# On $q$-Series Identities Related to Interval Orders

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## Abstract

We prove several power series identities involving the refined generating function of interval orders, as well as the refined generating function of the self-dual interval orders. These identities may be expressed as

$$\sum_{n \geq 0} \left( \frac{1}{p}; \frac{1}{q} \right)_n = \sum_{n \geq 0} pq^n \, (p;q)_n \, (q;q)_n$$

and

$$\sum_{n \geq 0} (-1)^n \left( \frac{1}{p}; \frac{1}{q} \right)_n = \sum_{n \geq 0} pq^n \, (p;q)_n \, (-q;q)_n = \sum_{n \geq 0} \left( \frac{q}{p} \right)^n (p;q^2)_n \,,$$

where the equalities apply to the (purely formal) power series expansions of the above expressions at $p = q = 1$, as well as at other suitable roots of unity.

## 1. Introduction and Combinatorial Motivation

Throughout this paper, we use the notation $(a;q)_n$ for the $q$-Pochhammer symbol, defined as

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

with $(a;q)_0 = 1$. Where $q$ is understood from the context, we write $(a)_n$ instead of $(a;q)_n$ for brevity.

The main goal of this paper is to prove new identities for the generating functions of interval orders and self-dual interval orders. The identities we

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derive may be stated as

$$\sum_{n\geq 0}\left(\frac{1}{p};\frac{1}{q}\right)_n = \sum_{n\geq 0} pq^n\,(p;q)_n\,(q;q)_n$$

and

$$\sum_{n\geq 0}(-1)^n\left(\frac{1}{p};\frac{1}{q}\right)_n = \sum_{n\geq 0} pq^n\,(p;q)_n\,(-q;q)_n = \sum_{n\geq 0}\left(\frac{q}{p}\right)^n\,(p;q^2)_n\,,$$

where the equalities mean that the corresponding expressions admit the same power series expansion as $p$ and $q$ approach 1. It follows from our argument that the equalities are in fact valid when $p$ and $q$ approach other roots of unity as well, provided the corresponding series expansions exist.

### 1.1. Interval Orders

Let $P$ be a poset with a strict order relation $\prec$. We say that $P$ is an *interval order* if we can assign to each element $x \in P$ a closed real interval $[l_x, r_x]$ in such a way that $x \prec y$ if and only if $r_x < l_y$. As shown by Fishburn [7], a poset is an interval order if and only if it does not contain a subposet isomorphic to the disjoint union of two chains of size two. In this paper, we are interested in *unlabelled* interval orders, i.e., we treat isomorphic posets as identical.

A *Fishburn matrix* is an upper-triangular square matrix of nonnegative integers with the property that every row and every column has at least one nonzero entry. The *size* of a matrix is defined as the sum of its entries. As implied in the work of Fishburn [8, 9], there is a bijective correspondence between unlabelled interval orders of $n$ elements and Fishburn matrices of size $n$. In fact, as pointed out in [5], there is a bijection that maps interval orders with $n$ elements having $r$ minimal and $s$ maximal elements to Fishburn matrices of size $n$, whose first row sums to $r$ and whose last column sums to $s$.

Let $f_n$ be the number of unlabelled interval orders on $n$ elements. The sequence $(f_n)_{n\geq 0}$ is known as the *Fishburn numbers* [10, sequence A022493]. Apart from counting interval orders and Fishburn matrices, the numbers $f_n$ have several other combinatorial interpretations. For instance, $f_n$ is the number of Stoimenow diagrams with $n$ arcs [17, 20], the number of ascent sequences of length $n$ [2], the number of certain pattern-avoiding permutations of order $n$ [2, 14], or the number of certain pattern-avoiding insertion tables [13].

Zagier [20] has shown that the generating function of Fishburn numbers may be expressed as

$$\sum_{n\geq 0} f_n x^n = \sum_{n\geq 0}\prod_{k=1}^{n}(1-(1-x)^k) = \sum_{n\geq 0}(1-x;1-x)_n\,,$$

and deduced the asymptotics

$$f_n = n!\left(\frac{6}{\pi^2}\right)^n\sqrt{n}\left(\alpha + O\left(\frac{1}{n}\right)\right), \quad \text{with } \alpha = \frac{12\sqrt{3}}{\pi^{5/2}}e^{\pi^2/12}.$$

Subsequently, several authors have obtained refinements of this generating function, enumerating interval orders with respect to various natural statistics, such as the number of minimal and maximal elements [11, 12], the number of indistinguishable elements [4] or the number of distinct down-sets [2, 11].

In this paper, we focus on the refined enumeration of interval orders by the number of maximal elements. Let $f_{m,\ell}$ be the number of interval orders of size $m$ with $\ell$ maximal elements. Recall that $f_{m,\ell}$ is also equal to the number of Fishburn matrices of size $m$ whose last column sums to $\ell$. Kitaev and Remmel [12] have shown that

$$
\begin{aligned}
\sum_{m\geq 0,\ell\geq 0} f_{m,\ell} x^{m-\ell} y^\ell &= 1 + \sum_{n\geq 0} \frac{y}{(1-y)^{n+1}} \prod_{k=1}^{n} \left(1 - (1-x)^k\right) \\
&= 1 + \sum_{n\geq 0} \left(\frac{1}{(1-y)^{n+1}} - \frac{1}{(1-y)^n}\right) \prod_{k=1}^{n} \left(1 - (1-x)^k\right) \\
&= \sum_{n\geq 0} \left(\frac{1-x}{1-y}\right)^{n+1} \prod_{k=1}^{n} \left(1 - (1-x)^k\right) \\
&= \sum_{n\geq 0} \left(\frac{1-x}{1-y}\right)^{n+1} (1-x; 1-x)_n .
\end{aligned}
\tag{1}
$$

Zagier [20], Yan [18] and Levande [13] have independently obtained another formula for the same generating function, which has also been conjectured by Kitaev and Remmel [12], namely

$$
\sum_{m\geq 0,\ell\geq 0} f_{m,\ell} x^{m-\ell} y^\ell = \sum_{n\geq 0} (1-y; 1-x)_n .
\tag{2}
$$

We remark that Jelínek [11] has derived a formula for the generating function counting interval orders by their size and the number of minimal and maximal elements, which simultaneously generalizes both (1) and (2), using the fact that the number of minimal elements has the same distribution as the number of maximal elements.

### 1.2. Self-Dual Interval Orders

Many families of objects enumerated by the Fishburn numbers admit a natural involutive symmetry map, which transforms an object into its 'mirror image'. In most cases, the known bijections between Fishburn-enumerated families commute with the corresponding symmetry maps. This suggests that the mirror symmetry is an inherent property of Fishburn families, and leads to a natural problem of enumerating symmetric Fishburn objects, i.e., objects that are fixed by the symmetry map.

For interval orders, the symmetry map corresponds to poset duality. For a poset $P$ with a strict order relation $\prec$, its *dual poset* $\overline{P}$ has the same elements as $P$ and its order relation $\overline{\prec}$ is defined by $x\overline{\prec}y \iff y \prec x$. Clearly, the

3

dual of an interval order is again an interval order. A poset is *self-dual* if it is isomorphic to its dual.

In the above-mentioned correspondence between interval orders and Fishburn matrices, the self-dual interval orders correspond to Fishburn matrices that are symmetric with respect to the north-east diagonal. We refer to such matrices as *self-dual* Fishburn matrices. Formally, an $n \times n$ matrix $M = (M_{ij})_{i,j=1}^{n}$ is *self-dual* if it satisfies $M_{i,j} = M_{n-j+1,n-i+1}$ for all $i$ and $j$. Of course, such a matrix $M$ is uniquely determined by the entries that lie on or below the north-east diagonal, i.e., by the entries $M_{i,j}$ with $i+j \geq n+1$; we refer to these entries as *the south-east entries* of $M$. Moreover, the entries lying on the north-east diagonal (i.e., the entries $M_{i,j}$ with $i + j = n + 1$) will be called *the diagonal entries* of $M$.

The *reduced size* of a matrix $M$ is defined as the sum of its south-east entries. For the purposes of enumerating self-dual Fishburn matrices, and thus also self-dual interval orders, the notion of reduced size seems to be more natural than the notion of size. Let $\mathcal{S}_m$ be the set of the self-dual Fishburn matrices of reduced size $m$, and let $\mathcal{S}_{m,\ell}$ be the set of those matrices in $\mathcal{S}_m$ whose last column has sum $\ell$. Let $s_m$ and $s_{m,\ell}$ be the cardinalities of $\mathcal{S}_m$ and $\mathcal{S}_{m,\ell}$, respectively.

The following two facts were proved by Jelínek [11] by means of generating functions, and a bijective proof was subsequently found by Yan and Xu [19].

**Fact 1.1.** *For every $m \geq 1$ and every $\ell$, the set $\mathcal{S}_{m,\ell}$ contains precisely $s_{m,\ell}/2$ matrices whose diagonal entries are all zero, and therefore also $s_{m,\ell}/2$ matrices with at least one nonzero diagonal entry.*

**Fact 1.2.** *Let us call a matrix $M$ a* row-Fishburn matrix *if $M$ is an upper-triangular matrix of nonnegative integers such that every row has at least one positive entry. Let $r_{m,\ell}$ be the number of row-Fishburn matrices with (non-reduced) size $m$ and the sum of the last column equal to $\ell$. For $m \geq 1$ and any $\ell$, we have $r_{m,\ell} = s_{m,\ell}/2$.*

This shows that enumerating self-dual Fishburn matrices by their reduced size is essentially equivalent to enumerating row-Fishburn matrices by their size. It is not hard to observe (see [11, Theorem 4.1]) that the generating function of $r_{m,\ell}$ may be expressed as

$$\sum_{m,\ell \geq 0} r_{m,\ell} x^{m-\ell} y^{\ell} = \sum_{n \geq 0} \prod_{k=1}^{n} \left( \frac{1}{(1-y)(1-x)^{k-1}} - 1 \right)$$

$$= \sum_{n \geq 0} (-1)^n \left( \frac{1}{1-y}; \frac{1}{1-x} \right)_n . \tag{3}$$

Denoting by $r_m$ the number of row-Fishburn matrices of size $m$, we then get

$$\sum_{m \geq 0} r_m x^m = \sum_{n \geq 0} \prod_{k=1}^{n} \left( \frac{1}{(1-x)^k} - 1 \right) = \sum_{n \geq 0} (-1)^n \left( \frac{1}{1-x}; \frac{1}{1-x} \right)_n . \tag{4}$$

4

The sequence $(r_m)_{m \geq 0}$ is listed as A158691 in the OEIS [10]. Peter Bala, who is the author of the OEIS entry, has pointed out that apparently the same coefficient sequence arises from expanding a different expression, namely

$$\sum_{n \geq 0} \prod_{k=1}^{n} \left(1 - (1-x)^{2k-1}\right) = \sum_{n \geq 0} \left(1 - x; (1-x)^2\right)_n. \qquad (5)$$

He conjectured that (4) and (5) indeed determine the same power series.

In this paper, we prove the identity conjectured by Bala. We actually extend this identity to the bivariate generating function of $r_{m,\ell}$ from (3), and moreover, we derive yet another, third way of expressing this generating function. Apart from that, we derive similar identities for the generating function of $f_{m,\ell}$, providing a third expression for this generating function, different from those given in (1) and (2).

It is remarkable that the identities involving the generating function of $r_{m,\ell}$ turn out to be analogous to those involving the generating function of $f_{m,\ell}$. In fact, in some cases the identities for the two generating functions may be deduced from the same general rule by a different choice of a parameter.

In the course of preparation of our manuscript, we have been informed that Bringmann, Li and Rhoades [3] have independently obtained another proof of Bala's conjecture, as well as several other identities involving the generating functions of Fishburn and row-Fishburn matrices. Most, but not all, of the identities derived by Bringmann, Li and Rhoades also follow from our Theorem 2.1 by setting $y$ equal to $x$. Apart from the power series identities, Bringmann, Li and Rhoades have obtained an asymptotic estimate for the number of row-Fishburn matrices.

**Theorem 1.3** (Bringmann, Li, Rhoades [3]). *Let $r_m$ be the number of row-Fishburn matrices of size $m$. Then, as $m \to \infty$, we have*

$$r_m = m! \left(\frac{12}{\pi^2}\right)^m \left(\beta + O\left(\frac{1}{m}\right)\right), \qquad \text{with } \beta = \frac{6\sqrt{2}}{\pi^2} e^{\pi^2/24}.$$

## 2. The Results

Let us define six formal power series as follows:

$$F_1(x,y) = \sum_{n \geq 0} (1-y; 1-x)_n \, ,$$

$$F_2(x,y) = \sum_{n \geq 0} \frac{1}{(1-y)(1-x)^n} \left(\frac{1}{1-y}; \frac{1}{1-x}\right)_n \left(\frac{1}{1-x}; \frac{1}{1-x}\right)_n ,$$

$$F_3(x,y) = \sum_{n \geq 0} \left(\frac{1-x}{1-y}\right)^{n+1} (1-x; 1-x)_n \, ,$$

$$G_1(x,y) = \sum_{n \geq 0} (-1)^n \left(\frac{1}{1-y}; \frac{1}{1-x}\right)_n ,$$

$$G_2(x,y) = \sum_{n \geq 0} (1-y)(1-x)^n (1-y; 1-x)_n (-(1-x); 1-x)_n , \text{ and}$$

$$G_3(x,y) = \sum_{n \geq 0} \left(\frac{1-x}{1-y}\right)^n \left(1-y; (1-x)^2\right)_n .$$

It is not hard to see that all the six infinite sums in these definitions are convergent in the ring of formal power series in $x$ and $y$. For instance, to see that the sum in the definition of $F_1(x,y)$ is convergent, it suffices to note that each monomial in the expansion of $(1-y; 1-x)_n$ has degree at least $n$.

Note that $F_1(x,y)$ and $F_3(x,y)$ correspond to the two formulas for the generating function of $f_{m,\ell}$ given in (2) and (1), respectively. In particular, it is known that $F_1(x,y) = F_3(x,y)$. Note also that $G_1(x,y)$ is the generating function of $r_{m,\ell}$ given in (3), and that $G_3(x,x)$ is Bala's formula (5). In particular, Bala's conjecture corresponds to the identity $G_1(x,x) = G_3(x,x)$.

The next theorem is our main result.

**Theorem 2.1.** *In the ring of formal power series in $x$ and $y$, we have the identities*

$$F_1(x,y) = F_2(x,y) = F_3(x,y), \tag{6}$$

*and*

$$G_1(x,y) = G_2(x,y) = G_3(x,y). \tag{7}$$

As we pointed out in the introduction, the equality $F_1(x,y) = F_3(x,y)$ has been previously proven by the combined results of Kitaev and Remmel [12], Levande [13], Yan [18] and Zagier [20]. We decided to include $F_3$ in the statement of Theorem 2.1 anyway, for comparison with the identities involving the $G_i$'s.

Note that $F_1(x,x)$ is obviously equal to $F_3(x,x)$, but the remaining identities of Theorem 2.1 remain non-trivial even when restricted to the case of $x = y$.

By setting $p = 1/(1-y)$ and $q = 1/(1-x)$ in (6), and $p = 1-y$ and $q = 1-x$ in (7), the identities of Theorem 2.1 can be expressed concisely as

$$\sum_{n\geq 0} \left(\frac{1}{p}; \frac{1}{q}\right)_n = \sum_{n\geq 0} pq^n (p;q)_n (q;q)_n = \sum_{n\geq 0} \left(\frac{p}{q}\right)^{n+1} \left(\frac{1}{q}; \frac{1}{q}\right)_n, \text{ and } \quad (8)$$

$$\sum_{n\geq 0} (-1)^n \left(\frac{1}{p}; \frac{1}{q}\right)_n = \sum_{n\geq 0} pq^n (p;q)_n (-q;q)_n = \sum_{n\geq 0} \left(\frac{q}{p}\right)^n (p;q^2)_n. \quad (9)$$

Note, however, that the expressions in (8) and (9) are in general not power series in $p$ and $q$; they should instead be understood as power series in variables $p-1$ and $q-1$ to make the identities meaningful.

In fact, the identities (8) and (9) can be interpreted in a broader way. If $p$ and $q$ are complex values such that $pq^{2k} = 1$ for some integer $k$, then all the three summations in (9) involve only finitely many nonzero summands, and the sums are therefore well defined. A straightforward adaptation of our proof of Theorem 2.1 then shows that the values of the three sums are equal. Moreover, if we consider complex values $p_0$ and $q_0$ such that $p_0 q_0^{2k} = 1$ for infinitely many integers $k$, then one may easily check that all the three summations in (9) are convergent as power series in $(p - p_0)$ and $(q - q_0)$. An appropriate adaptation of our proof then shows that the three power series are equal.

With the identities in (8), we need to be more careful. The left equality is again valid for those values of $p$ and $q$ for which the sums are both terminating, i.e., for values that satisfy $pq^k = 1$ for an integer $k$. And moreover, if $p_0$ and $q_0$ satisfy $p_0 q_0^k = 1$ for infinitely many integers $k$, the two expressions are equal as power series in $(p - p_0)$ and $(q - q_0)$. This may again be proven by a straightforward modification of our proof.

On the other hand, it is not clear whether the identities involving the right-hand side of (8), i.e., the expression $F_3(1-q, 1-p) = \sum_{n\geq 0} (p/q)^{n+1} (1/q; 1/q)_n$, can also be extended to other complex values of $p$ and $q$. Since the equality of $F_1(x, y)$ and $F_3(x, y)$ is based on the combinatorial interpretation of the two expressions as generating functions, the proof is only applicable to expansions in powers of $(p - 1)$ and $(q - 1)$. However, we conjecture that even this last equality can be extended to those values where the two sides are defined (see Conjecture 4.1 for a precise statement).

We remark that the expression $\sum_{n\geq 0} (p/q)^{n+1} (1/q; 1/q)_n$ on the right-hand side of (8) also admits a combinatorially meaningful expansion into powers of $p$ and $1/q$. More precisely, it is not hard to see that the expression equals $\sum_{r,s\geq 1} a_{r,s} p^r q^{-s}$ where $a_{r,s}$ is the difference between the number of partitions of $s$ into an odd number of parts and the number of partitions of $s$ into an even number of parts, where we only consider partitions into distinct parts whose largest part is $r$. In particular, for $p = 1$ we get the following well known identities (see e.g. Corollary 1.7 on p. 11, and Ex. 10 on p. 29 in [1]):

$$\sum_{n\geq 0} \frac{1}{q^{n+1}} \left(\frac{1}{q}; \frac{1}{q}\right)_n = 1 - \prod_{n\geq 1} (1 - q^{-n}) = 1 - \sum_{n=-\infty}^{\infty} (-1)^n q^{-n(3n-1)/2},$$

where the second identity is a version of the classical Pentagonal Number Theorem of Euler.

### 2.1. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the following identity, which has been discovered by Rogers [15] and independently by Fine [6, eq. (14.1)].

**Theorem 2.2** (Rogers–Fine Identity). *For $a$, $b$, $q$ and $t$ such that $|q| < 1$, $|t| < 1$ and $b$ is not a negative power of $q$, we have*

$$\sum_{n\geq0} \frac{(aq)_n}{(bq)_n} t^n = \sum_{n\geq0} \frac{(aq)_n \left(\frac{atq}{b}\right)_n b^n t^n q^{n^2} (1 - atq^{2n+1})}{(bq)_n (t)_{n+1}} \tag{10}$$

We now show how Theorem 2.1 follows from the Rogers–Fine Identity. Since $F_1$ and $F_3$ are already known to be equal, we only need to show that $F_1$ equals $F_2$, and that $G_1$, $G_2$ and $G_3$ are all equal. As a first step, we derive a general power series identity which directly implies both $F_1 = F_2$ and $G_1 = G_2$.

**Proposition 2.3.** *For any $r$, we have the following identity of formal power series in $x$ and $y$:*

$$\sum_{n\geq0} \left(\frac{1}{1-y}; \frac{1}{1-x}\right)_n r^n = \sum_{n\geq0}(1-y)(1-x)^n (1-y; 1-x)_n (r(1-x); 1-x)_n. \tag{11}$$

*Proof.* Let us substitute $a = \frac{1-y}{1-x}$, $b = \frac{1-y}{(1-x)z}$, $t = r/z$ and $q = 1-x$ into the Rogers–Fine identity, to obtain

$$\sum_{n\geq0} \frac{(1-y)_n}{z^n \left(\frac{1-y}{z}\right)_n} r^n =$$

$$\sum_{n\geq0} \frac{(1-y)_n (r(1-x))_n r^n (1-y)^n (1-x)^{n^2-n} \left(z - r(1-y)(1-x)^{2n}\right)}{z^{2n+1} \left(\frac{1-y}{z}\right)_n \left(\frac{r}{z}\right)_{n+1}}. \tag{12}$$

Let $L(x,y,z)$ and $R(x,y,z)$ denote respectively the left-hand side and the right-hand side of (12). Let us verify that both $L(x,y,z)$ and $R(x,y,z)$ are well defined as formal power series in $x$, $y$ and $z$. To see that $L(x,y,z)$ is well defined, we first note that the denominator $z^n \left((1-y)/z\right)_n$ of the $n$-th summand on the left-hand side of (12) is a polynomial in $x$, $y$ and $z$ with nonzero constant term, showing that each summand can be expanded into a power series. It remains to verify that the sum on the left-hand side of (12) is convergent in the ring of formal power series. This follows from the fact that every monomial $x^i y^j$ appearing with nonzero coefficient in the expansion of $(1-y; 1-x)_n$ satisfies $i + j \geq n$, and therefore a monomial $x^i y^j z^k$ may only appear with nonzero coefficient in the first $i + j$ summands of $L(x,y,z)$. Thus, $L(x,y,z)$ is well defined. The same reasoning applies to $R(x,y,z)$ as well.

8

We now set $z = 0$ in $L$ and $R$, to obtain

$$L(x, y, 0) = \sum_{n \geq 0} \frac{(1-y)_n}{(-1)^n (1-y)^n (1-x)^{\binom{n}{2}}} r^n$$

$$= \sum_{n \geq 0} r^n (-1)^n \prod_{k=1}^{n} \left( \frac{1}{(1-y)(1-x)^{k-1}} - 1 \right)$$

$$= \sum_{n \geq 0} \left( \frac{1}{1-y}; \frac{1}{1-x} \right)_n r^n$$

and

$$R(x, y, 0) = \sum_{n \geq 0} \frac{-(1-y)_n \, (r(1-x))_n \, r^{n+1} (1-y)^{n+1} (1-x)^{n^2+n}}{(-1)^{2n+1} r^{n+1} (1-y)^n (1-x)^{n^2}}$$

$$= \sum_{n \geq 0} (1-y)(1-x)^n \, (1-y)_n \, (r(1-x))_n \,,$$

as claimed. □

**Corollary 2.4.** $F_1(x, y) = F_2(x, y)$ and $G_1(x, y) = G_2(x, y)$.

*Proof.* Taking $r = -1$ in (11) shows that $G_1(x, y) = G_2(x, y)$. By taking $r = 1$ and substituting $x = -x'/(1 - x')$ and $y = -y'/(1 - y')$ in (11), we prove that $F_1(x', y') = F_2(x', y')$. □

**Lemma 2.5.** $G_1(x, y) = G_3(x, y)$.

*Proof.* We again use the Rogers–Fine identity. This time, we substitute $a = (1-x)^2/(1-y)$, $b = (1-x)/z$, $q = (1-x)^{-2}$ and $t = 1/z$. This yields

$$\sum_{n \geq 0} \frac{\left( \frac{1}{1-y} \right)_n}{z^n \left( \frac{1}{z(1-x)} \right)_n}$$

$$= \sum_{n \geq 0} \frac{\left( \frac{1}{1-y} \right)_n \left( \frac{1}{(1-y)(1-x)} \right)_n (1-x)^{n-2n^2} \left( z - \frac{1}{(1-y)(1-x)^{4n}} \right)}{z^{2n+1} \left( \frac{1}{z(1-x)} \right)_n \left( \frac{1}{z} \right)_{n+1}}. \quad (13)$$

Let $L'(x, y, z)$ and $R'(x, y, z)$ denote the left-hand side and right-hand side of (13), respectively. As with $L$ and $R$ in the proof of Proposition 2.3, we easily observe that $L'$ and $R'$ are formal power series in $x$, $y$ and $z$. Putting $z$ equal to 0, we get

$$L'(x,y,0) = \sum_{n\geq 0} \frac{\left(\frac{1}{1-y}\right)_n}{\prod_{k=1}^{n} -\frac{1}{(1-x)^{2k-1}}}$$

$$= \sum_{n\geq 0} \prod_{k=1}^{n} \left(\frac{1-x}{1-y} - (1-x)^{2k-1}\right)$$

$$= G_3(x,y),$$

and

$$R'(x,y,0) = \sum_{n\geq 0} \frac{-\left(\frac{1}{1-y}\right)_n \left(\frac{1}{(1-y)(1-x)}\right)_n \frac{1}{1-y}(1-x)^{-2n^2-3n}}{-(1-x)^{-2n^2-n}}$$

$$= \sum_{n\geq 0} \frac{1}{(1-y)(1-x)^{2n}} \left(\frac{1}{1-y};\frac{1}{1-x}\right)_{2n}$$

$$= \sum_{n\geq 0} \left(\left(\frac{1}{1-y};\frac{1}{1-x}\right)_{2n} - \left(\frac{1}{1-y};\frac{1}{1-x}\right)_{2n+1}\right)$$

$$= \sum_{n\geq 0} (-1)^n \left(\frac{1}{1-y};\frac{1}{1-x}\right)_n$$

$$= G_1(x,y). \qquad \square$$

Theorem 2.1 is a direct consequence of Proposition 2.3 and Lemma 2.5.

## 3. A Generalization

We are able to prove the following generalization of the Rogers–Fine identity:

**Theorem 3.1** (Generalized Rogers–Fine Identity)**.**

$$\sum_{n\geq 0} \frac{\left(\frac{\beta\gamma}{\alpha qt}\right)_n (\alpha)_n}{(\beta)_n (\gamma)_n} t^n$$

$$= \sum_{n\geq 0} \frac{\left(\frac{\alpha qt}{\beta}\right)_n \left(\frac{\alpha qt}{\gamma}\right)_n (\alpha)_n \left(1 - \alpha t q^{2n}\right)(-1)^n q^{\binom{n}{2}-n}\left(\frac{\beta\gamma}{\alpha}\right)^n}{(\beta)_n (\gamma)_n (t)_{n+1}}. \quad (14)$$

To deduce the Rogers–Fine identity from Theorem 3.1, simply take the limit $\gamma \to 0$ and set $\alpha = aq$ and $\beta = bq$.

*Proof of Theorem 3.1.* Our argument is based on the following identity of Watson (see e.g. [16, eq. (3.4.1.5.)]), valid for $f = q^{-N}$ with $|q| < 1$ and $N$ a positive

10

integer:

$$\sum_{n\geq 0} \frac{(a)_n (b)_n (c)_n (d)_n (e)_n (f)_n (1-aq^{2n}) \left(\frac{a^2 q^2}{bcdef}\right)^n}{(q)_n \left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n \left(\frac{aq}{d}\right)_n \left(\frac{aq}{e}\right)_n \left(\frac{aq}{f}\right)_n (1-a)}$$
$$= \frac{(aq)_N \left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N \left(\frac{aq}{e}\right)_N} \sum_{n\geq 0} \frac{\left(\frac{aq}{bc}\right)_n (d)_n (e)_n (f)_n q^n}{(q)_n \left(\frac{def}{a}\right)_n \left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n}. \quad (15)$$

In (15), we put $d=q$ and take the limit as $N \to \infty$, to get

$$\sum_{n\geq 0} \frac{(b)_n (c)_n (e)_n q^{\binom{n}{2}-n}(1-aq^{2n})\left(-\frac{a^2}{bce}\right)^n}{\left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n \left(\frac{aq}{e}\right)_n (1-a)}$$
$$= \frac{1-\frac{a}{e}}{1-a} \sum_{n\geq 0} \frac{\left(\frac{aq}{bc}\right)_n (e)_n \left(\frac{a}{e}\right)^n}{\left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n}. \quad (16)$$

Putting $a=\alpha t$, $b=\alpha q t/\beta$, $c=\alpha q t/\gamma$, and $e=\alpha$ in (16), and multiplying the resulting identity by $(1-\alpha t)/(1-t)$, we obtain (14). □

From the generalized Rogers–Fine identity, we may deduce a generalization of Proposition 2.3 and Lemma 2.5, by following the same arguments we used to prove Proposition 2.3 and Lemma 2.5 from the Rogers–Fine identity. In particular, substituting $\alpha = 1-y$, $\beta = (1-y)/z$, $t = r/z$ and $q = 1-x$ into (14), and taking $z=0$, shows that

$$\sum_{n\geq 0} \frac{\left(\frac{\gamma}{r(1-x)}; 1-x\right)_n \left(\frac{1}{1-y}; \frac{1}{1-x}\right)_n r^n}{(\gamma; 1-x)_n}$$
$$= \sum_{n\geq 0} (1-y)(1-x)^n \frac{(1-y; 1-x)_n (r(1-x); 1-x)_n}{(\gamma; 1-x)_n}.$$

Similarly, by putting $\alpha = 1/(1-y)$, $\beta = 1/z(1-x)$, $t = 1/z$, and $q = 1/(1-x)^2$, we get

$$\sum_{n\geq 0} (-1)^n \frac{\left(\frac{1}{1-y}; \frac{1}{1-x}\right)_n}{\left(\gamma; \frac{1}{(1-x)^2}\right)_{\lfloor \frac{n}{2}\rfloor}}$$
$$= \sum_{n\geq 0} \left(\frac{1-x}{1-y}\right)^n \frac{(1-y;(1-x)^2)_n \left(\gamma(1-x)(1-y); \frac{1}{(1-x)^2}\right)_n}{\left(\gamma; \frac{1}{(1-x)^2}\right)_n}.$$

Unfortunately, we are not able to find a combinatorial interpretation for these generalized identities.

11

## 4. Open Problems

Let us recall the identities between the function $F_3$ and the two functions $F_1$ and $F_2$. With the notation $p = 1/(1 - y)$ and $q = 1/(1 - x)$, the identities may be stated as

$$\sum_{n \geq 0} \left( \frac{1}{p}; \frac{1}{q} \right)_n = \sum_{n \geq 0} \left( \frac{p}{q} \right)^{n+1} \left( \frac{1}{q}; \frac{1}{q} \right)_n, \text{ and} \tag{17}$$

$$\sum_{n \geq 0} pq^n (p; q)_n (q; q)_n = \sum_{n \geq 0} \left( \frac{p}{q} \right)^{n+1} \left( \frac{1}{q}; \frac{1}{q} \right)_n. \tag{18}$$

By Theorem 2.1, the identities (17) and (18) are valid in the ring of power series in $(p-1)$ and $(q-1)$. However, it seems that the identities are in fact valid for any value of $p$ and $q$ for which the sums are terminating, and they also appear to be valid as power series in $(p - p_0)$ and $(q - q_0)$ whenever the corresponding sums both converge as formal power series. We state this more precisely in the next conjecture.

**Conjecture 4.1.** *If $p_0$ and $q_0$ are two complex $k$-th roots of unity for some $k$, then the left-hand side and the right-hand side of* (17) *converge to the same power series in* $(p - p_0)$ *and* $(q - q_0)$.

*Similarly, if $q_0$ is a root of unity, then both sides of* (18) *converge to the same power series in* $(q - q_0)$.

Finally, we note that although the power series from Theorem 2.1 have a natural combinatorial interpretation as generating functions of combinatorial objects, our proof of Theorem 2.1 does not use this interpretation at all. One might ask whether there is a way to interpret the identities of Theorem 2.1 combinatorially and thus provide an alternative proof.

**Problem 4.2.** Apart from the (previously known) identity $F_1(x, y) = F_3(x, y)$, is there a combinatorial proof of the identities in Theorem 2.1?

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