Reduction of Linear Programming to Linear Approximation

It is well-known that every l^∞ linear approximation problem can be reduced to a linear program. In this paper we show that conversely every linear program can be reduced to an l^∞ linear approximation problem.

Now we recall definitions.

An *affine function* of variables $x_1, \ldots, x_n$ is $c_0 + c_1x_1 + \cdots + c_nx_n$ where c_i are given numbers.

A *linear constraint* is one of the following constraints: $f \le g, f \ge g, f = g$, where f, g are affine functions.

A *linear program* is an optimization (maximization or minimization) of an affine function subject to a finite system of linear constraints.

A l^∞ linear approximation problem, also known as Chebyshev approximation problem or finding the least-absolute-deviation fit, is the problem of minimization of

$$\max(|f_1|, \ldots, |f_m|) = \|(f_1, \ldots, f_m)\|_\infty,$$

where f_i are affine functions. In more detail, $f_i = b_i - c_ix$, where x is a column on n variables (unknowns), c_i are given numbers, and c_i are given rows.

Given any Chebyshev approximation problem, we can write it as the following linear program with one additional variable t:

$$t \to \min, \text{ subject to } -t \le f_i \le t \text{ for } i = 1, \ldots, m.$$

This is a linear program with $n+1$ variables and $2m$ linear constraints.

Now we want to reduce an arbitrary linear program to a Chebyshev approximation problem. First of all it is well-known [V] that every linear program can be reduced to solving a symmetric matrix game.

So we start with a matrix game, with the payoff matrix $M = -M^T$ of size n by n. Our problem is to find a column $x = (x_i)$ (an optimal strategy) such that

$$Mx \le 0, x \ge 0, \sum x_i = 1. \tag{1}$$

As usual, $x \ge 0$ means that every entry of the column x is ≥ 0, Later we write $y \le t$ for a column y and a number t if every entry of y is $\le t$. We go even further in abusing notation, denoting by $y - t$ the column obtaining from y by subtracting t from every entry. Similarly we denote by $M + c$ the matrix obtained from M by adding a number c to every entry.

This problem (1) (of finding an optimal strategy) is about finding a feasible solution for a system of linear constraints. It can be written as the following linear program with an additional variable t:

$$t \to \min, Mx \le t, x \ge 0, \sum x_i = 1. \tag{2}$$

Now we find the largest entry c in the matrix M. If $c = 0$, then $M = 0$ and the problem (1) is trivial (every mixed strategy x is optimal). So we assume that $c > 0$.

Adding c to every entry of M we obtain a matrix $M + c \geq 0$ (all entries ≥ 0). The linear program (2) is equivalent to

$$t \to \min, (M + c)x \leq t, x \geq 0, \sum x_i = 1 \tag{3}$$

in the sense that these two programs have the same feasible solutions and the same optimal solutions. The optimal value for (2) is 0 while the optimal value for (3) is c.

Now we can rewrite (3) as follows:

$$\|(M + c)x\|_\infty \to \min, x \geq 0, \sum x_i = 1 \tag{4}$$

which is a Chebyshev approximation problem with additional linear constraints. We used that $M+c \geq 0$, hence $(M+c)x \geq 0$ for every feasible solution x in (2). The optimal value is still c.

Now we rid off the constraints in(4) as follows:

$$\| \begin{pmatrix} (M + c)x \\ c - x \\ \sum x_i + c - 1 \\ -\sum x_i - c + 1 \end{pmatrix} \|_\infty \to \min . \tag{5}$$

Note that the optimization problems (4) and (5) have the same optimal value c and every optimal solution of (4) is optimal for (5). Conversely, for every x with a negative entry, the objective function in (5) is $> c$. Also, for every x with $\sum x_i \neq 1$, the objective function in (5) is $> c$. So every optimal solution for (5) is feasible and hence optimal for (4).