B)$.

Higher K-theory

The above exact sequence can be extended to an infinite in both direction exact sequence involving groups $\mathrm{K}_m A$ which can be defined for all integers m in several ways which agree with each other.

Quillen's [Q] definition of the functors K_m with $m \geq 3$ are rather complicated so we do not give them here. He computed the higher K-theory of finite fields. Namely, for a finite field F with q elements and $i > 0$, we have $\mathrm{K}_{2i}(F) = 0$ and the group $\mathrm{K}_{2i-1}(F)$ is cyclic of order $q^i - 1$.

Later Quillen [Q2] gave a different definition for the same functors K_m.

Meanwhile, Volodin [V] gave a different definition for the same K_m which we outline now. Let A be an associative ring with 1. A subgroup of $\mathrm{GL}A$ is called triangular if it is conjugated by a permutation matrix to the group of the upper triangular matrices with ones along the main diagonal. A finite subset S of $\mathrm{GL}A$ is called a simplex if gS belongs to a triangular subgroup for some $g \in \mathrm{GL}A$. Thus, $\mathrm{GL}A$ becomes an abstract simplicial complex. Then $\mathrm{K}_n A = \pi_{n-1}\mathrm{GL}A$ for $n \geq 0$. The homotopy groups can be defined using the geometric realization or combinatorially.

For example, the connected component $\mathrm{K}_1 A = \pi_0 \mathrm{GL}A$ is defined using paths where a path from g to h is a sequence of invertible matrices g_i with $i \geq 0$ such that $g_0 = g, g_i = h$ for sufficiently large i, and $g_{i+1}^{-1} g_i$ is triangular for all $i \geq 0$.

Since every triangular matrix is a product of elementary matrices and every elementary matrix is triangular, every path can be refined to a path with all $g_{i+1}^{-1} g_i$ being elementary matrices, hence two matrices g, h can be connected if and only if they can be brought to each other by column addition operations. Thus, $\pi_0 \mathrm{GL}A = \mathrm{GL}A/\mathrm{E}A = \mathrm{K}_1 A$.

In general, a loop on an abstract simplicial complex X with a base point 1 is a path from 1 to 1. A simplex on the set of ΩX of the loops is a finite set $\{(x_{0,i}), \ldots, (x_{d,i})\}$ of loops such that the subset $\{x_{0,i}, \ldots, x_{d,i}, x_{0,i+1}, \ldots, x_{d,i+1}\}$ is a simplex for every i. The selected loop is the constant loop.

Then $\pi_n X = \pi_0 \Omega^n X$. In the case $X = \mathrm{GL} A$, a loop is a relation $g_1 g_2 \cdots g_d = 1$ with triangular matrices g_i. A relation is trivial if all matrices in it belong to a triangular subgroup or it is a product of those relations. All defining relations for the Steinberg group are trivial, Every trivial relation follows from the defining relations for $\mathrm{K}_2 A$. Therefore $\pi_1 \mathrm{GL} A = \mathrm{K}_2 A$.

The higher K-theory is not completely computed even for the integers $\mathbf{Z}$. The group $\mathrm{K}_3 \mathbf{Z}$ is known [LS] to be cyclic of order 48, and $\mathrm{K}_4 \mathbf{Z}$ is trivial [Ro].

When A is the ring of integers in a number field F, i.e., $n = \dim_{\mathbf{Q}}(F) < \infty$, Quillen [Q1] proved that $\mathrm{K}_i A$ is finitely generated for every i and Borel [B] computed the rank of those groups for $i \geq 1$. Recall that for $i = 1$ we have $\mathrm{K}_1 A = \mathrm{GL}_1 A$ by Mennicke and Bass-Milnor-Serre [BMS] and that by the Dirichlet theorem on units, the rank of $\mathrm{GL}_1 A$ is $r_1 + r_2 - 1$ where r_1 is the number of embeddings of F to $\mathbf{R}$ and $2r_2$ is the number of embeddings of F to $\mathbf{C}$ so $n = r_1 + 2r_2$.

For any $i \geq 1$, the rank is:

0 for even $i \geq 2$,

$r_1 + r_2$ when $i \equiv 1 \bmod 4$,

r_2 when $i \equiv 3 \bmod 4$.

There are conjectures relating the groups $\mathrm{K}_i A$ with the values of the zeta-function of F.

Lower (negative) K-theory and the "fundamental theorem"

The functors K_m with $m \leq -1$ can be defined inductively (H. Bass [B2]) as follows: $\mathrm{K}_{m-1} A$ is the cokernel of the homomorphism

$$\mathrm{K}_m(A[t]) \oplus \mathrm{K}_m(A[1/t]) \to \mathrm{K}_m(A[t, 1/t]).$$

Then

$$\mathrm{K}_m(A[t, 1/t]) = \mathrm{K}_m A \oplus \mathrm{K}_{m-1} A \oplus \mathrm{Nil}_m A \oplus \mathrm{Nil}_m A$$

for all m where $\mathrm{Nil}_m A$ is the cokernel of

$$\mathrm{K}_m A \to \mathrm{K}_m(A[t])$$

(the second $\mathrm{Nil}_m A$ comes from $A[1/t]$). This formula for $\mathrm{K}_m(A[t, 1/t])$ is known as the fundamental theorem of algebraic K-theory [B2].

When $m = 1, \mathrm{Nil}_1 A$ is the subgroup of $\mathrm{K}_1 A$ consisting of the Whitehead determinants of unipotent matrices. When A is (right) *regular* in the sense that A is right Noetherian and every finitely generated right A-module admits a finite projective resolution, then $\mathrm{K}_m A = 0$ for all $m \leq -1$ and $\mathrm{Nill}_m A = 0$ for all m, hence $\mathrm{K}_m A[t] = \mathrm{K}_m A$ and $\mathrm{K}_m A[t, 1/t] = \mathrm{K}_m A \oplus \mathrm{K}_{m-1} A$ for all m.

Here is how the mapping $\mathrm{K}_0 A \to \mathrm{K}_1(A[t, 1/t])$ works. For $p = p^2 \in \mathrm{M}_n A$, we have $tp + (1_n - p) \in \mathrm{GL}_n A$. Taking the Whitehead determinant, we obtain an element of $\mathrm{K}_1(A[t, 1/t])$.

Here is a more general construction. For any A, any finitely generated projective A-module P, and any automorphism α of P, we consider P' such that $P \oplus P' = A^n$ and define an automorphism β of A^n, i.e., an invertible matrix, by $\beta(p \oplus p') = \alpha p \oplus p'$. So we get $\mathrm{wh}(\alpha) = \mathrm{wh}(\beta) \in \mathrm{K}_1 A$.

In particular, for any $\gamma \in \mathrm{GL}_n C$ (where C is the center of A) and any P we have an automorphism of P^n and hence an element of $\mathrm{K}_1 A$. This gives a "multiplication" map $\mathrm{K}_1 C \times \mathrm{K}_0 A \to \mathrm{K}_1 A$.

Linearization

We follow [BHS]. To prove that $\mathrm{K}_1 A[t] = \mathrm{K}_1 A$ for a regular A,, we want to do the followin reduction. Given a matrix $\alpha(t) \in \mathrm{GL} A[t]$ with $\alpha_i \in \mathrm{M}_n A$ we want to reduce it to $\alpha(0) \in \mathrm{GL} A$ by addition operations.

The first step, reduction to a linear matrix (i.e., a matrix of degree 1 in t) is very easy, and it works for any ring A (regular or not) and any matrix. (Moreover, linearization works also when the indeterminate t do not commute with A.) A way to do this is know in differential equations. Namely, the matrix $\alpha(x) = \alpha_0 + \alpha_1 t + \cdots + \alpha_d t^d \in \mathrm{M}_n Atx]$ where $\alpha_0 = \alpha(0)$ and $d \geq 2$ is easily reduced to a linear matrix by addition operations as follows:

$$\alpha(x) = \begin{pmatrix} \alpha_0 + \alpha_1 t + \alpha_2 t^2 & 0 \\ 0 & 1_n \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0 + \alpha_1 t + \alpha_2 t^2 & t \\ 0 & 1_n \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0 + \alpha_1 t & t \\ -\alpha_2 t & 1_n \end{pmatrix}$$

when $d = 2$,

$$\alpha(x) = \begin{pmatrix} \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & 1_n \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 & t & 0 \\ 0 & 1_n & t \\ 0 & 0 & 1_n \end{pmatrix}$$

$$\mapsto \begin{pmatrix} \alpha_0 + \alpha_1 t & t & 0 \\ -\alpha_2 t - \alpha_3 t^2 & 1_n & t \\ 0 & 0 & 1_n \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0 + \alpha_1 t & t & 0 \\ -\alpha_2 t & 1_n & t \\ \alpha_3 t & 0 & 1_n \end{pmatrix}$$

when $d = 3$, and

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1_{(d-1)n} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0 + \alpha_1 t & t & 0 & \dots & 0 & 0 \\ -\alpha_2 t & 1_n & t & 0 \dots & 0 & 0 \\ \alpha_3 t & 0 & 1_n & t & 0 \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ (-1)^{d-2} \alpha_{d-1} t & 0 & \dots & 0 & 1_n & t \\ (-1)^{d-1} \alpha_d t & 0 & \dots & 0 & 0 & 1_n \end{pmatrix}$$

when $d > 3$ where t stands for a scalar matrix in $\mathrm{M}_n A[t]$. The degree 1 part of the last matrix looks like a (block) companion matrix, but remember that the mtrices α_i do not necessary commute between themselves.

Now we assume that the matrix $\alpha(t) = \alpha_0 + \alpha_1 t$ is linear and invertible. Replacing it by $\alpha_0^{-1} \alpha$, we can assume that $\alpha_0 = 1_n$. Now we want to reduce $\alpha(t) = 1_n + \alpha_1 t$ to an identity matrix by addition operations.

It is easy to see that a matrix $1_n + \alpha_1 t \in \mathrm{M}_n A[t]$ is invertible if and only if the matrix $\alpha_1 \in \mathrm{M}_n A$ is nilpotent.

If α_1 is upper triangular with zeros on the main diagonal that it is clear that α_1 is nilpotent (namely $\alpha_1^n = 0$) and that $1_n + \alpha_1 t \in \mathrm{E}_n A[t]$. Therefore $1_n + \alpha_1 t \in \mathrm{E}A[t]$ also in the more general case when α_1 i is similar to an upper triangular with zeros on the main diagonal.

To reduce $1 + \alpha_1 t \in \mathrm{GL}A[t]$ to 1 by addition operation in general, we have to assume that every finitely generated right A-module has a finite projective resolutions. See [BHS] or [B2] for details.

Linerization works easily also for matrices over $A[t, 1/t]$, see [BHS] or [B2] for details.

Fredhiolm operators

While the fundamental theorem allows us to express K_{m-1} in terms of K_m, there is another (but related) way to do this. Namely, for any associative ring with 1, let $\tilde{\omega}A$ be the ring of all infinite matrices $(a_{i,j})_{i,j \in \mathbf{Z}}$ over A with finitely many nonzero entries in each row and column. The ring $\tilde{\omega}A$ contains the ideal $\mathrm{M}A$ consisting of matrices with finitely many nonzero entries. The factor ring $\omega A = \tilde{\omega}(A)\ /\mathrm{M}A$ is an algebraic analog of Fredholm operators. It is easy to show that $\mathrm{K}_m(\tilde{\omega}A) = 0$ for all m, hence the long exact sequence mentioned above gives that $\mathrm{K}_m(\omega A) = \mathrm{K}_{m-1}A$ for all m.

We used that for any ring A with 1 and any n, m, the group $\mathrm{K}_m(\mathrm{M}_n A)$ is isomorphic to $\mathrm{K}_m A$. Moreover, $\mathrm{K}_m A = \mathrm{K}_m(\mathrm{M}A)$ $=\mathrm{K}_m(\tilde{\omega}A,\ \mathrm{M}A)$.

Homotopy fiber

We have mentioned a long exact sequence of K-groups corresponding to $A \to A/B$. More generally, given any morphism $f : A \to A'$ of associative rings, there is a long exact sequence

$\cdots \to \mathrm{K}_m^f \to \mathrm{K}_m A \to \mathrm{K}_m A' \to \mathrm{K}_{m-1}^f \to \cdots.$

Now we describe the groups K_m^f in the case when $m \geq 1$ and f is a morphism of rings with 1. In this case $\mathrm{K}_m^f = \pi_{m-1}X$ where X is the homotopy fiber of the morphism $\mathrm{GL}A \to \mathrm{GL}A'$ of abstract simplicial complexes. So a vertex in X is a pair (α, β) where $\alpha \in \mathrm{GL}A$ and β is a path connecting $f(\alpha)$ with $1 \in \mathrm{GL}A'$. Simplexes in X are defined in obvious way. The first projection $X \to \mathrm{GL}A$ gives the group morphisms $\mathrm{K}_m^f \to \mathrm{K}_m A$ in the sequence ($m \geq 1$). Sending a loop on β on $\mathrm{GL}A'$ to the pair $(1, \beta) \in X$ gives the group morphisms $\mathrm{K}_{m+1}A' \to \mathrm{K}_m^f$ in the sequence. The group operation on K_m^f can be defined useng the direct sum of matrices.

For $m \leq 0$, the groups K_m^f can be defined using the Fredholm operators.

Concluding remarks

When $m \leq 0$, the group $\mathrm{K}_m(A, B)$ depends only on the ring B (excision theorem).

For a commutative ring A, we have multiplication $\mathrm{K}_i A \times\ \mathrm{K}_j A \to \mathrm{K}_{i+j}A$ for any integers i, j.

When A is regular, there is a simpler way to define higher K-theory using polynomial loops on $\mathrm{GL}A$ [NV], [KV]. An equivalent way involves unipotent subgroups instead of triangular subgroups of Volodin's definition. For example, the Karoubi-Nobile-Villamayor $\mathrm{K}_1'A$ for any associative ring A is $\mathrm{GL}A$ modulo the subgroup generated by unipotent matrices. Equivalently, two matrices $\alpha, \beta \in \mathrm{GL}A$ have the same image in $\mathrm{K}'A$ if and only if there is a polynomial matrix $\gamma(t) \in \mathrm{GL}(A)[t]$ (where the indeterminant t commutes with the coefficient ring A) such that $\gamma(0) = \alpha$ and $\gamma(0) = \beta$.

The higher K$'$-theory can be defined using the following functor Ω' on the category of associative rings: $\Omega' A = t(t-1)A[t]$. Then $\mathrm{K}'_n A = \mathrm{K}'_1(\Omega'^{n-1}A)$.

A commutative local ring A with 1 is regular if and only if $\dim_{A/p}(p/p^2) = \dim(A)$ where p is the maximal ideal of A and the second dim is the Krull dimension i.e., the length of the longest chain of prime ideals in A.

A commutative ring A is regular if and only if it is Noetherian and for every maximal ideal p of A its localization $B = A_p$ is regular.

When $A = \mathbf{C}[x_1, \ldots, x_n]/B$ with the ideal B generated by polynomials $f_1, \ldots, f_m$ with $m < n$ the following smoothnes condition implies the regularity: the m by n Jacobi matrix $\partial f_k / \partial x_l$ of the first partial derivatives has rank m at every solution of the system $f_k = 0$.

Problems.

1. Let $p = p^2 \in \mathrm{M}_m A, q = q^2 \in \mathrm{M}_n A$ with $m \le n$.

Suppose that the A-modules pA^m and qA^n are isomorphic ($pA^m \cong qA^n$). Show that the matrices $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_{m+nk} A$ are similar (conjugated).

2. Show that $\mathrm{K}_2 F = 0$ for every finite field F.

3. Show that

$\mathrm{K}_1(F[t]) = \mathrm{K}_1 F$, $\mathrm{K}_0(F[t]) = \mathrm{K}_0 F$,

$\mathrm{K}_0(F[t, 1/t]) = \mathrm{K}_0 F$,

and

$\mathrm{K}_1(F[t, 1/t]) = \mathrm{K}_1 F \oplus \mathrm{K}_0 F$

for any field F.

4. Let A be the ring of all continuous functions $X \to \mathbf{R}$ on a topological space X and $n \ge 1$. Prove that every $p = p^2 \in \mathrm{M}_n A$ is similar to a symmetric matrix.

5. For any $n \ge 1, d \ge 0$ let A be the ring of all continuous functions $\mathrm{R}^d \to \mathbf{R}$. Prove that every $p = p^2 \in \mathrm{M}_n A$ is similar to $\begin{pmatrix} 1_l & 0 \\ 0 & 0 \end{pmatrix}$ for some l.

6. Let $A = \mathbf{Z}[x]$. Show that for any $n \ge 2$ the group $\mathrm{SL}_n A$ has a subgroup H of finite index which do not contain any $\mathrm{SL}_n B$ with an ideal B of finite index. Moreover, for any integer N there is a subgroup H of finite index in $\mathrm{SL}_n A$ such that

$\mathrm{card}(\mathrm{SL}_n B)/(H \cap \mathrm{SL}_n B) \ge N$

for every ideal B of finite index in A.

7. Show that the ring $\mathbf{Z}/4\mathbf{Z}$ is not regular.

8. Show that if A is regular then both $A[t]$ and $A[t, 1/t]$ are regular.

9. Let A be a (commutative) Dedekind ring. Show that:

every ideal of A is projective;

for every nonzero $a \in A$, the factor ring A/aA is a principal ideal ring;

every ideal B of A is generated by 2 elements.

$\mathrm{sr}(A) \le 2$;

every projective A-module is isomorphic to a direct sum of an ideal of A and a free A-module.

10. (Bass) Let A be a commutative Noetherian domain with 1 . Show that every projective A-module is either finitely generated or free.

11. Let X be a set, F a field, F^X the ring of functions $X \to F$.
Show that $\mathrm{K}_0(F^X)$ is the set of bounded functions $X \to \mathbf{Z}$.
Show that $\mathrm{K}_1(F^X)$ is the set of all functions $X \to \mathrm{GL}_1 F$.

12. Let X be a set, $\mathbf{Z}^X$ the ring of functions $X \to \mathbf{Z}$.
Show that $\mathrm{K}_0(\mathbf{Z}^X)$ is the set of bounded functions $X \to \mathbf{Z}$.
Show that $\mathrm{K}_1(\mathbf{Z}^X)$ is the set of all functions $X \to \mathrm{GL}_1\mathbf{Z}$.

13. Let A be a commutative ring with $1 \neq 0$ such that every prime ideal of A is of finite index. Prove that $\mathrm{sr}(A) = 1$.

14. Let (X, Δ) be an abstract simplicial complex where X is a set and Δ is a set of simplices. Its topological realization is a topological space T consisting of the functions $p : X \to \mathbf{R}$ such that:
the support $\{x \in X : p(x) \neq 0\}$ of p is a simplex;
$\sum_{x \in X} p(x) = 1$.
A basis of the open neighborhoods of $p \in T$ consists of

$$U(p, \varepsilon, S) = \{q \in T : |q(x) - p(x)| < \varepsilon, q(x) = 0 \text{ for } x \notin S\}$$

where $\varepsilon > 0$ and S is a simplex containing the support of p. The vertices (points) of X (i.e., the simplexes of cardinality 1 in Δ) correspond to the (0-1)-functions p.

Prove that $\pi_i(X, \Delta, x_0) = \pi_i(T, x_0)$ for all $i \geq 0$ for any base point $x_0 \in X$ (which is a vertex).

15. Let (X, Δ) be an abstract simplicial complex and (X, Δ_k) be its k-skeleton (consisting of simplexes of cardinality $\leq k + 1$). Prove that $\pi_i(X, \Delta, x_0) = \pi_i(X, \Delta_{i+1}, x_0)$ for all $i \geq 0$ and any basic point x_0.