STIDENT'S SOLUTIONS MANUAL

Inroduction to Linear Algebra

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Contents

Chapter 6. Transportation Problems

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Preface

This manual includes

Answers to Selected Exercises

(pages 305–317 of the textbook) with some corrections and give more solutions and answers. Note that the exercises may have many correct solutions and even several correct answers.

Here are some other corrections.

Page 11. There are two Exercises 57. Both solved below.

Page 17. Replace Aarea in the table by Area.

Page 138. Replace dv in the second displayed line by $d = v$.

Page 165, Exercise 8. Replace $+-$ by $-$.

Page 210, Exercise 10. The empty entry means 0.

Page 240, Exercise 8. Replace p by p twice.

Page 240. There are two Exercises 12. Both solved below.

Page 249., line 6. Replace $p = 3$ by $p = 2$.

Page 250, Exercise 13. Replace $+1/\alpha^{t+1}$ by $-(-1/\alpha)^{t+1}$.

Chapter 1. Introduction

§1. What Is Linear Programming?

1. True.
3. True.
5. True. This is because for real numbers any square and any absolute value are nonnegative.
7. False. For $x = -1,\ 3(-1)^3 < 2(-1)^2$.
8. False (see Definition 1.5).
9. False (see Example 1.9 or 1.10).
11. False. For example, the linear program

$$\text{Minimize } x + y \text{ subject to } x + y = 1$$

has infinitely many optimal solutions.

13. True. It is a linear equation. A standard form is $4x = 8$ or $x = 2$.
15. No. This is not a linear form, but an affine function.
16. Yes, if z is independent of x, y.
17. Yes if a and z do not depend on x, y.
18. No (see Definition 1.1).
19. No. But it is equivalent to a system of two linear constraints.
21. Yes. We can write $0 = 0 \cdot x$, which is a linear form.
22. True if y is independent of x and hence can be considered as a given number; see Definition 1.3.
23. Yes if a, b are given numbers. In fact, this is a linear equation.
25. No. We will see later that any system of linear constrains gives a convex set. But we can rewrite the constraint as follows $x \geq 1$ OR $x \leq 1$. Notice the difference between OR and AND.
27. See Problem 6.12.
28. We multiply the first equation by 5 and subtract the result from the second equation:

$$\left\{ \begin{array}{rcrcr} x & + & 2y & = & 3 \\ & & -y & = & -11. \end{array} \right.$$

Multiplying the second equation by -1, we solve it for y. Substituting this into the first equation, we find x. The answer is

$$\left\{ \begin{array}{rcr} x & = & -19 \\ y & = & 11. \end{array} \right.$$

29. $x = 3 - 2y$ with an arbitrary y.

31. min $= 0$ at $x = y = 0, z = -1$. All optimal solutions are given as follows: $x = -y, y$ is arbitrary, $z = -1$.

33. min $= 1$ at $x = 0$.

35. min $= 0$ at $x = -y = 1/2, z = -1$.

37. No. This is a linear equation.

38. No. Suppose $x + y^2 = ax + by$ with a, b independent of x, y. Setting $x = 0, y = 1$ we find that $b = 1$. Setting $x = 0, y = -1$ we find that $b = -1$.

39. No.

41. Yes.

43. No.

44. No. Suppose that xy is an affine function $ax + by + c$ of x, y. Setting $x = y = 0$, we find that $c = 0$. Setting $x = 0, y = 1$, we find that $b = 0$. Setting $x = 1, y = 0$, we find that $a = 0$. Setting $x = y = 1$, we find that $1 = 0$.

45. Yes.

47. Yes.

49. No.

50. No, this is a linear form.

51. Yes.

53. Yes. In fact, this is a linear equation.

55. No. This is not even equivalent to any linear constraint with rational coefficients.

57. No, see Exercise 44.

57. Let $f(x, y) = cx + dy$ be a linear form. Then $f(ax, ay) = cax + day = a(cx + dy) = af(x, y)$ for all a, x, y and $f(x_1 + x_2, y_1 + y_2) = c(x_1 + x_2) + d(y_1 + y_2) = cx_1 + dy_1 + cx_2 + dy_2 = f(x_1, y_1) + f(x_2, y_2)$ for all x_1, x_2, y_1, y_2.

59. min $= 2^{-100}$ at $x = 0, y = 0, z = \pi/2, u = -100, v = -100$. In every optimal solution, x, y, u, v are as before and $z = \pi/2 + n\pi$ with any integer n such that $-32 \leq n \leq 31$. So there are exactly 64 optimal solutions.

§2. Examples of Linear Programs

2. min $= 1.525$ at $a = 0, b = 0.75, c = 0, d = 0.25$

4. Let x be the number of quarters and y the number of dimes we pay. The program is

$$25x + 10y \to \min,$$

subject to

$$0 \le x \le 100,\ 0 \le y \le 90, 25x + 10y \ge C \text{ (in cents)},\ x, y \text{ integers.}$$

This program is not linear because the conditions that x, y are integers. For $C = 15$, an optimal solution is $x = 0, y = 2$. For $C = 102$, an optimal solution is $x = 3y = 3$ or $x = 1, y = 8$. For $C = 10000$, the optimization problem is infeasible.

5. Let x, y be the sides of the rectangle. Then the program is

$$xy \to \min,$$

subject to

$$x \ge 0,\ y \ge 0, 2x + 2y = 100.$$

Since $xy = x(50 - x) = 625 - (x - 25)^2 \le 625$, max $= 625$ at $x = y = 25$.

7. We can compute the objective function at all 24 feasible solutions and find the following two optimal matchings: Ac, Ba, Cb, Dd and Ac, Bb, Ca, Dd with optimal value 7.

8. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9 + 9 + 7 + 9 + 9 = 43$ for the objective function. This bound cannot be achieved because of a conflict over c (the third column). So max ≤ 42. On the other hand, the matching aa, Bb, Cc, De, Ed achieved 42, so this is an optimal matching.

9. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9+9+9+9+8+9+6 = 59$ for the objective function. However looking at B and C, we see that they cannot get $9 + 9 = 18$ because of the conflict over g. They cannot get more than $7 + 9 =16$. Hence, we have the upper bound max ≤ 57. On the other hand, we achieve this bound 57 in the matching Ac, Bf, Cg, De, Eb, Fd, Ga.

11. Let c_i be given numbers. Let c_j be an unknown maximal number (with unknown j). The linear program is

$c_1x_1 + \cdots + c_nx_n \to \max$, all $x_i \ge 0$, $x_1 + \cdots + x_n = 1$.

Answer: max $= c_j$ at $x_j = 1, x_i = 0$ for $i \ne j$.

§3. Graphical Method

1. Let SSN be 123456789. Then the program is
$-x \to \max, 7x \le 5, 13x \ge -8, 11x \le 10.$
Answer: $\max = 8/13$ at $x = -8/13$.
Answer: $\min = -22$ at $x = -65/11, y = 59/11$.

3. Let SSN be 123456789. Then the program is
$x - 3y \to \min, |6x + 4y| \le 14, |5x + 7y| \le 8, |x + y| \le 17.$
Answer: $\min = 242/11$ at $x = 65/11, y = -59/11$.

4. The first constrain is equivalent to 2 linear constraints $-7 \le x \le 3$. The feasable region for the second constraint is also an interval, $-8 \le x \le 2$. The feasable region for the linear program is the interval $-7 \le x \le 2$. In Case (i), the objective function is an increasing function of x and reaches its maximum 14 at the right endpoint $x = 2$. In Case (ii), the objective function is a decreasing function of x and reaches its maximum 63 at the left endpoint $x = -7$. In Case (ii),

$$\max = \begin{cases} 2b \text{ at } x = 2 & \text{if } b > 0, \\ 0 \text{ when } -7 \le x \le 2 & \text{if } b = 0, \\ -7b \text{ at } x = -7 & \text{if } b < 0. \end{cases}$$

5. $\max = 135$ at $x = -9, y = 18$
6. The objective function is not defined when $y = 0$. When $y = -1$ and $x \to \infty$, we have $x/y \to -\infty$. So this minimization problem is unbounded, $\min = -\infty$.
7. $\min = -1/4$ at $x = 1/2, y = -1/2$ or $x = -1/2, y = 1/2$.
9. $\max = 1$ at $x = y = 0$
11. $\max = 22$ at $x = 4, y = 2$

13. The program is unbounded.

14. The feasible region van be given by 4 linear constraints: $-5 \le x \le 0, 2 \le y \le 3$. It is a rechtangle with 4 corners $[x, y] = [0, 3], [-5, 3][-5, -2], [0, -2]$. The objective function is not affine. Its level $|x| + y^2 = c$ is empty when $c < 0$, is a point when $c = 0$, and is made of 2 parabola pieces when $c > 0$. It is clear that $\max = 14$ at $x = -5, y = 3$. The optimal solution is unique.

15. $\max = 3$ at $x = y = 9, z = 1$. See the answer to Exercise 11 of §2.

Chapter 2. Background

§4. Logic

1. False. For $x = -1$, $|-1| = 1$.
3. False. For $x = -10$, $|-10| > 1$.
5. True. $1 \geq 0$.
7. True. $2 \geq 0$.
9. True. The same as Exercise 7.
11. False. $1 \geq 1$.
13. True. $5 \geq 0$.
15. False. For example, $x = 2$.
17. True. Obvious.
19. False. For example, $x = 1$.
21. True. $1 \geq 0$.
23. Yes. $10 \geq 0$.
25. No, it does not. $(-5)^2 > 10$.
27. True.
29. False. The first condition is stronger than the second one.
30. False. The converse is true.
31. True.
33. (i) $\Rightarrow$ (ii)$\Rightarrow$ (iv) $\Rightarrow$ (ii).
35. (i) $\Rightarrow$ (iii).
37. (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii)
39. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
40. given that, assuming that, supposing that, in the case when, granted that.
41. “only if”
42. This depends on the definition of $linear function \triangleright$
43. No. $x \geq 1, x \leq 0$ are two feasible constraints, but the system is infeasible.
44. False.
45. False. Under our conditions, $|x| > |y|$.
47. Correct (add the two constraints in the system and the constraint $0 \leq 1$).
49. No, it does not follow.
51. Yes, it does. It is the sum of the first two equations.

§5. Matrices

1. $[2, 1, -6, 6]$

3. -14

5. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 3 \\ -2 & -4 & 0 & 6 \\ 4 & 8 & 0 & -12 \end{bmatrix}$

7. 2744

8. $-14^{999}A^TB = \begin{bmatrix} 0 & 14^{999} & 2\cdot 14^{999} & -4\cdot 14^{999} \\ 0 & 2\cdot 14^{999} & 4\cdot 14^{999} & -8\cdot 14^{999} \\ 0 & 0 & 0 & 0 \\ 0 & -3\cdot 14^{999} & -6\cdot 14^{999} & 12\cdot 14^{999} \end{bmatrix}$

9. No. $1 \neq 4$.

10. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 5 & 2 & 3 & -1 \\ 1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 2 & 3 & 0 & -1 \\ 1 & -1 & -3 & 0 & -2 \\ 0 & 0 & -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

15. $b = a - 1, c = -1/3, d = 7a - 4, a$ arbitrary

16. We permute the columns of the coefficient matrix and multiply the first equation by -1:

$$\begin{bmatrix} 0 & 0 & 1 & -2 & -3 \\ 1 & 0 & -2 & -1 & -3 \\ 0 & 1 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ x \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Next we add the first row multiplied by 2 to the send row:

$$\begin{bmatrix} 0 & 0 & 1 & -2 & -3 \\ 1 & 0 & 0 & -5 & -9 \\ 0 & 1 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ x \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Now we write the answer:

$y = 2b + 3c - 1, a = 5b + 9c, x = 3c - 1$, where b, c are arbitrary.

17. We will solve the system for x, d, a. So we rewrite the system (see the solution to Exercise 14 above):

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & -1 & 1 & -1 & -3 & -2 \\ 1 & 0 & 5 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now we subtract the first row from the last one and multiply the second row by -1 :

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 5 & -2 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Then we multiply the last row by $1/5$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 1 & -2/5 & -3/5 & -2/5 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2/5 \end{bmatrix}$$

Finally, we add the last row to the second one:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 3/5 & 12/5 & 8/5 \\ 0 & 0 & 1 & -2/5 & -3/5 & -2/5 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2/5 \\ -2/5 \end{bmatrix}$$

So our answer is

$x = -3b - 3c + y + 1,$

$d = -0.6b - 2.4c - 1.6y - 0.4,$

$a = 0.4b + 0.6c + 0.4y - 0.4$

with arbitrary b, c, y.

18. $2A + 3B = [-6, 5, -2, 0, , 9, 10]$.

19. $AB^T = 5$

20. $BA^T = 5$

21. $A^TB = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 3 & 2 \\ 2 & -1 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 6 & 4 \end{bmatrix}$

22. $B^TA = \begin{bmatrix} 0 & -2 & 2 & 0 & 0 & -4 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 6 \\ 0 & 2 & -2 & 0 & 0 & 4 \end{bmatrix}$

23. $(A^TB)^2 = 25$

25. 5^{1000}

27. $AB^T = 4$

28. $BA^T = 4$

29. $A^TB = \begin{bmatrix} -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ 1 & -1 & 0 & -3 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 6 & 6 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

30. $B^TA = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & 0 \\ -1 & -1 & 1 & 0 & 0 & -2 & 0 \end{bmatrix}$

31. $(A^TB)^2 = 4A^TB = \begin{bmatrix} -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ 4 & -4 & 0 & -12 & -12 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 8 & 0 & 24 & 24 & 16 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

32. $(A^TB)^3 = 16A^TB$

$$= \begin{bmatrix} -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ 16 & -16 & 0 & -48 & -48 & -32 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 32 & 0 & 96 & 96 & 64 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

33. $4^{999} A^T B$

34. $AB^T = \begin{bmatrix} 89/4 & 23 \\ 341/8 & 107/2 \end{bmatrix}$, $BA^T = \begin{bmatrix} 89/4 & 341/8 \\ 23 & 107/2 \end{bmatrix}$,

$$A^T B = \begin{bmatrix} 9/2 & 21/4 & 23 \\ 1/2 & 9/4 & -3 \\ 27/2 & 63/4 & 69 \end{bmatrix}, \; B^T A = \begin{bmatrix} 9/2 & 1/2 & 27/2 \\ 21/4 & 9/4 & 63/4 \\ 23 & -3 & 69 \end{bmatrix},$$

$$(A^T B)^2 = \begin{bmatrix} 81/4 & 441/16 & 529 \\ 1/4 & 81/16 & 9 \\ 729/4 & 3969/16 & 4761 \end{bmatrix},$$

$$(A^T B)^3 = \begin{bmatrix} 729/8 & 9261/64 & 12167 \\ 1/8 & 729/64 & -27 \\ 19683/8 & 250047/64 & 328509 \end{bmatrix}.$$

35. $E_1 C = \begin{bmatrix} 3 & 6 & 9 \\ -8 & -10 & -12 \end{bmatrix}$, $E_2 C = \begin{bmatrix} 21 & 27 & 33 \\ 4 & 5 & 6 \end{bmatrix}$,

$$(E_1)^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}, \; (E_2)^n = \begin{bmatrix} 1 & 5n \\ 1 & 0 \end{bmatrix}.$$

36. $CE_1 = \begin{bmatrix} 2 & 6 & 12 \\ 8 & 15 & 24 \end{bmatrix}$, $CE_2 = \begin{bmatrix} -8 & 2 & 3 \\ -14 & 5 & 6 \end{bmatrix}$,

$$DE_1 = \begin{bmatrix} 18 & 24 & 28 \\ 12 & 15 & 16 \\ 6 & 6 & 4 \end{bmatrix}, \; DE_2 = \begin{bmatrix} -12 & 8 & 7 \\ -6 & 5 & 4 \\ 0 & 2 & 1 \end{bmatrix},$$

$$E_1 E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -12 & 0 & 4 \end{bmatrix}, E_2 E_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -6 & 0 & 4 \end{bmatrix},$$

$$(E_1)^n = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}, (E_2)^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3n & 0 & 1 \end{bmatrix}.$$

37. $\begin{bmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix}$

38. For $n \times n$ diagonal matrices

$$A = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}, B = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix},$$

we have

$$A + B = \begin{bmatrix} a_1 + b_1 & 0 & \dots & 0 \\ 0 & a_2 + b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n + b_n \end{bmatrix} \text{ and}$$

$$AB = BA = \begin{bmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n b_n \end{bmatrix}.$$

For $m < n$, diagonal $m \times n$ matrices have the form $[A, 0]$ and $[B, 0]$ with $m \times m$ diagonal matrices A, B, and $[A, 0] + [B, 0] = [A + B, 0]$, where 0 is the zero $m \times (n - m)$ matrix. Similarly, sum of diagonal $m \times n$ matrices is diagonal in the case $m > n$.

Any nondiagonal entry of the product of diagonal matrices is the dot product of two rows, each having at most one nonzero entry, and these entries are located at different positions. So the product is a diagonal matrix.

39. Let $A = [a_{ij}], B = [b_{ij}]$ be upper triangular, i.e., $a_{ij} = 0 = b_{ij}$ whenever $i > j$. Than $(A + B)_{ij} = a_{ij} + b_{ij} = 0$ whenever $i > j$, so $A + B$ is upper triangular. For $i > j$, the entry $(AB)_{ij}$ is the product of a row whose first $i - 1 > j$ entries are zero and a column whose entries are zero with possible exception of the fist j entries. So this $(AB)_{ij} = 0$. Thus, AB is upper triangular.

Take upper triangular matrices $A = [1, 2], B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB \neq BA$ (they have different sizes). Here is an example with square matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now $AB = B \neq A = BA$.

40. Solution is similar to that of Exercise 39, and these Exercises can be reduced to each other by matrix transposition.

41. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

42. Adding the first column to the third column we obtain the matrix
$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & -2 \\ 2 & 4 & 7 \\ 8 & -2 & 7 \end{bmatrix}.$$
Adding the second column multiplied by 2 to the third column we obtain a lower matrix:
$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 2 & 4 & 15 \\ 8 & -2 & 3 \end{bmatrix}.$$
Now we kill nondiagonal entries in the second, third, and fourth rows using multiples of previous rows. Seven row addition operations bring our matrix to the diagonal matrix
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix}.$$

43. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

44. We kill the entries 5, 5 in the first column by two row addition operations:
$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}.$$
Adding a multiple of the second row to the last row, we obtain the upper triangular matrix
$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix}.$$
Since the diagonal entries are nonzero, we can bring this matrix to its diagonal part
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & -73/11 & 0 \end{bmatrix}$$
by five column addition operations.

§6. Systems of Linear Equations

1. $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ is invertible; $\det(A) = -4$

3. The matrix is invertible if and only if $abc \neq 0$; $\det(A) = abc$.

5. $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is invertible; $\det(A) = -2$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 13/7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is invertible; $\det(A) = 13$

9. $0 = 1$ (no solutions)

10. We do one row addition operation with the augmented matrix and then drop the zero row:
$-2 \begin{bmatrix} 1 & 2 & | & 3 \\ 2 & 4 & | & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix} \mapsto [1, 2 \mid 3]$. Answer: $x = 3 - 3y,\ y$ being arbitrary.

11. $x = -z - 3b + 9,\ y = -z + 2b - 6$.

12. We perform two addition and one multiplication row operations on the augmented matrix:
$-1 \begin{bmatrix} 1 & 4 & | & 1 \\ 1 & 5 & | & -8 \end{bmatrix} \mapsto -2 \begin{bmatrix} 1 & 4 & | & 1 \\ 0 & 2 & | & -9 \end{bmatrix} \mapsto$
$1/2 \cdot \begin{bmatrix} 1 & 0 & | & 19 \\ 0 & 2 & | & -9 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & | & 19 \\ 0 & 1 & | & -9/2 \end{bmatrix}$.
Answer: $x = 19, y == 9/2$.

13. If $t \neq 6 + 2u$, then there are no solutions. Otherwise, $x = -2y + +v + 3$, y arbitrary.

15. If $t = 1$, then $x = 1 - y$, y arbitrary.
If $t = -1$, there are no solutions.
If $t \neq \pm 1$, then $x = (t^2 + t + 1)/(t + 1), y = -1/(t + 1)$.

17. It is convenient to write the augmented matrix corresponding to the variables y, z, x (rather than x, y, z). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$-5/3 \begin{array}{c} \begin{matrix} y & z & x & & \end{matrix} \\ \begin{bmatrix} 3 & 5 & 2 & | & 2 \\ 5 & 8 & 3 & | & b \end{bmatrix} \end{array} \mapsto$$

$$\begin{array}{r} 1/3\cdot \\ -3\cdot \end{array}\begin{bmatrix} 3 & 5 & 2 & | & 2 \\ 0 & -1/3 & -1/3 & | & b-10/3 \end{bmatrix} \mapsto$$

$$-5/3 \begin{bmatrix} 1 & 5/3 & 2/3 & | & 2/3 \\ 0 & 1 & 1 & | & -3b+10 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & -1 & | & 5b-16 \\ 0 & 1 & 1 & | & -3b+10 \end{bmatrix}.$$

Answer:$y = x + 5b - 16, z = -x - 3b + 10,$ x is arbitrary.

18. It is convenient to write the augmented matrix corresponding to the variables x, z, y (rather than x, y, z). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$\begin{array}{cc} & \begin{array}{ccccc} x & z & y & & \end{array} \\ -3/2 & \begin{bmatrix} 2 & 5 & 3 & | & 2 \\ 3 & 8 & 5 & | & b \end{bmatrix} \end{array} \mapsto$$

$$\begin{array}{r} 1/2\cdot \\ 2\cdot \end{array}\begin{bmatrix} 2 & 5 & 3 & | & 2 \\ 0 & 1/2 & 1/2 & | & b-3 \end{bmatrix} \mapsto$$

$$-5/2 \begin{bmatrix} 1 & 5/2 & 3/2 & | & 1 \\ 0 & 1 & 1 & | & 2b-6 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & -1 & | & -5b+15 \\ 0 & 1 & 1 & | & 2b-6 \end{bmatrix}.$$

Answer:$x = y - 5b + 15, z = -y + 2b - 6,$ y is arbitrary.

19. No. The halfsum of solutions is a solution.

21. $A^{-1} = \begin{bmatrix} 7/25 & 4/25 & -1/25 \\ 19/25 & -7/25 & 8/25 \\ -18/25 & 4/25 & -1/25 \end{bmatrix}$

23. $A^{-1} = \begin{bmatrix} -3/22 & -1/22 & -41/22 & 3/11 \\ -15/22 & -5/22 & -51/22 & 4/11 \\ 5/22 & 9/22 & 61/22 & -5/11 \\ 15/22 & 5/22 & 73/22 & -4/11 \end{bmatrix}$

25. If $a \neq 0$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d - bc/a \end{bmatrix}.$$

If the first column of A is zero, then $A = 1_2 A = LU$. If the first row of A is zero, then $A = A 1_2 = LU$. Finally, if $a = 0 \neq bc$, then $A \neq LU$.

27. This cannot be done. We have $0 = A_{1,1} = L_{1,1}U_{1,1} \neq 0$ since A is invertible, hence U, V are invertible.

28. This cannot be done, because A is invertible (see the solution of Exercise 5) and $A_{11} = 0$. See the solution of Exercise 27.

29. $A = LU = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix}.$

$UL = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -1 \\ 21 & 33 & 8 \\ -50 & -100 & -25 \end{bmatrix}.$

30. $A = LU$

$= \begin{bmatrix} 1 & -2 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix}.$

$UL = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix}$

$= \begin{bmatrix} -11 & -30/11 & -1 \\ 71 & 185/11 & 8 \\ 26/11 & 104/121 & 13/11 \end{bmatrix}.$

32. Answer:
$x = a + b^2 + c^3 - d,$
$y = a + b^2 - 3c^3 + 2 * d,$
$z = -a - b^2 + 2c^3 - d.$

33. $x = -3(19 + 2d)/8, y = (15 + 2d)/8, z = -(3 + 2d)/8$

35. $x = (15u + 4v)/16, y = (11u + 4v)/16, z = -3u/4$

37. $x = y = 1, z = 1$

39. $x = y = 0, z = 100$

40. It is clear that any nonzero column with at least two entries can be reduced to the first column of the indentity matrix by row addition operations. By induction on the number of columns, it follows that any $m \times n$ matrix with linearly independent columns can be reduced by row addition operations to the matrix of the first n columns of 1_m provided that $m > n$. Therefore any invertible $m \times m$ matrix can be reduced by by row addition operations to the diagonal matrix with the first $m - 1$ diagonal entries being ones, and the last entry being the determinant. One row multiplication operation applyed to this matrix gives 1_m.

Therefore multiplication by an invertible matrix on the left is equivalent to performing row addition operations and a row multiplication operation.

If $A = 0$, then $b = 0, A' = 0$. and $b' = 0$, so there is nothing to prove. Similarly, the case $A' = 0$ is trivial. assume now that $A \neq 0$ and $A' \neq 0$.

Let B be the submatrix in $[A, b]$ such that the rows of B form a basis for the row space of A, and let B' be a similar matrix for $[A', b']$. By Theorem 6.11, $B' = DB$ and $B = D'B'$ for some matrices D, D'. We have $B = D'DB$ and $B' = DD'$. Since the rows of B are linearly independent, $D'D$ is the identity matrix. Since the rows of B' are linearly independent, DD' is the identity matrix. So D is invertible, hence B, B' have the same size. So row operations on A allows to change the rows of B to the rows of B'. Now by row addition operations we can make the other rows of $[A, b]$ (if any) equal to remaining rows of $[A', b]'$ (if any). A row permutation operation finish the job.

41. We use the parts of the previous solution. In particular, it is clear that the rank of the matrices of $[A, b]$ and $[A', b']$ are the same. By row addition operations we can make the last $m - m'$ rows of $[A, b]$ to be zeros. Then, as shown in the previous solution, we can the first m' rows to be the rows of $[A', b']$ by row addition operations and a row multiplication operations.

The only thing remaining to show is how to replace a row multiplication operation by row addition operations in presence of a zero row. Here is how this can be done:

$$1 \begin{bmatrix} r \\ 0 \end{bmatrix} \mapsto (d-1) \begin{bmatrix} r \\ r \end{bmatrix} \mapsto -1/d \begin{bmatrix} dr \\ r \end{bmatrix} \mapsto \begin{bmatrix} dr \\ 0 \end{bmatrix},$$

where multiplication of a row r by a nonzero number d is accomplished by three addition operations.

Chapter 3. Tableaux and Pivoting

§7. Standard and Canonical Forms for Linear Programs

1. Set $u = y + 1 \geq 0$. Then $f = 2x + 3y = 2x + 3v - 3$ and $x + y = x + u - 1$. A canonical form is

$$-f = -2x - 3v + 3 \to \min, x + u \leq 6, u, x \geq 0.$$

A standard form is

$$-f = -2x - 3v + 3 \min, x + u + v = 6, u, v, x \geq 0$$

with a slack variable $v = 6 - x - u \geq 0$.

2. Excluding $y = x+1$ and using $y \geq 1$, we obtain the canonical form

$$-x \to \min,\ 2x \leq 8, x \geq 0.$$

Introducing a slack variable $z = 8 - 2x$, we obtain the standard form

$$-x \to \min,\ 2x + z = 8, x \geq 0, z \geq 0.$$

3. We solve the equation for x_3:

$x_3 = 3 - 2x_2 - 3x_4$

and exclude x_3 from the LP:

$$x_1 - 7x_2 + 3 \to \min,\ x_1 - x_2 + 3x_4 \geq 3, \text{ all } x_i \geq 0.$$

A canonical form is

$$x_1 - 7x_2 + 3 \to \min,\ -x_1 + x_2 - 3x_4 \leq -3,\ \text{ all } x_i \geq 0.$$

A standard form is

$$x_1 - 7x_2 + 3 \to \min,\ -x_1 + x_2 - 3x_4 + x_5 = -3,\ \text{ all } x_i \geq 0$$

with a slack variable $x_4 = x_1 - x_2 + 3x_4 - 3$.

5. Set $t = x+1 \geq 0, u = y-2 \geq 0, f = x = y+z = t+u+z+1$ (the objective function). Then a standard and canonical form for our problem is

$$x + u + z + 1 \to \min; t, u, z \geq 0.$$

6. This mathematical program has exactly two optimal solution, but the set of optimal solutions of any LP is convex and hence cannot consist of exactly two optimal solutions (cf. Exercise 19 in §6.). Each of two optimal solution can be the optimality region for a linear program. For example, min $= -26$ at $x = 1, y = -3, z = 0$ is the only answer for the linear program $-26 \to \min, x = 1, y = -3, z = 0$.

7. Using standard tricks, a canonical form is

$$-x \to \min, x \le 3, -x \le -2, x \ge 0.$$

A standard form is

$$-x \to \min, x + u + 3, -x + v = -2; x, u, v \ge 0$$

with two slack variables.

8. Excludin $y = 1 - x$ from the LP, we obtain

$$f = -x + z + 2 \to \max, z \ge 0.$$

Writing $x = u - v$ and replacing f by $-f$, we obtain a normal and standard form:

$$-f = u - v - z - 2 \to \min; u, v, z \ge 0.$$

It is clear that the program is unbounded.

9. One of the given equations reads

$-5 - x - z = 0,$

which is inconsistent with given constraints $x, z \ge 0$. So we can write very short canonical and standard forms:

$0 \to \min, 0 \le -1; x, y, z \ge 0$ and $0 \to \min, 0 = 1; x, y, z \ge 0$.

10. The first matrix product is not defined.

11. Set $x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}^T]$ and $c = [3, -1, 1, 3, 1, -5, 1, 3, 1]$. Using standard tricks, we obtain the canonical form

$cx \to \min, Ax \le b, x \ge 0$

with

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -2 & 3 & 1 \\ 2 & -2 & -2 & 2 & 3 & -1 & -2 & 1 & 1 \\ -2 & 2 & 2 & -2 & -3 & 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 & 3 & -1 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

and $b = [-3, -1, 2, -2, 0, 0]^T$.

Excluding a couple of variables using the two given equations, we would get a canonical form with two variables and two constraints less. A standard form can be obtained from the canonical form by introducing a column u of slack variables:

$$cx \to \min, Ax + u = b, x \ge 0, u \ge 0.$$

§8. Pivoting Tableaux

1.
$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & \\
\wedge\!| & \wedge\!| & \wedge\!| & \wedge\!| & \wedge\!| & \\
a & b & c & d & e & 1
\end{array}
$$
$$
\left[\begin{array}{cccccc}
.3 & 1.2 & .7 & 3.5 & 5.5 & -50 \\
73 & 96 & 20253 & 890 & 279 & 4000 \\
9.6 & 7 & 19 & 57 & 22 & -100 \\
10 & 15 & 5 & 60 & 8 & 0
\end{array}\right]
\begin{array}{l}
= u_1 \ge 0 \\
= u_2 \ge 0 \\
= u_3 \ge 0 \\
= C \to \min
\end{array}
$$

3. $A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ -1 & 0 & 0 & 2 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}$

5. Canonical form:
$y - 5z + 2 \to \min,$
$-3x - y + 5z \le 3,$
$-x - y \le -10,$
$x + y \le 10,$
$-2y + 10z \le -7;$
$x, y, z \ge 0.$
Standard form:
$y - 5z + 2 \to \min,$
$-3x - y + 5z + u = 3,$
$x + y = 10,$
$-2y + 10z + v = -7;$
$x, y, z, u, v \ge 0.$

7. Canonical form: $-2x + 2 \to \min,$
$-3x - 2y - z \le 2, x - 3y \le 1, 2y - 2z \le 0; x, y, z \ge 0.$
Standard form: $-2x + 2 \to \min,$
$-3x - 2y - z + u = 2, x - 3y + v = 1, 2y - 2z + w = 0;$
$x, y, z, u, v, w \ge 0.$

9. The matrix is not square.

11.
$$
\begin{array}{cccc}
z & a & 3 & x
\end{array}
$$
$$
\begin{bmatrix} 1 & 2 & b+3 & a+1 \\ -1 & 2 & 3 & 1 \end{bmatrix}
\begin{array}{l} = y \\ = 1 \end{array}
$$

12.
$$\begin{array}{cccc|l} 1 & a & 3 & z & \\ \hline \end{array}$$
$$\begin{array}{cccc} 1 & a & 3 & z \end{array}$$
$$\begin{bmatrix} 1+a & -2a & b-3a & a \\ 1 & -2 & -3 & 1 \end{bmatrix} \begin{array}{l} = y \\ = x \end{array}$$

13.
$$\begin{array}{c} 2 \end{array}$$
$$\begin{bmatrix} 1/5 \end{bmatrix} = x$$

14.
$$\begin{array}{ccccc} 1 & a & 0 & x & x \end{array}$$
$$\begin{bmatrix} 1 & 0 & b & a & -3 \\ -1 & 2^* & 3 & 1 & 0 \end{bmatrix} \begin{array}{l} = y \\ = z \end{array} \mapsto$$
$$\begin{array}{ccccc} 1 & z & 0 & x & x \end{array}$$
$$\begin{bmatrix} 1 & 0 & b & a & -3 \\ 1/2 & 1/2 & -3/2 & -1/2 & 0 \end{bmatrix} \begin{array}{l} = y \\ = a \end{array}$$

15.
$$\begin{array}{ccccc} 1 & u & 0 & x & 1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & b & a & -3 \\ 0 & 1 & 0 & 0 & -1 \\ 1/2 & 1/2 & -3/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{array}{l} = y \\ = z \\ = 0 \\ = v \end{array}$$

16.
$$\begin{array}{ccccc} x_1 & x_2 & x_8 & x_4 & 1 \end{array}$$
$$\begin{bmatrix} 4/3 & -2/3 & 1/3 & 2/3 & -3 \\ -1/3 & -4/3 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1/3 & -2/3 & 1/3 & -1/3 & 0 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{array}{l} = x_5 \\ = x_6 \\ = x_7 \\ = x_3 \\ = v \end{array}$$

17.
$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_7 & x_6 & 1 \end{array}$$
$$\begin{bmatrix} 1/3 & 0 & 1/3 & 1/3 & -1/3 & 1/3 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -2/3 & 2 & 10/3 & 4/3 & -1/3 & 0 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -2/3 & 1 & 1/3 & 4/3 & -1/3 & 7/3 & 3 \end{bmatrix} \begin{array}{l} = x_5 \\ = x_8 \\ = x_9 \\ = x_{10} \\ = v \end{array}$$

18. Let us show that every column b of A equals to the corresponding column b' of A'. We set the corresponding variable on the top to be 0, and the other variables on the top to be zeros. Then the variables on the side take certain values, namely, $y = b = b'$.

§9. Standard Row Tableaux

1. Passing from the standard row tableau on page 95 to the canonical form (i.e., dropping the slack variables), we obtain a Linear program with one variable: $y/2 - 15/2 \to \min$,

$16y - 26 \geq 0, -3y/2 + 15/2 \geq 0, 3y/2 - 15/2 \geq 0, y \geq 0.$

We rewrite our constraints: $y \geq 0, 13/8, 5; y \leq 5$, so $y = 5$. In terms of the standard tableau, our answer is

$\min(-z) = -5$ at $y = 5, w_1 = 54, w_2 = w_3 = 0.$

In terms of the original variables, our answer is

$\max(z) = 5$ at $u = -3, v = -4, x = -1, y = 5.$

2. $$\begin{array}{c} \begin{array}{ccc} x & y & 1 \end{array} \\ \begin{bmatrix} -4 & -5 & 7 \\ -2 & -3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ = u \\ = -P \to \min \end{array}$$

with a slack variable $u = 7 - 4x - 5y$

3. $$\begin{array}{c} \begin{array}{ccc} x & y & 1 \end{array} \\ \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ \to \min \end{array}$$

with slack variable u_i

5. We multiply the last row by -1 and remove the the second and third rows from the tableau

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & -1 & -2 & -3 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = x_2 \\ = -v \to \min \end{array}$$

$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 5x_3 + x_4 + x_5.$

The tableau is not standard because x_2 occurs twice. We pivot on 1 in the x_2-row and x_6-column:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1^* & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = x_2 \\ = v \to \max \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_2 & 1 \end{array} \\ \begin{bmatrix} 1 & -2 & -2 & 0 & -3 & 1 & -1 \\ 0 & -2 & -3 & -1 & 0 & 1 & -1 \\ -1 & -1 & -6 & -1 & 1 & 2 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = x_6 \\ = v \to \max \end{array} .$$

Now we combine two x_2 from the top and obtain the standard tableau

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 1 & \\ \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 & 0 & -3 & -1 \\ 0 & -1 & -3 & -1 & 0 & -1 \\ -1 & 1 & -6 & -1 & 1 & 1 \end{bmatrix} \begin{array}{l} = x_7 \\ = x_6 \\ = v \to \max \end{array} .$$

The equations

$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 5x_3 + x_4 + x_5$

relate this LP with the original LP.

7. We pivot on the first 1 in the first row and then on 3 in the second row:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} = x_2 \\ = x_4 \\ = v \to \min \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{ccccccc} x_2 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & 3^* & 2 & -3 & 2 & -2 \\ -1 & 2 & 4 & 2 & -2 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} = x_1 \\ = x_4 \\ = v \to \min \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{ccccccc} x_2 & x_2 & x_4 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 2/3 & 0 & -1/3 & -1/3 & 2 & -1/3 & -2/3 \\ 1/3 & 0 & 3 & -2/3 & 1 & -2/3 & 2/3 \\ 1/3 & 2 & 4/3 & -2/3 & 2 & -5/3 & 8/3 \end{bmatrix} \end{array} \begin{array}{l} = x_1 \\ = x_3 \\ = v \to \min \end{array} .$$

Now we combine two x_2-columns and two x_4-columns and obtain the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_2 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 2/3 & -2/3 & 2 & -1/3 & -2/3 \\ 1/3 & 7/3 & 1 & -2/3 & 2/3 \\ 7/3 & 2/3 & 2 & -5/3 & 8/3 \end{bmatrix} \end{array} \begin{array}{l} = x_1 \\ = x_3 \\ = v \to \min \end{array} .$$

Do not forget the constraints $x_7, x_8, x_9, x_{10} \geq 0$ outside the tableaux.

9. We pivot the three zeros from the right margin to the top and drop the corresponding columns:

$$\begin{array}{c} \begin{array}{ccccccc} x_7 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \end{array} \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ = x_1 \\ = v \to \min \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{cccccc} x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 3 & 0 & -3 & 2 & -2 \\ 2 & 4 & 2 & -2 & 1^* & 0 \\ 2 & 4 & 2 & -3 & 2 & 1 \\ 1 & 1 & 2 & -2 & 3 & 3 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_7 \\ = 0 \\ = 0 \\ = x_1 \\ = v \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{ccccc} x_2 & x_3 & x_4 & x_5 & 1 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ -4 & -5 & -4 & 1^* & -2 \\ -2 & -4 & -2 & 2 & 0 \\ -2 & 0 & -2 & 1 & 1 \\ -5 & -11 & -4 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_7 \\ = 0 \\ = x_6 \\ = x_1 \\ = v \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 6 & 8 & 5 & 2 \\ 4 & 5 & 4 & 2 \\ 6 & 6 & 6 & 4 \\ 2 & 5 & 2 & 3 \\ 11 & 9 & 12 & 11 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_7 \\ = x_5 \\ = x_6 \\ = x_1 \\ = v \to \min. \end{array}$$

Now we take the first equation

$x_7 = 6x_2 + 8x_3 + 5x_4 + 2$

outside the table (since x_7 is not required to be ≥ 0) and obtain the standard tableau

$$\begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 4 & 5 & 4 & 2 \\ 6 & 6 & 6 & 4 \\ 2 & 5 & 2 & 3 \\ 11 & 9 & 12 & 11 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_5 \\ = x_6 \\ = x_1 \\ = v \to \min. \end{array}$$

Chapter 4. Simplex Method

§10. Simplex Method, Phase 2

1. The tableau is optimal, so the basic solution is optimal:
min = 0 at $a = b = c = d = 0$,
$y_1 = 0.4, y_2 = 0.4, y_3 = 0, y_4 = 0.5, y_5 = 1, y_6 = 0.1$.

2. The y_2-row is bad. The program is infeasible.

3. The first column is bad. However since the tableau is not feasible, this is not sufficient to conclude that the program is unbounded. Still we set $z_2 = z_3 = z_4 = 0$, and see what happens as $z_1 \to \infty$. We have $y_1 = 0.4 \geq 0, y_2 = 3z_1 + 0.4 \geq 0, y_3 = 0.6z_1 \geq 0, y_4 = 0.6z_1 + 5 \geq 0, y_5 = 0.1z_1 - 0.1 \geq 0, y_6 = 0.1 \geq 0$ for $z_1 \geq 1$, and the objective function $-11z_1 \to -\infty$. So min $= -\infty$.

4. False. The converse is true.

5. True

6. True

7. First we write the program in a standard tableau and then we apply the simplex method (Phase 2):

$$\begin{array}{c|l} \begin{array}{cccc} x_1 & x_2 & x_3 & 1 \end{array} & \\ \begin{bmatrix} -40^* & -20 & -60 & 1200 \\ -4 & -1 & -6 & 300 \\ -0.2 & -0.7 & -2 & 40 \\ -100 & -100 & -800 & 8000 \\ -0.1 & -0.3 & -0.8 & 8 \\ -2 & -3 & -7 & 0 \end{bmatrix} & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \to \min \end{array} \end{array} \quad \mapsto$$

$$\begin{array}{c|l} \begin{array}{cccc} u_1 & x_2 & x_3 & 1 \end{array} & \\ \begin{bmatrix} -0.025 & -0.5 & -1.5 & 30 \\ 0.1 & 1 & 0 & 180 \\ 0.005 & -0.6 & -1.7 & 34 \\ 2.5 & -50 & -650 & 5000 \\ 0.0025 & -0.25^* & -0.65 & 5 \\ 0.05 & -2 & -4 & -60 \end{bmatrix} & \begin{array}{l} = x_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \to \min \end{array} \end{array} \quad \mapsto$$

$$\begin{array}{cl} \begin{matrix} u_1 & u_5 & x_3 & 1 \end{matrix} & \\ \begin{bmatrix} -0.03 & 2 & -0.2 & 20 \\ 0.11 & -4 & -2.6 & 22 \\ 2 & 200 & -0.14 & 34 \\ 2.5 & -50 & -520 & 4000 \\ 0.01 & -4 & -2.6 & 20 \\ 0.03 & 8 & 1.2 & -100 \end{bmatrix} & \begin{matrix} = x_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = x_2 \\ = -P \to \min . \end{matrix} \end{array}$$

This tableau is optimal, so

$$\max(P) = 100 \text{ at } x_1 = 20, x_2 = 20, x_3 = 0.$$

The zero values for the nonbasic slack variables u_1 and u_5 indicate that the corresponding resource limits are completely used (no slack there). The other resources are not completely used; some reserves left.

9. First we solve the system of linear equations for a, b and the objective function f and hence obtain the standard tableau

$$\begin{array}{cl} \begin{matrix} c & d & 1 \end{matrix} & \\ \begin{bmatrix} 1 & 2 & -0.5 \\ -2 & -3 & 1.5 \\ 0.1 & 0.1 & 1.5 \end{bmatrix} & \begin{matrix} = a \\ = b \\ = f \to \min . \end{matrix} \end{array}$$

The tableau is not row feasible so we cannot apply Phase 2. Until we learn Phase 1, we can pivot at random:

$$\begin{array}{cl} \begin{matrix} c & d & 1 \end{matrix} & \\ \begin{bmatrix} 1^* & 2 & -0.5 \\ -2 & -3 & 1.5 \\ 0.1 & 0.1 & 1.5 \end{bmatrix} & \begin{matrix} = a \\ = b \\ = f \to \min \end{matrix} \quad \mapsto \end{array}$$

$$\begin{array}{cl} \begin{matrix} a & d & 1 \end{matrix} & \\ \begin{bmatrix} 1 & -2 & 0.5 \\ -2 & 1 & 0.5 \\ 0.1 & -0.1 & 1.55 \end{bmatrix} & \begin{matrix} = c \\ = b \\ = f \to \min . \end{matrix} \end{array}$$

Now the tableau is feasible, and we can use Phase 2:

$$\begin{array}{cl} \begin{matrix} a & d & 1 \end{matrix} & \\ \begin{bmatrix} 1 & -2^* & 0.5 \\ -2 & 1 & 0.5 \\ 0.1 & -0.1 & 1.55 \end{bmatrix} & \begin{matrix} = c \\ = b \\ = f \to \min \end{matrix} \quad \mapsto \end{array}$$

$$\begin{array}{cl} \begin{matrix} a & c & 1 \end{matrix} & \\ \begin{bmatrix} 0.5 & -0.5 & 0.25 \\ -1.5 & -0.5 & 0.75 \\ 0.05 & 0.05 & 1.525 \end{bmatrix} & \begin{matrix} = d \\ = b \\ = f \to \min . \end{matrix} \end{array}$$

The tableau is optimal, so
min $= 1.525$ at $a = 0, b = 0.75, c = 0, d = 0.25$.

10. First we write the program in the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 1 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & 2 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_5 \\ = x_6 \\ \to \min \end{array}$$

Setting $x_1 = 2x_2$, we obtain a standard feasibly tableau

$$\begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 2 & 1 & 1 \\ -1 & 2 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_5 \\ = x_6 \\ \to \min \end{array}$$

with the first column bad. So this problem is unbounded, hence the original problem is unbounded.

11. Set $f = x_2 + 2x_3 - 2$ (the objective function). We write the program in the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_5 \\ = x_6 \\ = -f \to \min \end{array}$$

The tableau is feasible, and two columns are bad (namely, the x_2-column and x_3-column), so the program is unbounded ($\max(f) = \infty$).

12. Set $f = x_2 + 2x_3 + 2$ (the objective function). We write the program in the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & -3 \\ 1 & 1 & 2 & 1 & -1 \\ 0 & -1 & 2 & 0 & -2 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_5 \\ = x_6 \\ = -f \to \min \end{array}$$

We set $x_1 = 3$ and obtain a feasible tableau

$$\begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ -1 & 2 & 0 & -2 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_5 \\ = x_6 \\ = -f \to \min \end{array}$$

with the first column bad. So the program is unbounded.

13. If the row without the last entry is nonnegative, then the tableau is optimal; else the LP is unbounded.

§11. Simplex Method, Phase 1

1. The second row (v-row) is bad, so the LP is infeasible.

2. The tableau is optimal, so the basic solution is optimal:
min $= 0$ at $x = y = z = 0, u = 2, v = 0$.
This is the only optimal solution.

3. This is a feasible tableau with a bad column (the z-column). So the LP is unbounded (z and hence w can be arbitrary large).

5. The tableau is standard. According to the simplex method, we pivot on 1 in the first row:

$$\begin{array}{c} \begin{array}{cccc} a & b & c & 1 \end{array} \\ \begin{bmatrix} 1^* & 2 & 3 & -1 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ = d \\ = e \\ \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{cccc} d & b & c & 1 \end{array} \\ \begin{bmatrix} 1 & -2 & -3 & 1 \\ 2 & -4 & -5 & 5 \\ -1 & 3 & 3 & -1 \end{bmatrix} \end{array} \begin{array}{l} \\ = a \\ = e \\ \to \min \end{array}$$

The tableau is feasible, and the d-column is bad, so the program is unbounded (min $= -\infty$).

7. We scale the last column and then pivot on the first 1 in the first row to get both c on the top:

$$\begin{array}{c} \begin{array}{cccc} a & b & c & 1 \end{array} \\ \begin{bmatrix} 1^* & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ = c \\ = d \\ = f \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{cccc} c & b & c & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & -1 \end{bmatrix} \end{array} \begin{array}{l} \\ = a \\ = d \\ = f \to \min . \end{array}$$

Now we combine two c-columns and obtain the standard tableau

$$\begin{array}{c} \begin{array}{ccc} b & c & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \end{array} \begin{array}{l} \\ = a \\ = d \\ = f \to \min . \end{array}$$

The first row is bad, so the program is infeasible. In fact, the first constraint in the original tableau is inconsistent with the constraint $a \geq 0$.

9. True

10. False

11. We use the simplex method:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \left[\begin{array}{ccccc} 1^* & 0 & -2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & -1 & 1 & 0 & 2 \end{array}\right] \begin{array}{l} = x_5 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{ccccc} x_5 & x_2 & x_3 & x_4 & 1 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 3 & 1 \\ -1 & 1 & -1 & -2 & 0 \\ 2 & -1^* & 4 & 7 & 5 \\ 1 & -1 & 3 & 3 & 3 \end{array}\right] \begin{array}{l} = x_1 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{ccccc} x_5 & x_7 & x_3 & x_4 & 1 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 3 & 1 \\ 1 & -1 & 3 & 5 & 5 \\ 2 & -1 & 4 & 7 & 5 \\ -1 & 1 & -1 & -4 & -2 \end{array}\right] \begin{array}{l} = x_1 \\ = x_6 \\ = x_2 \\ \to \min . \end{array} \end{array}$$

Phase 1 was done in one pivot step, and Phase 2 also was done in one pivot step, because we obtain a feasible tableau with a bad column (x_5-column). The program is unbounded.

12. We use the simplex method:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & 1 \end{array} \\ \left[\begin{array}{cccc} 1^* & 0 & -1 & -1 \\ -1 & 3 & 1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \begin{array}{l} = x_4 \\ = x_5 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{cccc} x_4 & x_2 & x_3 & 1 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ -1 & 3^* & 0 & -1 \\ 3 & -1 & 5 & 4 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} = x_1 \\ = x_5 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \end{array} \mapsto$$

$$
\begin{array}{c}
\begin{array}{cccc} x_4 & x_5 & x_3 & 1 \end{array} \\
\left[\begin{array}{cccc}
1 & 0 & 1 & 1 \\
1/3 & 3 & 0 & 1/3 \\
8/3 & -1/3 & 5 & 11/3 \\
2/3 & -1/3^* & 2 & 2/3 \\
2/3 & -1/3 & 0 & 2/3
\end{array}\right]
\end{array}
\begin{array}{l} \\ = x_1 \\ = x_2 \\ = x_6 \\ = x_7 \\ \to \min \end{array}
\quad \mapsto
$$

$$
\begin{array}{c}
\begin{array}{cccc} x_4 & x_7 & x_3 & 1 \end{array} \\
\left[\begin{array}{cccc}
1 & 0 & 1 & 1 \\
19/3 & -9 & 18 & 19/3 \\
6/3 & 1 & 3 & 9/3 \\
2 & -1/3 & 6 & 2 \\
0 & 1 & -2 & 0
\end{array}\right]
\end{array}
\begin{array}{l} \\ = x_1 \\ = x_2 \\ = x_6 \\ = x_5 \\ \to \min\,. \end{array}
$$

The x_3-column is bad, so the program is unbounded.

13. We use the simplex method:

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{array} \\
\left[\begin{array}{cccccc}
1 & 0 & -1 & -1 & 0 & 1 \\
-1 & 3 & 1^* & 0 & -2 & -1 \\
3 & -1 & 2 & 1 & 2 & 1 \\
1 & -1 & 1 & 0 & -1 & 1 \\
0 & -1 & -1 & 0 & -1 & 2
\end{array}\right]
\end{array}
\begin{array}{l} \\ = x_6 \\ = x_7 \\ = x_8 \\ = x_9 \\ \to \min \end{array}
\quad \mapsto
$$

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_7 & x_4 & x_5 & 1 \end{array} \\
\left[\begin{array}{cccccc}
0 & 3 & -1 & -1 & -2 & 0 \\
1 & -3 & 1^* & 0 & 2 & 1 \\
5 & -7 & 2 & 1 & 6 & 3 \\
2 & -4 & 1 & 0 & 1 & 2 \\
-1 & 2 & -1 & 0 & -3 & 1
\end{array}\right]
\end{array}
\begin{array}{l} \\ = x_6 \\ = x_3 \\ = x_8 \\ = x_9 \\ \to \min\,. \end{array}
$$

The fist column in this feasible tableau is bad, so the program is unbounded.

§12. Geometric Interpretation

1. The diamond can be given by four linear constraints $\pm x \pm y \leq 1$.

2. Any convex combination of convex combinations is a convex combination.

3. We have to prove that if $u = [x_1, y_1], v2 = [x_2, y_2]$ are feasible, i.e.,

$$x_1^4 + y_1^4 \leq 1 \text{ and } x_2^4 + y_2^4 \leq 1,$$

then the point $au + (1 - a)v$ is also feasible for $0 \leq a \leq 1$, i.e.,

$$(ax_1 + (1 - a)x_2)^4 + (ay_1 + (1 - a)y_2)^4 \leq 1.$$

Clearly, it suffices to do this in the case when $x_1, x_2, y_1, y_2 \geq 0$. In other words, it suffices to prove that the region

$$x^4 + y^4 \leq 1, x \geq 0, y \geq 0$$

is convex. The function $y = (1 - x^4)^(1/4)$ is smooth on the interval $0 < x < 1$, so it suffices to show that its slope decreases. At the point $[x, y] = [x, (1 - x^4)^(1/4)]$ the slope is $-x^3/y^3$ so it decreases.

Similarly, we can prove that the region $|x|^p + |y|^p \leq 1$ is convex for any $p \geq 1$. In the case $p = 1$ the slope is -1, a constant function.

5. The points $[x, y] = [3, 1], [3, -1]$ belong to the circle but the halfsum $[3, 0]$ (the center of the circle) does not.

7. Both $x = 1$ and $x = -1$ belong to the feasible region, but $0 = x/2 + y/2$ does not.

8. The tangent to the disc at the point $[x, y] = [2t/(1+t^2), (1-t^2)/(1+t^2)]$ is $2tx+(1-t^2)y = 1+t^2$. The family of linear constraints $2tx + (1 - t^2)y \leq 1 + t^2$, where t ranges over all rational numbers, gives the disc.

9. A set S is called closed if it contains the limit points of all sequences in S. Any system of linear constraints gives a closed set, but the interval $0 < x < 1$ is not closed. Its complement is closed.

10. The rows of the identity matrix 1_6. If the vectors are written as columns, take the columns of 1_6.

11. One.

13. Since x_i are affine, (a) $\Rightarrow$ (b). It is also clear that (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) $\Rightarrow$ (a). So it remains to prove that (c) $\Rightarrow$ (d). The last implication follows from the well-known inequality

$$(|x_1| + \cdots + |x_n|)/n \leq ((x_1^2 + \cdots + x_n^2)/n)^{1/2}.$$

14. Let x be in S is not a vertex. We find distinct y, z in S such that $x = (y + z)/2$. The linear constraints giving S restricted on the

line $ay + (1-a)z$ give linear constraints on a. The interval $0 \le a \le 1$ is a part of the feasible set. Any of the constraints is either tight on the whole interval, or is tight only at am end point. So the tight constraints are the same for all points $ay + (1-a)z$ with $0 < a < 1$.

The "only if" part proven, consider now the "if" part. Here is a counter example with an infinite system of constraints: The linear constraints are $x \ge c$ where c runs over all negative numbers. The feasible set S is the ray $x \ge 0$. No defining constraint is tight for any feasible x but $x = 0$ is the only vertex.

So we assume the S is given by a *finite* system of linear constraints. Let x, y be in S and x is a vertex and y has the same tight constraints. If $y \ne y$ then the same constraints are tight for every point on the line $ax + (1-a)y$. For a number $a_0 > 1$ sufficiently closed to 1, all other constraints are also satisfied (here we use the finiteness). We pick such a number $a_0 < 2$. Then

$$x = ((a_0 x + (1-a_0)y) + ((2-a_0)x + (a_0-1)y)/2$$

is not a vertex because $a_0 x + (1-a_0)y$ and $(2-a_0)x + (a_0-1)y$ are distinct points in S.

15. Suppose that x is optimal, y, z are in the convex set S, and $x = (y+z)/2$. We have to prove that $y = z$.

Since f is affine, $f(x) = (f(y) + f(z))/2$. Since x is optimal, so are y and z. By uniqueness of optimal solution, $y = x = z$.

17. Our set S is a subset of R^n. Let x be a point of S such that it is the limit of a sequence $y^{(1)}, y^{(2)}, \ldots.$ of points outside S 9in other words, x belongs to the boundary of S). For example, x could be a vertex of S.

For each t, we find the point z in the closure of S closest to $y^{(t)}$. (We use the Euclidean distance $((y-z)\cdot(y-z))^{1/2}$ between points y, z in R^n.) We consider the linear constraint

$$(y^{(t)} - z^{(t)}) \cdot X \le (y^{(t)} - z^{(t)})(y^{(t)} + y^{(t)})/2.$$

All points in S satisfy this constraint, while the point $y^{(t)}$ does not. Now we scale this constraint so it takes the form $c^{(t)}X \le b^{(t)}$ with $c^{(t)} \cdot c^{(t)} = 1$. Then we take a limit constraint $c \cdot X \le b$. Then all points in S satisfy the latter constraint and $c \cdot x = b$. Thus, x is maximizer of the linear form $f = c \cdot X \ne 0$ over S.

When the set S is closed, and x is a vertex, we can arrange x to be an unique maximizer.

18. Suppose that x is not a vertex in S'. Then $x = (y+z)/2$ with distinct $y, z.$ in S'. Since $y, z.$ are in S, x is not a vertex in S.

Chapter 5. Duality

§13. Dual Problems

1. Let $f = 5x - 6y + 2z$ be the objective function. Here is a standard column tableau:

$$\begin{array}{cl} \begin{array}{r} -x \\ -y \\ -z \\ 1 \end{array} & \begin{bmatrix} 0 & -1 & 5 \\ 1 & -1 & -6 \\ 0 & 0 & 2 \\ 7 & -3 & 0 \end{bmatrix} \\ & \begin{array}{ccc} \shortparallel & \shortparallel & \downarrow \\ v_1 & v_2 & -f \end{array} \to \max \end{array}$$

3.
$$\begin{array}{cccc} x & y & z & 1 \end{array}$$
$$\begin{bmatrix} 0 & -1 & 0 & 7 \\ 1 & 1 & 0 & -3 \\ 5 & -6 & 2 & 0 \end{bmatrix} \begin{array}{l} = v_1 \\ = v_2 \\ = f \quad \to \min \end{array}$$

The matrix in Exercise 2 is so big that the transposed matrix may not fit on the page. So we reduce it as follows. The fift constraint follows from the fourth constraint because $b \geq 0$ so we drop the redundant constraint.

Given any feasible solution, we can replace g, h by $0, g + h$ and obtain a feasible solution with the same value for the objective function f. So setting $g = 0$ we do not change the optimal value.

$$\begin{array}{cccccccccc} a & b & c & d & e & f & h & i & j & 1 \end{array}$$
$$\begin{bmatrix} 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 10 & -100 \\ 25 & 25 & 25 & 0 & 6 & 25 & 25 & 0 & 10 & -100 \\ 1 & 25 & 25 & 25 & 0 & 30 & 25 & 25 & 25 & -100 \\ 1 & 0 & 25 & 25 & 30 & 25 & 25 & 25 & 25 & -100 \\ 3 & 2 & 1 & 2 & 4 & 2 & 2 & 3 & 3 & -70 \\ .39 & .11 & .18 & .21 & .35 & .44 & .25 & .23 & .24 & 0 \end{bmatrix} \begin{array}{l} = v_1 \\ = v_2 \\ = v_3 \\ = v_4 \\ = v_6 \\ \to \min \end{array}$$

5. Let $cx + d, cy + d$ be two feasible values, where x, y are two feasible solutions. We have to prove that

$\alpha(cx + d) + (1 - \alpha)(cy + d)$

is a feasible value for any α such that $0\alpha \leq 1$. But

$\alpha(cx + d) + (1 - \alpha)(cy + d) = c(\alpha x + (1 - \alpha)y) + d,$

where $\alpha x + (1 - \alpha)y$ is a feasible solution because the feasible region is convex.

7. The first equation does not hold, so this is not a solution.

8. First we check that this $X = [x_i]$ is a feasible solution (i.e., satisfies all constraints) with $z = 2$. We introduce the dual variables y_i corresponding to x_i, write the dual problem as the column problem, and set $y_i = 0$ whenever $x_i \neq 0$ (assuming that X is optimal, cf. Problem 13.10 and its solution).

$$\begin{array}{r} -y_6 \\ -y_8 \\ 1 \\ \end{array}\begin{array}{c} \left[\begin{array}{cccccc} 7 & -2 & -6 & 6 & -1 & 15 \\ -1 & 0 & 1 & 0 & -1 & -2 \\ 4 & 0 & -3 & 5 & 3 & 4 \end{array}\right] \\ \begin{array}{cccccc} = 0 & = y_2 & = 0 & = y_4 & = 0 & = w \end{array} \end{array} \begin{array}{c} \\ \\ \\ \to \max . \end{array}$$

We have a system of three linear equations for y_6, y_8 , and the system has no solutions. So X is not optimal.

9. Proceeding as in the solutions of Problem 13.10 and Exercise 8, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1[\begin{bmatrix} 7 & -2 & -6 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix} = 0.$$

The system has the unique solution $y_6, = 1, y_7 = 2, y_8 = 1$. Moreover, this solution is feasible (the basic y_i are nonnegative). Since we have feasible solutions for the primal and dual problems and $x_i y_i = 0$ for all i, both solutions are optimal.

10. Proceeding as in the solutions of Problem 13.10 and Exercises 8,9, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1[\begin{bmatrix} -2 & -6 & 6 & -1 \\ 1 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 5 & 3 \end{bmatrix} = 0.$$

The system has no solutions, so the answer is: This is not optimal.

§14. Sensitivity Analysis and Parametric Programming

1. The tableau is not standard, so we treat the row and column programs separately. We pivot the row program on -1 in the b-column:

$$\begin{array}{cccc} a & b & c & 1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & -1^* & 0 & -3 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{array}{l} = d \\ = c \\ = w \quad \to \min \end{array} \qquad \mapsto$$

$$\begin{array}{cccc} a & c & c & 1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & -3 \\ 4 & -2 & 1 & -6 \end{bmatrix} \begin{array}{l} = d \\ = b \\ = w \quad \to \min . \end{array}$$

Then we combine the two c-columns:

$$\begin{array}{ccc} a & c & 1 \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{bmatrix} \begin{array}{l} = d \\ = b \\ = w \quad \to \min . \end{array}$$

This tableau is standard. We use the simplex method:

$$\begin{array}{ccc} a & c & 1 \end{array}$$
$$\begin{bmatrix} 1^* & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{bmatrix} \begin{array}{l} = d \\ = b \\ = w \quad \to \min \end{array} \qquad \mapsto$$

$$\begin{array}{ccc} d & c & 1 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 3 & 2 \end{bmatrix} \begin{array}{l} = a \\ = b \\ = w \quad \to \min . \end{array}$$

This is an optimal tableau, so $\min = 2$ at $a = 2, b = 3, c = d = 0$.

Now we rewrite the column program in a standard column tableau:

$$\begin{array}{c} -g \\ -h \\ 1 \end{array} \begin{bmatrix} -1 & 0 & 1 & -2 \\ -2 & 1 & 0 & -3 \\ 0 & -2 & -1 & 0 \end{bmatrix}$$
$$\begin{array}{cccccc} = i & = j & = k & = -u & \to & \max . \end{array}$$

The k-column is bad, so this program is infeasible.

3. $\min = 0$ at $d = e = 0, a \geq 0$ arbitrary

§15. More on Duality

1. No, it is not redundant.
2. Yes, it is 2$\cdot$ (first equation) + (second equation).
3. No, it is not redundant.
4. No, it is not redundant.
5. No, it is not redundant.
7. Yes, it is.

11. Let y_i be the dual variable corresponding to x_i. The first 7 columns of the tableau give 7 linear constraints for y_8, y_9. Two additional constraints are $y_8, y_9 \geq 0$. We can plot the feasible region (given by these 9 constraints) in the (y_8, y_9)-plane. (The constraints corresponding to x_5, x_6, x_7 are redundant.) The answer is max $= 2.5$ at $y_8 = 0, y_9 = 1.25$. By complementary slackness, for any optimal solution $[x_i]$ of the primal problem, we have $x_i = 0$ for $i \neq 1, 8$. For such a solution, we have $3x_1 - 1 = x_8 \geq 0, 4x_1 - 2 = x_9 = 0$. Thus,
min $= 2.5$ at $x_1 = 0.5$, all other $x_i = 0$.

13. Let y_i be the dual variable corresponding to x_i, and let u, v be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for u, v. Two additional constraints are $u, v \geq 0$. We can plot the feasible region (given by these 11 constraints) in the (u, v)-plane. The answer is max $= 75/34$ at $u = 15/34, y_{10} = 15/17$. By complementary slackness, for any optimal solution $[x_i]$ of the primal problem, we have $x_i = 0$ for $i \neq 4, 6$. For such a solution, we have $6x_4 + 8x_6 - 1 = 0, 14x_4 + 13x_6 - 2 = 0$. Thus,
min $= 75/34$ at $x_4 = 3/34, x_6 = 1/17$ all other $x_i = 0$.

14. Let y_i be the dual variable corresponding to x_i, and let u, v be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for u, v. Two additional constraints are $u, v \geq 0$. The constraint corresponding to the x_5 column reads $-5u - 8v + 0 \geq 0$, hence $u = v = 0$. On the other hand, then the constraint corresponding to the x_1 column reads $-1 \geq 0$. So the column problem is infeasible.

On the other hand, it is easy to find a feasible solution for the row program. For example, $x_9 = 1$ and $x_i = 0$ for all other i. By the duality theorem, the row problem is unbounded.

Chapter 6. Transportation Problems

§16. Phase 1

1.

20	10	5		35
		5	15	20
20	10	10	15	

3. The total supply is 256, while the total demand is 260. So the problem is infeasible.

5. The balance condition $50 = 50$ holds. Each time, we pick a position with the minimal cost: $x_{2,1} = 2, x_{24} = 13, x_{33} = 4$ at zero cost, $x14 = 2, x_{17} = 3, x_{49} = 1, x_{42} = 7, x_{32} = 1, x_{38} = 4, x_{36} = 2$ at unit cost 1, and $x_{15} = 10, x_{35} = 1$ at unit cost 2. The total number of selected positions is 12, which equals $4 + 9 - 1$. Total cost is $0 \cdot 19 + 1 \cdot 20 + 2 \cdot 11 = 42$.

6. The balance condition $50 = 50$ holds. Each time, we pick a position with the minimal cost: $x_{2,1} = 5, x_{33} = 12$ at zero cost, $x14 = 5, x_{17} = 3, x_{49} = 1, x_{42} = 8, x_{41} = 7, x_{43} = 2$ at unit cost 1, $x_{15} = 1, x_{18} = 4, x_{46} = 0$ at unit cost 2, and $x_{16} = 2$ at unit cost 3. The total number of selected positions is 12, which equals $4 + 9 - 1$. Total cost is $0 \cdot 17 + 1 \cdot 26 + 2 \cdot 5 + 3 \cdot 2 = 42$.

7. The balance condition $130 = 130$ holds. Each time, we pick a position with the minimal cost: $x_{16} = 30$ at zero cost, $x12 = 10, x_{35} = 15$ at unit cost 30, $x_{21} = 25, x_{34} = 30$ at unit cost 35, $x_{13} = 10,$ at unit cost 40, $x_{33} = 5,$ at unit cost 95, and $x_{23} = 5,$ at unit cost 100. The total number of selected positions is 8, which equals $3+6-1$. Total cost is $0{\cdot}30+30{\cdot}25+35{\cdot}55+40{\cdot}10+95{\cdot}5+100{\cdot}5 = 3350$.

§17. Phase 2

1.

	1	2	2	
0	1 175	2 25	3 (1)	200
0	1 (0)	2 100	2 200	300
	175	125	200	

This is an optimal table, and the corresponding solutions are optimal. The minimal cost for the transportation problem is $1 \cdot 175 + 2 \cdot 25 + 2 \cdot 100 + 2 \cdot 200 = 825$. The maximal profit for the dual problem is $1 \cdot 175 + 2 \cdot 125 + 2 \cdot 200 - 0 \cdot 200 - 0 \cdot 300 = 825$.

3. We start with the basic feasible solution found in the solution of Exercise 4, §16 and compute the corresponding dual basic solution:

	0	2	1	0	1	2	1	2	1	
0	1 (1)	2 (0)	3 (2)	1 (1)	2 (1)	3 (1)	1 3	2 7	3 (2)	10
0	0 2	3 (1)	2 (1)	0 5	1 1	2 (0)	1 (0)	2 (0)	1 12	20
1	2 (3)	1 3	0 4	1 (2)	2 (2)	1 (0)	2 (2)	1 5	1 (1)	12
0	1 (1)	1 (-1)	1 (0)	2 (2)	2 (1)	2 2	2 (1)	2 2	1 4	8
	2	3	4	5	1	2	3	14	16	

There is only one negative $w_{42} = -1$. We select this position and get the loop (4, 2) (3,2), (3, 8), (3,8). The maximal $\varepsilon = 2$, and we deselect the position (4.8). The total cost decreases by 2. Here is the new basic feasible solution and the corresponding dual basis solution:

	1	2	1	1	2	3	1	2	2	
0	1 (0)	2 (0)	3 (2)	1 (0)	2 (0)	3 (0)	1 3	2 7	3 (1)	10
1	0 2	3 (2)	2 (2)	0 5	1 1	2 (0)	1 (1)	2 (1)	1 12	20
1	2 (2)	1 1	0 4	1 (1)	2 (1)	1 (-1)	2 (2)	1 7	1 (0)	12
1	1 (1)	1 2	1 (1)	2 (2)	2 (1)	2 2	2 (2)	2 (1)	1 4	8
	2	3	4	5	1	2	3	14	16	

Again we have a negative $w_{36} = -1$. The loop is (3, 6), (3, 2), (4, 2), (4,6). The total cost decreases by $-w_{36} \cdot \varepsilon = 1$. Here is the new basic feasible solution and the corresponding dual basis solution:

	0	1	1	0	1	2	1	2	1	
0	1 (1)	2 (1)	3 (2)	1 (1)	2 (1)	3 (1)	1 3	2 7	3 (2)	10
0	0 2	3 (2)	2 (1)	0 5	1 1	2 (0)	1 (0)	2 (0)	1 12	20
1	2 (3)	1 (1)	0 4	1 (2)	2 (2)	1 1	2 (2)	1 7	1 (1)	12
0	1 (1)	1 3	1 (0)	2 (2)	2 (1)	2 1	2 (1)	2 (0)	1 4	8
	2	3	4	5	1	2	3	14	16	

This table is optimal with total cost at min = 47.

5. We start with the basic feasible solution found in the solution of Exercise 5, §16 and compute the corresponding dual basic solution:

	2	2	2	1	2	3	1	2	2	
0	1 (-1)	2 (0)	3 (1)	1 5	2 1	3 2	1 3	2 4	3 (1)	15
2	0 5	3 (3)	2 (2)	0 (1)	1 (1)	2 (1)	1 (2)	2 (2)	1 (1)	5
2	2 (2)	1 (1)	0 12	1 (2)	2 (2)	1 (0)	2 (3)	1 (1)	1 (1)	12
1	1 7	1 8	1 2	2 (2)	2 (1)	2 0	2 (2)	2 (1)	1 1	18
	12	8	14	5	1	2	3	4	1	

There is only one negative $w_{11} = -1$. The loop is (1,1), (1, 6), (4, 6), (4, 1). The decrease in the total cost is 2. Here is the new basic solution and the corresponding dual basic solution:

	1	1	1	1	2	2	1	2	1	
0	1 2	2 (1)	3 (2)	1 5	2 1	3 (1)	1 3	2 4	3 (2)	15
1	0 5	3 (3)	2 (2)	0 (0)	1 (0)	2 (1)	1 (1)	2 (1)	1 (1)	5
1	2 (2)	1 (1)	0 12	1 (1)	2 (1)	1 (0)	2 (2)	1 (0)	1 (1)	12
0	1 5	1 8	1 2	2 (1)	2 (0)	2 2	2 (1)	2 (0)	1 1	18
	12	8	14	5	1	2	3	4	1	

This table is optimal, and min = 48.

7. When $t \geq 25$, see the previous solution. When $t < 0$, the total supply is less than the total demand, so the program is infeasible. So it remains to consider the case $0 \leq t \leq 25$.

8. In the optimal tableau of Table 17.22, we replace c_{23} by t and w_{23} by $t - 40$:

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	t $(t-40)$	45 (10)	60 (30)	0 5	30
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

When $t \geq 40$, this table is optimal, with min = 3475 (independent of t).

Assume now that $t < 40$. We select the position (2, 3) and get the loop (2, 3), (1,3), (1,6), (2,6). The corresponding $\varepsilon = 5$. The decrease in the total cost is $5(40 - t)$. Here is the new table:

	$75 - t$	30	40	35	30	0	
0	55 $(t-20)$	30 10	40 15	50 (15)	40 (10)	0 25	50
$40-t$	35 25	30 $(40-t)$	t 5	45 $(50-t)$	60 $(70-t)$	0 $(40-t)$	30
0	40 $(t-35)$	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

This table is optimal when $35 \leq t \leq 40$.

Assume now that $t < 35$. We select the position (3, 1) and get the loop (3, 1), (3, 6), (1,6), (1, 3), (2,3), (2,1). The corresponding $\varepsilon = 5$. The decrease in the total cost is $5(35 - t)$. Here is the new table:

	$75-t$	30	40	$70-t$	$65-t$	0	
0	55 $(t-20)$	30 10	40 10	50 $(t-20)$	40 $(t-25)$	0 30	
40 $-t$	35 20	30 $(40-t)$	t 10	45 (15)	60 (35)	0 $(40-t)$	
35 $-t$	40 5	60 $(65-t)$	95 $(90-t)$	35 30	30 15	0 $(35-t)$	
	25	10	20	30	15	30	

This table is optimal when $25 \le t \le 35$.

Assume now that $t < 25$. We select the position (1, 5) and obtain the loop (1, 5), (3, 5), (3, 1), (2, 1), (2, 3), (1, 3). The corresponding $\varepsilon = 10$. The decrease in the total cost is $10(25 - t)$. Here is the new table:

	50	30	$t+15$	45	40	0	
0	55 (5)	30 10	40 $(25-t)$	50 (5)	40 10	0 30	50
15	35 10	30 (15)	t 20	45 (15)	60 (35)	0 $40-t$	30
10	40 15	60 (40)	95 $(90-t)$	35 30	30 5	0 (10)	50
	25	10	20	30	15	30	

This table is optimal when $t \le 25$.

Thus, we solve the program for all t. The minimal cost is

$$\begin{cases} 2900 + 20t & \text{for } t \le 25 \\ 3400 + 5(t - 25) & \text{for } t \ge 25 \\ 3475 & \text{for } t \ge 40. \end{cases}$$

§18. Job Assignment Problem

1. min $= 7$ at $x_{14} = x_{25} = x_{32} = x_{43} = x_{51} = 1$, all other $x_{ij} = 0$.

3. min $= 7$ at $x_{12} = x_{25} = x_{34} = x_{43} = x_{51} = x_{67} = x_{76} = 1$, all other $x_{ij} = 0$.

4. We subtract: 1 from the rows $2, 3, 4, 5, 8$; 2 from the row 7; 1 from the column 8:

$$\begin{array}{|ccccccccc|}
0 & 2 & 2 & 4 & 0 & 1 & 5 & 0 & 4 \\
1 & 2 & 0 & 2 & 3 & 2 & 1 & 2 & 0 \\
1 & 0 & 3 & 2 & 3 & 2 & 0 & 0 & 3 \\
3 & 1 & 3 & 1 & 1 & 1 & 0 & 3 & 1 \\
1 & 2 & 4 & 0 & 1 & 3 & 1 & 2 & 3 \\
5 & 2 & 2 & 4 & 0 & 1 & 1 & 0 & 4 \\
0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 3 \\
4 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 0 \\
1 & 2 & 1 & 2 & 0 & 1 & 2 & 3 & 5
\end{array}\,.$$

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Haungarian method (Remark 17.25). Let us try to place the flow at positions with zero cost. In each of the following six lines there is only one zero: r4 (row 4), r5, r9, c2 (column 2), c3, c6 (we pass c4 because the conflict with r5: the position (5,4) is already selected in r5). We select the positions with these zeros and add the six lines c7, c4, c5, r3, r2, r8 to our list L of covering lines. The remaining matrix is

$$\begin{array}{cccc}
 & c1 & c8 & c9 \\
r1 & 0 & 0 & 4 \\
r2 & 5 & 0 & 4 \\
r3 & 0 & 1 & 3
\end{array}\,.$$

We cannot place all flow at positions with zero cost, but we can cover all zeros by $2 < 3$ lines, namely, c1 and c8. The complete list L consists of 8 lines c7, c4, c5, r3, r2, r8, c1, c8. The least uncovered number is $m = 1$. We subtract 1 from all uncovered entries and add

1 to all twice-covered entries:

$$\boxed{\begin{array}{ccccccccc}
0 & 1 & 1 & 4 & 0 & 0 & 5 & 0 & 3 \\
2 & 2 & 0 & 3 & 4 & 2 & 2 & 3 & 0 \\
2 & 0 & 3 & 3 & 4 & 2 & 1 & 1 & 3 \\
3 & 0 & 2 & 1 & 1 & 0 & 0 & 3 & 0 \\
1 & 1 & 3 & 0 & 1 & 2 & 1 & 2 & 2 \\
5 & 1 & 1 & 4 & 0 & 0 & 1 & 0 & 3 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 \\
5 & 1 & 3 & 3 & 2 & 0 & 2 & 3 & 0 \\
1 & 1 & 0 & 2 & 0 & 0 & 2 & 3 & 4
\end{array}}\,.$$

Now we can place the flow at the positions with zero costs:

$$\boxed{\begin{array}{ccccccccc}
 & & & & & * & & & \\
 & & * & & & & & & \\
 & * & & & & & & & \\
 & & & & & & * & & \\
 & & & * & & & & & \\
 & & & & & & & * & \\
* & & & & & & & & \\
 & & & & & & & & * \\
 & & & & * & & & &
\end{array}}\,.$$

For this program, min $= 0$. However we changed the objective function (without changing the optimal solutions). For the original problem, min $= 1+1+1+1+1+1+1+0 = 7$.

5. max $= 14$ at $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$, all other $x_{ij} = 0$.

6. First we convert the maximixation problem to a minimization problem by subtracting each entry from the maximal entry in its row:

$$\boxed{\begin{array}{cccccc}
0 & 2 & 2 & 2 & 3 & 2 \\
2 & 1 & 3 & 2 & 0 & 0 \\
2 & 4 & 3 & 3 & 3 & 0 \\
2 & 1 & 0 & 1 & 0 & 1 \\
1 & 2 & 3 & 2 & 0 & 3 \\
1 & 0 & 1 & 0 & 2 & 1
\end{array}}\,.$$

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Haungarian method (Remark 17.25). The following five lines cover all zeros: rows 1,4,6 and columns 5,6. Since $t = 5 < n = 6$, we cannot place all flow at positions with zero cost. The least uncovered number is $m = 1$. We subtract 1 from all uncovered entries and add 1 to all twice-covered entries:

$$\begin{array}{|cccccc|}\hline 0 & 2 & 2 & 2 & 4 & 3 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 3 & 2 & 2 & 3 & 0 \\ 2 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 3 & 2 \\ \hline \end{array}\,.$$

Now we can place all flow at positions with zero cost:

$$\begin{array}{|cccccc|}\hline * & & & & & \\ & * & & & & \\ & & & & & * \\ & & * & & & \\ & & & & * & \\ & & & * & & \\ \hline \end{array}\,.$$

For this program, min = 0. However we changed the objective function (without changing the optimal solutions). For the original problem, max $= 4 + 3 + 4 + 4 + 4 + 2 = 21$.

Note that the sum of maximal entries in rows is 22, and we cannot get this much because the conflict in the last column. This proves the the solution is optimal.

7. max = 29 at $x_{15} = x_{26} = x_{33} = x_{41} = x_{57} = x_{62} = x_{74} = 1$, all other $x_{ij} = 0$.

Chapter 7. Matrix Games

§19. What are Matrix Games?

1. max min $= -1$. min max $= 0$. There are no saddle points.
$[1/3, 2/3, 0]^T$ gives at least $-2/3$ for the row player.
$[1/2, 0, 0, 0, 1/2]$ gives at least $1/2$ for the column player.
So $-2/3 \leq$ the value of the game $\leq -1/2$.

3. max min $= -1$. min max $= 2$. There are no saddle points.
(second row + 2·third row)/3 $\geq -1/3$.
(third column + sixth column)/2 ≤ 1.
So $-1/3 \leq$ the value of the game ≤ 1.

5. We compute the max in each column (marked by *) and min in each row (marked by ▪).

$$\begin{array}{cc}
 & \begin{matrix} 4 & 2 & 3 & 5 & 4 & 3^{\blacksquare} & 7 & 6 & 3^{\blacksquare} \end{matrix} \\
\begin{matrix} -4 \\ -2 \\ -4 \\ 0^* \\ -9 \end{matrix} &
\begin{bmatrix}
4^* & -4^{\blacksquare} & 3^* & 0 & 0 & 0 & -1 & 1 & -2 \\
-1 & 0 & 2 & 1 & -2^{\blacksquare} & -2^{\blacksquare} & 1 & 0 & -2 \\
-4^{\blacksquare} & 0 & -2 & -2 & 1 & -1 & 1 & 6^* & 2 \\
1 & 2^* & 2 & 5^* & 3 & 3^* & 7^* & 2 & 0^{\blacksquare} \\
-4 & -9^{\blacksquare} & -8 & 0 & 4^* & 2 & 2 & 0 & 3^*
\end{bmatrix}
\end{array}$$

Thus, max min $= 0$. min max $= 3$. There are no saddle points.

7. Optimal strategies are
$[0, 1/3, 0, , 0, 7/15, 1/5]^T, [0, 2/3, 0, 0, 0, 0, 0, 0, 0, 0, 1/3]$,
and the value of game is $-2/3$.

9. Optimal strategies are
$[55/137, 0, 0, 44/137, 0, 0, 38/137]^T$,
$[0, 0, 20/137, 0, 65/137, 0, 0, 0, 0, 0, 0, 0, 52/137]$,
and the value of game is $-54/137$.

11. We have seen that $a_{i,j} = a_{i',j} = a_{i,j'} = a_{i',j'}$ because $a_{i,j} \geq a_{i',j} \geq a_{i',j'} \geq a_{i,j'} \geq a_{i,j}$. Since $a_{i,j}$ and $a'_{i',j}$ are maximal in their columns j and j', so are $a_{i',j}$ and $a_{i,j'}$. Since $a_{i,j}$ and $a'_{i',j}$ are minimal in their columns i and i', so are $a_{i',j}$ and $a_{i,j'}$. Thus, (i, j') and (i', j) are saddle points.

§20. Matrix Games and Linear Programming

1. The optimal strategy for the row player is $[0, 2/3, 1/3]^T$.
The optimal strategy for the column player is $[1/2, 1/2, 0$.
The value of the game is 2.

3. The optimal strategy for the row player is $[0.2, 0, 0.8]^T$.
An optimal strategy for the column player is $[0, 0.5, 05, 0, 0, 0]$.
The value of the game is 1.

5. The optimal strategy for the row player is $[1/3, 2/3, 0]^T$.
The optimal strategy for the column player is $[2/3, 0, 0, 1/3]$.
The value of the game is $-2/3$.

7. The optimal strategy for the row player: $[1/8, 0, 7/8, 0]^T$.
The optimal strategy for the column player: $[0, 1/4, 0, 0, 0, 3/4]$.
The value of the game is -0.25.

9. Optimal strategies are
$[0, 0, 0, 7/8, 1/8]^T, [3/8, 0, 0, 0, 0, 0, 0, 0, 5/8]$,
and the value of game is 3/8.

11. Note that the second constraint is redundant, because it follows from the first one. We solve the equation for x_6 and exclude this from our LP. We obtain an equivalent LP with all $x_i \geq 0$:
$10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5 - 9 \to \min$,
$3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4$,
$(x_6 + 3 =)\ 3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 3$.
Again, the second constrain is redundant.

Now we take advatage of the fact that in this problem all coefficients in the objective function and all right-hand parts of constraints are positive. We set $v = 1/(10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5) > 0$ on the feasible region and
$p_1 = 10x_1v, p_2 = 5x_2v, p_3 = 4x_3v, p_4 = 7x_4v, p_5 = 4x_5v$.
All $p_i \geq 0$ and $p_1 + p_2 + p_3 + p_4 + p_5 = 1$. The minimization of $1/v - 9$ is equivalent to the maximization of v.
The constraint $3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4$ take the form

$$(3p_1/10 + p_2/5 + p_3/4 + 2p_4/7 + p_5/4)/4 \geq v.$$

This is the row player program for the matrix game with the payoff matrix $\begin{bmatrix} 3/40 \\ 1/20 \\ 1/16 \\ 1/14 \\ 1/16 \end{bmatrix}$. Our effort to get a smaller game pays, because with can solve this game: the value of game is $v = 1/14$ and the optimal strategy is $[p_1, , p_2, p_3, p_4, p_5]^T = [0, 0, 0, 1, 0]^T$.

This translates to $\max(1/v - 9) = 5$ at $x_1 = x_2 = x_3 = x_5 = 0, x_4 = 2, x_6 = 2x_4 - 3 = 1$.

12. We write our LP in a canonical form $-Ax \le b, x \ge 0, cx \to$ min corresponding to ther standrad row tableau (13.4) with

$x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T$,

$c = [-1, -2, -4, -1, -1, -1, 0], b = [-5, -6, 7]^T$,

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & -2 & 1 \\ 3 & 1 & 1 & 2 & 1 & 1 & -1 \\ -3 & -1 & -1 & -2 & -1 & 1 & 0 \end{bmatrix}$$

Then

$$M = \begin{bmatrix} 0 & -A & -b \\ A^T & 0 & -c^T \\ b^T & c & 0 \end{bmatrix}$$

is our payoff matrix (see page 219 of the textbook).

13. We take advatage of the fact that all coefficients in the objective function and all right-hand parts of constraints are positive. We set $v = 1/(x_1 + 2x_2 + x_3 + x_4 + x_5 + 3x_6 + x_7 + x_8) > 0$ on the feasible region (because the point where all $x_i = 0$ is not a feasible solution) and $p_1 = x_1 v, p_2 = 2x_2 v, p_3 = x_3 v, p_4 = x_4 v, p_5 = x_5 v, p_6 = 3x_6 v, p_7 = x_7 v, p_8 = x_8 v$. Our constraints take the form

$3p_1 + p_2/2 + p_3 + 2p_4 + p_5 + p_6/3 + p_7 - 3p_8 \ge v$,

$3p_1/5 + p_2/10 + p_3/5 + 2p_4/5 - p_5/5 + p_6/15 + p_7/5 + 3p_8/5 \ge v$,

$3p_1 + p_2/2 + p_3 + 2p_4 + p_5 - p_6/3 + p_7 + p_8 \ge v$.

Other constraints are: $p_i \ge 0$ for all i and $\sum_{i=1}^{8} p_i = 1$. The minimization of $1/v$ is equivalent to the maximization of v.

Thus, we obtain the LP for the row player, with the payoff matrix $\begin{bmatrix} 3 & 3/5 & 3 \\ 1/2 & 1/10 & 1/2 \\ 1 & 1/5 & 1 \\ 1 & -1/5 & 1 \\ 1/3 & 1/15 & -1/3 \\ 1 & 1/5 & 1 \\ -3 & 3/5 & 1 \end{bmatrix}$.

At the position (1,2), we have a saddle point, so the value of game is 0.6 and an optimal strategy for the row player is

$[p_1, , p_2, p_3, p_4, p_5, p_6]^T = [1, 0, 0, 0, 0, 0]^T$.

This translate to min $= 5/3$ at $x_1 = 5/3$, the other $x_i = 0$.

§21. Other Methods

1. The first row and column are dominated. The optimal strategy for the column player is $[0, 0.5, 0.5]^T$. The optimal strategy for the row player is $[0, 0.25, 0.75$. The value ofthe game is 2.5.

2. The second row is dominated by the first row, and the second column is dominated by the third column. So we obtain a 2×2 game $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ which can be easily solved using slopes: optimal strategies are $[0, 4, 0.6]^T, [0, 4, 0.6]$, and the value of game is 1.2. For the original game, the answer is: optimal strategies are $[0, 4, 0, 0.6]^T, [0, 4, 0, 0.6]$, and the value of game is 1.2.

3. The optimal strategy for the column player is $[0, 0.4, 0, 0.6]^T$. The optimal strategy for the row player is $[0, 0.4, 0.6$. The value of the game is 2.8.

4. By domination, we can reduce our matrix to $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$. this game can be solved easily by graphical method. Answer: optimal strategies are $[0, 0, 0, 1, 0, 0]^T$ (a pure strategy corresponding to the fifth row) and $0, 0.5, 0.5]$, and the value of game is 2.

5. The optimal strategy for the column player is $[1/3, 1/3, 1/3]^T$. The optimal strategy for the row player is $[0, 0, 2/7, 3/7, 2/7, 0$. The value of the game is 0.

6. Using slopes, the value of game is 15/4.

7. The value of the game is 0 because the game is symmetric.

8. There is a saddle point at the position (1, 3). The value of game is 1 , and optimal strategies are $[1, 0, 0, 0, 0]^T, [0, 0, 3, 0, 0]$.

9. The first two columns and the first row go by domination. It is easy to solve the remaining 2×2 matrix game. The value of game is 11/7 , and optimal strategies are $[0, 5/14, 9/14]^T, [0, 0, 5/7, 2/7]$.

11. There is a saddle point at the position (1, 1). So the value of game is 0.

The last column is dominated by any other column. After dropping this column, we get a symmetric game. This is another way to see that the value of game is 0.

13. 0 at a saddle point (at position (1.1)0. The row player has also other optimal strategies, e.g., $[1/3, 1/3, 0, 0, 1/3]^T$

Chapter 8. Linear Approximation

§22. What is Linear Approximation?

1. The mean is $-2/5 = -0.4$. The median is 1. The midrange is $-5/2 = -2.5$.

3. The mean x_2 is 5/9. The median x_1 is 0. The midrange x_∞ is $1/2 = 0.5$.

5. (a) $0, 0, 2, 2, 5$.
(b) $1, 2, 9$.
(c) $0, 0, 2, 2, 3$.
(d). Exercise 1.
(e) $1, 2, 2, 3, 3$.
(f). Exercise 3.

7. The program

$$|65 - 1.6c| + |60 - 1.5c| + |70 - 1.7c| \to \min$$

can be easily solved by computing slopes. E.g., the slope of the objective function on the interval $69/1.5 = 40 < c < 65/1.6 = 40.625$ is $1.5 - 1.6 - 1.7 = -1.8$ while the slope on the interval $65/1.6 = 40.625 < c < 70/1.7 \approx 42.2$ is $1.6 + 1.5 - 1.7 = 1.4$. Thus, $\min = 1.875$ at $c = 40.625$.

Since $1.875 < 6.25$ the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^1-metric.

The program

$$(65 - 1.6c)^2 + (60 - 1.5c)^2 + (70 - 1.7c)^2 \to \min$$

can be easily solved by differentiation. We obtain

$$1.6(65 - 1.6c) + 1.5(60 - 1.5c) + 1.7(70 - 1.7c) = 0,$$

hence $c \approx 40.6494$, $\min \approx 1.75325$. So the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^2-metric.

The program

$$\max(|65 - 1.6c|, |60 - 1.5c|, |70 - 1.7c|) \to \min$$

can be easily solved by computing slopes. Near $c = 40.625$, the objective function is $\max(70 - 1.7, 1.5c - 60)$. So $\min = 0.9375$ at $c = 40.625$. So the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^∞-metric.

9. We enter the data $h = [1.6, 1.5, 1.7, 1.8], w = [65, 60, 70, 80]$ to *Mathematica* as

h = { 1.6, 1.5, 1.7, 1.8}
w = {65, 60, 70, 80}

For $p = 1$, the objective function (to be minimized) is

f=Apply[Plus,Abs[w-a*hˆ 2]]

An optimization command is

FindMinimum[f,{a,1,2}]

The answer is min ≈ 7.59259 at $a \approx 24.6914$. For comparison, the model $w = b$ gives min $= 25$ when $65 \leq b \leq 70$ (medians).

For $p = 2$, we enter

f=Apply[Plus,(w-a*hˆ 2)2̂]; FindMinimum[f,{a,1,2}]

and obtain min ≈ 21.0733 at $a \approx 25.0412$. For comparison, the model $w = b$ gives min $= 218.75$ when $b = 68.75$ (the mean).

For $p = \infty$, we enter

f=Max[Abs[w-a*hˆ 2]]; FindMinimum[f,{a,1,2}]

and obtain min ≈ 3.09339 at $a \approx 25.2918$. For comparison, the model $w = b$ gives min $= 10$ when $b = 70$ (the midrange).

So the model $w = ah^2$ gives better l^p-fits for our data than the model $w = b$ for $p = 1, 2, \infty$.

11. We enter data as in the previous exercise, and then we enter

f=Apply[Plus,Abs[w-a*hˆ 3]]; FindMinimum[f,{a,1,2}]

with the answer min ≈ 21.6477 at $a \approx 14.2479$ for $p = 1$;

f=Apply[Plus,(w-a*hˆ 3)ˆ 2]; FindMinimum[f,{a,1,2}]

with the answer min ≈ 167.365 at $a \approx 14.8198$ for $p = 2$;

f=Max[Abs[w-a*hˆ 3]]; FindMinimum[f,{a,1,2}]

with the answer min ≈ 8.68035, at $a \approx 15.2058$. for $p = \infty$.

13. These are not linear approximations. Taking log of both sides, we obtain a model $\log_2(F_t) = ct$. For this model,

the best l^1-fit is for $c \approx 0.680907$ (with min ≈ 9.5),
the best l^2-fit is for $c \approx 0.680096$ (with min ≈ 2.8),
the best l^∞-fit is for $c \approx 0.671023$ (with min ≈ 0.67).

The limit value for c when we take morw and more terms of the sequence is $\log_2(\sqrt{5} + 1) - 1 \approx 0.694242$.

§23. Linear Approximation and Linear Programming

1. min $= 0$ at $a = -15, b = 50$ for $w = a + bh$
and
min ≈ 19 at $a \approx 25.23$ for $w = ah^2$
2. $x + y + 0.3 = 0$
3. $a = 0.9, b \approx -0.23$
5. We enter the data in *Mathematica*:
$h = \{1.5, 1.6, 1.7, 1.7, 1.8\}; w = \{60, 65, 70, 75, 80\}$
Then we enter
FindMinimum[Apply[Plus,Abs[w-a*h^ 2]],{a,1,2}]
The answer is min ≈ 10.1367 at $a \approx 25.3906$.
7. We enter the data in *Mathematica* as in Exercise 5. Then we enter
FindMinimum[Apply[Plus,Abs[w-a*h-b]],{a,1,2}, {b,1,2}]
The answer is min ≈ 13.7647 at $a \approx 40.5882, b \approx 1$.
9. We enter the data in *Mathematica* as in Exercise 5. Then we enter
FindMinimum[Max[Abs[w-a*h^ 2]],{a,1,2}]
The answer is min ≈ 3.09339 at $a \approx 25.2918$.
11. We enter the data in *Mathematica* as in Exercise 5. Then we enter
FindMinimum[Max[Abs[w-a*h-b]],{a,1,2}, {b,1,2}]
The answer is min ≈ 3.72727 at $a \approx 41.8182, b \approx 1$.
12. We get the least squares fit to p_n by $an + b$ when $a \approx 5.53069, b \approx -37.9697$ with nin ≈ 11923 being the minimal value for

$$\sum_{i=1}^{100} (p_n - an - b)^2),$$

while

$$\sum_{i=1}^{100} (p_n - n\log(n))^2 \approx 161206.$$

13. This is not a linear approximation. Taking log of the both sides, we get a linear model $\log_2(F_n) = \log_2(a) + bn$. The least squares fit is
min ≈ 0.269904 at $b \approx 0.693535, \log_2(a) \approx -0.443517$,
so $a \approx 0.74$.

§24. More Examples

1. The model is $w = ah + b$, or $w - x_2 = a(h - 1988) + b'$ with $b = x_2 + -1988a$ and $x_2 = 37753/45 \approx 838.96$. Predicted production P in 1993 is $x_2 + 5a + b'$.

For $p = 1, wehavea \approx 16.54, b' \approx 31, P \approx 953.$

For $p = 2, wehavea \approx 0, b' \approx 32, P \approx 871.$

For $p = \infty, wehavea \approx 17.59, b' \approx 32, x_5 \approx 959.$

So in this example l^∞-prediction is the best.

3. $a = \$4875, b = \$1500$

4. We enter data for 1900-1998 in $Mathematica$ with t replaced by $t - 1990$:

t = $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$;

x = $\{421, 429, 445, 449, 457, 460, 481, 503, 514\}$;

y= $\{628, 646, 764, 683, 824, 843, 957, 1072, 1126\}$

To get the best l^1-fit, we enter

FindMinimum[Apply[Plus,Abs[y-a*t-b*x-c]],
{a,1,2}, {b,1,2}, {c,1,2}]

with responce

{280.599, {a -> 47.7205, b -> 1.39226, c -> 1.}}

Then we plug these values for a, b, c to $1070 - 9a - 523b - c$ (to get get the difference between actual value 1070 and the prediction) and obtain ≈ -89.

To get the best l^2-fit, we enter

FindMinimum[Apply[Plus,(y-a*t-b*x-c)^2],
{a,1,2}, {b,1,2}, {c,1,2}]

with responce

{8275.95, {a -> -0.0169939, b -> 5.63814, c -> -1767.27}}

hence $1070 - 9a - 523b - c \approx -111$.

To get the best l^∞-fit, we enter

FindMinimum[Max[Abs[y-a*t-b*x-c]],
{a,1,2}, {b,1,2}, {c,1,2}]

with responce

{8275.95, {a -> 90.521, b > 1.09929, c > 1.}}

164.201, a -¿ 90.521, b -¿ 1.09929, c -¿ 1.

hence $1070 - 9a - 523b - c \approx -321$.

So the l^1-fit gave the best prediction.