Partition Identities with Mixed Mock Modular Forms

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Abstract

There are countless partition identities of the type introduced by Basil Gordon. All the previously known ones were related to the infinite products usually modular forms. In this paper we identify a further class of partition functions of the Basil Gordon genre that are in fact eta-products times Hecke-Type series.

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1 Introduction

In 1962 Basil Gordon proved the following theorem:

Gordon's Theorem (analytic version)**.** *Let $B_{k,a}(n)$ denote the number of partitions of n of the form $f_1 1 + f_1 2 + \ldots + f_n n = n$, where $f_i + f_{i+1} \leq k-1$ and $f_1 \leq a-1$. Then*

$$\sum_{n=0}^{\infty} B_{k,a}(n)q^n = \frac{\sum\limits_{n=-\infty}^{\infty}(-1)^n q^{(k+\frac{1}{2})(n^2+n)+an}}{\prod_{n=1}^{\infty}(1-q^n)} \tag{1.1}$$

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To reduce this assertion to one purely involving partitions, one observes that the numerator on the right-hand side of (1.1) is equal to an infinite product (from Jacobi's Triple Product Identity [10, p.22]. As a result, Gordon's Theorem is most familiar in the form $A_{k,a}(n) = B_{k,a}(n)$, where $A_{k,a}(n)$ is the number of partitions of n into parts $\neq 0, \pm a \pmod{2k+1}$.

In the subsequent decade there were many papers devoted to proving theorems of a similar nature and deducing from them identities between quite dissimilar partition functions (cf. [1, 2, 3, 4, 5, 6, 19, 21]). For a general survey see [7].

As an example of the plethora of such results we note the following special case of Theorem 1 in [2, p. 426, $\lambda = 2$, $a = k$]:

Theorem A. *Let $A_k(n)$ denote the number of partitions of n that are either odd or $\not\equiv 0, \pm 2k \pmod{4k+2}$. Let $B(n)$ denote the number of partitions of n of the form $f_1 1 + f_1 2 + \ldots + f_n n = n$ where $\lfloor \frac{f_i}{2} \rfloor + \lfloor \frac{f_{i+1}}{2} \rfloor \leq k-1$. Then $A_k(n) = B_k(n)$.*

Recently in work on mock theta functions and, more generally, on mock modular forms, interest has returned to the identification of these more esoteric functions as generating functions for classes of partitions [14, 17, 18]. Indeed Bringmann, Lovejoy, and Mahlburg [20] have identified the partitions effectively generated by the universal mock theta function of Gordon and M^{c}Intosh [22].

Our object here is to provide a class of partitions closely related to those enumerated by $B(n)$ in Theorem A: however their generating function has the numerator in (1.1) replaced by the one of the Hecke-Type series involving an indefinite quadratic form.

Theorem 1. *Let $C_k(n)$ denote the number of partitions of n of the following form $f_1 1 + f_1 2 + \ldots + f_n n = n$ where (1) $0 \leq f_i \leq 2k+1$, (2) if $f_i < 2k$, then $\lfloor \frac{f_i}{2} \rfloor + \lfloor \frac{f_{i+1}}{2} \rfloor \leq k-1$, and (3) if $f_i = 2k$ or $2k+1$, then i is odd, say $2j+1$, each of $f_1, f_3, \ldots, f_{2j+1}$ is either $2k$ or $2k+1$, and each of $f_2, f_4, \ldots, f_{2j+2}$ is zero, then*

$$\sum_{n=0}^{\infty} C_k(n) q^n = \frac{1 + \sum_{j=1}^{\infty} \left(q^{(2k+3)j^2+j} s(j, q^2) - q^{(2k+3)j^2-j} s(j-1, q^2) \right)}{\prod_{n=1}^{\infty} (1-q^n)} \tag{1.2}$$

where

$$s(j,q) = \sum_{n=-j}^{j} (-1)^j q^{-j^2} \tag{1.3}$$

A more vivid interpretation of the partitions enumerated by $C_k(n)$ may be given by the ideas in [15] on partitions with initial repetitions and in [16] on partitions with early conditions. Namely, $C_k(n)$ is the number of partitions of n in which the first parts consist entirely of consecutive odd numbers starting from 1 appearing $2k$ or $2k+1$ times and the larger parts are required to appear fewer than $2k$ times and are subject to the conditions.

Corollary 1.

$$\sum_{n=0}^{\infty} C_1(n) q^n = f_0(q^2) \prod_{n=1}^{\infty} (1+q^n)$$

where $f_0(q)$ is one of Ramanujan's fifth order mock theta functions [12, p. 114]

$$f_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)} \tag{1.3}$$

In Section 2, we prove Theorem 1. In Section 3, we treat a couple of corollaries, and in the conclusion we discuss possible further work.

2 Proof of Theorem 1

The proof of Theorem 1 relies on three different sources. We require two very different results concerning

$$Q_{k,i}(x;q) = \frac{1}{(xq;q)_\infty} C_{k,i}(x;q), \tag{2.1}$$

where

$$C_{k,i}(x;q) = \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}((2k+1)n(n+1)-in)} \frac{(1-x^i q^{(2n+1)i})(xq;q)_n}{(q;q)_n} \tag{2.2}$$

and

$$(A;q)_n = (1-A)(1-A_q)\cdots(1-Aq^{n-1}) \tag{2.3}$$

First, by Lemma 1-3 of [1],

$$(-xq;q)_\infty Q_{k,k}(x^2;q^2) = \sum_{m,n\geq 0} b_{k,k}(m,n)x^m q^n, \tag{2.4}$$

where $b_{k,k}(m,n)$ is the number of partitions of n into exactly m parts subject to the condition that if the partitions in question are given by

$$f_1 1 + f_1 2 + \ldots + f_n n = n$$

then

$$\left\lfloor \frac{f_i}{2} \right\rfloor + \left\lfloor \frac{f_{i+1}}{2} \right\rfloor \leq k-1$$

Next we review the multiple series expansion of $Q_{k,k}(x;q)$ given by Theorem 1 in [9]

$$Q_{k,k}(x;q) = \sum_{\substack{n_{k-1}\geq n_{k-2} \\ \geq\cdots\geq n_1}} \frac{x^{n_1+n_2+\cdots+n_{k-1}} q^{n_1^2+n_2^2+\cdots+n_{k-1}^2}}{(q;q)_{n_{k-1}-n_{k-2}}(q;q)_{n_{k-2}-n_{k-3}}\cdots(q;q)_{n_2-n_1}} \tag{2.5}$$

In addition to the information about $Q_{k,k}(x;q)$, we need the Bailey chain machinery [11, sec. 4].

If $\{\alpha_n\}$,$\{\beta_n\}$ are two sequences (generally these are each rational functions in several variables) satisfying

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_n}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{2.6}$$

then we say $\{\alpha_n\}$,$\{\beta_n\}$ form a Bailey pair with respect to a. The iteration of Bailey's Lemma (weak form) implies [13, p.30, eq.(3.44)]

$$\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty} q^{kn^2}a^{kn}\alpha_n = \sum_{n_k\geq\cdots\geq n_1\geq 0} \frac{a^{n_1+\cdots+n_k} + q^{n_1^2+n_2^2+\cdots+n_k^2}\beta_{n_1}}{(q;q)_{n_k-n_{k-1}}(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}} \tag{2.7}$$

In [12] it is proved that if

$$\beta_n = \frac{1}{(-q;q)_n}$$

and

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0 \\ q^{n(3n+1)/2} s(n, q) - q^{n(3n-1)/2} s(n-1, q) & \text{if } n > 0 \end{cases} \tag{2.8}$$

(where $s(n, q)$ is given by (1.3)), then $\{\alpha_n\}$,$\{\beta_n\}$ forms a Bailey pair with respect to $a = 1$.

Consequently substituting the Bailey Pair from (2.8) into (2.7) replacing q by q^2 and setting $a = 1$, we see that

$$1 + \sum_{n=1}^{\infty} (q^{(2k+3)n^2+n} s(n, q^2) - q^{(2k+3)n^2-n} s(n-1, q^2)) \tag{2.9}$$

$$= (q^2; q^2)_\infty \sum_{n_k \geq \cdots \geq n_1} \frac{q^{2n_1^2+2n_2^2+\cdots+2n_k^2}}{(q^2, q^2)_{n_k-n_{k-1}} (q^2, q^2)_{n_{k-1}-n_{k-2}} \cdots (q^2, q^2)_{n_2-n_1} (-q^2, q^2)_{n_1}}$$

One further comment before we proceed with the actual proof of Theorem 1. We will be relying on the iteration method presented by N.J. Fine in [22, Ch.1,Sec 5,pp.3-4]. As Fine notes, this is an elementry method for solving first order non-linear difference equations. We shall refer to this fact again in the conclusion.

Proof of Theorem 1 We let

$$F_k(x; q) = \sum_{m,n \geq 0} c_k(m, n) x^m q^n, \tag{2.10}$$

where $c_k(m, n)$ is the number of those partitions enumerated by $C_k(n)$ that have exactly m parts.

We claim that

$$F_k(x; q) = (-xq; q)_\infty Q_{k,k}(x^2; q^2) + x^{2k} q^{2k} (1 + xq) F_k(xq^2; q) \tag{2.11}$$

To see this, we split the partitions generated by $F_k(x; q)$ into two classes: (1) those partitions in which each part appears $< 2k$ times, and (2) those partitions in which some part appears at least $2k$ times.

From the definition of $C_k(m, n)$ and $b_{k,k}(m, n)$, we see that the partitions in class (1) are enumerated by $b_{k,k}(m, n)$. Consequently the generating function for the partitions in class (1) is

$$(-xq; q)_\infty Q_{k,k}(x^2; q^2)$$

according to (2.4).

Concerning the partitions in class (2), we see that the definition of $C_k(n)$ requires that 1 appears either 2k times or 2k+1 times. Hence

$$(x^{2k}q^{2k} + x^{2k+1}q^{2k+1})$$

will be a factor of the generating function. In addition, all other parts must be ≥ 3 and indeed form a partition generated by

$$F_k(xq^2; q)$$

as is clear from the portion of the definition $C_k(n)$ that concerns partitions containing at least one part appearing at least $2k$ times.

In light of the fact that class (1) and class (2) exhaust all the partitions enumerated by $F_k(x; q)$ we see that (2.11) is proved.

We now iterate (2.11) to find that

$$\begin{aligned} F_k(x; q) = \sum_{j=0}^{N} & \frac{(-xq; q)_\infty x^{2jk} q^{2kj^2} Q_{k,k}(x^2q^{4j}; q^2)}{(-xq^2; q^2)_j} \\ & + x^{2(N+1)k} q^{2k(N+1)^2} F_k(xq^{2N+2}; q)(-xq; q^2)_{N+1} \end{aligned} \tag{2.12}$$

a result easily deduced from (2.11) by mathematical induction. Now let $N \to \infty$ in (2.12). This yields

$$F_k(x; q) = \sum_{j=0}^{\infty} \frac{(-xq; q)_\infty x^{2jk} q^{2kj^2} Q_{k,k}(x^2q^{4j}, q^2)}{(-xq^2, q^2)_j}$$

and hence,

$$\begin{aligned} \sum_{n=0}^{\infty} C_k(n)q^n &= F_k(1; q) \\ &= (-q; q)_\infty \sum_{j=0}^{\infty} \frac{q^{2kj^2} Q_{k,k}(q^{4j}, q^2)}{(-q^2; q^2)_j} \\ &= (-q; q)_\infty \sum_{j=0}^{\infty} \frac{q^{2kj^2}}{(-q^2; q^2)_j} \\ & \quad \times \sum_{n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{4j(n_1+\cdots+n_{k-1})+2(n_1^2+\cdots+n_{k-1}^2)}}{(q^2; q^2)_{n_{k-1}-n_{k-2}} \cdots (q^2; q^2)_{n_2-n_1} (q^2; q^2)_{n_1}} \end{aligned}$$

$$
\begin{aligned}
&= (-q;q)_\infty \sum_{j=0}^{\infty} q^{2j^2} \\
&\quad \times \sum_{n_{j-1}\ge\cdots\ge n_1\ge 0} \frac{q^{2((n_1+j)^2+(n_2+j)^2)+\cdots+(n_{k-1}+j)^2}}{(q^2;q^2)_{n_{k-1}-n_{k-2}}\cdots(q^2;q^2)_{n_2-n_1}(q^2;q^2)_{n_1}(-q^2;q^2)_j} \\
&= (-q;q)_\infty \sum_{n_{k-1}\ge n_{k-2}\ge\cdots\ge n_1\ge j} \frac{q^{2(n_1^2+n_2^2+\cdots+n_{k-1}^2+j^2)}}{(q^2;q^2)_{n_{k-1}-n_{k-2}}\cdots(q^2;q^2)_{n_1-j}(-q^2;q^2)_j}
\end{aligned}
$$

(by shifting each n_i to $n_i - j$)

$$
= (-q;q)_\infty \frac{1}{(q^2;q^2)_\infty}(1+\sum_{n=1}^{\infty}(q^{(2n+3)n^2+n}s(n,q^2) - q^{(2n+3)n^2-n}s(n-1,q^2)))
$$

(by (2.9))

$$
= \frac{1}{(q;q)_\infty}(1+\sum_{n=1}^{\infty}(q^{(2n+3)n^2+n}s(n,q^2) - q^{(2n+3)n^2-n}s(n-1,q^2))
$$

and Theorem 1 is proved.

3 Corollaries

First we note that Corollary 2 follows directly from Theorem 1 once we recall [12, p.143, fourth equation]

$$
f_0(q) = \frac{1}{(q;q)_\infty}(1+\sum_{n=1}^{\infty}(q^{n(5n+1)/2}s(n,q) - q^{n(5n-1)/2}s(n-1,q)
$$

So by (1.2)

$$
\begin{aligned}
\sum_{n=0}^{\infty} C_1(n)q^n &= \frac{(q^2;q^2)_\infty f_0(q^2)}{(q;q)_\infty} \\
&= (-q;q)_\infty f_0(q^2)
\end{aligned}
$$

Corollary 2.

$$
1+\sum_{n=1}^{\infty}(q^{3n^2+n}s(n,q^2) - q^{3n^2-n}s(n-1,q^2)) = (-q;-q)_\infty(q;q)_\infty
$$

Remark. This is essentially the degenerate $k=0$ case of Theorem 1 which does not lend itself to our partition interpretations; however we include it for completeness. Technically what is required here is to go back one stage in the Bailey chain [11, p.278,Sec.4].

Proof. From equation (5.15) of [12, p.124], we see that with

$$\begin{aligned} 1+\sum_{n=1}^{\infty}(q^{n(3n+1)}s(n,q^2)-q^{n(3n-1)}s(n-1,q^2)) &= (q^2;q^2)_\infty^2(q^2;q^4)_\infty \\ &= (-q;q^2)_\infty(q;q)_\infty(q^2;q^2)_\infty \\ &= (-q;-q)_\infty(q;q)_\infty \end{aligned}$$

□

4 Conclusion

We believe that N.J. Fine's [22, p.3,Sec 5] "elementary method" alluded to in our Section 2 should have much wider application than it previously found. It is quite conceivable that a careful analysis of how inhomogeneous q-difference equation (2.11) interact with (2.7) might lead to a large number of results analogous to Theorem 1.

The purely combinatorial aspects of this work have not been considered here. One would hope, at least, there might be purely bijective proofs of the two corollaries. This is especially appealing in light of the fact that one can rewrite Corollary 2 as follows

$$\begin{aligned} 1+\sum_{n=1}^{\infty}(q^{n(3n+1)/2}\sum_{j=-n}^{n}(-1)^j q^{-j^2}-q^{n(3n-1)/2}\sum_{j=-n+1}^{n-1}(-1)^j q^{-j^2}) \\ = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2} \end{aligned}$$

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