Math 486. Mathematical Theory of Games. Class notes. by L. Vaserstein.

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Contents

These notes do not replace textbooks. They are a work in progress, with many misprints.
To complement the textbook, the notes give additional information, more examples with solutions, and more exercises.
Also math articles in wiki we refer to are not replacement for good math books.

Here are some comments to the Contents above.
After giving a few examples of games, the notes give two definitions of game: extensive and normal forms.
Then we work with matrix games and closely related linear programming.
After this, I give two solution concepts: Nash bargaining (for finite games without side payment. and the Shapley values (for games with side payments).
Both exist and unique for any finite game. Both use matrix games.
In the last chapter some advance topics are mentioned.

I do not discuss some controversial topics like
what is meaning of life or how to pass a course without trying but
I do mention some unresolved philosophical issues related with Game Theory.

Here is a link to a free book in pdf:
Game Theory Alive , by Anna Karlin and Yuval Peres, to be published by the American Mathematical Society (2016).
I recommend this rather advanced book for additional reading.

Math 486. Mathematical Theory of Games

Game Theory is about
defining games,
defining solutions,
finding solutions (if exist).

Here is a definition from **Game theory** wiki:
Game theory is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers". Game theory is mainly used in economics, political science, and psychology, as well as logic, computer science and biology.[1]

This is not a mathematical definition. What is "conflict" here?
Game theory , like real life, os about cooperation. But what is "cooperation"?
iI there is only one player where is conflict? where is cooperation?
What is "intelligent"? What is "rational"?

Here is a story explaining what "mathematical" means.

Black Sheep
An engineer, a physicist, and a mathematician were on a train heading north, and had just crossed the border into Scotland.

- The engineer looked out of the window and said "Look! Scottish sheep are black!"
- The physicist said, "No, no. *Some* Scottish sheep are black."
- The mathematician looked irritated. "There is at least one field, containing at least one sheep, of which at least one side is black."

For history, see *Walker, Paul (2005).* *"History of Game Theory"*
and Chronology of Game Theory I

What is a number? Unless said otherwise, a number in this class is a real number.

Examples of numbers: 1, 2, 4, 7, 9, 0, -1, 2/3, 0.5, $-\overline{2}/3$, $\sqrt{2}$

Not numbers: ∞, 0/0, 1/0, $\sqrt{-1}$

To see better the difference between 1 and 2, 1 and 7
and between 4 and 9.
watch the video about how to write numbers.
The number 0 is special. Division by 0 is not defined.

Exercise. Compute the sum:
(a) 6123 + 6356 =
(b) 1/4 + 7/9 + (-13/36)=

Ch1. Examples of Games

§1. One-player games.

Ch1 means Chapter 1.
§1 means Section 1.

1.1. **Mother of all games.** A player, Pete has to choose between \$1 and \$2.
Here is the solution for this simple one-player game: Pete chooses the maximal payoff \$2.

Pete
↙ ↘
\$1 \$2

1.2. **The value of game and an optimal strategy**. For any one-player game. In general, it is clear what it means to solve it. The player chooses the maximal payoff, called *the value of game*, if it exists. A way to get the value of game is called an *optimal* way, move, alternative, choice, option, or strategy.
If chance is present, we maximize the *expected* payoff.

While it is easy to understand what it means to solve a one-player game, solving a game can be difficult for some games. Manny casino games, including the most popular (blackjack, roulette) are one-player games.

1.3. An infinite game. Here is a game with no value.
Karl can choose any payoff he wants.

Karl may find this game very attractive . Here is a version of the game by
Karl Marx in 1875 *Critique of the Gotha Program*.:
to each according to his needs
(in German, *jedem nach seinen Bedürfnissen,*
I do not know who should decide what you need.

1.4. Double or nothing. This is a more complicated situation with infinitely many outcomes.
It is very popular in the books and booklets advising you how to win every casino game.
You start with \$1 bet in an even money wagging. You double your bet until you win.

Your expected payoff is $1 if you ever win. For example, if you win in the third try, it is -1 - 2 +4 = 1 ($).
But what if you never win?

1.5. Raffle. 100 tickets are sold for $1 each. They have different numbers. Then numbers are chosen at random for the grand prize $25, two $10 prizes, and five $1 prizes. What is the fair value of a ticket? Answer:-$0.5. So if you buy a ticket, your expected payoff is -$0.5. If you buy k tickets, you lose $k/2.
For example, if you buy all tickets, then k = 100 and you return $50 = k/2.
The optimal solution is do not participate. (k = 0).

```
          Player
     ↙ k = 0      ↘ k = 1
  $0        1% ↙  2%↓  5%↘      ↓ 92%
                  $24    $9       0   -$1 no prize
```

More generally, when n tickets are distributed and the total value of the prizes is m, the fair value of a ticket is m/n.
This should be compared with the prize of ticket to decide how many tickets you want to buy.

In sweepstakes and lotteries, total payoff may depend on the number of tickets sold, in which case you may be given odds of winning different prizes. Sometimes, you chose the number of your entry.
There are many sweepstakes with free entries. Your cost is to reveal your personal information.
They use it for better marketing or for getting your money.

See **Raffle wiki,** Sweepstakes wiki , and Lottery wiki for more information

1.6. Car and goats (see Monty Hall problem wiki | car &goats toronto | 2 mind spring | 3 youtube |4 NYT| 5 youtube |)

There is a car and two goats behind three closed doors.
You chose a door. The host opens a door with a goat and offers you to switch (to the other closed door).
Should you?

```
                         You
         stay  /                   \switch
          /1/3   \2/3          / 1/3    \2/3
        car      goat          goat      car
```

It is not a game until we have numbers for payoffs. Suppose you (the player) value the car as $30K and the goats as $0.

```
                      You
         stay   /                  \switch
          /1/3  \2/3             / 1/3    \2/3
  $30K car   0 goat    0 goat      car $30K
```

```
If  you stay, the probability of getting the car is 1/3 and your
expected payoff is $10K.
If you switch, you get the car if and only if your original
choice was wrong. The probability of this is 2/3. So your
expected payoff is $20K
Thus, the value of game is $20K and the optimal strategy is to
switch. The original choice of door does not matter.
```

```
                      You  $20K = value of game
         stay   /                \  \switch optimal choice
              $10K                   20K
          /1/3  \2/3            / 1/3    \2/3
   $30K car    0 goat    0  goat      car $30K
```

If the host is not required to offer you to switch, it becomes a two-player game (like in the film "21"), after we specify the host's payoffs.
For example, if the host wants to minimize your payoff. you should stay.

1.7. Roulette wiki. American roulette has 38 squares. If you bet $1 on a group of n squares and win, your payoff is 36/n - 1 except for the case n = 5 (top line) when it is $6 (rather than $6.20). If you lose, you lose your bet.

The expected payoff is -$1/19 if $n \neq 5$ and it is -3/38 when n = 5. The value of game (for $1 bet) is -$1/19. Every choice except the top line, is optimal. If you bet $b instead $1, the payoff is multiplied by b. You can make several bets at once. You may place your bet even after the ball is rolling until the dealer stops betting.

```
Player bets b on group of n = 18 (e.g., black)

         18/38 /                  \ 20/38

          b                             -b             Even bet,
expected payoff  -b/19 optimal (for given b)
```

Player bets b on a number

1/38 / \ 37/38 probabilities

35b -b Expected payoff -b/19 optimal (for given b)

Player bets b on the top line (basket bet)

5/38 / \ 33/38

6b -b
Expected payoff -3b/38 not optimal

The top line is the only group of 5 of 5 squares, namely, 0, 00, 1, 2, 3. Instead of betting 5b on the top line, you can bet b on sach of 5 squares. The probability of winning is the same 5/38, by the payoff in case of winning is 31b instead of 30b.

French roulette has 37 square (no 00 square). The value of game (when you bet $1) is -$1/37 > -$1/19. and all choices are optimal for a fixed bet b.For example if you bet $1 on red (n = 18), you win $1 with probability 18/37 and lose $1 with probability 19/38, so your expected payoff is -$1/37. All bets with a fixed total bet b are optimal with expected payoff -b/37.

So the house edge for French roulette is 1/37 which is smaller than 1/19 for American roulette.

There are many other variations of roulettes.

A student asked me why they do not use 36/n - 1 for the top row bet. I think the casinos do not want any change (coins) on the table. In particular, they do not want to pay you $36/5 - 1 =$6.20. So they add $6 to your $1 bet on the table.
Your payoff is $6 if you win. It is better for you to bet $1 on each of 5 squares in the basket bet than bet $5 on the basket.

```
But it is even better do not play roulette or any other
game with negative value.

In casinos, the bet  b  should be an integer subject to the
minimum and maximal constraints (in local currency). In our
class, we usually require only that  b ≥ 0. The player  does
not place any bet when  b = 0. Dealers do not use coins.
```

There are many books and websites which offers you tips how to "win" in roulette and other casino games. After taking this course, you should realize that those tips do not work.
Casinos are there to take your money not to enrich you. There are laws and rules to guarantee this.
For example, you can make money on roulette if you use a computer or place your bet after the ball stops (both are illegal).
A popular tip is double or nothing strategy in one form or another (see 1.4 above).

An exception is that in blackjack, counting cards allows you find sometimes that the expected value of game is positive.

Another possible exception are promotions (like free coupons or even free cash for coming to casino, free food and drinks, etc).
This is like free gifts or free samples some stores offer you for coming or giving personal information.
This is bait. A smart fish eats worms without getting hooked.

1.8. [Blackjack](#) wiki | [Card Game Rules | Bicycle Playing Cards |](#)

There are many variations of game. For simplicity, we do not allow splitting, doubling, surrender, Charlie rules, any bonus for natuaral blackjack (i.e., 21 in two cards), or insurance. The insurance have a negative value unless you have specific information about the remaining cards (see Example 1.18). The other **embellishments** increase the value of game a little bit but they have many variations.

Another subtle point to specify is how the shoe is replenished. Among many ways to do it, here is our way. The game starts with a full deck of card (52 cards) in the shoe. Each time the shoe is empty, (out cards) the dealer adds a new deck of cards.

The dealer draws at ≤ 16 and stands at ≥ 17. At tie "push" (tie), the payoff is 0

Rarely in **blackjack** there is a **rule** that if the player reaches a certain number of cards, usually **5** to 7, without busting, the player will automatically win. This is caled a "**Charlie**." If you surrender, you get back the half of your bet.

Information on remaining cards may improve your odds. One assumption is that we have no information (many decks in the shoe). Another assumption is the we know everything (perfect card counting). In past, the used cards were used again, players could touch their cards, and others could not see your cards until the end of game. The "many decks" here means that the probability of getting a card valued n is 1/13 for $1 \le n \le 9$, and it is 4/13 for n = 10.

An ace is counted as 11 unless it takes you over 21 in which case it is counted as 1. You haft a soft hand if you have an ace which is counted as 11.

E.g., any soft n becomes hard n if you get 10.

1.9. Philosophical issues.

Is there free will? In Game Theory, a player sometimes has freedom of choice.

Is there chance **(ind**eterminism**,** randomness)**?** In Game Theory, chance moves are allowed in some games.

State laws require Roulette to be random. But a small computer can predict quite well outcome (when and where the ball stops) given 3 times of passing 0, and you can still make your bet after this.

What happens if a fortune teller or a fatalist plays Roulette? See **Fortune-telling wiki, Fatalism wiki,** Clairvoyance wiki, and Precognition wiki are beyond the scope of our class.

See opinions of two physicists about chance:
Gott würfelt nicht – Wikipedia
Maxwell's demon - Wikipedia

Chance is a hard conception to grasp. Here is an explanation of the value of roulette without any chance.
Suppose you put $1 on every square in American Roulette. You total bet is $38; You get back exactly $36 for sure
(the dealer adds $35 to your $1 on the winning square after taking your $37 from the other squares).
There is no uncertainty here. So you lose $2 (for sure) which is 1/19 of your bet.

Similarly, in Raffle, if you buy all tickets you know exactly what is your payoff. No chances here!
This us am explanation for m/n in 1.5 above.

1.10. Sum of games. Suppose we have two games with values u and v. We can play them

simultaneously or one after the other.
The resulting game has value u + v.
For example, in American roulette, you can put $1 on red and $2 on black at once or in two rounds.
The value (the expected payoff) of resulting game is -$3/19.

1.11. Mixed strategy. In 1.1, we can decide to take $1 with probability 0.2 and $2 with probability 0.8. The expected payoff is 0.2 × $1 + 0.8 × $2 = $1.80. This mixed strategy is not optimal.
More generally, the expected payoff a mixture pA + (1-p)B of two strategies A and B with payoffs x and y in any
one-player game, where $0 \le p \le 1$, is px + (1-p)y. It is optimal when p = 0 or 1.

Remark 1.12. Most of casino games are one-player games. This includes all games with slot machines and two most popular games with dealers, blackjack and roulette. This is because the strategies of machines and dealers are fixed.

Remark 1,13. Are there smaller games that Mother of A;; Games (Example 1.1(? Yes, the number of choices could be less than 2. The number of players can be less than 1.
For example, suppose Pete gets $1. We can say that the value of this game is $1. We can say that Pete has no choice and we can say that Pete has only one choice: get $1.
At this point we can realize we still have no [recise definition of game. Are Examples 8.6, 8.7, and 8.8 below.
We can also realize that some problems are difficult because they are too big and some problems are difficult because they are too small.

Example 1.14. Blackjack.
Dealer has 16 hard. You have 16.
Solution. You cannot draw because otherwise the dealer wold have one card and her hand was ≤ 11. Let your bet be b. You have no information on the remaining cards, o assume that the probability of the next card to be values c is 4/13 when c = 10 and the probability is 1/13 when c = 1,…,
The dealer goes over when c ≥ 6. This happens with probability 8/13 in which case your payoff is b. With probability 5/13, c ≤ 5 in which case you lose b.
The expected payoff for you s 3b/13.

Example 1.15 .The basic urn game.
The urn contains b black balls and w white balls. A ball is drawn from the urn,
You are paid x if the ball is black and y if it is white. What is your expected payoff?
Answer. (bx + wy)/(b+w).

Example 1.16 Drawing money. A black box (or wallet) has b1 of $1 bills, b5 of $5 bills, and b10 of $10 bills.
You are allowed to draw a banknote for yourself. What is your Expected payoff?
Answer.(b1 + 5b5 + 10 b10)/(b1+b2+b10).

Example 1.17 . Drawing money twice. , you may take 2 bills from the box containing b1 $1 bills
and b5 $5 bills. What is your expected payoff?
Solution. We assume that the number of bills, N = b1+b2) ≥ 2.
The number of pairs of bills is N(N-1).
Here are these pairs sorted by the value:

value in $	1+1 =2	1+5 = 6	5 + 1 = 6	5 + 5 = 10
# of pair	b1(b1-1)	b1*b5	b5*b1	b5(b5 - 1)

So the expected payoff is
((2b1(b1-1) + 12b1*b5+ 10b5*b10+ 10b5(b5-1))/(N(N-1))..

Example 1.18. In Blackjack, your bet is b> 0 and the dealer shows an ace. You are offered a side bet that the dealer has a natural blackjack,
the value of the hole card is 10. If you take the bet (imsurane)
If it is 10, you are paid b/ If it is not 10, you ;one b/2. Should you take the bet?

Solution. Let p be the probability of 10. Then your expected value of the insurance is
pb - (1-p)b/2 = (3p - 1)b/2. So you take insurance when p > 1/3.
For example iassume yjat the game started for a deck in the shoe.
If you saw 3 cards and non of them is 10, then p = 16/48 = 1/3 and the value of the side bet is 0.
If you saw 10 cards and non of them is 0, then p = 16/42 > 1/3 and you take insurance.
If you saw 13 card and 4 of them are valued 10, then p = 12/39 = 4/13 < 1/3 so you do not take insurance.

Remark. When the game 21 came to America, game halls started to give bonuses to players for getting a black jack.
This is how the name Blackjack appeared.

Exercises to §1

Exercise 1. A roulette coupon has face value $5. You can bet on red or black. You are required to bet an integer b ≥ 5 of your money with the coupon.
If you win, your payoff is b + 5. If you lose, you lose b. You lose the coupon in both cases. What is the value of the coupon and what is your optimal strategy?
Hint: your optimal strategy includes a choice of copayment b. The value. of a coupon is the value of game with the coupon.
Assume that the roulette is American .

Remark. When you use coupons with your own money, your money are at risk even when expected payoff is positive.
Can you avoid any risk? Yes, you can, for a small reduction in the expected payoff.
Here is a way to do it. This way uses 2 coupons and, when the coupon says “only one coupon per person,“ a friend.

is needed.
You place a coupon and \$5 on red and \$1 on 0. Your friend places a coupon and \$5 on black and \$1 on 00 for you.
So there are \$12 of your money on table.
If the outcome is black or red, you get \$15
so your net payoff is \$3.
If it is green,
you get \$36 so your net payoff is \$36 - \$12 = \$24.

Thus, you payoff is always positive.
Your expected payoff decreases by \$2/19 in comparison with the bet with \$2 on green not placed.
The dealer would not like this way because of confusion and cooperation.

Exercise 2. In the car and goats game, solve the game assuming that you value a goat as \$900 and a car as \$90.

Exercise 3. In blackjack, you have 20 hard, and dealers top card is 10. The remaining cards (including dealer's second card) are A and 10. Your bet is \$100. Solve the game (find the value of game and an optimal strategy).

Exercise 4 To get only one card left in the shoe, we need many players. However their actions do not affect you, so it is still a one-player game. Usually, when few cards left in the shoe, (or dealer suspects card counting) the dealer add a new deck of 52 cards or a few decks. In old times the used cards were used again.

Assume that after the last card is taken from the shoe, the dealer add a new deck. Solve this game (see Exercise 3 above).

Exercise 5. In the car and goats game, assume that there are 2 cars and 2 goats instead 1 car and 3 goats. So there are 4 doors Everything else is the same.
Solve the game.

More generally, suppose there are $c \geq 1$ cars, $g \geq 1$ goats, and the host must open $k \leq g - 1$ goors with gots, $k \geq 1$.
Assuming that the value of car is larger than that of goat, show that staying is not optimal and switching is optimal.

Exercise 6. In a game of Deal or No Deal, your are offered either quit and keep \$10K or choose one of of remaining 5 cases.You know that remaining 5 prizes are \$10K, \$10K, \$20K, \$30K,

$1K, one in each (unknown) cases.Solve the game (Do you quit or open a case)? What is the value of game?).

Exercise 7. In Blackjack, you have 10. The dealer shows 9. The remaining cards are 6,6, 8, and 9.The bet is $100.

Exercise 8. Blackjack. You have 17. Dealer has 16 hard. Solve the game.

§2. Two-player games

2.1. Tic-Tac-Toe: thespruce || wiki || exploratori || play

2.2. Nim :|| wiki || toronto || cornel

An example
3 4 5 A starts
3 4 2 B
3 2 A
2 2 B
1 2 A
1 1 B
1 A
0 B loses

Optimal strategy for two piles: make them even.

2.3. Heads & Tails = Matching Pennies 1 | 2 | 3 ||

2.4. Rock–paper–scissors 1 | | 2 | 3 ||

2.5. 1 **Prisoner's dilemma wiki** | **2** youtube || **3** Dibert | **Encyclopedia of Philosophy** | |

2.6. Battle of the Sexes. (see wikipedia and Battle of Buddies in the textbook),

Exercises to §2

Exercise 1. Make your move in Nim in the position: 1, 10, 100, 1000 (4 piles).

Exercise 2 (bonus). Solve Tic-Tac-Toe (it takes about an hour and 10 pages).
If you do not use symmetry to reduce the number of positions and repeat positions to arrange them in a tree, it might take thousands of pages.

Exercise 3 (bonus). A more general game then Tic-Tac-Toe is played on m by n board and line of m is required to win.
In Tic-Tac-Toe, n = m = k = 3. The case when m = n = 19 and k = 5 is known as Go-Moku;
See weijima. Show that the second player has no winning strategy.

§3. Solving some games

Example 3.1. Game of Life. (**Wikipedia**). This is a zero-player game.

Example 3.2. Nim.
Optimal strategy: make the checksum (aka Nim-sum) 0. In other words, replace a number by the checksum of all other numbers.
It may happen the the last checksum is bigger than the number.
We can chose the number which has 1 in the first (from the left) column with odd sum.
The solution for 3 piles was given in
Charles L. Bouton, Nim, A Game with a Complete Mathematical Theory,
Annals of Mathematics, Second Series, Vol. 3, No. 1/4 (1901 - 1902), pp. 35-39

For instance, what is a winning move in the initial position
6 piles, 1, 22, 33, 22,44 55 pebbles?
Solution. We know that we have to make the checksum 0. Note that n ✧ n = 0 for any natural number n. So

1,✧ 22,✧ 33 ✧ 22,✧ 44 ✧ 55 = 1, ✧ 33 ,✧ 44 ✧ 55.

We do not need to touch the piles with 22 [ebbles. Now we want to replace one of these 4 numbers be the checksum of the other 3 numbers. Let us compute 1, ✧ 33 ,✧ 44 and compare it with 55.

	32	16	8	4	2	1
1						1
33	1	0	0	0	0	1
44	1	0	1	1	0	0

____

12 1 1 0 0 checksum.

So we can replace 55 by 12 = 1, ✧ 33 ,✧ 44 to make the checksum 0.

Answer. A winning move is to take 43 from the pile of 55.

Remark. How to find all winning moves in Nim?
We write the numbers $a_1,\ldots, a_k$ of of pebbles in piles in base one below other and compute the checksum C.
If C = 0, there are no winning moves. Otherwise, we consider the first bit (=1) of S and the corresponding column.
A number a_i has 1 in this column if and only if it is bigger than the checksum S ✧ a_i of the others a_j.
Replacing such a_i by S ✧ a_i is a winning move.

3.3. How to solve Tic-Tac-Toe.

Draw all positions and connect them by moves. Use symmetry and do not repeat positions, so you can fit it to 10 pages.

3.4. Restricted Nim.

Example 3.4. In a movie, see[.com/watch?v=HkzMA1jrm00](.com/watch?v=HkzMA1jrm00)
Austin Powers stands at 5 in Blackjack. Is it possible that this is the only optimal move?
Here is an example. You have 5 and the dealer shows 10. Remaining cards are 6 and 6, and 7. Your bet is $1000. Solve the game.
Solution. If I stand at 5, the dealer gets 22, and I win.
If I draw once, I end up with 11 and the dealer ads a new deck or several new decks to the shoe and draws at 16.
He wins gets 1-5 and wins with probability 5/13/ He loses with probability 8/13.
My expected payoff is $3K/13. It is not optimal.
Also I lose with a positive probability if I draw more than once.
So standing is the only optimal initial move.

Example 3.5 In Blackjack, your hand is 18. Dealer has 16 hard.
Remaining 51 cards: a deck without one 2.
Solve the game.
Solution. Suppose that my payoff is b
The dealer's hand is > 11, so she opened her hole card and I stand.

I have no moves. She has ≤ 16 so she draws.

With probability 12/51, the card is 3, 4, or 5 so she wins against my 18 and my payoff is -b.
With probability 3/51, the card is 2, so my payoff is 0.
With remaining probability 36/51, the card is A or ≥ 6, so my payoff is b.
Therefore my ex[ected [ayoff is -12b/51 + 36b/51 = 24b/51 = 8b/17.

Example 3.6 In Blackjack, your hand is 18. Dealer has 16 hard.
Remaining 50 cards: a deck without one 2 and one ace.
Solve the game.
Solution. Suppose that my payoff is b
The dealer's hand is > 11, so she opened her hole card and I stand.
I have no moves. She has ≤ 16 so she draws.

With probability 12/50, the card is 3, 4, or 5 so she wins against my 18 and my payoff is -b.
With probability 3/50, the card is 2, so my payoff is 0.
With remaining probability 35/50, the card is A or ≥ 6, so my payoff is b.
Therefore my ex[ected [ayoff is -12b/50 + 35b/50 = 23b/50 = 0.46b.

Example 3.7. In Blackjack, your hand is 18. Dealer has 15 hard.
Remaining 51ncards: a deck without an ace.
Solve the game.
Solution. Suppose that my payoff is b
The dealer's hand is > 11, so she opened her hole card and I stand.
I have no moves. She has ≤ 16 so she draws.

With probability 12/51, the card is 4, 5, or 6 so she wins against my 18 and my payoff is -b.
With probability 4/51, the card is 3, so my payoff is 0.
With probability 4/51, the card is an ace, so Dealer have 15 and 50 cards left.
She draws again, and my expected payoff is 23b/50 by Example 3.6.
With remaining probability 31/51, the card is 2, or ≥ 7, so my payoff is b.

Therefore my expected payoff is
-12b/51 + (4/51)(23/50)b + 31b/51 = 9b/51 + (4/51)(23/50)b = (450+92)b/2550

= 542b/2550 ≈ 0.21b. .

Example 3.8 Ib Blackjack, your hand is 18. Dealer has 14 hard. The shoe is empty so

she add a new deck of cards.
Solve the game.
Solution. The dealer has > 11, so she already opened her hole card hence I am standing at 18.
I have no moves left. To solve the game I have to compute my expected payoff.
Suppose my bet is b/
The dealer has 14 ≤ 16 so she draws.

With probability 1/13 + 6/13 she gets either 3 or ≥ 8 and loses so my payoff is b.
With probability 3/13 she gets 5, 6, or 7 and wins so my payoff is -b.
With probability 1/13, she gets 4 and my payoff is 0.

With probability 1/13, she gets 2 She must draw again at 16.
51 cards left.
With probability (52-8)/51 = 44/51 she gets > 2 and goes over 21, so my payoff is b.
Wits probability 3/51 = 1/17 she gets 2 and my payoff is 0.
With probability 4/51, she gets A her 17 loses against my 18. My payoff is b.
So my expected payoff is
48b/51 = 16b/17

Finally, with probability 1/13 she gets A and must draw again at 15.
51 cards are left.
With probability 12/51 she gets 4,5, or 6 and wins against my 18. My payoff is -b.
With probability 4/51 she gets 3 and my payoff is 0.
With probability 4/51, she gets 2 and loses. My payoff is b.
Finally, with probability 3/51 she gets Ace again and must draw again at 16.
50 cards left. She wins by getting 3,4,or 5, with probability 12/50 and ;loses with the remaining probability 38/50/
The expected payoff is 26b/50.

Here us the picture where P stands for the player and D stands for the dealer:

\$542b/2550 by Example 3.7 D15 ←1/13← P18 D 14 →1/13→ D16 \$8b/17. by Example 3.5

7/13 ↙ 5/13 ↓ ↘ 1/13

D 17 or >21 \$b D 18 \$0 D 19, 20, or 21 -\$b

So the expected payoff is
(1/13)(542b/2550) + (1/13)8b/17 + 6b/13
= (1/13)(542b/2550) + (8 + 102)b/(17*13)

$$= (542*17 + 110*2550)/(17*13*2550)$$
$$= 289714b/563550 \approx 0.514b.$$

I used that 13*17=221, 542*17= 9214, 110*2550 = 280500
280500+ 9214 = 289714, and17*13*2550 = 221*2550 = 563550.

Example 3.9. In Blackjack, you have 21 soft (e.g., A, 4, 6). The dealer shows 7. The cards remaining are 7 and 7.
Solve the game when y the other players are standing or over 21 and your bet is $b > 0$.
Solution. Let $b > 0$ be my bet.
If I stand, the dealer gets 21 and my payoff is 0.

If I draw, my hand becomes1.8 hard. The show is empty so the dealer adds a new deck to the shoe.
She opens his hole card, 7 and her hand becomes 14. The show is empty so she adds a new deck of cards.
If I stand at 18, then by Example 3.8, my expected payoff is $289714b/563550 \approx 0.514b > 0.5b$.
So standing at 21 was not optimal.

Next I have to wonder drawing again.
The hole card is 7, so the dealer has 14,. The shoe has a full deck of cards.
If I draw, with probability 1/13 I have19, with probability 1/13 I have 20, with probability 1/13 I get 21,
nd with probability $10/13 > 1/2$ I go over 21 and lose.
So it is better do not draw the second time.

Thus, the optima; strategy is to draw once at 21.The value of game is
$289714b/563550 \approx 0.514b. > -.5b/$

Example 3.10 Deal or NO Deal). You can either take home 10K or choose one of 2 cases.
One has nothing, but the other one has $20K.
You do not know which case is empty. What os your choice?

Solution. If you say "no deal" your payoff is 10K.
If you choose a case instead, your expected payoff is $0.5*20K + 0.5*0 = 10K$.
So both choices are optimal.
Some people would prefer do not take chance and do not play.
Moreover they billions are spent billions on insurances to spread risk.
Some people lmore advantiorour. They ;ike exitement. Moreover
they spent billions on gambling playing games with negative values.

Exercises to §3

Exercise 1. Restricted Nim. Players can take up to 9 stones in a move.
Initial position: 10, 25.

Exercise 2. Nim but the last move loses. Initial position: 5, 10.

Exercise 3.In American Roulette, you bet $6 on red, $3 on black, and $10 on square 2. What is your expected payoff?

Exercise 4. Restricted Nim. The first player can take 1 or 2 pebbles in a move. The second player can take 1 or 3
pebbles in a move. The initial position is 10 pebbles in a pile.

Exercise 5. Restricted Nim. In one move, a player can take a prime number of pebbles. The initial position is 18 pebbles in a pile.

Ch2. Background

If you took Math 484 with me, you know this background.
You can read my textbook on linear programming in library.
You are supposed to know §4 stuff since elementary school
and §§5,6 stuff from your class on linear algebra.
Here is a link to the manual for my textbook: ol2.pdf .
It has solutions to exercises in the textbook and much more.

§4. Logic

What the words "and" "or", and "if" mean?
This is a kindergarten level logic.
There are no definition in simpler terms, but there are explanations in other terms.

There is nothing simpler than "and." but here are possible replacements for it: "&', ",". In geometric terms, "and" is the intersection
($\cap$ in mathematics and $\wedge$ in logic). For example,
$x = 0$ and $y = 0$
correspond to the intersection of the vertical line $x = 0$ and thehorizontal line $y = 0$ in (x,y)-plane, i.e., the origin.

The word "or" correspond to the union ($\cup$ in mathematics and $\vee$ in logic.) For example, $x = 0$ or $y = 0$ is the union
of the ;one $x = 0$ and $y = 0$. This cross can be given by the quadratic equation $xy = 0$.
There is "or" hidden in "≥ 0". So $x \geq 0$ is the union of the point $x = 0$ on the line and the open ray $x > 0$.

In the following examples x and y are real numbers. It is OK if you assume that x and y are integers (to stay at elementary school level).

Here are true statements:
1 and 2 are positive. 2 and 1 are positive.
$x \geq 0$ means $x = 0$ or $x > 0$.
$x \geq 0$ or $x \leq 0$. $x \leq 0$ or $x \geq 0$.
If $xy = 0$, then $x = 0$ or $y = 0$. $x = 0$ or $y = 0$ if $xy = 0$.
$xy > 0$ if $x > 0$ and $y > 0$. $xy > 0$ provided that $x > 0$ and $y > 0$.
$x = 2$ only if $x \geq 0$. $x \div 0$ if $x = 2$.

If you are in State College, PA then you are in PA. Note that this implication is true even when you are not in State College.
Here is a set is a part of another set.

$x \geq 1$ implies that $x \geq 0$. T
"implode that" can be replaced by "$\Rightarrow$," "is stronger than", and "only if".
he ≥ 1 is a part of the set $x \geq 0$. The mathematical sign for inclusion is $\subset$.

The implication
$x \geq 1 \Rightarrow x \geq 0$
can be expressed in many other ways. Here are examples.
If $x \geq 1$ then (hence, so, therefore, thus,) $x \geq 0$.
$x \geq 0$ if (when, because, given that, provided that, assuming that) $x \geq 1$.

Note that implication is not symmetric. Here is an example of a relationship in real life which is not symmetric.
Alice loves Bob
is NOT the same as
Bob loves Alice.

implication is transitive. Geometrically, if A is a part of B and B is a part of C, then A is a part of C.

Both "and" and "or" are symmetric and transitive.

In terms of feasible sets, a
and corresponds to intersection,
or corresponds to union,
and
if corresponds to inclusion
of sets.

The union of sets include both sets and, in particular, the intersection.
We use the inclusive or
So the set $x = 0$ or $y = 0$ (cross) includes the origin.
Exclusive or (xor() is used rarely presently.
Exclusive or in logic correspond to +, while "and" is the multiplication.

May you answer "both" to the question"coffee or tea?"
It depends. If you flying the first class, certainly you can get both plus several bottles of champagne.
You can get it even before boarding the plane.
In other cases you may have to pay for extra drink.
I flew only commercial class this year, 4 times, 3 airlines.

In some cases, I got a few bottles of free beer, but in other cases I had to pay for every drink.

So in college logic,
$A \vee B = A B + A + B$, $(A \Rightarrow B) \neq (B \Rightarrow A)$, $(B \Rightarrow A) = (A \Leftarrow B)$, $A \wedge B = AB = BA$,
and $A \Rightarrow A$.
where $\wedge$ = and = &, $\vee$ = or, + = xor, and $\Rightarrow$ = implies, so,thus., only if.
Note that "only if" is the converse of "if."
If we have both, we may use $\Leftrightarrow$, "if and only if", iff, and "is equivalent to."

In Mother of All Games above, the player cannot take both \$1 and \$2.
But we can use mixed strategies, e.g., to decide where we go using a coin toss.
See 1.11 above.

We talk here about math logic. Quantum logic , Fuzzy logic, and Dialectical logic
are different. Also everyday logic can be different.
Suppose Alice says about Bob: "He is always late."
Does it mean that he is never on time? Probably, she means that he is late very often.

In math, never means never. See the Black Sheep story above.
Male and female logic are topics or hashtags of many politically incorrect jokes and stories.

When you became older, you realize than some stories for children are not quite true.
An example is the story about a stork delivering you to your parents.
Later you will realize that some stories for adults are not quite true.
This include several stories about games.

In modern English “or” is usually inclusive. A possible counter example is ”Every entry includes a soup or salad.” For exclusive “or” (xor), we usually use “either
… or” construction. Compare the following two statements about men in a town with one barber:
“Every man shaves himself or is shaved by a barber”
and
“Every man either shaves himself or is shaved by the barber” (Barber Paradox).

Here are several examples involving implications from my textbook on linear programming:
- $x \geq 0$, because $x \geq 2$.
- $x \geq 0$ if $x \geq 2$.
- $x \geq 2$ only if $x \geq 0$
- If $x \geq 2$, then $x \geq 0$.
- The bound $x \geq 2$ is sharper $x \geq 0$.
- Given $x \geq 2$, we conclude that $x \geq 0$.
- The bound $x \geq 2$ is better than $x \geq 0$.
- The constraint $x \geq 0$ is less tight than $x \geq 2$.
- The bound $x \geq 2$ is more precise than $x \geq 0$.
- In the view of condition $x \geq 2$, we have $x \geq 0$.
- The constraint $x \geq 0$ is less severe than $x \geq 2$.

• The constraint $x \geq 0$ is less strict than $x \geq 2$.
• The condition $x \geq 2$ implies the constraint $x \geq 0$.
• The constraint $x \geq 0$ is less stringent than $x \geq 2$.
• The constraint $x \geq 0$ is less demanding than $x \geq 2$.
• The condition $x \geq 2$ is sufficient to conclude that $x \geq 0$.
• The linear constraint $x \geq 0$ follows from the condition $x \geq 2$.

Here are more examples:

- $x \geq 0$ as long as $x \geq 2$,
- $x \geq 0$ now that $x \geq 2$,
- $x \geq 0$ whenever $x \geq 2$,
- $x \geq 0$ once $x \geq 2$,
- $x \geq 0$ while $x \geq 2$,
- $x \geq 0$ as $x \geq 2$,
- $x \geq 0$ rather than $x \geq 2$,
- $x \geq 0$ after $x \geq 2$.
-

Bertrand Russell is the Pope

Remarks. stronger = sufficient.
weaker = necessary. stronger $\Rightarrow$ weaker.

Here is another example to show that real life logic is more complicated than mathematical logic.
We die because we breath.
Should we stop berating if we want to live?
Notice that it is it is true that we breath and that we die
But **causation** is not the same as implication.

If and only if means two implications, $\Longleftrightarrow$.
We can replace it by "i.e..", "that is", "is equivalent".

Exercises to §4.

Exercise 1. True or false:
(A) $x \ge 0$ if $x = 2$,
(B) $x \ge 0$ if $x > 2$,
(C) $x > 0$ if $|x| > 2$,
(D) $|x| + y^2 > 0$ for all x and y.

Exercise 2. Find all true implications between the following 4 statements.
(A) $x = 0$,
(B) $x \ge 0$,
(C) $xy = 0$,
(D) $x = y = 0$.
Hint: it may help if you draw the feasible sets for A, B, C, and D in (x,y)-plane.
An implication is an inclusion of sets.
"And" corresponds to intersection. "Or" corresponds to union.

Exercise 3. True or false:
(A) the equation $x + y = 3$ is redundant when $x = 1$ and $y = 2$.
(B) the condition $x > 1$ is stronger than the condition $x = 0$.
(C) $x = 1$ only if $x \ge 0$,
(D) the condition $x \ge 0$ is sufficient for the conclusion $x = 1$.
(E) If $x = 1$ and $x = 2$, then $x = 3$.
(F) If $x = 1$ or $x = 2$, then $x = 3$.
(G) If $x = 0$ and $y = 0$, then $x = 0$ or $y = 0$.

§5. Matrices

A matrix is a rectangle array. The entries are usually (known or unknown) numbers.

Matrix addition and subtraction is defined for matrices of the same size.

The matrix productAB is defined for matrices A and B such that the number of columns of A equals to the number of rows of B.
If the sizes of A and B are m by n and n by k, then AB has size m by k and its entries are the matrix products of the m rows of A with the k columns of B.

The matrix product is not commutative, but to as associative: A(BC) = (AB)C where A,B, and C have appropriate sizes).

Matrix addition and multiplication are related by the distribution laws:
A(B+C) = AB + AC, (B+C)A = BA + CA.

Matrices can be multiplied by scalars (numbers}.

— — — — — — — — — — W— F

Multiplication by elementary matrices on left and right results in row and column addition operations.

Multiplication by diagonal matrices on left and right results in row and column multiplication operations.

Multiplication by permutation matrices on left and right results in row and column permutation operations.

Exercises to §5

Let $A = [1,2,0]$, $B = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$

Exercise 1. Compute $A + B^T$ and $3A - 2B^T$.

Exercise 2. Compute AB and BA.

Exercise 3. Compute $(BA)^9$.

§6. Linear equations

We use the row addition and (invertible) multiplication operations, and column multiplications with the augmented matrix.
This does not change the solutions.
We drop zero rows
in the augmented matrix (they correspond to the redundant equation 0 = 0).
We either create the equation 0 = 1 (in which case the system is inconsistent, i.r., has no solutions) or obtain the identity matrix on the left (in which case we are done too).

Examples.

6.1. Solve $3x + 1 = 2 - x$ for x. Answer: $x = 1/4$.

6.2. Solve $2x + 3y = 1$ for x, y.
Answer: $x = 1/2 - 3x/2$ (y arbitrary) or $y = 1/3 - 2x/3$ (x is arbitrary).

6.3 Solve the system $x - y = 1$, $2x - 2y = 3$ for x, y.
Answer : 0 = 1 (there are no solutions.

6.4.Solve the system $2x + 3y = 1$, $4x + 6y = 2$ for x, y.
Answer : the same as in 6.2.

6.5. Solve for x, y:

$$\begin{cases} 3x + 2y = 5 \\ 8x + 5y = 13 \end{cases}$$

Solution. The augmented matrix is
3 2 5
8 5 13

We multiply the first row by 1/3 and obtain

1 2/3 5/3
8 5 13

We add the first row multiplied by -8 to the second row which gives

1 2/3 5/3
0 -1/3 -1/3

We multiply the second row by -3 and obtain
1 2/3 5/3
0 1 1

We add the second row multiplied by -2/3 to the first row which gives
1 0 1
0 1 1

So by 2 row addition operations and two row multiplication operations we obtain a terminal matrix hence
x = 1 and y = 1.

Notice that the numbers in the data and answer are in integers. But the augmented matrices on the way from the initial matrix to the terminal matrix contain
fractions and even a negative number. Examples like this can be found in circa 2K old Chinese texts.
By comparison, negative numbers appeared in European texts only in the 17th century, see wiki. Fractions were known in Egypt circa 3K ago, see wiki.

Given any linear system, we can write it in the matrix form Ax = b, where [A,b] is the augmented matrix of size m by n+1 and x is the column of distinct (names of) unknowns. Solving it results in one of the following outcomes:

0 = 0 (every x is a solution; this happens if and only if [A,b] = 0, the zero matrix);

0 = 1 (there are no solutions);

x = d (exactly one solution);

y = Cz + d where the column

y
z

consists of all n unknowns (i.e., it is the column x permuted), y contains at least one unknown and z contains at least one unknown.

The corresponding terminal augmented matrices are:

the 1 by n +1 zero matrix,
the 1 by n + 1 matrix [0, 1],
the m by n + 1 matrix $[1_m, d]$,
the k by n + 1 matrix

$$\begin{array}{ccc} y^T & z^T & \\ [1_k, & -C, & d] \end{array}$$

where $1 \le k \le n - 1$.

We obtain a terminal matrix starting from [A,b] by the following operations:

row addition operations,
row multiplication operations (with nonzero coefficients)
column permutations (columns permuted together with the names of unknowns on the top margin),
dropping redundant rows (a row is redundant if is either the zero row or any row in presence of the row [0, 1]).

Remark 6.6. Elementary operations can be used to find all integer solutions for systems of linear equations in with integer coefficients.
Sometimes column addition operations are needed. To keep track of column operations, we agent the augmented matrix [A, b0 by the identity matrix:

$$[A, b] \to \begin{array}{cc} A & b \\ 1_n & \end{array}$$

Here is a small example: Solve 3x + 5y = 1 in integers x, y.
Solution. The augmented matrix [A,b] = [3, 5, 1]. We do 3 column addition operations: with integer
coefficients (Euclidean algorithm on the pair (3, 5)):

```
   ↗-1↘                              ↙-1↖                   ↗-2↘
3      5    1                    3     2    1             1    2    1
1   0   1
1      0              →          1    -1          →       2   -1          →
2   -5
0      1                         0     1                 -1    1
-1   3
```

The terminal matrix [1, 0, 1] gives the answer in terms pot new unknowns u, v:

u = 1 (v arbitrary).
The changed 1_n recorded the connection between (x,y) and (u,v):

$$\begin{matrix} x \\ y \end{matrix} = \begin{matrix} 2 & -5 \\ -1 & 3 \end{matrix} \quad \begin{matrix} u \\ v \end{matrix}$$

So the final answer is
x = 2 - 5v, y = -1 + 3v.
Here v is an arbitrary integer. The answer gives all integer solutions of the equation.

Remark 6.7. To assign a rational number to the diagonal of the unit square, a Pythagorean (possibly Hippasus of Metapontum),
used what we call now Euclidean algorithm: given to positive numbers (or intervals) we subtract the smaller one from the bigger one until w they are equal.
We start with (1, a) where a is the diagonal, 1 < a < 1.5, and do column addition operations:
(1, a) -> (1, a -1) -> (2 - a, a - 1) -> (3 - 2a, a - 1).
After we subtracted a -1 twice from 1, we got 0 < 3 - 2a < a - 1.
Now (a - 1(/1 = (3- 2a)/(a-) using geometry or the fact that $a^2 = 2$.
Therefore we subtract the smaller number from the larger number twice forever.
This the algorithm neve
r terminates, which proves that a is not a rational number.
In modern notation,

$$a = \sqrt{2}.$$

The terminal augmented matrices for the systems of linear equations with two variables:
[0, 0, 0] (everything is a solution), [0,0, 1] (there are no solutions), [1, *] (there is one solution).

The terminal augmented matrices for the systems of linear equations with one variable x:
[0, 0] (every x is a solution), [0, 1] (there are no solutions),
[1, *, *] m and
1 0 *
0 1 *
(there is one solution)..

Exercises to §6

1. Solve the following system for x,y:

$$\begin{cases} x + 2y = 3 \\ 4x + 5y = 6 \end{cases}$$

2. Solve the following system for x,y:

$$\begin{cases} x + 2y = 3 \\ 2x + ay = b \end{cases}$$

where a and b are given numbers.

Hint: division by 0 is not allowed.

Ch3. Definition of game

§7. Graphs (networks). Extensive form. Strategy

7.1. Definition. A graph (or network} is a set (of vertices , nodes, or positions) and a set of pairs of vertices (arrows, links, moves).
For an arrow (a,b .) Wwe say that (a, b) goes from a to b or connect a with b. We also say that (a,b) is a link from the source a to the target b.

In a more general definition, which we do not need, arrows (links) can be repeated. Here are some conditions which sometimes imposed.
Graph is finite, i.e., the set of vertices is finite (then the set of arrows is finite too).
Graph is undirected, i.e., when (a,b) is an arrow then (b,a) is an arrow.
Graph has no arrows (a,a).
Graphs is connected.
Graph has no loops.
Graph is a tree, i.e., it is connected and has no loops. In this definition we are allowed to go against the arrow direction.
(Examples of trees.)

7.2. Definition. A terminal vertex (node, position) is a vertex without arrows out.

7.3. Definition. A game in extensive form consists of:
a set (of players);
a graph (vertices are called positions and the arrows are called moves);
a number (payoff) for every player at each terminal position;
every non-terminal position is marked by either a player or by "nature" (or "chance");
at every chance position, on all moves out, non-negative numbers with sum 1 are written;
a position selected as the initial position.

In most of textbooks and in **Extensive-form game - Wikipedia**, the graph in extensive form is assume to be a finite tree.
Then every position remembers its history.
This assumption makes some theoretical results easer to obtain, but it makes practical solution of games (such as
blackjack, Tc-tac-toe, Nim) more difficult by repeating positions.

Our definition is for so-called games with perfect information. A more general definition involves information sets which allows us to include games like Heads&Tails and Scissors-Rock-Paper. However the extensive form for such games is not useful.

Moreover it is possible to generalize the definition of game in a way to include practically every situation in every science and life. But then the game theory loses its usefulness.

7.4. Some examples of games above can be easily written in extensive form.
Game of Life is a 0-player game if we drop the condition that the graph is finite or consider only initial positions such that the life disappears after a few steps.
However without this condition we have trouble in defining the payoffs
in games like Double or Nothing.
|| youtube 1 | youtube 2 ||

7.5. A strategy for a player P consists of P choosing a move in every position which belongs to P.
A *strategy profile* or j*oint strategy* (or just a *strategy*) consists of a strategy for each player.

Here is a tricky question which only a mathematician can understand. Suppose that there are no positions which belong to a player P. Haw many strategies P has?
Answer: 1 (do nothing).

7.6. Definition. We call an extensive form *finite* if the set of positions is finite and there are no directed loops of moves.

E.g., when the positions form a tree as in the Moriss textbook, the extensive form is finite.
If we have a directed loop and the initial position is on the loop, the game may continue forever.

7.7. **Payoff for a joint strategy.**
In ay extensive form, the payoff is given for every terminal position. Is it always possible to assign a payoff for every joint strategy (strategy profile)?
(We assume that there s at least one player.)
Here is a counterexample with an infinite graph. Consider an extensive form with one player, one initial chance position, and the terminal positions numbered by integer $n \geq 1$.
Let probability of going from the initial position to the terminal position n be $1/2^n$. Let the payoff t n be 2^n.
Then the expected payoff is an infinite sum of ones which is not a number.

In general, given a joint strategy, we have the probability Pr(x) of ending up at any terminal position x.
The expected payoff is the sum the sun of Pr(x)Pay(x) over all terminal positions x where Pay(x) is the joint payoff at x.
So the question is when this sum makes sense.
Each of the following two conditions on the extensive form guarantee the existence of payoffs for every joint strategies:

If the set of terminal positions is finite or, more generally, the payoff for each player is bounded then we have a well-defined payoff for each strategy.

If there are no chance positions or, more generally, the probability of every chance move is 0 or 1,
then , then fr any joint strategy, Pr(x) is nonzero for at most one x, so we have a well-defined payoff.

Note that in absence of chance positions we have an unique path starting at the initial position for any joint strategy. This path terminators at a terminal position or game never terminates. The payoff in the second case is 0 for each player.

7.8. Example. Here is an extensive form with:
2 players, A and B,
the initial position belonging A with 2 moves A1 and A2 out,
a chance position marked by N with 3 random moves out,
and 5 terminal positions.

```
           A
     A1  /      \ A2
        A          B
  A3  /  \ A4   / B1   \ B2
   1,2      3,4          N
                 0.3/  0.2l   \0.5
                 0,1     -1,1    2,-2
```

The player A has 4 strategies: (A1, A3), (A1, A4), (A2, A3), and (A2, A4),
(We may say that the last two strategies are nor really so different.)
The player B has 2 strategies: B1 and B2.
There are 8 = 4× 2 strategy profiles (joint strategies).
If A chooses (A1, A3), and B chooses anything, then the payoff is (1,2).
For the strategy profile (joint strategy) (A2, B2), the (expected) payoff is
0.3(0, 1) + 0.2(-1, 1) + 0.5(2, -2) = (0.8, -.0.5).

Which strategy profile we call a solution?
A good candidate is ((A1, A4) , B2) where each player gets the maximal possible payoff.
Some would say this is the solution of game assuming that A is rational.
But what if A demands the side payment 1 threading to choose (A1, A3)?
Should B believe that A is capable to do it and pay the damson (bribe}?
The answer depends on whether players are allowed to communicate and whether side payments are allowed.
On the other hand, if A chooses A2, then player B has an opportunity to blackmail A.

We will find Nash and Shapley solutions in Examples and 20.9 below.

Example 7.9. Jn the extensive form
(1,3) <- 0.5- N initial position -0.5-> (5,7)
we have 2 players (because the payoff at the terminal position consists of 2 numbers) and two moves, both radon. The initial position is a chance position.
Ech player has no positions, so he/she/it has only one strategy, do nothing.
The expected payoff is 0.5(1,3) + 0.5(5,7) = (3,5).
The normal form is given by the 1 by 1 bimatrix
[(3,5)].
There is no controversy about solving this game.

Exercises to §7 ↘

Exercise 1. Solve the 1-player game in extensive form

A
↙ ↘
N N
0.5 ↙ ↘ 0.5 ↙ 0.6 ↘ 0.4
1 2 N
0.3/↙ 0.2l ↘\0.5
1 3 2

Exercise 2. Solve restricted Nim;: 2,3, or 5 stones in a move. Initial position: 1000 stones in a pile.

Exercise 3. In Blackjack, your bet is $1, you stand at 18,   the dealer got hard 16 in two cards. and you know nothing about remaining cards (many cards in the shoe). What is your expected payoff?

§8. Normal (strategic) form. Equilibrium

8.1. Normal (strategic) form consists of
a nonempty set (of players);
a nonempty set (of strategies) for each player;
a number (payoff) for each player for each joint strategy.

8.2. Going from finite extensive form to normal form. See 7.7 above.

Examples, Heads&Tails, Rock-Scissors-Paper, Prisoner's Dilemma, and Battle of Sexes above are 2-player games in normal form.
More generally, normal form with 2 player and finitely many strategies for each is the same as bimatrix (or bi-matrix) game, given by two matrices (payoffs for the first and the second players) or a matrix where every entry is a pair of numbers.

The zero-sum bimatrix games are known as matrix games. We will study them in detail later.
When the payoff matrix has only one row or only one column, it can be considered as an one-player game.

8.3. Definition. An *equilibrium* is a strategy profile such that no player can improve his (her, its, …) payoff by a unilateral change.

For any 1-player game an equilibrium is the same as an optimal strategy. So solving a game means to find an equilibrium and the value of game (if they exist).

If there is only one strategy profile (e.g., no position belongs to any player or, more generally, every position which belongs to a player has
only one move out), then this profile is an equilibrium.

For any 2-player 0-sum game, an equilibrium(if it exists) together with the value of game is the answer to "solve the game."
This is also true for all 2-player constant-sum games

For other games (not 1-player game or 2-player constant-sum game) , an equilibrium is not an answer which makes everybody happy.
Cooperation, side payments, coalitions, and threats may result in different solutions.
Some authors call players looking for equilibria "rational."
Then our parents were irrational. If you against any cooperations, you think that everybody is agains you and your life is probably miserable.

Example. 8.4. Find all equilibria in the bimatrix game

Players R, C	C1	C2	C3	C4
R1	5, 3	0,2	0, -2	0, 3
R2	1, 0	0, 0	0, 0	0, 0
R3	2, 2	-1, 0	4, 6	1, 1

Solution. We mark by * maximal entries in every row and column:

Players R, C	C1	C2	C3	C4
R1	5*, 3*	0*,2	0, -2	0, 3*
R2	1, 0*	0*, 0*	0, 0*	0, 0*
R3	2, 2	-1, 0	4*, 6*	1*, 1

Positions marked twice are equilibria. There are 3 of them, namely, (R1, C1), (R2, C2), and (R3, C3).
Does any of them solve the game?

8.4. A philosophical issue. What is infinity and does in exists in nature?
This is a deep philosophical question. See **Infinity - Wikipedia/**
The space-time is infinite in most of current physical theories. In mathematics, we have infinitely many natural numbers.
There many infinite sets and procedures in calculus.
On the other hand, every digital computer has a finite memory, so practical numerical computations are done in finite space-time.
The number of all elementary particles in the universe is estimated to be finite. See **Eddington number**

Example 8.5. Here is a 2 -player extensive form with 5 positions:

```
             N initial
            ↙            ↘
     0.5  ↙                ↘  0.5
        ↙                    ↘
   Player1       left       Player 2
                ⇆⇆⇆
                right          ↓
       ↓                 ↓
    (-1,3)                (-3, 1)
```

Here is the normal form (bimatrix game):

	left	down
right	0, 0	-3, 1
down	-1, 3	-2, 2

There is no equilibria in pure strategies. The number of moves is not bounded, so the extensive form is not finite (cf., Definition 7.6).
because there is a directed loop. The payoff is (0,0) when we do not termnate.
Compare with 9.1 below.

Example 8.6. Solve th extensive for with one player Pete:
Pete initial position -> \$1.
Solution.
The extensive form has only two position
Pete has only one move. It is optimal
The value of game is \$1.

Example 8.7. Solve th extensive for with one player Pete:
Pete i \$1.
Solution. There is only one position. It is both initial and terminal.
Pete has no moves. His only strategy is to do nothing.
The value of game is \$1.

Example 8.8. Solve th extensive for with one player Pete:
initial position N —1—> \$1.

Solution. There are two positions: a chance position and terminal position.
The only move is random, going from the initial position to the terminal position with probability 1.
Pete has no positions and no moves.
He has only one strategy: do nothing.
The value of game is \$1.

Example 8.9. Extensive form with 3 players:

1,2,3 ←0.2– N initial position –0.2→ 4,5,6
0.1 ↙ ↓ 0.1 ↘ 0.4
0,0,0 0,0,1 -1,-2,0

We have one chance position marked by N and 5 terminal positions/
The expected payoff is
0.2*(1, 2, 3) + 0.1*(0, 0, 0) + 0.1*(0, 0, 1) + 0.4*(-1, -2, 0) + 0.2*(4, 5, 6) = (0.6, 0.6, 1.7).
Whatever payoff is defined be the answer in terms of payoffs, the answer is
(0.6, 0.6, 1.7).

Exercises to §8

Exercise 1. Find all equilibria in the game with 2 players, A and B:

```
          A
      ↙   ↓   ↘
0,-1  0,0    0,1
```

Exercise 2. Restricted Nim. The number of stones taken should be 1, 4, or 5. The initial position is one pile of 1000 stones. Solve the game.

§9. Equilibrium. Its existence for finite extensive form with perfect information

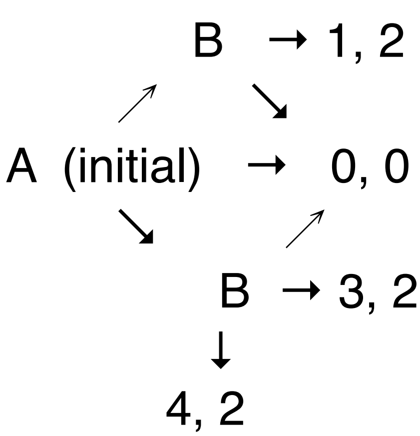
Every game in finite extensive form (see Definition 7.6) has an equilibrium. v

9.1. Finding the equilibrium in finite extensive forms. Dynamical programming (backward induction) finds equilibria for all initial positions.
We start with the terminal positions where the answer is given. Then we work with positions one move away from the terminal positions.
If such a position belongs to Nature, we just compute the expected payoff. If it belongs to a player, the player goes for maximal payoff.
Then we process the positions two moves away from the terminal positions. And so on.

Every equilibrium we find is subgame perfect, i.e., if restricted to any initial position, it gives an equilibrium for the corresponding subgame.
Conversely, every subgame perfect equilibrium can be found by dynamical programming.
However finding more than one equilibrium can be useful only as exercise or showing that equilibrium
is not the answer (except for 1-player or 2-player 0-sum games).

Example. In Nim (or, more generally, in any win-lose deterministic finite game in extensive form), an equilibrium exists and it consists of a winning strategy for a player combined with an arbitrary strategy for the other player/

Example. Find all equilibria in the extensive form with 2 players, A and B



Solution. We mark 5 moves of B from top down as B1, B2, B3, B4, and B5. Only B1, B4, and B5 maximizes B's payoff. However B2 and B3 can be a part of an equilibria if they are irrelevant do not change the terminal position).
B has 6 strategies.
We mark 3 moves by A from top down as A1, A2, and A3.

B → 1, 2 B1
↗ ↘ A1 B2
A (initial) → 0, 0 A2
↘ ↗ A3 B3
B → 3, 2 B4
↓ B5
4, 2

Here I wrote at the terminal positions the pathways to reach them:

B → 1, 2 A1, B1 (B3,B4,B5 irrelevant),
↗ ↘
A (initial) → 0, 0 A1, B2 (B3,B4,B5 irrelevant), A2 (B1-5 irrelevant) or A3 B3 (B1,B2 irrelevant)
↘ ↗
B → 3, 2 A3, B4 (B1, B2 irrelevant)
↓
4, 2 A3, B5 (B1, B2 irrelevant)

Here is the normal form

	B1,B3	B1,B4	B1,B5	B2,B3	B2,B4	B2,B5
A1	1,2	1,2	1,2	0,0	0,0	0,0
A2	0,0	0,0	0,0	0,0	0,0	0,0
A3	0,0	3,2	4,2	0,0	3,2	4,2

Now we mark by * maximal entries:

	B1,B3	B1,B4	B1,B5	B2,B3	B2,B4	B2,B5
A1	1*,2*	1,2*	1,2*	0*,0	0,0	0,0
A2	0,0*	0,0*	0,0*	0,*0*	0,0*	0,0*
A3	0,0	3*,2*	4*,2*	0*,0	3*,2*	4*,2*

The equilibria are the positions in the table marked twice. There are 6 of them.
Point of this example are to practice:
going from extensive form to normal form
and
finding all equilibria in bimatrix game;
Another point is to see again that an equilibrium or all equilibria in a bimatrix game do not give an answer which make everybody happy unless it is a matrix game.
Also equilibria which are not subgame perfect look ugly.

Equilibria for 2-player 0-sum games.

For these games, the equilibrium (if exists) have special properties, so an equilibrium (and the corresponding payoff) solves the game.
Consider a 2-player 0-sum game. Call the players He and She. His payoff plus her payoff is always 0.

9.2. At all equilibria, his payoff is the same, Indeed, consider two equilibria (R1, C1) and (R2, C2) where R1and R2 are his strategies and C1 and C2 are her strategies. Consider his payoffs:

	C1	C2
R1	a	b
R2	c	d

Since (R1, C1) is an equilibrium, a ≥ c and a ≤ b.
Since (R2, C2) is an equilibrium, d ≥ b and d ≤ c.
So a ≥ c ≥ d ≥ b ≥ a, hence a = c = d = b and a =d, QED.

For bimatrix game, the payoffs at equilibria can be different see Battle of Sexes.

9.3. If (R1, C1) and (R2, C2) are equilibria, then (R1, C2) and (R2, C1) are equilibria then so are (R1, C2!) and (R2, C1).

Consider his payoffs:

	C1	C2
R1	a	b
R2	c	d

As in the proof of 9.2, we obtain a = b = c = d.
Since (R1, C1) is an equilibrium, he cannot increase a in R1, hence he cannot improve c = a in R2.
Since (R2, C2!) is an equilibrium, she cannot decrease his payoff d in C2, hence she cannot improve c = d in C1.
So (R2, C1) is an equilibrium. Similarly, (R1, C2) is an equilibrium.

9.4. If his strategy is a part of an equilibrium, it is optimal in the following sense: it maximizes his worst-case payoff.
Indeed, consider any equilibrium (R1, C1) and any his strategy R2.
Consider her best strategy C2 against R2. Consider the corresponding payoffs:

	C1	C2
R1	*a*	*b*
R2	*c*	*d*

Since (R1, C1) is an equilibrium, a is the worst-case payoff for R1 and $a \geq c$.
We have chosen C2 such that d is his worst-case payoff for R2., hence c $\geq d$.
So $a \geq c \geq d$, hence $a \geq d$, QED.

9.5. If her strategy is a part of an equilibrium, it is optimal in the following sense: it maximizes her worst-case payoff.
Switching players in a 2-player 0-sum game (with switching strategies and payoffs) give a 2-player 0-sum game

9.6**. If there is an equilibrium, then (C2, R2) is also an equilibrium whenever R2 is optimal for him and C2 is optimal for her.**
Indeed, let (R1, C1) is an equilibrium. Consider his payoffs:

	C1	**C2**
R1	**a**	**b**
R2	**c**	**d**

Since (R1, C1) is an equilibrium, a ≥ c and a ≤ b.
On the other hand, c. d ≥ the worst-case payoff for R2
while a is the worst cease payoff for R1. Since R2 is optimal,
the worst-case payoff for R2 ≥ a.
hence c. d ≥ a. So a = c ≤ b. d.
Similarly, since C2 is optimal,
-b, -d ≥ the worst-case payoff for C2 ≥ the worst-case payoff for C1 = -a.
hence b,d ≤ a and a = b ≥ c, d. Therefore a = b = c = d.
Thus, d = the worst-case (minima;) payoff for R2
and
-d = the worst-case (minimal) payoff for C2.
i.e., (R2,C2) is an equilibrium/

9.7. Let R1 be an optimal strategy and u the minimal payoff for R1. So u is the maximal worst-case payoff for him.
Let C1 is her optimal strategy and -v is her maximal payoff for R1, so she pays him at most v

when she uses C1 and -v is her worst case payoff.
Then $u \le v$.
Indeed, let C2 is her worst case response to his R1 and R2 is his worst-case response to her C12/ Consider his payoffs

	C1	C2
R1	a	u
R2	v	d.

Then $a \ge u$ and $a \le v$ hence $u \le v$.

9.8. Let R1,C1, u, v be as in 9.7. If u = v, then (R1, C1) is an equilibrium.
Consider his payoff x for R1 against C1. As in 9.5, $u \le x \le v$. Since u = v, we have u = x = v.
So x is the minimal payoff for R1 and -x is the minimal payoff for C1, i.e., (R1, C1) is an equilibrium.

9.9. Let R1,C1, u, v be as in 9.7. If $u \ne v$, then there is no equilibrium.
It follows from 9.6.

9.10. Thus, for 2-player 0-sum games, the equilibrium exists if and only if
his maxima; the worst case payoff = - her maximal worst-case payoff.
J. von Neumann proved the existence of an equilibrium in mixed strategies for every matrix game (the minimax theorem, see below).
The example of Heads&Tails shows that matrix games need not to have equilibria in pure strategies.
Such games cannot be written in extensive forms (with perfect information).

9.11. Any mixture of his optimal strategies is optimal. Any mixture of her optimal strategies is optimal.

9.12. In the bimatrix game

1,5	2, 5	3, 5
1,9	2, 9	3,9

each of 6 joint strategies is an equilibrium.
So which of them is the non-cooperative "solution"?

Remark 9.13. For any game in extensive form with finitely many positions without chance positions there is an equilibrium. The backward induction does not work in presence of directed loops but we can convert the finite graph to a finite tree by repeating positions if necessary. If a play never ends, the payoff is 0 for every player.
A position in a tree is a directed path starting at the initial position which may return to a previous position only at the last move.
So a position in the tree remembers its history.
Moves in the tree are defined in an obvious way.
A termonal position in the tree is a path which either terminates in the graph (thne the payoff for the path is the payoff at the terminal position) or terminates at a repeated position (then the payoff for the path is 0 for every player).

Here are two examples with 2 players, A and B.

```
     A                                    A
   ↗   ↘                                ↗   ↘
A    ←    B → 1,1                    A    ←    B   →   1,-1
```

Here are the equilibria with the moves in the equilibria underlined:

```
      A  1, 1                 A 0, 0
   ↗        ↘              ↗       ↘
A  1, 1 ←   B 1, 1  →  1,1     A 0, 0  ←  B 0, 0 → 1,-1
```

In the first example, it is not a 0-sum game, so there is room for cooperation. the player , B may request a sidekick from A (who has no choice) fogging to (1, 1) rather then to (0,0) in the cycle.
In the second example, it is a matrix game so there is no much room for cooperation. But is A offers side payment b ≥ 1 to B, then B may agree to go to (1, -1) instead to (0,0).
For b = 1 we get another equilibrium with side payment.

Remark 9.14. Extensive forms based on trees are useful to prove the existence of equilibrium but not practical for solving games like Tic-Tac-Toe, Nimbuses, Restricted Non, Go, Chess.
For example, consider Nim with one pile with n ≥ 1 pebbles. It is obvious that the only winning move is to take all pebbles/
The extensive form used ub these notes for Nim or e Restricted Nim has n + 1 position. If we use a tree form, every position remembers its history and we have 2^n positions. The terminal position is repeated 2^{n-1} times.
Going from finite extensive form to finite normal form is possible but does not help to solve the game. For example, for Nim with n pebbles in 1 pile, the tree extensive for has 2^n positions and 2^{n-1} joint strategies. The corresponding norma; form is a matrix game with 2^{n-1} entries But the

matrix is much to big to be useful.
Our extensive forms could be much smaller but I do not go any example when going to normal form helps.
On the other hand, for extensive forms with imperfect information (not introduced in these notes) , going to normal form could be the only way to solve them in any sense.

Exercises to §9

Exercise 1. Find an equilibrium and the corresponding payoff for extensive form with 3 players, A, B, C:

```
                                   A

                         /              |            \

                     B                 N                C

                /        \      0.5  /  \  0.5    /    |    \

-1,0,0                      1,1,-1       3,3,3   0,0,0    0,0,5
```

Exercise 2. How many equilibrate are there for the extensive form with 3 players, A, B, C:

```
                    A
              /          I          \
         B              A                C
      /      \       /    \           /   I   \
  0,0,0      1,0,2           1,1,1      0,0,1   3,2,1
```

Exercise 3. In American Roulette, you start with $1 bet and use Double or Nothing until you either win or your bet exceed $100. Compute your expected payoff. You bet on red.

Remark. I met several students and even experts on game theory who believed that solving any game is about finding an equilibrium. They are right if it is a 1-player game or 2-player 0-sum game (the complete answer should include also the value of game). Otherwise, they are wrong. Any game which is not a 1-player game or 2-player constant-sum game shows this. See, for example, Prisoner's Dilemma (where the equilibrium is unique) or Battle of Sexes (where there are equilibria with different payoffs).

Often experts are wrong. Most of experts were wrong about the last US presidential elections. All experts were wrong giving stress as the reason for stomach ulcers. Many Nobel Prize are about somebody refuting the experts.
It is also possible that the experts giving out Nobel Prizes are sometimes wrong.

I gave the following 2 exercises as a part of a special final exam for my Math 486 student V. Lemin. He got A.
They are difficult even for experts on game theory.
Do not upload your solutions of bonus problems to Canvas but report them in class.

Ex

ercise 4. (bonus) Construct an extensive form with finite graph without equilibria.

Exercise 5 (bonus). Prove that any extensive form with finite graph without chance moves has an equilibrium.
When a strategy profile results in cycling rather than a terminal position, assign 0 payoff for each player.

An English issue. A *solution* of an equation is a set of values for unknowns satisfying the equations.
A *solution* for a problem is a way to solve the problem (to get an answer).
Later we will see feasible solutions, optimal solutions, and basic solutions.
Solutions in chemistry are something else.
On the other hand, *mixture* in math and chemistry are in harmony with each other.
E.g., typical vodka, whisky, gin, or brandy is a mixture, namely, 40% of ethanol + 60% of water.
(The flavors, colors, and poisons — less than 1%— not shown in contents.)
Another example is the composition of atmosphere at sea level: water H_2O about 4%;
dry air: = 79% of Nitrogen N_2 + 21% of Oxygen O_2 + 1% of Argon Ar (the other stuff is about 0.1%). Water is about 4%.
For Mars: 95% of CO_2 + 3% of N_2 + 2% of Ar
For Venus,: 96.5% of CO_2 + 3.5% of N_2.

Any 2-player constant-sum game can be reduced to 2-player 0-sum game by subtracting the half sum of payoffs for the payoffs of both players.
Therefore all nice properties of equilibria and optimal strategies can be extended to 2-player constant-sum games.

If you want to find an equilibrium in an extensive form, use dynamical programming rather than normal form.
However dynamical programming not always gives all equilibria.

Ch4. Matrix games

§10. Definition. Mixed strategies.

10.1. A matrix game is a 2-player 0-sum game in normal form with finitely many strategies.
In other words, it is given by an arbitrary matrix (payoff matrix for the first player), The first player choses a row and the second player chooses a column (row and column players).
The corresponding matrix entry is what the second player pays to the first player.

10.2. A mixed strategy for a player P s a probability distribution on P's original (pure) strategies, or a mixture of P's original (mixed) strategies.

10.3. For example in Heads&Tails,

He vs She	H	T
H	1	-1
T	-1	1

we see no equilibria. However if he uses (H+T)/2 and she uses (H+T)/2 , we get an equilibrium

He vs She	H	T	(H+T)/2
H	1	-1	0*
T	-1	1	0*
(H+T)/2	0^{Δ}	0^{Δ}	$0*^{\Delta}$

where * means that his payoff is maximal in its column and $^{\Delta}$ means that his payoff is minimal in its row (i.e., her payoff is maximal).

It is the only equilibrium in mixed strategies. In other words, if he uses any other mixed strategy his worst-case payoff will be negative and the same is true for her.
The value if game is 0.

10.4. If the payoff matrix has only one column, he (the row player) choses a maximal number. The number is the value of game. Its position is an equilibrium.

If the payoff matrix has only one row, she (the row player) choses a minimal number (to pay him). The number is the value of game. Its position is an equilibrium.

10.5. Graphical method. If the payoff matrix has only two rows or columns, the matrix game can be solved graphically.
There are many videos at web about this.
Here are examples from youtube:
Game Theory 2x3 graphical solution AQA Game Theory graphical method
Graphic Method of game theory by jolly coaching in hindi(GAME THEORY USING GRAPHIC METHOD)
Game theory graphical method
Finding Saddle Points

10.6. **Domination**. We say that a strategy S1 of a player P dominates a strategy S2 of P if S1 pays more or the same as S2 always.
For example. for a 2 by 5 matrix game

```
S1   1  1  0  3  0
S2   0  0  0  3 -1
```

S1 dominates S2, written as S2 ≺ S1.

For a matrix game

```
S1   S2
 0    1
 0    0
-2    0
```

we have S2 ≺ S1 (remember the matrix entries is what the second player pays to the first one.

When we look for an equilibrium (like for matrix games), domination can be used to reduce the size of problem.
For matrix games rows and columns which are dominated can be crossed out

For example, for 2 by 3 matrix game

```
      C1  C2  C3
 R1    1  -1   2
 R2   -1   1  -1
```

C3 ≺ C1. After crossing C3, we get a game we saw before.
So an equilibrium is (R1 + R2)/2, (C1+C2)/2) and the value is 0 for both games.

For any game, if we look for an equilibrium, strategies which are dominated can be eliminated.

But domination is not always there to help us. Also we not always interested in equilibria in which case it not OK to eliminate strategies which are dominated.

When we are looking for an equilibrium in extensive form by dynamical programming (backward induction), we discard some moves by domination.
We may lose some equilibria including all equilibria which are not subgame perfect.

For any 1-player game,
a strategy dominates any other strategy if and only if it is optimal,
a strategy is not optimal if and only if it is strictly dominated by another strategy.

For Heads&Tails, no mixed strategy is dominated by a different mixed strategy.

For any finite extensive form, any non optimal strategy is dominated by a different strategy. So elimination by domination allows us to find an equilibrium in pure strategies.

Philosophical issue. Domination is closely related with concept of "rational player." Some books give "rational interpretation" of equilibria and domination They claim that a rational player would necessarily have to pick an equilibrium as the solution of each game. But then they have to deal with "rationality paradox."
We do not define or use the concept of "rational player."
Cooperation is not irrational. We call player irrational if we do not like or understand them. This is not a mathematical definition.

Is this crocodile mom rational? **Crocodile Mom Scoops Up Babies in Mouth**

Are these penguin dads rational? **Baby Emperor Penguins Emerge from Their Shells | Nature on PBS**

Is rationality consistent with altruism?

10.7. **Symmetry**. If the payoff matrix M is skew-symmetric , i.e., $M^T = -M$, the game is called symmetric. If we switch the players and payoffs, the matrix stay the same.

The value of any symmetric matrix game is 0 and optimal strategies of the players are the same up to transposition.

So it suffices to find am optimal strategy for the first player.

Every m by n matrix game can be reduced to a symmetric mn by mn game (von Neumann) and even to a symmetric $m + n + 1$ by $m + n + 1$ game.

See **MR0039219 (12,513i)** Gale, D.; Kuhn, H. W.; Tucker, A. W. On symmetric games. *Contributions to the Theory of Games,* pp. 81–87. Annals of Mathematics Studies, no. 24. *rnceton University Press, Princeton, N. J.,* 1950. (Reviewer: J. Wolfowitz), 90.0X

Here is the reduction f or any m by n payoff matrix A. First we subtract the minimal entry min(A) of A from all entries of A and obtain a matrix B with all entries non-negative.

The equilibria (p, q) for A and B are the same, and the value $v' \ge 0$ for B is the value v for A minus min(A).

The $m+n+1$ by $m+n+1$ skew-symmetric matrix M is

$$\begin{matrix} 0 & -B^T & 1 \\ B & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$$

where the zeros stand for the zero matrices of sizes n by n, m by m and 1 by 1 and where

1 stands for rows and columns of m and n ones.

The p, q, and v' correspond to the optimal row $(q, p^T, v')/(2 + v')$ for the symmetric game.

Conversely, given an optimal row (x, y, w) for the symmetric game where x is an n-row and y is an m-row, we recover an equilibrium (p, q) and the value v' for B as follows.

Let c be the sum of entries in x, and c' be the sum of entries in y. Since $(x, y, w) M \ge 0$, we have $c \ge c'$.

If $w \ne 0$, then $c = c'$ and $(q, p^T, v') = (x/c, y/c, w/c)$.

If $w = 0$, then $v' = 0$, every nixed strategy p for the first player in B is optimal, and $q = x/c$.

Remark. If we start with any m by n matrix A, the entries of the von Neumann mn by mn matrix M are the differences of the entries of A/

From ancient times, coin flipping was used in sports and politic. Applied to an m by n matrix game, ut gives a symmetric mn by mn matrix game which the payoff matrix M/2.

10.8. Strict (strong) domination. We say that a strategy S1 of a player P strictly (strongly) dominates a strategy S2 of P if S1 pays more than S2 always.
Then S2 cannot be present in any equilibrium (even as a part of a mixed strategy).

A weak domination is a domination which is not strong.

Exercises to §10.

Exercise 1. In the matrix game with the payoff matrix M =

1	2	3	0	-1
3	-2	0	2	0
-1	1	0	0	0
1	2	-3	-1	1

compute his (row player) payoff for her (column player) strategies
S1 = [1,1,1,1,1]/5
and
S2 = [0,0,0,1,1]/2.
Which is better in the sense of the worst-case payoff?

Exercise 2. Solve the matrix game

1 3 -1 4 3

6 0 3 -2 0

0 0 -1 3 4.

Hint. Reduce size by domination and then use the graphical method.

Exercise 3 (bonus) Solve the matrix game

1 3 2
4 1 2
1 2 4

We worked on this problem in a class on Tue. It was easy to get approximate solutions. We also reduced the problem to linear programming.

§11. Optimal strategies. The minimax theorem.

To solve a matrix game is to find an equilibrium and the corresponding payoff for the row player.
A mixed strategy is optimal if it maximizes the worst-case payoff (the maximization is over all mixed strategies).
This happens if and only if it is a part of equilibrium.

J. von Neumann proved that every matrix game has an equilibrium in mixed strategies.
He reported this on December 7, 1926, to the Göttingen Mathematical Society.

John Nash proved that every finite normal form has an equilibrium in mixed strategies.

Reduction of solving matrix games to linear programming.

Exercises to §11.

Exercise 1. In the matrix gave with the payoff matrix

1	2	3	0	-1
3	2	0	0	0
-1	1	0	0	0
1	2	-3	-1	1

find all equilibria (saddle points) in pure strategies**.**

Exercise 2. Is S1 or S2 in Exercise 1 of §10 optimal?

§12. Examples.

Simplex method allows us to solve any matrix game (after reducing it to linear programming. We study linear programming and simplex method in the next 2 chapters.
It will be easier for students who took Math 484.

You have to know: what is a linear equation, what it means to solve it, and how to solve any system of linear equations
(This is a quote from Syllabus/Prerequisites.)
We reviewed this in class (see §6 above).

Here are some examples where matrix game can be solved without simplex method.

Example 12.1. Solve the matrix game

1 2 3 4 1 1
5 4 3 2 0 1
0 2 0 1 1 1

Solution. We mark maxima entries in columns by * and minimal entries in rows by $^{\Delta}$:

1^{Δ} 2 3^* 4^* $1^{*\Delta}$ $1^{*\Delta}$
5^* 4^* 3^* 2 0^{Δ} 1^*
0^{Δ} 2 0^{Δ} 1 1^* 1^*

There are two equilibria in pure strategies: row 1, columns 5 and 6.
The value of game is 1.
Any mixture of equilibria in any matrix game is an equilibrium/

Example 12.2. Solve the matrix game

2 0
0 2

Solution. Subtracting 1 from each entry, we get Heads&Tails, with the value 0.
For the original game, the value is 1. The equilibria are the same for both games.

Example 12.3. Solve the matrix game

3 2 0 0
1 0 2 3
0 2 0-1

Solution. We name strategies:

	C1	C2	C3	C4
R1	3	2	0	0
R2	1	0	2	3
R3	0	2	0	-1

R3 is dominated by R1 so we eliminate it:

	C1	C2	C3	C4
R1	3	2	0	0
R2	1	0	2	3

Now C1 is dominated by C2 and C4 is dominated by C3:

	C2	C3
R1	2	0
R2	0	2

We saw this game in Example 12.2. The equilibrium is ((R1 + R2)/2, (C2+C3)/2) and the value is 1 The same answer works for the original 3 by 4 matrix game.

Example 12.4. Solve the matrix game

2 6
5 1

Solution. This can be easily done by graphical method. We name strategies:

	C1	C2
R1	2	6
R2	5	1

We use slopes.
The equilibrium is ((R1 + R2)/2, (5C1 + 3C2)/8). and the value of game is 3.5.

Exercises to §12.

Exercise 1. Solve the matrix game

1 2 0 4 1 1 0 0
5 4 0 2 0 1 1 2
0 2 0 1 2 3 0 1

Exercise 2. Solve the matrix game
3 2 1
1 2 3

Exercise 3. Solve the matrix game
3 0
0 3
2 2

Ch 5, Linear. programming

§13. Definitions. Little tricks.

Terminology of optimization.
A number in our class is a real number unless said otherwise. Here are examples of numbers 0, 1, -1,1/2, -0.2. Concept of real numbers is complicated, but linear programming and most of game theory can be done in rational numbers.
Here is an example of a nonlinear mathematical program with rational data where the optimal solution is irrational:

minimize x^4 + x.

Calculus helps to solve this problem (and many other important problems) but we do not use it in this class.

In our class we use the following terms:

- optimization (maximization or minimization)
- feasible solution, feasible region
- feasible value,
- optimal (maximal or minimal) value, maximum or minimum
- optimal solution (optimizer)
- unbounded optimization problem
- infeasible optimization problem
- mathematical program

- linear form
- affine function
- linear equation
- linear constraint
- linear program

The definitions can be found on the first page of Linear Programming chapter of Handbook of Linear algebra.

Note that ∞, $-\infty$. and 1/0 are not numbers
and that
$x > 0$, $x \neq 0$ are not linear constraints.

Linear function means *linear form* in some publications and *affine function* in others.

In a linear program, we want satisfy all given linear constrain.

Example. $x^2 + |y - 1| \to$ min.
It is a mathematical program with two unknowns, x and y.
The feasible region: all pairs (x, y) of numbers. It is given by finitely many linear constraints.
The objective function: $x^2 + |y - 1|$. It is not an affine function (bonus homework).
The feasible values: nonnegative numbers.
The optimal value: min = 0.
The optimal solution: x = 0, y = 1.

Example. $x \to$ min, $x > 0$.
This mathematical program is bounded and feasible but there are no optimal solutions.

Example. $x \to$ max, $x \ge 0$.
This linear program is unbounded.

Example $x \to$ min, $0 = 1$.
This linear program is infeasible.

Historical issue. In the 1940s and 1950s, computers were humans and programming was not a computer science term.

Example of of optimization which are no mathematical programs. | soap bubbl

Little tricks. See the Linear Programming chapter of Handbook of Linear algebra.

Trick 1. The constraint $f \le b$ is equivalent to the constraint $-f \ge -b$.

Trick 2. The equation $f = b$ is equivalent to the system of two constraints:

$f = b \Leftrightarrow (f \le b, f \ge b)$

Trick 3. The optimization problems
$f \to$ min $-f \to$ max
with the same feasible region
have the same optimal solutions.
Optimal values differ by sign: min = -max.
Trick 4. The inequality $f \le b$ can be converted to equation $f + x_0 = b$ by new non=negativeunknown $x_0 = b - f \ge 0$.

Trick 5. Any unknown x can be written as the difference of two non-negative unknowns:

$x = x' - x''$ with $x', x'' \ge 0$.

Graphical Method. Linear programs with 1 or 2 unknowns can be solved graphically.
There are many videos at web.

What it means to solve an optimization problem? Usually, an optimal solution together with the optimal value is a complete answer. But what if there are no optimal solutions?
Then the answer should say explicitly whether the problem is unbounded or infeasible. We will see later that for a linear program we have exactly one of the following 3 outcomes: there is an optimal solution, the program is infeasible,
the program is unbounded.

Exercises to §13.
Exercise 1. Solve
$2x+ 3y \to \max$, $|x| + |y| \le 5$.
Exercise 2. Solve
$2x+ 3y \to \max$, $|x| + |y| \le 5.5$, x and y integers.
Exercise 3. Solve
$2x + 3y \to \min$, $x + y = 1$, $|x| \le 2$, $|y| \le 3$.

§14. Standard tableaux.

We use the standard tableaux of the Morris textbook.
Here is what a standard m by n row tableau looks like:

x	-1	
A	b	= -y
c	d	-> max

where
A b
c d

is an m by n matrix of given numbers, [c, d] is its last row,
b
d
is its last column,
x is a row of n -1 unknowns, y a column of m - 1 unknowns, all names of unknowns in x, y are distinct.
Such a tableau means the following linear program:

$Ax^T - b = -y$, $x \ge 0$, $y \ge 0$, $cx^T - d \to \max$.

Example 14.1. A 3 by 5 example is the tableau (3.15) on p.78 of the Morris textbook.

Every linear program can be written in standard tableau using 5 little tricks given above. Standard tableaux in different books and input forms for different linear programming software can be reduced to each other by the same tricks.

Example 14.2. Write in a standard row tableau:

$x1 + 2x2 - 3x3 \ge 4$,
$-5x1 + 6x2 - 7x4 \le 1$
$x1 + 2x3 + x4 = 2$
$x1, x2, x3, x4 \ge 0$,
$x2 - x3 + 2x4 \to \max$.

Solution. Using standard trick, we get the following answer:newer.

x1	x2	x3	x4	-1	
-1	-2	3	0	-4	= -x5
-5	6	0	-7	1	= -x6
1	0	2	1	2	= -x7
-1	0	-2	-1	-2	= -x8
0	1	-1	2	0	-> max

14.3. Pivoting a standard tableau. Here is the pivot rule:

$$\begin{array}{ccl} x_0 & & \\ \alpha^* & \beta & = -y_0 \\ \gamma & \delta & \end{array}$$

↓ pivot step$f$14

y_0		
$1/\alpha$	β/α	$= -x_0$
$-\gamma/\alpha$	$\delta - \beta\gamma/\alpha$	

where α (market by *) is a nonzero pivot entry,
β is any entry in the pivot row but not the pivot entry,
γ is any entry in the pivot column but not the pivot entry,
and δ is any entry not in the pivot row or column;
γ and δ and are in the same row; β and δ and are in the same column.

It is OK to use pivot steps to make a tableau standard, but if we want to keep our tableau standard, the pivot entry should not be in the last row or column.

14.4. Here is how a standard m by n column tableau looks like:

u	A	b
-1	c	d
	= v	->min

where A, b, c, d are as above,
u is a column of m-1 unknowns,
v is a row of n -1 unknowns, all names of unknowns in u, v are distinct.
Such a tableau means the following linear program:
$u^TA - c = v$, $u^Tb - d \to \min$, $u \geq 0$, $v \geq 0$.

Actually, we can write two linear programs sharing the same matrix:

	x	-1	
u	A	b	= - y
-1	c	d	->max
	=v	->min	

They are called *dual* to each other.

Here is our pivot step for both:

$$\begin{array}{cc|c|c|c}
 & x_0 & & \\
\hline
u_0 & \alpha^* & \beta & =-y_0 \\
\hline
 & \gamma & \delta & \\
\hline
 & =v_0 & &
\end{array}$$

↓ pivot step

$$\begin{array}{cc|c|c|c}
 & y_0 & & \\
\hline
v_0 & 1/\alpha & \beta/\alpha & =-x_0 \\
\hline
 & -\gamma/\alpha & \delta - \beta\gamma /\alpha & \\
\hline
 & =u_0 & &
\end{array}$$

Simplex method works with standard tableaux. We start with initial tableau and reach a terminal tableau in finitely many pivot steps.
A terminal tableau gives an answer to our linear program.

Here is how to switch the row and column problems:

$$\begin{array}{c|c|c|l}
 & x & -1 & \\
\hline
u & A & b & = - y \\
\hline
-1 & c & d & =f\text{->}\max \\
\hline
 & =v & =g\text{->}\min &
\end{array}$$

↓ ↑ transpose

$$\begin{array}{c|c|c|l}
 & u^T & -1 & \\
\hline
x^T & -A^T & -c^T & = - v^T \\
\hline
-1 & -b^T & -d & = -g\text{->}\max \\
\hline
 & =y^T & =-f\text{>}\min &
\end{array}$$

Examples.
Solve the linear programs given by standard tableaux.

Example 14.5

x1	x2	-1	
1	2	3	= -x3
0	-1	2	= -x4
0	1	-2	= -x5
-1	0	-2	-> max

The third row (x5-row) reads x2 +2 = -x5. Since all x's are ≥ 0 (the tableau is standard) this constraint is infeasible.

Answer: The linear program is infeasible.

More generally, we call a row in a standard tableau *bad* if it has the form

⊕	-1		
⊕	-	=	⊖

Here ⊕ stands for non-negative numbers, - stands for a negative number, and ⊖ stands for a nonpositive variables (but not for the objective variable which is not restricted in sign).

A bad row in any standard tableau makes the row problem infeasible .
The last row cannot be bad.

Example 14.6.

x1	x2	-1	
1	2	3	= -x3
0	-1	2	= -x4
0	1	2	= -x5
-1	0	-2	-> max

The object if function is f = -x1 + 2 ≤ 2. We can make f = 2 by setting x1 = x2 = 0, hence x3 = 3, x4 = 2, x5 = 2.
Answer: max = 2 at x1 = x2 = 0, x3 = 3, x4 = 2, x5 = 2.

More generally, we call a standard tableau

	x	-1	
u	A	b	= - y
-1	c	d	->max
	=v	->min	

optimal, if $b \ge 0$ and $c \le 0$. In other words, the matrix
A b
c d

of given numbers has the following signs:

⊕
⊖

For any optimal tableau
max = -d at x = 0, y = b (the *basic solution* for the row program is optimal)
and min = -d at u = 0, v = -c (the *basic solution* for the column program is optimal)
Note that the optimal values for the dual problems are the same.
The transpose of an optimal tableau is optimal.

```
Example 14.7. Solve the system   2x + 3y = 7, 3x+4y = 10
by 2 pivot steps.
Solution. We write the linear system in 2 by 2 column tableau

x  2   3
y  3   4
  =7  =10
```

We choose 2 as the first pivot entry and pivot:

```
x  2*   3                7    1/2   3/2
y  3    4     —>         y   -3/2  -1/2
  =7   =10                    =x    =10
```

where 4 ↦ 4 - 9/2 = -1/2. Now we chose -1/2 as the pivot entry and pivot:

```
7    1/2   3/2                 7   -4   3
y   -3/2  -1/2*   —>          10    3  -2
     =x    =10                     =x  =y
```

where -/2 ↠ 1/2 - (-9/4)//(-1.2) = 1/2 - 9/2 = -4. So x = -28 + 30 = 2 and y = 21 - 20 = 1.
Answer: x = 2, y = 1.

Note that we inverted the coefficient matrix
2 3
3 4
(its inverse is the matrix
-4 3
3 -2
of the last tableau) and that we can replace 7, 10 by any numbers a, b in the problem and the last tableau.

Also, like in ancient Chinese texts, all data and the final answer are in natural numbers but negative numbers and fractions appear between.

Exercises to §14.

Solve the linear programs given by standard tableaux.

Exercise 1.

```
x1   x2   -1
1    2     3    = -x3
0   -1     2    = -x4
-1   0    -2   -> max
```

Exercise 2.

```
x1   1   2     3
x2   0  -1     2
—1   -1  0    -2
    =x3  =x4    ->min
```

Exercise 3.

```
x1   x2   -1
1   2     3    = -x3
0  -1    -2    = -x4
-1 0     -2   -> max
```

§15. From matrix game to standard tableau.

Given a matrix game with players He and She and an m by n payoff matrix M,
we consider his mixed strategies as columns $p = [p1,\ldots, pm]^T$ and her mixed strategies as rows $[q1,\ldots, qn]$.
His optimization problem is
$\min(p^T M) \to \max$ subject to $p \geq 0$, $p1+\ldots+ pm = 1$
We convert it to a lineup program by introducing a new variable u:

$u \to \max$, $p^T M) \geq uJ_n$, $p \geq 0, p1+\ldots+ pm = 1$
where J_n is the row of n ones.

Her linear program is

$v \to \min$, $Mq^T \leq J'_m v$, $q \geq 0, q1 + \ldots \_ qn = 1$
where J'_m is the column of m ones.

Using the little tricks given above, we can write both problems in a standard tableau of size N + 3 by n + 3

$$
\begin{array}{cccccl}
 & q & v' & v'' & -1 & \\
p & M & -J'_m & J'_m & 0 & = -* \\
u'' & J_n & 0 & 0 & 1 & = -* \\
u' & -J_n & 0 & 0 & -1 & = -* \\
-1 & 0 & -1 & 1 & 0 & = - v \to \max \\
 & =* & =* & =* & =-u\to\min &
\end{array}
$$

or

$$
\begin{array}{cccccl}
 & p^T & u'' & u' & -1 & \\
q^T & -M^T & -J'_n & J'_n & 0 & = -* \\
v' & J_m & 0 & 0 & 1 & = -* \\
v'' & -J_m & 0 & 0 & -1 & = -* \\
-1 & 0 & -1 & 1 & 0 & = u \to \max \\
 & =* & =* & =* & =v\to\min &
\end{array}
$$

Now we see that his problem and her problem are dual to each other.

It is easy to see that both problems are feasible and bounded. So by the duality theorem (see the next chapter), they have the same optimal value hence we obtain the minimax theorem.

There is a trick to save two rows and two columns, see the Morris textbook.
It involves making the value v of game positive by adding a number to all entries of the payoff matrix (which do not change the optimal strategies) and then putting p/v and q/v into a standard tableau instead of p and q.
This makes sense if you have no computer (or a smart phone) around .

Exercises to §15.

Exercise 1. Write a standard tableau for Heads&Tails game.

Exercise 2. Write a standard tableau for the matrix game in Exercise 1 of **§12.**

Ch6. Simplex method

Scheme of simplex method.
All tableaux in simplex method are standard row tableaux.
In Phase 2, all tableaux are row feasible.

```
                                bad row  terminal
initial      Phase 1      ↗   (the row program is infeasible)
              ------->                                                 the basic solution is optimal
 tableau     pivot steps  ↘   a row feasible tableau     Phase 2           optimal tableau terminal
                                                                   ↗
                                terminal for Phase 1     ----------->
                                initial for Phase 2      pivot steps   ↘   feasible with a bad column
                                                                           terminal, program is inbounded
```

Comments on the simplex method.
We do not pivot terminal tableaux.
If a tableau is not terminal, we pivot.
If we have a bad row, the tableau is terminal and it is not rot feasible;
the converse is not true.
If a tableau is row feasible in Phase 1, we go to Phase 2 and never return to Phase 1.
If it not terminal in Phase 2, we pivot it until we get a terminal tableau.
See Example 14.5 for the definition of a bad row.
See Example 14.6 for definition of an optimal tableau.

Phase 1 in more detail

```
                  initial   tableau

                            ↓ pivot
                            ↓  steps
                            ↓
row feasible b ≥ 0   terminal             bad row
 Go to Phase 2   ←   for Phase 1   →       the row program
                     tableau               is infeasible
```

Phase 2 in more detail

```
initial row                    A b
 feasible tableau              c d          b ≥ 0

                              ↓ pivot
                              ↓  steps
                              ↓
optimal  c ≤ 0        terminal
                  ←    tableau      →    bad column
x  -1                                    The row
A   b   = -y ↙                           program
 max = -d                                is unbounded
at x = 0, y = b                          The column program
The basic solution for                   is unbounded
the column program is
also optimaland min = -d.
```

A bad column in Phase 2:

$$\begin{array}{llll} \underline{\ldots \oplus \ldots \quad -1} & & & \\ \ldots \ominus \ldots & \oplus & = & \ominus \\ \ldots + \ldots & & \to & \max \end{array}$$

16. Phase 2.

Given a standard tableau

	x	-1	
u	A	b	= - y
-1	c	d	=f->max
	=v	=g->min	

the basic solution for the row program is $x = 0, y = b$
and the basic solution for the column program is $u = 0, v = -c$.

The tableau is called *row feasible* if $b \geq 0$, i.e., the basic solution for the row program is feasible.
In this case, the current value $-d$ for f is feasible.
The tableau is called *column feasible i*f $c \leq 0$, i.e., the basic solution for the column program is feasible.

So the tableau is optimal if and only if it is both row and column feasible.
In this case, the basic solutions are optimal and $-d$ is the optimal value for both.

All tableaux in Phase 2 of simplex method are row feasible.
The strategy is to keep the tableau feasible and improve the current feasible value.
To keep the last entry in the pivot row positive, we want the pivot entry to be positive. To improve the current feasible value, we want the last entry in the pivot column to be positive.
We also have to check whether our tableau is terminal.

1. Is the tableau optimal, i.e., is $c \leq 0$?
If yes, we write the answer for the row program: max = $-d$ at $x = 0, y = b$..

2. Is there a bad column?
If yes, the row program is unbounded.

3. (choosing a pivot entry) We pick a positive entry c_j in the last row, but not the last entry.
(Such an entry exists. Otherwise the tableau is optimal and we terminated in 1.)
This entry is going to be the last entry in the pivot column.
We look for positive entries a_i above as potential pivot entries.
(At least one exists. Otherwise we have a bad column and we terminated in 2.)
Now we compute b_i/a_i for those a_i where b_i is the last entry in the pivot row.
We chose the minimal (closest to 0) ratio. This is our pivot entry a_i

4. Pivot and go to 1.

Note that $b_i \geq 0$ in the pivot row goes to $b_i/a_i \geq 0$.
Every other entry b_j in the last column (but not the last entry $-d$) goes to $b_j - a_j b_i/a_i$.
If $a_j \geq 0$, then $b_j/a_j - b_i/a_i \geq 0$ hence $b_j - a_j b_i/a_i \geq 0$.
If $a_j \leq 0$, then $b_j - a_j b_i/a_i \geq b_j \geq 0$.
So the tableau stays row feasible.

The last entry d goes to $d - c_i b_i/a_i \le d$ so the current feasible value -d either goes up (when $b_i \neq 0$) or stays the same (when $b_i = 0$).
In the second, when the last entry in the pivot row is 0, case not only the current feasible value stays the same, but the whole feasible solution stays the same, Such a pivot step is called *degenerate*.
However we switch a variable on top with a variable at right margin.

For a tableau of size m+1 by n +1, there are (m+n)! ways to place our m + n variables at the top and right margins. Not all of them can be always obtained from the initial tableau in Phase 2.
Once the positions for variables are fixed, the tableau is unique.
This is because all tableaux describe the same solutions for a system of linear equations.
So we cannot return to a table we had before unless all steps are degenerate.

So we terminate in finitely many steps (< (m+n)! steps) unless we have a cycle of degenerate pivot steps.
Bland gave a simple rule to prevent cycling.

Make a list of variables, i.e., sort them. Then whenever we have a choice, prefer the variable which happens first on the list.
The fact the rule works can be found in several textbooks on linear programming including my textbook.

This rule is rarely used in software implementations because cycling happens rarely. Also examples were constructed when this rule result in a very large number of pivot steps.
It is an open problem whether there is a modification of simplex method with the number of pivot steps bounded by a polynomial in m+n.

There are publications where an average number of pivot steps is bounded by an affine function of m + n. This number should be expected for real life linear programs.
In real life, linear programs as well as systems of linear equations) are solved approximately.
The most serious problem for simplex method is not cycling. It is error accumulation. Usual methods should be used to resolve this problem.

Example 16.1. Solve the linear program given by the standard row tableau

```
x1 x2 x3 x4 x5 -1
 0  1 -2  3  4  1 =-x6
 1  1 2  -3  4  0 =-x7
-3  1 0   3 -4  2 =-x8
 0 -1 -2  0 -1 -1 ->max
```

Solution. The tableau is optimal, so
max = 1 at x1=x2=x3=x4=x5 = 0, x6 = 1, x7 = 0, x8 = 2.
There are other optimal solutions.

An optimal solution for the dual problem is
y6 = y7 = y8 = 0, y1 = 0, y2 = 1, y3 = 2, y4 = 0, y5 = 1
where yi is dual to xi for i = 1,…,8.
The optimal value is the same.

Example 16.2. Solve the linear program given by the standard row tableau

```
x1 x2 x3 x4 x5 -1
 0  1 -2  3  4  1 =-x6
 1  1 2   3  4 -1 =-x7
-3  1 0   3 -4  2 =-x8
 0 -1 -2  0 -1 -1 ->max
```

Solution. The tableau is not row feasible so we cannot go to Phase 2.
The x7-row is bad so the row program is infeasible.

Example 16.3. Solve the linear program given by the standard row tableau

```
x1 x2 x3 x4 x5 -1
 0  1  2  3  4  1 =-x6
 1  1 —2  3  4  1 =-x7
-3  1  0  3 -4  0 =-x8
 0  1  2  0 -1 -1 ->max
```

Solution. The tableau is row feasible.
It is not optimal and there are no bad columns so the tableau is not terminal for Phase 2.
There are two choices for the pivot column, namely, x2-column and x3-column.
There are two choices for the pivot entry marked by *:

```
x1 x2 x3 x4 x5 -1
 0  1  2* 3  4  1 =-x6
 1  1 —2  3  4  1 =-x7
-3 1*  0  3 -4  0 =-x8
 0  1  2  0 -1 -1 ->max
```

The choice in x8-row is degenerate. We pivot on 2 in x6-row:

```
x1 x2 x3 x4 x5 -1
 0  1  2* 3  4  1 =-x6
 1  1 —2  3  4  1 =-x7
-3  1  0  3 -4  0 =-x8
 0  1  2  0 -1 -1 ->max
```

↓ pivot step

```
x1 x2 x6  x4 x5  -1
 01/2 1/2 3/2 2 1/2 =-x3
 1    —2          2 =-x7
-3  1  0   3 -4   0 =-x8
 0  0 -1  -3 -5  -2 ->max
```

We left 3 entries uncomputed.
Now the tableau is optimal.
Answer: max = 2 at x1= x2= x6 = x4 = x5 = 0, x3 = 1/2, x7 = 2, x8 = 0.

Example 16.4. Solve the linear program given by the standard row tableau
x1 x2 x3 x4 x5 -1
 0 1 -2 3 4 1 =-x6
 1 1 -2 3 4 1 =-x7
-3 1 0 3 -4 2 =-x8
 0 1 2 0 -1 -1 ->max

Solution. The tableau is row feasible, so we proceed with Phase 2.
The x3-column is bad so the row problem is unbounded and the column problem is infeasible.
To see that the row problem is unbounded, set x1 = x2 = x4 = x5 = 0. Then
x6 = 1 + 2x3 ≥ 0, x7 = 1 + 2x3 ≥ 0, and x8 = 2 ≥ 0 when x3 ≥ 0.
We increase x3 to see that the objective function 1 + 2x3 takes arbitrary large feasible values/
Answer. The program is unbounded.

Example 16.5. In the following standard tableau, mark by * the choices for the pivot entry consistent with Phase 2:

```
x1 x2 x3  x4  x5 -1
 0  2* -2  3  4   2 =-x6
 1* 1* -2  3* 4*  1 =-x7
-3 -1   0  3 -4   2 =-x8
 1  1  -2  1  1  -1 ->max
```

Solution.

```
x1 x2 x3  x4  x5 -1
 0  2* -2  3  4   2 =-x6
 1* 1* -2  3* 4*  1 =-x7
-3 -1   0  3 -4   2 =-x8
 1  1  -2  1  1  -1 ->max
```

There are 5 choices.

Example 16.6. Solve the linear program given by the standard row tableau

```
-1
 2   = -x1
 3   = -x2
-1   = -x3
      -> max
```

Solution. It is a 3 by 1 standard tableau. The last entry in the matrix is absent, but it does not matter in this example.
The x3-row is bad, so the row program is infeasible.
The column program is unbounded.
The tableau is not row feasible, so we cannot go to Phase 2.
It is column feasible. The transposed tableau is row feasible with a bad column.

In general, the simplex merged applied to the transposed tableau is known as the *dual simplex method*.

Example 16.7. Solve the linear program given by the standard row tableau

```
x1    x2    x3    -1
1      2    -1     1 = -  x4
1      2    -2     3  -> max
```

Solution. The tableau is feasible but not terminal. There are two choices for a pivot column and two choices for a pivot entry.
We pick the x1-column.

```
x1   x2    x3   -1
1*    2    -1    1 =- x4
1     2    -2    3  -> max
```

↓ pivot step

```
x4   x2    x3    -1
 1    2    -1     1   =- x1
-1    0    -1     2   -> max
```

The current feasible value improved from -3 to -2.
Now the tableau is optimal.
max = -2 at x4 = x2 = x3 = 0, x1 = 1.

Example 16.8. Solve the linear program given by the standard row tableau

```
x1    x2    x3   -1
1     2     -1    0   = -x4
1     2     -2    3   -> max
```

Solution. The tableau is not terminal. There are two choices for a pivot column and two choices for a pivot entry.
We pivot on one of them:

```
x1  x2   x3  -1
1*    2   -1   0 = -x4
1    2   -2    3  -> max
```

↓ pivot step

```
x4  x2   x3  -1
 1    2   -1   0 = -x1
-1   0   -1    3  -> max
```

It was a degenerate pivot step. The last column of the matrix did not change.
Now the tableau is optimal.
max = -3 at x4 = x2 = x3 = 0, x1 = 0.

Exercises to §16.

Exercise 1. Solve the linear program given by the standard row tableau

```
x1 x2 x3 x4 x5 -1
 0  1 -2  3  4  1 =-x6
 1  1 -2 -3  4  0 =-x7
-3  1  0  3 -4  2 =-x8
 0 -1  2  0 -1 -1 ->max
```

Exercise 2. Solve the linear program given by the standard row tableau

```
x1 x2 x3 x4 x5 -1
 0  1 -2  3  4  1 =-x6
 1  1  2  3  4  1 =-x7
-3  1  0  3 -4  2 =-x8
 0 -1 -2  0 -1 -1 ->max
```

Exercise 3. In the following standard tableau, mark by * the choices for the pivot entry consistent with Phase 2:

```
x1 x2 x3 x4 x5 -1
 0  1  4  3  4  2 =-x6
 1  1  2  3  0  1 =-x7
-3  1  0  3 -4  2 =-x8
 1  1  2  1  1  1 ->max
```

How many choices are there?

§17. Phase 1.

We start with a standard tableau with the matrix

A b
c d
of size m + 1 by n + 1.
The strategy is to increase the first negative entry in the last column b while keeping the entries above non-negative.
Also we check whether the tableau is terminal before pivoting.

1. Is the tableau row feasible, i.e., b ≥ 0?
If yes, we go to Phase 2.

2. Is there a bad row?

$$\frac{\oplus \quad -1}{\oplus \quad - \;=\; \ominus} \qquad \text{or} \qquad + \;=\; \ominus$$

If yes, the row program is infeasible.

3 (choosing a pivot entry). We find the first $a_i < 0$ in the last column any $a_i < 0$ in this row.

$$\begin{array}{ccl} & \underline{-1} & \\ & \oplus & | \\ a_i & b_i & |= \;\ominus \end{array}$$

This a_i is going to be the pivot column. The pivot entry is going to be $a_i < 0$ or some entry $a_j > 0$ above.
It may happen that there is nothing above a_i (i.e., a_i and b_i are in the first row) or all entries above a_i are ≤ 0 in which case a_i is the pivot entry.
We compare b_i / a_i and all b_j/a_j with $a_j > 0$ above a_i and choose a minimal (closest to 0) ratio.

4. Pivot and go to 1.

Cycling is possible. but unlikely.
Every pivot step in a cycle is degenerate.
Bland's rule prevents cycling.
Moreover under the rule, in Phase 1 or 2, we cannot return to a tableau with the same set of variables on top.
Therefore the total number of pivot steps in both phases is less than (n+m)!/(m!n!) which is an upper bound for the number of all basic solutions.
This bound can be improved but no polynomial in m+n bound is known. For a fixed m or n, the bound (n+m)!/(m!n!) w is polynomial.
If we know which n variables are on the top in a terminal tableau, we can reach a terminal tableau from any initial tableau in at most min(m,n) ≤ m + n
pivot steps (with all tableaux standard but with pivot entry choices not necessary consistent with simplex method).
If we are in Phase 2, it is unknown whether we can bound the number of pivot steps needed to reach a terminal tableau with feasible tableaux on the way
by a polynomial in m + n.

Note that we do not check for bad columns in Phase 1, because our primary program is the row program and presence of a bad column leave open the question whether the row
problem is unbounded or infeasible..
However if we reach a row feasible tableau, the question is resolved (the row problem is unbounded) so we do not need to follow with Phase 2.

Example 17.1. Solve the linear program given by the standard row tableau

x1	x2	x3	-1	
1	0	1	-1	= -x4
1	2	-2	3	-> max

Solution. The x4-row is bad, so the row problem is infeasible.
The x2-column is bad, so the column problem is infeasible.

Example 17.2. Solve the linear program given by the standard row tableau

x1	x2	x3	-1	
1	0	-1	-1	= -x4
1	2	-2	3	-> max

Solution. The tableau is not feasible and has no bad rows.
There is only one choice for a pivot entry consistent with Phase 1.

```
x1  x2    x3    -1
1    0    -1*   -1 = -x4
1    2    -2     3  -> max
```

↓ pivot step

```
x1   x2    x4  -1
-1    0    -1   1 = -x3
-1    2    -2   5  -> max
```

The tableau is row feasible so we go to Phase 2.
The x2-column is bad, so the row problem is unbounded.

The x2-column is bad in both tableaux. So the column program is infeasible.

Example 17.3. Solve the linear program given by the standard row tableau

```
x1  x2  x3  -1
1    2  -1   0 = -x4
1   -2  -1  -1 = -x5
-1  -2  -1  -1  -> max
```

Solution. The tableau is not feasible and has no bad rows.
There are 2 choices for a pivot entry consistent with Phase 1.

```
x1  x2    x3   -1
1    2*   -1    0 = -x4
1   -2    -1*  -1 = -x5
-1  -2    -1   -1  -> max
```

Pivoting on 2 (degenerate pivot step) would give an infeasible tableau with no bad rows.
Let us pivot on -1:

```
x1  x2    x3   -1
1    2    -1     0 = -x4
1   -2    -1*  -1 = -x5
-1  -2    -1   -1  -> max
```

↓ pivot step

```
x1  x2    x5   -1
 0   4    -1    1 = -x4
-1   2    -1    1 = -x3
-2   0    -1    0  -> max
```

The tableau is row feasible, so we go to Phase 2.
The tableau is optimal, so
max = 0 at x1 = x2 = x5 = 0, x3=x4 = 1.

'

The original tableau is column feasible so the dual simplex method looks attractive. Transposing the tableaus allows us to bypass Phase 1.
However the transposed tableau is not optimal so we have to pivot at least once.
Actually we would have only one choice for a pivot entry in the transposed tableau consistent with Phase 2. It would be to switch x3 and x5 like we did above.

Example 17.4. In the following standard tableau, mark by * the choices for the pivot entry consistent with the simplex method:

```
x1 x2 x3  x4 x5 -1
 0  1 -2  3   4  2 =-x6
 1  1 -2 -3   4  0 =-x7
-3 -1  0 -3  -4 -2 =-x8
 1 -1 -2  1   1 -1 -> max
```

Solution.

```
x1 x2 x3  x4 x5 -1
 0  1 -2  3*  4  2 =-x6
 1* 1*-2 -3   4* 0 =-x7
-3 -1  0 -3* -4 -2 =-x8
 1 -1 -2  1   1 -1 ->max
```

There are 5 choices.

Example 17.5. Solve the linear program given by the standard row tableau

```
 0  6  5
-3 -3 -2
 1  2  0
```

Solution. There are two choices compatible with simplex method:

```
 0  6  5
-3*-3*-2
 1  2  0
```

If the first column is the pivot column, the pivot step gives a feasible tableau with a bad column, so the row problem is unbounded.
Actually, the first column was bad in the initial tableau.

If we pivot on -3 in the second column, we obtain a feasible tableau without bad columns or rows. So we have to pivot more (unless we noticed that
the first column was bad in the initial tableau.)

By the way, choosing 6 as the pivot entry in the initial tableau is not consistent with Phase 1 because 2/3 < 5/6.
This choice would not produce a feasible tableau although it would keep the first entry in the last column positive and increase the second entry,

Example 17.6. Solve the linear program given by the standard row tableau

```
x1 x2 x3 -1
-3  6  5 -4  = -x4
 3 -3 -2  3  = -x5
-3  4  2 -3  = -x6
 1  2  0  1  ->max
```

Solution. Instead of simplex method, we use a trick.
We add the first two rows and obtain a bad row:

```
x1 x2 x3 -1
0   3  3 -1 = -x4 - x5.
```

So the row problem is infeasible.

In general, if the row m+1 by n + 1 problem is infeasible, there is a mixture of the first m rows which is a bad row.
However finding such a mixture could be difficult and the simplex method is the most common way to do this.

Example 17.7. Solve the linear program given by the standard row tableau

```
x1 x2 x3 -1
 3 -6  5 -2  = -x4
-3  3 -2  3  = -x5
-3 -1  2 -3  = -x6
-1  2  0  1  ->max
```

Solution. Instead of simplex method, we use a trick.
We set x1 = x2 and x3 = 0. The we have the standard tableau

```
x1   -1
-3   -2  = -x4
 0    3  = -x5
-4   -3  = -x6
 1    1  ->max
```

The x1-column is bad but the tableau is not row feasible.
Now we give x1 bigger and bigger values. Then

x4 = 3x1 - 2 ≥ 0 when x1 > 1,
x5 = 3 ≥ 0,
x6 = 4x1 - 3 ≥ 0 when x1 > 1,
and the objective function x1 - 1 takes arbitrary large feasible values. So the row program is unbounded.
Therefore the original row program is unbounded.

Geometric interpretation of Phase 2. (After Fourier). See my textbook on linear programming. Mixtures (=convex linear combinations, weighted centers of gravity), convex sets, corners (extreme points), neighbors (edges) . The corners of the feasible region and the basic feasible solutions.
In simplex method, Phase 2, we go from a corner by an edge to a better corner.
The number of corners is finite but could be very large.
E.g., for 100-cube which is the feasible region for a standard 101 by 101 feasible tableau, the number of corners is $2^{100} > 10^{30}$.

Exercises to §17.

Exercise 1. In the following standard tableau, mark by * the choices for the pivot entry consistent with the simplex method:

```
x1 x2 x3  x4 x5 -1
 0  1 -2  3   4  0 =-x6
 1  1 -2 -3   1  1 =-x7
 1  1 -2 -3   4  1 =-x8
-3 -1 -1 -3  -1 -1 =-x9
 1  1  2  1   1 -1 -> max
```

Exercise 2. Solve the linear program given by the standard row tableau

```
x1 x2 x3 x4  x5 -1
 0  1 -2  3   4  0 =-x6
 1  1 -2 -3   1  1 =-x7
 1  1  2  0   4  1 =-x8
-3 -4 -9 -3  -3 -4 =-x9
 1  1  2  1   1 -1 -> max
```

Exercise 3. Solve the linear program given by the standard row tableau

```
x1 x2 x3 x4  x5 -1
 0  1 -2  3   4  0 =-x6
 1  1 -2 -3   1  1 =-x7
 1  1  2  0   4 -1 =-x8
-3 -4 -9 -3  -3 -1 =-x9
 1  1  2  1   1 -1 -> max
```

§18. Duality. Theorem on 4 alternative

There are many connections between linear programs dual to each other.

18.1. Given a standard tableau

	x	-1	
u	A	b	= - y
-1	c	d	=f->max
	=v	=g->min	

every feasible value of the row program is ≤ every feasible value of the column program.
Moreover, for the difference of the feasible values $u^T b - d$ and $cx^T - d$, we have
$(u^T b - d) - (cx^T - d) = u^T y + v x^T \geq 0.$

Indeed, let (x, y) be a feasible solution for the row program and
(u, v) be a be a feasible solution for the column program .
We want to show that
$u^T b - cx^T = u^T y + v x^T.$

We have
$Ax^T - b = - y \leq 0, x \geq 0$
and
$u^T A - c = v \geq 0, u \geq 0,$
hence
$u^T A x^T - u^T b = - u^T y$
and
$u^T A x^T - cx^T = v x^T$.
So $u^T b - cx^T = u^T y + v x^T.$

Definition 18.2. The feasible solutions (x, y) and (u, v) are *complementary* if
$u^T y + v x^T = 0$, i.e.,
(the value of every variable) * (the value of the dual variable) = 0.

It follows from 17.1 that complementary feasible solutions are optimal.
The converse is also true. It is a part of the duality theorem which is a part of the following result.

18.3. Theorem on 4 alternatives.
For a linear program, there are 3 alternatives:
the program has an optimal solution.
the program is infeasible,
the program is unbounded,
Now we consider possible outcomes for the pair of linear programs dual to each other.
We write the programs in a standard tableau and apply the simplex method.
If the row program has an optimal solution, we terminate at an optimal tableau which also gives an optimal solution for the column program with the same optimal value.
This explains the first row
√ max=min — —
of the table

row \ column	optimal	infeasible	unbounded
optimal	√ max=min	—	—
infeasible	—	√	√
unbounded	—	√	—

√ possible
— impossible

The first column follows by transposing the tableau.
In particular we obtain the duality theorem: if a linear maximization problem has an optimal solution then the dual minimization problem has an optimal solution with the same optimal value.
f the row problem is unbounded, we terminate with a row feasible tableau with a bad column. The bad column shows that the column program is infeasible.
This explains the last row and hence the last column.
Finally, the following 2 by 2 standard tableau

0 -1
1 0
has both bad row and bad column so it may happen that both a linear program and its dual are infeasible.

Thus, of 9 potential outcomes, 4 are possible and 5 are not, hence the name "theorem on 4 alternatives.
It is also known as the theorem on 3 alternatives because 4 outcomes can be covered by 3 sentences.
E.g., given a pair of linear programs, one of following outcomes happens:
both programs have optimal solutions,
one is unbounded and the other is infeasible,
both are infeasible.

There are several ways to state the duality theorem. E.g.,
if both a linear program and its dual are feasible, then both have optimal solutions and max = min.

The duality theorem implies the minimax theorem for matrix games.
It also implies the complementary slackness theorem mentioned above:
feasible solutions for the row and column programs are optimal if and only if they are complementary.
This give a practical way to double-check optimality.

Here is another criterion for optimality:
feasible solutions for the row and column programs are optimal if and only if they have the same values for the objective functions.
However this criterion is not as useful for checking optimality when only one feasible solution is given.

We obtain the theorem on 4 alternatives from the fact that it is possible to avoid cycling in simple method (by perturbation or Bland's rule).

Example 18.4. Consider the row program given the standard tableau

```
x1 x2 x3  x4 x5 -1
 0  1 -2  3  -1  2 =-x6
 1  1 -2 -3  -1  0 =-x7
-3 -1  0 -3   0 -2 =-x8
 0  0  0  0   0 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 3, x4 = 4, x6 = 6?

Solution. For such a solution, the first row reads
2 - 6 + 12 - x5 - 2 = -6 hence x5 = 12.
The second row reads
1 + 2 - 6 -12 - 12 = -x7 hence x7 = 27.
The third row reads
-3 - 2 -12 + 2 = -x8 hence x8 = 15.
So we have a feasible solution.
Since the objective function is constant, every feasible solution is optimal
Any optimal solution for the column program is complementary so all dual variables take value 0.

Example 18.5. Consider the row program given the standard tableau

```
x1 x2 x3  x4 x5 -1
 0 -1 -2  3  -1  2 =-x6
 1  1 -2 -3  -1  1 =-x7
-3 -1  0 -3   0 -2 =-x8
 1  1 -2 -4  -1 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 0, x4 = 0, x6 = 6?

Solution. The first row reads -2 -x5 -2 = -6 hence x5 = 2.
The second row reads 1 + 2 -2 - 1 = -x7 hence x7 = 0.
The third row reads -3 - 2 +2 = - x8 hence x8 = 3.
So we have a feasible solution for the row program.
If it is optimal, the column program has an optimal solution.
Let yi be dual to xi. For optimal (yi) we have y1 = y2 = y5 = y6 = y8 = 0 by complementary slackness

```
     1    2  0   0   2 -1
 0   0   -1 -2   3  -1  2 =-2
y7   1    1 -2  -3  -1  1 = 0
 0  -3   -1  0  -3   0 -2 =-3
-1   1    1 -2  -4  -1 -1 -> max
    =0   =0 =y3 =y4 =0   -> min
```

The first column gives y7 = 1. The second column reads the same.
The third column gives y3 = 0. The y4-column gives y4 = 1.
The y5-column reads 0 = 0.
So we got a complementary feasible solution for the column program.
Therefore both feasible solutions are optimal.

Example 18.6. Consider the row program given the standard tableau

```
x1 x2 x3  x4 x5 -1
 0 -1 -2  3  -1  2 =-x6
 1  1 -2 -3  -1  0 =-x7
-3 -1  0 -3   0 -2 =-x8
 1  1 -2 -4  -1 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 0, x4 = 0, x5 = 0?

Solution. We compute x7 = -3. So there is no feasible solutions for the row program with given values.
There for Therefore there are no optimal values.

Example 18.7. Consider the row program given the standard tableau

```
x1 x2 x3  x4 x5 -1
 0 -1 -2  3  -1  2 =-x6
 1  1 -2 -3  -1  3 =-x7
-3 -1  0 -3   0 -2 =-x8
 1  1 -2 -4   0 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 0, x4 = 0, x5 = 0?
Solution. We compute x6 = 4, x7 = 0, and x8 = 3. This is a feasible solution for the row

program, and it is unique (with given values for the variables at the top margin)..
But is it optimal? The complementary slackness often allows us to answer this question without using simplex method. We are looking now for a complementary solution for the column program. The dual variables corresponding to the nonzero primal variables, namely y3, y4, y5, y6, and y8 dual to x3, x4, x5, x6, and x8, must to be 0:

```
     1   2   0    0    0 -1
0    0  -1  -2    3   -1  2 =-4
y7   1   1  -2   -3   -1  3 =-0
0   -3  -1   0   -3    0 -2 =-3
-1   1   1  -2   -4    0 -1 -> max
   =0  =0  =y3  =y4  =y5
```

Because of degeneracy (x7 = 0) we have more work to do than in generic case when all variables at the right margin are nonzero.
The first column reads y7 - 1 = 0, hence y7 = 1.
The second column reads the same.
The third column gives y3 = 0.
The fourth column gives y4 = 1
The fifth column gives y5 = -1.
This is infeasible.
So there is no complementary feasible solution.
Therefore our feasible solution for the row program is not optimal.

Remarks. In very degenerate case, when all variables for the row program take zero values, the complementary slackness gives no information about he dual variables.
A complementary feasible; e solution in this case is the same as a feasible solution for the column program which is the same as an optimal solution for the column program.

We have the complementary slackness for any pair
(optimal solution for the row program, an optimal solution for the column program).
That is, if a variable is nonzero then the dual variable iz zero.
But if such a pair exists, there is a pair with the following additional condition::
if a variable is zero then the dual variable is nonzero.
This is called the strict complementary slackness.I
could not find this in textbooks for undergraduate students.
A related problem is an explicit description of all optimal solutions.
It is also out the scope of undergraduate math.

Example 18.8. Is the basic solution for the row program in the following standard tableau optimal?

```
x1 x2 x3  x4 x5 -1
 0 -1 -2   3  -1  2 =-x6
 1  1 -2 -3  -1  3 =-x7
-3 -1  0 -3   0  0 =-x8
 1  1 -2 -4   0 -1 -> max
```

Solution. The tableau is row feasible, i.e., the
basic solution is feasible.

Let yi be the dual variable for xi.
For the complementary feasible solution,
we have y6 = y7 = 0 and y1 = -1 -3y8 < 0.
So there is no complementary feasible solution,

Another way to see that the basic solution is not optimal is to improve it.
This can be done by pivoting on 1 at (x7, x1)-posotion or by just increasing x1 a little bit.

Example 18.9. Is the basic solution for the row program in the following
standard tableau optimal?

```
x1 x2 x3  x4 x5 -1
 0 -1 -2  3  -1  2 =-x6
 1  1 -2 -3  -1  3 =-x7
 3  1  0 -3   0  0 =-x8
 1  1 -2 -4   0 -1 -> max
```

Solution. Let yi be the dual variable for xi.

For the complementary feasible solution,
we have y6 = y7 = 0 and
y1 = -1 + 3y8 ≥ 0,
y2 = -1 + y8 ≥ 0,
y3 = 2 ≥ 0,
y4 = 4 - 3y8 ≥ 0,
y5 = 0 ≥ 0,
y8 ≥ 0.
There is a solution, i.e. set y8 = 1.
So the basic solution is optima;.

Taking arbitrary y8 ≥ 1, we get all optimal solutions for the column program.

All optimal solutions for the row program are obtained by setting
x1 = x2 = x3 = x4 = 0 and allowing x5 to be any nonnegative number.

Remark. The feasible solutions of the row and column program are complementary if and only if the corresponding feasible values are equal.
So linear programming is about finding a non-nnegative solutions for systems of linear equations.

Exercises to §18.

Exercise 1. In the standard tableau

```
x1 x2 x3  x4 x5 -1
 0 -1 -2  3  -1 -2 =-x6
 1  1 -2 -3  -1  3 =-x7
-3 -1  0 -3   0 -2 =-x8
 1  1 -2 -4  -1 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 0, x4 = 0, x5 = 0?

Exercise 2. In the standard tableau

```
x1 x2 x3  x4 x5 -1
 0 -1 -2  3  -1 -3 =-x6
 1  1 -2 -3  -1  3 =-x7
-3 -1  0 -3   0 -5 =-x8
 1  1 -2 -4  -1 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 0, x4 = 0, x5 = 0?

Exercise 3. In the standard tableau

```
x1 x2 x3  x4 x5 -1
 0 -1 -2  3  -1 -3 =-x6
 1  1 -2 -3  -1  3 =-x7
-3 -1  0 -3   0 -5 =-x8
 1  1 -2 -4   0 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 0, x4 = 0, x5 = 0?

Exercise 4. In the standard tableau

```
x1 x2 x3  x4 x5 -1
 0  0 -2  3  -1 -2 =-x6
 1  1 -2 -3  -1  3 =-x7
-3 -1  0 -3   0 -2 =-x8
 1  1 -2 -4  -1 -1 -> max
```

Is there an optimal solution with x1 = 1, x2 = 2, x3 = 0, x4 = 0, x5 = 0?

Ch7. Cooperation

§19. Nash bargaining.

This is a cooperative approach to solving games (with finite normal forms). without side payments.
For every finite normal form , it gives a mixed joint strategy. The corresponding joint payoff is unique.

Nash suggested this approach in his Ph.D. for two players (i.e., for bimatrix games). But it can be generalized to any finite set of players.
Here we restrict ourselves to bimatrix games.
The Nash solution (the joint payoff) is called arbitration pair in the Morris textbook.
This is because sometimes bargaining involves an authority figure (**Arbiter, Justice of the peace, etc)**
who helps to reach an agreement (resolve a dispute) and enforce it.
In the case of any matrix game, the pair is (v, -v) where v is the value of game.

About "without side payments." sometimes side payments are called bribes or corruption and are illegal.
Sometimes it is difficult to transfer payoff from one player to another because payoff measures something like happiness or satisfaction.
One mysterious way to do this is known as love.
The next section consider the games with side payments.

By a mixed joint strategy we mean a mixture of strategy profiles (pure joint strategies). The corresponding joint payoff is the mixture of payoffs corresponding to the strategy profiles.

Example 19.1. Battle of Sexes

He & She	Ballet	Football
Ballet	1, 5	0, 0
Football	0,0	5, 1

We have 2 equilibria in pure strategies:(Ballet, Ballet) and (Football, Football).
There is another equilibrium in mixed strategies, ((Ballet+5Football)/6, (5Ballet + Football)/6):

He & She	Ballet	Football	(5Ballet + Football)/6
Ballet	1*, 5*	0, 0	5/6*, 25/6
Football	0,0	5*, 1*	5/6,* 1/6
(Ballet+5Football)/6	1/6, 5/6*	25/6,5/6*	5/6,* 5/6*

Can we call any of 3 a solution for the game? Obviously, none.

On the other hand, a mixed joint strategy ((Ballet, Ballet)+ (Football, Football))/2 with the payoff (3,3) is the Nash bargaining solution.
An interpretation of this solution is that they decide what to watch by tossing a fair coin each time.
So half of time he has his way, and half of time she has her way.

The Nash bargaining for any bimatrix game starts with computing an initial point (x_0, y_0) for bargaining, aka the disagreement point.
He computes x_0 = v(He) as the maximal payoff he gets in spite of her. In other words, he maximizes his worst-case payoff .
In other words, x_0 is the value of the matrix game where her payoff is replaced by the negative of his payoff.
Similarly, y_0 = v(She) is what she gets in spite of him.

In Example 19.1, x_0 is the value of the matrix game

1 0
0 5

so $x_0 = 5/6$. Similarly, $y_0 = 5/6$, because the game is symmetric.

In general, he wants to get at least x_0 and she wants to get at least y_0 .
They plot all joint payoffs (x,y) in plane. For m by n bimatrix game, they get a set P of ≤ mn points in plane
(some of them can coincide, e.g., for Battle of Sexes they have 3 points, (5, 1), (0,0), and (1,5).
Using mixed joint strategies, they get all mixtures of these points, which is called there convex hull of P.
The hull is a convex polygon whose corners are the points in P which are not mixtures of other points.
In the degenerate case, the hull could be an interval or even a point.

They consider the part H of the convex hull imposing the conditions $x \ge x_0$ and $y \ge y_0$.
This H is also a convex polygon.

A point (x,y) in H is called *Pareto optimal* if there is no other point (x', y') in H with (x,y) ≤ (x', y').

Since they cooperate, they want their joint payoff to be Pareto optimal.
Every Pareto optimal point is uniquely mixture of two points in P.
So once they found the joint payoff, (x*, y*) they want, they have a mixed joint strategy to achieve it.

It may happen that $x = x_0$ on H. In this case $x^* = x_0$ and y^* is the maximal value of y on H
It may happen that $y = y_0$ on H. In this case $y^* = y_0$ and x^* is the maximal value of x on H
In both cases (x*, y*) is the only Pareto optimal point on H.

More generally, when there is only one Pareto optimal solution it is the Nash solution.

Assume now that there is more than one Pareto optimal solution.
Then the Nash solution for the joint payoff (the arbitration pair in the textbook) is the optimal solution f(x*,y*) for
$(x - x_0)(y - y_0)$ -> max on the feasible set H.
The Nash solution (arbitration pair) always exists and is unique. It is always Pareto optimal.
InIn the textbook, it is defined by axioms.

For any matrix game, H consists of the single point (v(He), v(She)) so the Nash solution for joint payoff is
(v(He), v(She)) where v(he) = -v(She) is the value of game.

For Example 19.1 we have 3 joint payoffs: (0, 0), (1, 5), and (5, 1). The mixtures form a triangle. A
The Pareto optimal solutions are the mixtures of (5,1) and (1,5).
A picture of triangle with ignition point (3/5, 3/5) and the Nash bargaining solution (3, 3) can be found on p.133 of the textbook,
The Nash solution for the mixed joint strategy is
((Ballet, Ballet)+ (Football, Football)/2.
An implementation of this solution is that they toss a fair coin to decide whether he has his way or she does.

For the Prisoner's Dilemma, for usual choices of payoffs, the Nash bargaining solution for mixed joint strategy is the code of silence (the pure strategy profile where both prisoners do not cooperate with prosecutor) while the
equilibrium is the pure strategy profile where both prisoners go for plea bargaining (known as defection). The equilibrium , can be found by domination but cooperation between prisoner's give them better result.
To go for the Nash solutions they should trust each other or be afraid of punishment for breaking the code of silence.
The plea bargaining is used precisely to break the code of silence. Probably, it is better for use to break the code of silence and keep the prisoners where they belong.

On the other hand Prisoner's Dilemma can be interpreted as an arms race problem. The equilibrium means the unlimited arms race. The Nash solution is a treaty restricting the race. Since it is not an equilibrium the problem of trust and verification arises.
In some cases, arbitration may help.

Example. 19.2. Find the Nash bargaining solutions for Example 8.4, i.e., for the bimatrix game

Players R, C	C1	C2	C3	C4
R1	5, 3	0,2	0, -2	0, 3
R2	1, 0	0, 0	0, 0	0, 0
R3	2, 2	-1, 0	4, 6	1, 1

Solution. What R can get in spite of C? This is the value x_0 of the matrix game

5 0 0 0
1 0 0 0
2 -1 4 1

We have a saddle point (R1, C2) so $x_0 = 0$.

What C can get in spite of R? This is the value y_0 of the matrix game

	R1	R2	R3
C1	3	0	2
C2	2	0	0
C3	−2	0	6
C4	3	0	1

We have a saddle point (C1, R2) so $y_0 = 0$.

Now we draw the joint payoffs and their mixtures. We get the 5-gone with the corners (0, -2), (5, 3), (4, 6), (0, 3), (-1, 0).
The Pareto optimal solutions are the mixtures of (5, 3) and (4, 6). This is a side of the 5-gpn with negative slope.
(The other side with negative slope has no Pareto optimal points.)
Now we maximize xy on the mixtures X of (5, 3) and (4, 6).
Using the slopes, we see that $3x + y = 18$ on X.
To maximize $3xy$ on the straight line $3x + y = 18$ containing X, we set
$3x = y = 9$ hence $x = 3$ and $y = 9$. But this optimal solution on the line is outside X.
It is on the left of X, and (4,6) is the point of X closest to the point (3, 9).
Thus, (4.6) is the Nash solution (arbitration pair) in terms of joint payoffs.

The corresponding strategy profile (joint strategy) is (R3, C3).

This strategy profile happens to be an equilibrium.
It also happens to be the only optimal solution if we maximize the total payoff.
It is a coincidence in both cases.

Remark. The optimal solution to $xy \to \max$ subject to, $x + y = s$ with given $s \geq 0$ is $x = y = s/2$.
Indeed $xy = x(s - x) = -(x - s/2)^2 + s^2/4$ increases when we come closer to $x = s/2$.

Remark. If there is a point in H where both x and y are maximal, this point is a corner. and it is the only Pareto optimal solution and hence it is the arbitration pair (x^*, y^*).
Conversely , if there is only one Pareto optimal solution in H that maximizes both x and y, and it is (x^*, y^*). Thus, (x^*, y^*) can be easily found in this case.

The Pareto optimal solutions, if more than one, form the connected union of some sides of H with negative slopes.
If (x^*, y^*) is inside of a side, the slope of $(x^*, y^*) - (x_0, y_0)$ plus the slope of the side is 0.
If (x^*, y^*) is a common corner of two sides, then the slope of $(x^*, y^*) - (x_0, y_0)$ is between the absolute values of the slopes of the sides.
This allows us to find fast the side or two sides which contain (x^*, y^*) by computing the slopes of sides and
the slopes $(x, y) - (x_0, y_0)$ for corners (x, y).

Namely, let $(x_1, y_1), (x_2, y_2), \ldots$ be the Pareto optimal corners ordered by (increasing) x.
We compute the slopes
$(y_2 - y_1)/(x_2 - x_1), \ldots$
of sides and their absolute values
$s_1 < s_2 < , \ldots ,$
On the other hand, we compute the slopes
$t_i = (y_i - y_0)/(x_i - x_0)$ for $i = 2, \ldots$ (we missed $i = 1$ because it is possible that $x_0 = x_1$).
We have $t_2 > t_3 > \ldots$.

By bisection we find the first i such that $t_i \leq s_{i-1}$.
If $t_i = s_{i-1}$. then $(x^*, y^*) = (x_i, y_i)$.

If $t_i < s_{i-1}$, then (x^*, y^*) is inside the side $(x_{i-1}, y_{i-1}), (x_i, y_i)$ and can be easily found from the condition
that the slope of $(x^*, y^*) - (x_0, y_0)$ is s_{i-1}.

For instance,

i	0	1	2	3	5	6
x_i	0	5	9	12	14	15
y_i	1	21	19	17	14	10

We compute s and t :

i	0	1	2	3	5	6		
x_i	0	5	9	12	14	15		
y_i	1	21	19	17	14	10		
s_i		1/2	2/3	3/2	4			
t_i		4	<u>2</u>	<u>4/3</u>	13/14	3/5	=	$(y_i - 1)/x_i$

So the point (x*, y*) is inside the second side with ends (9, 19) and (12, 17) and the slope -2/3. The ends and hence the whole side belong to the line
2x + 3y = 75.
We can intersect it with the straight line passing through (0, 1) with the slope 2/3 which is the line 3(y - 1) = 2x.
An alternative , suggested above is solving the optimization problem
x(y-1) -> max on the line 2x + 3y = 75.
The optimal solution is the same for
(2x)(3y) -> max subject to 2x + 3y = 75.
We make the factors equal:
2x = 3y = 75/2 hence
x = 75/4, y = 25/2 is the optimal solution.
This point is a mixture of the ends (9, 19) and (12, 17) of the side:
(x*, y*) = (75/4, 25/2) = α(9, 19) + (1 - α)(12, 17) for α = ?.

Example. 19.3. For 1 by 5 bimatrix game

[(1, 6), (2, 6), (3,5), (0,0), (3,4)]

find the pure equilibria and the Nash bargaining solution.

Solution. There are two equilibria:
[(1*, 6*), (2*, 6*), (3*,5), (0*,0), (3*,4)].

Among given 5 (pure) payoffs, there are exactly two Pareto optimal payoffs, namely, (2, 6) and (3, 5).
The mixtures M of these two points are exactly the side of the convex hull of given 5 points with negative slope
The cornea of the hull are given 5 points.

The Nash solution belongs to M. Now we compute the starting point for Nash bargaining, $(x0, y0) = (0, 6)$.

The straight line containing M is $x + y = 8$. We maximize $x(y - 6)$ on the line.
Since $x + (y - 6) = 2$ on the line,
the optimal solution is $x = y - 6 = 1$, hence $x = 1$ and $y = 7$. The closest point on the side M is $(2, 6)$. It satisfies the condition
$(x, y) \ge (x0, y0)$, we have the arbitration pair $(x^*, y^*) = (2, 6)$.
In the corresponding joint strategy,
the second player chooses the second column and he first player has no choices.

Example 19.4. For any 1-player game, the Nash bargaining gives an optimal strategy and the value of game.

Example 19.5. Among the points (0, 0), (1,2), (4,5), (2,7), (4,6), (5,5), (6,4), (7,2) and (8, -5) on plane, find the Pareto optimal points.
Also find the Pareto optimal points in the convex hull of the points.

Solution. We have $(0,0) < (1,2) < (4,5)$ so (0,0) and (1,2) are not Pareto optimal.
Also $(4, 5) \le (4, 6)$ and $(4, 5) \le (4, 6)$, so (4,5) is not Pareto optimal.
The Pareto optimal are (2,7), (4,6), (5,5), (6,4), (7,2) and (8, -5).
In the convex hull, the Pareto optimal points are covered by the polygonal path
(2,7), (4,6), (5,5), (6,4), (7,2), (8, -5).
All the points in the Path are Pareto optimal including the point (5, 5) which is a mixture of (4,6), and (6,4),

Remark 19.6. For game with any number $N > 0$ players $P_1, \dots, P_N$, the Nash bargaining can be done the same way.
We each player P we compute P's maximal worst-case payoff v(P) against the other players who together try to minimize P's payoff. It is the value of a matrix game.
The N-tuple $x0 = [v(P_1), \dots, v(P_N)]$ is an initial point for bargaining.

We consider the mixtures (he convex hull) of all joint payoffs and its part H cut out by the constraints $x_i \ge v(P_i)$ for all i.
We consider the product f(x) of t all entries in x - xo which are not identically 0 on H and then maximize f(x) over H. This mathematical program always has exactly one optimal solution x*.
This x* is always Pareto optimal, i.e.,there is no other x' with $x' \ge x^*$. This is true because when x is not Pareto optimal, we can increase or keep the same every factor in f(x).
This x* is the Nash solution for joint payoff. It is a unique mixture of corners of H.

if the Pareto optimal solution is unique, it is the Nash bargaining solution x*.
Another case when it is easy to find x* is when only two factors stay in f(x). Then H is a convex polygon in a plane.
In general, computationally, it could be much more difficult to find x* for $N > 2$ than for $N = 2$

The number of corners in H is small - it is a part of given pure joint payoffs.
The Nash solution belongs to one of finitely many faces. Every face span a hyperplane given by a linear equation ax = b with $a \ge 0$, so the optimal solution on the hyperplane can be u found rather easily using the fact the product of nonnegative factors with given sum reaches its maximal value when all factors are equal.
But in higher dimensions it is not easy to get the optimal solution on the face, and the number of faces could be so big that we cannot even to list all faces.

Remark. The initial point $(x_0\ y_0)$ for Nash bargaining in a bimatrix game need not be a mixed joint strateg.
But what if for a bimatrix game there is no mixed joint payoff (x,y) such that $(x,y) \ge (x_0\ y_0)$?
Answer: this never happens.
Indeed, let the players be X and Y, mmA' is the payoff matrix for X and A" is the payoff matrix for Y.
Let p is and optimal strategy for the row player X for the matrix game with the payoff matrix A'. Then
$x_0 = \min(p^T A')$, the maximal worst case payoff for X.
Let q be an optimal strategy f for the row player Y or the matrix game with the payoff matrix A''^T.
Then $y_0 = \min(q^T A''^T)$, the worst-case payoff.
For the mixed joint strategy $(x, y) = p^T(A', A'')q$ we have $(x,y) \ge (x_0\ y_0)$.

Example 19.6. Do Nash bargaining in Example 7.8.
To see all pure joint payoff, we compute the payoff at the chance position

```
              A initial
        A1↙      ↘A2
         A             B
    A3↙   ↘A4   ↙B1 ↘B2
     1,2        3,4              0.8, -.0.5
```
.
There is only one Pareto optimal payoff, namely (3, 4).
So this is the arbitration pair.
It can be obtained by pure strategies (A1, A4); any strategy for B) or by (A2, B1).
We do not need to compute the initial point (3, 2) here, but it is computed in Example 20.9.

Example 19.7. Do Nash bargaining in Example 7.8.

To see all pure joint payoff, we compute the payoff at the chance position
A

```
        A1↙     ↘A2
         A            B
   A3↙  ↘A4   ↙B1 ↘B2
    1,2       3,4            0.8, -.0.5
```

Solution. The initial point is not given so this is not an extensive form.
Like with did for Nim and restricted Nims, we will solve the game for every initial position.
If the initial position is one of 3 terminal positions, the given payoff is the only possible payoff so it is the arbitration pair.
For the case when the initial position is the top position, see the previous example.
Here are two remaining cases.

```
                    A
 initial  A1↙     ↘A2
         A            B
   A3↙  ↘A4   ↙B1 ↘B2
    1,2       3,4            0.8, -.0.5
```

There are two pure payoffs possible, namely, (1, 2) and (3,4).
The initial point is (3,2).
(The first player A can get 3 in site of B and A can give 2 to B.)
Ut is not Pareto optimal.
The arbitration pair is (3,4).

```
              A
        A1↙     ↘A2 initial
         A            B
   A3↙  ↘A4   ↙B1 ↘B2
    1,2       3,4            0.8, -.0.5
```

Now the initial point is (0.8, 4).
Ut is not Pareto optimal.
The arbitration pair is (3,4).

Exercises to §19.

Exercise 1. Among the payoffs

(1,2,3), (0,0,0), (3,1,2), (2,1,2)   (1,0,1)

which are Pareto optimal?

Exercise 2 Do the Nash bargaining for the bimatrix game

5, 1 2, 1

0, 1   6, 2

Exercise 3. Do the Nash bargaining for the bimatrix game

1, 2   3, 4 -1,-3   9, 5

3,1 -1, 2   3,3   0,5

Exercise 4. Do the Nash bargaining for the extensive form

A initial

↙   ↖

B   B

/   \   /   \

(0, 1)   (0, 0)   (1, 2)

with 2 players, A and B.

Remark. Besides the bargaining scheme above, Nash suggested a more complicated scheme which takes in account threats. See

Bargaining problem - Wikipedia

Credible Threats in Negotiations : A Game-Theoretic Approach by Harold Houba , and Wilko Bolt
Kluwer Academic Publishers 20096 (ebook is available at the library).

The Bargaining Problem by : John F. Nash, Jr. Econometrica, Vol. 18, No. 2 (Apr., 1950), pp. 155-162

Two-Person Cooperative Games by : John Nash Econometrica, Vol. 21, No. 1 (Jan., 1953), pp. 128-140

Symmetric games. A bimatrix game is *symmetric* if transposing the bimatrix and switching the payoffs give the same bimatrix.
Examples include Prisoner;s Dilemma , Steal or Share, and all symmetric matrix games.
Battle of Sexes is symmetric in a more general sense: we also switch B and F strategies.
For any symmetric game, x0 = y0 for the disagreement point and x* = y* for the arbitration pair.
So the arbitration pair is the only Pareto optimal point on the line x = y.

For any one-player game, a payoff is Pareto optimal if and only if it is maximal, i.e., it is the value of game.

For any zero-sum game, any mixed joint payoff is Pareto optimal.

For any matrox game, the arbitration pair is
(the value of game, - the value of game).

For any bimatrix game, the initial point is the arbitration pair if it is Pareto optimal among the mixed joint payoffs.
The pair (x,y) is the only Pareto optimal pure payoff
if and only if each player reaches the maximal payoff; then it is also the only Pareto optimal mixed joint payoff and the arbitration pair.

§20. Coalitions. Shapley values.

Now we consider the situation when side payments are free and not restricted.

In real life, side payments are often possible but difficult and not free. Sometimes they are regulated by law.

We assume that the game is given in normal form with finite nonempty set of players and a a finite set of (pure(strategies for each player.
If the game [s given by an extensive form with a finite nonempty set of players, a finite set of positions, and no directed loops, the characteristic function (see below) and hence the Shapley values can be defined by going o the corresponding normal form.
However the characteristic function can be computed without going to normal form.

Since side payments are allowed and free, the players go for the maximal total payoff and then decide how to distribute (or redistribute)
(To enforce distribution, sometimes, arbitration is involved.)

Shapley suggested a way to do it. The Shapley values (how much each player gets) exist and unique for every finite normal form.

We want to find a fair value for each player. What did the player contribute to achieving the maximal total payoff?

Definition 20.1. A *coalition* is a set of players.

For a game with N players, there are 2^N coalitions.
One of them, has no players (empty coalition ∅). The *grand coalition* includes all players.

20.2. *Characteristic function*. For any coalition S, its "value" v(S) is the maximal total payoff the coalition can get in spite of the other players.
When S has a player but not all of them, v(S) is the value of an m by n matrix game where m is the number of joint strategies of players in S and n is the number of joint strategies of the other players. (We start with a finite normal form.)
For the empty S, v(S) = 0. (It is the value of a 0-player game/)
For the grand coalition S, v(S) is the maximal total payoff. (It is the value of a 1-player game.)
When S is not empty and is not the grand coalition, c(S) is the value of a matrix game where the first player is S and the second player is the coalition of the other players.

In the previous section, we had v(S) in the case when S consists of a single player.
This part of the characteristic function was used to start the Nash bargaining.

20.3. Contribution of a player P to a coalition S is defined as
$v(S) - v(S \setminus P)$.

20.4. For disjoint coalitions S and S', it is clear that $v(S \cup S') \geq v(S) + v(S')$.
In particular, the contribution of any player P to any coalition including P is $\geq v(P)$.

The Shapley value for a player P is P's average contribution.
Here is a way to define it.

Consider an N! by N table where columns correspond to the players and the rows correspond to the permutations of the players. The permutation is considered as a coalition which grows from the empty coalition to the grand coalition.
The number corresponding to a growing coalition and a player P is the P's contribution to the first coalition including P.
The sum of numbers in each row is v(grand coalition).
The Shapley value s(P) for a player P is the mean of numbers in P's column.
That is, it is the sum divided by N!.

It is clear that $s(P) \geq v(P)$ and that the sum of all Shapley values is v(grand coalition).

There are repetitions in the matrix so s(P) is a weighted average for a smaller than N! numbers.

20.5. An imputation is an N-tuple $T = (T_P)$ indexed by the players P such that $T_P \geq v(P)$ for all P
and the sum of all T_P is v(grand coalition).

So the Shapley values form an imputation.
Sometimes there are no other imputations. This happens if and only is the sum of all v(P) is v(grand coalition). In this case, s(P) = v(P) for all P.

The Shapley values are determined by the characteristic function and that different normal forms may have the same characteristic function.
They exist and are unique.

It is easy to define the Shapley values, but computing them for large N is a big challenge.

20.6. For 2 players, A and B.

	A	B
A, B	v(A)	v(A, B)- v(A)

B, A v(A, B) - v(B) v(B)
2s v(A, B)+v(A)-v(B) v(A, B)-v(A)+ v(B)

Example 20.7. Find the Shapley values for the bimatrix game

Players R, C	C1	C2	C3	C4
R1	5, 3	0,2	0, -2	0, 3
R2	1, 0	0, 0	0, 0	0, 0
R3	2, 2	-1, 0	4, 6	1, 1

(The game is the same as in Examples 8.4 and 19.2.)

Solution. We have v(R, C) = 10 (the maximal total payoff).
By 19.2, v(R) = v(C) = 0. By 20.6,
s(R) = s(C) = 5.

Example 20.8. Compute the Shapley values for the extensive form with 3 players A,B, and C:
(1,2,3) <— A —> (4,5,6).
First we compute the characteristic function.c = v:
c(empty) = 0, c(all) = 15;
c(A) = 4, c(B) = 2, c(C) = 3;
c(A,B) = 9, c(A,C) =10, c(B,C) = 5.

Notice that A has absolute power, so A maximizes the total payoff for any coalition S including A and minimizes the total payoff for S not containing A.
As usual, going to normal form you'd not help. not help
The Nash bargaining payoff is ((4,5,6), the only Pareto optimal payoff.

Now we compute the Shapley values.
Recall that
(s(A), s(B), s(C) ≥ (c(A), c(B), c(C) =(4, 2, 3)
and s(A) + s(B) + s(C) = c(all) = 15.
So the surplus is 15 - 9 = 6. The pirates way is to divide 6 into equal parts, but Shapley way could be different if the number of players is ≥ 3.
We make the table of contributions:

	A	B	C
ABC	4	5	6
ACB	4	5	6
BAC	7	2	6
BCA	10	2	3
CAB	7	5	3
CBA	10	2	3

6s	42	22	26
Shapley	7	3.5	4.5
Pirates	6	4	5
Nash	4	5	6

Which way you like better?

Example 20.9. Compute the Shapley values for Example 7.8.
First we compute the characteristic function c = v.
v(empty) = 0, v(all) =7.
To compute v(A), we replace the payoffs (x,y) in the extensive form by (x, -x):

```
          A
    A1↙       ↘A2
      A          B
A3↙   ↘A4   ↙B1  ↘B2
  1,-1   3,-3        0.8, -.0.8
```

Then we find the equilibria (A1,A2; any strategy for B) with the payoff (3, -3) by the backward induction:

```
          A 3,-3
    A1↙       ↘A2
      A  3,-3      B 0.8, -.0.8
A3↙   ↘A4   ↙B1  ↘B2
  1,-1   3,-3        0.8, -.0.8
```

So vA) = 3.
To compute v(B), we replace the payoffs (x,y) in the extensive form by (-y, y):

```
          A
    A1↙       ↘A2
      A          B
A3↙   ↘A4   ↙B1  ↘B2
  -2,2   -4,4        0.5, -.0.5
```

Then we find the equilibria (A1,A3; any strategy for B) with the payoff (-2,2) by the backward induction.

```
          A -2,2
    A1↙       ↘A2
      A  -2,2      B  -4,4
A3↙   ↘A4   ↙B1  ↘B2
  -2,2   -4,4        0.5, -.0.5
```

So vB) = 2.
The surplus v(A,B) - v(A) - v(B) =7 - 3 - 2 = 2.
So The Shapley values are (3, 2) + (1, 1) = (4, 3).

Remark 20.9. Juggling numbers
If we have an extensive form with P positions and M moves belonging to the players,
the number J of joint strategies is ≤ $(M/P)^P$. The upper bound is sharp when M/P is an integer.
So the corresponding normal form gives a joint payoff for each of J joint strategies.
When the number of players is N,to compute the characteristic function, we have to find the value of
2^N -2 matrix game. Each payoff matrix has J entries. It is impossible to write such a matrix when
J is bigger than the number of elementary particles in the u universe (estimated as 10^{80}).

However we can find each value solving an extensive for with P positions and M moves belonging to 2 players.

Exercises to §20.

Exercise 1. The characteristic function for 3 players A, B, C is

v(A) = 1, v(B) = 2, v(C) = 3, v(A, B) = 4, v(A, C) = 5, v(B, C) = 6, v(A, B, C) = 12.

Compute the Shapley values s(A) , s(B) , and s(C).

Exercise 2 Compute the characteristic function of the 3-player game given by the normal form

```
strategy      payoff
A   B   C     A  B  C
L   L   L     0  1  2
L   L   R     3  4  5
L   R   L     4  4  0
L   R   R     0  1  1
R   L   L     0  1  2
R   L   R     1  1  1
R   R   L     1  0  0
R   R   R     0  0  2
```

where players are A, B, and C and each has 2 strategies, L and R.

§21. Examples.

Examples 21.1-2. See Midterm 3| sol | for 2 examples with solutions.

Examples 21.3-4. See Midterm 3 | solutions | for 2 more examples with solutions.

Example 21.5. You are the 2nd player in the 1 by 2 bimatrix game

C1 C2
(1,0) , (-1, 0))

Explain your choice, C1 or C2.

Remarks. This was Problem 1 in Quiz 1 on Nov 8, 2017 for 5 pts.
Example 19.3 above is very similar but bigger.
Section 2 was worned on Monday about q1 on W.
I tried to warn Section 2 about a quiz on Th by a remarks on h7, but by my mistake, they did not get the warning.
But q1 participation in Section 1 was higher the in Section 2, I do not know why.
I excluded 21.5 from q1 in Section 1.

Solutions to 21.5. Here 1-10 are samples of student' solutions with the points they received and with my comments;
11-12 are Nash (nobel prize 1994) and Shapley (Nobel; prize 2012) solutions; 12-13 are hypothetical student solutions,
followed by my additional comment. This problem generated a big reverberation in students and hence in me.
So it was a good opportunity to learn (including for students in Section 1 who did not see it in their q1) which was what I hoped for.

Here are samples of student's solutions.

1. I choose C2 because then 1st player loses 1 and I get it.
My response: You do not get it. You get 0.

2. I choose C2 because I want to minimize my partner's payoff.
My response: 5 pts. I did not teach you to do this. But you gave an answer and an explanation, hence you get 5pts.
Your solution makes sense in some situations, e/g.. when you hate 1st player and want him to

know this.
He might hate you back for this.

3. I get nothing in both cases So It does not matter, C1 or C2.
My response: 5 pts. Sometimes, you have to choose, do not be a Buridan's ass.

4. Both are equilibria. So It does not matter, C1 or C2.
My response: 5 pts. Also all mixtures are equilibria. So the equilibrium is not a good answer.
We all strive for an equilibrium but is not always the answer.
Sometimes the answer is bargaining, compromise, coalition forming, and cooperation.
Sometimes you have faith and trust other players.
Sometimes somebody or the law enforces cooperation.

5. My choice is a mixed strategy (C1 + C2)/2 because it is fair. Both players get 0.
My response: 5 pts. I did not teach you to do this. But fairness could be an important issue in some cases. Fairness means different thing for different players.
It is not always the same payoff for everybody.
Some believe the ideal fairness is when everybody has nothing.
If you have no car and your neighbor has it, is it fair?
Is it OK to demand sharing the car?
How about bargaining for sharing, saying: I will choose C1 if you do share your payoff.
Each may get 1/2 instead 0 in your choice.
A version of a fair society is described in Animal Farm by a democratic socialist.
Here is a version of fairness by Karl Narx: (see 1.3 above):
to each according to his needs
But who determines you needs? Is it Karl Marx or Pol Pot?
Pol Pot decided that you do not need cars, electricity, or phone. Is it fair?
Should everyone have the same needs?
Fairness - Wikipedia

6. I chose to minimize other person output , if I receive 0 regardless.
My response: 0 pts. I do not understand what you mean by "output".
The first player has no choices. Maybe you and some other students have no idea what the bimatrix game is. Maybe, you made your choice at random and learned too little to write anything which makes sense.

7. I go for C2 because it is an equilibrium.
My response. 5 pts. (More precisely, (R1, C2) is an equilibrium.)
But (R1, C1) is also an equilibrium and so is every mixture of the two pure equilibria (saddle points).
Is the concept of equilibrium useful in this game?

8. I go for C2 because it is the equilibrium.
My response.. 0 pts. There are equilibria with different payoffs.

Here is the solution I expected. Let R and C be names of the players and R1 the name for the only strategy of the row player R.

9. The Nash bargaining solution in terms of joint strategies,is (R1, C1). The payoff there, the arbitration pair, is (1, 0).
is the Nash bargaining solution in terms of joint payoffs.
The arbitration pair is always Pareto optimal, and we have only one Pareto optimal mixed joint payoff, namely (1, 0).
There is no need to compute the starting point (v(R), v(C) = (-1, 0) for Nash bargaining.
So I (the second player C) choose C1 to get this joint payoff (1, 0).

10. For the Shapley values, (s(R), s(B)) = (0, 1), see the computation in 20.6.
There is only one way to get the maximal total payoff 1: I have to chose C1.
So the Shapley values also take me to C1.
The payoff (1, 0) at (R1, C1) is redistributed to the Shapley values (0, 1).
So I am happy with the Shapley solution.
The player R should be happy too because I could choose the payoff -1 for R.

Here is 2 hypothetical student's solutions without points or comments.

11. I chose C2 because the game describes the following situation. R attacked me so I knocked him down unconscious.
Then I prefer to finish him (C2) rather than call 911 to help him (C1).

12. I chose C2 because the game describes the following situation. On response to a question when bonus points would be posted,
the instructor asked class for volunteers to help him to process bonus points.
But it was his job. Moreover a student could see the grades of other students.
This is illegal and everybody knows this.
So instead of helping (C1 option) I went to his boss to protest (C2 option).
I protested not for a reward, but to enforce the law. It is illegal to pay a student for help.
Without my complaint, he would go further by requesting a student to grade my homework, another violation of law.
Also I have right to talk with the instructor until I get points I need to pass.
I need to pass so badly that I have priority over other students who want to talk with him.
It is OK with me if they learn the right way to get points because I do not try to minimize their grades.

The C2 choice is contagious, and no society can survive without cooperation and good will.
But it is not my job to force my opinions on you. My job is point out that there are different options, and sometimes there is no option which
is the best for everybody. Education may help you to make better choices.
It is not my job to teach you how to treat other humans and yourself. But it my job, starting this

semester, to inform you that Penn State provide help to students with mental problems.
So if you feel unhappy, mistreated, or betrayed, see the link in the syllabus.
It is also my job to point out that Penn State has penalty for
lying to instructor in order to improve your grade,
class direction,
unruly behavior,
and other violations of Penn State policies and rules.

It is natural to try to be best but hurting your competition can be punished, see Example 21.7 below.

I encourage you to help your classmates and me to learn, but doing their homework for them does not help them to learn.
See
Sportsmanship - Wikipedia
Sportsmanship is an aspiration or ethos that a sport or activity will be enjoyed for its own sake, with proper consideration for fairness, ethics, respect, and a.
and be a good sport [**Good sport].**

Example 21.6. Make your choice in 2 by 2 bimatrix game

C	D
3, 3	0, 5
5, 0	1, 1

Remarks. It was in qi in both sections in two versions: you choose your partner or you partner is the whole class, 5 pts for each version.
was in q1 in both sections.

We do not know what "solve
this game" means unless it is a 1-player game or a 2-player constant-sum game.

Both Nash bargaining and Shapley values lead to (3, 3).
However D dominates C. So if I do not trust my partner, I may go for D.
There are thousands publications on Prisoner's Dilemma, and more are coming.
Obviously, no solution makes everybody happy.

Our game is a Prisoner's Dilemma. However the numbers (payoffs) vary, sometimes even within the same book (e.g. our textbook).
A change in numbers may result in the Nash and Shapley solutions different from (C, C).

A student commented in class: My choice depends on my mood.
My comment. Do not pet a hungry crocodile.

More seriously, a student of mine got Ph.D. exploring games where players have different moods.

A student asked: What was the composed choice of class in 21.6?
My response. Approximately, it was (C + 2D)/3 in Section 2 at the Quiz 1 and (2C + D)/3 in Section 1 at Quiz 1
the next day.
Would not you like to know this before the quiz?
The average score was lower in Section 2 where D players are in majority even when they play with a partner the choose.
See Example 21.10 below.

A question. Does it help to have a little chat with my partner before playing a game?
My response. It may help you, your partner, or both, see Split or Steal in clips.

A complaint. I do not want to play Prisoner's Dilemma because my payoff depends on what my partner does.
I am not a prisoner, and you cannot make me to play.
My response. You cannot avoid games where your payoff depends on other players choices.
The Unabomber tried by living without electricity and phone. But he depended on US Postal Service.
His brother played D so the Unabomber is a prisoner now. There is electricity and phone there.
When you approach a 4-way stop, you life depends not only on your choices but also on another driver's choices.
When you walk on campus to a class, your life may depend on a terrorist driving a van.

Many instructors at Penn State use teams , peer reviews, curving or grading systems where your grade depends on other students actions.
When you want to register for a class you may find that the class is full and you cannot register.
If you want to be #1 in class, another student may stop your dream for coming true.

Big line to bathrooms in a movie theatre are possible when you really need a bathroom.
You can be a hermit in a cave but still depend on visitors bringing you food.
We are depend on each other so try to use cooperation rather than demanding total and complete independence.

If you got no points because your explanation did not make sense or was absent, it was independent of your choice or other student choices.
It is OK if you try to explain your q1 to me using what you learned after the test.

Your scores in all tests are independent on whether you wish As or Fs for other students or yourself.

Moreover I do not require you to tell me truth about wishes. In this country you can wish whatever you want. and keep your thoughts private.
In some countries, they use torture instead plea bargaining to extract truth or admission of guilt.

The game is a popular choice for publications on evolutionary game theory, see §23.
The D players take advantage of C players, so they proliferate (once one appears) when they are in minority and do not proliferate when they are in majority.
This explain why it is not often we see a population with D players in 2/3 majority.

Evolutionary game theory strives to address this issue as well as the story about the croc mom above and so do ethics and some religions.

21.7. Example based on a real life story. You are the column player C in Example 21.5 (real name Jeff Gillooly).
Your ex-wife TH (real name Tonya Harding) competes in figure skating.
Her completion, the row player R in Example 21.5 (real name Nancy Kerrigan) has a better chance to win.
You can do nothing about this (option C1) and allow R to win a medal. Her payoff is 1 but yours is 0.
The other option C2 is to participate in a plot to break the right leg of R. You estimate the expected payoff for R is -1, but your payoff is 0
because besides potential rewards, you face a prison term. Your competitive spirit or stupidity lead you to C2 choice
Later, under prosecution, you choose plea bargaining (option D in Prisoner's Dilemma) to reduce your prison term to 2 years.
R's leg is not broken but damaged. She wins
1994 Olympic silver medal. TH finishes eighth. Later she also chose D and received 3 years probation and other penalties.

21.8. Related to Example 21,5 is **Golden Rule**
which is the principle of treating others as one would wish to be treated. It is a maxim of altruism that is found in many religions and cultures
Ancient Egypt, circa 4K years ago. Do to the doer to make him do
Ancient Egypt, circa 2.5K years ago. That which you hate to be done to you, do not do to another
Moses. Whatever is hurtful to you, **do** not **do** to any other person.
JC, **Do unto others** as you would have them **do unto** you definition
Islam. None of you [truly] believes until he wishes for his brother what he wishes for himself
Hinduism By making *dharma* (right conduct) your main focus, treat others as you treat yourself
Buddha. One who, while himself seeking happiness, oppresses with violence other beings who also desire happiness, will not attain happiness hereafter.
Confucius What you do not wish for yourself, do not do to others

Maybe, some religions discouraging cooperation or procreations disappeared?
Golden Rule is discussed also in secular context, like philosophy and human rights; there are different interpretations and criticism.

To apply Golden Rule to the case of Prisoner's Dilemma, you have to replace your partner by

yourself and play the game for both players. So what is your choice, CC,CD, DC, or DD, in Example 21.6 below?
What if 5 in the example replaced by 7?

21.9. General matrix game with one row. The players are R and C.

C_1 C_2 ...
R1 $[(a_1, b_1), (a_2, b_2)$,...

The equilibria are the mixtures of the joint strategies $(R1, C_j)$ with $b_j = \max(b_k)$ $= v(C) = y0$.
The payoff for C is v(C) at each equilibrium.
The payoff for R varies between $x1 = \min(a_i \mid b_i = x0 = v(C)) \geq v(R)$ and and $x2 = \max(a_i \mid b_i = x0)$.
So concept of equilibria does not say much about the payoff of R.
The arbitration pair (x^*, y^*) is (x2 , v(R)) game without side payment).
The Shapley values are $(s(R), s(C)) = (v(R,C) + v(R) - v(C), v(A,B) - v(R) + v(C))/2 \geq (v(R), v(C))$.
So if $v(R) + v(C) < v(R,C)$ then $s(C) > v(C)$ and the player C should negotiate with R for redistribution of payoffs to get a better payoff (game with side payments).

Example 21.10. You play the prisoner Dilemma 21.6

C	D
3, 3	0, 5
5, 0	1, 1

against the class with composition
(a) (2C + D)/3,
((b) (C + 2D)/3.
What is your choice (C or D) ?
D strictly dominate C regardless of the class composition. So if you care only about your present payoff, you go for D ignoring religion, ethics, and the future games.
However if you care about your partners or about your future , you should consider C (both Nash and Shapley choice) which means cooperation.
Note that the expected payoff in the case (a) is
(4/9)3 + (2/9)(5,+0) + (1/9) = 23/9
vs that in the case (b):
(1/9)3 + (2/9)(5,+0) + (4/9) = 17/9.
Thus, cooperation pays in this game.
In more detail, you are better off in class (a) than in class (b) by
2 -1 = 1 if you play C
and by
11/3 - 7/3 = 4/3 if you play D.
So everybody wants to be in class (a), especially, D players.

In evolution game theory a bigger payoff is converted into better fitness (like smaller death rate and bigger birth rate).
Thus, class (a) has evolutionary advantage over class (b).
This gives an explanation why we still have a lot of C players around.

Some authors believe that your family is the best place to learn cooperation (family values).
Some learn cooperation in prison or army.
The military term for sticking together is the unit cohesion.
Prison gang is an example of Inmate cooperation.

Example 21.11. Find the equilibria in pure strategies, the pure Pareto optimal payoffs, the characteristic function v, the Shapley values, and the Nash bargaining solution.
Players: A, B, C.

strategies	payoffs
1 1 1	2 3 2
1 1 2	1 0 1
1 2 1	2 3 3
1 2 2	1 2 3
2 1 1	0 0 1
2 1 2	1 1 1
2 2 1	2 3 3
2 2 2	2 3 2
3 1 1	0 0 0
3 1 2	0 0 0
3 2 1	1 1 1
3 2 2	2 3 0

Solution.
Three equilibria and one Pareto optimal triple (2,3,3):

strategies	payoffs
1 1 1	2* 3* 2* equilibrium strategy
1 1 2	1 0 1
1 2 1	2* 3* 3* equilibrium & Pareto optimal
1 2 2	1 2 3
2 1 1	0 0 1
2 1 2	1 1 1
2 2 1	2* 3* 3*equilibrium & Pareto optimal
2 2 2	2 3 2
3 1 1	0 0 0
3 1 2	0 0 0
3 2 1	1 1 1
3 2 2	2 3 0

v(empty)=0, v(A,B,C)) = 8.
v(A) = 1 = the value of the matrix game

2	1*'	2	1
0	1	2	2
0	0	1	2

because we have and equilibrium in row 1, column 2 with the payoff 1.

v(B) = 1 = the value of the matrix game

3	0	0	1	0	0
3	2	3	3	1*'	3

v(C) = 0 = the value of the matrix game

2	3	1	3	0*'	1
1	3	1	2	0	0

v(A,B) = 5= the value of the matrix game

5	1
5	3
0	2
5	5*'
0	0
2	5

v(A,C) = 4= the value of the matrix game

4*'	5*
2	4
1	5
2	4
0	2
0	2

v(B,C) = 3 = the value of the matrix game

B&C vs A	c1	c2	c3
r11	5	1	0
r12	1	2	0
r21	6	6	2

order	contribution		
	A	B	C
ABC	1	4	3
ACB	1	4	3
BAC	4	1	3
BCA	5	1	2
CAB	4	4	0
CBA	5	3	0

	10/3	17/6	11/6 Shapley values.

The Nash solution is (2, 3, 3)., the only Pareto optimal payoff.

Exercises to §21.

Exercise 1. For the bimatrix game

Players R, C	C1	C2	C3	C4
R1	2, 3	0,2	0, -2	0, 3
R2	1, 2	0, 0	0, 0	0, 0
R3	2, 2	-1, 0	4, 2	1, 1

compute
all equilibria in pure strategies,
the Nash bargaining solution,
the Shapley values.

Exercise 2. For Exercise 2 in §20, find the Nash bargaining solution.

Ch8. Advanced topics

§22. Repeated game wiki. Fictitious play.

If the same game is played several times the players may remember the past (the previous strategy profiles and payoffs) and make some predictions
about the present round. In other words, your choice may depend on what happened in the past.
In the Morris textbook there is Section 5.2.4. *Supergames* where this issue is discussed in the case of Prisoner's Dilemma
(the arms race interpretation is mentioned too).

Here are some notations. We start with any game G in normal form. The payoff F_i for player i is a real-valued function on the set of all joint strategies.
The joint payoff F is the collection all F_i .

Repeated (or iterated) game RG (called supergame in the textbook) may have two parameters: the number of repetitions T (time horizon) which is an integer ≥ 1 or infinity (infinitely repeated game)
and discount factor δ which is a number in the interval $0 \leq \delta \leq 1$.
A joint strategy $S^{(t)}$ at round t may depend on all $S^{(t')}$ with $t' < t$. and the previous actual payoffs.
Note that when mixed strategies or chance are involved, $F(S^{(t)})$ is the expected joint payoff and does not necessarily determine the actual joint payoff.

We might want to find a dependence (learning) such that $S^{(t)}$ converges in some sense to a desired "solution" for original game G. For example, in the fictitious play, which is a method of solving any matrix game, $S^{(t)}$ converges to the set of equilibria.

We are interested in the repeated game RG which has the same set of players where a joint strategy RS consists of all $S^{(t)}$ where $S^{(t)}$ is a function of $S^{(t')}$ with
$t' < t$ and the ß previous actual joint payoffs and where the payoff RF is defined as
(22.1) $RF^{(T)}(RS) = F(S^{(1)}) + \delta F(S^{(2)} + \ldots + \delta^{T-1} F(S^{(T)})$
when T is finite,
(22.2) $RF(RS) = \lim\inf RF^{(t)}(RS)$ as $t \to \infty$
when T is infinity and $\delta < 1$. and
(22.3) $RF(RS) = \lim\inf RF^{(t)}(RS)/t$
when T is infinite and $\delta = 1$.

Note that RF is well-defined when F is bounded (e.g., G is finite) or T is finite. When RF is not defined, RG is not a game.
Often we scale RF replacing total by an average, so $\inf R_i \le RF_i \le \sup F_i$ for every player i.
Namely, we divide the right hand side of (22.1) by T
and multiply the right hand side of (22.2) by $1 - \delta$.
The right hand side of (22.31) is already scaled.

We would like to solve this repeated game RG on one sense or another.

Remark. Double or Nothing can be thought an example of repeated game when T is finite.
When T is infinite, we have a big trouble to define the payoff if we never win.

Fictitious play was introduced by G. Brown as a way to solve any matrix game.
The first pure joint strategy $S^{(1)}$ is arbitrary.
The strategy $S^{(t+1)}$ is a best pure response to the mixed strategy $(S^{(1)} + \ldots + S^{(t)})/t$.
J. Robinson proved that $(S^{(1)} + \ldots + S^{(t)})/t$. converges to the equilibria for any matrix game.
The method is simple, intuitive, and robust. It seems that animals use it in experiments.
But convergence is slow.

J. von Newman modified the method to improve convergence.
Recent interior point methods give much better convergence but they are much more complicated.
So the simplex method is still most common way to solve linear programs and matrix games.

[The Iterated Prisoner's Dilemma and The Evolution of Cooperation](#) video

The payoffs for Prisoner's Dilemma in 21.6 are taken from p.129 of the textbook. The general Prisoner's Dilemma

```
     C      D
C (a, a)  (b, c)
D (c, b)  (d, d)
```
with $c > a > d > b$ and $2a > b + c$

is discussed on this page too.
The arbitration pair and the Shapley values are both (a, a).
D strongly dominates C, so (D,D) is the only equilibrium, with the payoff (d, d).

If we study a repeated game where the number of rounds depends on the previous history and could be infinite, there are additional troubles as Double of Nothing strategy shows.

Exercises to §22.

Exercise 1. In Prisoners Dilemma (see 21.6 above) there are many complicated strategies for the repeated versions.

Here are 4 simple (with S4 possibly excepted) strategies:
S1. I always choose C.
S2. I always choose D.
S3. I use Tit for Tat, i.e. I start with C and then I play the previous choice of my partner.
S4. I play the best strategy I can find against Tit for Tat (describe your strategy).

Make the 4 by 4 table with the mean joint payoffs matching each strategy with each one assuming that
the game is played 100 times.

Bonus points, namely p - 300, will be given where p is your total payoff in S4 vs S3 match when your best response S4 to S3 is different from the other student's S4.
In the above notations, $T = 100$ and $\delta = 1$.

youtube
What is AUCTION THEORY? What does AUCTION THEORY mean? AUCTION THEORY meaning & explanation
3,049 view •Sep 17, 2017

Game Theory 101 MOOC (#41): Second Price Auctions
50,827 views •Jul 16, 2013

Auction Theory : Winner's Curse
YouTube · 566 views · 6/11/2017 · by Learn IOE

§23. Evolutionary games.

Evolutionary game theory is used in population biology, economics, sociology, and computer science.
It is a big area with many publication, c.f., e.g., Evolutionary games wiki 60+ books.

E.g , you want to write a good computer program to play chess.
There are many parameters in program, like values of chess pieces and evaluation method for chess positions.
You can ask a computer to change parameters at random (mutations), play different versions of the program with each other ,
and select better versions. You watch survival of better versions.

Here are some remarks connected with the presentation by the guest speaker.
If he was a student in class, I would give him 60 pts.
On the positive side, he generated some class participation.
On the other side, we ran out time.
The maximum score I gave, over 50 years of teaching, was 50 pts, to a student of Math 484 for a presentation on logic.

Fisher's principle looks dubious to me. By the way, Fisher did not discover Fisher's principle.
There is a much simpler explanation, for male/female ratio see the exercise below.
Here are some references.

Fisher's principle - Wikipedia
Historical research by A.W.F. Edwards[2][7] has shown that the argument is incorrectly attributed to Fisher (the name is in common use and is unlikely to change). Charles Darwin had originally formulated a similar but somewhat confused argument in the first edition of *The Descent of Man*[8] but withdrew it for the second edition[9] – Fisher only had a copy of the latter, and quotes Darwin in *The Genetical Theory of Natural Selection*.[1]
Specific

Human sex ratio - Wikipedia
In a study around 2002, the natural sex ratio at birth was estimated to be close to 1.06 males/ female.[9]

XY sex-determination system - Wikipedia
X chromosome (proposed by a student).

The same Fisher is responsible for 95% standard in the confidence level.
But Fisher was flexible about the number 95. He wrote that if you do not like the number, 95

use the number you like, e.g,, 90 or 99.
Some think that 95% came from the number 2 for standard deviation in normal distributions.
But this not exactly true. Others do not know why 95 is use so often.
More recently, 97% became fashionable to indicate a greater level of confidence or overwhelming support of the voters.
Politicians say sometimes that they are 110% or 200% sure but numbers ≥ 100% do not sound scientific.

Evolution game theory helps to understand evolution of species , genes. and the learning behavior.

Evolution game theory explains why many life forms not always rational in primitive sense.
Here is the link to a video on altruism:
TEDxTalpiot - Oren Harman - The Evolution of Altruism It has a lot of emotions but little math.

Nash and Shapley suggested two ways to solve games. Both agree with equilibrium approach in the case of matrix games.
Evolutionary game theory suggests a different framework. Let different strategies to compute and the best will survive.
In the simplest case, we converge to the best strategy.
Warning: Equilibrium and evolutionary stable point is not the same as (Nash) equilibrium in game theory above.
In a more complicated case we have a limit cycle.
But often we have a chaotic evolution which is hard to understand.

The speaker (in Section 1 on Nov 16) used the following game:
The **game of chicken**, also known as the **hawk–dove game** or **snowdrift game**,[1] is a model of conflict for two players in game theory.
The difference between Prisoner's Dilemma and Dove&Hawk is that D dominate C in the first game
whereas there is no domination in the second game.

Exercises to §23.

Exercise 1. Consider a population of a big size N with 80% of females and 20% of males.

So we have 0.8N females and 0.2N males.

Suppose the death rate is a% for females and b% for males per year.

Suppose that the number of babies born each year is c% of the number of females.

Assume 50/50 ratio of male/females for newborns.

So the next year we have

0.8N - (a/100)0.8N + (c/100)0.8N/2 females

and 0.2N - (b/100)0.2N + (c/100)0.8N/2 males.

Compute the number of males and females in T years when

(a) a = b = 2, c = 1 and T = 10,

(b) a = 1, b = 2, c = 3, and T = 10.

(c) (bonus) Compute the limit of the ratio #males/#females as T → ∞ in the case a< 2c.
(d) (bonus) Find a,b, and c producing the 1.06 males/female ratio.

Hint. If we have f females and m males this year, then the numbers f' and m' for the next year are obtained as follows:

$$\begin{pmatrix} f' \\ m' \end{pmatrix} = \begin{pmatrix} 1-a/100 + c/200 & 0 \\ c/200 & 1-b/100 \end{pmatrix} \begin{pmatrix} f \\ m \end{pmatrix}$$

Any population with f = 0, disappears (f stays 0 and m -> 0 as T → ∞ assuming 0 < b ≤ 100).
In case (a), the population with f = m is stable.
The limit ratio probably is the ratio of two entries of an eigenvector.

If we do not distinguish males and females, we have a simpler model but them we do not have male/female ration.
We simplify real life situation ignoring the fact that both birth and death rates depend on age. Also we inure many other factors like learning behavior (smoking, healthy food, etc) and interactions with other species (mosquitos, viruses, crocodiles, sharks, etc)
that effect your fitness.

In the first known example of population dynamics, Fibonacci [**Fibonacci number - Wikipedia**] wrote about rabbits. He took in account the age.
The golden ratio appears as a limit ratio.
His main goal was to promote the positional (numeral) system we use now rather than to alarm us by exponential growth of the rabbit population.
He did not know about rabbits in Australia (which we discussed in Section 2) because Australia was not discovered by Europeans yet, and had no rabbits at that time anyhow.
He did not know that the Fibonacci numbers were known in India for many years before he was born.

§24. Auctions

Auction theory - Wikipedia
Types of *auction*. There are traditionally four types of *auction* that are used for the allocation of a single item:
. Second-price sealed-bid *auctions* (Vickrey *auctions*) in which bidders place their bid in a sealed envelope and simultaneously hand them to the auctioneer.
General idea · Types of auction · Game-theoretic models · Revenue equivalence

Auction - Wikipedia
Sealed first-price *auction* or blind *auction*, also known as a first-price sealed-bid *auction* (FPSB). In this type of *auction* all bidders simultaneously submit sealed bids so that no bidder knows the bid of any other participant. The highest bidder pays the price they submitted.
History · Types · Common uses · Bidding strategy

Exercises to §24.

Exercise 1. It is a simple auction with the deadline at noon. You can see the previous offer and you can beat it by at least $1.
Do you
(A) bid now, at 9a the highest price you can afford and wait till noon to see whether somebody overbids you and pays a ridiculously high price,
(B) bid as often as possible until price becomes too high for you,
(C) wait for the last moment and overbid by $1 if the price is not too high.

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Rational interpretation again.
"Suppose that somebody even cleverer than Nash or Von Neumann had written a book that lists all possible games along with an authoritative recommendation on how each game should be played by rational players. Such a great book of game theory would necessarily have to pick a Nash equilibrium as the solution of each game. Otherwise it would be rational for at least one player to deviate from the book's advice, which would then fail to be authoritative."
Ken Binmore Game Theory: A Very Short Introduction Oxford 2007. pages 14-15.
Now when we know the ultimate truth about all games from somebody "even cleverer than Nash or Von Neumann " via the great prophet,
we have only few problems still open, like

How will the universe end?
Can machines think?
Who am I?

The idea of a great book of ultimate (absolute) truths (or secrets) is the theme of many films, book, and religions.
Binmore does not claim the book exists, but he knows what should be there.

So what is the best move, C or D in Prisoner's Dilemma?

Binmore says it is D. He and many others treat us like children or prisoners who should not cooperate with each other and never question the boss.
This is convenient if you are the boss. You do not want others to be assertive and aggressive.

On the other hand, I know an instructor who teaches his students: never take "no" as the answer.
He risks being harassed by students demanding a better grade.
He hopes this hawkish style will help students in life.

So what is my answer? Be smart and decide yourself when to play C and when to play D.
Do not be always C or always D player. Consider Tit for Tat and other strategies.
Sometimes, you do not want to be predictable.

A student said that cooperation cannot be enforced. But it can be. In usual interpretation of Prisoner's Dilemma, there is Code of Silence for C.
In general, there are contracts, agreements, and treaties which can be enforced legally.

Here are two videos by
Martin Andreas Nowak (born April 7, 1965) is the Professor of Biology and Mathematics and Director of the Program for Evolutionary Dynamics at Harvard University.
Both videos mention repeated Prisoner's Dilemma.

Martin Nowak: 'The Evolution of Cooperation' | 2015 ISNIE Annual Meeting - YouTube

Supercooperators: The mathematics of evolution, altruism and human behaviour

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================= end of class notes

120 past test problems for Math 486

F17 60 problems
m1 Section 1 Oct 5 15 problems | Section 2 Sep 25 10 problems
m2 Section 1 Nov 14 15 problems | Section 2 Nov 1. 10 problems
m3 Nov 17 Section 2 10 problems

S10. 12 problems
2/4 Midterm 1 5 problems | sol | sol by D.S. |
3/18 Midterm 2 | solutions | 5 problems
4/22 Midterm 3 2 problems | solutions | pictures

S8. 16 problems
Midterm 1 Th Febr 7, Ch. 1. solutions 5 problems| pictures |
Midterm 2 Th Mar 21. Ch. 2-4. problems | solutions | 5 problems | Sch ||
Midterm 3 Th. April 24. Ch. 5-6. 6 problems pictures

S7 15 problems
Midterm 1 Th Febr 8, Ch. 1. 5 problems solutions | pictures |
Students of Math 484.2 in Fall of 2006: midterm 3 pictures added.
2/9/7 1/26 4:40 p 106 MB. My second talk on games.
h2 solutions: html | msword || 3 problems on simplex method
Midterm 2 Th Mar 22. Ch. 2-4. 5 problems
Midterm 3 Th. April 26. Ch. 5-8. Ch 5-8 2 problems

S04. 12 problems
Homework 1, 10 pts, due Febr. 12, 8 am. Solve the matrix game
Midterm 1 Febr 5 5 problems
Midterm 2 (with solutions), Tue, March 2. This about linear programming and matrix games. 5 problems
Midterm 3 (with solutions and pictures), Tue April 20, 2004. 2 problems

F02 5 problems
Sept 25. W. midterm. 1 min = 14, average = 30, max = 42. 5 problems
midterm 2 Oct pictures.
midterm 3 November 20. pictures. |

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135 past test problems from Math 484

Standard tableaux in Math 484 and Math 486 were different:

484	486

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    ⊕  1                  ⊕  -1
⊖   A  b   = ⊕        ⊕  -A   b   = ⊖
1   c  d → min       -1  -c   d → max
    ||  ↓                ||    ↓
    ⊕  max                ⊕  min
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F16 45 problems

m1 Sept 30 25 problems

m2 Nov 2 10 problems

m3 Dec 2 10 problems

S16 90 problems

m1 Section 1 Section 2 Feb 19 25 + 25 problems

m2 Section 1 Section March 30 10 + 10 problems

m3 Section 1 Section 2 April 22 10 + 10 problems