4. Stable range

Stable rank

Let A be an associative ring with unity. An n-column (b_i) is called *unimodular* if $\sum Ab_i = A$, i.e. $\sum a_i b_i = 1$ for some $a_i \in A$. The set of all unimodular n-columns is denoted by $\mathrm{Um}_n A$. The group $\mathrm{GL}_n A$ acts on $\mathrm{Um}_n A$ by matrix multiplication.

All columns of an invertible matrix are unimodular. The converse is not always true.

The following condition was introduced by H. Bass.

($\mathbf{A}_n$) for every $(b_i) \in \mathrm{Um}_{n+1} A$, there are $c_i \in A$ such that $(b_i + c_i b_{n+1})_{1 \le i \le n} \in \mathrm{Um}_n A$.

Proposition 4.1. $(\mathrm{A}_m) \Rightarrow (\mathrm{A}_{m+1})$. Moreover, for any $n \ge m+1$ the condition (A_m) implies (A_n) with $c_i = 0$ for $i \ge m+1$.

Proof. Let $b \in \mathrm{Um}_{n+1} A$ so $ab = 1$ for an $(n+1)$-row a. We write $b = (b_i) = \begin{pmatrix} b' \\ b'' \\ b_{n+1} \end{pmatrix}$ with m-column b' and $(n-m)$-column b''. Similarly we write $a = (a', a'', a_{n+1})$, so $ab = a'b' + a''b'' + a_{n+1} b_{n+1} = 1$. By (A_m) applied to $\begin{pmatrix} b' \\ a''b'' + a_{n+1} b_{n+1} \end{pmatrix} \in \mathrm{Um}_{m+1} A$, there is an m-column d such that $b' + d(a''b'' + a_{n+1} b_{n+1}) \in \mathrm{Um}_m B$, hence

$$\begin{pmatrix} b' + d(a''b'' + a_{n+1} b_{n+1}) \\ b'' \end{pmatrix} \in \mathrm{Um}_n A.$$

Multiplying the last column by $\begin{pmatrix} 1 & -da'' \\ 0 & 1_{n-m} \end{pmatrix} \in E_n A$, we obtain that

$$\begin{pmatrix} b' + d a_{n+1} b_{n+1} \\ b'' \end{pmatrix} \in \mathrm{Um}_n A.$$ QED.

Definition 4.2. We denote $\mathrm{sr}(A)$ the least integer n such that (A_n) holds. If no such n exists, $\mathrm{sr}(A) = \infty$. It is not clear whether (A_n) makes sense when $n = 0$. It is reasonable to write $\mathrm{sr}(A) = 0$ if and only if $A = 0$. This is consistent with defining the dimension of the empty topological space to be -1.

Example 4.3. It is clear that $\mathrm{sr}(A) = 1$ for any local ring A (including any field or division algebra). More generally, Bass [B] showed that $\mathrm{sr}(A) = 1$ when $A/\mathrm{rad}(A)$ is a direct product of matrix rings over division rings.

Example 4.4. Bass showed that if A is finitely generated as module over its center C and the space of maximal ideals in C is a finite union of noetherian subspaces of dimension $\le d$, then $\mathrm{sr}(A) \le d+1$. Here the dimension is defined using chains of irreducible subspaces. A subspace is irreducible if it not a union of two closed proper subsets.

Example 4.5. It is an easy exercise, that $\mathrm{sr}(\mathbf{Z}) = 2$. More generally, $\mathrm{sr}(A) = 2$ for the ring of integers in any number field. Also $\mathrm{sr}(A) = 2$ when A is the ring of Hurvitz or Lipschitz quaternions.

Example 4.6. It is an easy exercise, that $\mathrm{sr}(F[x]) = 2$ for any field F. By [V14], $\mathrm{sr}(F[x_1 \ldots, x_d]] = d+1$ for all d when F is a subfield of $\mathbf{R}$. By [Su2], $\mathrm{sr}(\mathbf{C}[x_1 \ldots, x_d]) = d+1$ for all d. By [VS], $\mathrm{sr}(F[x_1 \ldots, x_d] =\le d$ for $d \ge 2$ if F is a finite field.

Example 4.7. Vaserstein [V14] showed that if $A = \mathbf{R}^X$ is the ring of continuous real functions on a topological space X of dimension d, then $\mathrm{sr}(A) = d+1$. Here the dimension is defined using maps $X \to \mathbf{R}^n$ with stable values. For example, for $X = \mathbf{R}^d$, A is the ring of continuous real functions in n variables and $\mathrm{sr}(A) = d + 1$. The subrings of bounded or smooth functions have the same stable rank $d + 1$.

For the ring $\mathbf{C}^X$ of complex-valued functions, we have $\mathrm{sr}(\mathbf{C}^X) = [d/2] + 1$ where $[]$ means the integer part. See [V14], Theorem 7.

Here are four other nontrivial examples.

Example 4.8. For the Weyl algebra $A = \mathbf{C}[p_1, q_1, \ldots, p_d, q_d]$ (where $p_i q_i - q_i p_i = 1$), $\mathrm{sr}(A) = 2$ (Stafford [St]).

Example 4.9. For the disc algebra A (i.e., the ring of holomorphic functions on open disc, continuous on the closed disc), $\mathrm{sr}(A) = 1$ [JMW].

Example 4.10. Let A be a right Bézout domain (see Example 2.9). We claim that $\mathrm{sr}(A) \le 2$. By [V14], the stable rank is right-left symmetric. So we have to prove that for any unimodular row (a_1, a_2, a_3) over A there are $c_1, c_2 \in A$ such that the row $(a_1 + a_3 c_1, a_2 + a_3 c_2)$ is unimodular. As in Example 2.9, we can find a matrix $\alpha \in \mathrm{GL}_2 A$ such that $(a_1, a_2)\alpha = (a_0, 0)$ where $a_1 A + a_2 A = a_0 A$. Then (a_0, a_3) is unimodular hence $(a_0, a_3)\alpha^{-1}$ is unimodular. But $(a_0, a_3)\alpha^{-1} = (a_1, a_2) + (0, a_3)\alpha^{-1}$ so we can take $(c_1, c_2) = (0, 1)\alpha^{-1}$ (the second row of α^{-1}).

Example 4.11. Let A be a C^*-algebra with 1 (if A is commutative, $A = \mathbf{C}^X$ for a compact Hausdorff topological space X). Then $\mathrm{sr}(A)$ is the maximum of d such that $\mathrm{Um}_d A$ is dense in A^d [HV].

We will give more examples in the end of section. Now we extend the definition of stable rank to rings without 1.

For any ring A with 1 and any ideal B of A, let $\mathrm{Um}_n B$ denote the set of $(b_i) \in \mathrm{Um}_n A$ such that $b_1 - 1, b_i \in B$ for $i \ge 2$. For such a column b the condition $\sum_{i=1}^m A b_i = A$ is equivalent to $\sum_{i=1}^m B b_i = B$ so it is independent of A.

We define $\mathrm{sr}(B)$ to be the least n such that the condition $(\mathbf{A}_n)$ holds for all $(b_i) \in \mathrm{Um}_{n+1} B$.

It is easy to check that:

the condition (A_n) holds for all $(b_i) \in \mathrm{Um}_{n+1} B$ and all $n \ge \mathrm{sr}(B)$,

$\mathrm{sr}(B)$ depends only on B (independent of embedding B as an ideal in a ring with unity);

$\mathrm{sr}(B_0) \le \mathrm{sr}(B)$ for any ideal B_0 of B;

$\mathrm{sr}(B') \le \mathrm{sr}(B)$ for any factor ring B' of B.

The following result is not so trivial. It shows that the concept of stable rank is right-left symmetric.

Proposition 4.12. For any associative ring B, $\mathrm{sr}(B) = \mathrm{sr}(B^0)$ where B^0 is the opposite ring (with the same additive group but the multiplication reversed).

Proof. Since $(B^0)^0 = B$, it suffices to show that $\mathrm{sr}(B) \ge \mathrm{sr}(B^0)$ Let $\mathrm{sr}(B) = m$. We have to prove that if $\sum_{i=1}^{m+1} a_i b_i = 1$ where $a_1 - 1, b_1 - 1, a_i, b_i \in B$ for $i \ge 2$ then there are $u_i \in B$ such that $\sum_{i=1}^{m} (a_i + a_{m+1} u_i) B = B$.

Consider the matrix

$$\alpha = \begin{pmatrix} 1 & a \\ 0 & 1_{m+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1_{m+1} \end{pmatrix} = \begin{pmatrix} 0 & a \\ -b & 1_{m+1} \end{pmatrix} \in \mathrm{GL}|_{m+2} A$$

(where A is an associative ring with 1 containing B as an ideal). Since $\mathrm{sr}(B) = m$, there are $v_i, c_i \in B$ such that

$$\sum_{i=1}^{m}(b_i + v_i a_{m+1} b_{m+1}) = -b_{m+1}.$$

Then the matrix

$$\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_m & 0 \\ 0 & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -v & 1_m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_m & v a_{m+1} \\ 0 & 0 & 1 \end{pmatrix} \alpha$$

has the form

$$\beta = \begin{pmatrix} 0 & a & \\ * & * & 0 \\ 0 & -u & 1 \end{pmatrix} \in \mathrm{GL}|_{m+2} A$$

where $v = (v_i) \in B^m$ is a column, $c = (c_i)$ is a row, and $u = (u_i)$ is a row with m entries in B. The matrix

$$\gamma = \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_m & 0 \\ 0 & u & 1 \end{pmatrix}$$

has the form

$$\gamma = \begin{pmatrix} 0 & a' & a_{m+1} \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $a' = (a_i + a_{m+1} u_i)_{1 \le i \le m}$ hence $\begin{pmatrix} 0 & a' \\ * & * \end{pmatrix} \in \mathrm{GL}|_{m+2} A$ so $\sum_{i=1}^{m} a'_i A = A$, i.e., $\sum_{i=1}^{m} a'_i B = B.$, QED.

Lemma 4.13. Let $n \ge \mathrm{sr}(B)$ and $(b_i) \in \mathrm{Um}_{n+1} B$. Then there are $c_i \in A$ such that $(b_i + c_i b_1)_{2 \le i \le n+1} \in \mathrm{Um}_n A$.

Proof. Let $\sum_{i=1}^{n+1} a_i b_i = 1$ with $a_i \in A$.

Using addition operation, we see that

$$(\, a_1, \quad a_{n+1}\,) \begin{pmatrix} b_1 \\ b_{n+1} \end{pmatrix} = (\, a_1 - a_{n+1}, \quad a_{n+1}\,) \begin{pmatrix} b_1 \\ b_{n+1} + b_1 \end{pmatrix}$$

$$= ((1 - b_1 - b_{n+1})(a_1 - a_{n+1}), (1 - b_1 - b_{n+1}) a_{n+1} + 1) \begin{pmatrix} b_1 \\ b_{n+1} + b_1 \end{pmatrix}$$

$$= (1, (a_1 - a_{n+1})(1 - b_1 - b_{n+1}) a_{n+1} + 1)) \begin{pmatrix} (1 - b_1 - b_{n+1})(a_1 - a_{n+1}) b_1 \\ b_{n+1} + b_1 \end{pmatrix}$$

$$= 1 - \sum_{i=2}^{n} a_i b_i.$$

Therefore $\begin{pmatrix} (1 - b_1 - b_{n+1})(a_1 - a_{n+1}) b_1 \\ b' \\ b_{n+1} + b_1 \end{pmatrix} \in \mathrm{Um}_{n+1} A$ where $b' = (b_i)_{2 \le i \le n}$. Since $\mathrm{sr}(B) \le n$, there is an n-column $d = \begin{pmatrix} d_1 \\ d' \end{pmatrix}$ such that

$$\begin{pmatrix} b' + d'(1-b_1-b_{n+1})(a_1-a_{n+1})b_1 \\ b_{n+1}+b_1+d_1(1-b_1-b_{n+1})(a_1-a_{n+1})b_1 \end{pmatrix} = \begin{pmatrix} b' \\ b_{n+1} \end{pmatrix} + cb_1 \in \mathrm{Um}_n A$$

where $c = \begin{pmatrix} d'(1-b_1-b_{n+1})(a_1-a_{n+1}) \\ 1+d_1(1-b_1-b_{n+1})(a_1-a_{n+1}) \end{pmatrix}$. QED

Normal subgroups in stable range

Theorem 4.14. (Bass [B1]). If $n \geq \mathrm{sr}(B)+1$, then

(a) $\mathrm{GL}_n B = \mathrm{E}_n(A,B)\mathrm{GL}_{n-1}B$;

(b) $[\mathrm{GL}_n B, \mathrm{GE}_n A] \subset \mathrm{E}_n(A,B)$.

If $n \geq \mathrm{sr}(A)+1$, then

(c) $\mathrm{E}_n(A,B)$ is normal in $\mathrm{GL}_n A$.

Proof. (a) Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_n B$ where $a - 1_n \in \mathrm{M}_n B, d-1 \in B$, etc. We want to reduce α to $\mathrm{GL}_{n-1}B$ by row addition operations.

By Lemma 4.13, there is an $(n-1)$-column b' such that $b+b'd \in \mathrm{Um}_{n-1}A$, so $c'(b+b'd)=1$ for an $(n-1)$-row c' over A. Then

$$\begin{pmatrix} 1_{n-1} & 0 \\ (1-d)c' & 1 \end{pmatrix}\begin{pmatrix} 1_{n-1} & b' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b'c & b+b'd \\ c+(1-d)c' & 1 \end{pmatrix}.$$

Set

$$\beta = \begin{pmatrix} 1_{n-1} & -b' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1_{n-1} & 0 \\ (1-d)c' & 1 \end{pmatrix}\begin{pmatrix} 1_{n-1} & b' \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_n(A,B).$$

Then $\beta\alpha = \begin{pmatrix} a'' & b'' \\ c'' & 1 \end{pmatrix}$ with $b'' = b+b'd-b', c'' = c+(1-d)c', a'' = a+b'c-b'c''$.

Now

$$\begin{pmatrix} 1_{n-1} & 0 \\ -c''(a''-b''c'')^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1_{n-1} & -b\text{"} \\ 0 & 1 \end{pmatrix}\beta\alpha \in \mathrm{GL}_{n-1}B \text{ and}$$

$$\begin{pmatrix} 1_{n-1} & 0 \\ -c''(a''-b''c'')^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1_{n-1} & -b'' \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_n B.$$

(b) Let $\alpha \in \mathrm{GL}_n B$ and $\beta \in \mathrm{GE}_n A$. We have to prove that $[\alpha,\beta] \in \mathrm{E}_n(A,B)$. Recall that $\mathrm{GE}_n A$ by definition is generated by diagonal and elementary matrices. Since the diagonal matrices normalize the elementary matrices and using Whitehead lemma, every matrix in $\mathrm{GE}_n A$ is a product of elementary matrices and a matrix of the form $\delta = \begin{pmatrix} 1_{n-1} & 0 \\ 0 & d \end{pmatrix}$ with $d \in \mathrm{GL}_1 A$. Since all permutation matrices normalize both $\mathrm{E}_n A$ and $\mathrm{E}_n(A,B)$ we can assume that either $\beta = a^{i,n}$ with $a \in A$ or $\beta = \delta$ as above.

By (a), $\alpha = \alpha_1\alpha_2$ with $\alpha_1 \in \mathrm{E}_n(A,B)$ and $\alpha_2 \in \mathrm{GL}_{n-1}B$. So

$[\alpha,\beta] = \alpha_1\alpha_2\beta\alpha_2^{-1}\alpha_1^{-1}\beta^{-1}$

$= \alpha_1[\alpha_2,\beta]\beta\alpha_1^{-1}\beta^{-1} \in \mathrm{E}_n(A,B)$

because $\alpha_1, \beta\alpha_1^{-1}\beta^{-1} \in \mathrm{E}_n(A,B)$ and $[\alpha_2,\beta]$ has the form $\begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_n B$ hence $[\alpha_2,\beta] \in \mathrm{E}_n(A,B)$ too.

(c) We have to prove that $\alpha b^{i,j}\alpha^{-1} \in \mathrm{E}_n(A,B)$ when $b \in B, 1 \leq i \neq j \leq n$ and $\alpha \in \mathrm{GL}_n A$. Since $\mathrm{E}_n(A,B)$ is invariant under conjugation by permutation matrices, it suffices to consider the case when $(i,j) = (1,n)$.

By (a) with $B = A, \alpha = \alpha_1\alpha_2$ with $\alpha_1 \in \mathrm{E}_n A$ and $\alpha_2 \in \mathrm{GL}_{n-1}A$. Since

$$\beta' := \alpha_2 b^{1,n} \alpha_2^{-1} = \begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_n B,$$

it is clear that $\beta' \in \mathrm{E}_n B$, so $\alpha b^{1,n} \alpha^{-1} = \alpha_1 \beta' \alpha_1^{-1} \in \mathrm{E}_n(A, B)$. QED.

Now we generalize Lemma 1.6:

Lemma 4.15. Let A be an associative ring with 1 and B an additive subgroup of A. Assume that either $n \geq 3$ or $n = 2$ and B is generated by its elements of the form $\gamma b \gamma - b$ where $b \in B, \gamma \in \mathrm{GL}_1 A$. Then $[\mathrm{E}_n B, \mathrm{E}_n A] = \mathrm{E}_n(A, B)$

Proof. If $n \geq 3$, our conclusion follows from the relations (1.7) and (1.11); in this case $\mathrm{E}_n(A, B) = \mathrm{E}_n(A, B')$ where B' is the ideal of A generated by B.

Let now $n = 2$. By (1.5), $\alpha = \begin{pmatrix} \gamma & 0 \\ 0 & 1/\gamma \end{pmatrix} \in \mathrm{E}_2 A$ hence

$(\gamma b \gamma - b)^{1,2} = [\alpha, b^{1,2}] \in [\mathrm{E}_2 A, \mathrm{E}_2 B]$ whenever $b \in B, \gamma \in \mathrm{GL}_1 A$. QED.

Corollary 4.16. Under the conditions of Lemma 4,5, assume that $n \geq \mathrm{sr}(B) + 1$. Then

$[\mathrm{E}_n A, \mathrm{E}_n B] = [\mathrm{E}_n A, \mathrm{GL}_n B] = \mathrm{E}_n(A, B)$.

Proof. Combine Theorem 4.4(b) and Lemma 4.5. QED.

Theorem 4.17. If $n \geq \mathrm{sr}(B) + 1$, then the kernel of the Whitehead determinant

wh: $\mathrm{GL}_n B \to \mathrm{K}_1(A, B)$

is $\mathrm{E}_n(A, B)$, so $\mathrm{GL}_n B / \mathrm{E}_n(A, B) = \mathrm{K}_1 B$.

This theorem will be proved in the next section.

Corollary 4.18. Assume that $n \geq \mathrm{sr}(B) + 1$ and that $\mathrm{E}_n A$ is perfect. Then

$[\mathrm{G}_n(A, B), \mathrm{E}_n A] \subset \mathrm{E}_n(A, B)$.

Therefore every subgroup H of $\mathrm{G}_n(A, B)$ containing $\mathrm{E}_n(A, B)$ is normalized by $\mathrm{E}_n A$.

Proof. We have to prove that $[\alpha, \beta] \in \mathrm{E}_n(A, B)$ when $\alpha \in \mathrm{G}_n(A, B)$ and $\beta \in \mathrm{E}_n A$.

We fix α and set

$f(\beta) = [\beta^{-1}, \alpha] \in \mathrm{GL}_n(A, B)$.

For $\beta_1, \beta_2 \in \mathrm{E}_n A$,

$f(\beta_1 \beta_2) = \beta_2 f(\beta_1) \beta_2^{-1} f(\beta_2)$.

Since $f(\beta_1) \in \mathrm{GL}_n B$, Theorem 4.4 (b) gives

$[\beta_2, f(\beta_1)] \in \mathrm{E}_n(A, B)$. So reduction of f modulo $\mathrm{E}_n(A, B)$ gives a homomorphism

$\mathrm{E}_n A \to \mathrm{GL}_n B / \mathrm{E}_n(A, B)$.

By our condition, $\mathrm{E}_n A$ is perfect, and by Theorem 4.7 the target group is $\mathrm{K}_1(A, B)$ which is a commutative group. Thus, the homomorphism is trivial, hence $f(\beta) = [\beta^{-1}, \alpha] \in \mathrm{E}_n(A, B)$. QED.

Lemma 4.19. Let $n \geq 3$ and a matrix $\alpha \in \mathrm{GL}_n A$ commutes with $1^{1,2}$ modulo the center $\mathrm{G}_n(A, 0)$. Then α commutes with $1^{1,2}$ hence all its off-diagonal entries in the first column and the second row are zeros.

Proof. Consider $1^{1,2} \alpha = \alpha 1^{1,2} c$ with $c \in C$, the center of A. Looking at the last column on the both sides, we conclude that

$v'c = v'$ where $v = \begin{pmatrix} v_1 \\ v' \end{pmatrix}$ is the last column of α and $v_1 \in A$. Similarly, $cu_1 = u_1$ for the fist entry u_1 of the last row $u = (u_1, u')$ of α^{-1}. Now $1 = uv = u_1 v_1 + u'v' = cu_1 v_1 + u'v'c = c$, so α commutes with $1^{1,2}$. Looking at the first row and the second column in $1^{1,2} \alpha = \alpha 1^{1,2}$, we complete our proof. QED.

Lemma 4.20. Assume that the group $\mathrm{E}_n A$ is perfect (e.g., $n \geq 3$) and that $[\beta, \alpha] \in \mathrm{G}_n(A, 0)$ (the center of $\mathrm{GL}_n A$) for a matrix $\alpha \in GL_n A$ and all $\beta \in \mathrm{E}_n A$. Then $\alpha \in \mathrm{G}_n(A, 0)$.

Proof. When $n \geq 3$, this is an easy consequence of Lemma 4.9. In general, we set $f(\beta) = [\beta, \alpha]$, so $\beta\alpha\beta^{-1} = f(\beta)\alpha$. Since $f(\beta)$ is center for $\beta \in \mathrm{E}_n A$, this gives a homomorphism $f : \mathrm{E}_n A \to \mathrm{G}_n(A, 0)$. Since the group $\mathrm{G}_n(A, 0)$ is commutative, $f([\mathrm{E}_n A, \mathrm{E}_n A]) = 0$, hence $f(\beta) = [\beta, \alpha] = 1$ for all $\beta \in \mathrm{E}_n A = [\mathrm{E}_n A, \mathrm{E}_n A]$. QED.

Proposition 4.21. Let $n \geq 3$, H a subgroup of $\mathrm{GL}_n A$ normalized by $\mathrm{E}_n A$. Suppose that H contains a non-central matrix $\alpha = (\alpha_{i,j})$ such that either

(a) $\alpha_{n,n} \in \mathrm{GL}_1 A$

or

(b) $\alpha_{n,n} - 1 \in \sum_{i=1}^{n-1} A\alpha_{i,n}$.

Then H contains $\mathrm{E}_n B$ for a nonzero ideal B of A.

Proof. In the case (a), if α has a zero in the last row or column we are done by Proposition 1.10. Otherwise we write

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1_{n-1} & b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ c' & 1 \end{pmatrix}$$

with

$$d = \alpha_{n,n}, b' = bd^{-1} \neq 0, c' = d^{-1}c, a' = a - bd^{-1}c \in \mathrm{GL}_{n-1}A.$$

Since $b \neq 0$ and $n \geq 3$, there is an elementary matrix $\beta \in \mathrm{GL}_{n-1}A$ such that $\beta b \neq b$, i.e., $\beta b' \neq b'$.

Now $\alpha_1 = [\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}, \alpha^{-1}]$

$$= \begin{pmatrix} 1_{n-1} & b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta a' \beta^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ c'\beta^{-1} - c' & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & d \end{pmatrix}^{-1} \begin{pmatrix} 1_{n-1} & -b' \\ 0 & 1 \end{pmatrix} \in H,$$

hence

$$\alpha_2 = \begin{pmatrix} 1_{n-1} & -b' \\ 0 & 1 \end{pmatrix} \alpha_1 \begin{pmatrix} 1_{n-1} & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1_{n-1} & \beta b' - b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\beta, a'] & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ c'' & 1 \end{pmatrix} \in H.$$

The last column $\begin{pmatrix} \beta b' - b' \\ 1 \end{pmatrix}$ of α_2 has exactly two nonzero entries, so we are done by Proposition 1.10.

The case (b) can be reduced to the case (a) by conjugating α with a matrix of the form $\begin{pmatrix} 1_{n-1} & 0 \\ * & 1 \end{pmatrix} \in \mathrm{E}_n A$. QED.

Theorem 4.22 (Bass). Let B' be an ideal of A, $\mathrm{sr}(B') = m$ and $n \geq \max(m+1, 3)$. Then for every subgroup $H \subset \mathrm{G}_n(A, B')$ which is normalized by $\mathrm{E}_n A$ we have

$$\mathrm{E}_n(A, B) \subset H \subset \mathrm{G}_n(A, B)$$

for an ideal B of A contained in B'.

Proof. Define $B = \{b \in A : b^{1,2} \in H\} \subset B_0$ (the lower level of H). By (1.7) and (1.11), B is an ideal of A and $\mathrm{E}_n(A, B) \subset H$. We have to prove that $H \subset \mathrm{G}_n(A, B)$.

Otherwise there is $\alpha \in H \setminus \mathrm{G}_n(A, B)$. The image H' of H in $\mathrm{GL}_n(A/B)$ is normalized by $\mathrm{E}_n(A/B)$ of $\mathrm{E}_n A$. The image α' of α in $\mathrm{GL}_n(A/B)$ is not central.

By Lemma 4.20 applied to $\alpha' \in \mathrm{GL}_n(A/B)$, we can assume that the commutator of α' with an elementary matrix is not central. Replacing α by a commutator, we can assume that $\alpha \in \mathrm{GL}_n B'$.

Using that $\mathrm{sr}(B') \le n-1$, we can conjugate α by a matrix of the form $\begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix}$ and arrange the following condition for $\alpha = (\alpha_{i,j})$:

$\sum_{i=1}^{n-1} A\alpha_{i,1} = A$.

Now we consider $\alpha_1 = [\alpha_1, 1^{1,2}]$ and its image α'_1 in $\mathrm{GL}_n(A/B)$.

Applying Lemma 4.9 to $H' \subset \mathrm{GL}_n(A, B)$, we conclude that either the $(n, 1)$-entry of α' is 0 or α'_1 is not central.

In the first case, $\mathrm{E}_n B_1 \subset H'$ for a nonzero ideal B'_1 of A/B by Proposition 1.10.

In the second case we have $\alpha_1 = (1_n + vu)(-1)^{1,2}$ where $v = \begin{pmatrix} v' \\ v_n \end{pmatrix}$ is the first column of α and $u = (u_1, \ldots, u_n)$ is the second row of α^{-1}. Thus, the last column of α_1 has the form $\begin{pmatrix} v'u_n \\ 1 + v_n u_n \end{pmatrix}$ with $v' \in \mathrm{Um}_n A$. Applying Proposition 4.11(b) to H', we conclude that $\mathrm{E}_n B'_1 \subset H'$ for a nonzero ideal B'_1 of A/B.

Thus, in both the cases H contains a matrix α_2 of the form $\alpha_2 = (b_1)^{3,2}\alpha_3$ with $b_1 \in B' \setminus B$ and $\alpha_3 \in \mathrm{GL}_n B$. We conclude our proof in th same way as that of Theorem 3.9 using the fact that $[\mathrm{E}_n A, \mathrm{GL}_n B] \subset \mathrm{E}_n(A, B) \subset H$. QED

Stable rank one rings

The rings A with $\mathrm{sr}(A) = 1$ have special properties which are not shared by rings of higher stable rank. Sometimes, it is convenient to embed a ring B to a ring with 1 as an ideal. Here is a way to do this: B_1 consists of the pairs (b, z) with $b \in B, z \in \mathbf{Z}$ with addition and multiplication given by

$$(b, z) + (b', z') = (b + b', z + z') \text{ and } (b, z)(b', z') = (bb' + zb' + bz', zz').$$

Proposition 4.23 (Kaplansky). Let B be an associate ring with $\mathrm{sr}(B) = 1$ and $b \in B$. If $B(1 + b) = B$ or $(1 + b)B = B$ then $1 + b \in \mathrm{GL}_1 B$.

Proof. Since $\mathrm{sr}(B) = \mathrm{sr}(B^0)$ by Proposition 4.12, it suffices to deal with the case $B(1 + b) = B$. Thus, we have to prove that $\mathrm{Um}_1 B = \mathrm{GL}_1 B$.

Set $x = 1 + b \in B_1$. Let $ax = 1$. For $d = 1 - xa$ we have $Ba + Bd = B$. So there is $t \in B$ such that $Bu = B$ for $u = a + td$. Since $dx = x - xax = x - x = 0$, we obtain that $1 = ux$, hence $u \in \mathrm{GL}_1 B$. Therefore $x, a \in \mathrm{GL}_1 B$.

Proposition 4.24. Let B be an associate ring with $\mathrm{sr}(B) = 1$ and J is a left or right ideal of B. Then $\mathrm{sr}(J) = 1$.

Proof. Since $\mathrm{sr}(B) = \mathrm{sr}(B^0)$ by Proposition 4.12, it suffices to deal with the case when $BJ \subset J$, i.e., $B_1 J = J$. We have to prove that $\mathrm{sr}(J) = 1$.

Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathrm{Um}_2(J)$, i.e., $a - 1, b \in J$ and $xa + yb = 1$ for some $x, y \in B_1$. Set

$$x' = 1 + (1 - a)x \in 1 + J \quad \text{and} \quad y' = (1 - a)y \in J.$$

Then $x'a + y'b = 1$. Since $\mathrm{sr}(B) = 1$, there are $s, t' \in B_1$ such that $s(a + t'y'b) = 1$. Set $t = t'y' \in B_1 J \subset J$. Then $a + tb - 1 \in J$. Since $s(a + tb) = 1$, it follows that $s - 1 \in J$, hence $s \in J_1$.

Proposition 4.25. Let B be an associate ring with $\mathrm{sr}(B) = 1$ and $p = p^2 \in B$. Then $\mathrm{sr}(pBp) = 1$.

Proof. Let $a - 1, b \in pBp = B'$ and $B'a + B'b = B'$. We claim that $\begin{pmatrix} a + 1 - p \\ b \end{pmatrix} \in \mathrm{Um}_2 B$. We have $B'(1 - p) = 0$, hence $p \in B'a + B'b \subset R(a + 1 - p) + Rb$. On the other hand, $(1 - p)a = 0 = (1 - p)b$. So

$$1 - p = (1 - p)(a + 1 - p) + (1 - p)b \in R(a + 1 - p) + Rb.$$

Thus, $1 = p + 1 - p \in R(a + 1 - p) + Rb$.

Since $\mathrm{sr}(B) = 1$, there is $t \in B$ such that $B(a + tb + 1 - p) = B$. We have

$$(1 - (1 - p)tb)(1 + (1 - p)tb) = 1 = (1 + (1 - p)tb)(1 - (1 - p)tb)$$

so $1 - (1 - p)tb)(1 + (1 - p)$ is a unit, hence

$$B = B(a + tb + 1 - p)(1 - (1 - p)tb) = RB(a + ptb + 1 - p).$$

Therefore $B'(a + ptpb) = B'$ with $ptb \in B'$.

Now we give 3 more examples of rings A with stable rank 1.

Example 4.26. For the ring A of all algebraic integers in $\mathbf{C}$, $\mathrm{sr}(A) = 1$. More generally [V51], let A be a commutative ring with 1 such that the multiplicative group of A/Aa is torsion for every nonzero $a \in A$ and such that the equation $x^n + cx^{n-1} + d = 0$ has a solution for x in A whenever n is a natural number, and $c, d \in A$. Then $\mathrm{sr}(A) = 1$.

We will show now that actually this A satisfies the following stronger condition

(4.27) If $b_1, b_2 \in A$ and $Ab_1 + Ab_2 = A$ then there is a unit $u \in \mathrm{GL}_1 A$ such that $A(b_1 + ub_2) = A$.

More generally, we will prove (4.27) for any commutative ring A with 1 such that:

the multiplicative group of A/Aa is torsion for every nonzero $a \in A$,

the equation $x^n + cx^{n-1} + dx + 1 = 0$ has a solution for x in A whenever $n \geq 3$ is a natural number, and $c, d \in A$.

Let $b_1, b_2 \in A$ and $Ab_1 + Ab_2 = A$. In the case $b_1 = 0$ or $b_2 = 0$ we have $A(b_1 + ub_2) = A$ with $u = 1$, so we assume now that $b_1 b_2 \neq 0$.

We find an even natural number $n \geq 4$ such that $b_1^n - 1 \in Ab_2$ and $b_2^n - 1 \in Ab_1$. Then $b_1^n + b_2^n - 1 = bb_1b_2$ with $b \in A$. Since $Ab_1^{n-2} + Ab_2^{n-2} = A$, we can find $c, d \in A$ such that $cb_1^{n-2} + db_2^{n-2} = -b$.

Now we find a zero $u \in A$ of the polynomial $f(x) = x^n + cx^{n-1} + dx + 1 \in A[x]$, so $u^n + cu^{n-1} + du + 1 = 0$. Clearly, $u \in \mathrm{GL}_1 A$ (namely, $-1/u = u^{n-1} + cu^{n-2} + d$).

Then $b_2u \in A$ is a zero of the polynomial

$$g(x) = b_2^n f(x/b_2) = x^n + b_2cx^{n-1} + b_2^{n-1}dx + b_2^n \in A[x],$$

i.e.,

$$(b_1u)^n + b_2c(b_1u)^{n-1} + b_2^{n-1}d(b_2u) + b_2^n = 0.$$

So $-b_1 + ub_2$ is a zero of the polynomial $h(x) = g(x + b_1)$. The constant term of $h(x)$ is

$$h(0) = g(-b_1) = b_1^n + cb_1^{n-1}b_2 + db_1b_2^{n-1} + b_2^n$$

$$= b_1^n + b_1b_2(cb_2^{n-2} + db_2^{n-2}) + b_2^n = b_1^n - bb_1b_2 + b_2^n = 1.$$

Thus, $b_1 + b_2u \in \mathrm{GL}_1 A$.

Example 4.28. Let A is the ring of all algebraic integers in $\mathbf{R}$. We will prove (4.27) for this A. Therefore $\mathrm{sr}(A) = 1$.

Let $b_1, b_2 \in A$ and $Ab_1 + Ab_2 = A$. In the case $b_1 = 0$ or $b_2 = 0$ we have $A(b_1 + ub_2) = A$ with $u = 1$, so we assume now that $b_1b_2 \neq 0$.

We find $n \geq 1$ such that $b_1^n - 1 \in Ab_2$ and $b_2^n - 1 \in Ab_1$. Replacing, if necessary, even n by $n/2$ and multiplying n by an odd number, we are reduced to the following two cases:

Case 1. $b_1^n + b_2^n - 1 \in Ab_1b_2$ with odd $n \geq 3$,

Case 2: $b_1^n - b_2^n \pm 1 \in Ab_1b_2$ with $n \geq 3$.

In Case 1, as in Example 4.25, $b_1^n + b_2^n - 1 = bb_1b_0$ and $cb_1^{n-2} + db_2^{n-2} = -b$ with $b, c, d \in A$.

Now we find a real zero $u \in A$ of the polynomial $f(x) = x^n + cx^{n-1} + dx + 1 \in A[x]$. Then as in Example 4.25, $ub_2 - b_1$ is a root of a polynomial $h(x)$ with constant term

$$h(0) = g(b_1) = b_1^n + cb_1^{n-1}b_2 + db_1b_2^{n-1} + b_2^n = 1,$$

hence $b_1 - ub_2 \in \mathrm{GL}_1 A$.

In Case 2, $b_1^n - b_2^n \pm 1 = bb_1b_0$ and $cb_1^{n-2} + db_2^{n-2} = -b$ with $b, c, d \in A$.

Now we find a real zero $u \in A$ of the polynomial $f(x) = x^n + cx^{n-1} + dx - 1 \in A[x]$. Then $ub_2 - b_1$ is a root of a polynomial $h(x)$ with constant term

$$h(0) = g(b_1) = b_1^n + cb_1^{n-1}b_2 + db_1b_2^{n-1} - b_2^n = \pm 1,$$

hence $b_1 - ub_2 \in \mathrm{GL}_1 A$.

Example 4.29. For the ring A of all entire functions in one complex variables, $\mathrm{sr}(A) = 1$ (L.A.Rubel [R]). However, this ring does not satisfy the stronger condition (4.27).

Exercises

1. Let A be a commutative ring with 1 and $n.m \geq 1$.
Prove that $(b_i^m) \in \mathrm{Um}_n A$ for any $(b_i) \in \mathrm{Um}_n A$

2. Let F be a field and A be the Grassmann algebra in x, y over F, i.e., $x^2 = y^2 = xy + yx = 0$. Set $B = Fx + Fy \subset A$. Show that B is not an ideal and that $H = \mathrm{E}_2 B$ is a subgroup of $\mathrm{GL}_2 A$ which is normalized by $\mathrm{E}_2 A$. Show that $\{a \in A : a^{1,2} \in \mathrm{E}_2 B\} = B$.

2. Let B be any ring, and $\mathrm{M}B = \cup \mathrm{M}_n B$ the ring of infinite matrices over B with finitely many nonzero entries in each. (So $\mathrm{GL}_1(\mathrm{M}B) = \mathrm{GL}B$.) Show that $\mathrm{sr}(\mathrm{M}B) = 1$ if and only if $\mathrm{sr}(B) = 1$. Show that for any $(\,1 + b_1, b_2\,) \in \mathrm{Um}_2 \mathrm{M}B$ there are $c_1, c_2 \in \mathrm{M}B$ such that $(1 + c_2)(b_2 + (1 + c_1)(1 + b_1)) = 1$.

4. Let $n \geq 2$. Prove that every matrix in $\mathrm{GL}_n A$ is $\alpha\beta\gamma$ with lower triangular α, γ and an upper triangular β if and only if $\mathrm{sr}(A) = 1$ (Vaserstein-Wheland).

5. Give an example of a local ring A, an ideal B, and a subgroup H such that $\mathrm{E}_n(A, B) \subset H \subset \mathrm{G}_n(A, B)$ but H is not normal in $\mathrm{GL}_n A$.

6. Show that the condition (A_n) for the first columns of all matrices in $\mathrm{E}_{n+1}(A, B)$ implies the unrestricted (A_n) (for all unimodular columns in $\mathrm{Um}_{n+1} B$).

7. For any natural number n and any ring $B \neq 0$, show that

$$\mathrm{sr}(\mathrm{M}_n B) - 1 = -[-(\mathrm{sr}(B) - 1)/n],$$

where [ ] means the integral part.

8. Let A be a commutative ring with 1 such that $f(a) = 0$ for all $a \in A$ where $f(x) \in \mathbf{Z}[x]$ is a primitive polynomial in one variable x with integer coefficients. An example is any Boolean ring where $f(x) = x^2 - x$. Show that $\mathrm{sr}(A) = 1$.

9. Show that the condition (4.27) implies that every element of A is a sum of two units which in its turn implies that A has no ideals of index two.

10. Let A be a semilocal ring without ideals of index two. Show that A satisfy (4.27).

11. Let a ring A be the direct product of a family A_i of rings. Show that $\mathrm{sr}(A) = \sup \mathrm{sr}(A_i)$.

12. Let A be a commutative ring with 1 and the row (a_1, a_2, a_3) is unimodular. Prove that the row (a_1^2, a_2, a_3) is the first row of an invertible matrix and that this row can be reduced to the row (a_1, a_2, a_3^2) by addition operations.

13. Let A be an associate ring with $1 \neq 0$. Show that the following condition is equivalent to the condition $\mathrm{sr}(A) = 1$:

for any $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathrm{Um}_2 A$ there are $a_1 \in \mathrm{GL}_1 A$ and $a_2 \in A$ such that $a_1 b_1 + a_2 b_2 = 1$.

14. Let A be an associate ring with 1. Show that the following condition is equivalent to the condition (4.27):

for any $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathrm{Um}_2 A$ there are $a_1, a_2 \in \mathrm{GL}_1 A$ such that $a_1 b_1 + a_2 b_2 = 1$.

15. Let A be a commutative principal ideal domain and $\mathrm{sr}(A) = 1$. Prove that A is a Euclidean ring. (Hint: define the Euclidean function N on A by $N(0) = 0$ and $N(a) = k+1$ when $a \neq 0$ and the product of k irreducible elements).