Chapter 5

Duality

§13. Dual Problems

Let us recall the canonical form of a linear programming problem:

$$cx + d \to \min, Ax \le b, x \ge 0.$$

Here x is a column of variables (unknowns) and the data consist of a row c, a number d, a matrix A, and a column b. In more detail, we want to

minimize the objective function

$$f(x) = c_1x_1 + c_2x_2 + \cdots + c_mx_m$$

subject to the linear constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &\le b_2 \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &\le b_n \end{aligned}$$

and the nonnegativity constraints

$$x_i \ge 0,\ i = 1, 2, \ldots, m.$$

The first n linear constraints can be written as linear equalities by introducing the slack variables, $u_j \ge 0$, where $j = 1, 2, \ldots, n$:

$$u_j = b_j - a_{j1}x_1 - a_{j2}x_2 - \ldots - a_{jm}x_m,\ j = 1, \ldots, n$$

(i.e., $u = b - Ax \ge 0$ in matrix form).

Thus, in matrix notation, we can write the linear programming problem as follows:

$$\begin{gathered}\text{minimize } cx\\ \text{subject to } b - Ax = u\\ x \geq 0,\ u \geq 0\end{gathered}$$

or, even shorter,

$$cx \to \min, b - Ax = u \geq 0, x \geq 0,$$

where c is the $1 \times m$ row matrix $c = [c_1\ c_2\ \ldots\ c_m],$ x is the $m \times 1$ column matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix},$$

b is the $n \times 1$ column matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

u is the $n \times 1$ column matrix

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

and A is the $n \times m$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

We can write our linear program in standard row tableau:

$$\begin{array}{cl} \begin{array}{cc} x^T & 1 \end{array} & \\ \begin{bmatrix} A & b \\ c & d \end{bmatrix} & \begin{array}{l} = u \\ = f \to \min, \quad x \geq 0, \quad u \geq 0. \end{array} \end{array}$$

We also can write the same problem in a column tableau:

$$\begin{array}{r} -x \\ 1 \end{array} \begin{bmatrix} -A^T & c^T \\ b^T & -d \end{bmatrix} \quad x \geq 0,\ u \geq 0$$
$$\begin{array}{cc} \| & \| \\ u^T & -f \end{array} \quad \to \max$$

We call such a column tableau *standard*. Notice that the matrix

$$\begin{bmatrix} -A^T & c^T \\ b^T & -d \end{bmatrix}$$

is an arbitrary given matrix, and all variables are distinct. We replaced x by $-x$ to keep the same pivot rule (8.6):

$$\begin{array}{r} -x \\ -y \end{array} \begin{bmatrix} \alpha^* & \beta \\ \gamma & \delta \end{bmatrix} \quad \mapsto \quad \begin{array}{r} -u \\ -y \end{array} \begin{bmatrix} 1/\alpha & -\beta/\alpha \\ \gamma/\alpha & \delta - \beta\gamma/\alpha \end{bmatrix}$$
$$\begin{array}{cc} \| & \| \\ u & v \end{array} \qquad\qquad \begin{array}{cc} \| & \| \\ x & v \end{array}$$

So we pivot standard column tableaux using the same pivoting rule we used for standard row tableaux. A pivot entry is always a nonzero entry of the matrix. We do not choose it in the last row or the last column to keep our tableaux standard. We change signs of the variables that are moved. This is the only difference from (8.6). There is no difference in the matrix, only at the margins.

Definition. The *basic solution* associated with the standard column tableau

$$\begin{array}{r} -y \\ 1 \end{array} \begin{bmatrix} A & b \\ c & d \end{bmatrix} \quad y \geq 0, v \geq 0$$
$$\begin{array}{cc} \| & \downarrow \\ v & \max \end{array}$$

is $y = 0,\ v = c$. ∎

This basic solution is feasible if and only if $c \geq 0$. In this case the column tableau is called (column) *feasible*. The corresponding value of the objective function is d.

Definition. A standard column tableau is called *optimal* if $b \geq 0$ and $c \geq 0$; that is, if it is both, row and column feasible. ∎

In this case the basic solution $y = 0,\ v = c$ is optimal, and $\max = d$.

Example 13.1. Let us write the diet problem (Example 2.1) in a standard column tableau:

$$\begin{array}{r} -a \\ -b \\ -c \\ -d \\ -e \\ 1 \\ \\ \end{array}\begin{array}{l}\left[\begin{array}{cccc} -0.3 & -73 & -9.6 & 10 \\ -1.2 & -96 & -7 & 15 \\ -0.7 & -20253 & -19 & 5 \\ -3.5 & -890 & -57 & 60 \\ -5.5 & -279 & -22 & 8 \\ -50 & -4000 & -1000 & 0 \end{array}\right] \\ \begin{array}{cccc} = u_1 & = u_2 & = u_3 & = -C \to \max . \end{array}\end{array}$$

Note that this tableau is row feasible, so the simplex method, Phase 2 can be applied. We apply it to an imaginary row problem, which later will be called the dual problem. In terms of the original (primal) diet problem, this is the dual simplex method.

In general, we can write any linear program in a standard column tableaux, and then use the simplex method. This is the dual simplex method. It may be more efficient than the simplex method that we discussed in Chapter 4.

Let a linear program be given using a standard row tableau:

$$\begin{array}{cc} x & 1 \\ \end{array}$$
$$\left[\begin{array}{cc} A & b \\ c & d \end{array}\right] \begin{array}{l} = u \\ = z \to \min, \quad x \geq 0, \quad u \geq 0 \end{array} \tag{13.2}$$

We will refer to this linear program as the *primal* or *row* problem. Note that in a standard row tableau we use the top and the right margins for the row of variables, x, and the column of variables, u, respectively. The *dual* or *column* problem of the primal problem given previously is defined to be the following linear program, which we write in the standard column tableau form:

$$\begin{array}{r} -y \\ 1 \end{array}\left[\begin{array}{cc} A & b \\ c & d \end{array}\right] \quad y \geq 0, \ v \geq 0 \tag{13.3}$$
$$\begin{array}{ccl} \| & \| & \\ v & w & \to \max, \end{array}$$

where all variables in the matrices y and v are distinct and different from the variables in the matrices x and u and the objective variable w. The variables in the matrices y and v are called *dual* variables.

The standard column tableau uses the remaining margins, i.e., the left and the bottom margins, for the variables. Both linear programming problems can be written in the same tableau, which now uses all the margins:

$$\begin{array}{cc}
 & \begin{array}{cc} \quad x & \quad 1 \end{array} \\
\begin{array}{c} -y \\ 1 \end{array} & \begin{bmatrix} A & b \\ c & d \end{bmatrix} \begin{array}{l} = u \\ = z \to \min \end{array} \\
 & \begin{array}{ccc} = v & = w & \to \max \end{array}
\end{array}
\qquad \begin{array}{l} x \geq 0, u \geq 0 \\ y \geq 0, v \geq 0. \end{array} \tag{13.4}$$

Note that every (primal) variable in the row (primal) problem is coupled with a (dual) variable in the column (dual) problem. For instance, the variables in the row matrix x are coupled with the variables in the row matrix v, and the variables in the column matrix u are coupled with the variables in the column matrix y. We will call this situation duality. A remarkable feature of the duality is that (13.2) and (13.3) are dual to each other. In other words, the dual problem of the primal problem (13.2) is problem (13.3) and vice versa (so we can say that the "dual of the dual is the primal"). This is because rewriting a standard row tableau as an equivalent column tableau and rewriting a standard column tableau as an equivalent row tableau both result in the same operation on the data matrix:

$$\begin{bmatrix} A & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} -A^T & c^T \\ b^T & -d \end{bmatrix},$$

and this operation repeated gives back the original matrix [recall that $(e^T)^T = e$ and $-(-e) = e$ for any matrix e].

Dropping the basic variables u in the row problem in (13.4), we obtain the canonical form

$$cx^T + d \to \min, -Ax^T \leq b, x \geq 0.$$

The canonical form for the dual problem is

$$b^T y^T - d \to \min, A^T y^T \leq c^T, y \geq 0.$$

So it is very easy to write the dual for a problem in canonical form, bypassing slack variables and tableaux.

For a linear program in standard form $cx^T + d \to \min, -Ax^T = b, x \geq 0$, its dual can be obtained via canonical form. A simpler dual has variables corresponding to the linear equations, which are not required to be nonnegative. An alternative is to solve the system $Ax = b$ in the standard form, which allows us to get a smaller canonical form and the dual problem.

Now we will find a relationship between the feasible values of a linear program and those of its dual. First we need a definition.

Definition 13.5. Given an optimization problem, the values of the objective function on the feasible region are called *feasible values.* ■

If the feasible region is empty (i.e., the system of constraints is inconsistent), then there are no feasible values. Now we apply this notion to the pair of linear programs written in (13.4).

Any feasible value for the row problem is $cx^T + d$, where the row x is a feasible solution (i.e., $x \geq 0$ and $Ax^T + b = u \geq 0$.) Any feasible value for the column problem is $-y^T A + d$, where the column y is a feasible solution (i.e., $y \geq 0$ and $-y^T A + c = v \geq 0$). Putting these equalities together, we obtain that

$$cx^T + d = (v + y^T A)x^T + d = vx^T + y^T(Ax^T) + d$$

$$= vx^T + y^T(u - b) + d = vx^T + y^T u - y^T b + d \geq -y^T b + d,$$

because both numbers vx^T and $y^T u$ are nonnegative.

Thus, we obtain the inequality $cx^T + d \geq -y^T b + d$ between the feasible values of our two problems and hence the following result.

Fact 13.6. Every feasible value for a linear minimization program is greater than or equal to every feasible value of the dual maximization problem. ■

This fact does not violate the symmetry between the problem and its dual because we multiply the objective functions by -1 when we transpose the situation. Here is a shorter way to write this fact: min $\geq$ max whenever both problems are feasible. In fact with our conventions about max and min for unbounded and infeasible problems, the inequality min $\geq$ max holds in all cases. Namely, when the row problem is unbounded, we obtain that the dual problem has no feasible values (i.e., it is infeasible; hence max $=$ min $= -\infty$). When the column problem is unbounded, then the row problem is infeasible; hence max $=$ min $= \infty$)

It is also clear that the inequality becomes an equality if and only if $vx^T + y^T u = 0$ (i.e., whenever a variable in one of the programs takes a nonzero value, the corresponding dual variable is zero). The last condition is called *complementary slackness.* So we obtain the following result, which is very useful to prove that a solution is optimal.

Fact 13.7. If we have a feasible solution for a linear program and a feasible solution for the dual problem and the feasible values are the same (i.e., the complementary slackness holds), then both solutions are optimal. ■

The duality theorem (included in Theorem 13.8) is a deeper result which replaces the inequality min $\geq$ max by the equality min $=$ max in the case when both problems are feasible. In particular, it gives the converse of the last fact (see Theorem 13.9). The duality theorem also relates properties of a given linear program, such as feasibility and boundness, with that of its dual problem. The idea is to apply pivoting on both the primal linear program and its dual in order to obtain outcomes for both problems. According to Theorem 11.3 there are three possible outcomes for each of these problems. However, as we will see, the outcomes of both problems are closely related.

To see how this theorem applies, we write both problems in the same tableau [as in (13.4)] and apply the simplex method. Then, in a finite number of pivoting steps, the following three outcomes are possible.

Case 1: We obtain an optimal tableau ($b \geq 0,\ \ c \geq 0$);

Case 2: We obtain a tableau with a bad row;

Case 3: We obtain a row feasible tableau ($b \geq 0$) with a bad column.

In Case 1, the optimal tableau ($b \geq 0,\ \ c \geq 0$) gives an optimal solution for either problem. An optimal solution for the primal problem is given by $u = b,\ x = 0$; an optimal solution for the dual problem is given by $v = c,\ y = 0$. Moreover, the optimal value is $d = \min(z) = \max(w)$ for both z and w.

In Case 3, $\min(z) = -\infty$ and the column problem (13.3) has no feasible solutions. So $\max(w) = -\infty$.

In Case 2, the row problem has no feasible solutions [that is, $\min(z) = +\infty$]. To see what happens with the column problem, we transpose the tableau and apply the simplex method. When we perform this operation, the original row problem becomes a column problem and the original column problem is now the row problem. Note that we cannot obtain an optimal tableau since this would give an optimal solution for the original row problem which is now the column problem in this new tableau. So only two outcomes are now possible:

a tableau with a bad row

or

a feasible tableau with a bad column.

In the first case neither problem has any feasible solutions. In the second case the new row problem (that is, the original column problem) is unbounded. It can be shown that by extra pivoting we can obtain a tableau with a bad row and a bad column.

Thus, we obtain the main theorem of linear programming—the theorem on four alternatives—which includes the duality theorem.

Theorem 13.8. Given a pair (13.4) of dual linear programs, one and only one of the following four situations occurs:

(1) Both problems have optimal solutions with the same optimal value [$\min(z) = \max(w)$].

(2) Both problems are infeasible.

(3) The row problem is infeasible and the column problem is unbounded [$\min(z) = \infty = \max(w)$].

(4) The column problem is infeasible and the row problem is unbounded [$\min(z) = -\infty = \max(w)$].

Proof. This theorem follows from the existence of a simplex method that always works (one that avoids cycling) and the previous discussion. ∎

Sometimes this is called the theorem on three alternatives, because (3) and (4) are symmetric. But usually the theorem on three alternatives refers to a less precise classification, namely, Cases 1–3. We prefer to stick to the name "theorem on four alternatives."

The duality theorem usually refers to the equality of the optimal values in the cases (1), (2), (4). But we will sometimes refer to Theorem 13.8 as the duality theorem for short.

The duality theorem is a deep fact with several interpretations and applications. Geometrically, the duality theorem means that certain convex sets can be separated from points outside them by hyperplanes. Traditionally, this is how the duality theorem is proved. But we chose another way of proving it: It follows from the existence of a simplex method that always works (one that avoids cycling). We will see another interpretation of the duality theorem in the theory of matrix games (cf. Chapter 7). For problems in economics, the dual problems and the duality theorem also have important economic interpretations (see examples in the next section).

As a corollary of the duality theorem, we obtain the following result called *the complementary slackness theorem*:

Theorem 13.9. Given a feasible solution for a linear problem and a feasible solution for the dual problem, they are both optimal if and only if the feasible values are the same (i.e., the complementary slackness condition holds). ■

Problem 13.10. Check whether $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 0$ is an optimal solution for the linear program

$$\begin{aligned}
&x_1 - x_2 + x_3 - 2x_4 + 6x_5 \to \min,\\
&x_1 + x_2 + x_3 + x_4 + x_5 \geq 10,\\
&x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \geq 20,\\
&x_1 - x_2 + x_3 - x_4 + x_5 \geq -2,\\
&\text{all } x_i \geq 0.
\end{aligned}$$

Solution. First we check that the given solution x is feasible. We find that it is, and the first, the third, and the last constraints hold as equalities (such constraints are called *active* or *tight*).

Then we put the problem in a standard row tableau with slack variables at the right margin, and we write dual variables y_i at the left and bottom margin in the same tableau, where all $x_i, y_i \geq 0$:

$$\begin{array}{c|cccccc|l}
 & x_1 & x_2 & x_3 & x_4 & x_5 & 1 & \\
\hline
-y_6 & 1 & 1 & 1 & 1 & 1 & -10 & = x_6 \\
-y_7 & 1 & 2 & 3 & 4 & 5 & -20 & = x_7 \\
-y_8 & 1 & -1 & 1 & -1 & 1 & 2 & = x_8 \\
1 & 1 & -1 & 1 & -2 & 6 & 0 & = z \ \to \min \\
\hline
 & = y_1 & = y_2 & = y_3 & = y_4 & = y_5 & = w & \to \quad \max .
\end{array}$$

Note that $x_6 = x_8 = 0$ for our slack variables in x corresponding to the active constraints. On the other hand, $x_7 = 10 \neq 0$. Assume now that our x is optimal and consider an optimal solution y for the dual problem that exists by the duality theorem. By complementary slackness, $y_1 = y_2 = y_3 = y_4 = 0$ and $y_7 = 0$. Thus, our column problem becomes

$$\begin{array}{c|cccccc|l}
-y_6 & 1 & 1 & 1 & 1 & 1 & -10 & \\
-y_8 & 1 & -1 & 1 & -1 & 1 & 2 & \\
1 & 1 & -1 & 1 & -2 & 6 & 0 & \\
\hline
 & = 0 & = 0 & = 0 & = 0 & = y_5 & = w & \to \max .
\end{array}$$

The second column says that $y_8 - y_6 = 1$ while the fourth column says that $y_8 - y_6 = 2$, which gives a contradiction. So the given solution is not optimal.

Exercises

1–2. Write in a standard column tableau:

1. $5x - 6y + 2z \to \min,\ [x, y, z] \ge 0, y \le 7, x + y \ge 3.$

2. Minimize
$.39a + .11b + .18c + .21d + .35e + .44f + .25g + .25h\ \ + .23i + .24j$
subject to
$15a + 15b + 15c + 15d + 15e + 15f + 10g + 15h + 15i + 10j \ge 100,$
$25a + 25b + 25c + 6f + 10g + 25h + 10j \ge 100,$
$a + 25b + 25c + 25d + 30f + 25g + 25h + 25i + 25j \ge 100,$
$a + 25c + 25d + 30e + 25f + 25g + 25h + 25i + 25j \ge 100,$
$a + 25b + 25c + 25d + 30e + 25f + 25g + 25h + 25i + 25j \ge 100,$
$3a + 2b + 1c + 2d + 4e + 2f + g + 2h + 3i + 3j \ge 70,$
$[a, b, c, d, e, f, g, h, i, j] \ge 0.$ ■

3. Rewrite the linear programs in Exercise 1 and Exercise 2 in standard row tableaux.

4. Solve the linear problem in Problem 13.10.

5. Prove that for any linear program, the set of feasible values is a convex set on line.

6–10. Check whether the following solution is optimal for the problem given by the standard row tableau

$$\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{array} \\
\left[\begin{array}{cccccc}
7 & -2 & -6 & 6 & -1 & 15 \\
-1 & 1 & 1 & -2 & 2 & -4 \\
-1 & 0 & 1 & 0 & -1 & -2 \\
4 & 0 & -3 & 5 & 3 & 4
\end{array}\right]
\begin{array}{l} = x_6 \\ = x_7 \\ = x_8 \\ = z \to \min . \end{array}
\end{array} .$$

6. The basic solution.
7. all $x_i = 1$.
8. $x_1 = 4, x_3 = 7, x_5 = x_7 = 1$, all other $x_i = 0$.
9. $x_1 = 1, x_2 = 2, x_3 = 3$, all other $x_i = 0$.
10. $x_3 = 3, x_2 = x_4 = x_5 = 1$, all other $x_i = 0$.

§14. Sensitivity Analysis and Parametric Programming

In real-life problems we often do not know exact values or these values may change with time. For example, RDAs in Example 2.1 (diet problem) are of necessity only vague estimates and they depend on country and year. Sensitivity analysis is concerned with how changes in data affect the optimal value and the optimal solutions. Most often, this is about small changes, while large changes are studied in *parametric programming.* In this section we consider both. We have already considered problems where some values in the data were not given explicitly, but we were asked to solve the problem for all possible values. These values are called *parameters*.

First we show how to find the changes in the optimal values caused by small changes in resource limits or required limits using optimal tableaux.

Suppose that in the diet problem (Example 2.1) we want to know the change in the minimal cost if

$$\text{the protein requirement is changed from 50 to 51;} \tag{14.1}$$

$$\begin{array}{c}\text{all three requirements are changed:}\\ \text{from 50, 4000, 1000 to}\\ 50+\varepsilon_1, 4000+\varepsilon_2, 1000+\varepsilon_3,\\ \text{respectively with } |\varepsilon_i| \le 1,\ i = 1,2,3;\end{array} \tag{14.2}$$

$$\begin{array}{c}\text{all five prices are changed by } \varepsilon_a, \varepsilon_b, \varepsilon_c, \varepsilon_d, \varepsilon_e,\\ \text{respectively, with absolute value of every change } \le 1\end{array} \tag{14.3}$$

$$\begin{array}{c}\text{all requirements are changed as in (14.2)}\\ \text{and all prices are changed as in (14.3)}\end{array} \tag{14.4}$$

Now we answer these four questions. Note that the second question includes the first one, and the last question includes all four questions.

First we write the diet problem in a standard row tableau:

$$\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & 1 \end{array} \\ \left[\begin{array}{cccccc} 0.3 & 1.2 & 0.7 & 3.5 & 5.5 & -50 \\ 73 & 96 & 20253 & 890 & 279 & -4000 \\ 9.6 & 7 & 19 & 57 & 22 & -1000 \\ 8 & 10 & 15 & 5 & 60 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ = u_1 \\ = u_2 \\ = u_3 \\ = C \to \min \end{array}$$

(cf. Example 13.1).

It takes at least two pivot steps to obtain an optimal tableau:

$$\begin{array}{cccccccc} a & b & u_1 & d & u_3 & 1 & \\ \end{array}$$
$$\begin{bmatrix} -0.519 & -0.136 & -0.2469 & -2.654 & 0.0617 & 49.38 \\ -10425 & -2710 & -4941 & -52951 & 1248 & 996931 \\ 0.011 & -0.20 & 0.21 & -0.30 & -0.0079 & 2.806 \\ 7.499 & 12.71 & 0.471 & 44.34 & 0.2458 & 269 \end{bmatrix} \begin{array}{l} = c \\ = u_2 \\ = e \\ = C \end{array}$$

with numbers given approximately; the exact value for the optimal value is $80,000/297 \approx 269.36$, not 269. Surplus variables u_1 and u_3 for protein and calcium are on the top, so the corresponding constraints are active (the constraints hold as equalities for the basic solution) while the constraint in vitamin A is not active (there is a surplus). So it is clear that a small change in the vitamin A requirement of 4000 would not affect the optimal solution.

One way to find what happens when 50 is replaced by another number is to replace 50 by a parameter, say, p, in the original tableau and pivot the modified tableau. The parameter would stay in the last column. However, there is another way to proceed that also allows us to get the modified optimal tableau without pivoting again. Instead of replacing 50 by p, we replace the restriction $u_1 \geq 0$ on the slack variable by the condition $u_1 \geq p - 50$. This makes our tableau nonstandard, but we do not do any additional pivoting.

To answer (14.1), our constraint on u_1 is $u_1 \geq 1$, so we cannot set it to 0 together with the other variables on the top to get an optimal solution. The best we can do is to set $u_1 = p - 50 = 1$, which gives the optimal value $\approx 0.47 + 269.36 = 269.83$. It is important to check the feasibility: The values for c and u_2 go down but stay positive. Thus, the answer to (14.1) is that the minimal cost increases by approximately 0.47 (the exact number is 140/297).

Similarly, if we replace the requirement 50 by, say, $p = 48$, we relax the condition $u_1 \geq 0$ to $u_1 \geq -2$ which allows us to take $u_1 = -2$ and get improvement in C equal 280/297 (now the minimal cost goes down). Again it is important to check the feasibility: the value for e goes down but stays positive.

Thus, we see that the minimal cost, $\mathrm{mc} = \mathrm{mc}(p)$, is an affine function of p when we replace the requirement 50 by a parameter p that stays close to 50 (so the optimal tableau for $p = 50$ produces feasible solutions when we change the basic value $u_1 = 0$ for the slack variable to the value $u_1 = p - 50$) and that the number $140/297 \approx 0.471$ is the slope of $\mathrm{mc}(t)$. This number is known as the *shadow* *price* of protein in our diet problem. We can compare it with prices for protein from alternative sources to make decisions about our diet.

Moreover, we can tell that if we replace all required limits 50, 4000, 1000 by parameters t_1, t_2, t_3 respectively, then for $|t_1 - 50|$, $|t_2 - 4000|$, and $|t_3 - 1000|$ sufficiently close to 0, the minimal cost, mc $\approx 0.471t_1 + 0.240t_3 + 269.36$. So it is an affine function of t_1, t_2, t_3 and it is independent of t_2. It is not a surprise that it is independent of t_2 because the corresponding constraint is not active for the basic solution—we have a surplus of vitamin A. Therefore, the answer to (14.2) is $\approx 0.471\varepsilon_1 + 0.240\varepsilon_3$.

Also, we can write the last column of the (standard) optimal tableau of the modified problem:

$$\begin{array}{c} 1 \\ \begin{bmatrix} 49.38 - 0.2469\varepsilon_1 + 0.0617\varepsilon_3 \\ 996931 - 4941\varepsilon_1 + 1248\varepsilon_3 + \varepsilon_2 \\ 2.806 + 0.21\varepsilon_1 - 0.0079\varepsilon_3 \\ 269 + 0.471\varepsilon_1 + 0.2458\varepsilon_3 \end{bmatrix} \end{array} \begin{array}{l} \\ = c \\ = u_2 \\ = e \\ = C \to \min. \end{array} \tag{14.5}$$

The rest of the tableau stays the same, independent of ε_i. It is now easy to compute the values of ε_i for which the tableau stays optimal. It is clear that the last column is positive when $|\varepsilon_i| \leq 1$.

Next we address the question of (14.3). Since the dual problem has the same optimal value and the coefficients of the objective function for the row problem are the constant terms in the constraints of the dual problem, we can use duality to get the answer. Also, we can argue directly that a small change in data would result in a small change in the optimal tableau, so the tableau stays optimal. Therefore, the answer is approximately $48.38\varepsilon_c + 2.806\varepsilon_e$ for ε_* sufficiently close to 0. But is the number 1 in (14.3) sufficiently close to 0?

To answer this question, it is time to write the last row of the modified tableau (approximately; the other rows do not depend on parameters):

$$\begin{bmatrix} 7.499 - 0.519\varepsilon_c + 0.011\varepsilon_e + \varepsilon_a \\ 12.71 - 0.136\varepsilon_c - 0.20\varepsilon_e + \varepsilon_b \\ 0.471 - 0.2469\varepsilon_c + 0.21\varepsilon_e \\ 44.34 - 2.654\varepsilon_c - 0.30\varepsilon_e + \varepsilon_d \\ 0.2458 + 0.0617\varepsilon_c - 0.0079\varepsilon_e \\ 269 + 49.38\varepsilon_c + 2.806\varepsilon_e \end{bmatrix}^T . \tag{14.6}$$

Now it is clear that this row is positive for $|\varepsilon_*| \leq 1$.

Finally, we have to face (14.4) when we modify both the requirements and prices. Then, in the modified optimal tableau, the first three entries of the last column are the same as in (14.5), the first five entries of the last row are the same as in (14.6), and the last entry in the last row or column is (approximately)

$$[\varepsilon_c, \varepsilon_e, 1] \begin{bmatrix} -0.2469 & 0.0617 & 49.38 \\ 0.21 & -0.0079 & 2.806 \\ 0.471 & 0.2458 & 269 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_3 \\ 1 \end{bmatrix}.$$

Dropping the constant term here, we get an approximate answer for (14.4).

The same trick works in general. Namely, suppose we have a linear program of the form $cx \to \min, Ax \geq t, x \geq 0$, and we ask how the optimal value depends on the requirements t when they are close to certain values $t = b$. Suppose that the program has an optimal solution for $t = b$ and that this optimal solution is the basic solution for an optimal tableau where all basic variables take positive values. (To put the problem in a standard row tableau, we introduce the surplus variables $u = Ax - b$.) Then the minimal value, $\mathrm{mv} = \mathrm{mv}(t)$, is an affine function $c_1(t - b) + \bar{d}$, where $\bar{d} = \mathrm{mv}(b)$ is the last entry in the last row in the optimal tableau and the row c_1 consists of entries of the last row of the optimal tableau corresponding to active constraints and zeros for the parameters corresponding to nonactive constraints. When the objective function is interpreted as the cost, the entries of the row c_1 are called the *shadow prices* corresponding to the active constraints. Note that small changes in the requirements in nonactive constraints do not change the optimal value. The corresponding shadow prices are zeros. When a basic slack variable takes the zero value, two different shadow prices for this constraint are possible: one corresponding to the increasing requirement, and the other corresponding to the decreasing requirement (see the discussion of parametric programming in §14).

Thus, the basic solution for the dual problem tells us how the optimal value depends on some changes in data. Similarly, the optimal tableau tells us how the optimal value depends on small changes in the objective function. Again the dependence is affine.

Remark. More generally, we can replace b and c by functions, not necessarily affine, $F(t)$ and $G(t)$ of parameters t. This makes the optimal value a function $\mathrm{mv}(t)$ of t (not necessarily defined for all t). Then the last row and column (if positive except the last entry) of the optimal tableau for the problem with $b = F(t_0)$ (if it exists) allows us to express the partial derivatives of mv(t) as a linear function of partial derivatives of $F(t)$ and $G(t)$ (if they exist).

Using the optimal tableau, it is also possible to find what happens with the optimal tableau under a small change of all entries in the initial tableau, not only those in the last row and column. For simplicity we assume that we have only one parameter t. So our setting is now as follows. We have a standard row tableau

$$\begin{array}{c} \begin{array}{cc} x & 1 \end{array} \\ \begin{bmatrix} A(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \end{array} \begin{array}{l} \\ = u \\ \to \min \end{array} \tag{14.7}$$

depending on a parameter t such that all functions are differentiable at $t = t_0$ and an optimal tableau for $t = t_0$

$$\begin{array}{c} \begin{array}{cc} \bar{x} & 1 \end{array} \\ \begin{bmatrix} \bar{A} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \end{array} \begin{array}{l} \\ = \bar{u} \\ \to \min \end{array} \tag{14.8}$$

has all entries in $\bar{b}$ and $\bar{c}$ positive. Then we show that the linear program has exactly one optimal solution $x = x(t)$ for every t sufficiently close to $t = t_0$, and we compute the derivative $z'(t_0)$ of the optimal value $z(t) = c(t)x(t)^T$ at $t = t_0$.

To simplify notations, we permute rows and columns in the tableaux to arrange that $x = [x_1, x_2]$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\bar{x} = [u_1, x_2]$, $\bar{u} = \begin{bmatrix} x_1^T \\ u_2 \end{bmatrix}$. The row x_2 consists of all variables that stay on the top, and it could be vacuous. The row x_1 consists of variables on top that go to the right margin at the optimal tableau, and it could be vacuous too, in which case the answer is trivial: $z'(t_0) = d'(t_0)$.

We rewrite the initial matrix (14.7) and the optimal tableau (14.8) accordingly:

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & 1 \end{array} \\ \begin{bmatrix} \alpha(t) & \beta(t) & b_1(t) \\ \gamma(t) & \delta(t) & b_2(t) \\ c_1(t) & c_2(t) & d(t) \end{bmatrix} \end{array} \begin{array}{l} \\ = u_1 \\ = u_2 \\ \to \min, \end{array} \tag{14.9}$$

$$\begin{array}{c} \begin{array}{ccc} u_1^T & x_2 & 1 \end{array} \\ \begin{bmatrix} * & * & \bar{b}_1 \\ * & * & * \\ \bar{c}_1 & * & \bar{d} \end{bmatrix} \end{array} \begin{array}{l} \\ = x_1^T \\ = u_2 \\ \to \min\,. \end{array} \tag{14.10}$$

The pivot steps, taking the tableau (14.9) with $t = t_0$ to (14.10), take the parametric tableau (14.9) to

$$\begin{array}{cccc} x_1 & x_2 & 1 & \\ \left[\begin{array}{c} \alpha(t)^{-1} \\ \gamma(t)\alpha(t)^{-1} \\ c_1(t)\alpha(t)^{-1} \end{array}\right. & \begin{array}{c} -\alpha^{-1}\beta(t) \\ * \\ * \end{array} & \left.\begin{array}{c} -\alpha(t)^{-1}b_1(t) \\ b_2(t) - \gamma(t)\alpha(t)^{-1}b_1(t) \\ d(t) - c_1(t)\alpha(t)^{-1}b_1(t) \end{array}\right] & \begin{array}{l} = u_1 \\ = u_2 \\ \to \min \end{array} \end{array} \tag{14.11}$$

(see Remark 8.11). Since all functions in (14.9) are continuous at $t = t_0$, we have the following. For all t sufficiently close to t_0 : the matrix $\alpha(t)$ is invertible [because $\alpha(t_0)$ is invertible]; the last column in (14.10) without the last entry is strictly positive (because $\bar{b} > 0$); the last row in (14.11) without the last entry is strictly positive (because $\bar{c} > 0$). So the tableau (14.11) is optimal and its basic solution is the only optimal solution (for t close to t_0). The last entry in the last row is the optimal value. Its derivative at $t = t_0$ is

$$z'(t_0) = d'(t) - c_1'(t_0)\bar{b}_1 - \bar{c}_1 b_1'(t_0) + \bar{c}_1\alpha'(t_0)\bar{b}_1.$$

In the case when $A(t)$ is constant, the last term drops. When both $A(t)$ and $c(t)$ are constant and both $b(t)$ and $d(t)$ are affine or both $A(t)$ and $b(t)$ are constant and both $c(t)$ and $d(t)$ are affine, the optimal value is an affine function of t for t close to t_0. ■

What happens when our optimal tableau has zero values for some basic variables (such tableaux are called *degenerate*)? In this case the slopes may depend on direction in the change of parameters.

Example 14.12. This example is adapted from Section 4-10 of *Linear programming and economic analysis,* by Dorfman, Samuelson and Solow (McGraw-Hill, 1958).

A chemical firm processes a certain raw material by the use of two major types of equipment, called stills and retorts. Four different production processes are available to the firm. If Process 1 is used to treat 100 tons of the raw material, it will utilize 7% of the weekly capacity of the stills and 3% of the weekly capacity of the retorts. The value of the product and the costs vary with the process used. If 100 tons are treated by Process 1, the net profit to the firm is \$60. The firm plans to process 1500 tons of raw material weekly. Obviously, the company wants to *maximize* the profit it will accrue by processing 1500 tons of raw material weekly, using an *optimal* combination of all four processes. The following table gives the pertinent information for all four processes:

Production processes	(1)	(2)	(3)	(4)	Available
Raw material (tons/week)	100	100	100	100	1500
Still capacity (%)	7	5	3	2	100
Retort capacity (%)	3	5	10	15	100
Profit ($/week)	60	60	90	90	

We can write these data in row and column standard tableaux, where x_i is the level (intensity) of Process i ($x_1 = 1$ means that a hundred tons of the raw material is treated by Process 1), and y_j is a slack variable. Here is the standard column tableau form of this linear program

$$\begin{array}{c} -x_1 \\ -x_2 \\ -x_3 \\ -x_4 \\ 1 \\ \\ \end{array} \begin{array}{c} \left[\begin{array}{cccc} 100 & 7 & 3 & -60 \\ 100 & 5 & 5 & -60 \\ 100 & 3 & 10 & -90 \\ 100 & 2 & 15 & -90 \\ 1500 & 100 & 100 & 0 \end{array}\right] \\ \begin{array}{cccc} = y_1 & = y_2 & = y_3 & = f \end{array} \to \max . \end{array} \qquad \begin{array}{c} x_1, x_2 \geq 0, \\ x_3, x_4 \geq 0; \\ y_1, y_2, y_3 \geq 0 \end{array}$$

We also include the standard row tableau, on which we will pivot:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array}$$
$$\left[\begin{array}{ccccc} -100 & -100 & -100 & -100 & 1500 \\ -7 & -5 & -3 & -2 & 100 \\ -3 & -5 & -10 & -15 & 100 \\ -60 & -60 & -90 & -90 & 0 \end{array}\right] \begin{array}{l} = y_1 \\ = y_2 \\ = y_3 \\ = -f \to \min. \end{array}$$

Let us work with the row form. It is (row) feasible, since the first three entries in the last column, 1500, 100, and 100, are ≥ 0. Therefore, we go to Phase 2 of the simplex method. Applying pivoting twice, we obtain an optimal tableau:

$$\begin{array}{ccccc} y_1 & x_2 & y_3 & x_4 & 1 \end{array}$$
$$\left[\begin{array}{ccccc} -1/70 & -5/7 & 1/7 & 5/7 & 50/7 \\ 61/700 & 6/7 & -4/7 & -13/7 & 185/7 \\ 3/700 & -2/7 & -1/7 & -12/7 & 55/7 \\ 33/70 & 60/7 & 30/7 & 150/7 & -7950/7 \end{array}\right] \begin{array}{l} = x_1 \\ = y_2 \\ = x_3 \\ = -f \to \min. \end{array}$$

So the (unique) answer is that maximal profit $f = \$7950/7$ per week (approximately \$1135.71 per week) at $x_1 = 50/7,\ x_2 = 0,\ x_3 = 55/7,\ x_4 = 0$. We used completely the raw material available ($y_1 = 0$) and the retorts ($y_3 = 0$), but the stills are underused.

Can we make more profit by buying and using more raw material? The answer depends on comparison of the price at which we can buy extra material and the *shadow price*, the change in the optimal value of the linear program when we replace 1500 by 1501 or 1499. So we consider the linear program with 1500 replaced by $1500 + \varepsilon$ as a function of the parameter ε. Here is the perturbed problem in a standard row tableau:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array}$$
$$\begin{bmatrix} -100 & -100 & -100 & -100 & 1500+\varepsilon \\ -7 & -5 & -3 & -2 & 100 \\ -3 & -5 & -10 & -15 & 100 \\ -60 & -60 & -90 & -90 & 0 \end{bmatrix} \begin{array}{l} = y_1 \\ = y_2 \\ = y_3 \\ = -f \to \min. \end{array}$$

Pivoting twice, we obtain an optimal tableau:

$$\begin{array}{ccccc} y_1 & x_2 & y_3 & x_4 & 1 \end{array}$$
$$\begin{bmatrix} -1/70 & -5/7 & 1/7 & 5/7 & 50/7+\varepsilon/70 \\ 61/700 & 6/7 & -4/7 & -13/7 & 185/7 - 61\varepsilon/700 \\ 3/700 & -2/7 & -1/7 & -12/7 & 55/7 - 3\varepsilon/700 \\ 33/70 & 60/7 & 30/7 & 150/7 & -7950/7 - 33\varepsilon/70 \end{bmatrix} \begin{array}{l} = x_1 \\ = y_2 \\ = x_3 \\ \to \min. \end{array}$$

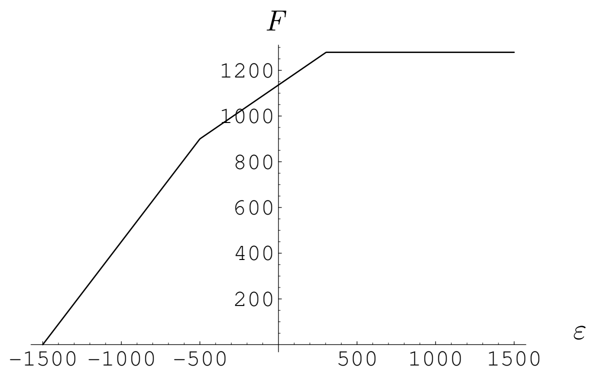
Here is how we obtain the answer (for small ε) *without* pivoting. We relax the condition $y_1 \ge 0$ replacing it by $y_1 \ge -\varepsilon$. From the optimal tableau, we see that the optimal value for f in the perturbed (relaxed) problem is $7950/7 + 33\varepsilon/70$ for small positive ε.

Note that in the optimal tableau of the perturbed problem, the last column is a perturbation of the last column of the original problem; that is, the difference between these two columns is given by a column matrix multiplied by ε. Moreover, this column matrix is precisely the negative of the first column of the original problem.

The answer to our question lies in the shadow price (i.e., the coefficient $33/70$ of ε in the optimal value of the perturbed problem; i.e., the slope of the optimal value as a function of the parameter). If the price of the raw material is less than the shadow price, then buy more raw material and increase the profit. Otherwise, do not buy any more raw material.

An economic interpretation of the slope, depending on the situation, can be the shadow price (i.e., the maximal price we are willing to pay for a small additional amount of a resource) or the marginal cost (i.e., the additional cost needed to produce a small additional amount of a product while we are minimizing the total cost).

Let us now consider the previous problem with $1500 + \varepsilon$ instead of 1500, where ε ranges over all real numbers (before we were interested only in small ε). This is an example of *parametric linear programming.* Note that ε is not a variable under our control, but a *parameter.* The optimal value (the maximal profit in our problem) is a function $F(\varepsilon)$ of the parameter. When $\varepsilon \leq -1500$, there are no feasible solutions. We do need the raw material to get any profit, so $F(-1500) = 0$. We have computed above $F(0) = 7950/7$. Here is a plot of $F(\varepsilon)$:



The graph of the function $F(\varepsilon)$ consists of line segments (such functions are often called piecewise linear; see the next definition). The function is nondecreasing because we are not required to use all resources available. The slope of the function is the marginal cost of the raw material. The slope changes at the *break points.* The slope decreases (such functions are called concave, see Definition 14.14 below)—the *law of diminishing returns.*

The optimal value of any linear program as a function of parameters in data has similar properties. To describe those properties we give a few definitions.

Definition 14.13. A function $f(t)$ of k variables $t = [t_1, \ldots, t_k]^T$ defined on a convex set P is called piecewise affine if the set P is the union of finitely many convex subsets such that $f(t)$ is an affine function on every subset. ■

The term *piecewise linear* is often used instead of *piecewise affine.* In most common case $k = 1$ the graph of a piecewise affine function is made from finitely many line segments. The convex sets on a line are easy to list: the empty set, points, intervals, rays, the whole line.

Remark. The following quote of Hermann Weyl (1885–1955) may help you to grasp the concept of piecewise linear function (The Mathematical Way of Thinking, an address given at the Bicentennial Conference at the University of Pennsylvania, 1940):

> Our federal income tax law defines the tax y to be paid in terms of the income x; it does so in a clumsy enough way by pasting several linear functions together, each valid in another interval or bracket of income. An archeologist who, five thousand years from now, shall unearth some of our income tax returns together with relics of engineering works and mathematical books, will probably date them a couple of centuries earlier, certainly before Galileo and Vieta.

Definition 14.14. A function $f(t)$ of k variables $t = [t_1, \ldots, t_k]^T$ defined on a convex set P is called convex if the set of points above its graph is convex. ∎

Sometimes the terms *concave upward* or *convex downward* are used instead of *convex*. We call a function f *concave* if the function $-f$ is convex (i.e., the set of points below the graph of f is convex).

There are several equivalent definitions of a convex function. For example, a function (defined on a convex set) is convex if and only if it is the maximum of a family of affine functions. In the case of a finite family, the maximum is a piecewise affine convex function.

Now we state a general result of parametric programming.

Theorem 14.15. Consider a linear program $cx \to \min, Ax \le b, x \ge 0$ in canonical form. Assume that all entries of either the row c or the column b are affine functions of k parameters $t_1, \ldots, t_k$. Consider the set P of values $t = [t_1, \ldots, t_k]$ of the parameter for which the linear program has an optimal solution, and let $f(t)$ be the optimal value. Then P is a convex set, and, when parameters are in c (resp., in b), $f(t)$ is the minimum (resp., maximum) of a finite set of affine functions on P. So $f(t)$ is a piecewise affine and concave (resp., convex) function. If b is a nondecreasing function of t or c is a nonincreasing function of t, then $f(t)$ is a nondecreasing function of t.

Proof. If P is empty, then we have nothing to prove so let us assume that our program has an optimal solution for at least one value of t.

Using the duality theorem, we see that it suffices to consider the case when the parameter is in the objective function $c = c' + c''t$.

Then the feasible region S is independent of t. To prove convexity of P, consider two numbers t', t'' in P and their convex combination $t = at' + (1-a)t''$. We have to show that t is in P, i.e., the function $c(t)x$ is bounded from below on S. Since $c(t)$ is an affine function of t, $c(t) = ac(t') + (1-a)c(t'')$. Since $c(t')x, c(t'')x \geq C$ for all x in S for a number C, we have $c(t)x = ac(t')x + (1-a)c(t'')x \geq aC + (1-a)C = C$ for all x in S; hence $c(t)x$ is bounded from below on S.

Each tableau for our LP has parameters only in the last row, and every entry in the last row is an affine function of parameters. This is true for the initial tableau, and pivot steps preserve this property. The total number of these tableaux is finite (it was bounded in §10).

Now we consider the subset of tableaux that are optimal for at least one value of (vector) parameter t (the value could be different for different tableaux) and the last entries in the last rows in these tableaux. For each t all these tableaux are feasible and at least one is optimal. So the optimal value $f(t)$ is the minimum over these last entries.

Finally, it is obvious that if all coefficients of the objective function increase or stay the same, then the minimal value $f(t)$ cannot improve. ■

Objective functions with parameters appear in *goal programming* when we want to combine several objectives or goals into one objective function. A way to do this is to take a linear combination of several objectives we want to minimize with nonnegative coefficients. The choice of coefficients (the weights) could be controversial, but they could be considered as parameters. If the functions we combine are affine, then the resulting objective function is also affine, and its coefficients are affine functions of parameters. So Theorem 14.15 can be applied.

Problem 14.16. Solve $P = 3x + 4y \to \max, 2x + 2y \leq 200, x + 3y \leq t; x, y \geq 0,$ t a given number.

Solution. This problem can be solved graphically (see §3). Here is an outline. When $t < 0$, the feasible region is empty, so the problem is infeasible. When $0 \leq t \leq 100$, the first constraint is redundant, the feasible region is a triangle, and $\max = 3t$ at $x = t, y = 0$. When $100 \leq t \leq 300$, the optimal solution is the intersection of the lines corresponding to the first two constraints: $2x + 2y = 200, x + 3y = t$; hence $x = 150 - t/2, y = t/2 - 50$, and $\max = 250 + t/2$. When $t \geq 300$, the second constraint is redundant, the feasible region is a triangle, and max $= 400$ at $x = 0, y = 100$.

Now we will solve the problem by the simplex method. We introduce two slack variables $u = 100 - x - y \geq 0$ and $v = t - x - 3y \geq 0$ and write the problem in a standard row tableau:

$$\begin{array}{c} \begin{array}{ccc} x & y & 1 \end{array} \\ \left[\begin{array}{ccc} -1 & -1 & 100 \\ -1 & -3 & t \\ -3 & -4 & 0 \end{array}\right] \begin{array}{l} = u \\ = v \\ = -P \quad \to \min. \end{array} \end{array} \tag{14.17}$$

When $t < 0$, the v-row is bad, so the problem is infeasible. Assume now that $t > 0$. The tableau is feasible, so we go to Phase 2. We choose the y-column as the pivot column.

If $t \geq 300$, then we pivot on -1 and obtain

$$\begin{array}{c} \begin{array}{ccc} x & u & 1 \end{array} \\ \left[\begin{array}{ccc} -1 & -1 & 100 \\ 2 & 3 & t - 300 \\ 1 & 4 & -400 \end{array}\right] \begin{array}{l} = y \\ = v \\ = -P \quad \to \min. \end{array} \end{array} \tag{14.18}$$

which is an optimal tableau; hence max $= 400$ at $x = 0, y = 100$.

Assume now that $0 < t < 300$. Then according to the simplex method we pivot the tableau (14.17) on -3 and obtain

$$\begin{array}{c} \begin{array}{ccc} x & v & 1 \end{array} \\ \left[\begin{array}{ccc} -2/3 & 1/3 & 100 - t/3 \\ -1/3 & -1/3 & t/3 \\ -5/3 & 4/3 & -4t/3 \end{array}\right] \begin{array}{l} = u \\ = y \\ = -P \quad \to \min. \end{array} \end{array} \tag{14.20}$$

Now the x-column is the pivot column. We have to compare $(100 - t/3)/(-2/3)$ and $(t/3)/(-1/3)$ to choose a pivot entry. When $0 < t \leq 100$, we pivot on -1/3 and obtain

$$\begin{array}{c} \begin{array}{ccc} y & v & 1 \end{array} \\ \left[\begin{array}{ccc} 2 & 1 & 100 - t \\ -3 & -1 & t \\ 5 & 3 & -3t \end{array}\right] \begin{array}{l} = u \\ = x \\ = -P \quad \to \min. \end{array} \end{array} \tag{14.21}$$

hence max $= 3t$ at $x = t, y = 0$.

When $100 \leq t \leq 300$, we pivot (14.20) on $-2/3$ and obtain

$$\begin{array}{c} \begin{array}{ccc} u & v & 1 \end{array} \\ \left[\begin{array}{ccc} -3/2 & 1/2 & 150 - t/2 \\ 1/2 & -1 & t/2 - 50 \\ 5/2 & 1/2 & -t/2 - 250 \end{array}\right] \begin{array}{l} = x \\ = y \\ = -P \quad \to \min; \end{array} \end{array} \tag{14.22}$$

hence max $= t/2 + 250$ at $x = 150 - t/2, y = t/2 - 50$.

Note that the slope is decreasing (the law of diminishing return):

interval	$0 \le t \le 100$	$100 \le t \le 300$	$300 \le t$
slope	3	1/2	0.

The last tableau is optimal when $t = 200$ (the case in a film on linear programming), and the slope is the last entry in the v-column. If we start to change the first limit 200 (instead of the second limit 200), then the slope would be 5/4.

Exercises

1–4. Solve the following linear programs, where all the variables $a, b, c, d, e, f, g, h, i, j$ and k are required to be nonnegative. *Hint.* The row and column problem in a nonstandard tableau need not be dual to each other.

1.
$$\begin{array}{c} \\ g \\ h \\ -1 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & -3 \\ 0 & 2 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccc} =i & =j & =k & =u \end{array} \end{array} \begin{array}{l} \\ = d \\ = c \\ = w \ \to \min \\ \to \quad \min . \end{array}$$

2.
$$\begin{array}{c} \\ g \\ h \\ -1 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & 1 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} =i & =j & =k & =u \end{array} \end{array} \begin{array}{l} \\ = d \\ = c \\ = w \ \to \min \\ \to \quad \min . \end{array}$$

3

$$\begin{array}{cccc} a & a & a & -1 \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} = d \\ = e \\ = w \to \min. \end{array}$$

4.

$$\begin{array}{ccc} a & b & -1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1+\varepsilon \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{array}{l} = c \\ = d \\ = w \to \max, \end{array}$$

where ε is a given number.

§15. More on Duality

The duality theorem has many facets and interpretations, and it was published in different forms by many authors including Fourier (1826), Gordan (1873), Minkowski (1896), and Farkas (1901).

In Theorem 6.16, we gave a version of duality for systems of linear equations. Here is another form of this: The system of linear equations $Ax = b$ has a solution if and only if $yb = 0$ for any row matrix y such that $yA = 0$. Here is how Fredholm (1903) stated this.

Exactly one of the following two systems is feasible:

$$\text{(a) } Ax = b; \quad \text{(b) } yA = 0, yb > 0.$$

We are going to state a version of duality for systems of linear inequalities. But first we show a way to deduce the dual problem from the primal problem. The main idea is to find a lower bound for the optimal value of the primal problem using the linear constraints of the primal problem. Namely, we combine linearly given constraints to obtain the lower bound. The dual program turns out to be the following: Maximize this lower bound.

Let us start with a simple example. Suppose that we want to solve the linear program:

$$\begin{cases} \text{Minimize} & x + y \\ \text{subject to} & x + y \geq 2. \end{cases}$$

It is clear that $\min\,(x + y) = 2$.

Here is a more complicated example:

$$\begin{cases} \text{Minimize} & 4x + 5y \\ \text{subject to} & x + 3y \geq 2, \\ & 2x - y \geq 3. \end{cases}$$

If we multiply the first constraint by 2 and add the result to the second constraint, then we obtain that $4x + 5y \geq 7$. This constraint, which is a linear combination of given constraints with positive coefficients, gives a low bound for the objective function: $\min(4x+5y) \geq 7$ (under our conditions). Moreover, it is easy to see in this example that this lower bound can be reached; that is, it is sharp. Thus,

$$\min(4x + 5y) = 7.$$

Let us try this approach to a bigger LP, given by the following standard row tableau:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & -2 & 0 & 4 & -1 \\ 0 & 2 & -3 & 1 & 0 \\ 1 & 2 & 3 & -4 & 5 \end{array}\right] \end{array} \begin{array}{l} \\ = u_1 \geq 0 \\ = u_2 \geq 0 \\ = u_3 \geq 0 \\ = z \to \min, \end{array} \qquad \text{all } \; x_i \geq 0$$

or, using matrices,

$$\begin{array}{c} \begin{array}{cc} x & 1 \end{array} \\ \left[\begin{array}{cc} A & b \\ c & d \end{array}\right] \end{array} \begin{array}{l} \\ = u \geq 0 \\ = z \to \min, \end{array} \qquad x \geq 0$$

or, using matrix multiplication,

$$Ax^T \geq -b, \; x \geq 0, \; cx^T + d \to \min.$$

If y is a column matrix such that $y \geq 0$ and $y^T \cdot A \leq c$, then we obtain that $cx^T \geq u^T \cdot Ax^T \geq -y^T \cdot b$. Thus, we obtain a lower bound

$$z = cx^T + d \geq -y^T \cdot b + d$$

for the objective function z using a linear combination of given constraints $Ax^T \geq -b$ with nonnegative coefficients y.

Next we try to pick y as before such that the bound is as sharp as possible:

$$-u^T \cdot b + d \to \max, \; y \geq 0, \; y^T \cdot A \leq c.$$

This is the dual problem, which can be written as the column problem in the same standard tableau:

$$\begin{array}{cc} & \begin{array}{cc} x & 1 \end{array} \\ \begin{array}{c} -y \\ 1 \end{array} & \left[\begin{array}{cc} A & b \\ c & d \end{array}\right] \\ & \begin{array}{cc} = v & = w \end{array} \end{array} \begin{array}{l} \\ = u \\ = z \to \min \\ \to \max. \end{array} \qquad \begin{array}{l} x \geq 0, u \geq 0 \\ y \geq 0, v \geq 0 \end{array}$$

In fact, passing to matrices made our computation easy, and they work for the LP given by an arbitrary standard row tableau.

Finding $y \geq 0$ such that $y^T \cdot A \leq c$ is equivalent to writing the linear function cx as a linear combination $y^T Ax + ux$ of the left-hand sides Ax, x of all given constraints $Ax \geq -b, x \geq 0$ with nonnegative coefficients y, u, where u is the row of basic variables for the dual (column) program.

So how is the duality theorem related to our situation? There is a possibility that our problem is unbounded. Then obviously we cannot get any bounds for z by any method, so the dual problem must be infeasible, which gives us a part of the theorem. There is a possibility that our problem is infeasible. Then the theorem says that the dual problem is either unbounded or infeasible. In terms of linear combinations, this means that either we can get arbitrary good bounds or no bound can be obtained as a linear combination. Finally, it may happen that our problem has an optimal solution. Then the theorem says that the optimal value is the best bound that can be obtained by linear combinations.

Now we ask ourselves for which numbers e the constraint $cx \geq d'$ follows from the given constraints $Ax \geq -b, x \geq 0$. In other words (see §4), does every feasible solution for the system satisfy the constraint as well? It is clear that the answer depends on the relation between min(cx) over the feasible region of the system and the number d'. If $\min(cx) < d'$ (including the case when $min = -\infty$, i.e., the program is unbounded), then the answer is no. If $\min(cx) \geq e$ (including the case when $\min = \infty$, i.e., the program is infeasible), then the answer is no.

Therefore, we can easily get our version of the duality theorem from the following version of the duality theorem.

Theorem 15.1. Given any system of inequalities $A'x' \geq b'$ and another inequality $c'x' \geq d'$ that follows from the system, then either the inequality is a linear combination of the constraints in the system together with the constraint $0 \geq -1$ with nonnegative coefficients, or the system is infeasible and the constraint $0 \geq 1$ is a linear combination of the constraints of the system. ■

We put primes into the condition of the theorem because the data are not the same as in the tableau. To apply the theorem, we take $A' = \begin{bmatrix} A \\ 1_n \end{bmatrix}$, n the number of variables in x, $x' = x$, $b' = -b$, and so on.

Theorem 15.1 can be obtained easily from our duality theorem using the standard trick $x = x' - x''; x', x'' \geq 0$ to write data in a standard row tableau.

Now we state a version of Theorem 6.16 for systems of inequalities. Suppose we are given a system of inequalities $Ax \geq b$ and another inequality $cx \geq d$ of the same type $\geq$. Does the latter constraint follow from the system? In other words (see §4), does every feasible solution for the system satisfy the constraint as well?

It is clear that the answer depends on the relation between $\min(cx)$ over the feasible region of the system and the number d. If $\min(cx) < d$ (including the case when $\min = -\infty$, i.e., the program is unbounded), then the answer is *no*. If $\min(cx) \geq d$ (including the case when $\min = \infty$, i.e., the program is infeasible), then the answer is *yes*.

Remark 15.2. The duality theorem implies that any primal-dual pair of linear programs

$$cx + d \to \min,\ Ax \geq -b,\ x \geq 0; \qquad -yb + d \to \max,\ yA \leq c,\ y \geq 0$$

can be written as a system of linear constraints:

$$Ax \geq -b,\ x \geq 0,\ cx + yb = 0,\ y \geq 0,\ yA \leq c.$$

Every feasible solution $[x^T, y]$ to this system of linear constraints gives optimal solutions x and y for both programs, and the optimal solutions x and u for the optimization problems give a solution $[x^T, y]$ for this system of constraints. In this sense linear programming is about solving systems of linear constraints. Another way to put it is that finding an optimal solution for an LP can be reduced to Phase 1 for another LP. ■

When a linear program comes from a real-life situation, its dual also can be interpreted in real-life terms. We consider now Examples 2.1, 2.2, 2.3, 2.4 (in more general forms) and give economic interpretations of the dual problems.

Example 15.3. Consider the general diet problem (a generalization of Example 2.1):

$$Ax \geq b, x \geq 0, C = cx \to \min,$$

where m variables in x represent different foods and n constraints in $Ax \geq b$ represent ingredients. We want to satisfy given requirements b in ingredients using given foods at minimal cost C.

On the other hand, consider a warehouse that sells the ingredients at prices $y_1, \ldots, y_n \geq 0$. Its objective is to maximize the profit $P = yb$, matching the price for each food: $yA \leq c$.

We can write both problems in a standard tableau using slack variables $u = Ax - b \geq 0$ and $v = c - yA \geq 0$:

$$\begin{array}{rcl} & \begin{array}{cc} x^T & 1 \end{array} & \\ \begin{array}{r} -y^T \\ 1 \end{array} & \left[\begin{array}{cc} A & -b \\ c & 0 \end{array}\right] & \begin{array}{ll} = u & \qquad x \geq 0, u \geq 0 \\ = C \to \min & \qquad y \geq 0, v \geq 0 \end{array} \\ & \begin{array}{cc} = v & = P \end{array} & \to \quad \max. \end{array}$$

So these two problems are dual to each other. In particular, the simplex method solves both problems, and if both problems are feasible, then $\min(C) = \max(P)$. The shadow prices mentioned in §14 turn out to be the optimal prices for the ingredients in the dual problem, so they are called *dual prices* as well.

Another economic interpretation for the same mathematical problem is that the variables in x are intensities of different industrial processes, the constraints correspond to different products, with b being the federal order to be fulfilled (or demand to be satisfied); $C = cx$ is the total cost that you want to minimize using given processes and satisfying given production quotas. With this interpretation for the primal problem, here is an interpretation for the dual problem: A competitor, Ann, who lost the government contract, says that you can buy the products from her at her low prices $y \geq 0$, matching unit cost for every process you got (i.e., $yA \leq c$) and maximizing her profit yb.

With this interpretation, the optimal prices y are *marginal costs* for you (i.e., they answer to the question what is the additional cost to produce an additional unit of each product). In §14, we saw that the marginal costs decrease with increase in volume (in linear programming). ∎

Example 15.4. Consider the general mixing problem (a generalization of Example 2.1):

$$Ax = b, x \geq 0, C = cx \to \min,$$

where m variables in x represent different alloys and n constraints in $Ax \geq b$ represent elements. We want to satisfy given requirements b in elements using given alloys at minimal cost C.

On the other hand, consider a dealer who buys and sells the elements at prices $y_1, \dots, y_n$. The positive price means that the dealer sells, and negative price means that the dealer buys. The dealer's objective is to maximize the profit $P = yb$, matching the price for each alloy: $yA \le c$.

To write the problems in standard tableaux, we use the standard tricks and artificial variables:

$u' = Ax - b \ge 0$, $u'' = -Ax + b \ge 0$;
$v = c - yA \ge 0; y = y' - y''$, $y' \ge 0, y'' \ge 0$.

Now we manage to write both problems in the same standard tableau:

$$\begin{array}{cc} & \begin{array}{cc} x^T & 1 \end{array} \\ \begin{array}{c} -y'^T \\ -y''^T \\ 1 \end{array} & \left[\begin{array}{cc} A & -b \\ -A & b \\ c & 0 \end{array}\right] \begin{array}{l} = u' \\ = u'' \\ = C \to \min \end{array} \\ & \begin{array}{ccc} = v & = P & \to \max. \end{array} \end{array} \qquad \begin{array}{l} x \ge 0, u', u'' \ge 0 \\ y', y'' \ge 0, v \ge 0 \end{array}$$

Remark 15.5. Adding an arbitrary constant d to the linear objective C, we get an arbitrary LP in standard form, and replacing the last zero in the last row by d, we have this LP in a standard row tableau.

Example 15.6. Consider a generalization of the manufacturing problem in Example 2.3:

$$P = cx \to \max, Ax \le b, x \ge 0,$$

where the variables in x are the amounts of products, P is the profit (or revenue) you want to maximize, constraints $Ax \le b$ correspond to resources (e.g., labor of different types, clean water you use, pollutants you emit, scarce raw materials), and the given column b consists of amounts of resources you have. Then the dual problem

$$yb \to \min, yA \ge c, y \ge 0$$

admits the following interpretation. Your competitor, Bob, offers to buy you out at the following terms: You go out of business, and he buys all resources you have at price $y \ge 0$, matching your profit for every product you may want to produce, and he wants to minimize his cost.

Again Bob's optimal prices are your resource shadow prices by the duality theorem. The shadow price for a resource shows the increase in your profit per unit increase in the quantity b_0 of the resource available or decrease in the profit when the limit b_0 decreases by one unit. While changing b_0 we do not change the limits for the other resources and any other data for our program. There are only finitely many values of b_0 for which the downward and upward shadow prices are different. One of these values could be the borderline between the values of b_0 for which the corresponding constraint is binding or nonbinding (in the sense that dropping this constraint does not change the optimal value).

In §14, we saw that the shadow price of a resource cannot increase when supply b_0 of this resource increases (the law of diminishing returns).

Example 15.7. This transportation problem is similar to Example 2.4, even though we have a different geographic setting. There are warehouses in Bedford and Scranton. They can supply 220 and 280 units, respectively. The retail stores are in State College, Altoona, and Harrisburg. These need 170, 120, 210 units, respectively. The shipping cost table is:

	State College	**Altoona**	**Harrisburg**
Bedford	77	39	105
Scranton	150	186	122

The constraints are

$$x_{ij} \geq 0, \;\; i = 1, 2; \;\; j = 1, 2, 3 \tag{i}$$

$$\begin{cases} x_{11} + x_{12} + x_{13} \leq 220 \\ x_{21} + x_{22} + x_{23} \leq 280 \end{cases} \tag{ii}$$

and

$$\begin{cases} x_{11} + x_{21} \geq 170 \\ x_{12} + x_{22} \geq 120 \\ x_{13} + x_{23} \geq 210. \end{cases} \tag{iii}$$

Notice that in this LP the sum of units available in the warehouses is 500, which equals the amount of units needed by the retail stores; this was not the case in the previous example (the warehouses had 130 widgets available, whereas the retail stores needed a total of 100). By looking at the constraints in (*ii*), we obtain

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} \leq 500. \tag{iv}$$

On the other hand, (*iii*) yields

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} \geq 500. \qquad (v)$$

Combining the inequalities (*iv*) and (*v*), we obtain the equality

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = 500.$$

This equality means that the total supply equals the total demand, and it is called the *balance condition.* This condition allows us, if we wish so, to replace all the inequality signs in (*ii*) and (*iii*) by equality signs. The balance condition forces the slack variables to be zero for all feasible solutions, so we obtain an equivalent problem if we replace all $\leq$ and $\geq$ in (*ii*) and (*iii*) by equality signs.

This small problem can be solved easily by the simplex method. In the next chapter we will see that the simplex method works particularly well for transportation problems. Our goal now is to give an economic interpretation for the dual problem. First we use Example 15.7 to introduce potentials. To put the problem in a standard row tableau, we introduce slack variables. Then we write the dual variables in the same tableau.

$$\begin{array}{c} \\ -u_1 \\ -u_2 \\ -v_1 \\ -v_2 \\ -v_3 \\ 1 \\ \\ \\ \end{array}
\begin{array}{c} \begin{array}{ccccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & 1 \end{array} \\
\left[\begin{array}{ccccccc}
-1 & -1 & -1 & 0 & 0 & 0 & 220 \\
0 & 0 & 0 & -1 & -1 & -1 & 280 \\
1 & 0 & 0 & 1 & 0 & 0 & -170 \\
0 & 1 & 0 & 0 & 1 & 0 & -120 \\
0 & 0 & 1 & 0 & 0 & 1 & -210 \\
77 & 39 & 105 & 150 & 186 & 122 & 0
\end{array}\right] \\
\begin{array}{ccccccc} \| & \| & \| & \| & \| & \| & \downarrow \\ w_{11} & w_{12} & w_{13} & w_{21} & w_{22} & w_{23} & \max \end{array} \end{array}
\begin{array}{l} \\ = y_1 \\ = y_2 \\ = z_1 \\ = z_2 \\ = z_3 \\ = C \\ \\ \\ \end{array}$$

with $u_i, v_j, w_{ij}, x_{ij}, y_i, z_j \geq 0$.

The objective function to be maximized by the dual problem is

$$-220u_1 - 280u_2 + 170v_1 + 120v_2 + 210v_3 \qquad (15.8)$$

The control variables u_i, v_j of the dual problem are called *potentials*. While the potentials correspond to the constraints on each retail store and each warehouse (or to the corresponding slack variables), there are other variables w_{ij} in the dual problem that correspond to the decision variable x_{ij} of the primal problem.

They are the slack variables for the six dual constraints

$$w_{ij} = c_{ij} + u_i - v_j \geq 0 \ \forall \ i, j, \tag{15.9}$$

where $i = 1, 2, \ \ j = 1, 2, 3$, and c_{ij} are the entries of the cost matrix

$$c = \begin{bmatrix} 77 & 39 & 105 \\ 150 & 186 & 122 \end{bmatrix}.$$

So what is a possible meaning of the dual problem?

Imagine that you want to be a mover and suggest a simplified system of tariffs. Namely, you assign a "zone" $u_i \geq 0 \ i = 1, 2$ to each of the warehouses and a "zone" $v_j \geq 0 \ j = 1, 2$ to each of the retail stores. To beat competition, you want the constraints (15.9). Your profit is (15.8), and you want to maximize it.

Remark 15.10. Observe that the problem is invariant under the change

$$u_i \to u_i + t, v_j \to v_j + t$$

using any fixed value of t for all i, j. This allows us to ignore the conditions $u, v \geq 0$ if we wish so.

Exercises

1–4. Find whether the last equation in the system is redundant (i.e., follows from the others).

1. $\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 6x + 8y + 10z = 0 \end{cases}$ **2.** $\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 7x + 10y + 13z = 16 \end{cases}$

3. $x = 6, y = 5, z = 0, 2x - 8y + 3z = 7$

4. $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 6 \\ 6x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 = 1 \\ x_1 - x_2 + x_3 - x_4 + x_5 = 0 \end{cases}$

5–8. Find whether the last constraint in the system is redundant (i.e., follows from the others).

5. $\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 6x + 8y + 10z \geq 0 \end{cases}$ **6.** $\begin{cases} x + 2y + 3z \geq 4 \\ 5x + 6y + 7z \geq 8 \\ 7x + 10y + 13z \geq 16 \end{cases}$

7. $x = 6, y = 5, z = 0, 2x - 8y + 3z \leq 7$

8. $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 6 \\ 6x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 = 1 \\ x_1 - x_2 + x_3 - x_4 + x_5 \geq 0 \end{cases}$

9–14. Solve the linear program, where all $x_i \geq 0$. *Hint*: Solve the dual problem by graphical method.

9.

$$\begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & 1 \end{array} \\ \begin{bmatrix} 0 & 8 & -5 & 6 & 7 & 8 & 3 & 5 & 4 & -1 \\ -1 & 2 & -2 & 1 & 1 & 1 & 2 & 2 & 5 & 0 \end{bmatrix} \begin{array}{l} \geq 0 \\ \to \quad \min \end{array} \end{array}$$

10.

$$\begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & 1 \end{array} \\ \begin{bmatrix} 6 & 8 & 5 & 6 & 7 & 8 & 3 & 5 & 4 & -1 \\ 1 & 2 & 5 & 1 & 1 & 1 & 5 & 0 & 5 & 0 \end{bmatrix} \begin{array}{l} \geq 0 \\ \to \quad \min \end{array} \end{array}$$

11.

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & 1 \end{array} \\ \begin{bmatrix} 3 & 4 & 1 & 1 & 1 & 2 & 3 & -1 \\ 4 & 3 & 1 & 0 & 0 & 2 & 4 & -2 \\ 5 & 5 & 1.4 & 5 & 6 & 3 & 6 & 0 \end{bmatrix} \begin{array}{l} = x_8 \\ = x_9 \\ = z \quad \to \min \end{array} \end{array}$$

12.

$$\begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & 1 \end{array} \\ \begin{bmatrix} 6 & 4 & 1 & 10 & 5 & 2 & 8 & 7 & -1 \\ 19 & 9 & 10 & 1 & 9 & 10 & 6 & 7 & -2 \\ 20 & 10 & 10 & 10 & 10 & 11 & 10 & 10 & 0 \end{bmatrix} \begin{array}{l} = x_9 \\ = x_{10} \\ = z \quad \to \min \end{array} \end{array}$$

13.

$$\begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & 1 \end{array} \\ \begin{bmatrix} 6 & 8 & 5 & 6 & 7 & 8 & 3 & 5 & 4 & -1 \\ 29 & 29 & 14 & 14 & 13 & 13 & 4 & 8 & 3 & -2 \\ 31 & 32 & 15 & 15 & 15 & 15 & 5 & 20 & 5 & 0 \end{bmatrix} \begin{array}{l} \geq 0 \\ \geq 0 \\ \to \quad \min \end{array} \end{array}$$

14.

$$\begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & 1 \end{array} \\ \begin{bmatrix} 6 & 8 & 5 & 6 & -7 & 8 & 3 & 5 & 4 & -1 \\ -9 & 0 & -4 & 1 & 13 & 13 & 4 & 8 & 3 & -2 \\ -1 & 2 & -5 & 15 & 15 & 15 & 5 & 0 & 5 & 0 \end{bmatrix} \begin{array}{l} \geq 0 \\ \geq 0 \\ \to \quad \min \end{array} \end{array}$$