3. Stable structure and the Whitehead determinant

Let A be an arbitrary associative ring with 1. For every integer $n \geq 1$ and any (two-sided) ideal B of A, we have the canonical homomorphism $\mathrm{GL}_n A \to \mathrm{GL}_n(A/B)$. Its kernel, the principal congruence subgroup, is denoted by $\mathrm{GL}_n B$. The inverse image of the center of $\mathrm{GL}_n(A/B)$ is denoted by $\mathrm{G}_n(A,B)$.

Both $\mathrm{GL}_n B$ and $\mathrm{G}_n(A,B)$ are normal subgroups of $\mathrm{GL}_n A$. The group $\mathrm{E}_n A$ need not be normal in general.

We denote by $\mathrm{E}_n(A,B)$ the normal subgroup of $\mathrm{E}_n A$ generated by $\mathrm{E}_n B$.

Theorem 3.1. When $n \geq 3$, $\mathrm{E}_n(A,B) = [\mathrm{E}_n A, \mathrm{E}_n B]$ and this group is generated by elements of the form $a^{j,i} b^{i,j} (-a)^{j,i}$ with $a \in A$ and $b \in B$.

Proof. Let H be the subgroup of $\mathrm{E}_n A$ generated by all $a^{j,i} b^{i,j} (-a)^{j,i}$ with $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$. Note that $\mathrm{E}_n B \subset H$. We want to prove that H is normal, i.e. it is normalized by all elementary matrices. We will use the relations (1.7) and (1.11) as well as the following relations between elementary matrices :

(3.2) $[x^{i,j}, y^{k,l}] = 1$ when $i \neq j \neq k \neq l \neq i$.

Let $a, c \in A$, $b \in B$, $1 \leq i \neq j \leq n$, and $1 \leq k \neq l \leq n$. We have to prove that $\alpha = c^{k,l} a^{j,i} b^{i,j} (-a)^{j,i} (-c)^{k,l} \in H$.

Case 1: The sets i,j and k,l are disjoint. Then by (3.2) we have $\alpha = a^{j,i} b^{i,j} (-a)^{j,i} \in H$.

Case 2: $(k,l) = (j,i)$. Then $\alpha = (a+c)^{j,i} b^{i,j} (-a-c)^{j,i} \in H$.

Case 3: $k = j$. Then $c^{k,l} = c^{j,l}$ commute with $a^{j,i}$, so α

$= a^{j,i} (c^{j,l} b^{i,j} (-c)^{j,l}) (-a)^{j,i}$

$= a^{j,i} (b^{i,j} [(-b)^{i,j}, c^{j,l}]) a^{j,i}$

$= a^{j,i} (b^{i,j} (-a)^{j,i}) (a^{j,i} (-bc)^{i,l} (-a)^{j,i}) \in H.$

Case 4: $(k,l) = (i,j)$. We find a number m distinct from i,j in the interval $1 \leq m \leq n$. Then $b^{i,j} = [b^{i,m}, 1^{m,j}]$, hence α

$= c^{i,j} [a^{j,i} b^{i,m} (-a)^{j,i}, a^{j,i} 1^{mj} (-a)^{j,i}] (-c)^{j,i}$

$= c^{i,j} [(ab)^{j,m} b^{i,m}, 1^{m,j} (-a)^{m,i}] (-c)^{i,j}$

$= [(cab + b)^{i,m} (-ab)^{i,m}, 1^{m,j} (-a)^{m,i}] (-c)^{m,i}.$

So it remains to notice that

$[(cab+b)^{i,m}, (1+ac)^{m,j} (-a)^{m.i}] \in (-a)^{m,i} \mathrm{E}_n B a^{m,i} \subset H$

and

$[(-ab)^{j,m}, (1+ac)^{m,j} (-a)^{m,i}] \in (1+ac)^{m,j} \mathrm{E}_n B (-1-ac)^{m,j} \subset H$

(we used that $[(1+ac)^{m,j}, (-a)^{m,i}] = 1_n$). QED.

Corollary 3.3. When $n \geq 3$,

$E_n(A, BB) \subset E_n B$

where BB is the ideal of A consisting of all sums of $b_1 b_2$ with $b_1, b_2 \in B$.

Proof. By Theorem 3.2, it suffices to show that the generators $a^{j,i} (b_1 b_2)^{i,j} (-a)^{j,i}$ of the group $E_n(A, BB)$ belong to $E_n B$. Here $a \in A$ and $b_1, b_2 \in B$.

Take $k \neq i, j$ in the interval $1 \leq k \leq n$. Then $(b_1 b_2)^{i,j} = [(b_1)^{i,k}, (b_2)^{k,j}]$, hence

$a^{j,i} (b_1 b_2)^{i,j} (-a)^{j,i}$

$= a^{j,i} [(b_1)^{i,k}, (b_2)^{k,j}] (-a)^{j,i}$

$= [a^{j,i} (b_1)^{i,k} (-a)^{j,i}, a^{j,i} (b_2)^{k,j} (-a)^{j,i}]$

$\in [\mathrm{E}_n B, \mathrm{E}_n B] \subset \mathrm{E}_n B$. QED.

Lemma 3.4. When $n \geq 3$, the subgroup $\begin{pmatrix} \mathrm{GL}_{n-1}A & 0 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{GL}_n A$ normalizes $\mathrm{E}_n(A,B)$.

Proof. Let $b \in B, 1 \leq i \neq j \leq n, \alpha = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_n A$ with $\beta \in \mathrm{GL}_{n-1}A$. We have to prove that $\alpha b^{i,j} \alpha^{-1} \in \mathrm{E}_n(A,B)$.

When $i = n$ or $j = n$, the matrix $b^{i,j}$ has the form $\begin{pmatrix} 1_{n-1} & 0 \\ * & 1 \end{pmatrix}$ or $\begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix}$ and the matrix $\alpha b^{i,j} \alpha^{-1}$ has the same form hence $\alpha b^{i,j} \alpha^{-1} \in \mathrm{E}_n B \subset \mathrm{E}_n(A,B)$.

When $i, j \leq n$,

$\alpha b^{i,j} \alpha^{-1}$

$= \alpha [b^{i,n}, 1^{n,j}] \alpha^{-1}$

$= [\alpha b^{i,n} \alpha^{-1}, \alpha 1^{n,j} \alpha^{-1}] \alpha^{-1}$

$\in [\mathrm{E}_n B, \mathrm{E}_n A] \subset \mathrm{E}_n(A,B)$. QED.

Now we generalize the Whitehead Lemma (1.5).

Lemma 3.5. Let $m, n \geq 1$ be integers, y an $m \times n$ matrix over a ring A, x an $n \times m$ matrix over an ideal B of A. Assume that $1_m + xy \in GL_m B$. Then $1_n + yx \in \mathrm{GL}_n B$ and $\begin{pmatrix} 1_m + xy & 0 \\ 0 & (1_n + yx)^{-1} \end{pmatrix}, \begin{pmatrix} 1_m + xy & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} (1_n + yx)^{-1} & 0 \\ 0 & 1_m \end{pmatrix} \in E_{m+n}(A,B)$.

Proof. We start with the matrix $\begin{pmatrix} 1_m + xy & 0 \\ 0 & 1_n \end{pmatrix}$ and perform three (block) row addition operations: $\begin{pmatrix} 1 + xy & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 + xy & 0 \\ y & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -x \\ y & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -x \\ 0 & 1 + yx \end{pmatrix}$.

The last matrix can be reduced to $\begin{pmatrix} 1 & 0 \\ 0 & 1 + yx \end{pmatrix}$ by a column addition operation which make it clear that $1_n + yx \in GL_n B$. But with this fact established, we use a row addition operation instead. Thus, we obtain

(3.6) $\begin{pmatrix} 1 + xy & 0 \\ 0 & (1 + yx)^{-1} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -y(1 + yx)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & (1 + yx)^{-1}x \\ 0 & 1 \end{pmatrix} \in E_{m+n}(A,B)$

because a block elementary matrix is a product of mn elementary matrices.

When $m \geq n$, we can replace x, y in (3.4) by $1_m, yx$ respectively and obtain that

$\beta = \begin{pmatrix} 1 & 1_n + yx & 0 \\ 0 & 1 & \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & (1_n + yx)^{-1} \end{pmatrix} \in \mathrm{E}_{m+n}(A,B)$,

hence

$\begin{pmatrix} 1_m + xy & 0 \\ 0 & 1_m \end{pmatrix} \begin{pmatrix} (1_n + yx)^{-1} & 0 \\ 0 & 1_m \end{pmatrix}$

$= \begin{pmatrix} 1 + xy & 0 \\ 0 & (1 + yx)^{-1} \end{pmatrix} \beta^{-1} \in \mathrm{E}_{m+n}(A,B)$. QED.

Corollary 3.7. For any $\alpha \in \mathrm{GL}_n A$ and $\beta \in \mathrm{GL}_n B$, we have $\begin{pmatrix} [\alpha, \beta] & 0 \\ 0 & 1_n \end{pmatrix} \in \mathrm{E}_{2n}(A,B)$.

Proof. Use Lemma 3.5. We let $x = \alpha(\beta - 1_n) \in \mathrm{M}_n B$, $y = \alpha^{-1} \in \mathrm{M}_n A$. Then $1 + xy = \alpha\beta\alpha^{-1}$ and $1 + yx = \beta$. QED.

Consider now the group GLA of infinite matrices α such that $\alpha = \begin{pmatrix} \beta & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$ for some $\beta \in \mathrm{GL}_n A$ with some $n \geq 1$. Abusing notations, we identify each $\mathrm{GL}_n A$ with the corresponding subgroup of GLA. In formal terms, GLA is an inductive limit of the groups $\mathrm{GL}_n A$ with respect to homomorphisms $\mathrm{GL}_m A \to \begin{pmatrix} \mathrm{GL}_m A & 0 \\ 0 & 1_{n-m} \end{pmatrix} \subset \mathrm{GL}_n A$ for $n \geq m$.

In the group GLA, we have a subgroup EA, the subgroup generated by all elementary matrices. Clearly, $\mathrm{E}A = \cup \mathrm{E}_n A$. More generally, for any ideal B of A we have $\mathrm{E}B = \cup \mathrm{E}_n B \subset \mathrm{E}A \subset \mathrm{GL}A$, $\mathrm{GL}B = \cup \mathrm{GL}_n B \supset \mathrm{E}B$, and $\mathrm{E}(A,B) = \cup \mathrm{E}_n(A,B) \subset \mathrm{GL}B \cap \mathrm{E}A$.

We do not define $\mathrm{G}(A,B)$ because $\mathrm{G}_n(A,B)$ is not a part of $\mathrm{G}_{n+1}(A,B)$ in general.

Theorem 3.8 (Whitehead, Bass [B1]). For every associative ring A with 1 and every ideal B of A,

$$\mathrm{E}(A,B) = [\mathrm{E}A, \mathrm{E}B] = [\mathrm{GL}A, \mathrm{GL}B].$$

In particular, when $B = A$,

$$\mathrm{E}A = [\mathrm{E}A, \mathrm{E}A] = [\mathrm{GL}A, \mathrm{GL}A].$$

Proof. The first equality follows from the first conclusion of Theorem 3.1. The second equality follows from Corollary 3.5. QED.

For any associative ring A with unity, we set $\mathrm{K}_1(A,B) = \mathrm{GL}B/\mathrm{E}(A,B)$ and $\mathrm{K}_1 A = \mathrm{K}_1(A,A) = \mathrm{GL}A/\mathrm{E}A$. The *Whitehead determinant* wh: $\mathrm{GL}A \to \mathrm{K}_1 A$ is a homomorphism onto a commutative group which is trivial on elementary matrices and such that

$\mathrm{wh}\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{wh}(\beta)$ for $\beta \in \mathrm{GL}_{n-1} A$.

By Section 1, wh = det when A is a field.

By Section 2, wh is the Dieudonné determinant when A is a division ring.

By Exercise 15 in Section 2, $\mathrm{K}_1 A = \mathrm{GL}_1 A/[\mathrm{GL}_1 A, \mathrm{GL}_1 A]$ when A is local.

For a commutative ring A with 1, we have $\mathrm{E}_n A \subset \mathrm{SL}_n A$, $\mathrm{EA} \subset \mathrm{SLA} = \cup \mathrm{SL}_n$ and $\mathrm{K}_1 A = \mathrm{SK}_1 A \times \mathrm{GL}_1 A$ where $\mathrm{SK}_1 A = \mathrm{SL}A/\mathrm{E}A$. In general, the homomorphism

$\mathrm{GL}_1 A/[\mathrm{GL}_1 A, \mathrm{GL}_1 A] \to \mathrm{K}_1 A$

is not surjective or injective.

For any A with 1 and any k, the group $\mathrm{GL}(\mathrm{M}_k A)$ is isomorphic to GLA, so $\mathrm{K}_1(\mathrm{M}_k A)$ is isomorphic to $\mathrm{K}_1 A$.

Theorem 3.9 (Bass [B1]). Let H be a subgroup of GLA. The following three conditions are equivalent:

(a) $\mathrm{E}(A,B) \subset H \subset \mathrm{GL}B$ for an ideal B of A,

(b) H is normal in GLA,

(c) H is normalized by EA.

Proof. The implication (a)$\Rightarrow$ (b) follows from Theorem 3.8 which gives

$[H, \mathrm{GL}A] \subset [\mathrm{GL}B, \mathrm{GL}A] = \mathrm{E}(A,B) \subset H$.

Since $\mathrm{E}A \subset \mathrm{GL}A$, (b) $\Rightarrow$ (c).

Let us now assume (c) and prove (a). Set $B = \{b \in A : b^{1,2} \in H\}$. By (1.7) and (1.11), B is an ideal of A and $\mathrm{E}(A, B) \subset H$. We have to prove that $H \subset \mathrm{GL}B$.

Otherwise there is $\alpha_1 \in H \setminus \mathrm{GL}B$. We have $\alpha_1 \in \mathrm{GL}_1 A$ for some n. Replacing n by $n+2$, if necessary, we have $n \geq 3$, the first entry in the last column of α_1 is 0, and the image of α_1 modulo B is not central.

The image H' of H in $\mathrm{GL}_n(A/B)$ is normalized by $\mathrm{E}_n(A/B)$. By Proposition 1.10, $\mathrm{E}_n(B_1/B) \subset H'$ for a nonzero ideal B_1/B of A/B where B_1 is an ideal of A containing B.

This means that there is a matrix $\alpha_2 \in H$ of the form $\alpha_2 = (b_1)^{3,2}\beta$ with $b_1 \in A \setminus B$ and $\beta \in \mathrm{GL}_n B$. Then $\alpha_3 := 1^{1,3}\alpha_1(-1)^{1,3}$
$= (1^{1,3} b_1^{3,2}(-1)^{1,3})(1^{1,3}\beta(-1)^{1,3})$
$= b_1^{1,2} b_1^{3,2}\beta[\beta^{-1}, 1^{1,3}]$
$= b_1^{1,2}\alpha_2[\beta^{-1}, 1^{1,3}] \in H$.

Since $\alpha_2 \in H$ and $[\beta^{-1}, 1^{1,3}] \in [\mathrm{GL}_n B, \mathrm{E}A] = \mathrm{E}(A, B) \subset H$, we conclude that $b_1^{1,2} \in H$, hence $b_1 \in B$ which contradicts our choice of $b_1 \in A \setminus B$. QED.

It is clear that the ideal B in Theorem 3.9 (a) (the lower level of H) is determined by H, namely, $B = \{b \in A : b^{1,2} \in H\}$.

Corollary 3.10. The group $\mathrm{E}A$ is simple if and only if the ring A is simple.

Remark 3.11. The Weyl algebras of differential operators with polynomial coefficients are simple rings, but not division rings.

Remark 3.12. Our notation $\mathrm{GL}_1 B$ for the multiplicative group of associative ring B becomes problematic when B has its own identity, different from $1 \in A$. The set $\mathrm{GL}_1 B - 1$ can be defined intrinsically as $\{x \in B : \exists y \in B : xy = yx, x + y + yx = 0\}$. So up to isomorphism the group $\mathrm{GL}_1 B$ depends only on the ring B.

For example $\mathrm{GL}B$ is the multiplicative group of ring $\mathrm{M}B = \cup \mathrm{M}_n B$ of infinite matrices with finitely many nonzero entries in each matrix.

Remark. In number theory, to solve an equation over $\mathbf{Z}$ we often pass from $\mathbf{Z}$ to other rings, e.g., the rational numbers, the real numbers, the complex numbers, the p-adic numbers, the p-adic integers, the algebraic integers, the adeles.

In differential geometry, we often pass from the ring of smooth functions to the local ring at a point (identifying two functions which agree at a neighborhood of the point), Taylor series at the point, the convergent Taylor series.

In all these cases we pass to "zero-dimensional" rings in the sense of the stable rank (see the next section). In algebraic K-theory we pass from A to the matrix ring $\mathrm{M}A$ with $\mathrm{sr}(\mathrm{M}A) = 1$. The ring $\mathrm{M}A$ is not commutative and has no 1 unless $A = 0$.

Example 3.13. When A is a division ring, then the Whitehead determinant coinsides with the the Dieudonné determinant. More generally, when A is local, we have

$\mathrm{GL}_n A/\mathrm{E}_n A = \mathrm{GL}_1 A/[\mathrm{GL}_1 A, \mathrm{GL}_1 A] = \mathrm{K}_1 A$ for $n \geq 2$.

Example 3.14. When $A = \mathbf{Z}$, the integers, the Whitehead determinant coincides with the usual determinant, so $\mathrm{K}_1\mathbf{Z}$ is a cyclic group of order 2.

More generally, for any commutative Euclidean ring A (e.g., the Gaussian integers),

$\mathrm{GL}_n A/\mathrm{E}_n A = \mathrm{GL}_1 A = \mathrm{K}_1 A$ for $n \geq 2$.

The rings A such that $\mathrm{GL}_n A = \mathrm{E}_n A \mathrm{GL}_1 A$ for all n hence $\mathrm{wh}(\mathrm{GL}_1 A) = \mathrm{K}_1 A$, are known as the GE-rings [C]. In other words, A is called generalized Euclidean, if every invertible matrix can be reduced to a diagonal matrix by row addition operations, or, equivalently, by column addition operaions. GE-rings include the rings of stable rank 1 (see the next section), as well as all right Euclidean rings (see Exercise 25 in Section 1). A commutative A is a GE-ring if and only if $\mathrm{SL}_n A = \mathrm{E}_n A$ for all n. Then $\mathrm{K}_1 A = \mathrm{GL}_1 A$.

Non-trivial examples include the ring if integers in the number fields except the imaginary quadratic fields, see [V19]. By [V19], any subring A with 1 of $\mathbf{R}$ consisting of algebraic numbers is a GE-ring. The same is true when A is a subring with 1 of $\mathbf{C}$ exept for the case when A consists of algebraic integers and cannot be embedded as a subring into $\mathbf{R}$.

See the next example for two non-commutative examples of GE-rings (one of them is Euclidean).

Example 3.15. The following well-known and important result (see Section 9 below) connects the algebraic functor K_1 with topological K-theory

Let X be a compact space and $A = \mathbf{R}^X$ the ring of all real-valued continuous functions on X. Then $\mathrm{E}_n A$ is the connected and path-connected component of 1_n in $\mathrm{SL}_n A$, hence it is normal in $\mathrm{GL}_n A$ and $\mathrm{SL}_n A/\mathrm{E}_n A$ is the set of the homotopy classes of continuous maps $X \to \mathrm{SL}_n \mathbf{R}$.

The Lie group $\mathrm{SL}_n \mathbf{R}$ here can be replaced by its compact subgroup $\mathrm{SO}_n \mathbf{R}$, see Lemma 3.17 below. (Similar results hold for complex-valued and quaternion-valued functions.)

Here is a more specific example of a comact space: $X = S^1$, the cicle. In this case, since $\mathrm{SO}_2 \mathbf{R} = S^1$, we have $\mathrm{SL}_2 \mathbf{R}^X/\mathrm{E}_2 \mathbf{R}^X = \pi_1(\mathrm{SO}_2 \mathbf{R}) = \pi_1 S^1 = \mathbf{Z}$ (an ifinite cyclic grou) and $\mathrm{SL}_n \mathbf{R}^X/\mathrm{E}_n \mathbf{R}^X == \pi_1(SO_n \mathbf{R}) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 3$. The two-shit covering of $\mathrm{SO}_n \mathbf{R}$) is the spinor group $\mathrm{Spin}_n \mathbf{R}$.

When $n = 3$, the spinor group is $\mathrm{SL}_1 \mathbf{H}$ (the quaternions of norm 1) acting on $\mathbf{R}i + \mathbf{R}j + \mathbf{R}k$ by

$$bi + cj + dk \to g^*(bi + cj + dk)g$$

where $g \in \mathbf{H}$ and $g^* g = 1$.

When $n = 4$, the spinor group is $\mathrm{SL}_1 \mathbf{H} \times \mathrm{SL}_1 \mathbf{H}$ acting on $\mathbf{H}$ by $x \to g^* x h$ where $g, h \in \mathrm{SL}_1 \mathbf{H}$ and $x \in \mathbf{H}$.

Now we extend this to arbitrary topological apace X. Recall that the Stone-Čech compactification βX of X is a compact Hausdorff topological space with a map $X \to \beta X$ such that $\mathbf{R}_0^X = \mathbf{R}^{\beta X}$.

Theorem 3.16. For any topological space, let $A_0 = \mathbf{R}_0^X \subset A = \mathbf{R}^X$ be a subring of bounded functions Then for every n, $\mathrm{SL}_n A_0/\mathrm{E}_n A_0 = \mathrm{SL}_n A/\mathrm{E}_n A$. So this group is the group of homotopy classes of continuous maps $\beta X \to \mathrm{SL}_n \mathbf{R}$.

Proof. Note that $E_n A_0$ is normal in $\mathrm{SL}_n A_0$ as the connected component of 1_n (see Exercise 15 in this section or Section 9). It is more difficult to show that $E_n A$ is normal in $\mathrm{SL}_n A$

We will use

Lemma 3.17. Every matrix $\alpha \in \mathrm{SL}_n A$ can be uniquely factored in $\mathrm{SL}_n A$ as $\alpha = \gamma\beta$ with an upper triangular matrix β with positive diagonal entries and an orthogonal matrix $\gamma = (\gamma^T)^{-1}$.

Proof. This is a version of the Gram-Schmidt process. To prove that the determinants of γ and β are 1, use that $\det(\beta) > 0$ and that $\det(\mathrm{O}_n(\mathbf{R}) = \pm 1$.

Now back to the normality of E_nA_0. We have to prove that $\delta=\alpha^{-1}a^{1,2}\alpha\in E_nA$ for any $a\in A$ and $\alpha\in \mathrm{SL}_nA$. We apply Lemma 3.,17 and write $\begin{pmatrix}1+|a| & 0\\ 0 & 1/(1+|a|)\end{pmatrix}\alpha=\gamma\beta$ with an upper triangular matrix $\beta\in \mathrm{E}_nA$ and $\gamma\in \mathrm{SO}_nA$.

Now $\delta=\beta^{-1}\gamma^{-1}(a/(1+|a|)^2)^{1,2}\gamma\beta$. Since $a/(1+|a|)^2\in A_0$ and $\mathrm{SO}_nA=\mathrm{SO}_nA_0$, we have $\gamma^{-1}(a/(1+|a|)^2)^{1,2}\gamma\in \mathrm{E}_nA_0$ hence $\delta v\in \mathrm{E}_nA$.

Using again Lemma 3.17 and the fact that $\mathrm{SO}_mA=\mathrm{SO}_nA_0$ we obtain that the homomorphism $\mathrm{SL}_nA_0/\mathrm{E}_nA_0\to \mathrm{SL}_nA/\mathrm{E}_nA$ is surjective. It remains to show that this homomorphism is injective, i.e., $\mathrm{SL}_nA_0\cap \mathrm{E}_nA=\mathrm{E}_nA_0$.

This follows from

Lemma 3.18. Every matrix $\alpha\in \mathrm{E}_nA$ can be uniquely factored in E_nA as $\alpha=\gamma\beta$ with an upper triangular matrix β with positive diagonal entries and $\gamma\in \mathrm{E}_nA_0\cap\ \mathrm{SO}_nA$.

We prove the lemma first in the case $n=2$. Proceeding by induction on the number of elementary matrices in α, it suffices to consider the case when $\alpha=a^{1,2}b^{2,1}$ with $a,b\in A$.

We write $\begin{pmatrix}1+ab\\ b\end{pmatrix}/((1+ab)^2+b^2)^{1/2}=\begin{pmatrix}\cos(c)\\ \sin(c)\end{pmatrix}$ with $c=c(x)\in A$ and $-\pi/2<c(x)<3\pi/2$ for all $x\in X$. We used that $\begin{pmatrix}1+ab\\ b\end{pmatrix}/((1+ab)^2+b^2)^{1/2}$ is never $\begin{pmatrix}-1\\ 0\end{pmatrix}$.

Now $\alpha=a^{1,2}b^{2,1}=\begin{pmatrix}\cos(c) & -\sin(c)\\ \sin(c) & \cos(c)\end{pmatrix}d^{1.2}$ with $d\in A$.

It remains to show that $\begin{pmatrix}\cos(c) & -\sin(c)\\ \sin(c) & \cos(c)\end{pmatrix}\in E_2A_0$.

But $\begin{pmatrix}\cos(c) & -\sin(c)\\ \sin(c) & \cos(c)\end{pmatrix}=\begin{pmatrix}\cos(c/4) & -\sin(c/4)\\ \sin(c/4) & \cos(c/4)\end{pmatrix}^4$ and $\begin{pmatrix}\cos(c/4) & -\sin(c/4)\\ \sin(c/4) & \cos(c/4)\end{pmatrix}$ is a product of 4 elementary matrices in E_2A_0 because $|c/4|<3\pi/8$ hence $\cos(c/4)\in \mathrm{GL}_1A_0$.

Consider now the case $n>2$. Since the group E_nA is generated by its upper triangular matrices together with matrices $1^{i,i-1}$ with $i=2,\dots,n$, it suffices to show Lemma 4.18 in the case when α has the form $\delta 1^{i,i-1}$ with an upper triangular matrix in SL_nA. But in this case the action takes place in the two by two submatrix (rows and columns $i,i-1$) so we are reduced to the case $n=2$. QED.

Now we consider a non-commutative example.

Example 3.19. The ring A of Lipschitz quaternions is a subring of the Hamilton quaternions $\mathbf{H}$ (see Exercise 1 in Section 2) consisting of $a+bi+cj+dk$ with $a,b,c,d\in\mathbf{Z}$. The norm of a Hamilton quaternion $a+bi+cj+dk$ is defined as $N(a)=aa^*=a^*a=a^2+b^2+c^2+d^2$ where $(a+bi+cj+dk)^*=a-bi-cj-dk$.

The ring A' of Hurvitz quaternions is a bigger subring of H consisting of A and the quaternions $a+bi+cj+dk$ with $a,b,c,d\in 1/2+\mathbf{Z}$.

We want to compute K_1A and K_1A'.

The ring A' is (right and, because of the involution, left) Euclidean [CS, Section 5.1] with the respect to the norm:

if $x,y\in A$ and $xy\neq 0$, then either there is $z\in A$ such that $N(y+xz)<N(x)$.

This "division with small remainder" allows us to reduce every matrix in GL_mA' to GL_1A' by column addition operations. So K_1A' is a factor group of the multiplicative group $\mathrm{GL}_1 A'$ which consists of 24 quaternions, namely $\pm 1, \pm i, \pm j, (\pm k, \pm 1 \pm i \pm j \pm k)/2$.

The commutators of the pairs of these 24 quaternions are the first 8 quaternions. So $\mathrm{K}_1(A')$ is a factor group of cyclic group of order 3.

On the other hand, there is a ring homomorphism $f : A' \to F_4$ onto a field F_4 of 4 elements. Namely, $f(a+bi+cj+dk) = a+b+c+d+2\mathbf{Z} \in \mathbf{Z}/2\mathbf{Z}$ on the subring $A \subset A'$, where $a, b, c, d \in \mathbf{Z}$, and $f((1+i+j+k)/2) = x + 2\mathbf{Z}[x] + (x^2 - x + 1)\mathbf{Z}[x] \in F_4$. So the image of $(1+i+j+k)/2$ in $\mathrm{GL}_1 F_4 = \mathrm{K}_1 F_4$ has order 3. Thus,

$\mathrm{K}_1 A'$ is a group of order 3 for the Hurwitz quaternions A"

The ring A of Lipschitz quaternions is not Euclidean (it is not even a principal ideal domain: the ideal $(1+i)A + (1+j)A$ is not principal) , but it is a GE-ring. Namely, $\mathrm{sr}(A) = 2$ (see Section 3) hence $\mathrm{E}_n A$ acts transitively on the unimodular columns in A^n for $n \geq 3$. So it remains to show that $\mathrm{E}_2 A$ acts transitively on the unimodular columns in A^2 or, equivalently, that $\mathrm{E}_2 A$ acts transitively on the unimodular rows (x, y) with $x, y \in A$.

By induction on $\max(N(x), N(y))$ we will show that (x, y) can be reduced to (1, 0) by (column) addition operations.

Since (x, y) is unimodular, $\max(N(x), N(y)) \geq 1$. If $\max(N(x), N(y)) = 1$ then x or y is a unit so we can reduce (x, y) to (1,0) by 3 addition operations. Assume now that $\max(N(x), N(y)) \geq 2$

Since the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{E}_2 A$, we can assume that $N(x) \leq N(y)$. and $N(y) \geq 2$. Since (x, y) is unimodular, $x \neq 0$. We write $y/x = a+bi+cj+dk$ with rational a, b, c, d. Then we find integers a', b', c', d' closest to a, b, c, d. so $|a-a'|, |b-b'|, |c-c'|, |d-d'| \leq 1/2$. If not all a, b, c, d. are in $1/2+\mathbf{Z}$ then for $q = a'+b'i+c'j+d'k \in A$ we have $N(y-xq) < N(x)$ hence $(x, y-xq)$ can be reduced to (1, 0) by addition operations by the induction hypothesis, hence (x, y) can be reduced to (1, 0) by addition operations.

Assume now that a, b, c, d. are in $1/2 + \mathbf{Z}$. We take

$$q = (a-1/2) + (b-1/2)i + (c-1/2)j + (d-1/2)k \in A.$$

Then $N(y - xq) = N(x)$ and $(y - xq)/x = 1/2 + i/2 + j/2 + k/2 \in A'$. Replacing (x, y) by $(x, y - xq)$ we can assume that $y/x = 1/2 + i/2 + j/2 + k/2$.

Since (x, y) is unimodular in A it is unimodular in A' hence $x \in \mathrm{GL}_1 A'$. But $A \cap \mathrm{GL}_1 A' = \mathrm{GL}_1 A = \{\pm 1, \pm i, \pm j, \pm k\}$. Since x is a unit in A and $y \in A$, $y/x = 1/2 + i/2 + j/2 + k/2 \in A$ which is a contradiction.

Thus, A is a GE-ring, hence $\mathrm{K}_1 A$ is a factor group of $\mathrm{GL}_1 A$, a group of order 8. Since $[i, j] = 1$, $\mathrm{K}_1 A$ is a factor group of a non-cyclic group of order 4

To get a lower bound on $\mathrm{K}_1 A$ we consider the ring homomorphism $A \to A/2A$ and the induced group homomorphism $\mathrm{K}_1 A \to \mathrm{K}_1(A/2A)$. The ring $A/2A$ is a local commutative ring of order 8. The group $\mathrm{K}_1 A/2A = \mathrm{GL}_1(A/2A)$ has order 4 and consists of the images of $1, i, j, k$. Thus,

$\mathrm{K}_1 A$ is a non-cyclic group of order 4 for the Lipschitz quaternions A

Example 3.20. Following [C], we prove that $\mathrm{E}_2 A \neq \mathrm{SL}_2 A$ when $A = F[x, y]$ where F is any field. (This is a part of Exercise 9 in Section 1.) In fact, we will prove that $\mathrm{E}_2 A$ is not normal in $\mathrm{SL}_2 A$.

The main tool is the following lemma which holds for an arbitrary ring A with 1.

Lemma 3.21. Let A be an associative ring with 1. Then every product g of $l \geq 1$ elementary matrices in $\mathrm{E}_2 A$ can be written as

$$\alpha a_1^{t(1)} a_2^{t(2)} \cdots a_k^{t(k)} \tag{3.22}$$

where either α or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \alpha$ is diagonal and where $1 \leq k \leq l, a_i \in A$ for $i = 1, \ldots, k$ $t(i)$ = (1,2) or (2,1) (so $a_i^{t(i)}$ is an elementary matrix), $t(i) \neq t(i+1)$ for all i, and $a_i \notin \mathrm{GL}_1 A$ for $2 \leq i \leq k-1$.

Clearly g can be writtern in the form (3.22) with $\alpha = 1_2$ if we drop the condition $a_i \notin \mathrm{GL}_1 A$ for $2 \leq i \leq k-1$. It remains to show that when the condition is not satisfied then we can rewrite g in the form (2.22) with a smaller k.

So we assume that we have (3.22) but $a_i \in \mathrm{GL}_1 A$ for some i such that $2 \leq i \leq k-1$. Set

$$\beta = (-1/a_i)^{t(i-1)} a_i^{t(i)} (-1/a_i)^{t(i-1)}$$

so $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \beta$ is diagonal.

Then

$$\alpha a_1^{t(1)} a_2^{t(2)} \cdots a_k^{t(k)} = \alpha\beta b_2^{t(2)} \cdots b_k^{t(k)}$$

where $b_j = a_j$ for $j > i+1, b_{i+1} = a_{i+1} + 1/a_i, b_i = -a_i(a_{i-1} + 1/a_i)a_i$, and $b_j = -a_i a_{j-1} a_i$ or $-a_i^{-1} a_{j-1} a_i^{-1}$ for $2 \leq j < i-1$.

Thus, we obtain a representation of the form (3.22) with $k-1$ elementary matrices and still either $\alpha\beta$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \alpha\beta$ is diagonal.

Now we return to the case $A = F[x, y]$ and claim that the matrix

$$g = \begin{pmatrix} 1 + xy & x^2 \\ -y^2 & 1 - xy \end{pmatrix} \in \mathrm{SL}_2 A$$

cannot be written in the form (3.22).

Otherwise, since $\mathrm{GL}_1 A = \mathrm{GL}_1 F$, we would have $\alpha \in \mathrm{SL}_2 F$ hence every entry of the matrix $\alpha^{-1} g$ has (total) degree 2. This is because the degree 2 terms in every column of g are linearly independent over F. In the case when $a_k \in F$ we multiply $\alpha^{-1} g$ by $(-a_k)^{t(k)}$ to obtain a matrix with every entry of degree 2.

We see now that a column of $\alpha^{-1} g$ is reduced to a column of 1_2 by $k > 1$ or $k-1 > 1$ row addition operations with non-constant coefficients. But each addition operation increases the degree of the corresponding entry making the degree greater than the degree of the other entry, so we never can get one of them constant.

Thus, the matrix g above is not in $\mathrm{E}_2 A$. The same proof works for any matrix in $\mathrm{SL}_2 A$ such that every entry has the same degree $d > 1$ and the degree d terms in every column are linearly independent over F. In particular the matrix $g 1^{1,2} g^{-1} = \begin{pmatrix} 1 - y^2 + xy^3 & (1 - xy)^2 \\ -y^4 & 1 + y^2 - xy^3 \end{pmatrix}$ does not belong to $\mathrm{E}_2 A$ hence $\mathrm{E}_2 A$ is not normal in $\mathrm{SL}_2 A$.

Exercises and problems.

1. Let α be an operator on a Banach space. Its spectrum is defined as the set of complex numbers λ such that $\alpha - \lambda$ is not invertible. For any two operators X, Y, prove that the spectra of XY and YX may differ only by one point 0.

2. Find a ring A with unity and a matrix $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{GL}_2 A$ such that $a \notin \mathrm{GL}_1 A$.

3. Find a ring A with unity and $x, y \in A$ such that $1 + xy \in \mathrm{GL}_1 A$ but $(1 + xy)(1 + yx)^{-1} \notin [\mathrm{GL}_1 A, \mathrm{GL}_1 A]$.

4. For any division ring D and integer $n \geq 1$, compute $\mathrm{K}_1 A$ for the ring A of all upper triangular matrices in $\mathrm{M}_n D$.

5. Suppose that $\begin{pmatrix} \mathrm{G}_n(A, B) & 0 \\ 0 & 1 \end{pmatrix} \subset \mathrm{G}_{n+1}(A, B)$ for some A, B, n. Prove that either $B = A$ or $2A = 0$.

6. Let A be an associative ring with $1 \neq 0$. Define $B_1 = \{x \in A : 1 + xy \in \mathrm{GL}_1 A \, \forall y \in A\}$ and $B_2 = \{x \in A : 1 + yx \in \mathrm{GL}_1 A \, \forall y \in A\}$. Prove that $B_1 = B_2$ and that $B_1 = B_2$ is the intersection of all maximal left ideal of A as well as the intersection of all maximal right ideals of A. (Hint: Use Lemma 3.3 with $m = n = 1$). Define $\mathrm{rad}(A) = B_1 = B_2$ (the Jacobson radical of A).

7. For any associative ring A with 1, prove that every element of $\mathrm{E}A = [\mathrm{E}A, \mathrm{E}A]$ is a product of two commutators (Dennis-Vaserstein).

8.(Open problem) Is there an associative ring A with 1 and an element of $\mathrm{E}A$ which is not a commutator?

9.[V63] Take $X = \mathbf{R}, A = \mathbf{R}^X$ (the continuous functions on the line), and $A_0 = \mathbf{R}_0^X$ (the bounded functions). Prove that

$\mathrm{SL}_2 A / \mathrm{E}_2 A = \mathrm{SL}_2 A_0 / \mathrm{E}_2 A_0 = A / A_0$

and

$\mathrm{SL}_n A / \mathrm{E}_n A = \mathrm{SL}_n A_0 / \mathrm{E}_n A_0 = 0$

for $n \geq 3$.

10. Show that the kernel of wh: $\mathrm{GL}_1 A \to \mathrm{K}_1 A$ is bigger than $[\mathrm{GL}_1 A, \mathrm{GL}_1 A]$ in the case when $A = \mathrm{M}_2(\mathbf{Z}/2\mathbf{Z})$.

11. Find a commutative ring A with $1 \neq 0$ such that the group $\mathrm{K}_1 A$ is trivial.

12. Let X be a compact topological space. We use the uniform norm $||f|| = \max_{x \in X} f(x)|$ for $f = f(x) \in A_0 = \mathbf{R}_0^X$. The induced topology on the matrix ring $M_n A_0$ is defined by the norm $||(a_{i,j})|| = \max_{i,j} ||a_{i,j}||$.

Show that if $\mu \in M_n A_0$ and $||\mu|| < 1/(n-1)$ then $1_n + \mu = LU$ where $L \in \mathrm{GL}_n A_0$ is a lower triangular matrix and $U \in \mathrm{GL}_n A_0$ is a upper triangular matrix.

Therefore $1_n + \mu \in \mathrm{GL}_n A_0$, and this matrix can be reduced to a diagonal matrix by $n(n-1)$ column addition operations. When $1_n + \mu \in \mathrm{SL}_n A_0$, show that this matrix can be reduced to the identity matrix 1_n by $(n+2)(n-1)$ column addition operations.

Show that $\mathrm{SK}_1 \mathbf{R}_0^X = \mathrm{SK}_1 \mathbf{R}^X$.

13. Let $A = \mathbf{C}^X$ be the ring of real-valued continuous functions on a topological space X and $A_0 = \mathbf{C}_0^X$ the subring of bounded functions.

Show that for any $n \geq 2$ every matrix $\alpha \in \mathrm{SL}_n A$ can be uniquely factored in $\mathrm{SL}_n A$ as $\alpha = \gamma\beta$ with an upper triangular matrix β with positive diagonal entries and a unitary matrix $\gamma = (\gamma^*)^{-1}$.

Show that $\mathrm{SL}_n A_0/\mathrm{E}_n A_0 = \mathrm{SL}_n A/\mathrm{E}_n A$.

So $\mathrm{SK}_1\mathbf{C}_0^X = \mathrm{SK}_1\mathbf{C}^X$.

14. Let A be the ring $\mathbf{H}^X$ of quaternion-valued continuous functions on a topological space X or the subring $\mathbf{H}_0^X$ of bounded functions.

Show that for any $n \geq 2$ every matrix $\alpha \in \mathrm{SL}_n A$ can be uniquely factored in $\mathrm{SL}_n A$ as $\alpha = \gamma\beta$ with an upper triangular matrix β with positive diagonal entries and an orthogonal matrix $\gamma = (\gamma^T)^{-1}$.

Show that $\mathrm{SL}_n A_0/\mathrm{E}_n A_0 = \mathrm{SL}_n A/\mathrm{E}_n A$.

So $\mathrm{SK}_1\mathbf{H}_0^X = \mathrm{SK}_1\mathbf{H}^X$.

Here $\mathrm{SL}_n A$ means the matrices of reduced norm 1. The reduced norm for a matrix $M_n\mathbf{H}$ can be defined by replacing every quaternion entry by the corresponding 2 by 2 complex matrix and taking the determinant of the obtained $2n$ by $2n$ complex matrix.

15. Let A be an associate ring with 1, A^∞ an infinitely-genrerated right A-module, $B = \mathrm{End}_A(A^\infty)$ its endomorphism ring.

Prove that $\mathrm{K}_1 B = 0$.

Let $\mathrm{M}A$ be a subring of B consisting of $b \in B$ such that bA^∞ is finitely generated A-module.

Prove that $\mathrm{K}_1(C,\mathrm{M}A) = \mathrm{K}_1 A$ for any ring C with 1 containing $\mathrm{M}A$ as an ideal (e.g., $C = B$).

16. Let A be an associate ring with 1 and $x \in A$. Let $m, n \in \mathbf{Z}$ and $m, n \geq 1$.

Prove that $(x^m - 1)A + (x^n - 1)A = (x^{\mathrm{GCD}(m,n)} - 1)A$.

17. Let $a_i \in \mathbf{Z}$ and $|a_i| \geq 2$ for all i. Prove that the matrix $a_1^{1,2}a_2^{2,1}\cdots a_{2k}^{2,1}$ is not a product of $< 2k$ elementary matrices in $\mathrm{SL}_2\mathbf{Z}$ and that the matrix $a_1^{1,2}a_2^{2,1}\cdots a_{2k+1}^{1,2}$ is not a product of $< 2k+1$ elementary matrices in $\mathrm{SL}_2\mathbf{Z}$ (Hint: use Lemma 3.21.)

18. Define a sequence in $\mathbf{Z}$ by $a_0 = 0, a_1 = 1, a_{i+1} = 2a_i + a_{i-1}$ for $i \geq 1$. Prove that for any $i \geq 1$ the pair (a_i, a_{i+1}) can be reduced to the pair (1,0) or (0, 1) by i addition operations, but cannot be reduced to (1,0) or (0, 1) by a smaller number of addition operations. (Hint: use the previous exercise1.)

19. [C] Let A be the ring of all algebraic integers in $Q(\sqrt{-d})$ where d is a square-free positive integer and $d \neq 1, 2, 3, 7, 11$. Prove that A is not a GE-ring. Hints: Let $a_i \in A \setminus GL_1 A$; then the matrix $a_1^{1,2}a_2^{2,1}\cdots a_{2k}^{2,1}$ is not a product of $< 2k$ elementary matrices in $\mathrm{SL}_2 A$ and the matrix $a_1^{1,2}a_2^{2,1}\cdots a_{2k+1}^{1,2}$ is not a product of $< 2k+1$ elementary matrices in $\mathrm{SL}_2 A$ (use Lemma 3.21 as well as the fact that the norm N has the following properties: there is no $a \in A$ with $1 < N(a) < 4$; $N(a+b)^{1/2} \leq N(a)^{1/2} + N(b)^{1/2}$ for any $a, b \in A$).

Note that when $d = -19$ the ring A is a principal ideal domain (PID). In the case when $d = 1, 2, 3, 7,$ or 11 the ring of algebraic integers in $Q(\sqrt{-d})$ is known to be Euclidean with respect to the norm so it is both PID and a GE-ring.

It is unknown whethere there are infinitely many square-free integers $d > 1$ such that $A = \mathbf{Z}[\sqrt{d})]$ is a PID (but, as mentioned above, all these A are GE-rings).