The Theory of Compositions:
IV. Compositions with designated summands

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Abstract. The theory of compositions has been applied and investigated by numerous authors since P.A. MacMahon first introduced them late in the 19th century. Recently a number of researchers have considered partitions in which certain parts are marked, i.e., "partitions with designated summands". Similar ideas may be applied to compositions. By doing so we discover a purely combinatorial proof of one of MacMahon's fundamental formulas. Additionally we note interesting parity results concerning these more general compositions.

1. Introduction

A *composition* of an integer n is an ordered sum of non-negative integers adding to n. Thus there are eight compositions of 4, namely $4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1$. We denote by $c(n)$, $c(m,n)$ and $c(N,m,n)$ the number of compositions of n, the number of compositions of n into m parts, and the number of compositions of n with m parts each $\leq N$, respectively.

P.A. MacMahon ([**PAM**], pp. 620 - 621) (cf. [**GA2**], Ch. 4) proved that

$$c(n) = 2^{n-1}, \tag{1.1}$$

$$c(m,n) = \binom{n-1}{m-1}, \tag{1.2}$$

and

$$c(N,m,n) = \sum_{j\geq 0}(-1)^j\binom{m}{j}\binom{n-jN-1}{m-1} \tag{1.3}$$

Z. Star [**ZS**] has given a thorough account of both symmetry properties of $c(N,m,n)$ and asymptotics. In addition, q analogues of $c(N,m,n)$ have found important applications in theoretical physics ([**GA3**], [**GA4**], and [**GA5**]). Previously ([**STS1**], [**STS2**], [**STS3**] and [**STS4**]), each of us has examined a variety of formulas for compositions.

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Each of the above formulas of MacMahon follows from the related generating functions. In particular,

$$\sum_{n\geq 0} c(N,m,n)q^n = (q+q^2+...+q^N)^m \tag{1.4}$$

$$= \frac{q^N(1-q^N)^m}{(1-q)^m}. \tag{1.5}$$

Now there are numerous combinatorial identities that are directly deductible from (1.3) when combined with algebraic variations of (1.4) and (1.5). For example, clearly $c(1,m,n+m)=\delta_{n,0}$. So for $n>0$,

$$0=\delta_{n,0}=\sum_{j\geq 0}(-1)^j\binom{m}{j}\binom{n+m-j-1}{m-1}=\sum_{j\geq 0}(-1)^{m-j}\binom{m}{j}\binom{n+j-1}{m-1},$$

and if $n=m$, we see that

$$0=\sum_{j\geq 0}(-1)^j\binom{m}{j}\binom{m+j-1}{j}. \tag{1.6}$$

Now in (1.6) we see that when $j=0$ the summand is 1. Hence there must be an odd number of the remaining terms that are odd. One of us (Soh) conjectured at the October 22, 2000 meeting of the Korean Mathematical Society that in fact exactly one other summand is odd. We shall establish this conjecture here:

THEOREM 1. *If $j>0$, then $\binom{m}{j}\binom{m+j-1}{j}$ is odd if and only if $j=2^\lambda$ where 2^λ is the largest power of 2 dividing m.*

In light of these observations we note that (1.1) and (1.2) have absolutely transparent combinatorial explanations. These were first given by MacMahon ([**PAM**], p. 621). However, MacMahon did not explain (1.3) combinatorially. In Section 2, we prove (1.3) combinatorially using the idea of "compositions with designated summands". This concept was first introduced for partitions in ([**ALL**]). In section 3, we prove Theorem 1.

As remarked above, Z. Star has fully developed the asymptotics of $c(N,m,n)$; however, the matter of computation of $c(N,m,n)$ has not been fully treated in the literature. We shall provide an account of the various algorithms available in computing $c(N,m,n)$ in Section 4.

2. The Combinatorics of Equation (1.3)

We begin with the idea of designating exactly j positions in compositions of n with m parts. As MacMahon has shown in equation (1.2) there are $\binom{n-1}{m-1}$ compositions of n with m parts. Owing to the fact that there are $\binom{m}{j}$ choices for the designated positions, there are clearly

$$\binom{m}{j}\binom{n-1}{m-1}$$

compositions of n into m parts with j designated parts. For example, there are twenty four compositions of 5 into 4 parts with two designated summands. Namely, $\bar{2}+\bar{1}+1+1,\ \bar{2}+1+\bar{1}+1,\ \bar{2}+1+1+\bar{1},\ 2+\bar{1}+\bar{1}+1,\ \ 2+\bar{1}+1+\bar{1},\ 2+1+\bar{1}+\bar{1},\ ...,1+1+\bar{1}+\bar{2}$.

Consequently we may now ask: How many compositions are there of n into m parts with j designated summands wherein each of the latter is $> N$? Subtracting N from each of these j summands, we are left with a composition of $n - jN$ with m parts and still j designated summands. Hence there are

$$\binom{m}{j}\binom{n-jN-1}{m-1}$$

such compositions.

we now wish to interpret

$$\sum_{j\geq 0}(-1)^j\binom{m}{j}\binom{n-jN-1}{m-1}.$$

This sum clearly gives a weighted count of each composition of n according to how many of its summands exceed N. Suppose S summands exceed N.Then the composition gets counted once by the $j = 0$ term. It is counted

$$-\binom{S}{1}$$

times by the $j = 1$ term;

$$+\binom{S}{2}$$

by the $j = 2$ term, etc.

Thus a composition of n with S summands $> N$ is counted

$$1-\binom{S}{1}+\binom{S}{2}-...+(-1)^S\binom{S}{S}=\begin{cases}1 & \text{if } S=0,\\ 0 & \text{if } S>0\end{cases}$$

times. Consequently

$$\sum_{j\geq 0}(-1)^j\binom{m}{j}\binom{n-jN-1}{m-1}=c(N,m,n),$$

as asserted by MacMahon in (1.3).

3. Proof of Theorem 1

In the same spirit that provided the combinatorial proof of MacMahon's formula in the preceding section, we note that the number of compositions of $2m$ into m parts with $m - j$ designated summands each > 1

$$\begin{aligned}&\binom{m}{j}\binom{2m-(m-j)-1}{m-1}\\ &=\binom{m}{j}\binom{m+j-1}{m-1}=\binom{m}{j}\binom{m+j-1}{j}.\end{aligned}$$

Consequently, we have an interpretation of Theorem 1 in terms of compositions with designated summands.

We now require three preliminary results for our proof of Theorem 1.

Lemma 1. [**L**] *Let p be a prime, $0 \leq m \leq n$. Then*

$$\binom{n}{m}\equiv\binom{a_r}{b_r}\cdots\binom{a_1}{b_1}\cdot\binom{a_0}{b_0}\pmod{p}$$

where $n = a_r p^r + ... + a_1 p + a_0,\ m = b_r p^r + ... + b_1 p + b_0,\ 0 \le a_i, b_i < p$ *are the* p*-nary representation of* n, m*, respectively.*

We call the product in the right hand "the Lucas product of $\binom{n}{m}$", and for simplicity, we also write the p-nary representation of n as follows:

$$n = (a_r, ..., a_1, a_0)_p.$$

Lemma 2. $\binom{2i}{i}$ *is even.*

Proof. For $\binom{2i}{i} = \frac{2i(2i-1)}{i(i-1)!i!} = 2\binom{2i-1}{i}$. □

Lemma 3. $\binom{2i-1}{i}$ *is even if and only if* $i \neq$ *some power of* 2.

Proof. Suppose first that $i = 2^\alpha$ for some $\alpha > 0$. Then since $2^{\alpha+1} - 1 = 2^\alpha + ... + 2^1 + 1$ we have by Lucas

$$\binom{2^{\alpha+1}}{2^\alpha} \equiv \binom{1}{1} \cdot \binom{1}{0} \cdots \binom{1}{0} \pmod 2 \equiv 1 \pmod 2.$$

Now suppose that $i \neq$ some power of 2.Then in its binary representation i must have at least two powers of 2, say 2^α and 2^β with $\alpha > \beta \ge 0$. To prove the claim in this case, we only have to consider those two in the lowest powers.

(Case 1: They are not adjacent).In this case, we have: $i = (..., 1, 0, ..., 0, 1, 0, ..., 0)_2$ (namely, $i = ...+2^\alpha+0+...+0+2^\beta+0+...+0$ where 2^α and 2^β are those two terms of the lowest powers in binary representation of i), and after geometric expansion of $2i-1$ we have $2i-1 = (..., 1, 0, ..., 0, 0, 1, 1, ..., 1)_2$, namely,

$$2i-1 = (...+2^{\alpha+1}+0+...+0+2^{\beta+1})-1 = (...+2^{\alpha+1}+0+...+0)+0+2^\beta+...+2^1+1;$$

hence we have by Lucas

$$\binom{2i-1}{i} \equiv \cdots \binom{1}{*}\binom{0}{1}\binom{0}{0} \cdots \binom{0}{0}\binom{1}{1}\binom{1}{0} \cdots \binom{1}{0} \equiv 0 \pmod 2,$$

where $* = 0$ or 1, because of being $\binom{0}{1} = 0$ in the product.

(Case 2: They are adjacent). In this case, we have: $i = (..., 1, 1, 0, ..., 0)_2$ (namely, $i = ... + 2^\alpha + 2^\beta + 0 + ... + 0$ where 2^α and 2^β are those two terms of the lowest powers in binary representation of i), and after geometric expansion of $2i-1$ we have $2i-1 = (..., 1, 0, 1, 1, ..., 1)_2$,namely,

$$2i - 1 = ... + 2^{\alpha+1} + 2^{\beta+1} - 1 = ... + 2^{\alpha+1} + 0 + 2^\beta + 2^{\beta-1} + ... + 2^1 + 1$$

; hence we have by Lucas

$$\binom{2i-1}{i} \equiv \cdots \binom{1}{*}\binom{0}{1}\binom{1}{1}\binom{1}{0} \cdots \binom{1}{0} \equiv 0 \pmod 2,$$

where $* = 0$ or 1, again because of being $\binom{0}{1} = 0$ in the product. □

3.1. Proof of Theorem 1. We first recall that $\binom{m+i-1}{i} = \binom{m+i}{i} - \binom{m+i-1}{i-1}$. Using this identity, we rewrite the product as follows:

$$\binom{m+i-1}{i}\binom{m}{i} = \binom{m+i}{2i}\binom{2i}{i} - \binom{m+i-1}{2i-1}\binom{2i-1}{i} \equiv \binom{m+i-1}{2i-1}\binom{2i-1}{i},$$

in modulo 2 and by Lemma 2, and hence this product is even by Lemma 3, with the possible exception of the case when $i =$ some power 2^λ of 2.

If $i = 2^\lambda$ then we still have

$$\binom{m+i-1}{i}\binom{m}{i} \equiv \binom{m+i-1}{2i-1}\binom{2i-1}{i} \pmod 2,$$

and $\binom{2i-1}{i}$ is odd by Lemma 3.

To finish the rest of the whole proof, namely when the exceptional case $i = 2^\lambda$, we also apply a similar argument to $\binom{m+i-1}{2i-1}$, which we have developed in the proof of Lemma 3 as follows:

(Case 1). Suppose the $i = 2^\lambda$ is the lowest power of 2 in the binary representation of m. Then since $m + (2^\lambda - 1) = (... + 2^\lambda) + 2^{\lambda-1} + ... + 1$ and $2^{\lambda+1} - 1 = 2^\lambda + 2^{\lambda-1} + ... + 1$, we have by Lucas

$$\binom{m+2^\lambda-1}{2^{\lambda+1}-1} \equiv \cdots \binom{1}{1} \cdots \binom{1}{1} \equiv 1 \pmod 2,$$

since the first $\cdots$ consists of a product of $\binom{*}{0}$ with $* = 1$ or 0.Thus, the product $\binom{m+i-1}{i}\binom{m}{i}$ is odd.

(Case 2). Suppose that the lowest power 2^α of 2 in the binary representation of m is strictly bigger than $i = 2^\lambda$.Then since $m+(2^\lambda-1) = (...+0)+2^{\lambda-1}+...+1$ and $2^{\lambda+1} - 1 = 2^\lambda + 2^{\lambda-1} + ... + 1$, in the Lucas product we must have $\binom{0}{1}$:

$$\binom{m+2^\lambda-1}{2^{\lambda+1}-1} \equiv \cdots \binom{0}{1}\binom{1}{1} \cdots \binom{1}{1} \equiv 0 \pmod 2,$$

and hence it is even; consequently the product $\binom{m+i-1}{i}\binom{m}{i}$ is even.

(Case 3). Suppose that the lowest power 2^α of 2 in the binary representation of m is strictly smaller than $i = 2^\lambda$. Then whereas 2^α appears once in the binary representation of $2^{\lambda+1} - 1 = 2^\lambda + 2^{\lambda-1} + ... + 1$, it does not appear in that of $m+(2^\lambda-1) = ...+2^{\lambda-1}+...+1$, since it shows up twice there. Therefore, at least one $\binom{0}{1}$ appears in the Lucas product of $\binom{m+2^\lambda-1}{2^{\lambda+1}-1}$; and hence $\binom{m+2^\lambda-1}{2^{\lambda+1}-1}$ is even; consequently the product $\binom{m+i-1}{i}\binom{m}{i}$ is again even.

This proves that the product $\binom{m+i-1}{i}\binom{m}{i}$ is odd if and only if $i = 2^\lambda$ is the lowest power of 2 in the binary representation of m. □

4. Computational Aspects

In this section, our main concern is the efficient computation of the $c(N, m, n)$. The first question is: how may of the $c(N, m, n)$ do we wish to compute? If we want only one, then the formula given in equation (1.3) is what one must use.

Suppose that we want to compute all the $c(N, m, n)$ for N and m fixed with $0 \le n \le mN$. In this case, equation (1.3) is not efficient. While it is easy to compute successively $\binom{m}{j}$ because

$$\binom{m}{j} = \frac{(m-j+1)}{j}\binom{m}{j-1},$$

it requires $N-1$ multiplications and $N-1$ divisions to pass recursively from

$$\binom{n-(j-1)N-1}{m-1}$$

to
$$\binom{n-jN-1}{m-1}.$$

However, from (1.5) we see that
$$\frac{\sum_{n\geq 0} c(N,m,n)q^n}{(1-q^N)^m} = \frac{q^N}{(1-q)^m};$$
so
$$\sum_{n\geq 0} c(N,m,n)q^n \sum_{s\geq 0}\binom{m+s-1}{s} q^{Ns} = \sum_{t\geq 0}\binom{m+t-1}{t} q^{t+N}.$$
Consequently
$$\sum_{s\geq 0}\binom{m+s-1}{s} c(N,m,t-Ns+N) = \binom{m+t-1}{t}.$$
So shifting t to $t-N$, we obtain
$$c(N,m,t) = \binom{m+t-N-1}{m-1} - \sum_{s\geq 1}\binom{m+s-1}{s} c(N,m,t-Ns). \tag{4.1}$$
In this recurrence, all the binomial coefficients are easily computed successively.

Finally, if we wish to compute all the $c(N,m,n)$ with both N and n varying and m fixed, then the natural recurrence
$$(q+q^2+...+q^m)^N = (q+q^2+...+q^m)(q+q^2+...+q^m)^{N-1}$$
yields immediately the formula
$$c(N,m,n) = c(N-1,m,n-1)+c(N-1,m,n-2)+...+c(N-1,m,n-m). \tag{4.2}$$

5. Conclusion

It would be very interesting to obtain a combinatorial proof of Theorem 1 using the relevant compositions described at the beginning of Section 3.

The product
$$\binom{m+j-1}{j}\binom{m}{j}$$
was the subject of Theorem 1. Additionally, we conjecture that for $j \geq 1$, all values of this product are distinct except for the cases $m=5, j=3$ which is 350 the same as $m=5, j=4$.

6. Appendix (Announcement of InetCompu Service)

For those who need all or some of the coefficients of the expression
$$\begin{aligned}(1+q+q^2+...+q^{N-1})^m &= \sum_{n=0}^{(N-1)m} s_n q^n \\ &= q^{-m}\sum_{n\geq 0} c(N,m,n)q^n,\end{aligned} \tag{6.1}$$
InetCompu Service (Internet Computing Service) is now available. For more about this, please visit

http://trinitas.mju.ac.kr/intro2rbf.html

For instance, by sending explicit values for N and m in (6.1) in the form:

input:
q:=101$
m:=200$
end input:

as a main body of an e-mail to the e-mail address

rbf@trinitas.mju.ac.kr

all the coefficients of (6.1) are automatically computed and the results are sent back to the e-mail sender even with a computation time stamp. For example, if one sends the above input to one of our InetCompu Services, **rbf@trinitas.mju.ac.kr**, it will not only report the computed results but also report that it takes only 1 Minutes and 10 Seconds to finish the whole computation for all the coefficients. If one visits the above homepage, one will find a few other computed results, as examples.

Here, rbf = Recursive Binomial Formula. We are using our recursive binomial formula in (6.1) for this service, and we hope that our InetCompu Service is very helpful to those who need such a computation. More information on InetCompu Service for other purposes is also available at

http://trinitas.mju.ac.kr/intro2InetCompu.html

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