A monic polynomial $G(x) \in R[x]$ will be called absolutely irreducible if $\bar{G}(x)$ is ir-ducible over the field $\bar{R}$. A polynomial $F(x) \in R[x]$ is called primary if there exists an bsolutely irreducible polynomial $G(x) \in R[x]$ such that $\bar{F}(x) = \bar{G}(x)^k$, $k \in \mathbf{N}$. By Hensel's mma (cf. [11, Sec. 5], [12]) every monic polynomial in $R[x]$ can be uniquely written as roduct of pairwise relatively prime primary polynomials. The following Theorem allows us use this result to simplify the matrices under consideration. Recall that every matrix er a commutative ring with identity is a root of its characteristic polynomial [10, Chap. V].

THEOREM 2. Let $A \in R_m$, $F(x)$ a monic polynomial in $R[x]$, $F(A) = 0$ and assume that

$$F(x) = F_1(x) \cdot \ldots \cdot F_t(x), \tag{1}$$

here the polynomials $F_1, \ldots, F_t$ are pairwise relatively prime and monic, and $F_i(A)$ is a ninvertible matrix for $i = \overline{1, t}$. Then we have

$$A \approx \mathrm{Diag}(A_1, \ldots, A_t), \tag{2}$$

here $F_i(A_i) = 0$, $i = \overline{1, t}$. Moreover, if $F(x) = \chi_A(x)$, then $\chi_{A_i}(x) = F_i(x)$. And if $B \in R_m$, $(B) = 0$ and $B \approx \mathrm{Diag}(B_1, \ldots, B_t)$ with $F_i(B_i) = 0$, $i = \overline{1, t}$ then $B \approx A$ if and only if for $i =$ t, $B_i \approx A_i$.

The proof of this Theorem uses the same methods as for matrices over a field, proceeding follows. The module $M(A)$ can be considered as module over the ring $Q = R[x]/F(x)$. In is ring one can select a system of pairwise orthogonal idempotents $e_1, \ldots, e_t$ such that $Qe_1 \dotplus \ldots \dotplus Qe_t$, $Qe_t \cong R[x]/F_i(x)$. To this decomposition of Q corresponds a decomposition $M(A) =$ $\dotplus \ldots \dotplus M_t$, where $M_i = e_i \circ M(A)$; here $M_i \neq 0$ since the matrix $F_i(A)$ is noninvertible. Each the modules M_i is a free R-module of the form $M_i \cong M(A_i)$ where A_i is a matrix over R such that $(A_i) = 0$. Using Theorem 1 we then obtain (2). If $F(x) = \chi_A(x)$, then $F_i(x) = \chi_{A_i}(x)$ follows virtue of (1) from $\bar{F}_i(x) = \chi_{\bar{A}_i}(x)$, $i = \overline{1, t}$, together with the fact that the polynomials $, \ldots, F_t$ are pairwise relatively prime. The last statement of Theorem 2 is (in view of eorem 1) a reformulation of the fact that the isomorphism $M(A) \cong M(B)$ is equivalent to the stem of isomorphisms $e_i \circ M(A) \cong e_i \circ M(B)$, $i = \overline{1, t}$.

In view of Theorem 2 the solution of problems 1-3 of the introduction may, wherever this convenient, be restricted to the case that the characteristic polynomial of the matrix A primary.

COROLLARY. If A is a normal matrix over R then

$$A \approx \mathrm{Diag}(S(G_1(x)), \ldots, S(G_r(x))), \tag{3}$$

re $G_1(x), \ldots, G_r(x)$ are primary polynomials. The matrix on the right hand side of (3) is iquely determined by A up to permutations of the blocks.

Proof. In view of Theorem 1 we have a decomposition $M(A) \cong M(S(F_1)) \dotplus \ldots \dotplus M(S(F_t))$. Each dule $M(S(F_i(x)))$ is cyclic and by Theorem 2 decomposes into a direct sum of primary cyclic dules: $M(S(F_i)) = M(S(F_{i1})) \dotplus \ldots \dotplus M(S(F_{ir_i}))$, where $F_{ij}(x)$ is a primary monic polynomial. m this one obtains the decomposition (3). Its uniqueness follows from the Krull–Schmidt orem [9, Chap. VIII, Sec. 2, Theorem 1], since every module $M(S(G_j))$ is primary and cyclic d hence irreducible.

As for matrices over a field the primary polynomials $G_1(x), \ldots, G_r(x)$ in (3) may be lled elementary divisors of the normal matrix A. In view of our Corollary we can then say at the set of all elementary divisors of a normal matrix determines this matrix uniquely to similarity.

As in the case of fields it is sometimes useful to pass to the language of characteristic rices when dealing with similarity transformations over R. The characteristic matrix of R_m is the matrix $xE - A$, where E is the identity matrix $E \in R_m$. Two matrices $\mathfrak{A}(x)$, $) \in R[x]_m$ are called equivalent, and we write $\mathfrak{A} \sim \mathfrak{B}$, if one is obtained from the other by inite sequence of elementary operations. The condition $\mathfrak{A}(x) \sim \mathfrak{B}(x)$ is equivalent to the istence of invertible matrices $U(x)$, $V(x) \in R[x]_m$ such that $U\mathfrak{A}V = \mathfrak{B}$.

With every matrix $\mathfrak{A}(x) \in R[x]_m$ we associate the $R[x]$-module $\mathfrak{M}(\mathfrak{A}(x)) = R[x]^{(m)}/\mathfrak{C}(\mathfrak{A}(x))$, re $R[x]^{(m)}$ is the module of column vectors and $\mathfrak{C}(\mathfrak{A}(x))$ is its submodule generated by the lumns of $\mathfrak{A}(x)$. One sees without difficulty that if $\mathfrak{A}(x) \sim \mathfrak{B}(x)$, then $\mathfrak{M}(\mathfrak{A}(x)) \sim \mathfrak{M}(\mathfrak{B}(x))$.

THEOREM 3. Let $A, B \in R_m$. Then $M(A) \cong \mathfrak{M}(xE - A)$ and the condition $A \approx B$ is equivalent $(xE - A) \sim (xE - B)$.