A Fine Dream

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Abstract

We shall develop further N. J. Fine's theory of three parameter non-homogeneous first order q-difference equations. The object of our work is to bring the Rogers Ramanujan identities within the purview of such a theory. In addition, we provide a number of new identities.

1 Introduction

One of the most charming works in the history of q-series is Nathan Fine's book, Basic Hypergeometric Series and Applications [1]. Published in 1988, the book was the product of a project that Fine began in the 1940's [10].

The focus is on

$$F(a, b; t : q) = \sum_{n=0}^{\infty} \frac{(aq; q)_n t^n}{(bq; q)_n} \ , \tag{1.1}$$

$$(A; q)_N = \prod_{n=0}^{\infty} \frac{(1 - Aq^n)}{(1 - Aq^{n+N})} \ . \tag{1.2}$$

In general it is assumed that $|t|, |a|, |b| \leqq 1$ and $|q| < 1$.

The first fifteen pages of [11] are devoted to finding functional equations (i.e. q-difference equations) relating $F(a, b; t : q)$ to the same function in which some of a, b and t are replaced by aq, bq, and tq respectively.

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¿From these deceptively simple considerations, Fine derives what has become known as the Rogers-Fine identity [11; eq. (14.1), p. 15]

$$F(a, b; t : q) = \sum_{n \geqq 0} \frac{(aq; q)_n \, (atq/b; q)_n \, (1 - atq^{2n+1}) b^n t^n q^{n^2}}{(bq; q)_n (t; q)_{n+1}}. \tag{1.3}$$

Indeed, this result follows from simple iteration of one of Fine's functional equations [11; eq. (4.1), p. 2]

$$(1 - t)F(a, b; t; q) = 1 - atq + \frac{(1 - aq)(b - atq)tq}{(1 - bq)} F(aq, bq; tq; q). \tag{1.4}$$

¿From (1.4), a variety of significant classical identities follow including Sylvester's extension of Euler's Pentagonal Number Theorem [4; Th. 9.2, p. 140]. Ramanujan clearly knew (1.3) although he never stated it. An entire chapter, Chapter 9, of the first volume of Ramanujan's Lost Notebook [8; Ch. 9] is devoted to corollaries of (1.3). Of course, the name of the identity correctly suggests that, in fact, L. J. Rogers [13; pp. 334-335] was the first to discover and prove (1.3). However, Rogers did not analyze the implications of (1.3) nearly as extensively as Fine did. It should also be noted that G. W. Starcher [14] proved many instances of (1.3).

Part of the reason that 40 years stretches between [10] and [11] is that Fine, as he told me, had hoped to extend his theory to a broader set of q-difference equations so that he could include the Rogers-Ramanujan identities [9, Ch. 4], [4, Ch. 7]. These are the following two identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}, \tag{1.5}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}, \tag{1.6}$$

Unfortunately Fine was never able to realize his dream, and [11] appears without mention of the Rogers-Ramanujan identities. In fact, when one recalls that the standard proof of these identities is focused on the second order q-difference equation [4; eq. (7.1.1), p. 104]

$$\mathcal{F}(x) = \mathcal{F}(xq) + xq\mathcal{F}(xy^2),$$

one suspects that the theory of first order q-difference equations will be inadequate in pursuing (1.5) and (1.6).

Our main object in this paper is to realize Fine's dream. We begin in Section 2 by reexamining (1.3) and (1.4) starting so to speak, from the wrong end. Section 3 treats a first order q-difference equation that is very similar to (1.4), in the course of this we are led to a new sequence of rational functions

$$\rho_m(a,b) = \sum_{0 \leqq 2j \leqq m} \left[\begin{array}{c} m-j \\ j \end{array} \right] \frac{q^{j^2+j} a^j (aq;q)_{m-j}}{(bq;q)_j}, \tag{1.7}$$

where

$$\left[\begin{array}{c} A \\ B \end{array} \right] = \begin{cases} 0 \text{ if } B < 0 & \text{or } B > A \\ \frac{(q;q)_B}{(q;q)_A (q;q)_{A-B}} & , \quad \text{otherwise.} \end{cases} \tag{1.8}$$

In Section 4, we deduce the Rogers-Ramanujan identities, and discover a new polynominal version thereof. Section 5 considers a variety of new identities implied by Theorem 1. We conclude with some comments relating our work to the classical literature.

2 Rogers-Fine Revisited.

Instead of starting with (1.1) as Fine did, let us start with (1.4). In other words, given (1.4) we would like to find the power series expansion in t for $F(a,b;t:q)$ assuming $F(a,b;0:q) = 1$.

Thus if we set

$$F(a,b;t:q) = \sum_{n=0}^{\infty} A_n(a,b) t^n \tag{2.1}$$

we see that $A_0(a,b) = 1$, $A_1(a,b) = \frac{(1-aq)}{(1-bq)}$ and by (1.4) for $n > 1$

$$A_n(a,b) - A_{n-1}(a,b) = \frac{(1-aq)}{(1-bq)} (bq^n A_{n-1}(aq,bq) - aq^n A_{n-2}(aq,bq)). \tag{2.2}$$

¿From this recurrence it is easy to compute further values for $A_n(a,b)$;

$$A_2(a,b) = \frac{(1-aq)(1-aq^2)}{(1-bq)(1-bq^2)}, \tag{2.3}$$

and

$$A_3(a,b) = \frac{(1-aq)(1-aq^2)(1-aq^3)}{(1-bq)(1-bq^2)(1-bq^3)}\,. \tag{2.4}$$

This leads to the conjecture that

$$A_n(a,b) = \frac{(aq;q)_n}{(bq;q)_n}\,, \tag{2.5}$$

and it is a matter of elementary algebra to demonstrate that this conjectured expression in fact satisfies the defining recurrence (2.2). Thus by starting with (1.4) we have deduced that (1.1) provides the unique power series solution. However, as we noted in the introduction, iteration of (1.4) yields (1.3).

Consequently, by starting with (1.4) we have obtained (1.3) without initially taking (1.1) into account.

3 The Extended Rogers-Fine Identity.

Theorem 1.

$$\sum_{n\geqq 0} \rho_n(a,b)t^n = \sum_{n\geqq 0} \frac{(aq;q)_n \left(\frac{atq}{b};q\right)_n a^n b^n t^{2n} q^{2n^2+n}(1-atq^{2n+1})}{(bq;q)_n(t;q)_{n+1}}\,, \tag{3.1}$$

where $\rho_n(a,b)$ is defined in (1.7).

Remark. The series on the right side of (3.1) differs from (1.3) only in that $2n^2$ has replaced n^2 in the exponent of q and $a^n t^n$ has been multiplied into each term.

Proof. Following the lead from Section 2, we begin with the q-difference equation

$$(1-t)f(a,b;t:q) = 1 - atq + \frac{(1-aq)(b-atq)at^2q^3}{(1-bq)} f(aq,bq;tq:q). \tag{3.2}$$

which together with $f(a,b;0;q) = 1$ uniquely defines the right-hand side of (3.1).

Let us define $A_n(a,b)$ by

$$f(a,b;t:q) = \sum_{n=0}^{\infty} A_n(a,b)t^n. \tag{3.3}$$

Clearly $A_0(a,b) = 1$, $A_1(a,b) = 1 - aq$, and for $n > 1$ by comparing coefficients of t^n in (3.2)

$$A_n(a,b) - A_{n-1}(a,b) = \frac{(1-aq)}{(1-bq)} aq^{n+1}(bA_{n-2}(aq,bq) - aA_{n-3}(aq,bq)). \quad (3.4)$$

Using this last recurrence, we easily deduce that

$$A_2(a,b) = (1-aq)(1-aq^2) + \tfrac{aq^2(1-aq)}{1-bq}$$

$$A_3(a,b) = (1-aq)(1-aq^2)(1-aq^3) + aq^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tfrac{(1-aq)(1-aq^2)}{(1-bq)}.$$

One or two more examples is enough to make the conjecture that

$$A_n(a,b) = \rho_n(a,b). \quad (3.5)$$

Clearly (3.5) is valid for $n \leqq 3$. So to prove (3.5) we need only show that the recurrence (3.4) holds for $\rho_n(a,b)$. We shall consider (3.4) multiplied throughout by $(1-bq)/(1-aq)$. Thus

$$\begin{aligned}
&\tfrac{(1-bq)}{(1-aq)}(\rho_n(a,b) - \rho_{n-1}(a,b)) \\
&= \tfrac{(1-bq)}{(1-aq)} \sum_{0 \leqq 2j \leqq n} \tfrac{q^{j^2+j}a^j}{(bq;q)_j}(aq;q)_{n-1-j} \\
&\times \left\{ \begin{bmatrix} n-j \\ j \end{bmatrix} (1-aq^{n-j}) - \begin{bmatrix} n-1-j \\ j \end{bmatrix} \right\} \\
&= \sum_{0 \leqq 2j \leqq n} \tfrac{q^{j^2+j}a^j(aq^2;q)_{n-2-j}}{(bq^2;q)_{j-1}} \left\{ \begin{bmatrix} n-1-j \\ j-1 \end{bmatrix} q^{n-2j} - aq^{n-j} \begin{bmatrix} n-j \\ j \end{bmatrix} \right\} \\
&\qquad \text{(by [4; eq. (3.3.3), p. 35])} \\
&= aq^n \sum_{0 \leqq 2j \leqq n} \tfrac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_{j-1}} \begin{bmatrix} n-2-j \\ j \end{bmatrix} \\
&-a^2q^{n+1} \sum_{-2 \leqq 2j \leqq n-2} \tfrac{q^{j^2+2j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \begin{bmatrix} n-j-1 \\ j+1 \end{bmatrix} = aq^n \sum_{0 \leqq 2j \leqq n-2} \tfrac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \begin{bmatrix} n-2-j \\ j \end{bmatrix} \\
&-a^2q^{n+1} \sum_{-2 \leqq 2j \leqq n-2} \tfrac{q^{j^2+2j}(aq^2;q)_{n-3-j}a^j}{(bq^2;q)_j} \\
&\times \left\{ \begin{bmatrix} n-j-3 \\ j \end{bmatrix} + q^{n-2j-2} \begin{bmatrix} n-j-3 \\ j-1 \end{bmatrix} + q^{j+1} \begin{bmatrix} n-j-2 \\ j+1 \end{bmatrix} \right\}
\end{aligned}$$

(by [4; eq's. (3.3.3) and (3.3.4), p. 35])

$$= -a^2q^{n+1}\rho_{n-3}(aq,bq) + aq^n \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \begin{bmatrix} n-2-j \\ j \end{bmatrix}$$

$$-a^2q^{2n-1} \sum_{-2\leqq 2j\leqq n-2} \frac{q^{j^2}(aq^2;q)_{n-3-j}a^j}{(bq^2;q)_j} \begin{bmatrix} n-j-3 \\ j-1 \end{bmatrix}$$

$$-a^2q^{n+2} \sum_{-2\leqq 2j\leqq n-2} \frac{q^{j^2+3j}(aq^2;q)_{n-3-j}a^j}{(bq^2;q)_j} \begin{bmatrix} n-j-2 \\ j+1 \end{bmatrix}$$

$$= -a^2q^{n+1}\rho_{n-3}(aq,bq) + aq^n \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \begin{bmatrix} n-2-j \\ j \end{bmatrix}$$

$$-a^2q^{2n-1} \sum_{-2\leqq 2j\leqq n-2} \frac{q^{j^2}(aq^2;q)_{n-3-j}a^j}{(bq^2;q)_j} \begin{bmatrix} n-j-3 \\ j-1 \end{bmatrix}$$

$$-aq^n \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2+j}(aq^2;q)_{n-2-j}a^j}{(bq^2;q)_{j-1}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}$$

$$= -a^2q^{n+1}\rho_{n-3}(aq,bq)$$

$$+aq^n \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \left(\begin{bmatrix} n-2-j \\ j \end{bmatrix} - aq^{n-1-j} \begin{bmatrix} n-j-3 \\ j-1 \end{bmatrix} \right)$$

$$-aq^n \sum_{0\leqq 2j\leqq n-1} \frac{q^{j^2+j}(aq^2;q)_{n-2-j}a^j}{(bq^2;q)_j}(1-bq^{j+1})$$

$$\times \left(\begin{bmatrix} n-2-j \\ j \end{bmatrix} + q^{n-2j-1} \begin{bmatrix} n-2-j \\ j-1 \end{bmatrix} \right)$$

(by [4; eq. (3.3.3), p. 35])

$$= -a^2q^{n+1}\rho_{n-3}(aq,bq) + abq^{n+1}\rho_{n-2}(aq,bq)$$

$$+aq^n \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \left(\begin{bmatrix} n-2-j \\ j \end{bmatrix} - aq^{n-1-j} \begin{bmatrix} n-j-3 \\ j-1 \end{bmatrix} \right)$$

$$-aq^n \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2+j}a^j(aq^2;q)_{n-2-j}}{(bq^2;q)_j} \begin{bmatrix} n-2-j \\ j \end{bmatrix}$$

$$-aq^{2n-1} \sum_{0\leqq 2j\leqq n-1} \frac{q^{j^2-j}(aq^2;q)_{n-2-j}a^j}{(bq^2;q)_{j-1}} \begin{bmatrix} n-2-j \\ j-1 \end{bmatrix}$$

$$= -a^2q^{n+1}\rho_{n-3}(aq,bq) + abq^{n+1}\rho_{n-2}(aq,bq)$$

$$+aq^n \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \begin{bmatrix} n-2-j \\ j \end{bmatrix} \{1-(1-aq^{n-j-1})\}$$

$$-a^2q^{2n-1} \sum_{0\leqq 2j\leqq n-2} \frac{q^{j^2}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \begin{bmatrix} n-3-j \\ j-1 \end{bmatrix}$$

$$-a^2q^{2n-1} \sum_{0\leqq 2j\leqq n-3} \frac{q^{j^2+j}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j} \begin{bmatrix} n-3-j \\ j \end{bmatrix}$$

$$= -a^2q^{n+1}\rho_{n-3}(aq,bq) + abq^{n+1}\rho_{n-2}(aq,bq)$$

$$
\begin{aligned}
&+a^2q^{2n-1}\sum_{0\leqq 2j\leqq n-2}\tfrac{q^{j^2}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j}\left[\begin{array}{c} n-2-j \\ j \end{array}\right] \\
&-a^2q^{2n-1}\sum_{0\leqq 2j\leqq n-2}\tfrac{q^{j^2}a^j(aq^2;q)_{n-3-j}}{(bq^2;q)_j}\left[\begin{array}{c} n-2-j \\ j \end{array}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by [4; eq. (3.3.4), p. 35])} \\
&= -a^2q^{n+1}\rho_{n-3}(aq,bq)+abq^{n+1}\rho_{n-2}(aq,bq).
\end{aligned}
$$

So we have proved that the $\rho_n(a,b)$ satisfy (3.4) and along with $\rho_0(a,b)=1$, $\rho_1(a,b)=1-aq$, we see that $A_n(a,b)=\rho_n(a,b)$ which establishes Theorem 1. □

4 The Rogers-Ramanujan Identities.

Theorem 2. For $N \geqq 0$

$$
\rho_N(a,0)=\sum_{n\geqq 0}(aq;q)_n(-1)^n a^{2n}q^{n(5n+3)/2}\left(\left[\begin{array}{c} N-2n \\ n \end{array}\right]-aq^{2n+1}\left[\begin{array}{c} N-2n-1 \\ n \end{array}\right]\right) \tag{4.1}
$$

Proof. Set $b=0$ in Theorem 1. Then expand $1/(t;q)_{n+1}$ into the q-binomial series [4; eq. (3.3.7). p. 36]

$$
\frac{1}{(t;q)_{n+1}}=\sum_{j=0}^{\infty}\left[\begin{array}{c} n+j \\ j \end{array}\right]t^j.
$$

Finally compare coefficients of t^N on each side. □

Corollary 3.

$$
\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}(-1)^n a^{2n}q^{n(5n+3)/2}(1-aq^{2n+1}). \tag{4.2}
$$

Proof. Let $N\to\infty$ in (4.1). □

Corollary 4. The Rogers-Ramanujan identities (i.e. (1.5) and (1.6), are valid.

Proof. Let $a \to q^{-1}$ in (4.2). Hence

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{n(5n-1)/2}(1+q^n)\right) \\ &= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \qquad \text{(cf. [9; §8.4])}.\end{aligned}$$

Let $a = 1$ in (4.2). Hence

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(5n+3)}(1-q^{2n+1}) \\ &= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \qquad \text{(cf. [9; §8.4])}.\end{aligned}$$

□

We note that the object in [7] was to provide a polynomial refinement of (4.2) that was obviously a polynomial in both a and q. While that goal was achieved in [7] through the introduction of a-Gaussian polynomials, we see that the same object has been accomplished here in Theorem 2 without recourse to a-Gaussian polynomials. We also note that the right-hand side of (4.1) is a special case of [6; eq. (6.3), p. 7 with $h = 1$, $k = 2$, $i = N - 1$].

5 Further Consequence of Theorem 1.

Several specializations of Theorem 1 provide instance of identities for series resembling theta functions.

Corollary 5.

$$(1-t)\sum_{n=0}^{\infty} \rho_n(a,a)t^n = \sum_{n\geqq 0} a^{2n}t^{2n}q^{2n^2+n}(1-atq^{2n+1}),$$

or equivalently

$$\begin{aligned}(1-t)\sum_{n,m=0}^{\infty} t^{n+2m}\begin{bmatrix} n+m \\ m \end{bmatrix} q^{m^2+m}a^m(aq^m;q)_n \\ = \sum_{n\geq 0} a^{2n}t^{2n}q^{2n^2+n}(1-atq^{2n+1}).\end{aligned}$$

Proof. Set $b = a$ in Theorem 1. □

We note the surprising fact that if we replace t by t/a in Corollary 5, then the right-hand side is independent of a while the left-hand side still appears to depend on a.

Corollary 6.

$$(1-t)\sum_{n=0}^{\infty}\rho_n(t,0)t^n=\sum_{n=0}^{\infty}(-1)^n t^{5n}q^{n(5n+3)/2}(1-t^2q^{2n+1}),$$

or equivalently

$$\begin{aligned}(1-t)\sum_{n,m=0}^{\infty} t^{n+3m}\begin{bmatrix} n+m \\ m\end{bmatrix} q^{m^2+m}(tq;q)_{n+m}\\ =\sum_{n=0}(-1)^n t^{5n}q^{n(5n+3)/2}(1-t^2q^{2n+1}).\end{aligned}$$

Proof. Set $b=0$, $a=t$ in Theorem 1. $\square$

We note that the right-hand side of Corollary 6 explicitly appears in Theorem 2 of [3; p. 290]. This suggests the possibility of relating the proof of the second Rogers-Ramanujan identity given in [3] to our Corollary 6.

Corollary 7.

$$\sum_{n=0}^{\infty}\rho_n\left(\frac{t}{q},0\right)t^n=\sum_{n=0}^{\infty}(-1)^n t^{5n}q^{n(5n-1)/2}(1+tq^n),$$

or equivalently

$$\begin{aligned}\sum_{n,m=0}^{\infty} t^{n+3m}\begin{bmatrix} n+m \\ m\end{bmatrix} q^{m^2}(t;q)_{n+m}\\ =\sum_{n=0}(-1)^n t^{5n}q^{n(5n-1)/2}(1+tq^n).\end{aligned}$$

Proof. Set $b=0$, $a=\frac{t}{q}$ in Theorem 1. $\square$

We note that the right-hand side of Corollary 7 multiplied by t^{-2} explicitly appears in Theorem 1 of [3; p. 290]. Again this suggests the possibility of a possible connection between Corollary 7 and Theorem 1 of [3; p. 290].

Corollary 8.

$$\sum_{n\geqq 0}\rho_n\left(\frac{t}{q},\frac{t}{q^{1/2}}\right)=\sum_{n=0}^{\infty}t^{4n}q^{2n^2-n/2}(1+tq^n),$$

or equivalently

$$\sum_{n,m=0}^{\infty} t^{n+3m} \left[\begin{array}{c} n+m \\ m \end{array} \right] q^{m^2} \frac{(t;q)_{n+m}}{(tq^{1/2};q)_m}$$
$$= \sum_{n=0}^{\infty} t^{4n} q^{2n^2-n/2}(1+tq^n).$$

Proof. Set $a = t/q$, $b = t/q^{1/2}$ in Theorem 1. $\square$

6 Conclusion.

First let us put Theorem 1 into its place in the q-hypergeometric hierarchy. In [1; p. 564], it is noted that the Rogers-Fine identity is, in fact, a special case of Watson's q-analog of Whipple's theorem (cf. [12; eq. (2.5.1), p. 43], [2, eq. (3.1), p. 198], [6; eq. (2.10), p. 3]). Theorem 1 can be deduced from the generalization of Whipple's theorem [2; Th. 4, p. 199, with $k = 3$; a, b_3, c_3 and b_1, replaced by atq, q, aq and $\frac{atq}{b}$ resp., and finally c_1, b_2, c_2 $N \to \infty$]. As mentioned in Section 4, it was recognized in [6; Section 6] that the right-hand side of our Theorem 1 could be obtained from the very well-poised ${}_{10}\phi_9$; however, the proofs of the Rogers-Ramanujan identities described there lack the eleqance of $\rho_N(a, b)$ and do not hint at its existence. Indeed, our Theorem 1 gives reasonable hope that one may be able to answer affirmatively Question 1 of [6; p. 1] which effectively asked whether it would be possible to gather all the known proofs of the Rogers-Ramanujan identities as special cases of Theorem 4 of [2; p. 199].

Finally the simplicity of the results in Sections 3-5 suggests that there may be nice combinatorial proofs of these results. This seems especially plausible in light of the quite straight forward bijective proof of (1.3) given in [1; Section 4].

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