

# **INSTRUCTOR'S SOLUTIONS MANUAL**

**Introduction to Linear Programming**

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# Contents

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<b>Preface</b> .....	iv
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## **Chapter 1. Introduction**

1. What Is Linear Programming? .....	1
2. Examples of Linear Programs .....	3
3. Graphical Method .....	7

## **Chapter 2. Background**

4. Logic .....	9
5. Matrices .....	10
6. Systems of Linear Equations .....	17

## **Chapter 3. Tableaux and Pivoting**

7. Standard and Canonical Forms for Linear Programs .....	26
8. Pivoting Tableaux .....	28
9. Standard Row Tableaux.....	31

## **Chapter 4. Simplex Method**

10. Simplex Method, Phase 2 .....	38
11. Simplex Method, Phase 1 .....	41
12. Geometric Interpretation .....	45

## **Chapter 5. Duality**

13. Dual Problems.....	48
14. Sensitivity Analysis and Parametric Programming.....	50
15. More on Duality .....	51

## **Chapter 6. Transportation Problems**

<b>16.</b>	Phase 1 .....	54
<b>17.</b>	Phase 2 .....	55
<b>18.</b>	Job Assignment Problem .....	61

## **Chapter 7. Matrix Games**

<b>19.</b>	What are Matrix Games? .....	66
<b>20.</b>	Matrix Games and Linear Programming .....	68
<b>21.</b>	Other Methods .....	71

## **Chapter 8. Linear Approximation**

<b>22.</b>	What is Linear Approximation? .....	73
<b>23.</b>	Linear Approximation and Linear Programming .....	76
<b>24.</b>	More Examples .....	78

# Preface

We give solutions or answers to all exercises in the textbook. Note that the exercises may have many correct solutions and even several correct answers. Some misprints in Answers to Selected Exercises (pages 305–317 in the first print of textbook) are corrected in the solutions below.

Here are some other corrections to the first print.

Dedication page (page iv). Replace *my* by *our*.

In **Contents**, pages v and vi, replace page numbers:

§6 52 → 54, §17 180 → 179,, §20 210 → 211, §21 220 → 221,

In **Preface**, page vii, the last row, replace *Thus* by *Thus*, . On page ix, update the URL for Mathematical Programming Glossary to

<http://glossary.computing.society.informs.org/>  
and the URL for SIAM Activity Group on Optimization to  
<http://www.siam.org/activity/optimization/>.

Page 4. On line 15 from below, add minus before 7. On line 3 from below., add a period after 1.3.

Page 5, line 14. Remove one comma from “,,”.

Page 6, line 9 from below. Add period after 1.8.

Page 7, line 6. Replace 2 by 3.

Page 9. On line 1, delete “s” in “differents”. On line 4 from below, replace *rediscovered* by *rediscovered*

Page 11. There are two Exercises 57. Both solved below. Replace **36–42** by **36-43**. On line 5 from below, replace *linear* by a linear form.

Page 13, line 8 after the table. Remove space before the question mark.

Page 14, the last line. Insert *dual* after *the* and replace 4 by 5.

Page 17. Add period in the end of display. In the table, replace *Aarea* by *Area* and .6 by .8.

Page 21, line 14. Replace *to worker* by *for worker*.

Page 22, line 14. Insert it after *Although*.

Page 23. On lines 2 and 5, replace §1 of Chapter 2 by §3. On line 16, replace 2.6 by 2.5.

Page 27. Add a period after **Figure 3.5**.

Page 28. On the first line after Figure 3.7, replace §3 by §12. Remove the periods after the names of Figures 3.7 and 3.8.

Page 30, line 12 from below. Replace min by max.

Page 31. On the last line, replace §3 by §12. Remove the period after the name of Figure 3.14.

Page 33, last line. Replace  $y + y$  by  $y + z$ .

Page 35, line 14. Remove space between  $\wedge$  and  $)$ .

Page 38, line 18. Replace that by than.

Page 39, line 14. Insert  $:$  in the end.

Page 40, line 8 from below. Replace 30 by 31

Page 41, Ex. 31 Replace if and only if by means that

Page 42, line 8 from below. Remove the comma before  $($ .

Page 43, line 2 from below. Replace uses by use.

Page 46, line 9 from below. Delete the second period after 5.8.

Page 47, line 3 Replace prededing i by preceding

Page 49. In 2 dispayed matrices, replace  $I$  by 1. In the proof of Proposition 5.14, replace  $I_n$  by  $1_n$  six times. On line 2 from below, replace multiple by scalar multiple and drop “by a number”.

Page 50. On line 3, replace  $B$  by  $A$ . Chage the last row of the matrix  $P$  to  $[1\ 0\ 0]$ . In the last row of the matrix  $D$ , drop 4. In the last row of the matrix  $D^{-1}$ , replace 8 by 2.

Page 51. Replace the matrices  $E$  and  $E^{-1}$  by there transposes.

Page 52, Ex.12–14. Replace “in the sense of Definition 1.3” by “(see page 4)”.

Page 53, Ex.44. Delete !!.

Page 54. On line 5, replace system by systems. On line 17 from bottom, replace correspond by corresponds. On line 10 from bottom, delete set of.

Page 56. On line 19, replace  $cz$  by  $-cz$ . On line 4 from bottom, replace a by an. On the last line, delete the first comma.

Page 57. line 2 from bottom Replace system by systems.

Page 58. On line 4, replace system by systems. On line 14 from bottom, replace time by times. On line 13 from bottom, replace a by an.

Page 59. On line 9 replace a by an.

Page 61. On line 12 (left of 2nd row of the third matrix), replace  $-3/2$  with  $1/2$ . On line 9 from bottom, replace  $-71$  by  $7$ . On line 5 from bottom, replace  $-71$  by  $+7$ .

Page 62. On lines 4 and 18, replace  $-71$  by  $7$ . On line 7, drop to. On line 11, switch exactly and means. On line 14 from bottom, replace  $-7/8$  by  $+7/8$ .

Page 64, line -3. Replace 3 by 2.

Page 66, line 19. Replace **36–38** by **36–39**.

Page 68. On line 7, delete the second “of.” On line 15, replace necessary by necessarily.

Page 70. On line 15 from bottom, replace  $7,4$  by  $7.4$ . On line 13 from bottom, delete the after equivalent. On line 5 from bottom, add  $s$  after eliminate.

Page 71. On line 3, replace Thick by Trick. On line 16, add  $s$  after constraint. On line 18, insert  $by$  before adding. On line 5 from bottom, replace in by is.

Page 72. On line 1, replace rid off by get rid of. On line 4, replace know any upper or lower bonds by do not know any bounds. On line 19 from bottom, replace preeding by preceding.

Page 73, lines 18 and 20. Replace  $\geq 1$  by  $\leq -1$ .

Page 74, lines 17. Replace  $7,1$  by  $7.1$ .

Page 75, line 10. Replace method by methods.

Page 76, line 6. Delete space between  $b$  and  $\therefore$ .

Page 77, line 2. Replace system by systems.

Page 78,. On line 5., add the after one of. In (8.5), replace  $7 - 2$  by  $6 - 3$ .

Page 80. In the first paragraph, replace  $\beta$  by  $\gamma$ . On line 15, replace  $-1/17$  by  $1$ .

Page 81, line 3. Insert of after column.

Page 82, line 5. Insert our before system.

Page 85, line 5 from below. Replace is by are.

Page 86, line 1. Delete the period after  $e$ .

Page 86, line 1. Delete the period after  $e$ .

Page 87. In Ex.7, remove “=.”

Page 89. Insert a period after **9.1**.

**vii** *Corrections*

Page 90,. On line 18., replace  $, z$  by  $, z$ . On line 6 from the bottom, remove the semicolon after  $\text{sign}$ .

Page 91, the first tableau. Replace  $= w$  by  $= -w$ .

Page 93, two lines above the last tableau. Delete “is.”

Page 95, line 3. Delete space between  $-1$  and  $:$ .

Page 96. On lines 2,6, 7, 7, replace  $y$  by  $u$ . On line 11 from bottom, replace  $y$  by  $u, v$ . On line 9 from bottom, replace Trick 7.7 by Trick 7.8.

Page 97. Insert period after the first tableau. On line 9, replace  $Ax - u = b$  by  $Ax - u = -b$ .

Page 98. Delete the first line after the first tableau.

Page 99, line 4 from bottom. Replace  $x_7$  by  $= x_7$ .

Page 100, the last line. Replace  $\geq$  by  $\leq$ .

Pages 102–132, headings on even pages. Delete “:” after Chapter 4.

Page 102, line 5 from bottom. Replace  $=\rightarrow$  by  $\rightarrow$ .

Page 105. On line 3, replace  $\text{isthe}$  by  $\text{is the}$ . On line 15 from bottom, replace  $-2\text{and}$  and  $-2$  and. On line 8 from bottom, replace  $2x_2$  by  $2x_3$ . On line 6 from bottom, replace  $\text{min=}$  by  $\text{min} =$ .

Page 106, the last row in the 4th tableau. Replace  $1\ 0\ 1\ 1$  by  $1\ 4\ 1\ 1$ .

Page 107,. On line 11, drop  $-$  after  $x_3$  to  $\text{and}$  and replace  $x_2$  by  $x_3$ . On line 21, replace  $\text{previosly}$  by  $\text{previously}$ .

Page 110., Exercises. Put the periods after **1** and **2** in boldface.

Page 111. Put the period after **3** in boldface. On line 2 from bottom, delete the first “one.”

Page 112. On the last row of (11.2) and on the last line of the page, drop the last  $x$ . On line 9 from bottom, replace “column” by “row.”

Page 113, line 6. from bottom. Put the colon in boldface and remove space before it

Page 115, line 18. Replace “not necessary follows” by “does not necessarily follow.”

Page 116. In Problem 11.6, replace in 10.4 by in 10.10. In the solution, replace Phase 2 by Phase 1.

Page 118. On the line above **Remark**, replace  $x_4 = 1$  by  $x_1 = 1$ . On the second line from bottom, insert i a space Before If.

Page 119. On line 9, replace  $z$  be by  $z$  by.

Page 120. In Exercise 10, remove `has`. In Ex. 11, 12, 13 put the periods in boldface.

Page 121. Remove the colon after the name of Figure 12.3.

Page 122. Remove the period after the name of Figure 12.4.

Page 123. On 13 from bottom, add `an` after `As`. On line 4 from bottom, replace `constraint` by `constraints`.

Page 124, line 1. Remove the period after Definition.

Page 125. On line 12 from the bottom, replace  $Ax^t$  by  $Ax^T$ . On line 5 from the bottom, insert `is` before `empty`.

Page 127. On line 11, drop `of`. On line 14, replace `could be now` by `now could be`. On line 16, replace `that is` by `stated as`. Add a period after **12.19**.

Pages 134–164, headings on even pages. Delete “.” after Chapter 5.

Page 135. In Definition, replace `associated to` by `associated with`. On the last line, replace `min` by `max`.

Page 136, line 9 from bottom. Replace `previosly` to by `previously`.

Page 137, line 9 from bottom. Replace  $t$  by  $T$  twice.

Page 138. Replace  $dv$  in the second displayed line (line 14) by  $d = v$ . On line 16, replace  $\leq$  by  $\geq$ . On line 5 from bottom, add `an` before `equality`.

Page 139. On line 6, add `the` before `equality`. Switch Case 2 and Case 3 on lines 11 and 9 from bottom.

Page 140, line 17 from bottom. Replace `in` by `on`.

Page 141. Replace  $-2$  in the third line of the first matrix and the second line of the second matrix by  $2$ .

Page 143. On line 11, insert `comma` before `but`. On last line in (14.2), replace `respectively,` by `respectively`.

Page 144,. On line 16 from bottom. replace  $-1$  by  $-50$ . On line 9 from bottom. Insert “an” in between “get” and “improvement”; insert “to” between “equal” and “280/297.”

Page 145. On the first line after (14.5), replace `tableu` by `tableau`. On the first and second lines after (14.5), replace “It is easy to compute now for which values of  $\varepsilon_i$  the tableau stays optimal” by “It is now easy to compute the values of  $\varepsilon_i$  for which the tableau stays optimal.” On the ninth line after (14.5), replace `stay` by `stays`.



Page 146, line 10 and 11 from bottom. Insert the after to.

Page 150, line 14 from bottom. Replace  $e$  by  $\varepsilon$ .

Page 151, line 12 from bottom. Replace row by raw.

Page 152, Theorem 14.15. Replace the lines 6–7 by:

value. Then  $P$  is a convex set, and, when parameters are in  $c$  (resp., in  $b$ ),  $f(t)$  is the minimum (resp., maximum) of a finite set of affine functions on  $P$ . So  $f(t)$  is a piecewise affine and concave (resp., convex)

Page 153, line 10 from bottom. Replace  $y > \geq$  by  $y \geq$ .

Page 154. On line 3 after (17.7), replace chose by choose. On line  $u$  in (14.20), replace  $-1/3$  by  $1/3$ .

Page 155, line 10. Replace  $i$ , and  $j$  by  $i, j$ , and  $k$ .

Page 156, line 4 from bottom. Replace low by lower.

Page 157, on right from the first tableau. Add space between all and  $x_i \geq 0$ .

Page 158, line 12 from bottom. Add the before duality.

Page 159. On line 6, replace Is by Does. On line 7, replace follows by follow. Add a period after **Remark 15.2**. Two lines later, replace  $yA \geq c$  by  $yA \leq c$ .

Page 160. On line 14 from bottom, replace van you to by can. On line 10 from bottom, delete to. On line 5 from bottom, replace 2.1 by 2.2.

Page 161. On the right of tableau, replace  $u \geq = 0$  by  $u', u'' \geq 0$ . On line 5 from bottom, add a comma after Bob.

Page 162. On line 4, replace  $b_o$  by  $b_0$ . On line 10, delete of after dropping.

Page 163, last line of the tableaux. Replace 105 by 122.

Page 164. Remove the period after **Remark**.

Page 165. In Exercise 8, replace  $+-$  by  $-$ . On line 4, delete the period before *Hint*. Reduce the hight of brackets in Exercises 9 and 10.

Page 166, line 6 from bottom. Replace transpcrtation by transportation.

Page 168, line 3 above the last table. Replace 2-by-3 table by 2-by-2 table.

Page 169, line 1 above the last table. Add a colon in the end.

Page 170. On line 2, insert a period in the end of the displayed formula. On line 3, replace 15.2 by 15.7. On line 9, replace  $=$  by  $\geq$ . On line 10, replace  $=$  by  $\leq$ .

Page 171, line 17 from bottom. Delete the comma before “).”

Page 172, line 4 after the 1st table. Replace row by column.

Page 173, the last matrix. Replace it by  $\begin{bmatrix} 77 & 39 & 105 \\ 150 & 186 & 122 \end{bmatrix}$ .

Page 174, **Figure 16.7**. Replace cost 1, 2, 2, 2 on the arrows by 77, 39, 186, 122.

Page 175. In **Table 16.6**, replace 15.2 by 15.7. In **Figure 16.5**, replace the cost 1, 2, 2, 2 on the arrows by 77, 39, 186, 122 and the potentials 0, 0, 1, 2, 2 at the nodes by 0, -147, 77, 39, -25.

Page 176, line 1 after **Table 16.9**. Replace previosly by previously.

Page 180. On line 4, replace  $30) =$  by  $50) =$ . On line 3 above Table 17.4, replace the second and by but. In the first row of Table 17.4, center (65), (60), and (50).

Page 181, the figure title. Add a period after **17.5**.

Page 183, the line above Figure 17.10. Add space between Figure and 17.10.

Page 184, line 4 after **Table 17.11**. Replace  $e$  by  $\varepsilon$ .

Page 189. On top of Figure 17.23, replace  $c = 25$  by  $c = 35$ . On the last line, replace ficticious by fictitious.

Page 188, the second figure title. Add a period after **17.21**.

Page 189. On top of Figure 17.23, replace  $c = 25$  by  $c = 35$ . On the last line, replace ficticious by fictitious.

Page 192. On line 7, delete then twice. On line 16 from bottom, switch “)” and “.”.

Page 193. On line 8, insert space between . and The. On line 9, replace problem we may by problem may. On line 3 in **Solution**, remove . between that is and , the. On line 4 in **Solution**, switch “)” and “.”.

Page 194, first table. Insert \* as the last entry in the first row. Move \* in the last row from the second position to the first position.

Page 195. In the third table, delete the last two asterisks. In the next line, replace “column does produce four” by “row does produce two”. On the last line, replace  $+ 2$  by  $+3$ .

Page 196. In the second row in first two tables, replace the first 3 entries (2) (3) (1) by (1) (2) (0). On line 4 after the first table, replace 6 by 4. On the line above the last table, drop of after along. In the last table, replace the potentials 6, 2, 1 at the left margin by 5, 3, 2. In the second row, replace the first 3 entries (3) (3) (2) by (2) (2) (1).

Page 197. In the first table, replace  $1(-1)0-\varepsilon$  in the first row by  $1-\varepsilon(-1)0$  and  $3|(0)(0)(-7)$  in the second row by  $2|(-1)\varepsilon(-1)(-2)$ . In the next line, replace (2, 3). Again  $+\varepsilon = 0$ . by (2, 1), and  $\varepsilon = 1$ . In the next line, replace “(1, 3) (no other choice this time).” by (2, 4). In the second table, replace  $1(0)(1)0$  in the first line by  $0(0)(1)1$  and  $(0)(1)01$  in the second line by  $1(1)0(1)$ . In the last table replace the first two lines  $\begin{matrix} * & & * \\ & * & \end{matrix}$  by  $\begin{matrix} * & & * \\ & * & \end{matrix}$ . Replace the next tree lines by: The optimal value is  $\min = 10$ . On line 4 from bottom, replace  $[n$  by  $(n$ .

Page 200. On line 13, replace  $\text{hin}$  by  $\text{him}$ . On line 7 from bottom, replace  $\text{paoff}$  by  $\text{payoff}$  and delete the last period. On the second line from bottom, insert  $\text{of}$  before  $\text{as}$ .

Page 201. On line 2, replace the period by a colon. On line 8 from bottom, replace  $\text{an}$  by  $\text{a}$ . On the next line remove space between  $($  and  $\text{If}$ .

Page 202, line 2 from bottom. Drop  $\text{is}$  after  $\text{is}$ .

Page 203. Remove the period after **Definition**.

Page 204, the first two displayed formulas. Place  $p \in P$  under  $\max$  and  $q \in Q$  under  $\min$ .

Page 205, line (2,3) in two matrices. Replace  $1\ 1\ -1$  by  $1\ 0\ 0$ . This makes both matrices skew symmetric.

Page 208. On line 3, replace  $\text{previosly}$  by  $\text{previously}$ .

Page 210, In Exercise 10., the empty entry means 0. On line 10 from bottom, replace  $(0, 3)$  by  $(0, 3)$ . On line 9 from bottom, replace  $(1,2)$  by  $(1, 2)$ .

Page 211, On line 20, replace  $)$  by  $]$ . On line 8 from bottom, replace  $\text{Player 1}$  by  $\text{She}$ . On line 4 from bottom, replace  $\text{win } 1/4$  from  $\text{by lose } 1/4$  to. On line 3 from bottom, add  $^T$  after  $]$  and replace  $\text{Scissors}$  by  $\text{Rock}$ .

Page 212. On line 14, add  $e$  to  $\text{th}$ . On line 11 from bottom, insert space before  $\text{by}$  in  $-1/2\text{by}$ .

Page 213, line 5 from bottom. Replace nodes by node.

Page 215, line 2 from bottom. Replace  $\mu'$  by  $\mu'$ .

Page 216, line 19. Insert is between problem and solved.

Page 217. On line 4, insert the before simplex. On line 5 after the first matrix. , replace lows by lows us.

Page 218. On line 7, add the before game. On the last line, replace solutions by strategies.

Page 219. On line 12., replace A by M. On line 6 from bottom, replace ).] b y )].

Page 221. On line, remove the before blackjack. On line 12 from bottom, remove space before the question mark. On line 6 from bottom, replace loose by lose.

Page 223. On line 2, replace c3+ by c3 +. On line 2 from the last matrix., replace with r3 by with c3.

Page 224, line 2. Replace c2 by with c3.

Page 226. On line 3, drop of. On line 17, replace c4,c5 by c4. c5. On line 4 from bottom, replace the second 1/4 by 1/2. On line 2 from bottom, replace 0.1 by 0, 1.

Page 227. On line 2, replace: [1,1,1] by [1,2,1] ; [0 by [1 . On line 5, replace 0.1 by 0, 1. In Exercise 5, replace games by game.

Page 229. On line 6, switch we and can. On line 8, insert is between it and not.

Pages 230–256, headings on even pages. Delete “.” after Chapter 8.

Page 230, line 17. Replace number by numbers.

Page 231, line 4 from bottom. Replace tells that by tells us that.

Page 232. On line 9, replace to with by to do with. On line 13, replace suma by sumo. On line 16, replace “Survival” TV show by TV show ”Survivor”. On line 3 from bottom, replace “ $5h+75$  for women and  $w = 6h+76$ ” by “ $5h-200$  for women and  $w = 6h-254$ ”. On the next line, also replace 75 by -200 and 76 by -254.

Page 233,. On line 13 from bottom, switch we and can. On line 7 from bottom, replace Three by The three.

Page 234. On line 6, insert h after d twice. On line 7, replace function by functions. On line 6 from bottom, replace NHI by NIH .

Page 236,. On line 10, insert the before  $l^1$ -approach. On line 8 from bottom, replace 18 by 19.

Page 238. On line 5, insert of in front of one. On line 8, replace kind by kinds. On line 10 from bottom, insert a between solve and system.

Page 239, line 1. Replace on by of.

Page 240. In **Remark**, insert the before literature twice. In Exercise 9, replace  $p$  by  $p$  twice. In Exercise 9, drop the last sentence (which repeat the previous one). There are two Exercises 12. Both solved below. In the last exercise, replace **13** by **14**.

Page 241. On the line above **Example 23.2**, replace Otherwise by When the columns of  $A$  are linearly independent. On line 4 from bottom, replace  $A =$  by  $A^T =$ . On line 3 from bottom, replace  $w =$  by  $w^T =$ . On the last line, switch  $a$  and  $b$ .

Page 242. On line line 5, replace  $A =$  by  $A^T =$ . On line 6, insert  $^T$  after  $w$ . On line 8, replace  $\begin{bmatrix} b \\ a \end{bmatrix}$  by  $\begin{bmatrix} a \\ c \end{bmatrix}$ . On lines 16 and 12 from bottom, replace  $a$  by  $X$ .

Page 243,. On line 10, replace  $a =$  by  $b =$ . On line 15, replace consider by considered. On line 12 from bottom, replace  $e_i|$  by  $|e_i|$ . On lines 17 from bottom, replace know by known. On lines 15 from bottom, replace **best** by **Best**. On lines 13, 10, and 8 from bottom, replace  $a$  and  $a_j$  by  $X$ . On line 8 from bottom, replace  $t$  by  $u$ . On line 6 from bottom, add a period after 23.5.

Page 244. On lines 6 and 8 from bottom, delete `vskip-5pt` three times.

Page 246, line 8. Replace the last  $B$  by  $C$ .

Page 247. On line 11, add , before etc. On line 18, replace allowe by allow. On line 17, replace semicolumns by semicolons On lines 11 (7) and 2 from bottom, replace  $A^t$  by  $A^T$ .

Page 248, On line 4, drop ( and replace ) by }. On line 11, delete the space after Maple. On line 9 from bottom, replace of by at.

Page 249. On line 6, replace  $p = 3$  by  $p = 2$ . On line 9, replace not so by not as. On line 14, replace in trash by in the trash.

Page 250. In Ex. 9, replace best $l$  best  $l$ . In Exercise 13, replace  $+1/\alpha^{t+1}$  by  $-(-1/\alpha)^{t+1}$ . In Exercise 14, replace the first four periods by commas.

Page 251. On line 4, remove space before the comma. On line 16, replace questions by question. On line 17, insert of before \$5K. On line 22, replace those by these.

Page 252. On line 9 from bottom., put the period after **24.3** in boldface. On line 8 from bottom, replace billions by billion. Remove the period after the name of Figure 24.3.

Page 253. On line 16, replace **Example 24.3** by **Example 24.4**. On line 14 from bottom, drop one from. On line 7 from bottom, replace this year by of the year. On line 5 from bottom, replace intitial by initial. On line 4 from bottom, replace sufficiently by sufficiently.

Page 254, line 8. Replace accept by except.

Page 255. On line 6, replace liner by linear. On line 11, replace date by data. On line 19, italicize x and y.

Page 256. In Ex. 3, replace 24.3 by 24.4. On line 4 from bottom., replace  $l_p$  by  $l^p$ .

Page 258. On line 15 from bottom, drop "a". On line 14 from bottom, replace "then" by "than".

Page 259. On line 14, drop a from a a. On line 16, drop of.

Page 260. On line 8, switch . and ). On line 10, switch . and ].

Page 262. On line 10, remove the last ). On line 10 from bottom, replace  $= 0.]$  by  $= 0]$ . On line 5 from bottom, replace otherwise by otherwise.

Page 263. On line 6 and 3 from bottom, replace Lipshitz by Lipschitz.

Page 264. On lines 17, replace Lipshitz by Lipschitz and add is after or. On line 14 from bottom, replace  $[V2]$  by  $[V]$ . On line 2 from bottom, insert the second ) before  $/$ .

Page 265. On line 15, replace  $(0 - g(x_1))$  by  $(0 - g(x_1))$ . On line 7 from bottom, replace Lipshitz by Lipschitz.

Page 267. On the last line, insert the second ) before  $+$ .

Page 268. On line 8 from bottom, drop ) . On line 5 from bottom, insert ) after  $x_{t+1}$ .

Page 269. Replace the heading by that from page 267.

Page 271. Add period in the end of the second display.

Page 273. In head, replace A3. ... by A4. ... On line 11 from bottom, delete to.

Page 275. On line 7, insert ) before the comma. In (A4.6), replace the second  $F(w)$  by  $w$ .

Page 276. On line 8, delete the comma in the end. On line 9 from bottom, replace `get better` by `get a better`.

Page 277. On line 16 from bottom, delete the comma after `methods`. On line 14 from bottom, insert the second `]` after `L`.

Page 282. On line 11 from bottom, delete the third `)`.

Page 283, line 11. Replace `,` `The` by `.` `The` `.` On line 18 from bottom, insert `)` between `|` and `/` `.`

Page 284. On line 16, delete the last `)`.

Page 285. In head, replace `A6. ...` by `A5. ...`. On line 4, replace `f` by `h`. On lines 8 and 9, drop `)` from `))`. On line 9, drop `-`. On line 10, insert the `before` setting.

Page 286, line 1 and 2. Replace `Pertubation` by `Perturbation` `.`

Page 287. In head, replace `A6. ...` by `A7.Goal Programming`. On line 12 from bottom, add `)` in the end of line.

Page 289, line 5 from below. Replace `,` `While` by `.` `While` `.`

Page 290, line 6 from below. Replace `point ,and` by `point, and`.

Page 291, line 10. Replace `(ee` by `(see` `.`

Page 293, line 5 from below. Replace `by` by `be` `.`

Page 295, Theorem A10.2. Replace `s` by `n`. Replace `"n] = m"` by `"i] = n."`

Page 296. On line 6, replace `**` by `*`. On line 18 from bottom, replace `previosly` by `previously`. On line 10 from bottom, replace `.` `then` by `,` `then`. On line 5 from bottom, replace `o(11)` by `o(1)`.

Page 297. In head, replace `A11. ...` by `A10. ...`. On line 3, replace `tupple` by `tuple`. On line 2 from bottom, replace `F]` by `F0`.

Page 299. On line 12, replace `Transportation` by `The transportation`. On line 15 from bottom, remove space before the semicolon.

Page 301. Remove space before comma in `[B1]`. Remove space after period in `[DL]`. Insert comma after `C`. in `[C1]`. Add space before `"and"` in `[DL]`.

Page 302. In `[FMP]`, insert space before `C`. In `[FSS]`, replace `Forg` by `Forgó` and `Szp` by `Szép`.

Page 303. Remove space after period in `[K4]`. Remove space before comma in `[K5]`. In `[L]`, insert space before `G`. Remove space after period in `[NC]`.

Page 304. In `[S3]`, insert space before `M`. In `[VCS]`, insert space before `I`. In `[V]`, replace `V`. by `N`.

Page 318 (index). Replace assignment problem,, by assignment problem,. Delete space after Dantzig.

Page 319. Replace “48 ,” by “48,”, “.” by “,”, inconsistant by inconsistent, “Karush-Kuhn-Tucker, 280” by “Karush-Kuhn-Tucker, 260”, “KKT conditions, 280” by “KKT conditions, 260”, klein by Klein, Lagrange multiplies by Lagrange multipliers.

Page 320. Switch “operations research” and “operational research” and replace “.” by “,”. Add “,” after “payoff matrix”. Add space before = on lines 3 and 4 in the right column. Add space after = on lines 3 in the right column. Replace the period by a comma on the last row in the right column.

Page 321. Replace the period by a comma on the first row in the left column. Replace “.” by “,” 1144 by 144, and TCP by TSP.



## Chapter 1. Introduction

### §1. What Is Linear Programming?

1. True.
2. True.
3. True.
4. True.  $-35 \geq -36$ .
5. True. This is because for real numbers any square and any absolute value are nonnegative.
6. False. Take  $x = -1$ .
7. False. For  $x = -1$ ,  $3(-1)^3 < 2(-1)^2$ .
8. False (see Definition 1.5).
9. False (see Example 1.9 or 1.10).
10. False.
11. False. For example, the linear program
$$\text{Minimize } x + y \text{ subject to } x + y = 1$$
has infinitely many optimal solutions.
12. False.
13. True. It is a linear equation. A standard form is  $4x = 8$  or  $x = 2$ .
14. Yes.
15. No. This is not a linear form, but an affine function.
16. Yes, if  $z$  is independent of  $x, y$ .
17. Yes if  $a$  and  $z$  do not depend on  $x, y$ .
18. No (see Definition 1.1).
19. No. But it is equivalent to a system of two linear constraints.
20. Yes, this is a linear equation.
21. Yes. We can write  $0 = 0 \cdot x$ , which is a linear form.
22. True if  $y$  is independent of  $x$  and hence can be considered as a given number; see Definition 1.3.
23. Yes if  $a, b$  are given numbers. In fact, this is a linear equation.
24. Yes, it is equivalent to the system of two linear constraints  $-1 \leq x \leq 1$ .
25. No. We will see later that any system of linear constraints gives a convex set. But we can rewrite the constraint as follows  $x \geq 1$  OR  $x \leq -1$ . Notice the difference between OR and AND.
26. Yes, it is equivalent to  $x = 0$ .
27. See Problem 6.12.

## 2 §1. What Is Linear Programming?

28. We multiply the first equation by 5 and subtract the result from the second equation:

$$\begin{cases} x + 2y = 3 \\ -y = -11. \end{cases}$$

Multiplying the second equation by -1, we solve it for  $y$ . Substituting this into the first equation, we find  $x$ . The answer is

$$\begin{cases} x = -19 \\ y = 11. \end{cases}$$

29.  $x = 3 - 2y$  with an arbitrary  $y$ .

30. Multiplying the first equation by 3 and subtract the result from the second equation we obtain  $0 = -9$  which can be simplified to  $0 = 1$ . So the system has no solutions.

31.  $\min = 0$  at  $x = y = 0, z = -1$ . All optimal solutions are given as follows:  $x = -y, y$  is arbitrary,  $z = -1$ .

32.  $\min = 1$  when  $-2 \leq x \leq -2.5$ .

33.  $\max = 1$  at  $x = 0$ .

34. The problem is unbounded.

35.  $\min = 0$  at  $x = -y = 1/2, z = -1$ .

36. Yes, it is.

37. No. This is a linear equation.

38. No. Suppose  $x + y^2 = ax + by$  with  $a, b$  independent of  $x, y$ . Setting  $x = 0, y = 1$  we find that  $b = 1$ . Setting  $x = 0, y = -1$  we find that  $b = -1$ .

39. No.

40. Yes,  $y = 0 \cdot x + 1 \cdot y$ .

41. Yes.

42. Yes,  $(x + 1)^2 + 2y - x^2 - 1 = 2x + 2y$ .

43. No.

44. No. Suppose that  $xy$  is an affine function  $ax + by + c$  of  $x, y$ . Setting  $x = y = 0$ , we find that  $c = 0$ . Setting  $x = 0, y = 1$ , we find that  $b = 0$ . Setting  $x = 1, y = 0$ , we find that  $a = 0$ . Setting  $x = y = 1$ , we find that  $1 = 0$ .

45. Yes.

46. No.

47. Yes.

48. No.

49. No.

50. No, this is a linear form.

51. Yes.

52. Yes, this is a linear equation because  $0 = 0 \cdot x + 0 \cdot y$  is a linear form.

53. Yes. In fact, this is a linear equation.

54. Yes.

55. No. This is not even equivalent to any linear constraint with rational coefficients.

56. Yes.

57. No, see Exercise 44.

57 (58 in the next print). Let  $f(x, y) = cx + dy$  be a linear form. Then  $f(ax, ay) = cax + day = a(cx + dy) = af(x, y)$  for all  $a, x, y$  and  $f(x_1 + x_2, y_1 + y_2) = c(x_1 + x_2) + d(y_1 + y_2) = cx_1 + dy_1 + cx_2 + dy_2 = f(x_1, y_1) + f(x_2, y_2)$  for all  $x_1, x_2, y_1, y_2$ .

58 (59 in the next print). We set  $c = f(1, 0)$  and  $d = f(0, 1)$ . Then  $f(x, y) = f(x + 0, 0 + y) = f(x, 0) + f(0, y) = f(1, 0)x + f(0, 1)y = cx + dy$  for all  $x, y$ .

59 (60 in the next print).  $\min = 2^{-100} - 1$  at  $x = 0, y = 0, z = 3\pi/2, u = -100, v = -100$ . In every optimal solution,  $x, y, u, v$  are as before and  $z = 3\pi/2 + 2n\pi$  with any integer  $n$  such that  $-16 \leq n \leq 15$ . So there are exactly 32 optimal solutions.

## §2. Examples of Linear Programs

1. Date: Nov 19, 2002. Store: <http://www.peapod.com>.

	Cereal by General Mills	Box Size	Price per oz	Price per box
A	APPLE CINNAMON			
	CHEERIOS	15 OZ	\$.27	\$3.99
B	BASIC 4	16.2 OZ	\$.26	\$4.29
C	CHEERIOS	15 OZ	\$.25	\$3.79
F	FIBER ONE	16 OZ	\$.25	\$3.99
G	GOLDEN GRAHAMS	13 OZ	\$.31	\$3.99
H	HARMONY	16.7 OZ	\$.25	\$4.19
K	KIX	12.7 OZ	\$.28	\$3.59
L	LUCKY CHARMS	14 OZ	\$.30	\$4.19
M	Multi-Bran CHEX	15.6 OZ	\$.26	\$3.99
T	TOTAL Corn Flakes	10 OZ	\$.40	\$3.99

## 4 §2. Examples of Linear Programs

Food composition and Dietary Reference Intakes (DRIs) are taken from <http://www.nal.usda.gov/fnic/etext/000020.html> (Food and Nutrition Information Center, US Dept of Agriculture). The DRIs are actually a set of four reference values: Estimated Average Requirements (EAR), Recommended Dietary Allowances (RDA), Adequate Intakes (AI), and Tolerable Upper Intake Levels, (UL) that have replaced the 1989 Recommended Dietary Allowances (RDAs). DRIs for the vitamins are for a male of age 21. The RDA for protein is taken for 62.5 kg of body weight. Vitamin  $B_2$  is also known as thiamin. Data are per serving unit. The DRIs represents daily requirements.

	serving size	Protein g	A IU	$B_1$ mg	C mg	$B_6$ mg	$B_{12}$ mcg
A	30 g	1.8	500.1	0.375	6	0.501	1.5
B	55 g	4.3	392.7	0.297	0	0.39	1.155
C	30 g	3.3	500.1	0.375	6	0.501	1.5
F	30 g	2.4	0	0.375	6	0.501	1.5
G	30 g	1.5	500.1	0.375	6	0.501	1.5
H	100 g	11	909	2.73	55	1.82	7.6
K	30 g	1.8	529.2	0.375	6.3	0.501	1.5
L	30 g	2.1	500.1	0.375	6	0.501	1.5
M	49 g	3.43	445.41	0.333	5.39	0.446	1.323
T	30 g	1.815	428.7	1.5	60	2.001	6
DRI / day		50	900	1.2	90	1.3	2.4

Our variables are amounts of foods. Since most of data are per serving size, we measure all variables in serving sizes. Now we compute prices per serving size. We use prices per box rather than less precise prices per ounce. We use the US standard: 1 pound = 453.59237 grams = 16 ounces, hence 1 oz = 28.3495 g. We round off prices to 4 digits after the decimal point. For example, the price in dollars of one serving of APPLE CINNAMON CHEERIOS (A) is

$$30 \cdot 3.99 / (15 \cdot 28.3495) = 0.281486 \approx 0.2815.$$

serial serial	variable (servings)	price per serving
A	$a$	\$ 0.2815
B	$b$	\$ 0.5138
C	$c$	\$ 0.2674
F	$f$	\$ 0.2640
G	$g$	\$ 0.3448
H	$h$	\$ 0.8850
K	$k$	\$ 0.2991
L	$l$	\$ 0.3167
M	$m$	\$ 0.4421
T	$t$	\$ 0.4222

Now we can write our objective function, the total cost (in dollars per day):

$$0.2815a + 0.5138b + 0.2674c + 0.2640f + 0.3448g + 0.8850h \\ + 0.2991k + 0.3167l + 0.4421m + 0.4222t \rightarrow \min.$$

The constraints are

$$a, b, c, f, g, h, k, l, m, t \geq 0,$$

$$\begin{bmatrix} 1.8 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 4.3 & 392.7 & 0.297 & 0 & 0.39 & 1.155 \\ 3.3 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 2.4 & 0 & 0.375 & 6 & 0.501 & 1.5 \\ 1.5 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 11 & 909 & 2.73 & 55 & 1.82 & 7.6 \\ 1.8 & 529.2 & 0.375 & 6.3 & 0.501 & 1.5 \\ 2.1 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 3.43 & 445.41 & 0.333 & 5.39 & 0.446 & 1.323 \\ 1.815 & 428.7 & 1.5 & 60 & 2.001 & 6 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \\ f \\ g \\ h \\ k \\ l \\ m \\ t \end{bmatrix} \geq \begin{bmatrix} 50 \\ 900 \\ 1.2 \\ 90 \\ 1.3 \\ 2.4 \end{bmatrix}.$$

$$2. \min = 1.525 \text{ at } a = 0, b = 0.75, c = 0, d = 0.25$$

$$3. \max(P) = 100 \text{ at } x_1 = 20, x_2 = 20,$$

4. Let  $x$  be the number of quarters and  $y$  the number of dimes we pay. The program is

$$25x + 10y \rightarrow \min,$$

subject to

$$0 \leq x \leq 100, 0 \leq y \leq 90, 25x + 10y \geq C \text{ (in cents), } x, y \text{ integers.}$$

This program is not linear because the conditions that  $x, y$  are integers. For  $C = 15$ , an optimal solution is  $x = 0, y = 2$ . For

## 6 §2. Examples of Linear Programs

$C = 102$ , an optimal solution is  $x = 3, y = 3$  or  $x = 1, y = 8$ . For  $C = 10000$ , the optimization problem is infeasible.

5. Let  $x, y$  be the sides of the rectangle. Then the program is

$$\begin{aligned} xy &\rightarrow \max, \\ \text{subject to} \\ x &\geq 0, y \geq 0, 2x + 2y = 100. \end{aligned}$$

Since  $xy = x(50 - x) = 625 - (x - 25)^2 \leq 625$ ,  $\max = 625$  at  $x = y = 25$ .

6. If we choose the best worker for each work, we obtain the upper bound

$$\max \leq 20 + 70 + 90 + 67$$

for the objective function. However, this is not a matching, because B has two jobs, “a” and “d”. If “a” is done by somebody else, the objective function drops by at least 5. If “d” is done by somebody else, the objective function drops by at least 12. So Ab,Bd,Cc,Da is the optimal matching with  $\max = 15 + 70 + 90 + 67 = 242$ .

7. We can compute the objective function at all 24 feasible solutions and find the following two optimal matchings: Ac, Ba, Cb, Dd and Ac, Bb, Ca, Dd with optimal value 7.

8. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound  $9 + 9 + 7 + 9 + 9 = 43$  for the objective function. This bound cannot be achieved because of a conflict over c (the third column). So  $\max \leq 42$ . On the other hand, the matching aa, Bb, Cc, De, Ed achieved 42, so this is an optimal matching.

9. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound  $9 + 9 + 9 + 9 + 8 + 9 + 6 = 59$  for the objective function. However looking at B and C, we see that they cannot get  $9 + 9 = 18$  because of the conflict over g. They cannot get more than  $7 + 9 = 16$ . Hence, we have the upper bound  $\max \leq 57$ . On the other hand, we achieve this bound 57 in the matching Ac, Bf, Cg, De, Eb, Fd, Ga.

10. We find the best match for A, then the best remaining match for B, and so on. We obtain matching Aa, Bb, Cg, Df, Ec, Fd, Ge with  $8 + 9 + 9 + 5 + 1 + 9 + 1 = 42$  (years) total. A similar procedure for columns yield the matching Aa, Bb, Cc, Fd, De, Ef, Gg with  $8 + 9 + 7 + 9 + 1 + 5 + 4 = 43$  total, which is a better matching.

11. Let  $c_i$  be given numbers. Let  $c_j$  be an unknown maximal number (with unknown  $j$ ). The linear program is

$$c_1x_1 + \cdots + c_nx_n \rightarrow \max, \text{ all } x_i \geq 0, x_1 + \cdots + x_n = 1.$$

Answer:  $\max = c_j$  at  $x_j = 1, x_i = 0$  for  $i \neq j$ .

12.  $|a - x| + |b - x| + |c - x| \rightarrow \min$ .

### §3. Graphical Method

1. Let SSN be 123456789. Then the program is

$-x \rightarrow \max, 7x \leq 5, 13x \geq -8, 11x \leq 10$ .

Answer:  $\max = 8/13$  at  $x = -8/13$ .

2. Let SSN be 123456789. Then the program is  $f = x - 3y \rightarrow \min, |6x + 4y| \leq 14, |5x + 7y| \leq 8, |x + y| \leq 17$ .

Answer:  $\min = -22$  at  $x = -65/11, y = 59/11$ .

3. Let SSN be 123456789. Then the program is

$x + 2y \rightarrow \min, |12x + 4y| \leq 10, |5x + 15y| \leq 10, |x + y| \leq 24$ .

Answer:  $\min = -25/16$  at  $x = -11/16, y = -7/16$ .

4. The first constraint is equivalent to 2 linear constraints  $-7 \leq x \leq 3$ . The feasible region for the second constraint is also an interval,  $-8 \leq x \leq 2$ . The feasible region for the linear program is the interval  $-7 \leq x \leq 2$ . In Case (i), the objective function is an increasing function of  $x$  and reaches its maximum 14 at the right endpoint  $x = 2$ . In Case (ii), the objective function is a decreasing function of  $x$  and reaches its maximum 63 at the left endpoint  $x = -7$ . In Case (ii),

$$\max = \begin{cases} 2b \text{ at } x = 2 & \text{if } b > 0, \\ 0 \text{ when } -7 \leq x \leq 2 & \text{if } b = 0, \\ -7b \text{ at } x = -7 & \text{if } b < 0. \end{cases}$$

5.  $\min = -72$  at  $x = 0, y = -9$

6. The objective function is not defined when  $y = 0$ . When  $y = -1$  and  $x \rightarrow \infty$ , we have  $x/y \rightarrow -\infty$ . So this minimization problem is unbounded,  $\min = -\infty$ .

7.  $\min = -1/4$  at  $x = 1/2, y = -1/2$  or  $x = -1/2, y = 1/2$ .

8. There is only one feasible solution. Answer:  $\max = 3$  at  $x = 1, y = 1, z = 1$ .

9.  $\max = 1$  at  $x = y = 0$

10. This is a linear equation for  $b$ . If  $x \neq 0$ , then the unique solution is  $b = c/x$ . If  $x = 0 = c$ , then every  $b$  is a solution. If  $x = 0, c \neq 0$ , then there are no solutions.

11.  $\max = 22$  at  $x = 4, y = 2$

12. Our variables,  $a$  and  $b$  are amounts of the foods in grams (including refuse). The constraints are

## 8 §3. Graphical Method

$$a \geq 0, b \geq 0,$$

$$0.4 \cdot 5.78a + 0.98 \cdot 0.56b \geq 2000 \text{ (energy in kcal),}$$

$$0.4 \cdot 0.0024a + 0.98 \cdot 0.0005b \geq 1.1 (B_1 \text{ in mg}),$$

$$0.4 \cdot 0.0081a + 0.98 \cdot 0.0005b \geq 1.1 (B_2 \text{ in mg}).$$

The objective function in dollars to be minimized is

$$2a/454 + 3b/454.$$

We took 1 pound  $\approx 454$  g (the exact value is 453.59237). The program can be solved graphically. However, it is clear that almond are both cheaper and have more contents for all 3 ingredients, so  $b = 0$  in any optimal solution. Answer:

$$\min \approx \$5.05 \text{ at } a \approx 1146 \text{ g} \approx 2.5 \text{ lb}, b = 0.$$

13. The program is unbounded.

14. The feasible region can be given by 4 linear constraints:  $-5 \leq x \leq 0, 2 \leq y \leq 3$ . It is a rectangle with 4 corners  $[x, y] = [0, 3], [-5, 3], [-5, -2], [0, -2]$ . The objective function is not affine. Its level  $|x| + y^2 = c$  is empty when  $c < 0$ , is a point when  $c = 0$ , and is made of 2 parabola pieces when  $c > 0$ . It is clear that  $\max = 14$  at  $x = -5, y = 3$ . The optimal solution is unique.

15.  $\max = 3$  at  $x = y = 9, z = 1$ . See the answer to Exercise 11 of §2.



## Chapter 2. Background

### §4. Logic

1. False. For  $x = -1$ ,  $|-1| = 1$ .
2. False. Take  $y = 1 \neq 0, x = 0$ .
3. False. For  $x = -10$ ,  $|-10| > 1$ .
4. True.
5. True.  $1 \geq 0$ .
6. False. Take  $x = -2$ .
7. True.  $2 \geq 0$ .
8. True.
9. True. The same as Exercise 7.
10. True.
11. False.  $1 \geq 1$ .
12. False. Take  $x = -1$ .
13. True.  $5 \geq 0$ .
14. True.
15. False. For example,  $x = 2$ .
16. True.
17. True. Obvious.
18. True.
19. False. For example,  $x = 1$ .
20. True. The product of two nonzero numbers is nonzero.
21. True.  $1 \geq 0$ .
22. Yes, we can.
23. Yes.  $10 \geq 0$ .
24. True.
25. No, it does not.  $(-5)^2 > 10$ .
26. True.
27. True.
28. True.
29. False. The first condition is stronger than the second one.
30. False. The converse is true.
31. True.
32. (i)  $\Rightarrow$  (ii), (iii), (iv).
33. (i)  $\Rightarrow$  (ii), (iii), (iv).
34. (i)  $\Leftrightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (ii).
35. (i)  $\Rightarrow$  (iii).

## 10 §4. Logic

36. (i)  $\Rightarrow$  (ii) (add the constraints in (i) to obtain (ii)); (iv)  $\Rightarrow$  (iii).
37. (i)  $\Leftrightarrow$  (ii)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii)
38. (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv).
39. (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii).
40. given that, assuming that, supposing that, in the case when, granted that.
41. “only if”
42. This depends on the definition of *linear function*.
43. No.  $x \geq 1, x \leq 0$  are two feasible constraints, but the system is infeasible.
44. False.
45. False. Under our conditions,  $|x| > |y|$ .
46. Correct.
47. Correct (add the two constraints in the system and the constraint  $0 \leq 1$ ).
48. Correct (add the two constraints in the system).
49. No, it does not follow.
50. Yes, it does. It is the sum of the first two equations.
51. Yes, it does. Multiply the first equation by -2 and add to the second equation to obtain the third equation.
52. Yes, it does. It is the difference of the first two equations.

## §5. Matrices

1.  $[2, 1, -6, 6]$
2.  $-14$
3.  $-14$
4. 
$$\begin{bmatrix} 0 & -1 & -2 & 4 \\ 0 & -2 & -4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & -12 \end{bmatrix}$$
5. 
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 3 \\ -2 & -4 & 0 & 6 \\ 4 & 8 & 0 & -12 \end{bmatrix}$$
6.  $-14A^TB = \begin{bmatrix} 0 & 14 & 28 & -56 \\ 0 & 28 & 56 & -116 \\ 0 & 0 & 0 & 0 \\ 0 & -42 & -84 & 168 \end{bmatrix}$

$$7. (-14)^2 \cdot A^T B = \begin{bmatrix} 0 & -196 & -392 & 784 \\ 0 & -392 & -784 & 1568 \\ 0 & 0 & 0 & 0 \\ 0 & 588 & 1176 & -2352 \end{bmatrix}$$

$$8. -14^{999} A^T B = \begin{bmatrix} 0 & 14^{999} & 2 \cdot 14^{999} & -4 \cdot 14^{999} \\ 0 & 2 \cdot 14^{999} & 4 \cdot 14^{999} & -8 \cdot 14^{999} \\ 0 & 0 & 0 & 0 \\ 0 & -3 \cdot 14^{999} & -6 \cdot 14^{999} & 12 \cdot 14^{999} \end{bmatrix}$$

$$9. \text{No. } 1 \neq 4.$$

$$10. A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$11. A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$12. \begin{bmatrix} 5 & 2 & 3 & -1 \\ 1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$13. \begin{bmatrix} 0 & 2 & 3 & 0 & -1 \\ 1 & -1 & -3 & 0 & -2 \\ 0 & 0 & -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$14. \begin{bmatrix} 0 & 2 & 3 & 0 & 1 & -1 \\ 1 & -1 & -3 & -1 & 0 & -2 \\ 5 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$15. b = a - 1, c = -1/3, d = 7a - 4, a \text{ arbitrary}$$

$$16. \text{We permute the columns of the coefficient matrix:}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & -2 & -1 & -3 \\ 1 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ a \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Next we subtract the first row from the last row, and then add the third row to the first and the second rows with coefficients 1 and 2:

12 §5. Matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & -5 & -15 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \begin{bmatrix} x \\ a \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$$

Now we write the answer:  $x = 3c - 1, a = 5b + 15c - 4, y = 2b + 6c - 2$ , where  $b, c$  are arbitrary.

17. We will solve the system for  $x, d, a$ . So we rewrite the system (see the solution to Exercise 14 above):

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & -1 & 1 & -1 & -3 & -2 \\ 1 & 0 & 5 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now we subtract the first row from the last one and multiply the second row by  $-1$ :

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 5 & -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Then we multiply the last row by  $1/5$ :

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 1 & -2/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2/5 \end{bmatrix}$$

Finally, we add the last row to the second one:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 3/5 & 12/5 & 2 \\ 0 & 0 & 1 & -2/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2/5 \\ -2/5 \end{bmatrix}$$

So our answer is

$$x = -2b - 3c + y + 1,$$

$$d = -0.6b - 2.4c - 2y - 0.4,$$

$$a = 0.4b + 0.6c - 0.4$$

with arbitrary  $b, c, y$ .

18.  $2A + 3B = [-6, 5, -2, 0, 9, 10]$ .

19.  $AB^T = 5$

20.  $BA^T = 5$

21.  $A^T B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 3 & 2 \\ 2 & -1 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 6 & 4 \end{bmatrix}$

22.  $B^T A = \begin{bmatrix} 0 & -2 & 2 & 0 & 0 & -4 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 6 \\ 0 & 2 & -2 & 0 & 0 & 4 \end{bmatrix}$

23.  $(A^T B)^2 = 5A^T B$ , and see Answer to 21.

24.  $(A^T B)^3 = 125$

25.  $(A^T B)^{1000} = A^T (BA^T)^{999} B = 5^{999} A^T B$ , and see Answer to 21.

26.  $2A + 3B = [-1, 5, -2, 9, 9, 10, -3]$ .

27.  $AB^T = 4$

28.  $BA^T = 4$

29.  $A^T B = \begin{bmatrix} -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ 1 & -1 & 0 & -3 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 6 & 6 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

30.  $B^T A = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & 0 \\ -1 & -1 & 1 & 0 & 0 & -2 & 0 \end{bmatrix}$

14 §5. Matrices

$$31. (A^T B)^2 = 4A^T B = \begin{bmatrix} -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ 4 & -4 & 0 & -12 & -12 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 8 & 0 & 24 & 24 & 16 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$32. (A^T B)^3 = 16A^T B \\ = \begin{bmatrix} -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ 16 & -16 & 0 & -48 & -48 & -32 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 32 & 0 & 96 & 96 & 64 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$33. 4^{999} A^T B$$

$$34. AB^T = \begin{bmatrix} 89/4 & 23 \\ 341/8 & 107/2 \end{bmatrix}, BA^T = \begin{bmatrix} 89/4 & 341/8 \\ 23 & 107/2 \end{bmatrix},$$

$$A^T B = \begin{bmatrix} 9/2 & 21/4 & 23 \\ 1/2 & 9/4 & -3 \\ 27/2 & 63/4 & 69 \end{bmatrix}, B^T A = \begin{bmatrix} 9/2 & 1/2 & 27/2 \\ 21/4 & 9/4 & 63/4 \\ 23 & -3 & 69 \end{bmatrix},$$

$$(A^T B)^2 = \begin{bmatrix} 2667/8 & 6363/16 & 6699/4 \\ -297/8 & -633/16 & -809/4 \\ 8001/8 & 19089/16 & 20097/4 \end{bmatrix},$$

$$(A^T B)^3 = \begin{bmatrix} 777861/32 & 1857429/64 & 1952517/16 \\ -93351/32 & -222039/64 & -235047/16 \\ 2333583/32 & 5572287/64 & 5857551/16 \end{bmatrix}.$$

$$35. E_1 C = \begin{bmatrix} 3 & 6 & 9 \\ -8 & -10 & -12 \end{bmatrix}, E_2 C = \begin{bmatrix} 21 & 27 & 33 \\ 4 & 5 & 6 \end{bmatrix},$$

$$(E_1)^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}, (E_2)^n = \begin{bmatrix} 1 & 5n \\ 1 & 0 \end{bmatrix}.$$

$$36. CE_1 = \begin{bmatrix} 2 & 6 & 12 \\ 8 & 15 & 24 \end{bmatrix}, CE_2 = \begin{bmatrix} -8 & 2 & 3 \\ -14 & 5 & 6 \end{bmatrix},$$

$$DE_1 = \begin{bmatrix} 18 & 24 & 28 \\ 12 & 15 & 16 \\ 6 & 6 & 4 \end{bmatrix}, DE_2 = \begin{bmatrix} -12 & 8 & 7 \\ -6 & 5 & 4 \\ 0 & 2 & 1 \end{bmatrix},$$

$$E_1 E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -12 & 0 & 4 \end{bmatrix}, E_2 E_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -6 & 0 & 4 \end{bmatrix},$$

$$(E_1)^n = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}, (E_2)^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3n & 0 & 1 \end{bmatrix}.$$

$$37. \begin{bmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix}$$

38. For  $n \times n$  diagonal matrices

$$A = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}, B = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix},$$

we have

$$A + B = \begin{bmatrix} a_1 + b_1 & 0 & \dots & 0 \\ 0 & a_2 + b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n + b_n \end{bmatrix} \text{ and}$$

$$AB = BA = \begin{bmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n b_n \end{bmatrix}.$$

For  $m < n$ , diagonal  $m \times n$  matrices have the form  $[A, 0]$  and  $[B, 0]$  with  $m \times m$  diagonal matrices  $A, B$ , and  $[A, 0] + [B, 0] = [A + B, 0]$ , where  $0$  is the zero  $m \times (n - m)$  matrix. Similarly, sum of diagonal  $m \times n$  matrices is diagonal in the case  $m > n$ .

Any nondiagonal entry of the product of diagonal matrices is the dot product of two rows, each having at most one nonzero entry, and these entries are located at different positions. So the product is a diagonal matrix.

39. Let  $A = [a_{ij}], B = [b_{ij}]$  be upper triangular, i.e.,  $a_{ij} = 0 = b_{ij}$  whenever  $i > j$ . Then  $(A + B)_{ij} = a_{ij} + b_{ij} = 0$  whenever  $i > j$ , so  $A + B$  is upper triangular. For  $i > j$ , the entry  $(AB)_{ij}$  is the product of a row whose first  $i - 1 > j$  entries are zero and a column whose entries are zero with possible exception of the first  $j$  entries. So this  $(AB)_{ij} = 0$ . Thus,  $AB$  is upper triangular.

Take upper triangular matrices  $A = [1, 2], B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Then  $AB \neq BA$  (they have different sizes). Here is an example with square matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

16 §5. Matrices

Now  $AB = B \neq A = BA$ .

40. Solution is similar to that of Exercise 39, and these Exercises can be reduced to each other by matrix transposition.

41. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

42. Adding the first column to the third column we obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & -2 \\ 2 & 4 & 7 \\ 8 & -2 & 7 \end{bmatrix}.$$

Adding the second column multiplied by 2 to the third column we obtain a lower matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 2 & 4 & 15 \\ 8 & -2 & 3 \end{bmatrix}.$$

Now we kill nondiagonal entries in the second, third, and fourth rows using multiples of previous rows. Seven row addition operations bring our matrix to the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix}.$$

43. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

44. We kill the entries 5, 5 in the first column by two row addition operations:

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}.$$

Adding a multiple of the second row to the last row, we obtain the upper triangular matrix

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix}.$$

Since the diagonal entries are nonzero, we can bring this matrix to its diagonal part



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & -73/11 & 0 \end{bmatrix}$$

by five column addition operations.

### §6. Systems of Linear Equations

1.  $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$  is invertible;  $\det(A) = -4$

2. If  $a \neq 0$ , then by a row and a column addition operations we can reduced  $A$  to  $\begin{bmatrix} a & 0 \\ 0 & d - bc/a \end{bmatrix}$ . Similarly if  $d \neq 0$ , by two addition operations we can get  $\begin{bmatrix} a - bc/d & 0 \\ 0 & d \end{bmatrix}$ . If  $a = d = 0, b \neq 0$ , then adding the second column to the first column we obtain  $\begin{bmatrix} b & b \\ c & 0 \end{bmatrix}$ .

Two more addition operations produce  $\begin{bmatrix} b & 0 \\ 0 & -c \end{bmatrix}$ . In all three cases, the product of diagonal entries in the diagonal form is  $ad - bc$ , so matrix is invertible if and only if  $ad - bc \neq 0$ . Finally, when  $a = b = 0$ , then the matrix is not invertible because it has a zero row, and  $ad - bc = 0$ . So in all possible cases the matrix is invertible if and only if  $\det(A) = ad - bc \neq 0$ .

3. The matrix is invertible if and only if  $abc \neq 0$ ;  $\det(A) = abc$ .

4.  $1 \begin{matrix} \nearrow \\ \uparrow \\ \nwarrow \end{matrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mapsto -1 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mapsto$

$1 \begin{matrix} \nearrow \\ \uparrow \\ \nwarrow \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ . The matrix is invertible,

$\det(A) = -1$ .

5.  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$  is invertible;  $\det(A) = 2$

6.  $-2 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} -5 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} \mapsto -4 \begin{matrix} \nwarrow \\ \downarrow \\ \searrow \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 4 & 7 \end{bmatrix} \mapsto$

$\nearrow \xrightarrow{1} \searrow$

18 §6. Systems of Linear Equations

$$\begin{array}{c} \nearrow \quad \searrow \\ -8 \end{array} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix}.$$

The matrix is invertible,  $\det(A) = -25$ .

7.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 13/7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$  is invertible;  $\det(A) = 13$

8.  $\begin{array}{c} \nearrow \quad \xrightarrow{1} \quad \searrow \\ \nearrow \quad \xrightarrow{2} \quad \searrow \end{array} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 5 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 6 & 0 & 7 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 16 & 5 & 7 \end{bmatrix}$ . By two row addition

operations, we obtain  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}$ . Now we perform three more

row addition operations:

$$\begin{array}{c} -10 \nearrow \\ -16 \nearrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix} \mapsto -1 \nwarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & -9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -9 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & -9 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 5 & -9 \end{bmatrix}. \text{ By two column addition operations, we clear}$$

the second row and obtain  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 5 & -9 \end{bmatrix}$ . Now we work on the

submatrix  $\begin{bmatrix} -8 & 9 \\ 5 & -9 \end{bmatrix}$ . Adding the last column to the previous

one, we obtain  $\begin{bmatrix} 1 & 9 \\ -4 & -9 \end{bmatrix}$ . Now we use the entry 1 to clear the

first row and the first column. By two addition operations we obtain  $\begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix}$ . Thus, by addition operations with integral coefficients we

reduced the matrix  $A$  to the diagonal matrix  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 27 \end{bmatrix}$ . So

the matrix  $A$  is invertible, and  $\det(A) = 27$ .

9.  $0 = 1$  (no solutions)

10. We do one row addition operation with the augmented matrix and then drop the zero row:

$-2 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & 2 & | & 3 \\ 2 & 4 & | & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix} \mapsto [1, 2 | 3]$ . Answer:  $x = 3 - 3y$ ,  $y$  being arbitrary.

11.  $x = -z - 3b + 10$ ,  $y = -z + 2b - 6$ .

12. We perform two addition operations on the augmented matrix:

$$-1 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & 4 & | & 1 \\ 1 & 5 & | & -8 \end{bmatrix} \mapsto -4 \begin{array}{c} \nearrow \\ \swarrow \end{array} \begin{bmatrix} 1 & 4 & | & 1 \\ 0 & 1 & | & -9 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & | & 37 \\ 0 & 1 & | & -9 \end{bmatrix}.$$

Answer:  $x = 37$ ,  $y = -9$ .

13. If  $t \neq 6 + 2u$ , then there are no solutions. Otherwise,  $x = -2y + u + 3$ ,  $y$  arbitrary.

14. We do a row addition operation on the augmented matrix:

$$-2 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & t & | & 3 \\ 2 & 4 & | & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & t & | & 3 \\ 0 & 4 - 2t & | & -5 \end{bmatrix}.$$

If  $t = 2$ , the second equation shows that the system has no solutions. Otherwise we perform a multiplication and an addition operation:

$$\begin{aligned} 1/(4 - 2t) \begin{bmatrix} 1 & t & | & 3 \\ 0 & 4 - 2t & | & -5 \end{bmatrix} &\mapsto -t \begin{array}{c} \nearrow \\ \swarrow \end{array} \begin{bmatrix} 1 & t & | & 3 \\ 0 & 1 & | & 5/(2t - 4) \end{bmatrix} \\ &\mapsto \begin{bmatrix} 1 & 0 & | & 3 - 5t/(2t - 4) \\ 0 & 1 & | & 5/(2t - 4) \end{bmatrix}. \end{aligned}$$

Answer:  $0 = 1$  when  $t = 2$ ;  $x = (t - 12)/(2t - 4)$ ,  $y = 5/(2t - 4)$  when  $t \neq 2$ .

15. If  $t = 1$ , then  $x = 1 - y$ ,  $y$  arbitrary.

If  $t = -1$ , there are no solutions.

If  $t \neq \pm 1$ , then  $x = (t^2 + t + 1)/(t + 1)$ ,  $y = -1/(t + 1)$ .

16. We do a row addition operation on the augmented matrix:

$$-2 \begin{array}{c} \swarrow \\ \searrow \end{array} \begin{bmatrix} 1 & t & | & t^2 \\ 2 & u & | & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & t & | & t^2 \\ 0 & u - 2t & | & 1 - 2t^2 \end{bmatrix}.$$

## 20 §6. Systems of Linear Equations

If  $u - 2t = 0 = 1 - 2t^2$ , then the second row is zero. We drop this row and the answer is  $x = t^2 - ty$ ,  $y$  arbitrary. If  $u - 2t = 0 \neq 1 - 2t^2$ , there are no solutions.

If  $u - 2t \neq 0$ , we perform a multiplication and an addition operation:

$$\begin{aligned} & 1/(u - 2t) \left[ \begin{array}{cc|c} 1 & t & t^2 \\ 0 & u - 2t & 1 - 2t^2 \end{array} \right] \mapsto \\ & \quad -t \begin{array}{l} \nearrow \\ \nwarrow \end{array} \left[ \begin{array}{cc|c} 1 & t & t^2 \\ 0 & 1 & (1 - 2t^2)/(u - 2t) \end{array} \right] \\ & \mapsto \left[ \begin{array}{cc|c} 1 & 0 & t^2 - t(1 - 2t^2)/(u - 2t) \\ 0 & 1 & (1 - 2t^2)/(u - 2t) \end{array} \right]. \end{aligned}$$

Answer:  $0 = 1$  when  $u = 2t \neq \sqrt{2}$ ;  $x = t^2 - ty$ ,  $y$  arbitrary when  $u = 2t = \sqrt{2}$ ;  $x = t^2 - t(1 - 2t^2)/(u - 2t)$ ,  $y = (1 - 2t^2)/(u - 2t)$  when  $u \neq 2t$ .

17. It is convenient to write the augmented matrix corresponding to the variables  $y, z, x$  (rather than  $x, y, z$ ). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$\begin{aligned} & -5/3 \begin{array}{l} \nwarrow \\ \searrow \end{array} \left[ \begin{array}{ccc|c} y & z & x & \\ 3 & 5 & 2 & 2 \\ 5 & 8 & 3 & b \end{array} \right] \mapsto \\ & 1/3 \cdot \left[ \begin{array}{ccc|c} 3 & 5 & 2 & 2 \\ 0 & -1/3 & -1/3 & b - 10/3 \end{array} \right] \mapsto \\ & -5/3 \begin{array}{l} \nearrow \\ \nwarrow \end{array} \left[ \begin{array}{ccc|c} 1 & 5/3 & 2/3 & 2/3 \\ 0 & 1 & 1 & -3b + 10 \end{array} \right] \mapsto \\ & \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 5b - 16 \\ 0 & 1 & 1 & -3b + 10 \end{array} \right]. \end{aligned}$$

Answer:  $y = x + 5b - 16$ ,  $z = -x - 3b + 10$ ,  $x$  is arbitrary.

18. It is convenient to write the augmented matrix corresponding to the variables  $x, z, y$  (rather than  $x, y, z$ ). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$\begin{aligned} & -3/2 \begin{array}{l} \nwarrow \\ \searrow \end{array} \left[ \begin{array}{ccc|c} x & z & y & \\ 2 & 5 & 3 & 2 \\ 3 & 8 & 5 & b \end{array} \right] \mapsto \\ & 1/2 \cdot \left[ \begin{array}{ccc|c} 2 & 5 & 3 & 2 \\ 0 & 1/2 & 1/2 & b - 3 \end{array} \right] \mapsto \end{aligned}$$

$$\begin{array}{c} \nearrow \\ -5/2 \swarrow \end{array} \left[ \begin{array}{ccc|c} 1 & 5/2 & 3/2 & 1 \\ 0 & 1 & 1 & 2b-6 \end{array} \right] \mapsto \left[ \begin{array}{ccc|c} 1 & 0 & -1 & -5b+16 \\ 0 & 1 & 1 & 2b-6 \end{array} \right].$$

Answer:  $x = y - 5b + 16$ ,  $z = -y + 2b - 6$ ,  $y$  is arbitrary.

19. No. The halfsum of solutions is a solution.

20. We augment  $A$  by the identity matrix  $1_4$  and then make it  $1_4$  by elementary row operations. First we permute rows:

$$\begin{array}{c} \left[ \begin{array}{cccc|cccc} 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \mapsto \\ \left[ \begin{array}{cccc|cccc} -1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 & 0 & 0 \end{array} \right].
 \end{array}$$

Now we perform two addition, four multiplication operations, and four addition operations:  $[A|1_4] =$

$$\begin{array}{c} \swarrow \quad \nwarrow \\ 1 \mid \quad 3 \swarrow \quad \nwarrow \end{array} \left[ \begin{array}{cccc|cccc} -1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 & 0 & 0 \end{array} \right] \mapsto \\ \begin{array}{c} -1 \cdot \\ -1 \cdot \\ -1 \cdot \\ 1/2 \cdot \end{array} \left[ \begin{array}{cccc|cccc} -1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 1 & 1 & 0 \end{array} \right] \mapsto \\ \begin{array}{c} \nearrow \quad \nwarrow \\ 2 \mid \quad 1 \swarrow \quad \nwarrow \end{array} \left[ \begin{array}{cccc|cccc} 1 & -1 & -2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \end{array} \right] \mapsto \\ \begin{array}{c} \nearrow \quad \nwarrow \\ 2 \downarrow \quad 1 \downarrow \swarrow \quad \nwarrow \end{array} \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & -2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \end{array} \right] \mapsto \\ \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \end{array} \right] = [1_4, A^{-1}]
 \end{array}$$

## 22 §6. Systems of Linear Equations

$$21. A^{-1} = \begin{bmatrix} 7/25 & 4/25 & -1/25 \\ 19/25 & -7/25 & 8/25 \\ -18/25 & 4/25 & -1/25 \end{bmatrix}$$

$$22. [A, 1_3] =$$

$$\begin{array}{l} \begin{array}{c} \swarrow \quad \swarrow \\ -2 \downarrow \quad -5 \searrow \\ \quad \searrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 5 & 1 & 3 & | & 0 & 1 & 0 \\ 2 & 4 & 5 & | & 0 & 0 & 1 \end{bmatrix} \mapsto \\ \\ -8/11 \begin{array}{c} \swarrow \quad \swarrow \\ \quad \searrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 11 & 8 & | & -5 & 1 & 0 \\ 0 & 8 & 7 & | & -2 & 0 & 1 \end{bmatrix} \mapsto \\ \\ 1/11 \cdot \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 11 & 8 & | & -5 & 1 & 0 \\ 0 & 0 & 13/11 & | & 18/11 & -8/11 & 1 \end{bmatrix} \mapsto \\ 11/13 \cdot \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 11 & 8 & | & -5 & 1 & 0 \\ 0 & 0 & 13/11 & | & 18/11 & -8/11 & 1 \end{bmatrix} \mapsto \\ \\ -8/11 \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 8/11 & | & -5/11 & 1/11 & 0 \\ 0 & 0 & 1 & | & 18/13 & -8/13 & 11/13 \end{bmatrix} \mapsto \\ \\ 1 \uparrow \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array} 2 \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array} \begin{bmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -19/13 & 7/13 & -8/13 \\ 0 & 0 & 1 & | & 18/13 & -8/13 & 11/13 \end{bmatrix} \mapsto \\ \\ \begin{bmatrix} 1 & 0 & 0 & | & -7/13 & 6/13 & -5/13 \\ 0 & 1 & 0 & | & -19/13 & 7/13 & -8/13 \\ 0 & 0 & 1 & | & 18/13 & -8/13 & 11/13 \end{bmatrix} = [1_3, A^{-1}]. \end{array}$$

$$23. A^{-1} = \begin{bmatrix} -3/22 & -1/22 & -41/22 & 3/11 \\ -15/22 & -5/22 & -51/22 & 4/11 \\ 5/22 & 9/22 & 61/22 & -5/11 \\ 15/22 & 5/22 & 73/22 & -4/11 \end{bmatrix}$$

$$24. [A, 1_2] =$$

$$-6 \begin{array}{c} \swarrow \quad \swarrow \\ \quad \searrow \end{array} \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 6 & 8 & | & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -4 & | & -6 & 1 \end{bmatrix} = [U, L^{-1}]$$

$$\text{with } A = LU. \text{ So } A = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$$

25. If  $a \neq 0$ , then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d - bc/a \end{bmatrix}.$$

If the first column of  $A$  is zero, then  $A = 1_2 A = LU$ . If the first row of  $A$  is zero, then  $A = A 1_2 = LU$ . Finally, if  $a = 0 \neq bc$ , then  $A \neq LU$ .

26. The matrix  $U = A$  is upper triangular, and  $A = I_3 A = LU$ .

27. This cannot be done. Suppose  $A = LU$ . At the position (1,1), we have  $0 = A_{11} = L_{11}U_{11}$ . Since the first row of  $A$  is nonzero, we conclude that  $L_{11} \neq 0$ . Since the first column of  $A$  is nonzero, we conclude that  $U_{11} \neq 0$ . Thus,  $0 = A_{11} = L_{11}U_{11} \neq 0$ .

28. This cannot be done, because  $A$  is invertible (see the solution of Exercise 5) and  $A_{11} = 0$ . See the solution of Exercise 27.

$$29. A = LU = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix}.$$

$$UL = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -1 \\ 21 & 33 & 8 \\ -50 & -100 & -25 \end{bmatrix}.$$

$$30. A = LU$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix}.$$

$$UL = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -30/11 & -1 \\ 71 & 185/11 & 8 \\ 26/11 & 104/121 & 13/11 \end{bmatrix}.$$

31. We multiply  $A$  by a lower triangular matrix  $E_1^{-1}$  on left and obtain  $E_1^{-1}A$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 5 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 6 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}.$$

Then we multiply by a lower triangular matrix  $E_2^{-1}$  on left and obtain  $E_2^{-1}E_1^{-1}A$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/11 & 1 & 0 \\ 0 & 16/11 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -8/11 & 10/11 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix}.$$

## 24 §6. Systems of Linear Equations

Then we multiply by a lower triangular matrix  $E_3^{-1}$  on left and obtain an upper triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 73/8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -8/11 & 10/11 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -8/11 & 10/11 \\ 0 & 0 & 0 & -11/4 \end{bmatrix} = R.$$

So  $A = LR$  with  $L = E_1 E_2 E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 1/11 & 1 & 0 \\ 5 & 16/11 & 73/8 & 1 \end{bmatrix}.$

32. Answer:

$$x = a + b^2 + c^3 - d,$$

$$y = a + b^2 - 3c^3 + 2d,$$

$$z = -a - b^2 + 2c^3 - d.$$

33.  $x = -3(19 + 2d)/8, y = (15 + 2d)/8, z = -(3 + 2d)/8$

34. Answer:

$$x = u_1 + u_2 - u_3,$$

$$y = u_1 - 3u_2 + 2u_3,$$

$$z = -u_1 + 2u_2 - u_3.$$

35.  $x = (15u + 4v)/16, y = (11u + 4v)/16, z = -3u/4$

36.  $x = y = 0, z = 5.$

37.  $x = y = 1, z = 1$

38.  $x = y = 1, z = 0.$

39.  $x = y = 0, z = 100$

40. It is clear that any nonzero column with at least two entries can be reduced to the first column of the identity matrix by row addition operations. By induction on the number of columns, it follows that any  $m \times n$  matrix with linearly independent columns can be reduced by row addition operations to the matrix of the first  $n$  columns of  $I_m$  provided that  $m > n$ . Therefore any invertible  $m \times m$  matrix can be reduced by row addition operations to the diagonal matrix with the first  $m - 1$  diagonal entries being ones, and the last entry being the determinant. One row multiplication operation applied to this matrix gives  $I_m$ .



Therefore multiplication by an invertible matrix on the left is equivalent to performing row addition operations and a row multiplication operation.

If  $A = 0$ , then  $b = 0$ ,  $A' = 0$ , and  $b' = 0$ , so there is nothing to prove. Similarly, the case  $A' = 0$  is trivial. Assume now that  $A \neq 0$  and  $A' \neq 0$ .

Let  $B$  be the submatrix in  $[A, b]$  such that the rows of  $B$  form a basis for the row space of  $A$ , and let  $B'$  be a similar matrix for  $[A', b']$ . By Theorem 6.11,  $B' = DB$  and  $B = D'B'$  for some matrices  $D, D'$ . We have  $B = D'DB$  and  $B' = DD'B'$ . Since the rows of  $B$  are linearly independent,  $D'D$  is the identity matrix. Since the rows of  $B'$  are linearly independent,  $DD'$  is the identity matrix. So  $D$  is invertible, hence  $B, B'$  have the same size. So row operations on  $A$  allows to change the rows of  $B$  to the rows of  $B'$ . Now by row addition operations we can make the other rows of  $[A, b]$  (if any) equal to remaining rows of  $[A', b']$  (if any). A row permutation operation finish the job.

41. We use the parts of the previous solution. In particular, it is clear that the rank of the matrices of  $[A, b]$  and  $[A', b']$  are the same. By row addition operations we can make the last  $m - m'$  rows of  $[A, b]$  to be zeros. Then, as shown in the previous solution, we can the first  $m'$  rows to be the rows of  $[A', b']$  by row addition operations and a row multiplication operations.

The only thing remaining to show is how to replace a row multiplication operation by row addition operations in presence of a zero row. Here is how this can be done:

$$1 \begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} \mapsto (d-1) \begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix} \begin{bmatrix} r \\ r \end{bmatrix} \mapsto -1/d \begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix} \begin{bmatrix} dr \\ r \end{bmatrix} \mapsto \begin{bmatrix} dr \\ 0 \end{bmatrix},$$

where multiplication of a row  $r$  by a nonzero number  $d$  is accomplished by three addition operations.

## Chapter 3. Tableaux and Pivoting

### §7. Standard and Canonical Forms for Linear Programs

1. Set  $u = y + 1 \geq 0$ . Then  $f = 2x + 3y = 2x + 3v - 3$  and  $x + y = x + u - 1$ . A canonical form is

$$-f = -2x - 3v + 3 \rightarrow \min, x + u \leq 6, u, x \geq 0.$$

A standard form is

$$-f = -2x - 3v + 3 \rightarrow \min, x + u + v = 6, u, v, x \geq 0$$

with a slack variable  $v = 6 - x - u \geq 0$ .

2. Excluding  $y = x + 1$  and using  $y \geq 1$ , we obtain the canonical form

$$-x \rightarrow \min, 2x \leq 8, x \geq 0.$$

Introducing a slack variable  $z = 8 - 2x$ , we obtain the standard form

$$-x \rightarrow \min, 2x + z = 8, x \geq 0, z \geq 0.$$

3. We solve the equation for  $x_3$ :

$$x_3 = 3 - 2x_2 - 3x_4$$

and exclude  $x_3$  from the LP:

$$x_1 - 7x_2 + 3 \rightarrow \min, x_1 - x_2 + 3x_4 \geq 3, \text{ all } x_i \geq 0.$$

A canonical form is

$$x_1 - 7x_2 + 3 \rightarrow \min, -x_1 + x_2 - 3x_4 \leq -3, \text{ all } x_i \geq 0.$$

A standard form is

$$x_1 - 7x_2 + 3 \rightarrow \min, -x_1 + x_2 - 3x_4 + x_5 = -3, \text{ all } x_i \geq 0$$

with a slack variable  $x_5 = x_1 - x_2 + 3x_4 - 3$ .

4. We can solve the given equation for  $x_4$ :

$$x_4 = 1 - 2x_2/3 - x_3/3.$$

Then exclude  $x_4$  from our LP. The objective function  $f$  becomes  $x_1 - 7x_2 + 3$ . The constraint  $x_1 - x_2 + 3x_4 \geq 3$  becomes  $x_1 - 3x_2 - x_3 \geq 0$ . So a canonical form for our LP is

$$-f = -x_1 + 7x_2 - 3 \rightarrow \min, -x_1 + 3x_2 + x_3 \leq 0; x_1, x_2, x_3 \geq 0.$$

To get a standard form we introduce a slack variable  $x_5$ :

$$-f = -x_1 + 7x_2 - 3 \rightarrow \min, -x_1 + 3x_2 + x_3 + x_5 = 0; x_1, x_2, x_3, x_5 \geq 0.$$

Note that we cannot have the variable  $x_4$  in a normal or standard form unless we know that it takes only nonnegative values.

5. Set  $t = x + 1 \geq 0, u = y - 2 \geq 0$ . The objective function is  $x + y + z = t + u + z + 1$ . Then a standard and a canonical form for our problem is

$$x + u + z + 1 \rightarrow \min; t, u, z \geq 0.$$

6. This mathematical program has exactly two optimal solution, but the set of optimal solutions of any LP is convex and hence cannot consist of exactly two optimal solutions (cf. Exercise 19 in §6.). Each of two optimal solution can be the optimality region for a linear program. For example,  $\min = -26$  at  $x = 1, y = -3, z = 0$  is the only answer for the linear program  $-26 \rightarrow \min, x = 1, y = -3, z = 0$ .

7. Using standard tricks, a canonical form is

$$-x \rightarrow \min, x \leq 3, -x \leq -2, x \geq 0.$$

A standard form is

$$-x \rightarrow \min, x + u = 3, -x + v = -2; x, u, v \geq 0$$

with two slack variables.

8. Excluding  $y = 1 - x$  from the LP, we obtain

$$f = -x + z + 2 \rightarrow \max, z \geq 0.$$

Writing  $x = u - v$  and replacing  $f$  by  $-f$ , we obtain a normal and standard form:

$$-f = u - v - z - 2 \rightarrow \min; u, v, z \geq 0.$$

It is clear that the program is unbounded.

9. One of the given equations reads

$$-5 - x - z = 0,$$

which is inconsistent with given constraints  $x, z \geq 0$ . So we can write very short canonical and standard forms:

$$0 \rightarrow \min, 0 \leq -1; x, y, z \geq 0 \text{ and } 0 \rightarrow \min, 0 = 1; x, y, z \geq 0.$$

10. The first matrix product is not defined.

11. Set  $x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}^T]$  and  $c = [3, -1, 1, 3, 1, -5, 1, 3, 1]$ . Using standard tricks, we obtain the canonical form

$$cx \rightarrow \min, Ax \leq b, x \geq 0$$

with

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -2 & 3 & 1 \\ 2 & -2 & -2 & 2 & 3 & -1 & -2 & 1 & 1 \\ -2 & 2 & 2 & -2 & -3 & 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 & 3 & -1 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

## 28 §8. Pivoting Tableaux

and  $b = [-3, -1, 2, -2, 0, 0]^T$ .

Excluding a couple of variables using the two given equations, we would get a canonical form with two variables and two constraints less. A standard form can be obtained from the canonical form by introducing a column  $u$  of slack variables:

$$cx \rightarrow \min, Ax + u = b, x \geq 0, u \geq 0.$$

## §8. Pivoting Tableaux

1.

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \wedge & \wedge & \wedge & \wedge & \wedge \\ a & b & c & d & e & 1 \\ \left[ \begin{array}{ccccc} .3 & 1.2 & .7 & 3.5 & 5.5 & -50 \\ 73 & 96 & 20253 & 890 & 279 & -4000 \\ 9.6 & 7 & 19 & 57 & 22 & -1000 \\ 10 & 15 & 5 & 60 & 8 & 0 \end{array} \right] & \begin{array}{l} = u_1 \geq 0 \\ = u_2 \geq 0 \\ = u_3 \geq 0 \\ = C \rightarrow \min \end{array} \end{array}$$

2.

$$\begin{array}{cccc} a & b & c & d \\ \left[ \begin{array}{cccc} .9 & .8 & .7 & .6 \\ 1 & 1 & 1 & 1 \\ 1.2 & 1.4 & 1.7 & 1.9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] & \begin{array}{l} = .75 \\ = 1 \\ = C \rightarrow \min \\ \geq 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \end{array} \end{array}$$

$$3. A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ -1 & 0 & 0 & 2 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

4. Canonical form:

$$z + 2 \rightarrow \min,$$

$$-3x - 2y - z - 3 \leq 0,$$

$$-x + 3y \leq 3,$$

$$x - 3y \leq -3,$$

$$-x - 2y + 2z \leq 0;$$

$$x, y, z \geq 0.$$

Standard form:

$$z + 2 \rightarrow \min,$$

$$-3x - 2y - z - 3 + u = 0,$$

$$-x + 3y = 3,$$

$$-x - 2y + 2z + v = 0;$$

$$x, y, z, u, v \geq 0.$$

5. Canonical form:

$$y - 5z + 2 \rightarrow \min,$$

$$-3x - y + 5z \leq 3,$$

$$-x - y \leq -10,$$

$$x + y \leq 10,$$

$$-2y + 10z \leq -7;$$

$$x, y, z \geq 0.$$

Standard form:

$$y - 5z + 2 \rightarrow \min,$$

$$-3x - y + 5z + u = 3,$$

$$x + y = 10,$$

$$-2y + 10z + v = -7;$$

$$x, y, z, u, v \geq 0.$$

6. Canonical form:  $-z + 6 \rightarrow \min$ ,

$$-2y + z \leq 0, 2y - z \leq 0, x + 3y \leq -3, 2y + 2z \leq 0; x, y, z \geq 0.$$

Standard form:  $-z + 6 \rightarrow \min$ ,

$$-2y + z = 0, x + 3y + u = -3, 2y + 2z + v = 0; x, y, z, u, v \geq 0.$$

It is clear that the program is infeasible (the constraints  $x \geq 0, y \geq 0, x + 3y \leq -3$  are inconsistent).

7. Canonical form:  $-x + z + 2 \rightarrow \min$ ,

$$-3x - 2y - z \leq 2, x - 3y \leq 1, 2y - 2z \leq 0; x, y, z \geq 0.$$

Standard form:  $-x + z + 2 \rightarrow \min$ ,

$$-3x - 2y - z + u = 2, x - 3y + v = 1, 2y - 2z + w = 0;$$

$$x, y, z, u, v, w \geq 0.$$

8.

$$(a) \quad \begin{array}{cccc|l} x' & x'' & u & 1 & \\ \hline -3 & -2 & 1 & -b & = y \\ -1 & 3 & 0 & 1 & = a \\ -6 & -6 & 2 & -2b & = x' \\ -3 & -2 & 1 & 2 - b & = z \end{array}$$

(b)

**30** §8. Pivoting Tableaux

$$\begin{array}{cccc} x' & x'' & z & 1 \\ \left[ \begin{array}{cccc} 3 & 2 & 1 & b-2 \\ -1 & 3 & 0 & 1 \\ 0 & -2 & 2 & -4 \\ 0 & 0 & 1 & -2 \end{array} \right] & \begin{array}{l} = u \\ = a \\ = x' \\ = y \end{array} \end{array}$$

9. The matrix is not square.

$$10. \begin{array}{cccc} 1 & a & 3 & x \\ \left[ \begin{array}{cccc} 1^* & 0 & b & a \\ -1 & 2 & 3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = z \end{array} & \mapsto \end{array}$$

$$\begin{array}{cccc} y & a & 3 & x \\ \left[ \begin{array}{cccc} 1 & 0 & -b & -a \\ -1 & 2 & 3+b & 1+a \end{array} \right] & \begin{array}{l} = 1 \\ = z \end{array} \end{array}$$

$$11. \begin{array}{cccc} z & a & 3 & x \\ \left[ \begin{array}{cccc} 1 & 2 & b+3 & a+1 \\ -1 & 2 & 3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = 1 \end{array} \end{array}$$

$$12. \begin{array}{cccc} 1 & a & 3 & z \\ \left[ \begin{array}{cccc} 1+a & -2a & b-3a & a \\ 1 & -2 & -3 & 1 \end{array} \right] & \begin{array}{l} = y \\ = x \end{array} \end{array}$$

$$13. \begin{array}{c} 2 \\ \left[ 1/5 \right] \end{array} = x$$

$$14. \begin{array}{cccc} 1 & a & 0 & x & x \\ \left[ \begin{array}{ccccc} 1 & 0 & b & a & -3 \\ -1 & 2^* & 3 & 1 & 0 \end{array} \right] & \begin{array}{l} = y \\ = z \end{array} & \mapsto \end{array}$$

$$\begin{array}{ccccc} 1 & z & 0 & x & x \\ \left[ \begin{array}{ccccc} 1 & 0 & b & a & -3 \\ 1/2 & 1/2 & -3/2 & -1/2 & 0 \end{array} \right] & \begin{array}{l} = y \\ = a \end{array} \end{array}$$

$$15. \begin{array}{ccccc} 1 & u & 0 & x & 1 \\ \left[ \begin{array}{ccccc} 1 & 0 & b & a & -3 \\ 0 & 1 & 0 & 0 & -1 \\ 1/2 & 1/2 & -3/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1 \end{array} \right] & \begin{array}{l} = y \\ = z \\ = 0 \\ = v \end{array} \end{array}$$

$$\begin{array}{l}
16. \quad \begin{array}{ccccc} x_1 & x_2 & x_8 & x_4 & 1 \\ \left[ \begin{array}{ccccc} 4/3 & -2/3 & 1/3 & 2/3 & -3 \\ -1/3 & -4/3 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1/3 & -2/3 & 1/3 & -1/3 & 0 \\ -1 & 1 & 0 & 1 & 1 \end{array} \right] & \begin{array}{l} = x_5 \\ = x_6 \\ = x_7 \\ = x_3 \\ = v \end{array} \end{array} \\
\\
17. \quad \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_7 & x_6 & 1 \\ \left[ \begin{array}{ccccccc} 1/3 & 0 & 1/3 & 1/3 & -1/3 & 1/3 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -2/3 & 2 & 10/3 & 4/3 & -1/3 & 1/3 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -2/3 & 1 & 1/3 & 4/3 & -1/3 & 7/3 & 3 \end{array} \right] & \begin{array}{l} = x_5 \\ = x_8 \\ = x_9 \\ = x_{10} \\ = v \end{array} \end{array}
\end{array}$$

18. Let us show that every column  $b$  of  $A$  equals to the corresponding column  $b'$  of  $A'$ . We set the corresponding variable on the top to be 0, and the other variables on the top to be zeros. Then the variables on the side take certain values, namely,  $y = b = b'$ .

#### §9. Standard Row Tableaux

1. Passing from the standard row tableau on page 95 to the canonical form (i.e., dropping the slack variables), we obtain a Linear program with one variable:  $y/2 - 15/2 \rightarrow \min$ ,

$$16y - 26 \geq 0, -3y/2 + 15/2 \geq 0, 3y/2 - 15/2 \geq 0, y \geq 0.$$

We rewrite our constraints:  $y \geq 0, 13/8, 5; y \leq 5$ , so  $y = 5$ . In terms of the standard tableau, our answer is

$$\min(-z) = -5 \text{ at } y = 5, w_1 = 54, w_2 = w_3 = 0.$$

In terms of the original variables, our answer is

$$\max(z) = 5 \text{ at } u = -3, v = -4, x = -1, y = 5.$$

$$2. \quad \begin{array}{ccc} x & y & 1 \\ \left[ \begin{array}{ccc} -4 & -5 & 7 \\ -2 & -3 & 0 \end{array} \right] & \begin{array}{l} = u \\ = -P \rightarrow \min \end{array} \end{array}$$

with a slack variable  $u = 7 - 4x - 5y$

$$3. \quad \begin{array}{ccccc} x' & x'' & y' & y'' & 1 \\ \left[ \begin{array}{ccccc} -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \end{array} \right] & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = -x \rightarrow \min \end{array} \end{array}$$

with  $x = x' - x'', y = y' - y''$  and slack variables  $u_i$ .

## 32 §9. Standard Row Tableaux

4. We introduce a slack variable  $u_1$  for the first constraint and slack variables  $u_2, u_3$  for the third constraint (replaced by two inequalities). Here is the resulting standard tableau:

$$\begin{array}{ccccccc|l} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & -1 & \\ \hline \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -3 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} & \begin{array}{l} = u_1 \\ = x_8 \\ = u_2 \\ = u_3 \\ = v \rightarrow \min \end{array} \end{array}$$

$x_9, x_{10} \geq 0.$

To get a smaller standard tableau, we can solve the equation for  $x_5$  and exclude  $x_5$  from the first and fourth rows. In other words we pivot on third entry 1 in the  $x_5$ -column and then drop the column with 0 on top. This gives the standard tableau

$$\begin{array}{cccccc|l} x_1 & x_2 & x_3 & x_4 & x_6 & -1 & \\ \hline \begin{bmatrix} -2 & 6 & 10 & 4 & 1 & 0 \\ -1 & 0 & 2 & 1 & 1 & -2 \\ 1 & -2 & -3 & -1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 2 & 3 \end{bmatrix} & \begin{array}{l} = u_1 \\ = x_8 \\ = x_5 \\ = v \rightarrow \min \end{array} \end{array}$$

$x_9, x_{10} \geq 0.$

5. We multiply the last row by -1 and remove the the second and third rows from the tableau

$$\begin{array}{cccccc|l} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 & \\ \hline \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & -1 & -2 & -3 \end{bmatrix} & \begin{array}{l} = x_7 \\ = x_2 \\ = -v \rightarrow \min \end{array} \end{array}$$

$$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 3x_3 + x_4 + x_5.$$

The tableau is not standard because  $x_2$  occurs twice. We pivot on 1 in the  $x_2$ -row and  $x_6$ -column:

$$\begin{array}{cccccc|l} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 & \\ \hline \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1^* & 1 \\ 1 & -1 & 0 & -1 & -1 & -2 & -3 \end{bmatrix} & \begin{array}{l} = x_7 \\ = x_2 \\ = -v \rightarrow \min \end{array} & \mapsto \\ \hline \begin{array}{cccccc|l} x_1 & x_2 & x_3 & x_4 & x_5 & x_2 & 1 & \\ \hline \begin{bmatrix} 1 & -2 & -2 & 0 & -3 & 1 & -1 \\ 0 & -2 & -3 & -1 & 0 & 1 & -1 \\ 1 & 3 & 6 & 1 & -1 & -2 & -1 \end{bmatrix} & \begin{array}{l} = x_7 \\ = x_6 \\ = -v \rightarrow \min \end{array} \end{array} \end{array}$$

Now we combine two  $x_2$  from the top and obtain the standard tableau



$$\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & 1 \\
\left[ \begin{array}{cccccc} 1 & -1 & -2 & 0 & -3 & -1 \\ 0 & -1 & -3 & -1 & 0 & -1 \\ 1 & 1 & 6 & 1 & -1 & -1 \end{array} \right] & & & & & \\
& & & & & = x_7 \\
& & & & & = x_6 \\
& & & & & = -v \rightarrow \min
\end{array}$$

The equations

$$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 5x_3 + x_4 + x_5$$

relate this LP with the original LP. After we solve the program without  $x_8$  and  $x_9$ , we complete the answer with the values for  $x_8$  and  $x_9$ . By the way, looking at the equation for  $x_6$  (the second row of the standard tableau), we see the the program is infeasible.

6. We combine the first and the third column, subtract the first row from the second one, and multiply the third row by -1:

$$\begin{array}{cccccc}
x_1 & x_2 & x_4 & x_5 & x_6 & 1 \\
\left[ \begin{array}{cccccc} -1 & 0 & 1 & -3 & 1 & 0 \\ -4 & 0 & 0 & 3 & 0 & -2 \\ 9 & -2 & -1 & -1 & 0 & 0 \\ -11 & 2 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 & 2 & 3 \end{array} \right] & & & & & \\
& & & & & = x_7 \\
& & & & & = 0 \\
& & & & & \geq 0 \\
& & & & & = x_3 \\
& & & & & = v \rightarrow \min
\end{array}$$

We introduce a slack variable  $u$  for the third row, pivot on  $-4$ , and drop the first column, and obtain the standard tableau

$$\begin{array}{cccccc}
x_2 & x_4 & x_5 & x_6 & 1 \\
\left[ \begin{array}{cccccc} 0 & 1 & -15/4 & 1 & 1/2 \\ 0 & 0 & 3/4 & 0 & -1/2 \\ -2 & -1 & 23/4 & 0 & -9/2 \\ 2 & 1 & -33/4 & 1 & 13/2 \\ 1 & 1 & 1/4 & 2 & 7/2 \end{array} \right] & & & & & \\
& & & & & = x_7 \\
& & & & & = x_1 \\
& & & & & = u \\
& & & & & = x_3 \\
& & & & & = v \rightarrow \min
\end{array}$$

7. We pivot on the first 1 in the first row and then on 3 in the second row:

$$\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\
\left[ \begin{array}{cccccc} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \end{array} \right] & & & & & & \\
& & & & & & = x_2 \\
& & & & & & = x_4 \\
& & & & & & = v \rightarrow \min
\end{array} \mapsto$$

$$\begin{array}{cccccc}
x_2 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\
\left[ \begin{array}{cccccc} 1 & 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & 3^* & 2 & -3 & 2 & -2 \\ -1 & 2 & 4 & 2 & -2 & 1 & 0 \end{array} \right] & & & & & & \\
& & & & & & = x_1 \\
& & & & & & = x_4 \\
& & & & & & = v \rightarrow \min
\end{array} \mapsto$$

34 §9. Standard Row Tableaux

$$\begin{array}{ccccccc}
 x_2 & x_2 & x_4 & x_4 & x_5 & x_6 & 1 \\
 \left[ \begin{array}{ccccccc}
 2/3 & 0 & -1/3 & -1/3 & 2 & -1/3 & -2/3 \\
 1/3 & 0 & 1/3 & -2/3 & 1 & -2/3 & 2/3 \\
 1/3 & 2 & 4/3 & -2/3 & 2 & -5/3 & 8/3
 \end{array} \right] & \begin{array}{l} = x_1 \\ = x_3 \\ = v \rightarrow \min \end{array} .
 \end{array}$$

Now we combine two  $x_2$ -columns and two  $x_4$ -columns and obtain the standard tableau

$$\begin{array}{ccccccc}
 x_2 & x_4 & x_5 & x_6 & 1 \\
 \left[ \begin{array}{ccccccc}
 2/3 & -2/3 & 2 & -1/3 & -2/3 \\
 1/3 & -1/3 & 1 & -2/3 & 2/3 \\
 7/3 & 2/3 & 2 & -5/3 & 8/3
 \end{array} \right] & \begin{array}{l} = x_1 \\ = x_3 \\ = v \rightarrow \min \end{array} .
 \end{array}$$

Do not forget the constraints  $x_7, x_8, x_9, x_{10} \geq 0$  outside the tableaux.

8. We make three pivot steps:

$$\begin{array}{ccccccc}
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & -4 \\
 \left[ \begin{array}{ccccccc}
 1 & 0 & 1 & 1 & -3 & 1 & 0 \\
 -1 & 0 & 2 & 1 & 0 & 1^* & -2 \\
 -1 & 2 & 3 & 1 & 1 & 0 & 0 \\
 -1 & 2 & 3 & 1 & 0 & 1 & 1 \\
 -1 & 1 & 0 & 1 & 1 & 2 & 3
 \end{array} \right] & \begin{array}{l} \geq 0 \\ = 1 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \quad \mapsto
 \end{array}$$

$$\begin{array}{ccccccc}
 x_1 & x_2 & x_3 & x_4 & x_5 & 1 & -4 \\
 \left[ \begin{array}{ccccccc}
 2 & 0 & -1 & 0 & -3 & 1 & 2 \\
 1 & 0 & -2 & -1 & 0 & 1 & 2 \\
 -1 & 2 & 3 & 1 & 1^* & 0 & 0 \\
 0 & 2 & 1 & 0 & 0 & 1 & 3 \\
 1 & 1 & -4 & -1 & 1 & 2 & 7
 \end{array} \right] & \begin{array}{l} \geq 0 \\ = x_6 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \quad \mapsto
 \end{array}$$

$$\begin{array}{ccccccc}
 x_1 & x_2 & x_3 & x_4 & 0 & 1 & -4 \\
 \left[ \begin{array}{ccccccc}
 -1 & 6 & 8 & 3 & -3 & 1 & 2 \\
 1 & 0 & -2 & -1 & 0 & 1 & 2 \\
 1 & -2 & -3 & -1 & 1 & 0 & 0 \\
 0 & 2 & 1^* & 0 & 0 & 1 & 3 \\
 2 & -1 & -7 & -2 & 1 & 2 & 7
 \end{array} \right] & \begin{array}{l} \geq 0 \\ = x_6 \\ = x_5 \\ = x_1 \\ = v \rightarrow \min \end{array} \quad \mapsto
 \end{array}$$

$$\begin{array}{ccccccc}
 x_1 & x_2 & x_1 & x_4 & 0 & 1 & -4 \\
 \left[ \begin{array}{ccccccc}
 -1 & -10 & 8 & 3 & -3 & -7 & -22 \\
 1 & 4 & -2 & -1 & 0 & 3 & 8 \\
 1 & 4 & -3 & -1 & 1 & 3 & 9 \\
 0 & -2 & 1 & 0 & 0 & -1 & -3 \\
 2 & 13 & -7 & -2 & 1 & 9 & 28
 \end{array} \right] & \begin{array}{l} \geq 0 \\ = x_6 \\ = x_5 \\ = x_3 \\ = v \rightarrow \min . \end{array}
 \end{array}$$

Now we introduce a slack variable  $u$  for the first row, combine some columns, and obtain the following standard tableau:

$$\begin{array}{cccc|l} x_1 & x_2 & x_4 & 1 & \\ \hline \begin{bmatrix} 7 & -10 & 3 & -29 \\ -1 & 4 & -1 & 11 \\ -2 & 4 & -1 & 12 \\ 1 & -2 & 0 & -4 \\ -5 & 13 & -2 & 37 \end{bmatrix} & & & & \begin{array}{l} = u \\ = x_6 \\ = x_5 \\ = x_3 \\ = v \rightarrow \min. \end{array} \end{array}$$

9. We pivot the three zeros from the right margin to the top and drop the corresponding columns:

$$\begin{array}{cccccc|l} x_7 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 & \\ \hline \begin{bmatrix} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} & & & & \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \end{array} \mapsto$$

$$\begin{array}{cccccc|l} x_2 & x_3 & x_4 & x_5 & x_6 & 1 & \\ \hline \begin{bmatrix} 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 3 & 2 & -3 & 2 & -2 \\ 2 & 4 & 2 & -2 & 1^* & 0 \\ 2 & 4 & 2 & -3 & 2 & 1 \\ 1 & 1 & 2 & -2 & 3 & 3 \end{bmatrix} & & & & \begin{array}{l} = x_7 \\ = 0 \\ = 0 \\ = x_1 \\ = v \rightarrow \min \end{array} \end{array} \mapsto$$

$$\begin{array}{cccc|l} x_2 & x_3 & x_4 & x_5 & 1 & \\ \hline \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ -4 & -5 & -2 & 1^* & -2 \\ -2 & -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 1 & 1 \\ -5 & -11 & -4 & 4 & 3 \end{bmatrix} & & & & \begin{array}{l} = x_7 \\ = 0 \\ = x_6 \\ = x_1 \\ = v \rightarrow \min \end{array} \end{array} \mapsto$$

$$\begin{array}{cccc|l} x_2 & x_3 & x_4 & 1 & \\ \hline \begin{bmatrix} 6 & 8 & 3 & 2 \\ 4 & 5 & 2 & 2 \\ 6 & 6 & 2 & 4 \\ 2 & 1 & 0 & 3 \\ 11 & 9 & 4 & 11 \end{bmatrix} & & & & \begin{array}{l} = x_7 \\ = x_5 \\ = x_6 \\ = x_1 \\ = v \rightarrow \min. \end{array} \end{array}$$

Now we take the first equation

$$x_7 = 6x_2 + 8x_3 + 3x_4 + 2$$

outside the table (since  $x_7$  is not required to be  $\geq 0$ ) and obtain the standard tableau

36 §9. Standard Row Tableaux

$$\begin{array}{cccc} x_2 & x_3 & x_4 & 1 \\ \left[ \begin{array}{cccc} 4 & 5 & 2 & 2 \\ 6 & 6 & 2 & 4 \\ 2 & 1 & 0 & 3 \\ 11 & 9 & 4 & 11 \end{array} \right] & \begin{array}{l} = x_5 \\ = x_6 \\ = x_1 \\ = v \rightarrow \min. \end{array} \end{array}$$

10. We take the row

$$x_7 = -x_1 + 2x_2 + 3x_3 + x_4 + x_5$$

out the tableau, multiply the last row and column by -1, and add the first row to the second row with coefficient -2. We obtain

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\ \left[ \begin{array}{cccccc} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ -3 & 0 & 0 & -1 & 6 & -2 & 2 \\ -1 & 2 & 3 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & -1 & -2 & 3 \end{array} \right] & \begin{array}{l} = x_1 \\ = 0 \\ = x_2 \\ = -v \rightarrow \min. \end{array} \end{array}$$

Now we take three pivot steps

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\ \left[ \begin{array}{cccccc} 1 & 0 & 1^* & 1 & -3 & 1 & 0 \\ -3 & 0 & 0 & -1 & 6 & -2 & 2 \\ -1 & 2 & 3 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & -1 & -2 & 3 \end{array} \right] & \begin{array}{l} = x_1 \\ = 0 \\ = x_2 \\ = -v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_1 & x_4 & x_5 & x_6 & 1 \\ \left[ \begin{array}{cccccc} -1 & 0 & 1 & -1 & 3 & -1 & 0 \\ -3 & 0 & 0 & -1^* & 6 & -2 & 2 \\ -4 & 2 & 3 & -2 & 9 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & 3 \end{array} \right] & \begin{array}{l} = x_3 \\ = 0 \\ = x_2 \\ = -v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_1 & 0 & x_5 & x_6 & 1 \\ \left[ \begin{array}{cccccc} 2 & 0 & 1 & 1 & -3 & 1 & -2 \\ -3 & 0 & 0 & -1 & 6 & -2 & 2 \\ 2 & 2 & 3 & 2 & -3 & 2^* & -5 \\ -2 & -1 & 0 & -1 & 5 & -4 & 5 \end{array} \right] & \begin{array}{l} = x_3 \\ = x_4 \\ = x_2 \\ = -v \rightarrow \min \end{array} \quad \mapsto \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_1 & 0 & x_5 & x_2 & 1 \\ \left[ \begin{array}{cccccc} 1 & -1 & -1/2 & 0 & -3/2 & 1/2 & 1/2 \\ -1 & 2 & 3 & 1 & 3 & -1 & -3 \\ -1 & -1 & -3/2 & -1 & 3/2 & 1/2 & 5/2 \\ 2 & 3 & 6 & 3 & -1 & -2 & -5 \end{array} \right] & \begin{array}{l} = x_3 \\ = x_4 \\ = x_6 \\ = -v \rightarrow \min. \end{array} \end{array}$$

Now we drop the column with 0 on top, combine two columns with  $x_1$  on top, combine two columns with  $x_2$  on top, which gives the following standard tableau:

$$\begin{array}{cccc}
x_1 & x_2 & x_5 & 1 \\
\left[ \begin{array}{cccc}
-1/2 & -1/2 & -3/2 & 1/2 \\
2 & 1 & 3 & -3 \\
-5/2 & -1/2 & 3/2 & 5/2 \\
8 & 1 & -1 & -5
\end{array} \right] & & & \\
& & & = x_3 \\
& & & = x_4 \\
& & & = x_6 \\
& & & = -v \rightarrow \min .
\end{array}$$

## Chapter 4. Simplex Method

### §10. Simplex Method, Phase 2

1. The tableau is optimal, so the basic solution is optimal:

$\min = 0$  at  $a = b = c = d = 0$ ,

$y_1 = 0.4, y_2 = 0.4, y_3 = 0, y_4 = 0.5, y_5 = 1, y_6 = 0.1$ .

2. The  $y_2$ -row is bad. The program is infeasible.

3. The first column is bad. However since the tableau is not feasible, this is not sufficient to conclude that the program is unbounded. Still we set  $z_2 = z_3 = z_4 = 0$ , and see what happens as  $z_1 \rightarrow \infty$ . We have  $y_1 = 0.4 \geq 0, y_2 = 3z_1 + 0.4 \geq 0, y_3 = 0.6z_1 \geq 0, y_4 = 0.6z_1 + 0.5 \geq 0, y_5 = 0.1z_1 - 0.1 \geq 0, y_6 = 0.1 \geq 0$  for  $z_1 \geq 1$ , and the objective function  $-11z_1 \rightarrow -\infty$ . So  $\min = -\infty$ .

4. False. The converse is true.

5. True

6. True

7. First we write the program in a standard tableau and then we apply the simplex method (Phase 2):

$$\begin{array}{cccc|cl}
 x_1 & x_2 & x_3 & 1 & & \\
 \left[ \begin{array}{cccc}
 -40^* & -20 & -60 & 1200 \\
 -4 & -1 & -6 & 300 \\
 -0.2 & -0.7 & -2 & 40 \\
 -100 & -100 & -800 & 8000 \\
 -0.1 & -0.3 & -0.8 & 8 \\
 -2 & -3 & -7 & 0
 \end{array} \right] & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \rightarrow \min
 \end{array} & \mapsto \\
 \\
 u_1 & x_2 & x_3 & 1 & & \\
 \left[ \begin{array}{cccc}
 -0.025 & -0.5 & -1.5 & 30 \\
 0.1 & 1 & 0 & 180 \\
 0.005 & -0.6 & -1.7 & 34 \\
 2.5 & -50 & -650 & 5000 \\
 0.0025 & -0.25^* & -0.65 & 5 \\
 0.05 & -2 & -4 & -60
 \end{array} \right] & \begin{array}{l} = x_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \rightarrow \min
 \end{array} & \mapsto
 \end{array}$$

$$\begin{array}{cccc}
u_1 & u_5 & x_3 & 1 \\
\left[ \begin{array}{cccc}
-0.03 & 2 & -0.2 & 20 \\
0.11 & -4 & -2.6 & 200 \\
0.001 & 2.4 & -0.14 & 22 \\
2 & 200 & -520 & 4000 \\
0.01 & -4 & -2.6 & 20 \\
0.03 & 8 & 1.2 & -100
\end{array} \right] & & & \\
& & & & = x_1 \\
& & & & = u_2 \\
& & & & = u_3 \\
& & & & = u_4 \\
& & & & = x_2 \\
& & & & = -P \rightarrow \min.
\end{array}$$

This tableau is optimal, so

$$\max(P) = 100 \text{ at } x_1 = 20, x_2 = 20, x_3 = 0.$$

The zero values for the nonbasic slack variables  $u_1$  and  $u_5$  indicate that the corresponding resource limits are completely used (no slack there). The other resources are not completely used; some reserves left.

8.  $\min = -\infty$ . For example, we set  $x = y = 0, v = -1$ . Then the objective function  $z \rightarrow -\infty$  as  $z \rightarrow -\infty$ .

9. First we solve the system of linear equations for  $a, b$  and the objective function  $f$  and hence obtain the standard tableau

$$\begin{array}{ccc}
c & d & 1 \\
\left[ \begin{array}{ccc}
1 & 2 & -0.5 \\
-2 & -3 & 1.5 \\
0.1 & 0.1 & 1.5
\end{array} \right] & & \\
& & = a \\
& & = b \\
& & = f \rightarrow \min.
\end{array}$$

The tableau is not row feasible so we cannot apply Phase 2. Until we learn Phase 1, we can pivot at random:

$$\begin{array}{ccc}
c & d & 1 \\
\left[ \begin{array}{ccc}
1^* & 2 & -0.5 \\
-2 & -3 & 1.5 \\
0.1 & 0.1 & 1.5
\end{array} \right] & & \\
& & = a \\
& & = b \\
& & = f \rightarrow \min
\end{array} \mapsto$$

$$\begin{array}{ccc}
a & d & 1 \\
\left[ \begin{array}{ccc}
1 & -2 & 0.5 \\
-2 & 1 & 0.5 \\
0.1 & -0.1 & 1.55
\end{array} \right] & & \\
& & = c \\
& & = b \\
& & = f \rightarrow \min.
\end{array}$$

Now the tableau is feasible, and we can use Phase 2:

$$\begin{array}{ccc}
a & d & 1 \\
\left[ \begin{array}{ccc}
1 & -2^* & 0.5 \\
-2 & 1 & 0.5 \\
0.05 & -0.1 & 1.55
\end{array} \right] & & \\
& & = c \\
& & = b \\
& & = f \rightarrow \min
\end{array} \mapsto$$

40 §10. Simplex Method, Phase 2

$$\begin{array}{ccc|c} a & c & 1 & \\ \hline \left[ \begin{array}{ccc} 0.5 & -0.5 & 0.25 \\ -1.5 & -0.5 & 0.75 \\ 0.15 & 0.05 & 1.525 \end{array} \right] & & & \\ \hline & & & = d \\ & & & = b \\ & & & = f \rightarrow \min. \end{array}$$

The tableau is optimal, so

$\min = 1.525$  at  $a = 0, b = 0.75, c = 0, d = 0.25$ .

10. First we write the program in the standard tableau

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & 1 & \\ \hline \left[ \begin{array}{ccccc} 1 & -1 & 0 & 1 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & -2 \end{array} \right] & & & & & \\ \hline & & & & & = x_5 \\ & & & & & = x_6 \\ & & & & & \rightarrow \min \end{array}$$

The tableau is optimal, so  $\min = -2$  at  $x_1 = x_2 = x_3 = x_4 = 0, x_5 = 3, x_6 = 1$ .

11. Set  $f = x_2 + 2x_3 - 2$  (the objective function). We write the program in the standard tableau

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & 1 & \\ \hline \left[ \begin{array}{ccccc} 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 & 2 \end{array} \right] & & & & & \\ \hline & & & & & = x_5 \\ & & & & & = x_6 \\ & & & & & = -f \rightarrow \min \end{array}$$

The tableau is feasible, and two columns are bad (namely, the  $x_2$ -column and  $x_3$ -column), so the program is unbounded ( $\max(f) = \infty$ ).

12. Set  $f = x_2 + 2x_3 + 2$  (the objective function). We write the program in the standard tableau

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & 1 & \\ \hline \left[ \begin{array}{ccccc} 1 & 1 & 0 & 1 & -3 \\ 1 & 1 & 2 & 1 & -1 \\ 0 & -1 & 2 & 0 & -2 \end{array} \right] & & & & & \\ \hline & & & & & = x_5 \\ & & & & & = x_6 \\ & & & & & = -f \rightarrow \min \end{array}$$

We set  $x_1 = 3$  and obtain a feasible tableau

$$\begin{array}{ccccc|c} x_2 & x_3 & x_4 & 1 & & \\ \hline \left[ \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ -1 & 2 & 0 & -2 \end{array} \right] & & & & & \\ \hline & & & & & = x_5 \\ & & & & & = x_6 \\ & & & & & = -f \rightarrow \min \end{array}$$

with the first column bad. So the program is unbounded.

13. If the row without the last entry is nonnegative, then the tableau is optimal; else the LP is unbounded.

14. Either the tableau is optimal, or the row is bad, or one pivot step according to the simplex method makes the tableau optimal.



## §11. Simplex Method, Phase 1

1. The second row ( $v$ -row) is bad, so the LP is infeasible.
2. The tableau is optimal, so the basic solution is optimal:  
 $\min = 0$  at  $x = y = z = 0, u = 2, v = 0$ .  
 This is the only optimal solution.

3. This is a feasible tableau with a bad column (the  $z$ -column).  
 So the LP is unbounded ( $z$  and hence  $w$  can be arbitrarily large).

4. First we make the tableau standard by scaling the second and fourth columns:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & -1 & -4 \\ 2 & 1 & 0 & 6 \\ 0 & -2 & -1 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array}$$

According to the simplex method, we pivot on 1 in the first row:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1^* & 0 & -1 & -4 \\ 2 & 1 & 0 & 6 \\ 0 & -2 & -1 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} a & b & d & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & 1 & 4 \\ 2 & 1 & 2 & 14 \\ 0 & -2 & -1 & 0 \end{array} \right] & = c \\ & = e \\ & \rightarrow \min \end{array}.$$

Now we have a feasible tableau with two bad columns ( $b$ - and  $d$ -columns), so the program is unbounded ( $\min = -\infty$ ).

5. The tableau is standard. According to the simplex method, we pivot on 1 in the first row:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1^* & 2 & 3 & -1 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 0 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} d & b & c & 1 \\ \left[ \begin{array}{cccc} 1 & -2 & -3 & 1 \\ 2 & -4 & -5 & 5 \\ -1 & 3 & 3 & -1 \end{array} \right] & = a \\ & = e \\ & \rightarrow \min \end{array}$$

The tableau is feasible, and the  $d$ -column is bad, so the program is unbounded ( $\min = -\infty$ ).

6. We scale the last column to get the standard tableau

42 §11. Simplex Method, Phase 1

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1 & 2 & 3 & 3 \\ 2 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{array} \right] & = d \\ & = e \\ & \rightarrow \min \end{array} .$$

This is a feasible tablea with a bad column (the fist column), so  $\min = -\infty$ .

7. We scale the last column and then pivot on the first 1 in the first row to get both  $c$  on the top:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1^* & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 \end{array} \right] & = c \\ & = d \\ & = f \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} c & b & c & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & -1 \end{array} \right] & = a \\ & = d \\ & = f \rightarrow \min . \end{array}$$

Now we combine two  $c$ -columns and obtain the standard tableau

$$\begin{array}{ccc} b & c & 1 \\ \left[ \begin{array}{ccc} 0 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{array} \right] & = a \\ & = d \\ & = f \rightarrow \min . \end{array}$$

The first row is bad, so the program is infeasible. In fact, the first constraint in the original tableau is inconsistent with the constraint  $a \geq 0$ .

8. We scale the last column and obtain the standard tableau

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 \end{array} \right] & = e \\ & = d \\ & = f \rightarrow \min . \end{array}$$

Then we pivot on the first 1 in the second row:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1^* & 1 & -1 \\ 1 & 2 & 3 & 0 \end{array} \right] & = e \\ & = d \\ & = f \rightarrow \min \end{array} \mapsto$$

$$\begin{array}{cccc} a & d & c & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 1 & 2 \end{array} \right] & = e \\ & = b \\ & = f \rightarrow \min . \end{array}$$

The tableau is optimal, so the basic solution is optimal:

$\min = 2$  at  $a = d = c = 0, e = 1, b = 1$ .

9. True

10. False

11. We use the simplex method:

$$\left[ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \\ 1^* & 0 & -2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & -1 & 1 & 0 & 2 \end{array} \right] \begin{array}{l} = x_5 \\ = x_6 \\ = x_7 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[ \begin{array}{ccccc} x_5 & x_2 & x_3 & x_4 & 1 \\ 1 & 0 & 2 & 3 & 1 \\ -1 & 1 & -1 & -2 & 0 \\ 2 & -1^* & 4 & 7 & 5 \\ 1 & -1 & 3 & 3 & 3 \end{array} \right] \begin{array}{l} = x_1 \\ = x_6 \\ = x_7 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[ \begin{array}{ccccc} x_5 & x_7 & x_3 & x_4 & 1 \\ 1 & 0 & 2 & 3 & 1 \\ 1 & -1 & 3 & 5 & 5 \\ 2 & -1 & 4 & 7 & 5 \\ -1 & 1 & -1 & -4 & -2 \end{array} \right] \begin{array}{l} = x_1 \\ = x_6 \\ = x_2 \\ \rightarrow \min. \end{array}$$

Phase 1 was done in one pivot step, and Phase 2 also was done in one pivot step, because we obtain a feasible tableau with a bad column ( $x_5$ -column). The program is unbounded.

12. We use the simplex method:

$$\left[ \begin{array}{cccc} x_1 & x_2 & x_3 & 1 \\ 1^* & 0 & -1 & -1 \\ -1 & 3 & 1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array} \right] \begin{array}{l} = x_4 \\ = x_5 \\ = x_6 \\ = x_7 \\ \rightarrow \min \end{array} \mapsto$$

44 §11. Simplex Method, Phase 1

$$\left[ \begin{array}{cccc} x_4 & x_2 & x_3 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 3^* & 0 & -1 \\ 3 & -1 & 5 & 4 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} = x_1 \\ = x_5 \\ = x_6 \\ = x_7 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[ \begin{array}{cccc} x_4 & x_5 & x_3 & 1 \\ 1 & 0 & 1 & 1 \\ 1/3 & 1/3 & 0 & 1/3 \\ 8/3 & -1/3 & 5 & 11/3 \\ 2/3 & -1/3^* & 2 & 2/3 \\ 2/3 & -1/3 & 0 & 2/3 \end{array} \right] \begin{array}{l} = x_1 \\ = x_2 \\ = x_6 \\ = x_7 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[ \begin{array}{cccc} x_4 & x_7 & x_3 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 6/3 & 1 & 3 & 9/3 \\ 2 & -3 & 6 & 2 \\ 0 & 1 & -2 & 0 \end{array} \right] \begin{array}{l} = x_1 \\ = x_2 \\ = x_6 \\ = x_5 \\ \rightarrow \min . \end{array}$$

The  $x_3$ -column is bad, so the program is unbounded.

13. We use the simplex method:

$$\left[ \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 3 & 1^* & 0 & -2 & -1 \\ 3 & -1 & 2 & 1 & 2 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & -1 & 2 \end{array} \right] \begin{array}{l} = x_6 \\ = x_7 \\ = x_8 \\ = x_9 \\ \rightarrow \min \end{array} \mapsto$$

$$\left[ \begin{array}{cccccc} x_1 & x_2 & x_7 & x_4 & x_5 & 1 \\ 0 & 3 & -1 & -1 & -2 & 0 \\ 1 & -3 & 1^* & 0 & 2 & 1 \\ 5 & -7 & 2 & 1 & 6 & 3 \\ 2 & -4 & 1 & 0 & 1 & 2 \\ -1 & 2 & -1 & 0 & -3 & 1 \end{array} \right] \begin{array}{l} = x_6 \\ = x_3 \\ = x_8 \\ = x_9 \\ \rightarrow \min . \end{array}$$

The first column in this feasible tableau is bad, so the program is unbounded.

## §12. Geometric Interpretation

1. The diamond can be given by four linear constraints  $\pm x \pm y \leq 1$ .
2. Any convex combination of convex combinations is a convex combination.
3. We have to prove that if  $u = [x_1, y_1], v = [x_2, y_2]$  are feasible, i.e.,

$$x_1^4 + y_1^4 \leq 1 \text{ and } x_2^4 + y_2^4 \leq 1,$$

then the point  $au + (1 - a)v$  is also feasible for  $0 \leq a \leq 1$ , i.e.,

$$(ax_1 + (1 - a)x_2)^4 + (ay_1 + (1 - a)y_2)^4 \leq 1.$$

Clearly, it suffices to do this in the case when  $x_1, x_2, y_1, y_2 \geq 0$ . In other words, it suffices to prove that the region

$$x^4 + y^4 \leq 1, x \geq 0, y \geq 0$$

is convex. The function  $y = (1 - x^4)^{1/4}$  is smooth on the interval  $0 < x < 1$ , so it suffices to show that its slope decreases. At the point  $[x, y] = [x, (1 - x^4)^{1/4}]$  the slope is  $-x^3/y^3$  so it decreases.

Similarly, we can prove that the region  $|x|^p + |y|^p \leq 1$  is convex for any  $p \geq 1$ . In the case  $p = 1$  the slope is  $-1$ , a constant function.

4. The halfsum of the top point of the diamond and the top point of the triangle does not belong to the union.

5. The points  $[x, y] = [3, 1], [3, -1]$  belong to the circle but the halfsum  $[3, 0]$  (the center of the circle) does not.

6. The halfsum of the lowest points of the intervals does not belong to the union.

7. Both  $x = 1$  and  $x = -1$  belong to the feasible region, but  $0 = x/2 + y/2$  does not.

8. The tangent to the disc at the point  $[x, y] = [2t/(1 + t^2), (1 - t^2)/(1 + t^2)]$  is  $2tx + (1 - t^2)y = 1 + t^2$ . The family of linear constraints  $2tx + (1 - t^2)y \leq 1 + t^2$ , where  $t$  ranges over all rational numbers, gives the disc.

9. A set  $S$  is called closed if it contains the limit points of all sequences in  $S$ . Any system of linear constraints gives a closed set, but the interval  $0 < x < 1$  is not closed. Its complement is closed.

10. The rows of the identity matrix  $1_6$ . If the vectors are written as columns, take the columns of  $1_6$ .

11. One.

12. In terms of two variables on the top, we have  $m + 1$  linear constraints (including two sign restrictions). So the feasible region in plane has at most  $m + 1$  corners (vertices). Therefore Phase 2

terminates in at most  $m$  pivot steps. In the case of  $m$  steps, the initial and the terminal vertex are on one side (edge), so one pivot step would be sufficient. Thus, a better choice for the first pivot step results in at most  $m - 1$  pivot steps.

13. Since  $x_i$  are affine, (a)  $\Rightarrow$  (b). It is also clear that (b)  $\Rightarrow$  (c) and (d)  $\Rightarrow$  (e)  $\Rightarrow$  (a). So it remains to prove that (c)  $\Rightarrow$  (d). The last implication follows from the well-known inequality

$$(|x_1| + \cdots + |x_n|)/n \leq ((x_1^2 + \cdots + x_n^2)/n)^{1/2}.$$

14. Let  $x$  be in  $S$  is not a vertex. We find distinct  $y, z$  in  $S$  such that  $x = (y + z)/2$ . The linear constraints giving  $S$  restricted on the line  $ay + (1 - a)z$  give linear constraints on  $a$ . The interval  $0 \leq a \leq 1$  is a part of the feasible set. Any of the constraints is either tight on the whole interval, or is tight only at an end point. So the tight constraints are the same for all points  $ay + (1 - a)z$  with  $0 < a < 1$ .

The “only if” part proven, consider now the “if” part. Here is a counter example with an infinite system of constraints: The linear constraints are  $x \geq c$  where  $c$  runs over all negative numbers. The feasible set  $S$  is the ray  $x \geq 0$ . No defining constraint is tight for any feasible  $x$  but  $x = 0$  is the only vertex.

So we assume the  $S$  is given by a *finite* system of linear constraints. Let  $x, y$  be in  $S$  and  $x$  is a vertex and  $y$  has the same tight constraints. If  $y \neq x$  then the same constraints are tight for every point on the line  $ax + (1 - a)y$ . For a number  $a_0 > 1$  sufficiently closed to 1, all other constraints are also satisfied (here we use the finiteness). We pick such a number  $a_0 < 2$ . Then

$$x = ((a_0x + (1 - a_0)y) + ((2 - a_0)x + (a_0 - 1)y))/2$$

is not a vertex because  $a_0x + (1 - a_0)y$  and  $(2 - a_0)x + (a_0 - 1)y$  are distinct points in  $S$ .

15. Suppose that  $x$  is optimal,  $y, z$  are in the convex set  $S$ , and  $x = (y + z)/2$ . We have to prove that  $y = z$ .

Since  $f$  is affine,  $f(x) = (f(y) + f(z))/2$ . Since  $x$  is optimal, so are  $y$  and  $z$ . By uniqueness of optimal solution,  $y = x = z$ .

16. Every vertex is the basic solution for a feasible tableau. Then we can change the last row of this tableau, so its entries (except the last entry whose value does not matter) are positive. For the LP given by such an optimal tableau the optimal solution is unique.

Let  $S$  be the open diamond  $|x| + |y| \leq 1$  with all convex combinations of the points  $[1/3, 2/3], [2/3, 1/3]$  added. Then these two points are vertices. A linear form  $ax + by$  is minimized at  $[1/3, 2/3]$

if and only if  $a = b \leq 0$ . But in this case it is also minimized at every convex combination of  $[1/3, 2/3], [2/3, 1/3]$ .

17. Our set  $S$  is a subset of  $R^n$ . Let  $x$  be a point of  $S$  such that it is the limit of a sequence  $y^{(1)}, y^{(2)}, \dots$  of points outside  $S$  (in other words,  $x$  belongs to the boundary of  $S$ ). For example,  $x$  could be a vertex of  $S$ .

For each  $t$ , we find the point  $z$  in the closure of  $S$  closest to  $y^{(t)}$ . (We use the Euclidean distance  $((y - z) \cdot (y - z))^{1/2}$  between points  $y, z$  in  $R^n$ .) We consider the linear constraint

$$(y^{(t)} - z^{(t)}) \cdot X \leq (y^{(t)} - z^{(t)})(y^{(t)} + y^{(t)})/2.$$

All points in  $S$  satisfy this constraint, while the point  $y^{(t)}$  does not. Now we scale this constraint so it takes the form  $c^{(t)}X \leq b^{(t)}$  with  $c^{(t)} \cdot c^{(t)} = 1$ . Then we take a limit constraint  $c \cdot X \leq b$ . Then all points in  $S$  satisfy the latter constraint and  $c \cdot x = b$ . Thus,  $x$  is maximizer of the linear form  $f = c \cdot X \neq 0$  over  $S$ .

When the set  $S$  is closed, and  $x$  is a vertex, we can arrange  $x$  to be an unique maximizer.

18. Suppose that  $x$  is not a vertex in  $S'$ . Then  $x = (y + z)/2$  with distinct  $y, z$  in  $S'$ . Since  $y, z$  are in  $S$ ,  $x$  is not a vertex in  $S$ .

## Chapter 5. Duality

### §13. Dual Problems

1. Let  $f = 5x - 6y + 2z$  be the objective function. Here is a standard column tableau:

$$\begin{array}{r} -x \\ -y \\ -z \\ 1 \end{array} \left[ \begin{array}{ccc} 0 & -1 & 5 \\ 1 & -1 & -6 \\ 0 & 0 & 2 \\ 7 & -3 & 0 \end{array} \right]$$

$$\begin{array}{ccc} \parallel & \parallel & \downarrow \\ v_1 & v_2 & -f \end{array} \rightarrow \max$$

2. Let  $f$  be the objective function. Here is a standard column tableau:

$$\begin{array}{r} -a \\ -b \\ -c \\ -d \\ -e \\ -f \\ -g \\ -h \\ -i \\ -j \\ 1 \end{array} \left[ \begin{array}{cccccc} -15 & -25 & -1 & -1 & -1 & -3 & .39 \\ -15 & -25 & -25 & 0 & -25 & -2 & .11 \\ -15 & -25 & -25 & -25 & -25 & -1 & .18 \\ -15 & 0 & -25 & -25 & -25 & -2 & .21 \\ -15 & 0 & 0 & -30 & -30 & -4 & .35 \\ -15 & -6 & -30 & -25 & -25 & -2 & .44 \\ -10 & -10 & -25 & -25 & -25 & -1 & .25 \\ -15 & -25 & -25 & -25 & -25 & -2 & .25 \\ -15 & 0 & -25 & -25 & -25 & -3 & .23 \\ -10 & -10 & -25 & -25 & -25 & -3 & .24 \\ -100 & -100 & -100 & -100 & -100 & -70 & 0 \end{array} \right]$$

$$\begin{array}{cccccc} \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \downarrow \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & -f \end{array} \rightarrow \max$$

$$3. \quad \begin{array}{ccc} x & y & z & 1 \\ \left[ \begin{array}{ccc} 0 & -1 & 0 & 7 \\ 1 & 1 & 0 & -3 \\ 5 & -6 & 2 & 0 \end{array} \right] & = v_1 \\ & = v_2 \\ & = f \end{array} \rightarrow \min$$

The matrix in Exercise 2 is so big that the transposed matrix may not fit on the page. So we reduce it as follows. The fifth constraint follows from the fourth constraint because  $b \geq 0$  so we drop the redundant constraint.



Given any feasible solution, we can replace  $g, h$  by  $0, g + h$  and obtain a feasible solution with the same value for the objective function  $f$ . So setting  $g = 0$  we do not change the optimal value.

$$\begin{array}{cccccccccc}
 a & b & c & d & e & f & h & i & j & 1 \\
 \left[ \begin{array}{cccccccccc}
 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 10 & -100 \\
 25 & 25 & 25 & 0 & 6 & 25 & 25 & 0 & 10 & -100 \\
 1 & 25 & 25 & 25 & 0 & 30 & 25 & 25 & 25 & -100 \\
 1 & 0 & 25 & 25 & 30 & 25 & 25 & 25 & 25 & -100 \\
 3 & 2 & 1 & 2 & 4 & 2 & 2 & 3 & 3 & -70 \\
 .39 & .11 & .18 & .21 & .35 & .44 & .25 & .23 & .24 & 0
 \end{array} \right] & \begin{array}{l} = v_1 \\ = v_2 \\ = v_3 \\ = v_4 \\ = v_6 \\ \rightarrow \min
 \end{array}
 \end{array}$$

4.  $\min = -\infty$ . Namely,  $z = -0.5x_4 \rightarrow -\infty$  as  $x_1 = x_2 = x_5 = 0, x_3 = 1.5x_4, x_6 = -10 + 2.5x_4, x_7 = -20 + .5x_4, x_8 = -2 + 0.5x_4, x_4 \rightarrow \infty$ .

5. Let  $cx + d, cy + d$  be two feasible values, where  $x, y$  are two feasible solutions. We have to prove that

$$\alpha(cx + d) + (1 - \alpha)(cy + d)$$

is a feasible value for any  $\alpha$  such that  $0\alpha \leq 1$ . But

$$\alpha(cx + d) + (1 - \alpha)(cy + d) = c(\alpha x + (1 - \alpha)y) + d,$$

where  $\alpha x + (1 - \alpha)y$  is a feasible solution because the feasible region is convex.

6. In the basic solution,  $x_7 = -4$ , so it is not feasible.

7. The first equation does not hold, so this is not a solution.

8. First we check that this  $X = [x_i]$  is a feasible solution (i.e., satisfies all constraints) with  $z = 2$ . We introduce the dual variables  $y_i$  corresponding to  $x_i$ , write the dual problem as the column problem, and set  $y_i = 0$  whenever  $x_i \neq 0$  (assuming that  $X$  is optimal, cf. Problem 13.10 and its solution).

$$\begin{array}{ccccccc}
 -y_6 & \left[ \begin{array}{cccccc}
 7 & -2 & -6 & 6 & -1 & 15 \\
 -1 & 0 & 1 & 0 & -1 & -2 \\
 4 & 0 & -3 & 5 & 3 & 4
 \end{array} \right] \\
 -y_8 & & & & & & \\
 1 & & & & & & \\
 & = 0 & = y_2 & = 0 & = y_4 & = 0 & = w & \rightarrow \max.
 \end{array}$$

We have a system of three linear equations for  $y_6, y_8$ , and the system has no solutions. So  $X$  is not optimal.

9. Proceeding as in the solutions of Problem 13.10 and Exercise 8, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1] \begin{bmatrix} 7 & -2 & -6 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix} = 0.$$

## 50 §13. Dual Problems

The system has the unique solution  $y_6 = 1, y_7 = 2, y_8 = 1$ . Moreover, this solution is feasible (the basic  $y_i$  are nonnegative). Since we have feasible solutions for the primal and dual problems and  $x_i y_i = 0$  for all  $i$ , both solutions are optimal.

10. Proceeding as in the solutions of Problem 13.10 and Exercises 8,9, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1] \begin{bmatrix} -2 & -6 & 6 & -1 \\ 1 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 5 & 3 \end{bmatrix} = 0.$$

The system has no solutions, so the answer is: This is not optimal.

## §14. Sensitivity Analysis and Parametric Programming

1. The tableau is not standard, so we treat the row and column programs separately. We pivot the row program on  $-1$  in the  $b$ -column:

$$\begin{array}{cccc} a & b & c & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & -1 & -2 \\ 2 & -1^* & 0 & -3 \\ 0 & 2 & 1 & 0 \end{array} \right] & \begin{array}{l} = d \\ = c \\ = w \end{array} & \mapsto & \begin{array}{l} \\ \\ \rightarrow \min \end{array} \end{array}$$

$$\begin{array}{cccc} a & c & c & 1 \\ \left[ \begin{array}{cccc} 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & -3 \\ 4 & -2 & 1 & -6 \end{array} \right] & \begin{array}{l} = d \\ = b \\ = w \end{array} & \rightarrow \min. \end{array}$$

Then we combine the two  $c$ -columns:

$$\begin{array}{ccc} a & c & 1 \\ \left[ \begin{array}{ccc} 1 & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{array} \right] & \begin{array}{l} = d \\ = b \\ = w \end{array} & \rightarrow \min. \end{array}$$

This tableau is standard. We use the simplex method:

$$\begin{array}{ccc} a & c & 1 \\ \left[ \begin{array}{ccc} 1^* & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{array} \right] & \begin{array}{l} = d \\ = b \\ = w \end{array} & \mapsto \begin{array}{l} \\ \\ \rightarrow \min \end{array} \\ \\ d & c & 1 \\ \left[ \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 4 & 3 & 2 \end{array} \right] & \begin{array}{l} = a \\ = b \\ = w \end{array} & \rightarrow \min. \end{array}$$

This is an optimal tableau, so  $\min = 2$  at  $a = 2, b = 1, c = d = 0$ .

Now we rewrite the column program in a standard column tableau:

$$\begin{array}{r} -g \\ -h \\ 1 \end{array} \left[ \begin{array}{cccc} -1 & 0 & 1 & 2 \\ -2 & 1 & 0 & 3 \\ 0 & -2 & -1 & 0 \end{array} \right] \\ \begin{array}{cccc} = i & = j & = k & = -u \end{array} \rightarrow \max.$$

The  $k$ -column is bad, so this program is infeasible.

2. The tableau is not standard, so we treat the row and column programs separately. The second constraint in the row program reads  $2a + c + 3 = c$ , hence  $a = -3/2$ . So the row program is infeasible.

Now we rewrite the column program in a standard column tableau:

$$\begin{array}{r} -g \\ -h \\ 1 \end{array} \left[ \begin{array}{cccc} -1 & -2 & -3 & -1 \\ -2 & 0 & -1 & 3 \\ 1 & -1 & 0 & 0 \end{array} \right] \\ \begin{array}{cccc} = i & = j & = k & = -u \end{array} \rightarrow \max.$$

The first row is bad, so the column program is either infeasible or unbounded. It is clear without pivoting that the program is unbounded: as  $g \rightarrow \infty, g \geq 1/2, h = 0$ , we have

$$-w = g \rightarrow \infty, i = g + 1 \geq 0, j = 2g - 1 \geq 0, k = 3g \geq 0.$$

3.  $\min = 0$  at  $d = e = 0, a \geq 0$  arbitrary

4. We rewrite the program in a standard row tableau:

$$\begin{array}{ccc} a & b & 1 \\ \left[ \begin{array}{ccc} 1 & 0 & -1 - \varepsilon \\ 0 & 1 & -1 \\ -1 & -2 & 0 \end{array} \right] & = & \begin{array}{l} c \\ d \\ -w \end{array} \end{array} \rightarrow \min.$$

Since we have a bad column (in fact, both columns are bad), our program is either infeasible or unbounded. Without pivoting, it is easy to see that the program is unbounded. Namely, let both  $a, b \rightarrow \infty$ . Then  $w \rightarrow \infty$  while  $c \geq 0$  when  $a \geq 1 - \varepsilon$ , and  $d \geq 0$  when  $b \geq 1$ .

## §15. More on Duality

1. No, it is not redundant.
2. Yes, it is  $2 \cdot (\text{first equation}) + (\text{second equation})$ .
3. No, it is not redundant.

52 §15. More on Duality

4. No, it is not redundant.
5. Yes, it is Adding the first two equations, we obtain

$$6x + 8y + 10z = 12$$

which implies the last constraint because  $12 \geq 0$ .

6. It is redundant because it can be obtained as follows: multiply the first constraint by 2 and add it to the second constraint.

7. Yes, it is.

8. We solve the system

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 6$$

$$6x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 = 1$$

$$x_1 - x_2 + x_3 - x_4 + x_5 = f$$

for  $x_1, x_2, f$ :

$$x_1 = -4 + x_3 + 2x_4 + 3x_5$$

$$x_2 = 5 - 2x_3 - 3x_4 - 4x_5$$

$$f = -9 + 4x_3 + 4x_4 + 8x_5.$$

Setting  $x_3 = x_4 = x_5 = 0, x_1 = -4, x_2 = 5$ , we satisfy the equations but  $f = -9 < 0$ , so the constraint  $f \geq 0$  is not redundant.

9. For the dual problem, the first column reads  $-1 \geq 0$ , hence the dual program is infeasible. So the row program (the row program with a bad column) is either infeasible or unbounded. Pivoting on  $-5$  in the  $x_3$ -column yield a row-feasible tableau (and  $x_1$ -column stays bad), so the row program is unbounded.

10. Let  $u$  be the nonbasic dual variable (corresponding to the first row). The dual constraints are:  $-6u + 1 \geq 0, -8u + 2 \geq 0, -5u + 5 \geq 0, -6u + 1 \geq 0, -7u + 1 \geq 0, -8u + 1 \geq 0, -3u + 2 \geq 0, -5u \geq 0, -4u + 5 \geq 0$  and, of course,  $u \geq 0$ . The only feasible solution is  $u = 0$ , which is optimal (for any objective function). By complementary slackness,  $x_i = 0$  for  $i \neq 8$  for any optimal solution for the primal problem. The first row reads  $x_8 \geq 1/5$ . So  $\min = 0$  when  $x_8 \geq 1/5$  and all other  $x_i = 0$ .

11. Let  $y_i$  be the dual variable corresponding to  $x_i$ . The first 7 columns of the tableau give 7 linear constraints for  $y_8, y_9$ . Two additional constraints are  $y_8, y_9 \geq 0$ . We can plot the feasible region (given by these 9 constraints) in the  $(y_8, y_9)$ -plane. (The constraints corresponding to  $x_5, x_6, x_7$  are redundant.) The answer is  $\max = 2.5$  at  $y_8 = 0, y_9 = 1.25$ . By complementary slackness, for any optimal solution  $[x_i]$  of the primal problem, we have  $x_i = 0$  for  $i \neq 1, 8$ . For such a solution, we have  $3x_1 - 1 = x_8 \geq 0, 4x_1 - 2 = x_9 = 0$ . Thus,

$\min = 2.5$  at  $x_1 = 0.5$ , all other  $x_i = 0$ .

12. Let  $y_i$  be the dual variable corresponding to  $x_i$ . The first 8 columns of the tableau give 8 linear constraints for  $y_9, y_{10}$ . Two additional constraints are  $y_9, y_{10} \geq 0$ . We can plot the feasible region (given by these 10 constraints) in the  $(y_9, y_{10})$ -plane. The answer is  $\max = 90/41$  at  $y_9 = 10/41, y_{10} = 40/41$ . By complementary slackness, for any optimal solution  $[x_i]$  of the primal problem, we have  $x_i = 0$  for  $i \neq 1, 5$ . For such a solution, we have  $6x_1 + 5x_5 - 1 = x_9 = 0, 19x_1 + 9x_5 - 2 = x_{10} = 0$ . Thus,

$\min = 90/41$  at  $x_1 = 1/41, x_5 = 7/41$  all other  $x_i = 0$ .

13. Let  $y_i$  be the dual variable corresponding to  $x_i$ , and let  $u, v$  be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for  $u, v$ . Two additional constraints are  $u, v \geq 0$ . We can plot the feasible region (given by these 11 constraints) in the  $(u, v)$ -plane. The answer is  $\max = 75/34$  at  $u = 15/34, v = 15/17$ . By complementary slackness, for any optimal solution  $[x_i]$  of the primal problem, we have  $x_i = 0$  for  $i \neq 4, 6$ . For such a solution, we have  $6x_4 + 8x_6 - 1 = 0, 14x_4 + 13x_6 - 2 = 0$ . Thus,

$\min = 75/34$  at  $x_4 = 3/34, x_6 = 1/17$  all other  $x_i = 0$ .

14. Let  $y_i$  be the dual variable corresponding to  $x_i$ , and let  $u, v$  be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for  $u, v$ . Two additional constraints are  $u, v \geq 0$ . The constraint corresponding to the  $x_8$  column reads  $-5u - 8v + 0 \geq 0$ , hence  $u = v = 0$ . On the other hand, then the constraint corresponding to the  $x_1$  column reads  $-1 \geq 0$ . So the column problem is infeasible.

On the other hand, it is easy to find a feasible solution for the row program, for example,  $x_9 = 1$  and  $x_i = 0$  for all other  $i$ . By the theorem on four alternatives, the row problem is unbounded.

## Chapter 6. Transportation Problems

§16. Phase 1

1.

20	10	5		35
		5	15	20
20	10	10	15	

2. By the northwest method,

35				35
5	4			9
	6	5		11
		5	15	20
40	10	10	15	

3. The total supply is 256, while the total demand is 260. So the problem is infeasible.

4. The balance condition  $50 = 50$  holds. Each time, we pick a position with the minimal cost:  $x_{2,1} = 2, x_{2,4} = 5, x_{3,3} = 4$  at zero cost,  $x_{1,7} = 3, x_{2,5} = 1, x_{2,9} = 12, x_{3,2} = 3, x_{3,8} = 5, x_{4,9} = 4$  at unit cost 1, and  $x_{4,5} = 2, x_{1,8} = 7, x_{4,8} = 2$  at unit cost 2. The total number of selected positions is 12, which equals  $4 + 9 - 1$ . Total cost is  $0 \cdot 11 + 1 \cdot 28 + 2 \cdot 11 = 50$ .

5. The balance condition  $50 = 50$  holds. Each time, we pick a position with the minimal cost:  $x_{2,1} = 2, x_{2,4} = 13, x_{3,3} = 4$  at zero cost,  $x_{1,4} = 2, x_{1,7} = 3, x_{4,9} = 1, x_{4,2} = 7, x_{3,2} = 1, x_{3,8} = 4, x_{3,6} = 2$  at unit cost 1, and  $x_{1,5} = 10, x_{3,5} = 1$  at unit cost 2. The total number of selected positions is 12, which equals  $4 + 9 - 1$ . Total cost is  $0 \cdot 19 + 1 \cdot 20 + 2 \cdot 11 = 42$ .

6. The balance condition  $50 = 50$  holds. Each time, we pick a position with the minimal cost:  $x_{21} = 5, x_{33} = 12$  at zero cost,  $x_{14} = 5, x_{17} = 3, x_{49} = 1, x_{42} = 8, x_{41} = 7, x_{43} = 2$  at unit cost 1,  $x_{15} = 1, x_{18} = 4, x_{46} = 0$  at unit cost 2, and  $x_{16} = 2$  at unit cost 3. The total number of selected positions is 12, which equals  $4 + 9 - 1$ . Total cost is  $0 \cdot 17 + 1 \cdot 26 + 2 \cdot 5 + 3 \cdot 2 = 42$ .

7. The balance condition  $130 = 130$  holds. Each time, we pick a position with the minimal cost:  $x_{16} = 30$  at zero cost,  $x_{12} = 10, x_{35} = 15$  at unit cost 30,  $x_{21} = 25, x_{34} = 30$  at unit cost 35,  $x_{13} = 10$ , at unit cost 40,  $x_{33} = 5$ , at unit cost 95, and  $x_{23} = 5$ , at unit cost 100. The total number of selected positions is 8, which equals  $3 + 6 - 1$ . Total cost is  $0 \cdot 30 + 30 \cdot 25 + 35 \cdot 55 + 40 \cdot 10 + 95 \cdot 5 + 100 \cdot 5 = 4050$ .

8. Suppose our solution  $x = (y + z)/2$  where  $y, z$  are feasible solution. We compare the values of  $x, y, z$  at the first selected position. Since the values for  $x$  was maximal possible, the values of  $y, z$  are the same. Then we consider the second selected position and conclude that  $x, y, z$  agree there. Continuing the process, we conclude that  $x, y, z$  agree at all selected positions. Then  $x = y = z$ . thus,  $x$  is a vertex.

#### §17. Phase 2

1.

	1	2	2	
0	$\begin{matrix} 1 \\ 175 \end{matrix}$	$\begin{matrix} 2 \\ 25 \end{matrix}$	$\begin{matrix} 3 \\ (1) \end{matrix}$	200
0	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ 100 \end{matrix}$	$\begin{matrix} 2 \\ 200 \end{matrix}$	300
	175	125	200	

This is an optimal table, and the corresponding solutions are optimal. The minimal cost for the transportation problem is  $1 \cdot 175 + 2 \cdot 25 + 2 \cdot 100 + 2 \cdot 200 = 825$ . The maximal profit for the dual problem is  $1 \cdot 175 + 2 \cdot 125 + 2 \cdot 200 - 0 \cdot 200 - 0 \cdot 300 = 825$ .

2.

	1	2	3	
0	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ 0 \end{matrix}$	$\begin{matrix} 3 \\ 200 \end{matrix}$	200
0	$\begin{matrix} 1 \\ 175 \end{matrix}$	$\begin{matrix} 2 \\ 125 \end{matrix}$	$\begin{matrix} 2 \\ (-1) \end{matrix}$	300
	175	125	200	

56 §17. Phase 2

The table is not optimal. Adjusting the flow along a cycle may improve the total cost:

	1	2	3	
0	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ 0 + \varepsilon \end{matrix}$	$\begin{matrix} 3 \\ 200 - \varepsilon \end{matrix}$	200
0	$\begin{matrix} 1 \\ 175 \end{matrix}$	$\begin{matrix} 2 \\ 125 - \varepsilon \end{matrix}$	$\begin{matrix} 2 \\ \varepsilon (-1) \end{matrix}$	300
	175	125	200	

Namely, taking  $\varepsilon = 125$  we decrease the total cost by 125.

3. We start with the basic feasible solution found in the solution of Exercise 4, §16 and compute the corresponding dual basic solution:

	0	2	1	0	1	2	1	2	1	
0	$\begin{matrix} 1 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ (0) \end{matrix}$	$\begin{matrix} 3 \\ (2) \end{matrix}$	$\begin{matrix} 1 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 3 \\ (1) \end{matrix}$	$\begin{matrix} 1 \\ 3 \end{matrix}$	$\begin{matrix} 2 \\ 7 \end{matrix}$	$\begin{matrix} 3 \\ (2) \end{matrix}$	10
0	$\begin{matrix} 0 \\ 2 \end{matrix}$	$\begin{matrix} 3 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 0 \\ 5 \end{matrix}$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 2 \\ (0) \end{matrix}$	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ (0) \end{matrix}$	$\begin{matrix} 1 \\ 12 \end{matrix}$	20
1	$\begin{matrix} 2 \\ (3) \end{matrix}$	$\begin{matrix} 1 \\ 3 \end{matrix}$	$\begin{matrix} 0 \\ 4 \end{matrix}$	$\begin{matrix} 1 \\ (2) \end{matrix}$	$\begin{matrix} 2 \\ (2) \end{matrix}$	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ (2) \end{matrix}$	$\begin{matrix} 1 \\ 5 \end{matrix}$	$\begin{matrix} 1 \\ (1) \end{matrix}$	12
0	$\begin{matrix} 1 \\ (1) \end{matrix}$	$\begin{matrix} 1 \\ (-1) \end{matrix}$	$\begin{matrix} 1 \\ (0) \end{matrix}$	$\begin{matrix} 2 \\ (2) \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ 2 \end{matrix}$	$\begin{matrix} 2 \\ (1) \end{matrix}$	$\begin{matrix} 2 \\ 2 \end{matrix}$	$\begin{matrix} 1 \\ 4 \end{matrix}$	8
	2	3	4	5	1	2	3	14	16	

There is only one negative  $w_{42} = -1$ . We select this position and get the loop  $(4, 2), (3, 2), (3, 8), (4, 8)$ . The maximal  $\varepsilon = 2$ , and we deselect the position  $(4, 8)$ . The total cost decreases by 2. Here is the new basic feasible solution and the corresponding dual basis



solution:

	1	2	1	1	2	3	1	2	2	
0	1 (0)	2 (0)	3 (2)	1 (0)	2 (0)	3 (0)	1 3	2 7	3 (1)	10
1	0 2	3 (2)	2 (2)	0 5	1 1	2 (0)	1 (1)	2 (1)	1 12	20
1	2 (2)	1 1	0 4	1 (1)	2 (1)	1 (-1)	2 (2)	1 7	1 (0)	12
1	1 (1)	1 2	1 (1)	2 (2)	2 (1)	2 2	2 (2)	2 (1)	1 4	8
	2	3	4	5	1	2	3	14	16	

Again we have a negative  $w_{36} = -1$ . The loop is (3, 6), (3, 2), (4, 2), (4,6). The total cost decreases by  $-w_{36} \cdot \varepsilon = 1$ . Here is the new basic feasible solution and the corresponding dual basis solution:

	0	1	1	0	1	2	1	2	1	
0	1 (1)	2 (1)	3 (2)	1 (1)	2 (1)	3 (1)	1 3	2 7	3 (2)	10
0	0 2	3 (2)	2 (1)	0 5	1 1	2 (0)	1 (0)	2 (0)	1 12	20
1	2 (3)	1 (1)	0 4	1 (2)	2 (2)	1 1	2 (2)	1 7	1 (1)	12
0	1 (1)	1 3	1 (0)	2 (2)	2 (1)	2 1	2 (1)	2 (0)	1 4	8
	2	3	4	5	1	2	3	14	16	

This table is optimal with total cost at  $\min = 47$ .

4. We start with the basic feasible solution found in the solution

58 §17. Phase 2

of Exercise 5, §16 and compute the corresponding dual basic solution:

	1	1	0	1	2	1	1	1	1	
0	1 (0)	2 (1)	3 (3)	1 2	2 10	3 (2)	1 3	2 (1)	3 (2)	15
1	0 2	3 (3)	2 (3)	0 13	1 (0)	2 (2)	1 (1)	2 (2)	1 (1)	15
0	2 (1)	1 1	0 4	1 (0)	2 1	1 2	2 (1)	1 4	1 (0)	12
0	1 (0)	1 7	1 (1)	2 (1)	2 (0)	2 (1)	2 (1)	2 (1)	1 1	8
	2	8	4	15	11	2	3	4	1	

The table is optimal,  $\min = 42$ .

5. We start with the basic feasible solution found in the solution of Exercise 6, §16 and compute the corresponding dual basic solution:

	2	2	2	1	2	3	1	2	2	
0	1 (-1)	2 (0)	3 (1)	1 5	2 1	3 2	1 3	2 4	3 (1)	15
2	0 5	3 (3)	2 (2)	0 (1)	1 (1)	2 (1)	1 (2)	2 (2)	1 (1)	5
2	2 (2)	1 (1)	0 12	1 (2)	2 (2)	1 (0)	2 (3)	1 (1)	1 (1)	12
1	1 7	1 8	1 2	2 (2)	2 (1)	2 0	2 (2)	2 (1)	1 1	18
	12	8	14	5	1	2	3	4	1	

There is only one negative  $w_{11} = -1$ . The loop is (1,1), (1, 6), (4, 6), (4, 1). The decrease in the total cost is 2. Here is the new basic

solution and the corresponding dual basic solution:

	1	1	1	1	2	2	1	2	1	
0	1 2	2 (1)	3 (2)	1 5	2 1	3 (1)	1 3	2 4	3 (2)	15
1	0 5	3 (3)	2 (2)	0 (0)	1 (0)	2 (1)	1 (1)	2 (1)	1 (1)	5
1	2 (2)	1 (1)	0 12	1 (1)	2 (1)	1 (0)	2 (2)	1 (0)	1 (1)	12
0	1 5	1 8	1 2	2 (1)	2 (0)	2 2	2 (1)	2 (0)	1 1	18
	12	8	14	5	1	2	3	4	1	

This table is optimal, and  $\min = 40$ .

6. We start with the optimal solution with  $t = 0$  given in Table 17.22:

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	100 (60)	45 (10)	60 (30)	0 5	30
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

Now we replace 30 by  $30 + t$  at the right margin and at the bottom margin (remember that the last column correspond to the fictitious store). We also replace the corresponding entry 5 in the table by  $5 + t$ .

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	100 (60)	45 (10)	60 (30)	0 $5+t$	$30+t$
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	$30+t$	

60 §17. Phase 2

When  $t \geq -5$  (e.g.,  $|t| \leq 1$ ), this is an optimal table, The total cost is at  $\min = 3475$  (independent of  $t$ ).

7. When  $t \geq 25$ , see the previous solution. When  $t < 0$ , the total supply is less than the total demand, so the program is infeasible. So it remains to consider the case  $0 \leq t \leq 25$ .

8. In the optimal tableau of Table 17.22, we replace  $c_{23}$  by  $t$  and  $w_{23}$  by  $t - 40$ :

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	$t$ ( $t - 40$ )	45 (10)	60 (30)	0 5	30
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

When  $t \geq 40$ , this table is optimal, with  $\min = 3475$  (independent of  $t$ ).

Assume now that  $t < 40$ . We select the position (2, 3) and get the loop (2, 3), (1,3), (1,6), (2,6). The corresponding  $\varepsilon = 5$ . The decrease in the total cost is  $5(40 - t)$ . Here is the new table:

	$75 - t$	30	40	35	30	0	
0	55 ( $t - 20$ )	30 10	40 15	50 (15)	40 (10)	0 25	50
40 $-t$	35 25	30 ( $40 - t$ )	$t$ 5	45 ( $50 - t$ )	60 ( $70 - t$ )	0 ( $40 - t$ )	30
0	40 ( $t - 35$ )	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

This table is optimal when  $35 \leq t \leq 40$ .

Assume now that  $t < 35$ . We select the position (3, 1) and get the loop (3, 1), (3, 6), (1,6), (1, 3), (2,3), (2,1). The corresponding  $\varepsilon = 5$ . The decrease in the total cost is  $5(35 - t)$ . Here is the new

table:

	$75 - t$	30	40	$70 - t$	$65 - t$	0	
0	55 ( $t - 20$ )	30 10	40 10	50 ( $t - 20$ )	40 ( $t - 25$ )	0 30	
40 $-t$	35 20	30 ( $40 - t$ )	$t$ 10	45 (15)	60 (35)	0 ( $40 - t$ )	
35 $-t$	40 5	60 ( $65 - t$ )	95 ( $90 - t$ )	35 30	30 15	0 ( $35 - t$ )	
	25	10	20	30	15	30	

This table is optimal when  $25 \leq t \leq 35$ .

Assume now that  $t < 25$ . We select the position (1, 5) and obtain the loop (1, 5), (3, 5), (3, 1), (2, 1), (2, 3), (1, 3). The corresponding  $\varepsilon = 10$ . The decrease in the total cost is  $10(25 - t)$ . Here is the new table:

	50	30	$t + 15$	45	40	0	
0	55 (5)	30 10	40 ( $25 - t$ )	50 (5)	40 10	0 30	50
15	35 10	30 (15)	$t$ 20	45 (15)	60 (35)	0 $40 - t$	30
10	40 15	60 (40)	95 ( $90 - t$ )	35 30	30 5	0 (10)	50
	25	10	20	30	15	30	

This table is optimal when  $t \leq 25$ .

Thus, we solve the program for all  $t$ . The minimal cost is

$$\begin{cases} 2900 + 20t & \text{for } t \leq 25 \\ 3400 + 5(t - 25) & \text{for } t \geq 25 \\ 3475 & \text{for } t \geq 40. \end{cases}$$

### §18. Job Assignment Problem

1.  $\min = 7$  at  $x_{14} = x_{25} = x_{32} = x_{43} = x_{51} = 1$ , all other  $x_{ij} = 0$ .

2. Subtract 1 from the rows 1,2,5 and the column 4. Subtract 2 from the row 4. Now we can place the flow at positions with zero

## 62 §18. Job Assignment Problem

cost, namely, positions (1, 4), (2, 3), (3,2), (4, 1), (5, 6), (6, 5), So  $\min = 6$ .

3.  $\min = 7$  at  $x_{12} = x_{25} = x_{34} = x_{43} = x_{51} = x_{67} = x_{76} = 1$ , all other  $x_{ij} = 0$ .

4. We subtract: 1 from the rows 2,3,4,5,8; 2 from the row 7; 1 from the column 8:

0	2	2	4	0	1	5	0	4
1	2	0	2	3	2	1	2	0
1	0	3	2	3	2	0	0	3
3	1	3	1	1	1	0	3	1
1	2	4	0	1	3	1	2	3
5	2	2	4	0	1	1	0	4
0	1	2	1	2	1	0	1	3
4	1	3	2	1	0	1	2	0
1	2	1	2	0	1	2	3	5

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Hungarian method (Remark 17.25). Let us try to place the flow at positions with zero cost. In each of the following six lines there is only one zero: r4 (row 4), r5, r9, c2 (column 2), c3, c6 (we pass c4 because the conflict with r5: the position (5,4) is already selected in r5). We select the positions with these zeros and add the six lines c7, c4, c5, r3, r2, r8 to our list  $L$  of covering lines. The remaining matrix is

	$c1$	$c8$	$c9$
$r1$	0	0	4
$r6$	5	0	4
$r3$	0	1	3

We cannot place all flow at positions with zero cost, but we can cover all zeros by  $2 < 3$  lines, namely,  $c1$  and  $c8$ . The complete list  $L$  consists of 8 lines  $c7, c4, c5, r3, r2, r8, c1, c8$ . The least uncovered number is  $m = 1$ . We subtract 1 from all uncovered entries and add

1 to all twice-covered entries:

0	1	1	4	0	0	5	0	3
2	2	0	3	4	2	2	3	0
2	0	3	3	4	2	1	1	3
3	0	2	1	1	0	0	3	0
1	1	3	0	1	2	1	2	2
5	1	1	4	0	0	1	0	3
0	0	1	1	2	0	0	1	2
5	1	3	3	2	0	2	3	0
1	1	0	2	0	0	2	3	4

Now we can place the flow at the positions with zero costs:

						*		
		*						
*							*	
			*					
							*	
*								*
				*				
							*	
								*

For this program,  $\min = 0$ . However we changed the objective function (without changing the optimal solutions). For the original problem,  $\min = 1 + 1 + 1 + 1 + 1 + 1 + 2 + 1 + 0 = 9$ .

5.  $\max = 14$  at  $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$ , all other  $x_{ij} = 0$ .

6. First we convert the maximization problem to a minimization problem by subtracting each entry from the maximal entry in its row:

0	2	2	2	3	2
2	1	3	2	0	0
2	4	3	3	3	0
2	1	0	1	0	1
1	2	3	2	0	3
1	0	1	0	2	1

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Hungarian

64 §18. Job Assignment Problem

method (Remark 17.25). The following five lines cover all zeros: rows 1,4,6 and columns 5,6. Since  $t = 5 < n = 6$ , we cannot place all flow at positions with zero cost. The least uncovered number is  $m = 1$ . We subtract 1 from all uncovered entries and add 1 to all twice-covered entries:

0	2	2	2	4	3
1	0	2	1	0	0
1	3	2	2	3	0
2	1	0	1	1	2
0	1	2	1	0	3
1	0	1	0	3	2

Now we can place all flow at positions with zero cost:

*					
	*				
					*
		*			
			*		
				*	

For this program,  $\min = 0$ . However we changed the objective function (without changing the optimal solutions). For the original problem,  $\max = 4 + 3 + 4 + 4 + 4 + 2 = 21$ .

Note that the sum of maximal entries in rows is 22, and we cannot get this much because the conflict in the last column. This proves the the solution is optimal.

7.  $\max = 30$  at  $x_{15} = x_{26} = x_{33} = x_{41} = x_{54} = x_{62} = x_{77} = 1$ , all other  $x_{ij} = 0$ .

8. To convert the maximization problem to a minimization problem, we subtract every entry from the maximal number in its row;

5	3	3	1	5	4	0	4	1
2	1	3	1	0	1	2	0	3
2	3	0	1	0	1	3	2	0
1	3	1	3	3	3	4	0	3
3	2	0	4	3	1	3	1	1
0	3	3	1	5	4	4	4	1
3	2	1	2	1	2	3	1	0
0	3	1	2	3	4	3	1	4
4	3	4	3	5	4	3	1	0



Now we subtract 1 from columns 2,4, and 6 to create a zero in each column:

5	2	3	0	5	3	0	4	1
2	0	3	0	0	0	2	0	3
2	2	0	0	0	0	3	2	0
1	2	1	2	3	2	4	0	3
3	1	0	3	3	0	3	1	1
0	2	3	0	5	3	4	4	1
3	1	1	1	1	1	3	1	0
0	2	1	1	3	3	3	1	4
4	2	4	2	5	3	3	1	0

We are ready to apply the Hungarian method (Remark 17.25). We cannot place the flow at the positions with zero cost, so we can cover all zeros by  $< 9$  lines. Our 8 lines are the rows 1–5 and the columns 1, 3, 9. Now we subtract 1 from uncovered numbers and add 1 to twice-covered numbers (this correspond to decreasing by 1 the potentials for the covering columns and uncovering rows):

6	2	3	1	5	3	0	4	2
3	0	3	1	0	0	2	0	4
3	2	0	1	0	0	3	2	1
2	2	1	3	3	2	4	0	4
4	1	0	4	3	0	3	1	2
0	1	2	0	4	2	3	3	1
3	0	0	1	0	0	2	0	0
0	1	0	1	2	2	2	0	4
4	1	3	2	4	2	2	0	0

Now we can place the flow at the positions with zero cost:

							*	
*								
	*							
							*	
			*		*			
				*				
*								*

The maximal efficiency for the original problem is  
 $5 + 3 + 4 + 5 + 4 + 4 + 4 + 5 + 5 = 39$ .

## Chapter 7. Matrix Games

§19. What are Matrix Games?

1.  $\max \min = -1$ .  $\min \max = 0$ . There are no saddle points.

$[1/3, 2/3, 0]^T$  gives at least  $-2/3$  for the row player.

$[1/2, 0, 0, 0, 1/2]$  gives at least  $1/2$  for the column player.

So  $-2/3 \leq$  the value of the game  $\leq -1/2$ .

2. Computing  $\max \min$  and  $\min \max$ , we find that the value  $v$  of game belongs to the interval  $1 \leq v \leq 2$ , the best pure for the row player (giving him at least 1) is the second row, and the best pure strategies for the column player (when she pays at most 2 are columns 1, 2, 5. there are no saddle points.

The columns 3 and 4 are dominated by column 2, so we obtain

a smaller game: 
$$\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 2 \\ -1 & 2 & 0 \\ -4 & -8 & 2 \end{bmatrix}.$$

Now the first and the last rows are dominated by the second row, so we obtain a smaller game: 
$$\begin{bmatrix} 2 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}.$$

Now the last column is dominated by the first column, so we obtain a smaller game: 
$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

This game can be easily solved using slopes: optimal strategies are  $[3/4, 1/4]^T$ ,  $[1/4, 3/4]$ , and  $v = 5/4$ . For the original big game, optimal strategies are  $[0, 3/4, 1/4, 0]^T$ ,  $[1/4, 3/4, 0, 0]$ , and  $v = 5/4$ .

3.  $\max \min = -1$ .  $\min \max = 2$ . There are no saddle points.

(second row + 2·third row)/3  $\geq -2/3$ .

(third column + sixth column)/2  $\leq 1$ .

So  $-2/3 \leq$  the value of the game  $\leq 1$ .

4. We compute  $\max \min = -1$  and  $\min \max = 0$  in pure strategies. So  $-1 \leq v \leq 0$  for the value  $v$  of game. There are no saddle points.

Choosing the row 4, he (the row player) gets at least -1 (i.e., he pays at most 1), but using the mixed strategy  $[0, 1/3, 1/3, 1/3]^T$  he gets at least  $-1/3$ . Thus,  $-1/3 \leq v \leq 0$ .

5. We compute the max in each column (marked by \*) and min in each row (marked by ■).

$$\begin{array}{c} \begin{array}{cccccccccc} 4 & 2^{\blacksquare} & 3 & 5 & 4 & 3^{\blacksquare} & 7 & 6 & 3 \\ -4 & 4^* & -4^{\blacksquare} & 3^* & 0 & 0 & 0 & -1 & 1 & -2 \\ -2 & -1 & 0 & 2 & 1 & -2^{\blacksquare} & -2^{\blacksquare} & 1 & 0 & -2^{\blacksquare} \\ -4 & -4^{\blacksquare} & 0 & -2 & -2 & 1 & -1 & 1 & 6^* & 2 \\ 0^* & 1 & 2^* & 2 & 5^* & 3 & 3^* & 7^* & 2 & 0^{\blacksquare} \\ -9 & -4 & -9^{\blacksquare} & -8 & 0 & 4^* & 2 & 2 & 0 & 3^* \end{array} \end{array}$$

Thus,  $\max \min = 0$ ,  $\min \max = 2$ . There are no saddle points.

6. We compute  $\max \min = -1$  and  $\min \max = 0$  in pure strategies. So  $-1 \leq v \leq 0$  for the value  $v$  of game. There are no saddle points.

Choosing the row 4, he (the row player) gets at least -1 (i.e., he pays at most 1), but using the mixed strategy  $[0, 1/3, 0, 2/3, 0]^T$  he gets at least  $-2/3$ . Thus,  $-2/3 \leq v \leq 0$ . Choosing column 2 or 10, she pays him at most 0, but choosing the halfsum of this columns, she pays him at most  $-1/2$  (i.e., she gets at least  $1/2$ ). Thus,  $-2/3 \leq v \leq -1/2$ .

7. Optimal strategies are

$$[0, 1/3, 0, 0, 7/15, 1/5]^T, [0, 2/3, 0, 0, 0, 0, 0, 0, 0, 1/3],$$

and the value of game is  $-2/3$ .

8. Optimal strategies are

$$[0, 0, 0, 19/121, 94/121, 8/121]^T, \\ [11/121, 25/121, 0, 0, 0, 0, 0, 0, 0, 85/121],$$

and the value of game is  $-14/121$ .

9. Optimal strategies are

$$[55/137, 0, 0, 44/137, 0, 0, 38/137]^T, \\ [0, 0, 20/137, 0, 65/137, 0, 0, 0, 0, 0, 0, 52/137],$$

and the value of game is  $-54/137$ .

10. Optimal strategies are

$$[0, 1/4, 0, 0, 0, 0, 0, 3/4, 0]^T, \\ [0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0, 1/2],$$

and the value of game is  $-1/2$ .

11. We have seen that  $a_{i,j} = a_{i',j} = a_{i,j'} = a_{i',j'}$  because  $a_{i,j} \geq a_{i',j} \geq a_{i',j'} \geq a_{i,j'} \geq a_{i,j}$ . Since  $a_{i,j}$  and  $a_{i',j}$  are maximal in their columns  $j$  and  $j'$ , so are  $a_{i',j}$  and  $a_{i,j'}$ . Since  $a_{i,j}$  and  $a_{i',j}$  are minimal in their columns  $i$  and  $i'$ , so are  $a_{i',j}$  and  $a_{i,j'}$ . Thus,  $(i, j')$  and  $(i', j)$  are saddle points.

12. We compute all payoffs corresponding to the mixed strategies

68 §20. Matrix Games and Linear Programming

$$p = [9/68, 23/68, 28/68, 8/68]^T, q = [5/12, 0, 2/12, 0, 5/12] :$$

$$\begin{array}{c|cccccc} A \setminus B & (4,0) & (3,1) & (2,2) & (1,3) & (0,4) & q \\ \hline (3,0) & -3 & 0 & 4 & 3 & 2 & 1/4 \\ (2,1) & 0 & -2 & -1 & 2 & 1 & 1/4 \\ (1,2) & 1 & 2 & -1 & -2 & 0 & 1/4 \\ (0,3) & 2 & 3 & 4 & 0 & -3 & 1/4 \\ \hline p & 1/4 & 1/2 & 1/4 & 1/4 & 1/4 & 1/4^{*'} \end{array} .$$

Since the last  $1/4$  is both maximal in its column and minimal in its row, we got a saddle point. there are other optimal strategies, e.g.,  $[5/12, 0, 1/6, 0, 5/12]$ .

§20. Matrix Games and Linear Programming

1. The optimal strategy for the row player is  $[2/3, 1/3, 0]^T$ .

The optimal strategy for the column player is  $[1/2, 1/2, 0]$ .

The value of the game is 2.

2. Let  $r_1, r_2, r_3$  (resp.,  $c_1, c_2, c_3$ ) denote the strategies of the row (resp., column) player. Since  $r_2$  is dominated by  $(2r_1 + 3r_3)/5$ ,  $r_2$  can be dropped. Since  $c_1$  is dominated by  $(c_2 + c_3)/2$ ,  $c_1$  can be dropped. The remaining  $2 \times 2$  matrix game can be easily solved using slopes. The optimal strategies are  $[2/9, 0, 7/9]^T$  and  $[0, 4/9, 5/9]$ , and the value of the game is  $8/9$ .

3. The optimal strategy for the row player is  $[0.2, 0, 0.8]^T$ .

An optimal strategy for the column player is  $[0, 0.5, 0.5, 0, 0, 0]$ .

The value of the game is 1.

4. By domination, we can remove repeated rows and columns and obtain the matrix

$$\begin{bmatrix} 1 & -3 & 5 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} .$$

Using domination by mixed strategies as in the solution of Exercise 2 above, we can eliminate the second row and the first column. The remaining  $2 \times 2$  matrix game with the payoff matrix  $\begin{bmatrix} -3 & 5 \\ 2 & 0 \end{bmatrix}$  can be easily solved using slopes: the optimal strategies are  $[0.2, 0.8]^T, [0.5, 0.5]$ , and the value of game is 1. For the original game, the answer is: optimal strategies are

$$[0.2, 0, 0, 0, 0.8]^T, [0, 0, 0, 0.5, 0.5],$$

and the value of game is 1.

5. The optimal strategy for the row player is  $[1/3, 2/3, 0]^T$ .  
 The optimal strategy for the column player is  $[2/3, 0, 0, 0, 1/3]$ .  
 The value of the game is  $-2/3$ .
6. Using slopes, the optimal strategy for the column player is  $[7/12, 5/12]$  and the value of game is  $15/4$ .
7. The optimal strategy for the row player:  $[1/8, 0, 7/8, 0]^T$ .  
 The optimal strategy for the column player:  $[0, 1/4, 0, 0, 0, 3/4]$ .  
 The value of the game is  $-0.25$ .
8. Optimal strategies are  
 $[0, 1/2, 1/2, 0, 0]^T, [0, 1/2, 1/4, 0, 0, 0, 1/4]$ ,  
 and the value of game is 0.
9. Optimal strategies are  
 $[0, 0, 0, 7/8, 1/8]^T, [3/8, 0, 0, 0, 0, 0, 0, 0, 5/8]$ ,  
 and the value of game is  $3/8$ .
10. Optimal strategies are  
 $[0, 0, 1/3, 2/3, 0]^T, [0, 2/3, 0, 0, 0, 0, 0, 0, 1/3]$ ,  
 and the value of game is  $-2/3$ .

11. Note that the second constraint is redundant, because it follows from the first one. We solve the equation for  $x_6$  and exclude this from our LP. We obtain an equivalent LP with all  $x_i \geq 0$ :

$$10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5 - 9 \rightarrow \min,$$

$$3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4,$$

$$(x_6 + 3 =) 3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 3.$$

Again, the second constraint is redundant.

Now we take advantage of the fact that in this problem all coefficients in the objective function and all right-hand parts of constraints are positive. We set  $v = 1/(10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5) > 0$  on the feasible region and

$$p_1 = 10x_1v, p_2 = 5x_2v, p_3 = 4x_3v, p_4 = 7x_4v, p_5 = 4x_5v.$$

All  $p_i \geq 0$  and  $p_1 + p_2 + p_3 + p_4 + p_5 = 1$ . The minimization of  $1/v - 9$  is equivalent to the maximization of  $v$ .

The constraint  $3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4$  take the form

$$(3p_1/10 + p_2/5 + p_3/4 + 2p_4/7 + p_5/4)/4 \geq v.$$

This is the row player program for the matrix game with the

payoff matrix  $\begin{bmatrix} 3/40 \\ 1/20 \\ 1/16 \\ 1/14 \\ 1/16 \end{bmatrix}$ . Our effort to get a smaller game pays, be-

## 70 §20. Matrix Games and Linear Programming

cause with can solve this game: the value of game is  $v = 1/14$  and the optimal strategy is  $[p_1, p_2, p_3, p_4, p_5]^T = [0, 0, 0, 1, 0]^T$ .

This translates to  $\max(1/v - 9) = 5$  at  $x_1 = x_2 = x_3 = x_5 = 0, x_4 = 2, x_6 = 2x_4 - 3 = 1$ .

12. We write our LP in a canonical form  $-Ax \leq b, x \geq 0, cx \rightarrow \min$  corresponding to their standard row tableau (13.4) with

$$x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T, \\ c = [-1, -2, -4, -1, -1, -1, 0], b = [-5, -6, 7]^T,$$

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & -2 & 1 \\ 3 & 1 & 1 & 2 & 1 & 1 & -1 \\ -3 & -1 & -1 & -2 & -1 & 1 & 0 \end{bmatrix}$$

Then

$$M = \begin{bmatrix} 0 & -A & -b \\ A^T & 0 & -c^T \\ b^T & c & 0 \end{bmatrix}$$

is our payoff matrix (see page 219 of the textbook).

13. We take advantage of the fact that all coefficients in the objective function and all right-hand parts of constraints are positive. We set  $v = 1/(x_1 + 2x_2 + x_3 + x_4 + x_5 + 3x_6 + x_7 + x_8) > 0$  on the feasible region (because the point where all  $x_i = 0$  is not a feasible solution) and  $p_1 = x_1v, p_2 = 2x_2v, p_3 = x_3v, p_4 = x_4v, p_5 = x_5v, p_6 = 3x_6v, p_7 = x_7v, p_8 = x_8v$ . Our constraints take the form

$$3p_1 + p_2/2 + p_3 + 2p_4 + p_5 + p_6/3 + p_7 - 3p_8 \geq v, \\ 3p_1/5 + p_2/10 + p_3/5 + 2p_4/5 - p_5/5 + p_6/15 + p_7/5 + 3p_8/5 \geq v, \\ 3p_1 + p_2/2 + p_3 + 2p_4 + p_5 - p_6/3 + p_7 + p_8 \geq v.$$

Other constraints are:  $p_i \geq 0$  for all  $i$  and  $\sum_{i=1}^8 p_i = 1$ . The minimization of  $1/v$  is equivalent to the maximization of  $v$ .

Thus, we obtain the LP for the row player, with the payoff

$$\text{matrix} \begin{bmatrix} 3 & 3/5 & 3 \\ 1/2 & 1/10 & 1/2 \\ 1 & 1/5 & 1 \\ 1 & -1/5 & 1 \\ 1/3 & 1/15 & -1/3 \\ 1 & 1/5 & 1 \\ -3 & 3/5 & 1 \end{bmatrix}.$$

At the position (1,2), we have a saddle point, so the value of game is 0.6 and an optimal strategy for the row player is

$$[p_1, p_2, p_3, p_4, p_5, p_6]^T = [1, 0, 0, 0, 0, 0]^T.$$

This translates to  $\min = 5/3$  at  $x_1 = 5/3$ , the other  $x_i = 0$ .

## §21. Other Methods

1. The first row and column are dominated. The optimal strategy for the row player is  $[0, 0.5, 0.5]^T$ . The optimal strategy for the column player is  $[0, 0.25, 0.75]$ . The value of the game is 2.5.

2. The second row is dominated by the first row, and the second column is dominated by the third column. So we obtain a  $2 \times 2$  game  $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$  which can be easily solved using slopes: optimal strategies are  $[0.4, 0.6]^T$ ,  $[0.4, 0.6]$ , and the value of game is 1.2. For the original game, the answer is: optimal strategies are  $[0.4, 0, 0.6]^T$ ,  $[0.4, 0, 0.6]$ , and the value of game is 1.2.

3. The optimal strategy for the row player is  $[0, 0.4, 0, 0.6]^T$ . The optimal strategy for the column player is  $[0, 0.4, 0.6]$ . The value of the game is 2.8.

4. By domination, we can reduce our matrix to  $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$ . this

game can be solved easily by graphical method. Answer: optimal strategies are  $[0, 0, 0, 1, 0, 0]^T$  (a pure strategy corresponding to the fifth row) and  $[0, 0.5, 0.5]$ , and the value of game is 2.

5. An optimal strategy for the row player is  $[1/3, 1/3, 1/3]^T$ . An optimal strategy for the column player is  $[0, 0, 2/7, 3/7, 2/7, 0]$ . The value of the game is 0.

6. Using slopes, the value of game is  $15/4$ .

7. The value of the game is 0 because the game is symmetric.

8. There is a saddle point at the position (1, 3). The value of game is 1, and optimal strategies are  $[1, 0, 0, 0, 0]^T$ ,  $[0, 0, 1, 0, 0]$ .

9. The first two columns and the first row go by domination. It is easy to solve the remaining  $2 \times 2$  matrix game. The value of game is  $11/7$ , and optimal strategies are  $[0, 5/14, 9/14]^T$ ,  $[0, 0, 5/7, 2/7]$ .

10. The value of game is 5, and optimal strategies are  $[0, 1, 0, 0, 0]^T$  (the second row) and  $[0.5, 0, 0.5, 0]$ .

11. There is a saddle point at the position (1, 1). So the value of game is 0.

The last column is dominated by any other column. After dropping this column, we get a symmetric game. This is another way to see that the value of game is 0.

12. The value of game is  $1/2$ , and optimal strategies are

**72** §22. What is Linear Approximation?

$[0, 1/6, 1/2, 1/3, 0]^T$  and  $[1/4, 0, 3/4, 0, 0]$ .

13. 0 at a saddle point (at position (1, 1)). The row player has also other optimal strategies, e.g.,  $[1/3, 1/3, 0, 0, 1/3]^T$

14. The value of game is 1, and optimal strategies are  $[21/59, 33/295, 32/295, 14/59, 11/59, 0]^T$  and  $[0, 0, 0, 0, 0, 1, 0]$  (the seventh column).



## Chapter 8. Linear Approximation

§22. What is Linear Approximation?

1. The mean is  $-2/5 = -0.4$ . The median is 1. The midrange is  $-5/2 = -2.5$ .

2. The mean is  $148/7 \approx 21.1429$ . The median is 8. The midrange is  $61/2 = 20.5$ .

3. The mean  $x_2$  is  $5/9$ . The median  $x_1$  is 0. The midrange  $x_\infty$  is  $1/2 = 0.5$ .

4.  $x_2 = 3.1; 3 \leq x_1 \leq 5; x_\infty = 1$ .

5. (a) 0, 0, 2, 2, 5.

(b) 1, 2, 9.

(c) 0, 0, 2, 2, 3.

(d). Exercise 1.

(e) 1, 2, 2.2, 3, 3.

(f). Exercise 3.

6. Consider the model  $w = ch^2$  with  $h = [1, 1.5, 2]$  and  $w = [45, 50, 200]$ . The median of  $[w_1/h_1^2, w_2/h_2^2, w_3/h_3^2] = [45, 200/9, 50]$  is 45, while the optimal solution to

$$|w_1 - ch_1^2| + |w_2 - ch_2^2| + |w_3 - ch_3^2| \rightarrow \min$$

is  $c = 50$ . This is because the slope of the objective function is

$$\begin{cases} 7.25 & \text{for } c > 50 \\ -0.75 & \text{for } 45 < c < 50. \end{cases}$$

7. The program

$$|65 - 1.6c| + |60 - 1.5c| + |70 - 1.7c| \rightarrow \min$$

can be easily solved by computing slopes. E.g., the slope of the objective function on the interval  $60/1.5 = 40 < c < 65/1.6 = 40.625$  is  $1.5 - 1.6 - 1.7 = -1.8$  while the slope on the interval  $65/1.6 = 40.625 < c < 70/1.7 \approx 42.2$  is  $1.6 + 1.5 - 1.7 = 1.4$ . Thus,  $\min = 1.875$  at  $c = 40.625$ .

Since  $1.875 < 6.25$  the model  $w = ch$  is better than the model  $w = ch^2$  in Example 22.7 when we use  $l^1$ -metric.

## 74 §22. What is Linear Approximation?

The program

$$(65 - 1.6c)^2 + (60 - 1.5c)^2 + (70 - 1.7c)^2 \rightarrow \min$$

can be easily solved by differentiation. We obtain

$$1.6(65 - 1.6c) + 1.5(60 - 1.5c) + 1.7(70 - 1.7c) = 0,$$

hence  $c \approx 40.6494$ ,  $\min \approx 1.75325$ . So the model  $w = ch$  is better than the model  $w = ch^2$  in Example 22.7 when we use  $l^2$ -metric.

The program

$$\max(|65 - 1.6c|, |60 - 1.5c|, |70 - 1.7c|) \rightarrow \min$$

can be easily solved by computing slopes. Near  $c = 40.625$ , the objective function is  $\max(70 - 1.7c, 1.5c - 60)$ . So  $\min = 0.9375$  at  $c = 40.625$ . So the model  $w = ch$  is better than the model  $w = ch^2$  in Example 22.7 when we use  $l^\infty$ -metric.

8. The program

$$|65 - 1.6^3a| + |60 - 1.5^3a| + |70 - 1.7^3a| \rightarrow \min$$

can be easily solved by computing slopes. E.g., the slope of the objective function near the point  $a = 65/1.6^3 \approx 15.8691$  are  $-1.6^3 + 1.7^3 - 1.5^3 < 0$  on the left and  $1.6^3 + 1.7^3 - 1.5^3 > 0$  on the right. Thus,  $\min = 14.4067$  at  $a = 65/1.6^3 \approx 15.8691$ .

The model  $w = ch^2$  is better than the model  $w = ch^3$  in Example 22.7 when we use  $l^1$ -metric.

The program

$$(65 - 1.6^3a)^2 + (60 - 1.5^3a)^2 + (70 - 1.7^3a)^2 \rightarrow \min$$

can be easily solved by differentiation. We obtain

$$1.6^3(65 - 1.6^3a) + 1.5^3(60 - 1.5^3a) + 1.7^3(70 - 1.7^3a) = 0,$$

hence  $a \approx 15.5366$ ,  $\min \approx 99.1546$ . So the model  $w = ch^2$  is better than the model  $w = ch^3$  in Example 22.7 when we use  $l^2$ -metric.

The solve the program

$$\max(|65 - 1.6^3a|, |60 - 1.5^3a|, |70 - 1.7^3a|) \rightarrow \min$$

with *Mathematica*. The command is

```
FindMinimum[Max[Abs[65 - 1.6^3 a], Abs[60 - 1.5^3 a],
Abs[70 - 1.7^3 a]], {a, 1, 2}]
```

The answer is  $\min \approx 7.06202$  at  $a \approx 15.6853$ . So the model  $w = ch^3$  gives a worse  $l^1$ -fit than the model  $w = ch^2$  in Example 22.7.

9. We enter the data  $h = [1.6, 1.5, 1.7, 1.8]$ ,  $w = [65, 60, 70, 80]$  to *Mathematica* as

```
h = { 1.6, 1.5, 1.7, 1.8}
w = {65, 60, 70, 80}
```

For  $p = 1$ , the objective function (to be minimized) is

```
f=Apply[Plus,Abs[w-a*h^ 2]]
```

An optimization command is

```
FindMinimum[f,{a,1,2}]
```

The answer is  $\min \approx 7.59259$  at  $a \approx 24.6914$ . For comparison, the model  $w = b$  gives  $\min = 25$  when  $65 \leq b \leq 70$  (medians).

For  $p = 2$ , we enter

```
f=Apply[Plus,(w-a*h^ 2)^ 2]; FindMinimum[f,{a,1,2}]
```

and obtain  $\min \approx 21.0733$  at  $a \approx 25.0412$ . For comparison, the model  $w = b$  gives  $\min = 218.75$  when  $b = 68.75$  (the mean).

For  $p = \infty$ , we enter

```
f=Max[Abs[w-a*h^ 2]]; FindMinimum[f,{a,1,2}]
```

and obtain  $\min \approx 3.09339$  at  $a \approx 25.2918$ . For comparison, the model  $w = b$  gives  $\min = 10$  when  $b = 70$  (the midrange).

So the model  $w = ah^2$  gives better  $l^p$ -fits for our data than the model  $w = b$  for  $p = 1, 2, \infty$ .

10. We enter data as in the previous exercise, and then we enter:

```
f=Apply[Plus,Abs[w-a*h]]; FindMinimum[f,{a,1,2}]
```

with the answer  $\min \approx 3.63636$  at  $a \approx 42.4242$  for  $p = 1$ ;

```
f=Apply[Plus,(w-a*h)^ 2]; FindMinimum[f,{a,1,2}]
```

with the answer  $\min \approx 34.5978$  at  $a \approx 41.77332$  for  $p = 2$ ;

```
f=Max[Abs[w-a*h]]; FindMinimum[f,{a,1,2}]
```

with the answer  $\min \approx 3.63636$  at  $a \approx 42.4242$  for  $p = \infty$ .

11. We enter data as in the previous exercise, and then we enter

```
f=Apply[Plus,Abs[w-a*h^ 3]]; FindMinimum[f,{a,1,2}]
```

## 76 §22. What is Linear Approximation?

with the answer  $\min \approx 21.6477$  at  $a \approx 14.2479$  for  $p = 1$ ;

$f = \text{Apply}[\text{Plus}, (w - a * h^3)^2]; \text{FindMinimum}[f, \{a, 1, 2\}]$

with the answer  $\min \approx 167.365$  at  $a \approx 14.8198$  for  $p = 2$ ;

$f = \text{Max}[\text{Abs}[w - a * h^3]]; \text{FindMinimum}[f, \{a, 1, 2\}]$

with the answer  $\min \approx 8.68035$ , at  $a \approx 15.2058$  for  $p = \infty$ .

12. For the model  $p_n = cn$ ,

the best  $l^1$ -fit is for  $c \approx 4.98611$  (with  $\min \approx 1968$ ),

the best  $l^2$ -fit is for  $c \approx 4.96398$  (with  $\min \approx 47428$ ),

the best  $l^\infty$ -fit is for  $c \approx 5.03077$  (with  $\min \approx 38$ ).

For the model  $p_n = cn \log(n)$ ,

the best  $l^1$ -fit is for  $c \approx 1.16302$  (with  $\min \approx 251$ ),

the best  $l^2$ -fit is for  $c \approx 1.16038$  (with  $\min \approx 1056$ ),

the best  $l^\infty$ -fit is for  $c \approx 1.15849$  (with  $\min \approx 8.5$ ).

We can see that  $cn \log(n)$  is a better  $l^p$ -fit to  $p_n$  than  $cn$  for all three values of  $p = 1, 2, \infty$ . The model  $p_n = cn \log(n + 1)$  gives slightly better fits.

13. These are not linear approximations. Taking log of both sides, we obtain a model  $\log_2(F_t) = ct$ . For this model,

the best  $l^1$ -fit is for  $c \approx 0.680907$  (with  $\min \approx 9.5$ ),

the best  $l^2$ -fit is for  $c \approx 0.680096$  (with  $\min \approx 2.8$ ),

the best  $l^\infty$ -fit is for  $c \approx 0.671023$  (with  $\min \approx 0.67$ ).

The limit value for  $c$  when we take morw and more terms of the sequence is  $\log_2(\sqrt{5} + 1) - 1 \approx 0.694242$ .

## §23. Linear Approximation and Linear Programming

1.  $\min = 0$  at  $a = -15, b = 50$  for  $w = a + bh$

and

$\min \approx 19$  at  $a \approx 25.23$  for  $w = ah^2$

2.  $x + y + 0.3 = 0$

3.  $a = 0.9, b \approx -0.23$

4. We set

$w = [85, 60, 70, 80], h = [1.6, 1.5, 1.7, 1.8]; c = [1, 1, 1, 1]$ .

We want to find a least squares solution to  $[c^T, h^T][a, b]^T = w^T$ ,

i.e., a solution to  $\begin{bmatrix} c \cdot c & c \cdot h \\ c \cdot h & h \cdot h \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \cdot w \\ h \cdot w \end{bmatrix}$ .

The solution is  $a \approx 37.2348, b \approx 13.3511$ .

5. We enter the data in *Mathematica*:

$h = \{1.5, 1.6, 1.7, 1.7, 1.8\}; w = \{60, 65, 70, 75, 80\}$

Then we enter

$\text{FindMinimum}[\text{Apply}[\text{Plus}, \text{Abs}[w - a * h^2]], \{a, 1, 2\}]$

The answer is  $\min \approx 10.1367$  at  $a \approx 25.3906$ .

6. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Apply[Plus,Abs[w-a*h^ 3]],{a,1,2}]
```

The answer is  $\min \approx 24.9797$  at  $a \approx 15.2656$ .

7. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Apply[Plus,Abs[w-a*h-b]],{a,1,2},{b,1,2}]
```

The answer is  $\min \approx 13.7647$  at  $a \approx 40.5882, b \approx 1$ .

8. It makes sense to compare the best  $l^p$ -fits for various models with the same  $p$ . For instance, for data in Problem 23.2 Exercises 5–7 show that the model  $w = ah + b$  gives a better fit than the model  $w = ah^2$ , and the model  $w = ah^2$  is worst (all in  $l^1$ -sense). It is harder to compare the best  $l^2$ -fit and the best  $l^1$ -fit of the form  $w = ah^2$ —either is best in its own sense. We can compare the  $l^1$ -average error  $\approx 13.7647/5 \approx 2.75294$  in Exercise 7 with the  $l^2$ -average error  $\approx ((\sum_{i=1}^5 (w_i + 41.7 - 67.3h_i)^2)/5)^{1/2} \approx (67.3077/5)^{0.5} \approx 3.669$ . So in some sense  $l^1$ -fit is better than  $l^2$ -fit for our data. However one should recall that  $l^p$ -fit of positive numbers cannot be worse than  $l^q$ -fit of the same numbers when  $1 \leq p < q$ .

9. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Max[Abs[w-a*h^ 2]],{a,1,2}]
```

The answer is  $\min \approx 3.09339$  at  $a \approx 25.2918$ .

10. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Max[Abs[w-a*h^ 3]],{a,1,2}]
```

The answer is  $\min \approx 8.68035$  at  $a \approx 15.2058$ .

11. We enter the data in *Mathematica* as in Exercise 5. Then we enter

```
FindMinimum[Max[Abs[w-a*h-b]],{a,1,2},{b,1,2}]
```

The answer is  $\min \approx 3.72727$  at  $a \approx 41.8182, b \approx 1$ .

12. We get the least squares fit to  $p_n$  by  $an + b$  when  $a \approx 5.53069, b \approx -37.9697$  with  $\min \approx 11923$  being the minimal value for

$$\sum_{i=1}^{100} (p_n - an - b)^2,$$

while

$$\sum_{i=1}^{100} (p_n - n \log(n))^2 \approx 161206.$$

## 78 §23. Linear Approximation and Linear Programming

13. This is not a linear approximation. Taking log of the both sides, we get a linear model  $\log_2(F_n) = \log_2(a) + bn$ . The least squares fit is

$\min \approx 0.269904$  at  $b \approx 0.693535$ ,  $\log_2(a) \approx -0.443517$ ,  
so  $a \approx 0.74$ .

14. We introduce three new variables  $a, b, c$ , and here is our linear program with 9 variables and 15 linear constraints:

$$a + b + c \rightarrow \min$$

subject to

$$e_1 = 2x_1 + 3x_2 - 1, -a \leq e_1 \leq a,$$

$$e_2 = x_1 - 2x_2 - 2, -b \leq e_2 \leq b,$$

$$e_3 = -x_1 + x_2 + 3, -c \leq e_3 \leq c,$$

$$e_4 = x_1 - x_2 - 4 - c \leq e_4 \leq c,$$

$$-e_3 \leq e_1 \leq e_3 \leq 4.$$

The variables  $e_1, e_2, e_3, e_4$  can be easily excluded.

Answer;

$\min \approx 2.42857$  at  $x_1 \approx 1.14286, x_2 \approx -0.428571$ .

## §24. More Examples

1. The model is  $w = ah + b$ , or  $w - x_2 = a(h - 1988) + b'$  with  $b = x_2 + 1988a$  and  $x_2 = 37753/45 \approx 838.96$ . Predicted production  $P$  in 1993 is  $x_2 + 5a + b'$ .

For  $p = 1$ , we have  $a \approx 16.54, b' \approx 31, P \approx 953$ .

For  $p = 2$ , we have  $a \approx 0, b' \approx 32, P \approx 871$ .

For  $p = \infty$ , we have  $a \approx 17.59, b' \approx 32, x_5 \approx 959$ .

So in this example  $l^\infty$ -prediction is the best.

2. For our data, the equation  $w = ar + b$  is the system of linear equations  $AX = w$ , where

$$w = [5, 13, 29, 45, 81, 113, 149, 197, 253]^T,$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T, \text{ and } X = \begin{bmatrix} a \\ b \end{bmatrix}.$$

The least-squares solutions are solutions to  $A^TAX = A^Tw$ , i.e.,

$$\begin{bmatrix} 285 & 45 \\ 45 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6277 \\ 885 \end{bmatrix}.$$

The solution is  $a = 463/15, b = -56$ . The error

$$\sum_{i=1}^9 (w_i - 463i/15 + 56)^2 = 48284/15 \approx 3218.93.$$

For  $w = ar^2$ , the system is  $Ba = w$ , where  $w$  is as above and  $B = [1^2, 3^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2]^T$ . The equation  $B^T Ba = B^T w$  is  $415398a = 47598$ , hence  $a = 23799/7699 \approx 3.09118$ . The error

$$\sum_{i=1}^9 (w_i - 23799i^2/7699)^2 = 2117089/7699 \approx 274.982.$$

So the model  $w = ar^2$  gives a much better least-squares fit than the model  $w = ar + b$  although it has one parameter instead of two.

The model  $w = \pi r^2$  without parameters gives the error

$$\sum_{i=1}^9 (w_i - \pi i)^2 = 147409 - 95196\pi + 15398\pi^2 \approx 314.114.$$

It is known that  $w_i/h_i^2 \rightarrow \pi$  as  $i \rightarrow \infty$ . Moreover,  $|w_i - \pi r_i^2|/r_i$  is bounded as  $i \rightarrow \infty$ . It follows that  $a \rightarrow \pi$  for the the best  $L_p$ -fit  $w = ar^2$  for any  $p \geq 1$  as we use data for  $r = 1, 2, \dots, n$  with  $n \rightarrow \infty$ .

3.  $a = \$4875, b = \$1500$

4. We enter data for 1900-1998 in *Mathematica* with  $t$  replaced by  $t - 1990$ :

$t = \{0, 1, 2, 3, 4, 5, 6, 7, 8\};$

$x = \{421, 429, 445, 449, 457, 460, 481, 503, 514\};$

$y = \{628, 646, 764, 683, 824, 843, 957, 1072, 1126\}$

To get the best  $l^1$ -fit, we enter

`FindMinimum[Apply[Plus, Abs[y-a*t-b*x-c]],  
{a,1,2}, {b,1,2}, {c,1,2}]`

with response

`{280.599, {a -> 47.7205, b -> 1.39226, c -> 1.}}`

Then we plug these values for  $a, b, c$  to  $1070 - 9a - 523b - c$  (to get the difference between actual value 1070 and the prediction) and obtain  $\approx -89$ .

To get the best  $l^2$ -fit, we enter

`FindMinimum[Apply[Plus, (y-a*t-b*x-c)^2],  
{a,1,2}, {b,1,2}, {c,1,2}]`

with response

`{8275.95, {a -> -0.0169939, b -> 5.63814, c -> -1767.27}}`

hence  $1070 - 9a - 523b - c \approx -111$ .

To get the best  $l^\infty$ -fit, we enter

`FindMinimum[Max[Abs[y-a*t-b*x-c]],  
{a,1,2}, {b,1,2}, {c,1,2}]`

**80** §24. More Examples

with response

$$\{8275.95, \{a \rightarrow 90.521, b \rightarrow 1.09929, c \rightarrow 1.\}\}$$

hence  $1070 - 9a - 523b - c \approx -321$ .

So the  $l^1$ -fit gave the best prediction.