ON SHAPIRO'S CATALAN CONVOLUTION

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Dedicated to my friend Dennis Stanton

ABSTRACT. L. Shapiro found an elegant formula for the self-convolution of the even subscrtipted terms in the Catalan sequence. This paper provides a natural q-analog of Shapiro's formula together with three proofs, one of which is purely combinatorial.

1. INTRODUCTION

The Catalan numbers are the famous sequence

$$(1.1) \qquad C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{4n+2}\binom{2n+2}{n+1},$$

(note that $C_{-1} = -1/2$). In 2002, L. Shapiro found and proved the first formula in the following (cf. [6; p. 123, eq. (5.12)], [8; p. 31, ex. 6.C.14])

Theorem 1. *For non-negative integers n,*

$$(1.2) \qquad \sum_{j=0}^{n} C_{2j}C_{2n-2j} = 4^n C_n,$$

$$(1.3) \qquad \sum_{j=0}^{n+1} C_{2j}C_{2n+1-2j} = 0,$$

$$(1.4) \qquad \sum_{j=0}^{n} C_{2j+1}C_{2n-1-2j} = -4^n C_n$$

The absence of (1.3) and (1.4) from the literature may well be owing to the unfamiliarity of $C_{-1} = -1/2$, an observation which allows the familiar Catalan convolution to be written

$$(1.5) \qquad \sum_{j=-1}^{n} C_j C_{n-1-j} = 0.$$

T. Koshy provides the natural proof of (1.2) [6; p. 123] via generating functions. R. P. Stanley [8; p. 31 ex. 6.C.14 (b)] asks for a bijective proof of (1.2).

In this paper, we shall consider a q-analog of Theorem 1 involving the q-Catalan numbers introduced in [3], namely

$$(1.6) \qquad \mathcal{C}_n(\lambda, q) = \frac{q^{2n}\left(-\frac{\lambda}{q}; q^2\right)_n}{(q^2; q^2)_n},$$

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where

$$(1.7) \qquad (A_1, A_2, \ldots, A_r; q)_n = \prod_{i=1}^{r} \prod_{j=0}^{n-1} \left(1 - A_i q^j \right).$$

Note that

$$(1.8) \qquad \mathcal{C}_n(\lambda, -q) = \mathcal{C}_n(-\lambda, q),$$

and as noted in [3; p. 268, last paragraph of Sec. 1]

$$(1.9) \qquad \lim_{q \to 1} \mathcal{C}_{n+1}(-1, q) = -2^{-2n-1} C_n.$$

Our full proof of our q-analog of Theorem 1 will be given in Section 2. The result is

Theorem 2. *For positive integers n,*

$$(1.10) \quad \sum_{j=0}^{n} q^{2j} \mathcal{C}_{2j+1}(1, -q) \mathcal{C}_{2n+1-2j}(1, -q) = \frac{-q^{2n+1}(-q^2; q^2)_n \mathcal{C}_{n+1}(1, -q)}{(-q; q^2)_{n+1}},$$

$$(1.11) \quad \sum_{j=0}^{n} q^{2j} \mathcal{C}_{2j}(1, -q) \mathcal{C}_{2n+1-2j}(1, -q) = \frac{q^{2n+2}(1 - q^{2n-1})(-q^2; q^2)_{n-1} \mathcal{C}_n(1, -q)}{(-q; q^2)_{n+1}},$$

$$(1.12) \quad \sum_{j=0}^{n} q^{2j} \mathcal{C}_{2j}(1, -q) \mathcal{C}_{2n-2j}(1, -q) = \frac{q^{2n}(-q^2; q^2)_{n-1} \mathcal{C}_n(1, -q)}{(-q; q^2)_n}.$$

If we let $q \to 1$ in each of (1.10), (1.11) and (1.12) invoke (1.9) and multiply by an appropriate power of 2, we obtain (1.2), (1.3) and (1.4) respectively. Hence Theorem 1 is a direct corollary of Theorem 2.

It turns out that (1.10) and (1.12) are, in fact, specializations of a new q-hypergeometric series summation. Indeed, this is one of the most interesting aspects of this paper. It is seldom that a new q-hypergeometric summation turns up. Recall [5; p. 4]

$$(1.13) \qquad {}_{r+1}\phi_r \left(\begin{matrix} a_0, a_1, \ldots, a_r; q, z \\ b_1, \ldots, b_r \end{matrix} \right) = \sum_{n=0}^{\infty} \frac{(a_0; q)_n (a_1, q)_n \cdots (a_r; q)_n z^n}{(q; q)_n (b_1; q)_n \cdots (b_r; q)_n}.$$

The summation in question is

Theorem 3. *For n a non-negative integer,*

$$(1.14) \qquad {}_4\phi_3 \left(\begin{matrix} q^{-2n}, a, b, \frac{q^{1-2n}}{ab}; q^2, q^2 \\ \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}, abq \end{matrix} \right) = \frac{q^{-n}(a; q)_n (b, q)_n (-q; q)_n (ab; q^2)_n}{(ab; q)_n (a; q^2)_n (b; q^2)_n}.$$

We prove Theorem 3 in Section 3, and in the next section, we deduce (1.10) and (1.12) from (1.14). In the final section we look at further implications of Theorem 3 as well as an open problem.

2. Generating function proof of Theorem 3

Lemma 4. *For $a = 0, 1$ and $b = 0, 1$*

$$\left(\frac{(-xq^2; q^2)_\infty}{(xq^3; q^2)_\infty} + (-1)^a \frac{(xq^2; q^2)_\infty}{(-xq^3; q^2)_\infty} \right) \left(\frac{(-xq; q^2)_\infty}{(xq^2; q^2)_\infty} + (-1)^b \frac{(xq; q^2)_\infty}{(-xq^2; q^2)_\infty} \right)$$

$$= \frac{(-xq; q)_\infty}{(xq^2; q)_\infty} + (-1)^{a+b} \frac{(xq; q)_\infty}{(-xq^2; q)_\infty} + (-1)^b(1 - xq) + (-1)^a(1 + xq).$$

Proof. The product of the first terms in each parenthesis is the first term on the right-hand side. The product of the second terms in each parenthesis is the second term on the right-hand side.

The other two products after simplification become

$$(-1)^b(1 - xq) + (-1)^a(1 + xq). \qquad \square$$

Next we note that the generating funciton for $\mathcal{C}_n(\lambda, q)$ is an infinite product. Namely

$$(2.1) \qquad \sum_{n=0}^\infty \mathcal{C}_n(\lambda, q)x^n = \sum_{n=0}^\infty \frac{\left(-\frac{\lambda}{q}; q^2\right)_n (xq^2)^n}{(q^2; q^2)_n}$$

$$= \frac{(-\lambda xq; q^2)_\infty}{(xq^2; q^2)_\infty} \qquad \text{(by [5; p. 7 eq. (1.3.2)])}$$

Therefore

$$(2.2) \qquad \sum_{n=0}^\infty \mathcal{C}_{2n+1}(1, -q)x^{2n+1} = \frac{1}{2} \left(\frac{(xq; q^2)_\infty}{(xq^2; q^2)_\infty} - \frac{(-xq; q^2)_\infty}{(-xq^2; q^2)_\infty} \right),$$

and

$$(2.3) \qquad \sum_{n=0}^\infty \mathcal{C}_{2n}(1, -q)x^{2n} = \frac{1}{2} \left(\frac{(xq; q^2)_\infty}{(xq^2; q^2)_\infty} + \frac{(-xq; q^2)_\infty}{(-xq^2; q^2)_\infty} \right),$$

Consequently

$$(2.4) \quad 4 \sum_{n=0}^\infty \left(\sum_{j=0}^n q^{2j} \mathcal{C}_{2j}(1, -q) \mathcal{C}_{2n-2j}(1, -q) \right) x^{2n}$$

$$= \left(\frac{(xq^2; q^2)_\infty}{(xq^3; q^2)_\infty} + \frac{(-xq^2; q^2)_\infty}{(-xq^3; q^2)_\infty} \right) \left(\frac{(xq; q^2)_\infty}{(xq^2; q^2)_\infty} + \frac{(-xq; q^2)_\infty}{(-xq^2; q^2)_\infty} \right)$$

$$= \frac{(xq; -q)_\infty}{(xq^2; -q)_\infty} + \frac{(-xq; -q)_\infty}{(-xq^2; -q)_\infty} + 2 \quad \text{(by Lemma 4 with } q \to -q\text{)}$$

$$= 4 + \sum_{n=1}^\infty \frac{\left(\frac{1}{q}; -q\right)_n x^n q^{2n}}{(-q; -q)_n}(1 + (-1)^n) \quad \text{(by [5; p. 7, eq. (1.3.2)])}$$

$$= 4 + 2 \sum_{n=1}^\infty \frac{\left(\frac{1}{q}; -q\right)_{2n} x^{2n} q^{4n}}{(-q; -q)_{2n}}.$$

By comparing coefficients of x^{2n} in the extremes of (2.4), we see that

$$(2.5) \qquad \sum_{j=0}^{n} q^{2j} \mathcal{C}_{2j}(1,-q)\mathcal{C}_{2n-2j}(1,-q) = \frac{q^{4n}\left(\frac{1}{q};-q\right)_{2n}}{2(-q;-q)_{2n}}$$

$$= \frac{q^{4n}\left(\frac{1}{q};q^2\right)_{n}(-q^2;q^2)_{n-1}}{(q^2;q^2)_n(-q;q)_n}$$

$$(2.6) \qquad = \frac{q^{2n}(-q^2;q^2)_{n-1}}{(-q;q^2)_n}\mathcal{C}_n(1,-q),$$

which is (1.12).

Next

$$(2.7)\ \ 4\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} q^{2j}\mathcal{C}_{2j}(1,-q)\mathcal{C}_{2n+1-2j}(1,-q)\right)x^{2n+1}$$

$$= \left(\frac{(xq^2;q^2)_\infty}{(xq^3;q^2)_\infty} + \frac{(-xq^2;q^2)_\infty}{(-xq^3;q^2)_\infty}\right)\left(\frac{(xq;q^2)_\infty}{(xq^2;q^2)_\infty} - \frac{(-xq;q^2)_\infty}{(-xq^2;q^2)_\infty}\right)$$

$$= \frac{(xq;-q)_\infty}{(xq^2;-q)_\infty} - \frac{(-xq;-q)_\infty}{(-xq^2;-q)_\infty} - (1+xq)+(1-xq) \quad \text{(by Lemma 4)}$$

$$= -2xq + \sum_{n=0}^{\infty}\frac{\left(\frac{1}{q};-q\right)_{n}x^n q^{2n}}{(-q;-q)_n}(1+(-1)^n) \quad \text{(by [5; p. 7, eq. (1.3.2)])}$$

$$= -2xq + 2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{q};-q\right)_{2n+1}x^{2n+1}q^{4n+2}}{(-q;-q)_{2n+1}}.$$

By comparing coefficients of x^{2n+1} in the extremes of (2.7), we see that for $n>0$,

$$(2.8)\ \ \sum_{j=0}^{n} q^{2j}\mathcal{C}_{2j}(1,-q)\mathcal{C}_{2n+1-2j}(1,-q) = \frac{q^{4n+2}\left(\frac{1}{q};-q\right)_{2n+1}}{2(-q;-q)_{2n+1}}$$

$$= \frac{q^{4n+2}\left(\frac{1}{q};q^2\right)_{n}(1-q^{2n-1})(-q^2;q^2)_{n-1}}{(q^2;q^2)_n(-q;q^2)_{n+1}}$$

$$(2.9) \qquad = \frac{q^{2n+2}(1-q^{2n-1})(-q^2;q^2)_{n-1}\mathcal{C}_n(1,-q)}{(-q;q^2)_{n+1}},$$

which is (1.11).

Finally

$$(2.10)\ \ 4\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} q^{2j+1}\mathcal{C}_{2j+1}(1,-q)\mathcal{C}_{2n+1-2j}(1,-q)\right)x^{2n+2}$$

$$= \left(\frac{(xq^2;q^2)_\infty}{(xq^3;q^2)_\infty} - \frac{(-xq^2;q^2)_\infty}{(-xq^3;q^2)_\infty}\right)\left(\frac{(xq;q^2)_\infty}{(xq^2;q^2)_\infty} - \frac{(-xq;q^2)_\infty}{(-xq^2;q^2)_\infty}\right)$$

$$= -\frac{(-xq;-q)_\infty}{(-xq^2;-q)_\infty} - \frac{(-xq;-q)_\infty}{(-xq^2;-q)_\infty} + (1+xq)+(1-xq)$$

$$= 2 - \sum_{n=0}^{\infty} \frac{\left(\frac{1}{q};-q\right)_n x^n q^{2n}}{(-q;-q)_n}(1+(-1)^n) \quad \text{(by [5; p. 7, eq. (1.3.2)])}$$

$$= 2 - 2\sum_{n=0}^{\infty} \frac{\left(\frac{1}{q};-q\right)_{2n} x^{2n} q^{4n}}{(-q;-q)_{2n}}.$$

By comparing coefficients of x^{2n+2} in the extremes of (2.10), we see that for $n \geq 0$

$$(2.11) \quad \sum_{j=0}^{n} q^{2j} \mathcal{C}_{2j+1}(1,-q)\mathcal{C}_{2n+1-2j}(1,-q) = \frac{-q^{4n+3}\left(\frac{1}{q};-q\right)_{2n+2}}{2(-q;-q)_{2n+2}}$$

$$= \frac{-q^{2n+1}q^{2n+2}\left(\frac{1}{q};q^2\right)_{n+1}(-q^2;q^2)_n}{(q^2;q^2)_{n+1}(-q;q^2)_{n+1}}$$

$$(2.12) \qquad\qquad = \frac{-q^{2n+1}(-q^2;q^2)_n \mathcal{C}_{n+1}(1,-q)}{(-q;q^2)_{n+1}},$$

which is (1.10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Proof of Theorem 3

We begin with an identity that is a natural companion of [2; mid-page 21], but different from that result. Using the standard notation [5; p. 6, eqs. (1.2.41) and (1.2.42)] extending (1.7), we wish to show that

$$(3.1) \quad \sum_{j=0}^{\infty} \frac{(A^2,B;q^2)_j \left(\frac{q}{B}\right)^j}{\left(q^2,\frac{A^2q^2}{B};q^2\right)_j} = \frac{1}{2}\frac{\left(A,-\frac{Aq}{B};q\right)_\infty (q;q^2)_\infty}{\left(\frac{A^2q^2}{B},\frac{q}{B};q^2\right)_\infty} + \frac{1}{2}\frac{\left(-A,\frac{Aq}{B};q\right)_\infty (q;q^2)_\infty}{\left(\frac{A^2q^2}{B},\frac{q}{B};q^2\right)_\infty}.$$

To establish (3.1), we replace q by q^2 and then set $a = A^2$, $b = B$, $c = \frac{A^2q^2}{B}$, $z = \frac{q}{B}$ in [5; p. 10, eq. (1.4.5)]. Thus

$$\sum_{j=0}^{\infty} \frac{(A^2,B;q^2)_j \left(\frac{q}{B}\right)^j}{\left(q^2,\frac{A^2q^2}{B};q^2\right)_j} = \frac{\left(\frac{A^2q^2}{B^2},q;q^2\right)_\infty}{\left(\frac{A^2q^2}{B},\frac{q}{B};q^2\right)_\infty} \sum_{j=0}^{\infty} \frac{\left(\frac{B}{q};q\right)_{2j}\left(\frac{Aq}{B}\right)^{2j}}{(q;q)_{2j}}$$

$$= \frac{1}{2}\frac{\left(\frac{A^2q^2}{B^2},q;q^2\right)_\infty}{\left(\frac{A^2q^2}{B},\frac{q}{B};q^2\right)_\infty} \sum_{j=0}^{\infty} \frac{\left(\frac{B}{q};q\right)_j\left(\frac{Aq}{B}\right)^j}{(q;q)_j}(1+(-1)^j)$$

$$= \frac{1}{2}\frac{\left(\frac{A^2q^2}{B^2},q;q^2\right)_\infty}{\left(\frac{A^2q^2}{B},\frac{q}{B};q^2\right)_\infty}\left(\frac{(A;q)_\infty}{\left(\frac{Aq}{B};q\right)_\infty} + \frac{(-A;q)_\infty}{\left(\frac{-AB}{q};q\right)_\infty}\right)$$

$$= \frac{1}{2}\frac{\left(A,-\frac{Aq}{B};q\right)_\infty (q;q^2)_\infty}{\left(\frac{A^2q^2}{B},\frac{q}{B};q^2\right)_\infty} + \frac{1}{2}\frac{\left(-A,\frac{Aq}{B};q\right)_\infty (q;q^2)_\infty}{\left(\frac{A^2q^2}{B},\frac{q}{B};q^2\right)_\infty}.$$

In (3.1) we set $A = q^{-n}$ where n is a non-negative integer and $B = c$. As a result, the term containing $(A;q)_\infty$ vanishes, and as a result

$$(3.2) \quad \sum_{j=0}^{n} \frac{\left(q^{-2n}, c; q^2\right)_j \left(\frac{q}{c}\right)^j}{\left(q^2, \frac{q^{2-2n}}{c}; q^2\right)_j} = \frac{1}{2} \frac{\left(-q^{-n}, \frac{q^{1-n}}{c}; q\right)_\infty (q; q^2)_\infty}{\left(\frac{q^{2-2n}}{c}, \frac{q}{c}; q^2\right)_\infty}$$

$$= \frac{\left(-q^{-n}, \frac{q^{1-n}}{c}; q\right)_n}{\left(\frac{q^{2-2n}}{c}; q^2\right)_n} \qquad \text{(by cancellation and}$$
$$[1; \text{p. 5, eq. (1.2.5)}])$$

$$= \frac{q^{-n}(-q, c; q)_n}{(c; q^2)_n}$$

by simplifying.

In addition, we need two special cases of the q-Pfaff-Saalschütz summation [5; p. 13, eq. (1.7.2)]. First

$$(3.3) \qquad \sum_{j=0}^{n} \frac{\left(q^{-n}, q^{1-n}, \frac{q^{1-2n}}{ab}; q^2\right)_j q^{2j}}{\left(q^2, \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}; q^2\right)_j} = \frac{q^{\binom{n}{2}}(a, b; q)_n}{(a, b; q^2)_n}.$$

Second,

$$(3.4) \qquad \sum_{j=0}^{n} \frac{\left(\frac{Aq}{BC}, Aq^n, q^{-n}; q\right)_j q^j}{\left(q, \frac{Aq}{B}, \frac{Aq}{C}; q\right)_j} = \frac{\left(\frac{Aq}{BC}\right)^n (B, C; q)_n}{\left(\frac{Aq}{B}, \frac{Aq}{C}; q\right)_n}.$$

With these results in hand, we are ready to prove Theorem 3.

$$(3.5) \quad {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, \frac{q^{1-2n}}{ab}; q^2, q^2 \\ \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}, qab \end{matrix} \right)$$

$$= \sum_{j=0}^{n} \frac{\left(q^{-2n}, \frac{q^{1-2n}}{ab}; q^2\right)_j (abq^{2n})^j}{(q^2, abq; q^2)_j} \sum_{i=0}^{j} \frac{\left(\frac{q^{2-2n}}{ab}, q^{2j-2n}, q^{-2j}; q^2\right)_i q^{2i}}{\left(q^2, \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}; q^2\right)_i} \quad \text{(by (3.4))}$$

$$= \sum_{j \geqq i \geqq 0} \frac{\left(q^{-2n}; q^2\right)_{j+i} \left(\frac{q^{1-2n}}{ab}; q^2\right)_j \left(\frac{q^{2-2n}}{ab}; q^2\right)_i a^j b^j (-1)^i q^{i^2+i+2j(n-i)}}{(q^2; q^2)_{j-i}(abq; q^2)_j \left(q^2, \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}; q^2\right)_i}$$

$$= \sum_{i \geqq 0} \frac{\left(q^{-2n}; q^2\right)_{2i} \left(\frac{q^{1-2n}}{ab}, \frac{q^{2-2n}}{ab}; q^2\right)_i a^i b^i (-1)^i q^{-i^2+i}}{\left(q^2, \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}, abq; q^2\right)_i}$$

$$\times \sum_{j=0}^{n-2i} \frac{\left(q^{-2n+4i}, q^{1-2n+2i}; q^2\right)_j a^i b^j q^{j(2n-2i)}}{(q^2, abq^{2i+1}; q^2)_j} \qquad \text{(shifting } j \text{ to } j+i\text{)}$$

$$= q^{-n} \sum_{i \geqq 0} \frac{\left(q^{-2n}; q^2\right)_{2i} \left(\frac{q^{1-2n}}{ab}; q\right)_{2i} \left(-q, \frac{q^{1-2n+2i}}{ab}; q\right)_{n-2i} a^i b^i (-1)^i q^{2ni-i^2+3i}}{\left(q^2, \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}, abq; q^2\right)_i \left(\frac{q^{1-2n+2i}}{ab}; q^2\right)_{n-2i}}$$

$$\text{(by (3.2))}$$

$$= \frac{q^{-\binom{n+1}{2}+2i}(ab;q^2)_n(-q;q)_n}{(ab;q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{1-n}, \frac{q^{1-2n}}{ab}; q^2, q^2 \\ \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b} \end{matrix} \right)$$

$$= \frac{q^{-n}(a,b,-q;q)_n(ab;q^2)_n}{(a,b,ab;q^2)_n},$$

by (3.3) as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Second proofs of (1.10) and (1.12)

First we treat (1.12).

$$(4.1)\sum_{j=0}^{n} q^{2j}\mathcal{C}_{2j}(1,-q)\mathcal{C}_{2n-2j}(1,-q)$$

$$= q^{2n}\sum_{j=0}^{n} q^{-2j} \frac{q^{4j}\left(\frac{1}{q};q^2\right)_{2j}}{(q^2;q^2)_{2j}} \frac{q^{4n-4j}\left(\frac{1}{q};q^2\right)_{2n-2j}}{(q^2;q^2)_{2n-2j}} \qquad \text{(where we have reversed the order of summation)}$$

$$= \frac{q^{6n}\left(\frac{1}{q};q^2\right)_{2n}}{(q^2;q^2)_{2n}}\sum_{j=0}^{n} \frac{\left(\frac{1}{q},q;q^4\right)_j\left(\frac{1}{q};q^2\right)_{2n}}{(q^2,q^4;q^4)_j(q^2;q^2)_{2n}} \frac{q^{4j}(q^{-4n},q^{-4n+2};q^4)_j}{(q^{-4n+3},q^{-4n+5};q^4)_j}$$

$$= \frac{q^{6n}\left(\frac{1}{q};q^2\right)_{2n}}{(q^2;q^2)_{2n}} {}_4\phi_3 \left(\begin{matrix} q^{-4n}, \frac{1}{q}, q, q^{-4n+2}; q^4, q^4 \\ q^{-4n+5}, q^{-4n+3}, q^2 \end{matrix} \right)$$

$$= \frac{q^{6n}\left(\frac{1}{q};q^2\right)_{2n} q^{-2n}\left(\frac{1}{q};q^2\right)_n (q;q^2)_n(q^4;q^4)_{n-1}(-q^2;q^2)_n}{(q^2;q^2)_{2n}\left(\frac{1}{q};q^4\right)_n (q;q^4)_n(q^2;q^2)_{n-1}} \qquad \text{(by Theorem 3)}$$

$$= \frac{q^{4n}\left(\frac{1}{q};q^2\right)_n (q;-q)_{2n}(-1;q^2)_n}{2(q;-q)_{2n}(-q;-q)_{2n}}$$

$$= \frac{q^{4n}\left(\frac{1}{q};-q\right)_{2n}}{2(-q;-q)_{2n}}.$$

Therefore

$$(4.2) \qquad \sum_{j=0}^{n} q^{2j}\mathcal{C}_{2j}(1,-q)\mathcal{C}_{2n-2j}(1,-q) = \frac{q^{4n}\left(\frac{1}{q},q^2\right)_n}{(q^2;q^2)_n} \frac{(-q^2,q^2)_{n-1}}{(-q;q^2)_n}$$

$$= \frac{q^{2n}(-q^2;q^2)_{n-1}}{(-q;q^2)_n}\mathcal{C}_n(1,-q),$$

which is the assertion in (1.12).

Now we treat (1.10).

$$(4.3) \sum_{j=0}^{n} q^{2j} \mathcal{C}_{2j+1}(1,-q) \mathcal{C}_{2n+1-2j}(1,-q)$$

$$= q^{2n} \sum_{j=0}^{n} q^{-2j} \frac{q^{4n+2-4j}\left(\frac{1}{q};q^2\right)_{2n+1-2j}}{(q^2;q^2)_{2n+1-2j}} \frac{q^{4j+2}\left(\frac{1}{q};q^2\right)_{2j+1}}{(q^2;q^2)_{2j+1}} \quad \begin{array}{l}\text{(where we have reversed}\\ \text{the order of summation)}\end{array}$$

$$= \frac{q^{6n+4}\left(1-\frac{1}{q}\right)\left(\frac{1}{q};q^2\right)_{2n+1}}{(1-q^2)(q^2;q^2)_{2n+1}} \sum_{j=0}^{n} \frac{(q;q^2)_{2j}}{(q^4;q^2)_{2j}} \frac{(q^{-4n-2};q^2)_{2j}\, q^{4j}}{(q^{1-4n};q^2)_{2j}}$$

$$= \frac{q^{6n+4}\left(1-\frac{1}{q}\right)\left(\frac{1}{q};q^2\right)_{2n+1}}{(1-q^2)(q^2;q^2)_{2n+1}} \,{}_4\phi_3\!\left(\begin{array}{c} q^{-4n},q,q^3,q^{-4n-2};q^4,q^4 \\ q^{3-4n},q^{1-4n},q^6 \end{array}\right)$$

$$= \frac{q^{6n+4}\left(1-\frac{1}{q}\right)\left(\frac{1}{q};q^2\right)_{2n+1} q^{-2n}(q;q^2)_n(q^3;q^2)_n(-q^2;q^2)_n(q^4;q^4)_n}{(1-q^2)(q^2;q^2)_{2n+1}(q^4;q^2)_n(q;q^4)_n(q^3;q^4)_n} \quad \text{(by Theorem 3)}$$

$$= \frac{-q^{4n+3}\left(\frac{1}{q};-q\right)_{2n+2}}{2(-q;-q)_{2n+2}}.$$

Thus we have proved (2.11) which is equivalent to (2.12) which proves (1.10) once again.

5. Combinatorial Proof of Theorem 2

We shall only provide details for the proof of (1.10) which is the q-analog of (1.2). Stanley asked for a bijective proof of (1.2). It is not difficult (only tedious) to transform our proof of (1.10) into a bijective proof.

Reexamining (2.7) wherein (1.10) is proved, we see that (1.10) is equivalent to the assertion that

$$\left(\frac{(-xq^2;q^2)_\infty}{(xq^3;q^2)_\infty} - \frac{(xq^2;q^2)_\infty}{(-xq^3q^2)_\infty}\right)\left(\frac{(-xq;q^2)_\infty}{(xq^2;q^2)_\infty} - \frac{(xq;q^2)_\infty}{(-xq^2;q^2)_\infty}\right)$$
$$= \frac{(-xq;q)_\infty}{(xq^2;q)_\infty} + \frac{(xq;q)_\infty}{(-xq^2;q)_\infty} - 2.$$

The expression inside the first set of parentheses is twice the generating function for partitions with an odd number of parts (x marks the number of parts), no repeated evens and no 1's. The expression inside the second set of parentheses is twice the generating function for partitions with an odd number of parts with no repeated odds.

On the other hand, the right-hand side is twice the generating function for bipartitions into an even number of parts wherein the first component has no repetitions and the second has no 1's. We call these R-partitions.

Consequently this last type of bipartitions for any given n should be double the bipartitions of n in which the first component has an odd number of parts, no repeated evens and no 1's while the second component has an odd number of parts and no repeated odds. We call these latter partitions L-partitions.

We now ask: Given an ordinary partition π with no repeated 1's, how many different R-partitions can be produced from it, and how many L-partitions?

If π has s different parts > 1 of which g are even and u are odd, then an R-partition can be formed by putting some subset of distinct parts taken from the s possibilities into the first component. I.e. π produces 2^s R-partitions.

Now we consider how many L-partitions can be produced from π. Clearly if there is a 1 it must go into the second component. We must select a subset of j distinct evens for the first component from the g possibilities (the remaining evens go to the second component) and also a subset of k distinct odds from the u possibilities (the remaining odds > 1 go to the first component). Suppose G is the total number of evens in π, and U is the total number of odds > 1, then the number of parts in the first component is $j + (U - k)$ and the number in the second component is $\epsilon + k + (G - j)$ (where $\epsilon = 1$ if π has a 1 and $\epsilon = 0$ otherwise).

Case 1. U even. Since $j + U - k$ must be odd. Therefore $j \not\equiv k \pmod 2$. The total number of L-partitions produced by π is

$$\sum_{j \not\equiv k \pmod 2} \binom{g}{j}\binom{u}{k} = \sum \binom{g}{j}\binom{u}{k}\frac{(1-(-1)^{j+k})}{2} = 2^{g+u-1} = 2^{s-1}.$$

Case 2. U odd. Again $j + U - k$ must be odd. Therefore $j \equiv k \pmod 2$. The total number of L-partitions produced by π is

$$\sum_{j \equiv k \pmod 2} \binom{g}{j}\binom{u}{k} = \sum \binom{g}{j}\binom{u}{k}\frac{(1+(-1)^{j+k})}{2} = 2^{g+u-1} = 2^{s-1}.$$

Consequently each ordinary partition π with an even number of parts and no repeated 1's produces exactly twice as many R-partitions as L-paritions. This proves (1.12).

To treat (1.10) and (1.11) we need only provide analogous interpretations of the initial generating function identities appearing in each of (2.4) and (2.7) respectively.

6. Conclusion

We note that both (1.2) and (1.4) may be deduced from the identity of Whipple [9; eq. (6.6)], [4; p. 33, Sec. 4.7] transforming a nearly poised $_4F_3$ into a $_5F_4$. This proved to be quite intriguing because this particular theorem has no natural q-analog. Nonetheless a comparison of (1.2) with an empirically discovered (1.10) suggested that while a full q-analog of Whipple's formula didn't exist there might well be a q-analog of some specialization. Theorem 3 was thus discovered.

While Theorem 3 provides alternative proofs of (1.10) and (1.12) it does not imply (1.11). Equation (1.11) would follow from

$$(6.1) \quad {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, \frac{q^{3-2n}}{ab}; q^2, q^2 \\ \frac{q^{2-2n}}{a}, \frac{q^{4-2n}}{b}, abq \end{matrix}\right)$$
$$= \frac{-q^{-1-n}(a;q)_n(b;q)_{n-1}(-q;q)_n(ab;q^2)_{n-1}\left(abq^{2n-2}(q^2-b)+abq^{n-1}(1-q)-q+b\right)}{(ab;q)_{n-1}(1-abq^{2n-1})(a;q^2)_n\left(\frac{b}{q^2};q^2\right)_n}.$$

Identity (6.1) has been verified for $1 \leqq n \leqq 6$. Presumably a full verification could be provided using one of the computer algebra summation packages. It would be interesting to understand it along the lines of the proof of Theorem 3. P. Larcombe [7] has also studied the Shapiro convolution and has made a number of interesting connections with hypergeometric identities.

References

[1] G. E. Andrews, The Theory of Partitions, Encycl. Math. and its Appli., Vol. 2, Addison-Wesley, Reading, MA, 1976 (Reprinted: Cambridge University Press, Cambridge, 1998).

[2] G. E. Andrews and R. Askey, *Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials*, Higher Combinatorics (M. Aigner, ed.), 1977, Rediel, Boston, pp. 3-26.

[3] G. E. Andrews, Catalan numbers, q-Catalan numbers and hypergeometric series, *J. Comb. Th.* (A), **44** (1987), 267-273.

[4] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Math. Tract No. 32, Cambridge University Press, Cambridge, 1935 (Reprinted: Stechert-Hafner, New York, 1964).

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encycl. of Math. and Its Appl., Vol. 35, Cambridge University Press, Cambridge, 1990.

[6] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, New York, 2009.

[7] P. J. Larcombe, On some Catalan identities of Shapiro, J. Comb. Math. and Combinatorial Computing, **54**(2005), 165-174.

[8] R. P. Stanley, Catalan Addendum, `http://www-math.mit.edu/~rstan/ec/catadd.pdf`, version 6 October 2008.

[9] F. J. Whipple, Some transformations of generalized hypergeometric series, *Proc. London Math. Soc.* (2), **26** (1927), 257-272.

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