BASIS PARTITIONS AND THE ROGERS-RAMANUJAN IDENTITIES

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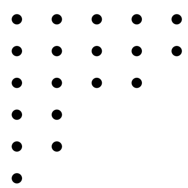
Abstract. In this paper, a common generalization of the Rogers-Ramanujan series and the generating function for basis partitions is studied. This leads naturally to a sequence of polynomials, called BsP-polynomials. In turn, the BsP-polynomials provide simultaneously a proof of the Rogers-Ramanujan identities and a new, more rapidly converging series expansion for the basis partition generating function.

1. Introduction

The late Hansraj Gupta [6] introduced the concept of basis partitions. Basis partitions are defined in terms of successive ranks [4] or the "rank vector" of a partition.

Namely, each partition, π, of a positive integer contains a largest square of nodes in its Ferrers graph. This square is called the Durfee square. If the Durfee square has side d, we define the i^{th} rank r_i of π $(1 \le i \le d)$ as the difference between the number of nodes in the i^{th} row of the Ferrers graph of π and the number in the i^{th} column. The rank vector for π is $(r_1, r_2, \ldots, r_d)$.

For example, if π is the partition $5+5+4+2+2+1$, then it Ferrers graph is:



Its rank vector is $(-1, 0, 1)$

Gupta [6] showed that for every rank vector, $\vec{r}$, there is a smallest integer that has a partition with rank vector $\vec{r}$, and that parition is unique. This partition is called the basis partition for $\vec{r}$. We let $B(n)$ denote the number of basis partitions of n.

For example, the basis partition for $(-1, 0, 1)$ is $4+4+4+2+1$.

In [9], Nolan, Savage and Wilf showed that

$$\sum_{n=0}^{\infty} B(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(q;q)_n}, \tag{1.1}$$

where

(1) Partially supported by National Science Foundation Grant DMS-0801184.

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{1.2}$$

Hirschhorn [8] gave a new proof of (1.1) and related basis partitions to the Rogers-Ramanujan series from the first Rogers-Ramanujan identity [3, p. 113]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{1.3}$$

Our central object is to study

$$G(a,x;q) := \sum_{n=0}^{\infty} \frac{a^n q^{n^2}(x;q)_n}{(q;q)_n}. \tag{1.4}$$

Notice that if we set $x = -q$, we get the series in (1.1), and if we set $x = 0$ we get the series in (1.3). We want to find an identity for $G(a,x;q)$ which both leads directly to the Rogers-Ramanujan identities and also provides a new representation of the series in (1.1).

We shall prove

Theorem 1.

$$G(a,x;q) = \frac{1}{(aq;q)_\infty}\left(1 + \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}(1-aq^{2n})(-1)^n q^{n(3n-1)/2} a^n B_n(a,x)}{(q;q)_n}\right) \tag{1.5}$$

where

$$B_n(a,x) = \sum_{j=0}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix} (x;q)_j a^j x^{n-j} q^{nj}, \tag{1.6}$$

and

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{cases} 0 & \text{if } j < 0 \text{ or } j > n \\ \frac{(q;q)_n}{(q;q)_j (q;q)_{n-j}} & \text{if } 0 \le j \le n \end{cases}$$

It is immediate from (1.6) that $B_n(a,0) = a^n q^{n^2}$, which implies, by setting $x = 0$ in (1.5),

Corollary 2.

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} = \frac{1}{(aq;q)_\infty}\left(1 + \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(1-aq^{2n})(-1)^n a^{2n} q^{n(5n-1)/2}}{(q;q)_n}\right).$$

Corollary 2 is one of the standard identities (e.g. see [7, ch. 19]) used to deduce both Rogers-Ramanujan identities.

We shall call the $B_n(a,x)$, the Basis Partition polynomials, abbreviated BsP-Polynomial.

The BsP-polynomials have a variety of representations which we explore in Section 3.

2. Proof of Theorem 1.

We recall the weak form of Bailey's lemma [2, Th.2, p. 273]:
If

$$\beta_n = \sum_{r=0}^{n} \frac{a_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{2.1}$$

then

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \sum_{r=0}^{\infty} a^r q^{r^2} a_r, \tag{2.2}$$

Furthermore the relation (2.1) can be inverted [2, p. 278, eq (4.1)]:

$$a_n = \frac{(-1)^n q^{\binom{n}{2}} (a;q)_n (1-aq^{2n})}{(q;q)_n (1-a)} \sum_{j=0}^{n} (q^{-n};q)_j (aq^n;q)_j q^j \beta_j. \tag{2.3}$$

Consequently comparing (2.2) with (1.5) we see that to establish (1.5) we need only show that

$$B_n(a,x) = \sum_{j=0}^{n} \frac{(q^{-n};q)_j (aq^n;q)_j (x;q)_j q^j}{(q;q)_j} \tag{2.4}$$

Now by [5, p. 242, eq. (III.13), with b replaced by d/b and then $d=0$, $e \to \infty$] we see that

$$\sum_{j=0}^{n} \frac{(q^{-n};q)_j (c;q)_j (b;q)_j q^j}{(q;q)_j} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (c;q)_j b^j c^{n-j}, \tag{2.5}$$

and if we set $c = x$, $b = aq^n$ in (2.5) we deduce (2.4). □

3. Properties of $B_n(a,x)$

There are several representations of $B_n(a,x)$ as well as a second order linear recurrence which we include in the following results.

Theorem 3.

$$B_n(a,x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (aq^n;q)_{n-j} x^{n-j} a^j q^{nj} \tag{3.1}$$

$$= \sum_{r,s \ge 0} \begin{bmatrix} n \\ r,s \end{bmatrix} (-1)^{n+r+s} (aq^n)^{n-s} x^{n-r} q^{\binom{n-r-s}{2}}, \tag{3.2}$$

where

$$\begin{bmatrix} n \\ r,s \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix} = \frac{(q;q)_n}{(q;q)_r (q;q)_s (q;q)_{n-r-s}}$$

Proof. From (2.4) it is clear that $B_n(a,x)$ is symmetric in x and aq^n. Hence (3.1) is valid. To prove (3.2), we first prove

$$\sum_{j=0}^{n} \frac{(q^{-n};q)_j(c;q)_j(b;q)_jq^j}{(q;q)_j} = \sum_{r,s\geq 0} \begin{bmatrix} n \\ r,s \end{bmatrix} (-1)^{n+r+s}c^{n-r}b^{n-s}q^{\binom{n-r-s}{2}} \tag{3.3}$$

This is proved as follows

$$\begin{aligned}
\sum_{j=0}^{n} \frac{(q^{-n};q)_j(c;q)_j(b;q)_jq^j}{(q;q)_j} &= \sum_{j=0}^{n} \frac{(q^{-n};q)_jq^j}{(q;q)_j} \sum_{r\geq 0} \begin{bmatrix} j \\ r \end{bmatrix} (-1)^r c^r q^{\binom{r}{2}} \sum_{s\geq 0} \begin{bmatrix} j \\ s \end{bmatrix} (-1)^s b^s q^{\binom{s}{2}} \\
&= \sum_{r\geq 0} \frac{(-1)^r c^r q^{\binom{r}{2}}}{(q;q)_r} \sum_{s\geq 0} \frac{(-1)^s b^s q^{\binom{s}{2}}}{(q;q)_s} \sum_{j\geq 0} \frac{(q^{-n};q)_j(q;q)_jq^j}{(q;q)_{j-r}(q;q)_{j-s}} \qquad (3.4)
\end{aligned}$$

In light of the fact that the inner sum is symmetric in r and s, we can without loss of generality assume that $r \geq s$. Hence

$$\begin{aligned}
\sum_{j\geq 0} \frac{(q^{-n};q)_j(q;q)_jq^j}{(q;q)_{j-r}(q;q)_{j-s}} &= \frac{(q^{-n};q)_r(q;q)_rq^r}{(q;q)_{r-s}} \sum_{j\geq 0} \frac{(q^{-n+r};q)_j(q^{r+1};q)_jq^j}{(q;q)_j(q^{r-s+1};q)_j} \\
&= \frac{(q^{-n};q)_r(q;q)_rq^r}{(q;q)_{r-s}} \frac{q^{(r+1)(n-r)}(q^{-s};q)_{n-r}}{(q^{r-s+1};q)_{n-r}} \qquad (3.5)
\end{aligned}$$

(by [5, p. 11, eq. (1.5.3)]).
Substituting (3.5) in (3.4) and simplifying, we find

$$\begin{aligned}
\sum_{j=0}^{n} \frac{(q^{-n};q)_j(c;q)_j(b;q)_jq^j}{(q;q)_j} &= \sum_{r,s\geq 0} \frac{(q;q)_n(-1)^{n+r+s}q^{\binom{s+r-n}{2}}c^rb^s}{(q;q)_{n-r}(q;q)_{n-s}(q;q)_{r+s-n}} \\
&= \sum_{r,s\geq 0} \frac{(q;q)_n(-1)^{n+r+s}q^{\binom{n-r-s}{2}}c^{n-r}b^{n-s}}{(q;q)_r(q;q)_s(q;q)_{n-r-s}}
\end{aligned}$$

(where $r \to n-r$ and $s \to n-s$), which is (3.2). □

Theorem 4.

$$(x-1)B_n(a,x) = A_nB_{n+1}(a,x) - (A_n - C_n)B_n(a,x) - C_nB_{n-1}(a,x) \tag{3.6}$$

where

$$A_n = \frac{(1-aq^n)}{(1-aq^{2n})(1-aq^{2n+1})},$$

and

$$C_n = \frac{(1-q^n)a^2q^{3n-1}}{(1-aq^{2n-1})(1-aq^{2n})}$$

Proof. This follows immediately from [5, p. 186, Ex. 7.10] with b replaced by $\frac{b}{aq}$, then a and c are set to 0 and then b is replaced by a new a. □

As a final result, we deduce the following identity for the generating function for $B(n)$ given in (1.1). Note the acceleration of convergence.

Corollary 5.

$$\sum_{n=0}^{\infty} B(n)q^n := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(q;q)_n} = \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(1+q^n)(-1)^n q^{n(3n-1)/2} b_n\right) \tag{3.7}$$

where b_n is given by the recurrence

(3.8)

$$(1-q^{2n-1})b_{n+1} = -(1+q^n)(q+q^n-2q^{2n}-q^{2n+1}-q^{2n+2}+q^{3n}+q^{4n+1})b_{n-1} + q^{3n-1}(1-q^{2n+1})b_{n-2},$$

Proof. In Theorem 1, set $a=1$, $x=-q$, then simplify and invoke Theorem 4 to determine the recurrence for $b_n = B_n(1,-q)$. □

4. Conclusion

The BsP-Polynomials are actually specializations of the Al-Salam and Carlitz polynomials [1]. Unfortunately, while (2.4) reveals that BsP-polynomials to be a limiting case of the Big q-Jacobi polynomials [5, p. 167, eq. (7.3.10)], the limiting process destroys orthogonality on the real line.

It seems plausible that

$$G(1,-q;q^2) = \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q;q^2)_n}{(q^2;q^2)_n}$$

may well have combinatorial interest.

References

[1] W. A. Al-Salam and L. Carlitz, Some orthogonal q-polynomials, Math. Nachr., **30**(1965), 47–61.

[2] G. E. Andrews, Multimple series Rogers-Ramanujan type identities, Pac. J. Math., **114**(1984), 267–283.

[3] G. E. Andrews, The Theorey of Partitions, Vol. 2, Encycl. Math. and Its Appl., Addison-Wesley, Reading, 1976 (Re-issued: Cambridge University Press, Cambridge, 1984).

[4] A. O. L. Atkin, A note on ranks and conjugacy of partitions, Quarterly J. Math., **17**(1966), 335-338.

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Vol. 35, Encycl. of Math. and Its Appl., Cambridge University Press, Cambridge, 1990.

[6] H. Gupta, The rank vector of a partition, Fibo. Quarterly, **16**(1978), 548–552.

[7] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 5^{th} ed., Oxford University Press, Oxford, 1979.

[8] M. D. Hirschhorn, Basis partitions and Rogers-Ramanujan partitions, Discr. Math., **205**(1999), 241–243.

[9] J. M. Nolan, C. D. Savage, and H. S. Wilf, Basis partitions, Discr. Math., **179**(1998), 277-283.

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