Bressoud Polynomials, Rogers-Ramanujan Type Identities, and Applications

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In memory of a wonderful mathematician, my friend Marvin Knopp.

Abstract

In his paper providing an easy proof of the Rogers-Ramanujan identities, D. Bressoud extended his work to multiple series identities. Intrinsic in his works are polynomials with diverse application to several aspects of q-series. This paper provides an initial exploration of these polynomials.

1 Introduction

In [9], D. Bressoud provides a stunningly elementary proof of the celebrated Rogers-Ramanujan identities [2, Ch.7]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{1.2}$$

where

$$(A)_n = (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}) = \prod_{j=0}^{\infty} \frac{(1-Aq^j)}{(1-Aq^{j+n})}, \tag{1.3}$$

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and

$$(A_1, A_2, \ldots, A_k; q)_N = \prod_{j=1}^{k} (A_j; q)_N, \tag{1.4}$$

In fact, the Rogers-Ramanujan identities are the cases $k = 2$, $i = 1, 2$ of the following general identity [1]:

$$\sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2+s_2^2+\cdots+s_{k-1}^2+s_i+s_{i-1}+\cdots+s_{k-1}}}{(q)_{s_1-s_k}(q)_{s_i-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty} \tag{1.5}$$

Note that it is unnecessary to require that the s_j's are nonincreasing because $\frac{1}{(q)_M} = 0$ if $M < 0$ by the final expression in (1.3).

Now in Bressoud's paper [9], after providing his elegant, easy proof, he provides a generalization.

Bressoud's Theorem. *Given positive integers k and N, we have that*

$$\sum_{s_1,\ldots,s_k} \frac{q^{s_1^2+s_2^2+\cdots+s_k^2}}{(q)_{N-s_1}(q)_{s_1-s_3}\cdots(q)_{s_{k-1}-s_k}(q)_{2s_k}} \prod_{1}^{s_k}(1+xq^m)(1+x^{-1}q^{m-1}) \tag{1.6}$$
$$= (q)_{2N}^{-1} \sum_m x^m q^{((2k+1)m^2+m)/2} \begin{bmatrix} 2N \\ N-m \end{bmatrix},$$

where

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{(q)_N}{(q)_m (q)_{N-m}}. \tag{1.7}$$

Bressoud notes that the case $k = 2$ is the instance from which the Rogers-Ramanujan identities follow, and he points out that the case $x = -1$ provides a finite version of (1.5) when $i = k$. Indeed, when $i = k$, then (1.5) follows with $x = -1$ and $N \to \infty$; all that is necessary is to invoke Jacobi's triple product [2, p.22, Cor. 2.9]

If we explore (1.6) a little further, something quite surprising is in store. Namely, setting $x = -q^{k-i}$ and letting $N \to \infty$, we find (see Theorems 2 and 3 in Section 4)

$$\sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2+s_2^2+\cdots+s_{k-1}^2} B_{k-i}(q^{s_{k-1}})}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}} \tag{1.8}$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)\binom{n+1}{2}-in} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_n}{(q)_\infty},$$

where

$$B_n(z,q) := \sum_{j=0}^{n} \begin{bmatrix} n+j \\ 2j \end{bmatrix} (-1)^j q^{j(3j-1)/2-nj} (zq^{1-j})_j. \tag{1.9}$$

Now the series in (1.8) is identical with the series in (1.5) when $i = k$ and $i = k-1$, because $B_0(z) = 1$, and $B_1(z) = z$. However, while the right-hand sides of (1.5) and (1.8) are identical for $1 \leq i \leq k$, the left-hand sides differ for $i < k-1$.

In light of the fact that (1.8) is a special case of (1.5), we shall call the $B_n(z,q)$ Bressoud polynomials. We shall, in Section 2, introduce three further polynomial sequences closely related to $B_n(z,q)$ and derive some of their basic properties.

In Section 3, we put the Bressoud polynomials into the theory of Bailey pairs and Bailey chains. This will be applied in Sections 5 and 6.

In Section 4, we shall reconsider Bressoud's Theorem from the viewpoint of Bailey chains and derive related results.

Section 5 will look at applications of Bressoud polynomials to Rogers-Ramanujan type identities, and Section 6 will consider their relationship to the mock theta functions.

The final two sections provide a discussion of open questions connected with the Bressoud polynomials.

2 Recurrences. The Four Families of Bressoud Polynomials

In addition to the $B_n(z,q)$ defined by (1.9), we shall also consider

$$\overline{B}_n(z,q) = B_n(z, \tfrac{1}{q}) = \sum_{j=0}^{n} \begin{bmatrix} n+j \\ 2j \end{bmatrix} (-1)^j q^{j(j+1)/2-nj} (z)_j, \tag{2.1}$$

$$D_n(z,q) \sum_{j=0}^{n} \begin{bmatrix} n+j+1 \\ 2j+1 \end{bmatrix} (-1)^j q^{j(3j+1)/2-n(j+1)} (zq^{1-j})_j, \tag{2.2}$$

and

$$\overline{D}_n(z,q) = D_n(z,\tfrac{1}{q}) = \sum_{j=0}^{n} \begin{bmatrix} n+j+1 \\ 2j+1 \end{bmatrix} (-1)^j q^{j(j+1)/2-nj}(z)_j. \qquad (2.3)$$

The following theorem is fundamental to each application of the Bressoud polynomials:

Theorem 1. *If we define for $n < 0$*

$$B_n(z,q) = \overline{B}_n(z,q) = D_n(z,q) = \overline{D}_(z,q) = 0, \qquad (2.4)$$

and

$$B_0(z,q) = \overline{B}_0(z,q) = D_0(z,q) = \overline{D}_0(z,q) = 1, \qquad (2.5)$$

then for $n \geq 3$,

$$B_n(z,q) = zB_{n-1}(z,q) + zq^{1-n}B_{n-2}(z,q) - q^{1-n}B_{n-3}(z,q) \qquad (2.6)$$

$$\overline{B}_n(z,q) = z\overline{B}_{n-1}(z,q)?zq^{n-1}\overline{B}_{n-2}(z,q) - q^{n-1}\overline{B}_{n-3}(z,q), \qquad (2.7)$$

$$D_n(z,q) = zD_{n-1}(z,q) + zq^{-n}D_{n-2}(z,q) - q^{-n-1}D_{n-3}(z,q), \qquad (2.8)$$

$$\overline{D}_n(z,q) = z\overline{D}_{n-1}(z,q) + zq^{n}\overline{D}_{n-2}(z,q) - q^{n+1}\overline{D}_{n-3}(z,q) \qquad (2.9)$$

Proof. We shall prove (2.7) and (2.9). Recurrence (2.6) follows from (2.7) by (2.1), and (2.8) follows from (2.9) by (2.3).

First we note that

$$z(z)_j = q^{-j}(1-(1-zq^j))(z)_j = q^{-j}(z)_j - q^{-j}(z)_{j+1}. \qquad (2.10)$$

So if we compare coefficients of $(z)_j$ on each side of (2.7), we see that we need to prove

$$\begin{aligned} t(n,j) = {} & q^{-j}t(n-1,j) - q^{1-j}t(n-1,j-1) \\ & + q^{n-1-j}t(n-2,j) - q^{n-j}t(n-2,j-1) \\ & - q^{n-1}t(n-3,j), \end{aligned} \qquad (2.11)$$

where

$$t(n,j) = \begin{bmatrix} n+j \\ 2j \end{bmatrix} (-1)^j q^{j(j+1)/2-nj}. \qquad (2.12)$$

Now multiplying both sides of (2.11) by $(q)_{2j}(q)_{n-j}/(q)_{n+j-3}$ reduces the expression to an easily verified polynomial identity.

Similarly if we compare coefficients of $(z)_j$ on each side of (2.9), we see that we need to prove

$$u(n,j) = q^{-j}u(n-1,j) - q^{1-j}u(n-1,j-1) - q^{n-j}u(n-2,j) \qquad (2.13)$$
$$+ q^{n-j+1}u(n-2,j-1) + q^{n+1}u(n-3,j),$$

where

$$u(n,j) = \begin{bmatrix} n+j+1 \\ 2j+1 \end{bmatrix} (-1)^j q^{j(j+1)/2-nj}. \qquad (2.14)$$

Finally mulitplying both sides of (2.13) by $(q)_{2j+1}(q)_{n-j}/(q)_{n+j-2}$ reduces this expression to an easily verified polynomials identity. □

3 Bressoud Polynomials in Bailey Pairs

Let us recall fundamental aspects of the theory of Bailey chains and Bailey pairs [3]. We start with a pair of sequences of rational functions (α_n, β_n) defined to be a Bailey pair provided [3, p.278]

$$\alpha_n = \frac{(1-aq^{2n})(-1)^n q^{n(n-1)/2}(a)_n}{(1-a)(q)_n} \sum_{j=0}^{n} (q^{-n})_j (aq^n)_j q^j \beta_j, \qquad (3.1)$$

or equivalently

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \qquad (3.2)$$

Whenever you have a Bailey pair then the following holds for integers $K \geq 1$ [3, p.273, eq.(3.1)]

$$\frac{1}{(aq)_\infty} \sum_{n\geq 0} q^{Kn^2} a^{Kn} \alpha_n \qquad (3.3)$$
$$= \sum_{s_1 \geq s_2 \geq \cdots \geq s_K \geq 0} \frac{a^{s_1+\cdots+s_K} q^{s_1^2+\cdots+s_K^2} \beta_{s_K}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{K-1}-s_K}},$$

and in particular when $K=1$

$$\frac{1}{(aq)_\infty} \sum_{n=0}^{\infty} q^{n^2} a^n \alpha_n = \sum_{s=0}^{\infty} a^s q^{s^2} \beta_n. \qquad (3.4)$$

Now let us consider four Bailey pairs associated with the Bressoud polynomials. We begin with $B_n(z,q)$.

$$\begin{aligned} B_n(z,q) &= \sum_{j=0}^{n} \begin{bmatrix} n+j \\ 2j \end{bmatrix} (-1)^j q^{j^2+j(j-1)/2-nj}(zq^{1-j})_j \\ &= \sum_{j=0}^{n} (q^{-n})_j (q^{n+1})_j q^j \frac{q^{j(j+1)/2} z^j (z^{-1})_j}{(q)_{2j}} \end{aligned}$$

Hence with

$$\alpha 1_n = \frac{(1-q^{2n+1})(-1)^n q^{n(n-1)/2}}{(1-q)} B_n(z,q), \tag{3.5}$$

we see that

$$\left(\alpha 1_n, \frac{(-1)^n z^n q^{n(n-1)/2}(z^{-1})_n}{(q)_{2n}}\right) = \left(\alpha 1_n, \frac{q^{n^2-n}(zq^{1-n})_n}{(q)_{2n}}\right) \tag{3.6}$$

is a Bailey pair for $a = q$.

By exactly the same reasoning, with

$$\alpha 2_n = \frac{(1-q^{2n+1})}{(1-q)} (-1)^n q^{n(n-1)/2} \overline{B}_n(z,q), \tag{3.7}$$

we see that

$$\left(\alpha 2_n, \frac{(z)_n}{(q)_{2n}}\right) \tag{3.8}$$

is a Bailey pair for $a = q$.

Similarly,

$$\alpha 3_n = \frac{(1-q^{2n+2})(-1)^n q^{n(n-1)/2}}{(1-q)(1-q^2)} D_n(z,q), \tag{3.9}$$

we see that

$$\left(\alpha 3_n, \frac{(-1)^n q^{n(n+1)/2} z^n (z^{-1})_n}{(q)_{2n+1}}\right) = \left(\alpha 3_n, \frac{q^{n^2}(zq^{1-n})_n}{(q)_{2n+1}}\right) \tag{3.10}$$

is a Bailey pair for $a = q^2$.

Finally,

$$\alpha 4_n = \frac{(1-q^{2n+2})(-1)^n q^{n(n-1)/2}}{(1-q)(1-q^2)} \overline{D}_n(z,q), \tag{3.11}$$

we see that

$$\left(\alpha 4_n, \frac{(z)_n}{(q)_{2n+1}}\right) \tag{3.12}$$

is a Bailey pair for $a = q^2$.

4 Bressoud polynomials and the generalized Rogers-Ramanujan Series

The object of this section is first to show how Bressoud's theorem relates to the generalized Rogers-Ramanujan series [1] via Bailey chains, and second to provide a companion result in which the $D_n(z)$ appears.

Theorem 2. *For* $1 \le i \le k$,

$$\begin{aligned}
&\sum_{s_1 \ge \cdots \ge s_{k-1} \ge 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2+s_i+s_{i+1}+\cdots+s_{k-1}}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_k}} \\
&= \sum_{s_1 \ge \cdots s_{k-1} \ge 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2} B_{k-i}(q^{s_{k-1}})}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_k}} \\
&= \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}.
\end{aligned} \tag{4.1}$$

Remark. This result may also be derived from Bressoud's original proof of Bressoud's Theorem. Our goal here is to put the proof into the Bailey chain scenario.

Proof. The q-binomial theorem [1, p.36, eq.(3.3.6)] asserts that

$$(xq^{1-n})_{2n} = \sum_{j=0}^{2n} \begin{bmatrix} 2n \\ j \end{bmatrix} (-1)^j x^j q^{\binom{j+1}{2}-nj}, \tag{4.2}$$

and this result may be rewritten as

$$\frac{(x^{-1})_n(xq)_n}{(q)_{2n}} = \frac{1}{(q)_n^2} + \sum_{r=1}^{n} \frac{(-1)^r(q^{\binom{r+1}{2}}x^r + q^{\binom{r}{2}}x^{-r})}{(q)_{n-r}(q)_{n+r}}. \tag{4.3}$$

Thus (α_n, β_n) is a Bailey pair for $a = 1$, where

$$\beta_n = \frac{(x^{-1})_n (xq)_n}{(q)_{2n}}, \tag{4.4}$$

and

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0 \\ (-1)^n \left(q^{\binom{n+1}{2}} x^n + q^{\binom{n}{2}} x^{-n} \right) & \text{if } n > 0. \end{cases} \tag{4.5}$$

We now insert this pair into (3.3) with $a = 1$ and $K = k$.

$$\frac{1}{(q)_\infty} \left(1 + \sum_{n=1}^{\infty} q^{kn^2} (-1)^n \left(q^{\binom{-n}{2}} x^n + q^{\binom{n}{2}} x^{-n} \right) \right) \tag{4.6}$$
$$= \sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2 + \cdots s_{k-1}^2}}{(q)_{s_1 - s_2} \cdots (q)_{s_{k-2} - s_{k-1}} (q)_{s_{k-1}}} \sum_{s_k = 0}^{s_{k-1}} \frac{(q)_{s_{k-1}} (x^{-1})_{s_k} (xq)_{s_k} q^{s_k^2}}{(q)_{2s_k} (q)_{s_{k-1} - s_k}}.$$

Now set $x = q^{k-i}$. The left side of (4.6) becomes

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2 + n(n+1)/2 + (k-i)n} \tag{4.7}$$
$$= \frac{1}{(q)_\infty} \left(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1} \right)_\infty$$

(by Jacobi's Triple product [1, p.22, Cor. 2.9])

which is the third expression in (4.1), and thus equal to the first expression in (4.1) by (1.5).

On the right-hand side of (4.6), the inner sum on s_k becomes

$$\sum_{s_k=0}^{s_{k-1}} \frac{\left(q^{-(k-i)}\right)_{s_k} \left(q^{k-i+1}\right)_{s_k} q^{s_k}}{(q)_{2s_k}} \left(1 - q^{s_{k-1}}\right) \cdots \left(1 - q^{s_{k-1} - s_k + 1}\right)$$
$$= \sum_{S=0}^{k-i} \begin{bmatrix} S + k - i \\ 2S \end{bmatrix} (-1)^S q^{S(3S-1)/2 - (k-i)S} \left(q^{s_k + 1 - S}\right)_S$$
$$= B_{k-i}\left(q^{s_k}\right),$$

and this confirms that the resulting right side of (4.6) equals the expression in (4.7) and the theorem is proved. □

Theorem 3. *For $1 \leq i \leq k$,*

$$\sum_{s_1 \geq \cdots s_{k-1} \geq 0} \frac{q^{s_1^2+\cdots s_{k-1}^2+s_i+s_{i+1}+\cdots s_{k-1}}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_k}} \tag{4.8}$$

$$= \sum_{s_1 \geq \cdots s_{k-1} \geq 0} \frac{q^{s_1^2+\cdots s_{k-1}^2+s_1+s_2+\cdots s_{k-1}} D_{i-1}\left(q^{s_{k-1}}\right)}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_k}}$$

$$= \frac{\left(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1}\right)_\infty}{(q)_\infty}.$$

Proof. The q-binomial theorem [1, p.36, eq.(3.3.6)] asserts that

$$(xq^{1-n})_{2n+1} = \sum_{j=0}^{2n+1} \begin{bmatrix} 2n+1 \\ j \end{bmatrix} (-1)^j x^j q^{\binom{j+1}{2}-nj}, \tag{4.9}$$

and the result may be rewritten as

$$\frac{(x^{-1})_n (xq)_{n+1}}{(q^2;q)_{2n}} = \sum_{r=0}^{n} \frac{(-1)^r\left(x^{-r}q^{\binom{r}{2}} - x^{r+1}q^{\binom{-r-1}{2}}\right)}{(q)_{n-r}(q^2;q)_{n+r}}. \tag{4.10}$$

Thus $(\overline{\alpha}_n, \overline{\beta}_n)$ is a Bailey pair for $a = q$ where

$$\overline{\beta}_n = \frac{(x^{-1})_n (xq)_{n+1}}{(q^2;q)_{2n}}, \tag{4.11}$$

and

$$\overline{\alpha}_n = (-1)^n \left(x^{-n}q^{\binom{n}{2}} - x^{n+1}q^{\binom{-n-1}{2}}\right). \tag{4.12}$$

We now insert this pair into (3.3) with $a = q$ and $K = k$ (and we multiply both sides by $\frac{1}{(1-q)}$). □

$$\frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{kn^2+kn}(-1)^n \left(x^{-n}q^{\binom{n}{2}} - x^{n+1}q^{\binom{-n-1}{2}}\right) \tag{4.13}$$

$$= \sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}}$$

$$\times \sum_{s_k=0}^{s_{k-1}} \frac{(q)_{s_{k-1}} (x^{-1})_{s_k} (xq)_{s_k+1} q^{s_k^2}}{(q)_{2s_k+1} (q)_{s_{k-1}-s_k}}.$$

Now set $x = q^{i-1}$. The left side of (4.13) becomes

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+n(n-1)/2+(k-i+1)n} \tag{4.14}$$

$$= \frac{1}{(q)_\infty} \left(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1}\right)_\infty$$

(by Jacobi's Triple product [2, p.22, Cor 2.9]),

which is the third expression in (4.8), and thus equal to the first expression in (4.8) by (1.5).

On the right-hand side of (4.13), the inner sum on s_k becomes

$$\sum_{s_k=0}^{s_{k-1}} \frac{\left(q^{-(i-1)}\right)_{s_k} (q^i)_{s_k+1}\, q^{s_k^2}}{(q)_{2s_k+1} (q)_{s_{k-1}-s_k}}$$

$$= \sum_{S=0}^{i-1} \begin{bmatrix} S+i \\ 2S+1 \end{bmatrix} (-1)^S q^{S(3S+1)/2-iS} \left(q^{s_{k-1}+1-S}\right)_S$$

$$= D_{i-1}\left(q^{s_{k-1}}\right).$$

This confirms that the resulting right side of (4.13) equals the expression in (4.14), and the theorem is proved.

To close this section, we note that the first two expressions in Theorem 3 are identical at $i = 1$ because $D_0(x) = 1$.

5 Rogers-Ramanujan-Slater Identities

In this short section, we note that eight identities from [11] are direct corollaries of the results in Section 2.

Theorem 4.

$$B_n(0,q) = \begin{cases} 0 & \text{if } n = 3\nu + 1 \\ (-1)^\nu q^{-\nu(3\nu+1)/2} & \text{if } n = 3\nu \\ (-1)^\nu q^{-\nu(3\nu-1)/2} & \text{if } n = 3\nu - 1 \end{cases} \tag{5.1}$$

$$B_n\left(-\frac{1}{q},q\right)=\begin{cases}(-1)^\nu q^{-\nu^2} & \textit{if } n=2\nu\\ (-1)^\nu q^{-\nu^2} & \textit{if } n=2\nu-1\end{cases} \tag{5.2}$$

$$D_n(0,q)=\begin{cases}0 & \textit{if } n=3\nu-1\\ (-1)^\nu q^{-(\nu+1)(3\nu+2)/2+1} & \textit{if } n=3\nu\\ (-1)^\nu q^{-(\nu+1)(3\nu+4)/2+1} & \textit{if } n=3\nu+1\end{cases} \tag{5.3}$$

$$D_n\left(-\frac{1}{q},q\right)=\begin{cases}0 & \textit{if } n=2\nu+1\\ (-1)^\nu q^{-(\nu+1)^2+1} & \textit{if } n=2\nu\end{cases} \tag{5.4}$$

$$\overline{B}_n(0,q)=B_n\left(0,\tfrac{1}{q}\right)=\begin{cases}0 & \textit{if } n=3\nu+1\\ (-1)^\nu q^{\nu(3\nu+1)/2} & \textit{if } n=3\nu\\ (-1)^\nu q^{\nu(3\nu-1)/2} & \textit{if } n=3\nu-1\end{cases} \tag{5.5}$$

$$\overline{B}_n(-q,q)=B_n(-q,\tfrac{1}{q})=\begin{cases}(-1)^\nu q^{\nu^2} & \textit{if } n=2\nu\\ (-1)^\nu q^{\nu^2} & \textit{if } n=2\nu-1\end{cases} \tag{5.6}$$

$$\overline{D}_n(0,q)=D_n(0,\tfrac{1}{q})=\begin{cases}0 & \textit{if } n=3\nu-1\\ (-1)^\nu q^{(\nu+1)(3\nu+2)/2-1} & \textit{if } n=3\nu\\ (-1)^\nu q^{(\nu+1)(3\nu+4)/2-1} & \textit{if } n=3\nu+1\end{cases} \tag{5.7}$$

$$\overline{D}_n(-q,q)=D(-q,\tfrac{1}{q})=\begin{cases}0 & \textit{if } n=2\nu+1\\ (-1)^\nu q^{(\nu+1)^2-1} & \textit{if } n=2\nu\end{cases} \tag{5.8}$$

$$\overline{B}_n(q,q^2)=B_n(q,\tfrac{1}{q^2})=q^{n(n+1)/2} \tag{5.9}$$

Proof. We only need prove (5.1) - (5.4) in light of the fact that (5.5)- (5.8) follow by replacing q by $\frac{1}{q}$.

As for (5.1) - (5.4) these follow directly by mathematical induction using the recurrences in Theorem 1. For example, by (2.6)

$$B_n(0,q)=-q^{1-n}B_{n-3}(0,q), \tag{5.10}$$

and one verifies directly that the right-hand side of (5.1) satisfies this same recurrence.

Equations (5.2)-(5.4) follow in the same manner. □

Seven of these eight identities imply identities listed in Slater's compendium [11]. In each case, one combines one of the above with the corresponding Bailey pair from section 3 and inserts the result in (3.4). Our (5.1) implies equation (83) of [11].

Our (5.2) implies equation (46) of [11].

Our (5.3) implies equation (86) of [11].

Our (5.4) implies equation (44) of [11].

Our (5.5) implies equation (99) of [11].

Our (5.7) implies equation (96) of [11].

Our (5.8) implies equation (59) of [11].

The first entry in (5.9) implies equation (32) of [11], and the second entry in (5.9) implies equation (19) of [11].

The missing case (which turns out to be a linear combination of two of Slater's identities) can be written out in full to illustrate the method.

Corollary 5.

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q;q)_n}{(q)_{2n}} = \frac{(q^6,q^8,q^{14};q^{14})_\infty}{(q)_\infty} - \frac{q\,(q^2,q^{12},q^{14};q^{14})_\infty}{(q)_\infty}. \tag{5.11}$$

Proof. Substituting (5.6) into (3.7) we see that (α_n,β_n) is a Bailey pair for $a=q$, with

$$\alpha_n = \frac{(1-q^{2n+1})}{(1-q)}(-1)^{n+\lfloor\frac{n+1}{2}\rfloor}q^{n(n-1)+\lfloor\frac{n+1}{2}\rfloor^2}$$

$$\beta_n = \frac{(-q)_n}{(q)_{2n}}.$$

Inserting this pair into (3.4) with $a=q$ we find

$$\frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{7n^2-n} - \sum_{n=-\infty}^{\infty}(-1)^n q^{7n^2-5n+1}\right) = \sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q)_n}{(q)_{2n}}, \tag{5.12}$$

and if we apply Jacobi's triple product to the two series on the left we obtain the desired result. □

Turning to Slater's list we find that the two infinite products in Corollary 5 correspond to equations (51) and (59) of [11]. Not surprisingly, Slater's identities imply that the left side of (5.11) is

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^{n^2}(-q)_n}{(q)_{2n}} - q\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q)_n}{(q)_{2n+1}} \\
&\quad = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q)_n}{(q)_{2n}} - \sum_{n=1}^{\infty}\frac{q^{n^2}(-q)_{n-1}}{(q)_{2n-1}} \\
&\quad = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q)_{n-1}}{(q)_{2n}}\left((1+q^n)-(1-q^{2n})\right) \\
&\quad = 1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}(-q)_n}{(q)_{2n}}.
\end{aligned}$$

6 Mock Theta Functions

It should be noted that the Bailey pair (3.8) was suggested in [5] and studied extensively in [6]. Thus it is natural that it will arise here among the five instances we shall exhibit that yield the Hecke-type expansions involving indefinite quadratic forms. The cases we have chosen illustrate how often Bressoud polynomials arise in this context. We emphasize that this is only a sample.

Theorem 6.

$$B_n(-1,q) = (-1)^n \sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} (-1)^j q^{-j^2} \tag{6.1}$$

$$\overline{B}_n(-1,q) = B_n(-1,\tfrac{1}{q}) = (-1)^n \sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} (-1)^j q^{j^2} \tag{6.2}$$

$$\overline{D}_n(q,q) = \begin{cases} q^{\nu^2+2\nu}\sum_{j=-\nu}^{\nu} q^{-j^2} & \text{if } n=2\nu \\ 2q^{\nu(\nu+1)-1}\sum_{j=0}^{\nu-1} q^{-j^2-j} & \text{if } n=2\nu-1 \end{cases} \tag{6.3}$$

$$D_n(\tfrac{1}{q},q) = \overline{D}_n(\tfrac{1}{q},\tfrac{1}{q}) = \begin{cases} q^{-\nu^2-2\nu}\sum_{j=-\nu}^{\nu} q^{j^2} & \text{if } n=2\nu \\ 2q^{-\nu^2-\nu-1}\sum_{j=0}^{\nu-1} q^{j^2+j} & \text{if } n=2\nu-1 \end{cases} \tag{6.4}$$

$$\overline{D}_n(q,q^2) = q^n \sum_{j=0}^{n} q^{j(j+1)/2} \tag{6.5}$$

$$D_n(\tfrac{1}{q},q^2) = \overline{D}_n(\tfrac{1}{q},\tfrac{1}{q^2}) = q^{-n} \sum_{j=0}^{n} q^{-j(j+1)/2} \tag{6.6}$$

Proof. As in Section 5, these results are easily deduced from the recurrences in Theorem 1. As an example, we treat (6.5). It is immediate by inspection that (6.5) is valid for $n = 0, 1, 2$. Now let Δ_n denote the right-hand side of (6.5). Clearly

$$\begin{aligned}\Delta_n - q\Delta_{n-1} &= q^n \sum_{j=0}^{n} q^{j(j+1)/2} - q^n \sum_{j=0}^{n-1} q^{j(j+1)/2} \\ &= q^{n(n+3)/2}\end{aligned}$$

while

$$\begin{aligned}&q^{1+2n}\Delta_{n-2} - q^{2n+2}\Delta_{n-3} \\ &\quad = q^{1+2n}\left(\Delta_{n-2} - q\Delta_{n-3}\right) \\ &\quad = q^{1+2n+(n-2)(n+1)/2} \\ &\quad = q^{\frac{n^2}{2}+\frac{3n}{2}}\end{aligned}$$

so

$$\Delta_n - q\Delta_{n-1} = q^{1+2n}\Delta_{n-2} - q^{2n+2}\Delta_{n-3},$$

and the relevant recurrence is established. □

We may immediately use this results in the Bailey pairs in section 3, and thus obtain the following identities of mock theta or false theta type.

Theorem 7.

$$\sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}(-1;q)_n}{(q)_{2n}} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{n(3n+1)/2}\left(1-q^{2n+1}\right) \sum_{|j|\le\lfloor\frac{n}{2}\rfloor} (-1)^j q^{-j^2}, \tag{6.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-1;q)_n}{(q)_{2n}} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{n(3n+1)/2}(-1)^n\left(1-q^{2n+1}\right) \sum_{|j|\le\lfloor\frac{n}{2}\rfloor} (-1)^j q^{j^2}, \tag{6.8}$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^{n+1};q)_{n+1}} \\
&\qquad = \frac{1}{(q)_\infty} \left\{ \sum_{n=0}^{\infty} q^{7n^2+4n} \left(1-q^{4n+2}\right) \sum_{|j|\le n} q^{-j^2} \right. \\
&\qquad\qquad \left. - 2\sum_{n=0}^{\infty} q^{7n^2+12n+4} \left(1-q^{4n+4}\right) \sum_{j=0}^{n} j^{-j^2-j} \right\}
\end{aligned} \tag{6.9}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{(q^{n+1};q)_{n+1}} \\
&\qquad = \frac{1}{(q)_\infty} \left\{ \sum_{n=0}^{\infty} q^{5n^2+3n} \left(1-q^{4n+2}\right) \sum_{|j|\le n} q^{j^2} \right. \\
&\qquad\qquad \left. - 2\sum_{n=0}^{\infty} q^{5n^2+8n+1} \left(1-q^{4n+4}\right) \sum_{j=0}^{\infty} q^{j^2+j} \right\} \\
&\qquad = \sum_{n=0}^{\infty} (-1)^n q^{n(5n+7)/2} \left(1+q^{3n+1}+q^{5n+2}+q^{8n+4}\right)
\end{aligned} \tag{6.10}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{2n^2+4n}\,(q;q^2)_n}{(q^2;q^2)_{2n+1}} \\
&\qquad = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{3n^2+4n} \left(1-q^{4n+4}\right) \sum_{j=0}^{n} q^{j(j+1)/2}
\end{aligned} \tag{6.11}
$$

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+4n}\,(q;q^2)_n}{(q^2;q^2)_{2n+1}} \tag{6.12}
$$

$$
= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{3n^2+4n} \left(1-q^{4n+4}\right) \sum_{j=0}^{n} q^{-j(j+1)/2}. \tag{6.13}
$$

Proof. In each of the six identities we take the corresponding evaluation of the Bressoud polynomial in Theorem 6 and then apply it to the corresponding Bailey pair and the Bailey Lemma from section 3.

We should remark that (6.9) is a small variation of the seventh order mock theta function identity in [5, p.132, eq. (7.23)]. Also the last line of (6.10) is due to L. J. Rogers [10, Sec.9]. □

7 Bressoud's Theorem and the Generalized Rogers-Ramanujan Series

The object of this section is to reconsider the left-sides of (1.5) and (1.8).

Theorem 8. *For* $1 \leq i \leq k$,

$$\sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2+s_i+s_{i+1}+\cdots+s_{k-1}}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_k}} \tag{7.1}$$
$$= \sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2} B_{k-i}\left(q^{s_{k-1}}\right)}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}}$$

Proof. The above assertion follows immediately from the fact that the right sides of (1.5) and (1.8) are identical. □

The object here is to consider whether (7.1) can be proved directly without using either (1.5) or (1.8).

The cases $i = k$ and $i = k-1$ are, in fact tautologies because $B_0(z, q) = 1$ and $B_1(z, q) = z$.

We go through the $i = k-2$ case step by step because it illustrates the general case.

For simplicity, we write

$$T(s_1, \ldots, s_{k-3}) = \frac{q^{s_1^2+\cdots+s_{k-3}^2}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_{k-3}}}.$$

Then

$$\sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2+s_{k-2}+s_{k-1}}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}}$$

$$
\begin{aligned}
&= \sum_{\underline{s}} T(s_1, \ldots, s_{k-3}) \frac{q^{s_{k-2}^2+s_{k-1}^2+s_{k-2}+s_{k-1}}}{(q)_{s_{k-3}-s_{k-2}}(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}} \\
&= \sum_{\underline{s}} T(s_1, \ldots, s_{k-3}) \frac{q^{s_{k-2}^2+s_{k-1}^2+2s_{k-1}}\left(1-\left(1-q^{s_{k-2}-s_{k-1}}\right)\right)}{(q)_{s_{k-3}-s_{k-2}}(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}} \\
&= \sum_{\underline{s}} T(s_1, \ldots, s_{k-3}) \frac{q^{s_{k-2}^2+s_{k-1}^2+2s_{k-1}}}{(q)_{s_{k-3}-s_{k-2}}(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}} \\
&\quad - \sum_{\underline{s}} T(s_1, \ldots, s_{k-3}) \frac{q^{s_{k-2}^2+(s_{k-1}-1)^2+2(s_{k-1}-1)}\left(1-q^{s_{k-1}}\right)}{(q)_{s_{k-3}-s_{k-2}}(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{where } s_{k-1} \to s_{k-1}-1 \text{ in second sum}) \\
&= \sum_{\underline{s}} \frac{T(s_1, \ldots, s_{k-3})\, q^{s_{k-2}^2+s_{k-1}^2}}{(q)_{s_{k-3}-s_{k-2}}(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}} \left(q^{2s_{k-1}} + q^{s_{k-1}-1} - q^{-1}\right) \\
&= \sum_{\underline{s}} \frac{q^{s_1^2+\cdots+s_{k-1}^2} B_2\left(q^{s_{k-1}}\right)}{(q)_{s_1-s_2} \cdots (q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}},
\end{aligned}
$$

because

$$
B_2(z, q) = z^2 + \frac{z}{q} - \frac{1}{q}.
$$

The simplification applied in the $i = k-2$ case can be made applicable to the general problem of reducing the left side of (7.1) to the right-side.

Lemma 9. *Let*

$$
T_{R,S}(u, t, n) = \sum_{i_1, i_2, i_3 \geq 0} \frac{q^{i_1^2+i_2^2+i_3^2+ui_1+ti_2+ni_3}}{(q)_{R-i_1}(q)_{i_1-i_2}(q)_{i_2-i_3}(q)_{i_3-S}}. \tag{7.2}
$$

Then

$$
T_{R,S}(u,t,n) = T_{R,S}(u-1,t+1,n) - q^{-t} T_{R,S}(u-1,t-1,n) - q^{-t} T_{R,S}(u-1,t,n-1). \tag{7.3}
$$

Proof. We note that

$$
1 = q^{-i_1+i_2}\left(1 - \left(1 - q^{i_1-i_2}\right)\right).
$$

Hence

$$
\begin{aligned}
T_{R,S}(u,t,n) &= \sum_{i_1,i_2,i_3\geq 0} \frac{q^{i_1^2+i_2^2+i_3^2+ui_1+ti_2+ni_3-i_1-i_2}\left(1-(1-q^{i_1-i_2})\right)}{(q)_{R-i_1}(q)_{i_1-i_2}(q)_{i_2-i_3}(q)_{i_3-S}} \\
&= T_{R,S}(u-1,t+1,n) - \sum_{i_1,i_2,i_3\geq 0} \frac{q^{i_1^2+i_2^2+i_3^2+(u-1)i_1+(t+1)i_2+ni_3}}{(q)_{R-i_1}(q)_{i_1-i_2-1}(q)_{i_2-i_3}(q)_{i_3-S}} \\
&= T_{R,S}(u-1,t+1,n) \\
&\quad - \sum_{i_1,i_2,i_3\geq 0} \frac{q^{i_1^2+(i_2-1)^2+i_3^2+(u-1)i_1+(t+1)(i_2-1)+ni_3}\left(1-(1-q^{i_2-i_3})\right)}{(q)_{R-i_1}(q)_{i_1-i_2}(q)_{i_2-i_3}(q)_{i_3-S}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{where } i_2\to i_2-1) \\
&= T_{R,S}(u-1,t+1,n) - q^{-t}T_{R,S}(u-1,t-1,n) - q^{-t}T_{R,S}(u-1,t,n-1).
\end{aligned}
$$

□

I believe that successive applications of Lemma 9 to the left-hand side of (7.1) will yield the right-hand side of (7.1) just as we did in the case $i=k-2$ with effectively one application of Lemma 9.

In particular, let us define

$$
\begin{aligned}
&\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+s_2^2+\cdots+s_k^2+s_{j+1}+s_{j+2}+\cdots+s_{k-1}+hs_k}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} \\
&\qquad = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+s_2^2+\cdots+s_k^2}f_k(k-j,h;q^{s_k})}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}
\end{aligned}
$$

Then successive application of Lemma 9 reveal that

$$
\begin{aligned}
f_k(k-1,h;z) &= z^h, \\
f_k(k-2,h;z) &= z^{h+1} - \frac{z^{h-1}}{qh} + \frac{1}{q^h},
\end{aligned}
$$

and for $j>2$

$$
\begin{aligned}
f_k(k-j,h;z) &= f_k(k-j+1,h+1;z) \\
&\quad + (q^{-1}-q^{-h})f_k(k-j+1,h-1;z) \\
&\quad - q^{-1}f_k(k-j+2,h;z) + q^{-h}f_k(k-j+1,h;z).
\end{aligned}
$$

Conjecture.

$$f_k(k-j,1;z) = B_j(z,q).$$

We have already shown that this is true for $j = 0, 1, 2$. The assertion has been verified for $0 \le j \le 10$.

8 Conclusion

There are many questions yet to be examined in connection with the topics explored in this paper. The fact that $B_n(z,q)$ (actually a normalized version) played a central role in the recent study of the seventh order mock theta functions [3] suggests that the other three Bressoud polynomials might well reveal interesting results from a similar study.

The work in Section 7 is effectively dual to the work of Berkovich and Paule [7], [8]. In their work they were able to represent

$$\sum_{s_1 \ge \cdots \ge s_{k-1} \ge 0} \frac{q^{s_1^2+s_2^2+\cdots+s_{k-1}^2+M_1 s_1+M_2 s_2+\cdots+M_{k-1}s_{k-1}}}{(q)_{s_1-s_2}(q)_{s_2-s_3}\cdots(q)_{s_{k-2}-s_{k-1}}(q)_{s_{k-1}}}$$

as linear combinations of the series given in (1.5). In section 8, we examine when various linear combinations are identical. Presumably identical combinations could be identified by means of Lemma 9.

Finally the fact that the Bressoud polynomials have such diverse applications as those considered here suggests that they merit further study in their own right.

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