q-ORTHOGONAL POLYNOMIALS, ROGERS-RAMANUJAN IDENTITIES, AND MOCK THETA FUNCTIONS

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In honor of Professor A.A Karatsuba's 75th birthday

ABSTRACT. In this paper, we examine the role that q-orthogonal polynomials can play in the application of Bailey pairs. The use of specializations of q-orthogonal polynomials reveals new instances of mock theta functions.

1. INTRODUCTION

This paper arose from a careful examination of the following theorem taken from [5, Th. 18]:

Theorem 1.

$$\sum_{n\geq 0}\frac{q^{n^2}a^n(-yq;q)_n}{(q^2;q^2)_n} = \frac{1}{(aq;q)_\infty}\sum_{n\geq 0}\frac{(-1)^n a^n q^{2n^2}(a^2;q^2)_n(1-aq^{2n})}{(q^2;q^2)_n(1-a)}p_n(y;-\frac{a}{q},-1:q), \tag{1.1}$$

where

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}),$$
$$(A_1,A_2,\dots,A_r;q)_n = (A_1;q)_n(A_2;q)_n\cdots(A_r;q)_n,$$
$${}_{r+1}\phi_r\left(\begin{matrix}a_0,a_1,\dots,a_r;q,t\\ b_1,\dots,b_r\end{matrix}\right) = \sum_{n=0}^{\infty}\frac{(a_0,a_1,\dots,a_r;q)_n t^n}{(q,b_1,\dots,b_r;q)_n},$$

and

$$p(y;A,B:q) = {}_2\phi_1\left(\begin{matrix}q^{-n},ABq^{n+1};q,yq\\ Aq\end{matrix}\right), \tag{1.2}$$

the little q-Jacobi polynomials [17, p. 27].

This theorem appears in Section 12 of [5]. It was included there to show that a certain combinatorial generating function (the left side of (1.1)) could be expanded in a series that would reveal simultaneously the classical Rogers-Ramanujan identities and the Hecke-type expansion of the one of the fifth order mock theta functions [3, eq. (1.4)].

The thing that stands out in (1.1) is this: Identity (1.1) is a classic expansion of a function (of y) in a series of orthogonal polynomials (namely, $p_n(y;-\frac{a}{q},-1;q)$).

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In Section 2, we provide a proof of Theorem 1 in the spirit of orthogonal polynomial expansions.

In 1976 [7], Richard Askey and I considered expansions in series of little q-Jacobi polynomials. In particular, we proved the following fundamental expansion theorem:

Theorem 2 ([7, Th. 10]). *If $a_{k,n}$ is defined for all n with $0 \le k \le n$ by*

$$p_n(x;\gamma,\delta:q) = \sum_{k=0}^{n} a_{k,n} p_k(x;\alpha,\beta:q),$$

then

$$a_{k,n} = \frac{(-1)^k q^{k(k+1)/2}(\gamma\delta q^{n+1};q)_k (q^{-n};q)_k (\alpha q;q)_k}{(q;q)_k(\gamma q;q)_k(\alpha\beta q^{k+1};q)_k} \times {}_3\phi_2\left(\begin{matrix} q^{-n+k}, \gamma\delta q^{n+k+1}, \alpha q^{k+1}; q, q \\ \gamma q^{k+1}, \alpha\beta q^{2k+2} \end{matrix}\right)$$

As a corollary of Theorem 2, we proved the following celebrated theorem of G. N. Watson [19], which, incidentally, contains numerous Rogers-Ramanujan type identites as limiting cases.

Theorem 3. *For $n \ge 0$, a non-negative integer,*

$${}_8\phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}; q, \frac{a^2q^{n+2}}{bcde} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{n+1} \end{matrix}\right) = \frac{(aq, \frac{aq}{de};q)_n}{\left(\frac{aq}{d}, \frac{aq}{e};q\right)_n} {}_4\phi_3\left(\begin{matrix} \frac{aq}{bc}, d, e, q^{-n}; q, q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{deq^{-n}}{a} \end{matrix}\right)$$

In the early 1980's, the powerful iterative Bailey chain method [2] was discovered. With it one could not only prove Theorem 3 but also infinite families of such identities. As a result, the use of q-orthogonal polynomials in studying Rogers-Ramanujan type identities faded into disuse.

The upshot of this neglect is that a number of Bailey pairs of great significance in the study of mock theta functions [3], [4], [9], [10] had a purely empirical discovery and rather ad hoc proofs.

We shall phrase the main theorem of this paper in terms of classical, q-hypergeometric functions. However, what is really asserted is the expansion of the general, terminating, balanced ${}_5\phi_4$ in a series of Askey-Wilson polynomials. The power of this theorem in applications lies in the fact that the Askey-Wilson polynomials satisfy an elegant three-term recurrence relation as indicated [17, p. 185].

Theorem 4. *If $qabc = efg$, then*

(1.3)
$${}_5\phi_4\left(\begin{matrix} q^{-N}, \rho_1, \rho_2, b, c; q, q \\ \frac{\rho_1\rho_2 q^{-N}}{a}, e, f, g \end{matrix}\right) = \frac{\left(\frac{aq}{\rho_1}\right)_N \left(\frac{aq}{\rho_2}\right)_N}{(aq)_N \left(\frac{aq}{\rho_1\rho_2}\right)_N} \sum_{n=0}^{N} \frac{(\rho_1)_n(\rho_2)_n(q^{-N})_n(a)_n(1-aq^{2n})}{\left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n (aq^{N+1})_n (q)_n(1-a)} \times \left(\frac{aq^{N+1}}{\rho_1\rho_2}\right) P_n(a,b,c,e,f,g;q),$$

where

$$P_n = P_n(a,b,c,e,f,g;q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, aq^n, b, c; q, q \\ e, f, g \end{matrix}\right) \tag{1.4}$$

and

$$A_n P_{n+1} + B_n P_n + C_n P_{n-1} = 0. \tag{1.5}$$

with

$$A_n = aq^n(1-aq^n)(q^{1-n}-aq^n)(q^{-n}-e)(q^{-n}-f)(q^{-n}-g), \tag{1.6}$$
$$C_n = -q^{-n}(1-q^{-n})(q^{-n}-aq^{n+1})(e-aq^n)(f-aq^n)(g-aq^n), \tag{1.7}$$
$$B_n = a(q^{-n}-aq^{n+1})(q^{-n}-aq^n)(q^{-n+1}-aq^n)(1-b)(1-c) - A_n - C_n. \tag{1.8}$$

If we let N, ρ_1, and $\rho_2 \to \infty$ in Theorem 4, we find

Corollary 5.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a^n q^{n^2} (b)_n (c)_n}{(q)_n (e)_n (f)_n (g)_n} \\ = \frac{1}{(aq)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{(3n^2-n)/2} a^n (a)_n (1-aq^{2n})}{(q)_n (1-a)} P_n(a,b,c,e,f,g;q).\end{aligned} \tag{1.9}$$

As we shall see in Section 3, Theorem 4 is effectively a routine application of the strong form of Bailey's lemma [2, p. 270]. In the classical work of Bailey [13], [14], [15] and Slater, [18] the object was to find the summable instances of P_n, i.e. to reduce the three-term recurrence (1.5) to a two-term recurrence.

Our object is to find instances where the recurrence still has three terms, but the A_n, B_n and C_n can be sufficiently simplified to yield new representations of P_n as arise in the Hecke-type expansions related to mock theta functions.

Thus we find, for example, a new representation for Ramanujan's third order mock theta function, $\psi(q)$ [20, p. 62]

$$\psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{2n^2+n}(1-q^{6n+6}) \sum_{j=0}^{n} q^{-\binom{j+1}{2}}, \tag{1.10}$$

with a companion

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n^2}{(q)_{2n}} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{2n^2+n}(1-q^{6n+6}) \sum_{j=0}^{n} q^{-\binom{j+1}{2}}, \tag{1.11}$$

and two relatives

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2;q^2)_n}{(q)_{2n}} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \psi(-q), \tag{1.12}$$

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q^2;q^2)_n}{(q)_{2n+1}} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \big(1-\psi(-q)\big). \tag{1.13}$$

While $\psi(q)$ has been studied extensively with an alternative representation [20, p. 65], it would appear that (1.10)–(1.13) are new.

There are a number of mock theta results to be found. We present just three of the most appealing.

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2+n}\left(1-q^{6n+3}\right) \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \tag{1.14}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(-q;q)_{2n+1}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2+3n}\left(1-q^{2n+1}\right) \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \tag{1.15}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(q;q^2)_n}{(q^2;q^2)_n(-q)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2+2n}\left(1-q^{4n+2}\right) \sum_{j=-n}^{n} (-1)^j q^{-j^2}. \tag{1.16}$$

Of course, these identities may be estabished by the means used in [3]. Indeed, these results can be obtained by application of the Bailey pairs listed in [11, p. 75]. The point being emphasized here is the switch from transformations of q-hypergeometric series (cf. [3], [10] and [11]) to recurrence relations provided by the q-orthogonal polynomials. The recurrence approach will turn out to be essential in a subsequent work [6] where transformations of q-hypergeometric series are unavailable (at least currently). Furthermore, it is perfectly suited to the numerous computer algebra summation packages now available.

The mock theta results we have found arise from the special case of Corollary 5 when c and g are set to 0. In the conclusion we indicate some of the results arising from Corollary 5 in full generality.

2. q-Orthogonal polynomial expansions

Our object here is to provide a new proof of Theorem 1 to illustrate the parallel nature of q-orthogonal polynomial expansions and the strong form of Bailey's Lemma.

When considering the little q-Jacobi polynomials, we are dealing with a discrete measure

$$\frac{x^{\log_q \alpha}(xq)_\infty}{(\beta xq)_\infty}$$

at the discrete mass points q^i, $i = 0, 1, 2, \ldots$ where $0 < q < 1$. Thus it is convenient to phrase things in terms of the q-integral

$$\int_0^1 f(x)\, d_q x = \sum_{i=0}^{\infty} f(q^i)(q^i - q^{i+1}) = (1-q) \sum_{i=0}^{\infty} f(q^i) q^i.$$

Hence

$$\begin{aligned}
(2.1)\quad &\int_0^1 (x\beta q)_m p_n(x;\alpha,\beta:q)\frac{x^{\log_q \alpha}(xq)_\infty}{(\beta xq)_\infty}\,d_qx \\
&= \sum_{i=0}^{\infty}\frac{\alpha^i q^i (q^{i+1})_\infty}{(\beta q^{i+m+1})_\infty}\sum_{j=0}^{n}\frac{(q^{-n})_j(\alpha\beta q^{n+1})_j (q^{i+1})^j}{(q)_j(\alpha q)_j} \\
&= \frac{(q)_\infty}{(\beta q^{m+1})_\infty}\sum_{j=0}^{n}\frac{(q^{-n})_j(\alpha\beta q^{n+1})_j q^j}{(q)_j(\alpha q)_j}\sum_{i=0}^{\infty}\frac{(\beta q^{m+1})_i(\alpha q^j+1)^i}{(q)_i} \\
&= \frac{(q)_\infty}{(\beta q^{m+1})_\infty}\sum_{j=0}^{n}\frac{(q^{-n})_j(\alpha\beta q^{n+1})_j q^j}{(q)_j(\alpha q)_j}\frac{(\alpha\beta q^{m+j+2})_\infty}{(\alpha q^{j+1})_\infty} \\
&= \frac{(q)_\infty(\alpha\beta q^{m+2})_\infty}{(\beta q^{m+1})_\infty(\alpha q)_\infty}\sum_{j=0}^{n}\frac{(q^{-n})_j(\alpha\beta q^{n+1})_j q^j}{(q)_j(\alpha\beta q^{m+2})_j} \\
&= \frac{(q)_\infty(\alpha\beta q^{n+m+2})_\infty(\alpha\beta q^{n+1})^n(1-q^m)(1-q^{m-1})\cdots(1-q^{m-n+1})}{(\beta q^{m+1})_\infty(\alpha q)_\infty}
\end{aligned}$$

(by [17, p. 11, eq. (1.5.3)])

In a proof exactly like the evaluation given in (2.1), Askey and I gave the full orthogonality relation for the little q-Jacobi polynomials in Theorem 9 of [7]:

$$\begin{aligned}
(2.2)\quad &\int_0^1 p_n(x,\alpha,\beta:q)p_m(x;\alpha,\beta:q)\frac{x^{\log_q \alpha}(xq)_\infty}{(\beta xq)_\infty}\,d_qx \\
&= \begin{cases} 0 & \text{if } m\neq n \\ \frac{\alpha^n q^n (q)_\infty(\alpha\beta q^{n+1})_\infty (q)_n}{(\beta q^{n+1})_\infty(\alpha q)_\infty(\alpha q)_n(1-\alpha\beta q^{2n+1})}, & \text{if } m=n. \end{cases}
\end{aligned}$$

Thus we now envision Theorem 1 as the following orthogonal expansion

$$(2.3)\qquad \sum_{m=0}^{\infty}\frac{a^m q^{m^2}(-xq)_m}{(q^2;q^2)_m} = \sum_{m=0}^{\infty} C_m p_m(\alpha;-\frac{a}{q},-1;q).$$

The problem is to find each C_n, so we multiply both sides of (2.3) by $p_n(x;-\frac{a}{q},-1;q)$ and integrate over the mass distribution listed at the beginning of this section. For the left-hand side, we find by (2.1) that the result is

$$\begin{aligned}
(2.4)\quad &\sum_{m\geq n}\frac{a^m q^{m^2}}{(q^2;q^2)_n}\cdot\frac{a^n q^{n^2}(q)_m(aq^{m+n+1})_\infty(q)_\infty}{(q)_{m-n}(-q^{m+1})_\infty(-a)_\infty} \\
&= \frac{a^{2n}q^{2n^2}(q)_\infty(aq^{2n+1})_\infty}{(-a)_\infty(-q)_\infty}\sum_{m=0}^{\infty}\frac{a^m q^{m^2+2mn}}{(q)_m(q^{2n+1})_m} \\
&= \frac{a^{2n}q^{2n^2}(q)_\infty}{(-a)_\infty(-q)_\infty} \qquad \text{(by [1, p. 20, Cor. 2.6])}
\end{aligned}$$

On the right-hand side, the result by (2.2) is

$$\frac{C_n(-a)^n(q)_\infty(aq^n)_\infty(q)_n}{(-q^{n+1})_\infty(-a)_\infty(-a)_n(1-aq^{2n})}.$$

Consequently this last expression must be equal to the final expression in (2.4). Solving for C_n, we find that

$$C_n = \frac{(-1)^n a^n q^{2n^2}(a^2;q^2)_n(1-aq^{2n})}{(aq)_\infty (q^2;q^2)_n(1-a)},$$

which is the necessary expression for C_n to conclude the proof of Theorem 1. □

The utility of Theorems 1 and 4 lies in the fact that the classical q-orthogonal polynomials satisfy beautiful three term recurrence relations (e.g. (1.6)–(1.8)).

Thus in [5], several lemmas were constructed to show that for $n > 0$,

$$2p_n(-1;-\frac{1}{q},-1:q) = (-1)^n q^{\binom{n}{2}} \left(q^n \sum_{j=-n}^{n} (-1)^j q^{-j^2} - \sum_{j=-n+1}^{n-1} (-1)^j q^{-j^2} \right). \tag{2.5}$$

However, this assertion is a simple exercise in mathematical induction once we recall the three term recurrence for the little q-Jacobi polynomials [17, p. 167, eq's (7.3.6)–(7.3.8)] where $p_n := p_n(x;a,b:q)$.

$$xp_n = A_n(p_{n+1}-p_n) - C_n(p_n - p_{n-1}),$$

where

$$A_n = \frac{-(1-abq^{n+1})(1-aq^{n+1})q^n}{(1-abq^{2n+1})(1-abq^{2n+2})},$$

$$C_n = \frac{-a(1-q^n)(1-bq^n)q^n}{(1-abq^{2n})(1-abq^{2n+1})}.$$

Now with $a = -\frac{1}{q}$ and $b = -1$, we find

$$A_n = -C_{n+1} = \frac{-q^n}{1-q^{2n+1}},$$

and, as mentioned above, (2.5) becomes an easy application of mathematical induction.

3. The Proofs of Theorem 4 and Corollary 5

Theorem 4 follows from an immediate application of the Strong Bailey Lemma. Namely, we are given the Baily pair $(\alpha_n,\beta_n)_{n\geq 0}$ of sequences of rational functions related by

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q;q)_{n-j}(aq;q)_{n+j}}. \tag{3.1}$$

Then

$$\sum_{n=0}^{N} \beta_n \delta_n = \sum_{n=0}^{N} \alpha_n \gamma_n \tag{3.2}$$

where

$$\gamma_n = \frac{\left(\frac{aq}{\rho_1}\right)_N \left(\frac{aq}{\rho_2}\right)_N}{(aq)_N \left(\frac{aq}{\rho_1\rho_2}\right)_N} \frac{(\rho_1)_n(\rho_2)_n(q^{-N})_n \left(\frac{-aq}{\rho_1\rho_2}\right)^n q^{nN-\binom{n}{2}}}{\left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n (aq^{N+1})_n}, \tag{3.3}$$

and

$$\delta_n = \frac{(\rho_1)_n(\rho_2)_n(q^{-N})_n q^n}{\left(\frac{\rho_1\rho_2 q^{-N}}{a}\right)_n}. \tag{3.4}$$

If we set

$$\beta_n = \frac{(b)_n(c)_n}{(e)_n(f)_n(g)_n}, \tag{3.5}$$

then α_n is completely determined by (3.1).

Now it is possible to invert the relation (3.1), and, in fact, we find [2, p. 278]

$$\alpha_n = \frac{(a)_n(1-aq^{2n})(-1)^n(q^{\binom{n}{2}})}{(q)_n(1-a)}\sum_{j=0}^{n}(q^{-n})_j(aq^n)_j q^j\beta_j. \tag{3.6}$$

Substituting (3.5) into (3.6) and then inserting the resulting expressions into (3.2), we have (1.3). The assertion (1.5) now follows from the strong form of Bailey's lemma. □

Corollary 5 now follows if we let N, ρ_1, and ρ_2 all $\to\infty$ in Theorem 4. □

4. Big q-Jacobi Polynomials

While Theorem 4 is the most general result we shall consider it is useful to restate Corollary 5 in the case $c = g = 0$.

Corollary 6.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{a^n q^{n^2}(b)_n}{(q)_n(e)_n(f)_n} \\ = \frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(3n^2-n)/2}a^n(a)_n(1-aq^{2n})}{(q)_n(1-a)}\,{}_3\phi_2\left(\begin{matrix}q^{-n}, aq^n, b; q, q\\ e, f\end{matrix}\right),\end{aligned} \tag{4.1}$$

where

$$bp_n = \mathcal{A}_n p_{n+1} + \mathcal{B}_n p_n + \mathcal{C}_n p_{n-1}, \tag{4.2}$$

$$\mathcal{A}_n = \frac{(1-aq^n)(1-eq^n)(1-fq^n)}{(1-aq^{2n})(1-aq^{2n+1})}, \tag{4.3}$$

$$\begin{aligned}\mathcal{B}_n = q^{n-1}\Big(&a^2q^{2n+1} + afq^{2n+1} + aeq^{2n+1} + aefq^{2n} - aq^{n+2} - afq^{n+1} \\ &- aeq^{n+1} - aq^{n+1} - efq^{n+1} - afq^n - aeq^n - efq^n + aq + eq + fq + ef\Big) \\ &/\left((1-aq^{2n-1})(1-aq^{2n+1})\right).\end{aligned} \tag{4.4}$$

$$\mathcal{C}_n = \frac{-q^{n-1}(1-q^n)(e-aq^n)(f-aq^n)}{(1-aq^{2n-1})(1-aq^{2n})}, \tag{4.5}$$

and

$$p_n = {}_3\phi_2\left(\begin{matrix}q^{-n}, aq^n, b; q, q\\ e, f\end{matrix}\right). \tag{4.6}$$

We note that p_n is a big q-Jacobi polynomial in the variable b [8, p. 593].

5. $\psi(q)$ and Its Companions

We first prove (1.10). In Corollary 6, we set $a = 1$, $b = q$, $e = -f = q^{1/2}$. Consequently

$$\psi(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \frac{1}{(q)_\infty}\left(1 + \sum_{n=1}^{\infty} (-1)^n q^{(3n^2-n)/2}(1+q^n)\,{}_3\phi_2\left(\begin{matrix} q^{-n}, q^n, q; q, q \\ q^{1/2}, -q^{1/2} \end{matrix}\right)\right). \tag{5.1}$$

By (4.2)–(4.5), the ${}_3\phi_2$ in the right-hand sum (abbreviated as p_n) satisfies for $n > 1$

$$q(1+q^n)p_n = p_{n+1} + q^n p_{n-1}, \tag{5.2}$$

with $p_0 = 1$, $p_1 = q$. Now we define $\overline{p}_0 = 1$, $\overline{p}_1 = q$, and form $n > 1$

$$\overline{p}_n = q^{\binom{n+1}{2}} \sum_{j=0}^{n-1} q^{-\binom{j+1}{2}} - q^{\binom{n}{2}} \sum_{j=0}^{n-2} q^{-\binom{j+1}{2}}.$$

So

$$\overline{p}_n = q^{\binom{n+1}{2}} S_{n-1} - q^{\binom{n}{2}} S_{n-2},$$

where

$$S_n = \sum_{j=0}^{n} q^{-\binom{j+1}{2}}.$$

Thus

$$S_n - S_{n-1} = q^{-\binom{n+1}{2}}.$$

Consequently

$$\begin{aligned} \overline{p}_{n+1} - q\overline{p}_n - q^n\overline{p}_{n+1} &= q^{\binom{n+2}{2}}(S_n - S_{n-1}) - q^{\binom{n+1}{2}}(S_{n-1} - S_{n-2}) \\ &\quad + q^{\binom{n+1}{2}+1}(S_{n-2} - S_{n-1}) + q^{\binom{n}{2}+1}(S_{n-2} - S_{n-3}) \\ &= q^{n+1} - q^n - q^{n+1} + q^n = 0. \end{aligned}$$

Hence

$$\begin{aligned} p_n = \overline{p}_n &= q^{\binom{n+1}{2}} S_n - q^{\binom{n}{2}} S_{n-1} \\ &= q^{\binom{n+1}{2}} S_{n-1} - q^{\binom{n}{2}} S_{n-2}, \end{aligned} \tag{5.3}$$

and

$$
\begin{aligned}
&\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{(3n^2-n)/2}(1+q^n)p_n\right)\\
&\qquad=\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{(3n^2-n)/2}(1+q^n)\left(q^{\binom{n+1}{2}}S_n-q^{\binom{n}{2}}S_{n-1}\right)\right)\\
&\qquad=\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{2n^2+n}S_n+\sum_{n=1}^{\infty}(-1)^n q^{2n^2}S_n\right.\\
&\qquad\qquad\left.-\sum_{n=1}^{\infty}(-1)^n q^{2n^2}S_{n-1}-\sum_{n=1}^{\infty}(-1)^n q^{2n^2-n}S_{n-1}\right)\\
&\qquad=\frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{2n^2+n}S_n+\sum_{n=1}^{\infty}(-1)^n q^{2n^2-\binom{n+1}{2}}\right.\\
&\qquad\qquad\left.-\sum_{n=1}^{\infty}(-1)^n q^{2n^2-n}S_{n-1}\right)\\
&\qquad=\frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{2n^2+n}S_n-\sum_{n=2}^{\infty}(-1)^n q^{2n^2-n}S_{n-2}\right)\\
&\qquad=\frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{2n^2+n}S_n-\sum_{n=0}^{\infty}(-1)^n q^{2n^2+7n+6}S_n\right)\\
&\qquad=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+n}(1-q^{6n+6})S_n,
\end{aligned}
$$

which establishes (1.10). □

Next we treat (1.11). In Corollary 6 we set $a=1$, $b=-q$, $e=-f=q^{1/2}$. Consequently, after multiplying numerator and denominator in the sum on the left-hand side by $(-q;q)_n$, we find

(5.4)
$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q)_n^2}{(q)_{2n}}=\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{(3n^2-n)/2}(1+q^n)\,{}_3\phi_2\left(\begin{matrix}q^{-n},q^n,-q;q,q\\ q^{1/2},-q^{1/2}\end{matrix}\right)\right)$$

By (4.2)–(4.5), the ${}_3\phi_2$ in the right-hand sum (abbreviated as π_n) satisfies for $n>1$

$$-q^n(1+q^n)\pi_n=\pi_{n+1}+q^n\pi_{n-1}, \tag{5.5}$$

with $\pi_0=1$, $\pi_1=-q$. Now we define $\overline{\pi}_0=1$, $\overline{\pi}_1=-q$, and for $n>1$

$$\overline{\pi}_n=(-1)^n p_n$$

The remainder of the proof of (1.13) is exactly like that of (1.12) taking into account at each step the introduction of $(-1)^n$.

Finally we consider (1.12) and (1.13) together where the series involved are very close in appearance to that of (1.10) and (1.11). In fact, we begin by proving the following three identites:

$$F(q) - G(q) = -H(q), \tag{5.6}$$

$$G(q) = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}, \tag{5.7}$$

and

$$H(q) = -\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\big(\psi(-q) - 1\big), \tag{5.8}$$

where

$$F(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2;q^2)_n}{(q)_{2n}}, \tag{5.9}$$

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2;q^2)_n}{(q)_{2n+1}}, \tag{5.10}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q^2;q^2)_n}{(q)_{2n+1}}. \tag{5.11}$$

Equation (5.6) follows immediately:

$$\begin{aligned} F(q) - G(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2;q^2)_n}{(q)_{2n+1}} \big((1-q^{2n+1}) - 1\big) \\ &= -H(q). \end{aligned}$$

As for (5.7), this is a special case of the q-analog of Gauss's theorem. Namely, in Corollary 2.4 of [1, p. 20], replace q by q^2, then set $c = q^3$, $b = -q^2$, $a = -\frac{1}{\tau}$, let $\tau \to 0$ and multiply by $q/(1-q)$.

For (5.8), we note that if in the second Heine transformation [17, p. 10, eq. (1.4.5)] we replace q by q^2, the set $a = -q/\tau$, $b = -q^2$, $c = q^3$, $z = q^2\tau$ and then let $\tau \to 0$ after multiplying by $q/(1-q)$, we obtain

$$\begin{aligned} H(q) &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} q \sum_{n=0}^{\infty} (-q^2;q^2)_n(-q)^n \\ &= -\frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \left(\psi(-q) - 1\right). \end{aligned}$$

By [16, p. 58, eq. (26.53) observing that Fine's $\psi(q)$ is our $\psi(q) - 1$], and (5.8) is equivalent to (1.13).

We now combine (5.6)–(5.8) to represent $F(q)$ in terms of $\psi(-q)$

$$F(q) = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\psi(-q),$$

which is (1.12).

Obviously the right-hand side of (1.12) and (1.13) can be converted into infinite products times Hecke-type series just by invoking (1.10).

6. Proofs of (1.14), (1.15) and (1.16)

The proofs of these results resemble those of (1.10) and (1.11). For this reason we shall supress some of the computational details.

First we consider (1.14). In Corollary 6, we replace q by q^2, and set $a = 1$, $b = q^2$, $e = -q$, and $f = -q^2$. The resulting left-hand side is the left-hand side of (1.14). The resulting right-hand side is

$$\frac{1}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{3n^2-n}(1+q^{2n})\,{}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n},q^2;q^2,q^2\\ -q,-q^2\end{matrix}\right)\right). \tag{6.1}$$

If we define $\rho_0 = 1$, and for $n > 0$

$$\rho_n = (1+q^{2n})\,{}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n},q^2;q^2,q^2\\ -q,-q^2\end{matrix}\right) \tag{6.2}$$

then by Corollary 6, we see that ρ_n satisfies

$$q^2(1-q^{2n-1})\rho_n = \rho_{n+1} + \frac{q^{2n}(1-q^2)}{1-q^{2n+1}}\rho_n - q^{2n+1}\rho_{n-1}. \tag{6.3}$$

We now define

$$t_n := \sum_{j=-n}^{n}(-1)^j q^{-j^2}, \tag{6.4}$$

and

$$\overline{\rho}_n = \begin{cases}1 & \text{if } n=0,\\ (-1)^n\left(q^{n^2+2n}t_n - q^{n^2}t_{n-1}\right) & \text{if } n>0.\end{cases} \tag{6.5}$$

Using the same procedure given in the proof of (5.1), we deduce that for $n \geq 0$,

$$\rho_n = \overline{\rho}_n. \tag{6.6}$$

Consequently,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-q;q)_{2n}} &= \frac{1}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{3n^2-n}(-1)^n\left(q^{n^2+2n}t_n - q^{n^2}t_{n-1}\right)\right)\\ &= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{n=0}^{\infty}q^{4n^2+n} - \sum_{n=1}^{\infty}q^{4n^2-n}t_{n-1}\right)\\ &= \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{4n^2+n}\left(1-q^{6n+3}\right)t_n,\end{aligned}$$

which is (1.14).

Next we treat (1.15). In Corollary 6, we replace q by q^2, and set $a = q^2$, $b = q^2$, $e = -q^3$, $f = -q^2$. The resulting left-hand side is the left-hand side of (1.15) multiplied by $(1+q)$. The resulting right-hand side is

$$\frac{(1+q)}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+n}\left(1-q^{2n+1}\right)\frac{(1+q^{2n+1})}{(1+q)}\,{}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n+2},q^2;q^2,q^2\\ -q^3,-q^2\end{matrix}\right). \tag{6.7}$$

If we define $\sigma_0 = 1$ and for $n > 0$

$$\sigma_n = \frac{(1+q^{2n+1})}{(1+q)}\,{}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n+2},q^2;q^2,q^2\\ -q^3,-q^2\end{matrix}\right), \tag{6.8}$$

then by Corollary 6 we see that σ_n satisfies

$$q^2\left(1-q^{2n+1}\right)\sigma_n=\sigma_{n+1}-q^{2n+3}\sigma_{n-1}. \tag{6.9}$$

We define

$$\overline{\sigma}_n=(-1)^n q^{n^2+2n}t_n. \tag{6.10}$$

As before, we deduce that for $n\geq 0$,

$$\sigma_n=\overline{\sigma}_n. \tag{6.11}$$

Consequently,

$$\sum_{n\geq 0}\frac{q^{2n^2+2n}}{(-q^2;q)_{2n}}=\frac{(1+q)}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{4n^2+3n}\left(1-q^{2n+1}\right)t_n,$$

which is equivalent to (1.15).

Finally, we shall prove (1.16). In Corollary 6, we replace q by q^2, and set $a=q^2$, $b=q$, $e=-q$, and $f=-q^2$. The resulting left-hand side is the left-hand side of (1.16). The resulting right-hand side is

$$\frac{1}{(q^2;q^2)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{3n^2+n}\left(1-q^{4n+2}\right){}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n+2},q;q^2,q^2\\ -q,-q^2\end{matrix}\right)\right) \tag{6.12}$$

If we define $\tau_0=1$ and for $n>0$

$$\tau_n={}_3\phi_2\left(\begin{matrix}q^{-2n},q^{2n+2},q;q^2,q^2\\ -q,-q^2\end{matrix}\right), \tag{6.13}$$

then by Corollary 6, we see that τ_n satisfies

$$q(1-q^{2n+1})\tau_n=\tau_{n+1}-q^{2n+1}\tau_{n-1}. \tag{6.14}$$

We define

$$\overline{\tau}_n=(-1)^n q^{n^2+n}t_n. \tag{6.15}$$

As before, we deduce that for $n\geq 0$

$$\tau_n=\overline{\tau}_n. \tag{6.16}$$

Consequently

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}(q;q^2)_n}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{1}{(q^2;q^2)_n}\sum_{n=0}^{\infty}q^{4n^2+2n}\left(1-q^{4n+2}\right)t_n,$$

which is (1.16).

As a final comment on the results in this section, we note that by replacing $(-q;q)_{2n}$ or $(-q;q)_{2n+1}$ by $(q;q)_{2n}$ or $(q;q)_{2n+1}$ respectively, we obtain familiar results. Namely (1.14) is replaced by Slater's identity (39) [18, p. 156]

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}. \tag{6.17}$$

Equation (1.15) is replaced by Slater's identity (38) [18, p. 155]

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q;q)_{2n+1}}=\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty}. \tag{6.18}$$

Equation (1.16) is replaced by a false theta function identity

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n^2+n}, \tag{6.19}$$

(cf. [6, Lemma 12], [12, p. 681, eq. (10)]).

7. Conclusion

This paper is really just the beginning of a further study of q-series that have Hecke-type mock theta expansion. In this paper, we have restricted ourselves to series where the power of orthogonal polynomials could be used to good effect. Their value here lies in the three term recurrence relation that they satisfy. In practice, such relations are instances of contiguous relations. Clearly other contiguous relations will be useful in extending this approach to objects like the seventh order mock theta functions (which do not fall within the scope of this work).This becomes crucial in [6, Section 4].

The applications of orthogonal polynomials to mock theta functions seem to be restricted to instances of Corollary 6. The more general Corollary 5 does yield other results such as Ramanujan's identity [9, p. 99, eq. (5.3.1)]

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-a^{-1}q;q^2)_n q^{2n^2}}{(q^2;q^2)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{3n^2} a^n. \tag{7.1}$$

There are also much more esoteric results such as

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}(-q^3;q^4)_n(q^{-1};q^4)_n}{(q^2;q^2)_{2n}(q^4;q^8)_n} = \frac{1}{(q^4;q^4)_\infty} \sum_{n=0}^{\infty} q^{4n^2} \alpha_n,$$

where

$$\alpha_n = \frac{(-1)^n (q^6;q^8)_{\lfloor \frac{n}{2} \rfloor} q^{2n^2-3n} \varepsilon(n)}{(q^2;q^8)_{\lfloor \frac{n+1}{2} \rfloor}}$$

and

$$\varepsilon(n) = \begin{cases} 1+q^{4n} & \text{if } n \text{ is even} \\ 1-q^{8n} & \text{if } n \text{ is odd.} \end{cases}$$

This result and other strange results like it fall within the purview of Corollary 5 and the methods developed here; however, none of these results appears to be directly related to the identification of q-series with either mock theta functions or modular forms.

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