The Bailey Transform and False Theta Functions

by

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Abstract

An empirical exploration of five of Ramanujan's intriguing false theta function identities leads to unexpected instances of Bailey's transform which, in turn, lead to many new identities for both false and partial theta functions.

1 Introduction

On page 13 of Ramanujan's Lost Notebook [7] (c.f. [3, Sec. 9.3, pp. 227–232], we find the following five identities:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n} (q;q^2)_n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n^2+n}. \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2}. \tag{1.3}$$

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$$\sum_{n=0}^{\infty} \frac{(q;-q)_{2n} q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}. \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{(q;-q)_n (-q^2;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)}, \tag{1.5}$$

where $(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$.

From the right-hand sides of these identities we see that each is an instance of the false theta series (c.f. [8, §9])

$$\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \tag{1.6}$$

with q replaced respectively by q, q^2, q^3, q^4 and q^6. However, this regularity is completely obscured in the left-hand sides.

On the other hand, each of the five left-hand sides is of the form

$$\sum_{n=0}^{\infty} \frac{t_1 t_2 \cdots t_n q^n}{(-q;q)_{2n+1}}, \tag{1.7}$$

and one naturally asks: Did Ramanujan find all the instances of (1.7) in which the resulting power series is a false theta series (not necessarily an instance of (1.6))?

A computer algebra search using MACSYMA revealed the instances $z = 1, -1, i, e^{2\pi i/3}$, and $e^{\pi i/3}$ of the following identity which apparently escaped Ramanujan's attention:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-zq;q^2)_n (-z^{-1}q;q^2)_n q^n}{(-q;q)_{2n+1}} &= \sum_{n=0}^{\infty} \frac{(1-z^{2n+1})}{(1-z)} z^{-n} q^{n^2+n} \\ &= \sum_{n=0}^{\infty} \frac{(1+z+z^2+\cdots+z^{2n})}{z^n} q^{n^2+n}. \end{aligned} \tag{1.8}$$

Once (1.8) has been discovered, the next questions are: (1) How do you prove it, and (2) Where does it fit in the classical theory of q-series?

In Section 2, we lay the ground work for answers to these questions by considering instances of a bilateral version of the classic Bailey Transform. In Section 3, we derive three identities that will yield a number of applications of the results in Section 2 to false and partial theta functions. In Section 4 we prove ((1.1)–(1.4)), and in Section 5 we prove a number of related results including

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(-q^2;q^2)_n(1+q^{2n+1})} = \sum_{n=0}^{\infty}(2n+1)q^{n^2+n}, \tag{1.9}$$

and

$$(1-q)\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(-q^2;q^2)_{n+1}} = 1. \tag{1.10}$$

We conclude with a few comments about unexplored aspects of the Bailey Transform.

2 The Bilateral Bailey Transforms

In [5, p. 1], W. N. Bailey first presented the basic result which has become known as Bailey's Transform [10, Sec. 2.4, pp. 58–74].

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r},$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n},$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n,$$

subject to conditions on the four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the infinite series absolutely convergent.

For our purposes we require two bilateral versions of Bailey's Transform.

Symmetric Bilateral Bailey Transform.

If

$$\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} u_{n+r}, \tag{2.1}$$

and

$$\gamma_n = \sum_{r \geq |n|} \delta_r u_{r-n} v_{r+n} \tag{2.2}$$

then

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.3}$$

subject to conditions on the four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the relevant infinite series absolutely convergent.

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n \delta_n &= \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_r u_{n-r} u_{n+r} \\
&= \sum_{r=-\infty}^{\infty} \alpha_r \sum_{n \geqq |r|} \delta_n u_{n-r} u_{n+r} \\
&= \sum_{r=-\infty}^{\infty} \alpha_r \gamma_r.
\end{aligned}$$

$\square$

Asymmetric Bilateral Bailey Transform.

If

$$\beta_n = \sum_{r=-n-1}^{n} \alpha_r u_{n-r} u_{n+r+1} \tag{2.4}$$

and

$$\gamma_n = \sum_{r \geqq n} \delta_r u_{r-n} u_{r+n+1}, \tag{2.5}$$

then

$$\sum_{n=o}^{\infty} (\alpha_n + \alpha_{-n-1}) \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.6}$$

subject to conditions on the four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the relevant infinite series absolutely convergent.

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n \delta_n &= \sum_{n=0}^{\infty} \delta_n \sum_{r=-n-1}^{n} \alpha_r u_{n-r} u_{n+r+1} \\
&= \sum_{n=0}^{\infty} \delta_n \left(\sum_{r=0}^{n} + \sum_{r=-n-1}^{-1} \right) \alpha_r u_{n-r} u_{n+r+1} \\
&= \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r u_{n-r} u_{n+r+1} \\
&\quad + \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_{-r-1} u_{n+r+1} u_{n-r} \\
&= \sum_{r=0}^{\infty} (\alpha_r + \alpha_{-r-1}) \sum_{n \geqq r} \delta_n u_{n-r} u_{n+r+1} \\
&= \sum_{r=0}^{\infty} (\alpha_r + \alpha_{-r-1}) \gamma_r .
\end{aligned}$$

□

3 (γ_n, δ_n) Pairs

We shall obtain three (γ_n, δ_n) pairs. Two are related to the Symmetric Bilateral Bailey Transforms; the third is related to the Asymmetric Transform.

Theorem 1. *If we take $u_n = v_n = 1/(q^2; q^2)_n$ and*

$$\delta_n = (q^2; q^2)_{2n} q^n / (-q; q)_{2n+1} \tag{3.1}$$

in the Symmetric Bilateral Bailey Transform, then

$$\gamma_n = q^{|n|} \sum_{y \geqq 0} q^{j^2 + (2|n|+1)j} = q^{-n^2} \sum_{j \geqq |n|} q^{j^{-2}+j} \tag{3.2}$$

Proof. By (2.2)

$$\gamma_n = \sum_{r \geqq |n|} \frac{(q^2;q^2)_{2r} q^r}{(-q;q)_{2r+1}} \frac{1}{(q^2;q^2)_{r-n}(q^2;q^2)_{r+n}}.$$

Clearly $\gamma_n = \gamma_{-n}$, so for the remainder of the proof we assume that $n \geqq 0$. Hence

$$\begin{aligned}
\gamma_n &= \sum_{r=0}^{\infty} \frac{(q^2;q^2)_{2r+2n} q^{r+n}}{(-q;q)_{2r+2n+1}(q^2;q^2)_r(q^2;q^2)_{r+2n}} \\
&= \frac{q^n}{(-q;q)_{2n+1}} \sum_{r=0}^{\infty} \frac{(q^{4n+2};q^2)_{2r} q^r}{(q^2;q^2)_r (q^{4n+2};q^2)_r (-q^{2n+2};q)_{2r}} \\
&= \frac{q^n}{(-q;q)_{2n+1}}\, {}_3\phi_2\left(\begin{matrix} q^{2n+1}, -q^{2n+1}, q^{2n+2}; q^2, q \\ -q^{2n+3}, q^{4n+2} \end{matrix}\right) \\
&\qquad \text{(in the notation of [6, p. 4, eq. (1.2.22)])} \\
&= \frac{q^n}{(-q;q)_{2n+1}} \frac{(1+q^{2n+1})(q^{2j+2};q)_\infty}{(q^{4n+2};q^2)_\infty (q;q^2)_\infty}\, {}_3\phi_2\left(\begin{matrix} q^2, -q^{2n+1}, q; q^2, -q^{2n+1} \\ q^{2n+3}, q^{2n+2} \end{matrix}\right)
\end{aligned}$$

(by [6, p. 359, eq. (III.10)] with $a = q^{2n+1}, b = -q^{2n+1}, c = q^{2n+2}, d = -q^{2n+3}, e = q^{4n+2}$)

$$\begin{aligned}
&= \frac{q^n}{(1-q^{2n+1})} \sum_{j=0}^{\infty} \frac{(-q^{2n+1};q^2)_j (q;q^2)_j (-q^{2n+1})^j}{(q^{2n+3};q^2)_j (q^{2n+2};q^2)_j} \\
&= \frac{q^n (q;q)_{2n}}{(-q;q^2)_n (q^{1-2n};q^2)_n} \sum_{j=0}^{\infty} \frac{(-q;q^2)_{n+j} (q^{1-2n};q^2)_{n+j} (-q^{2n+1})^j}{(q;q)_{2n+2j+1}} \\
&= \frac{(-q)^n (q;q)_{2n}}{(-q;q^2)_n q^{-n^2} (q;q^2)_n} \sum_{j=n}^{\infty} \frac{(-q;q^2)_j (q^{1-2n};q^2)_j (-q^{2n+1})^{j-n}}{(q;q)_{2j+1}} \\
&= \frac{q^{-n^2} (q^2;q^2)_n}{(1-q)(-q;q^2)_n} \sum_{j=n}^{\infty} \frac{(-q;q^2)_j (q^{1-2n};q^2)_j (-q^{2n+1})^j}{(q^2;q^2)_j (q^3;q^2)_j} \\
&= \frac{q^{-n^2} (q^2;q^2)_n}{(1-q)(-q;q^2)_n} \left({}_2\phi_1\left(\begin{matrix} -q, q^{1-2n}; q^2, -q^{2n+1} \\ q^3 \end{matrix}\right) - \sum_{j=0}^{n-1} \frac{(-q;q^2)_j (q^{1-2n};q^2)_j (-q^{2n+1})^j}{(q^2;q^2)_j (q^3;q^2)_j} \right)
\end{aligned}$$

$$= \frac{q^{-n^2}(q^2;q^2)_n}{(1-q)(-q;q^2)_n} \left(\frac{(-q^2;q^2)_\infty (q^{2n+2};q^2)_\infty}{(q^3;q^2)_\infty(-q^{2n+1};q^2)_\infty} - \sum_{j=0}^{n-1} \frac{(-q;q^2)_j (q^{1-2n};q^2)_j (-q^{2n+1})^j}{(q^2;q^2)_j (q^3;q^2)_j} \right)$$

$$= q^{-n^2} \left(\sum_{j=0}^{\infty} q^{j^{-2}+j} - \frac{(q^2;q^2)_n}{(-q;q^2)_n} \sum_{j=0}^{n-1} \frac{(-q;q^2)_j (q^{1-2n};q^2)_j (-q^{2n+1})^j}{(q^2;q^2)_j (q;q^2)_{j+1}} \right)$$

(by [3, p. 11, eq. (1.1.7)])

If we could show that

$$\frac{(q^2;q^2)_n}{(-q;q^2)_n} \sum_{j=0}^{n-1} \frac{(-q;q^2)_j (q^{1-2n};q^2)_j (-q^{2n+1})^j}{(q^2;q^2)_j (q;q^2)_{j+1}} = \sum_{j=0}^{n-1} q^{j^{-2}+j}, \tag{3.3}$$

then it would follow immediately that

$$\gamma_n = q^{-n^2} \sum_{j \geqq n} q^{j^{-2}+j} = q^n \sum_{j=0}^{\infty} q^{j^2+(2n+1)j},$$

and our theorem would be proved.

Now (3.3) is a specialization of Watson's q-analog of Whipple's theorem [6, p. 360, eq. (III.17)], where n is replaced by $n-1$, then $b = aq^n, d \to \infty$; finally replace q by q^2 and then set $a = c = q$, $e = -q$. □

Theorem 2. *If we take $u_n = v_n = 1/(q^2;q^2)_n$ and*

$$\delta_n = (q;q)_{2n} q^n \tag{3.4}$$

in the Symmetric Bilateral Bailey Transform, then

$$\gamma_n = \sum_{j \geqq 0} q^{\binom{j+1}{2}+(2j+1)|n|} = q^{-2n^2} \sum_{j \geqq 2|n|} q^{\binom{j+1}{2}} \tag{3.5}$$

Proof. By (2.2)

$$\gamma_n = \sum_{r \geqq |n|} (q;q)_{2r} q^r \frac{1}{(q^2;q^2)_{r-n}(q^2;q^2)_{r+n}}.$$

Clearly $\gamma_n = \gamma_{-n}$, so for the remainder of the proof we assume that $n \geqq 0$. Hence

$$\begin{aligned}
\gamma_n &= \sum_{r=0}^{\infty} \frac{(q;q)_{2r+2n} q^{r+n}}{(q^2;q^2)_r (q^2;q^2)_{r+2n}} \\
&= \frac{q^n}{(-q;q)_{2n}} \; {}_2\phi_1 \left(\begin{matrix} q^{2n+1}, q^{2n+2}; q^2, q \\ q^{4n+2} \end{matrix} \right) \\
&= \frac{q^n}{(-q;q)_{2n}} \frac{(q^{2n+1};q^2)_\infty (q^{2n+2};q^2)_\infty}{(q^{4n+2};q^2)_\infty (q;q^2)_\infty} \; {}_2\phi_1 \left(\begin{matrix} q^2, q^{2n+1}; q^2, q^{2n+1} \\ q^{2n+2} \end{matrix} \right)
\end{aligned}$$

(by [6, p. 359, eq. (III.2), $q \to q^2$, then $a = q^{2n+2}, b = q^{2n+1} c = q^{4n+2}, z = q$])

$$\begin{aligned}
&= \frac{q^n}{(-q;q)_\infty (q;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(q^{2n+1};q^2)_j q^{(2n+1)j}}{(q^{2n+2};q^2)_j} \\
&= q^n \sum_{j=0}^{\infty} \frac{(q^{2n+1};q^2)_j q^{(2n+1)j}}{(q^{2n+2};q^2)_j}
\end{aligned}$$

(by [6, p. 241, eq. (8.10.9)])

$$= q^n \sum_{j \geqq 0} (1 + q^{2j+2n+1}) q^{2j^2+4nj+j}$$

(by the Rogers-Fine identity [3, p. 223, eq. (9.1.1) with $q \to q^2$, then $\alpha = q^{2n+1}, \beta = q^{2n+2}, \tau = q^{q^2 n+1}$])

$$= \sum_{j \geqq 0} q^{\binom{j+1}{2} + (2j+1)n}$$

$\square$

Theorem 3. *If we take $u_n = 1/(q^2;q^2)_n$ and*

$$\delta_n = (q;q)_{2n+1} q^n$$

in the Asymmetric Bilateral Bailey Transform, then

$$\gamma_n = \sum_{j \geqq 0} q^{j(j+3)/2 + n(2j+1)} = q^{-2n^2-2n} \sum_{j \geqq 2|n|} q^{j(j+3)/2}$$

Proof. By (2.5)

$$\begin{aligned}
\gamma_n &= \sum_{r \geqq n} (q;q)_{2r+1}\ q^r\ \frac{1}{(q^2;q^2)_{r-n}(q^2;q^2)_{r+n+1}} \\
&= \sum_{r \geqq 0} \frac{(q;q)_{2r+2n+1}\, q^{r+n}}{(q^2;q^2)_r (q^2;q^2)_{r+2n+1}} \\
&= \frac{q^n}{(-q;q)_{2n+1}}\ {}_2\phi_1\begin{pmatrix} q^{2n+2}, q^{2n+3}; q^2, q \\ q^{4n+4} \end{pmatrix} \\
&= \frac{q^n}{(-q;q)_{2n+1}}\ \frac{(q^{2n+2};q^2)_\infty (q^{2n+3};q^2)_\infty}{(q^{4n+4};q^2)_\infty (q;q^2)_\infty}\ {}_2\phi_1\begin{pmatrix} q^2, q^{2n+2}; q^2, q^{2n+2} \\ q^{2n+3} \end{pmatrix}
\end{aligned}$$

(by [6, p.359, eq.(III.2), $q \to q^2$, then $a = q^{2n+3}, b = q^{2n+2}, c = q^{4n+4}, z = q$])

$$\begin{aligned}
&= \frac{q^n}{(-q;q)_\infty (q;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(q^{2n+2};q^2)_j q^{(2n+2)j}}{(q^{2n+3};q^2)_j} \\
&= q^n \sum_{j=0}^{\infty} \frac{(q^{2n+2};q^2)_j q^{(2n+2)j}}{(q^{2n+3};q^2)_j} \\
&= q^n \sum_{j=0}^{\infty} (1 + q^{2n+2j+2}) q^{2j^2-j+4(n+1)j}
\end{aligned}$$

(by the Rogers-Fine identity [3, p. 223, eq. (9.1.1) with $q \to q^2$, then $\alpha = q^{2n+2}, \beta = q^{2n+3}, \tau = q^{2n+2}$])

$$= \sum_{j \geqq 0} q^{j(j+3)/2+(2j+1)n}.$$

□

4 Ramanujan's False Theta Identities

Theorem 1 now allows us to give uniform proofs of (1.1)–(1.4). Namely, by combining the Symmetric Bilateral Bailey Transform with Theorem 1, we obtain

Theorem 4. *If for $n \geqq 0$*

$$\beta_n = \sum_{r=-n}^{n} \frac{\alpha_n}{(q^2;q^2)_{n-r}(q^2;q^2)_{n+r}}, \tag{4.1}$$

then

$$\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^n \beta_n}{(-q;q)_{2n+1}} = \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} \alpha_n q^{-n^2} \tag{4.2}$$

subject to conditions on α_n and β_n that make the series absolutely convergent.

Proof. We use the γ_n and δ_n of Theorem 1 in the Symmetric Bilateral Bailey Transform with $u_n = v_n = 1/(q^2;q^2)_n$. Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^n \beta_n}{(-q;q)_{2n+1}} &= \sum_{n=0}^{\infty} \beta_n \delta_n \\ &= \sum_{n=-\infty}^{\infty} \alpha_n \gamma_n \\ &= \sum_{n=-\infty}^{\infty} \alpha_n q^{-n^2} \sum_{j \geqq |n|} q^{j^2+j} \\ &= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} \alpha_n q^{-n^2}. \end{aligned}$$

□

Before proceeding we remark that, in the language of [2], $(\overline{\alpha}_n, \beta_n)$ is a Bailey pair with $a = 1$ and $q \to q^2$ where $\overline{\alpha}_0 = \alpha_0$ and for $n > 0$, $\overline{\alpha}_n = \alpha_n + \alpha_{-n}$. Hence we may utilize L. J. Slater's compendium of Bailey pairs [9] to obtain false theta identities as well as other sources [2], [5].

Theorem 5. *Identity (1.1) is valid. I.e.*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}(q;q^2)_n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n \, q^{n(n+1)/2}. \tag{4.3}$$

Proof. We apply Theorem 4 with $\alpha_n = (-1)^n q^{n(n+1)/2}$ and

$$\beta_n = \frac{(-1)^n q^{n^2} (q;q^2)_n}{(q^2;q^2)_{2n}}.$$

The fact that these α_n and β_n satisfy (4.1) follows by specializing [5, p. 5, Sec. 6, (ii)] with $a = 1, b \to 0$ and x replaced by q. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n} (q;q^2)_n}{(-q;q)_{2n+1}} &= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^n \beta_n}{(-q;q)_{2n+1}} \\ &= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} \alpha_n q^{-n^2} \\ &= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} (-1)^n q^{-n(n-1)/2}\end{aligned}$$

It is an easy exercise in mathematical induction on j to show that

$$\sum_{n=-j}^{j} (-1)^n q^{-n(n-1)/2} = (-1)^j q^{-j(j+1)/2}; \tag{4.4}$$

therefore

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n} (q;q^2)_n}{(-q;q)_{2n+1}} = \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)/2}.$$

□

Theorem 6. *Identities (1.2) and (1.8) are valid.*

Proof. Clearly (1.2) is the special case $z = -1$ of (1.8). Furthermore with $\alpha_n = z^n q^{n^2}$ in (4.1), we see that

$$\begin{aligned}\beta_n &= \sum_{r=-n}^{n} \frac{z^n q^{n^2}}{(q^2;q^2)_{n-r} (q^2;q^2)_{n+r}} \\ &= \frac{(-zq;q^2)_n (-z^{-1}q;q^2)_n}{(q^2;q^2)_{2n}}\end{aligned}$$

by [1, p. 49, Ex. 1]. Consequently by Theorem 4

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-zq;q^2)_r(-z^{-1}q;q^2)_n q^n}{(-q;q)_{2n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^n \beta_n}{(-q;q)_{2n+1}} \\
&= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} \alpha_n q^{-n^2} \\
&= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} z^n \\
&= \sum_{j=0}^{\infty} q^{j^2+j} \frac{(1+z+z^2+\cdots+z^{2j})}{z^j}.
\end{aligned}
$$

□

Theorem 7. *Identity (1.3) is valid.*

Proof. We apply Theorem 4 with $\alpha_n = (-1)^n q^{n(3n+1)/2}$ and

$$\beta_n = \frac{(q;q^2)_n}{(q^2;q^2)_{2n}}.$$

The fact that these α_n and β_n satisfy (4.1) follows by specializing again [5, p. 5, Sec. 6, (ii)] with $a = 1, b \to \infty$ and x replaced by q.

Thus

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n+1}} &= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^n \beta_n}{(-q;q)_{2n+1}} \\
&= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} \alpha_n q^{-n^2} \\
&= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} (-1)^n q^{n(n+1)/2} \\
&= \sum_{j=0}^{\infty} (-1)^j q^{3j(j+1)/2},
\end{aligned}
$$

by (4.4) with q replaced by $1/q$. □

Theorem 8. *Identity (1.4) is valid.*

Proof. We apply Theorem 4 with $\alpha_n = (-1)^n q^{2n^2+n}$

$$\beta_n = \frac{1}{(-q;-q)_{2n}}.$$

The fact that these α_n and β_n satisfy (4.1) follows from [9, p. 468, (F1)] with q replaced by q^2 and then $q \to -q$.

So

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(q;-q)_{2n} q^n}{(-q;q)_{2n+1}} &= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^n \beta_n}{(-q;q)_{2n+1}} \\ &= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} \alpha_n q^{-n^2} \\ &= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} (-1)^n q^{n^2+n} \\ &= \sum_{j=0}^{\infty} (-1)^j q^{2j^2+2}\end{aligned}$$

by (4.4) with q replaced by $1/q^2$. □

Identity (1.5) apparently does not follow as easily from Theorems 1–3.

5 Further applications

The number of possible specializations of Theorems 1–3 is huge. Indeed each of these theorems will take a standard Bailey pair and produce a new identity. In light of the tremendous number of known Bailey pairs (cf. [5], [2]), we will only provide a minimal sample in this section.

We begin with noting that (1.9) is merely (1.8) with $z = 1$. Also in (1.8) if we set $z = q^2$, we see that

$$\frac{1}{1+q} + \sum_{n=1}^{\infty} \frac{(-q^{-1};q^2)_n(-q^3;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(1-q^{4n+2})q^{n^2-n}}{1-q^2}$$
$$= \frac{1}{1-q^2}\left(\sum_{n=0}^{\infty} q^{n^2-n} - \sum_{n=0}^{\infty} q^{(n+1)(n+2)}\right)$$
$$= \frac{2}{1-q^2} \; .$$

Hence
$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(-q^2;q^2)_{n+1}} \; = \; \frac{2}{1-q^2} - \frac{1}{1+q} = \frac{1}{1-q} \; ,$$
which is equivalent to (1.9).

Each of Theorems 2 and 3 can be combined with the relevant Bailey transform to produce analogs of Theorem 4. In each case, the proof is a mirror image of that given for Theorem 4 and is consequently omitted.

Theorem 9. *If for* $n \geqq 0$

$$\beta_n = \sum_{r=-n}^{n} \frac{\alpha_n}{(q^2;q^2)_{n-r}(q^2;q^2)_{n+r}} \, , \tag{5.1}$$

then

$$\sum_{n=0}^{\infty} (q;q)_{2n} q^n \beta_n = \sum_{j=0}^{\infty} q^{\binom{j+1}{2}} \sum_{n=-\lfloor \frac{j}{2} \rfloor}^{\lfloor \frac{j}{2} \rfloor} \alpha_n q^{-2n^2} \; . \tag{5.2}$$

Theorem 10. *If for* $n \geqq 0$

$$\beta_n = \sum_{r=-n-1}^{n} \frac{\alpha_r}{(q^2;q^2)_{n-r}(q^2;q^2)_{n+r+1}} \, , \tag{5.3}$$

then

$$\sum_{n=-0}^{\infty} (q;q)_{2n+1} q^{n+1} \beta_n = \sum_{j=0}^{\infty} q^{j(j+3)/2} \sum_{n=-\lfloor \frac{j}{2} \rfloor - 1}^{\lfloor \frac{j}{2} \rfloor} \alpha_n q^{-2n^2-2n} \; . \tag{5.4}$$

In order to indicate the scope of these theorems, we close this section with two results. The first is one of the elegant formulas from Ramanujan's Lost Notebook [3, p. 238, Entry 9.5.2].

Theorem 11.

$$\sum_{n=0}^{\infty} (q;q^2)_n q^n = \sum_{n=0}^{\infty} (-1)^n q^{3n^2+n}(1+q^{2n+1})\,.$$

Proof. In Theorem 9, we use the Bailey pair from the first Rogers-Ramanujan identity [9, p. 468, B(1)] namely $\alpha_n = (-1)^n q^{3n^2+n}$, $\beta_n = 1/(q^2;q^2)_n$ with $q \to q^2$ and $a = 1$. Hence in (5.2)

$$\begin{aligned}
\sum_{n=0}^{\infty} (q;q^2)_n q^n &= \sum_{n=0}^{\infty} \beta_n \delta_n \\
&= \sum_{j=0}^{\infty} q^{\binom{j+1}{2}} \sum_{n=-\lfloor \frac{j}{2} \rfloor}^{\lfloor \frac{j}{2} \rfloor} (-1)^n q^{n^2+n} \\
&= \sum_{j=0}^{\infty} q^{\binom{j+1}{2}} (-1)^{\lfloor \frac{j}{2} \rfloor} q^{\lfloor \frac{j}{2} \rfloor^2 + \lfloor \frac{j}{2} \rfloor} \\
&= \sum_{n=0}^{\infty} (-1)^n q^{3n^2+n}(1+q^{2n+1})\ .
\end{aligned}$$

□

As the next result reveals, there is no need to restrict ourselves to single fold series.

Theorem 12.

$$\sum_{n,j \geqq 0} \frac{q^{2j^2+j+n}(q^2;q^2)_{n+j}}{(-q;q)_{2n+2j+1}(q^2;q^2)_j(q^2;q^2)_n} = \sum_{n=0}^{\infty} (-1)^n q^{5n^2+3n}(1+q^{4n+2})\,.$$

Proof. In Theorem 4, we require Bressoud's Bailey pair [2, p. 280, eq.'s (5.6) and (5.7) with $a = 1$, $q \to q^2$]:

$$\begin{aligned}
\alpha_{2m} &= (-1)^m q^{5m^2+m} \\
\alpha_{2m+1} &= 0 \\
\beta_n &= \frac{1}{(q^2;q^2)_{2n}} \sum_{j=0}^{n} \frac{(q^2;q^2)_n q^{2j^2}}{(q^2;q^2)_j (q^2;q^2)_{n-j}}\ .
\end{aligned}$$

Thus by (5.2)

$$
\begin{aligned}
&\sum_{n,j\geqq 0} \frac{q^{2j^2+j+n}(q^2;q^2)_{n+j}}{(-q;q)_{2n+2j+1}(q^2;q^2)_j(q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n}}{(-q;q)_{2n+1}}\, q^n \,\frac{1}{(q^2;q^2)_{2n}} \sum_{j=0}^{n} \frac{(q^2;q^2)_n q^{2j^2}}{(q^2;q^2)_j(q^2;q^2)_{n-j}} \\
&= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^n}{(-q;q)_{2n+1}}\, \beta_n \\
&= \sum_{j=0}^{\infty} q^{j^2+j} \sum_{n=-j}^{j} \alpha_n q^{-n^2} \\
&= \sum_{j=0}^{\infty} q^{4j^2+2j} \sum_{m=-j}^{j} (-1)^m q^{5m^2+m-(2m)^2} \\
&\quad + \sum_{j=0}^{\infty} q^{(2j+1)^2+(2j+1)} \sum_{m=-j}^{j} (-1)^m q^{5m^2+m-(2m)^2} \\
&= \sum_{j=0}^{\infty} (-1)^j q^{5j^2+3j} + \sum_{j=0}^{\infty} (-1)^j q^{(2j+1)^2+(2j+1)+j^2+j} \\
&= \sum_{j=0}^{\infty} (-1)^j q^{5j^2+3j}(1+q^{4j+2})\,.
\end{aligned}
$$

□

6 Conclusion

This paper provides a new direction for the Bailey transform. In works such as [2], the emphasis has been on picking up new Bailey pairs, and the sequences γ_n and δ_n have been restricted to one specific instance. There have been some investigations of alternatives to the standard γ_n and δ_n by Bailey, Bressoud, Berkovich, Warnaar and others; however none seem to capture the types of results given in (1.1)–(1.4).

In this initial work, we have restricted ourselves to false theta series applications. However, the Hecke type series involving indefinite quadratic forms clearly fall within the purview of these methods and will be dealt with in a subsequent investigation.

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