

# Singular Overpartitions

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## Abstract

The object in this paper is to present a general theorem for overpartitions analogous to Rogers-Ramanujan type theorems for ordinary partitions with restricted successive ranks.

## 1 Introduction

L. J. Slater's list of 120 Rogers-Ramanujan type identities [20] has been a source for numerous applications in the theory of partitions. The celebrated Rogers-Ramanujan identities themselves are equations (14) and (18). The Göllnitz-Gordon identities (preceding Gordon [12] and Göllnitz [11] by 10 years) are equations (34) and (36). Identities (1)-(3) are due to Euler, (4) is a special case of the  $q$ -Gauss summation [2, Cor. 2.4, p. 20], and (5) is easily deduced from identities of Euler. The reader should consult the excellent paper by McLaughlin, Sills, and Zimmer [18] for the full background on these identities (cf. Sills [19]).

The first identity in Slater's list that is not easily pigeon-holed in the literature is (6) (corrected):

$$\sum_{n=0}^{\infty} \frac{(-1; q)_n q^{n^2}}{(q; q)_n (q; q^2)_n} = \frac{\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}}{(q; q)_{\infty}} \quad (1.1)$$

Most of the first twenty identities apart from (6) especially those related to the modulus 5 arise in the work of Rogers (as noted by Sills [19, p. 100]);

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however identity (21) seems the next entry in a family beginning with (1.1), namely

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (-q^{2n+1}; q)_{\infty}}{(q^2; q^2)_n (q^{2n+1}; q^2)_{\infty}} = \frac{\sum_{n=-\infty}^{\infty} q^{n(5n+1)/2}}{(q; q)_{\infty}}. \quad (1.2)$$

This is likely the only modulus 5 identity listed by Slater that Rogers missed.

Indeed, even Slater missed the following two companions to (1.2): The first

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n} (-q^{2n+1}; q)_{\infty}}{(q^2; q^2)_n (q^{2n+1}; q^2)_{\infty}} = \frac{\sum_{n=-\infty}^{\infty} q^{n(5n+3)/2}}{(q; q)_{\infty}} \quad (1.3)$$

was found and proved by Bowman, McLaughlin and Sills [6]. The second

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n} (-q^{2n+2}; q)_{\infty}}{(q^2; q^2)_n (q^{2n+1}; q^2)_{\infty}} = \frac{\sum_{n=-\infty}^{\infty} q^{n(5n+5)/2}}{(q; q)_{\infty}} \quad (1.4)$$

was found and proved by McLaughlin, Sills and Zimmer [18].

A natural question that arises from these observations is this: Are there Rogers-Ramanujan partition theorems where the function

$$\frac{\sum_{n=-\infty}^{\infty} q^{n((2k+1)n+(2i-1))/2}}{(q; q)_{\infty}} \quad (1.5)$$

is the generating function for one of the sets of partitions?

Indeed, the central motivation of this paper is to show that interesting combinatorial and arithmetic results are related to the function given by (1.5).

Obviously, the coefficients in the power series for (1.5) will, after a few initial values, be larger than  $p(n)$ . So something other than ordinary partitions will be required.

S. Corteel and J. Lovejoy in [9] laid the groundwork for a flourishing new component of the theory of partitions – overpartitions.

An overpartition in  $n$  is a partition of  $n$  in which the final occurrence of a part may be overlined. Thus the eight overpartitions of 3 are  $3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1, 1 + 1 + \bar{1}$ . While there had been fragmentary hints of such a subject going as far back as Hardy and Ramanujan [14, p. 304], it was Corteel and Lovejoy who truly revealed the generality, depth and beauty of the subject in a series of papers beginning with [9]. Papers [9], [10], [15], [16], [17] among others reveal how much this new aspect of the theory of

partitions has developed, and overpartitions are just what is required for applications of (1.1), (1.2), and (1.3).

Two of the early papers in this series [15], [16] prove theorems for overpartitions analogous to B. Gordon's celebrated generalization of the Rogers-Ramanujan identities [13]:

**Theorem.** *Let  $B_{k,i}(n)$  denote the number of partitions of  $n$  where at most  $i - 1$  of the parts are equal to 1 and the total number of occurrences of  $j$  and  $j + 1$  together is at most  $k - 1$ . Let  $A_{k,i}(n)$  denote the number of partitions of  $n$  into parts not congruent to  $0, \pm i$  modulo  $2k + 1$ . Then  $A_{k,i}(n) = B_{k,i}(n)$ .*

There is a related theorem that is most easily stated in terms of Frobenius symbols. A Frobenius symbol for  $n$  is a two-rowed array:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

where  $\sum(a_i + b_i + 1) = n$  and  $a_1 > a_2 > \cdots > a_r \geq 0, b_1 > b_2 > \cdots > b_r \geq 0$ . There is a natural mapping [3] that reveals a one-to-one correspondence between the Frobenius symbols for  $n$  and the ordinary partitions of  $n$ . The theorem of Bressoud [7] (originally done for  $k$  odd in [1]) asserts:

**Successive Ranks Theorem.** *Given  $1 \leq i < k/2$ , the number of Frobenius symbols for  $n$  in which  $-i + 2 \leq a_j - b_j \leq k - i - 2$  for  $1 \leq j \leq r$  equals  $A_{k,i}(n)$  (given above in Gordon's theorem).*

The theorem from [15] that will be of most relevance in this paper is the following:

**Theorem.** *Let  $\overline{B}_k(n)$  denote the number of overpartitions of  $n$  of the form  $y_1 + y_2 + \cdots + y_s$ , where  $y_j - y_{j+k-1} \geq 1$  if  $y_{j+k-1}$  is overlined and  $y_j - y_{j+k-1} \geq 2$  otherwise. Let  $\overline{A}_k(n)$  denote the number of overpartitions of  $n$  into parts not divisible by  $k$ . Then  $\overline{A}_k(n) = \overline{B}_k(n)$ .*

In Section 2 we shall introduce "singular overpartitions" which will be Frobenius symbols for  $n$  with at most one overlined entry in each row, thus the name "singular."

We shall let  $\overline{Q}_{k,i}(n)$  denote the number of such singular overpartitions subject to additional overlining conditions prescribed by  $k$  and  $i$  (see Section 2). Our main result (to be proved in Section 3) is:

**Theorem 1.**  $\overline{Q}_{k,i}(n) = \overline{C}_{k,i}(n)$ , where  $\overline{C}_{k,i}(n)$  is the number of overpartitions of  $n$  in which no part is divisible by  $k$  and only parts  $\equiv \pm i \pmod{k}$  may be overlined.

We note that  $\overline{C}_{3,1}(n) = \overline{A}_3(n)$ . Section 3 will be devoted to a proof of Theorem 1. There are some interesting congruences for  $\overline{Q}_{3,1}(n)$ . Namely,

**Theorem 2.**  $\overline{Q}_{3,1}(9n+3) \equiv \overline{Q}_{3,1}(9n+6) \equiv 0 \pmod{3}$ .

Section 4 will be devoted to this result. In section 5, we look briefly at how instances of the function in (1.5) may be identified with a multiple  $q$ -hypergeometric series. Also we shall show the  $k=3, i=1$  case is related to work in Ramanujan's Lost Notebook. We conclude with a look at open problems.

## 2 Singular Overpartitions

We shall be considering Frobenius symbols [3]

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

where the rows are strictly decreasing sequences of nonnegative integers. We say such a symbol is a Frobenius symbol for  $n$  if  $\sum(a_i + b_i + 1) = n$ .

We shall say that a column  $\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}$  in a Frobenius symbol is  $(k, i)$ -positive if  $a_j - b_j \geq k - i - 1$  and we shall say the column is  $(k, i)$ -negative if  $a_j - b_j \leq -i + 1$ . If  $-i + 1 < a_j - b_j < k - i + 1$ , we shall say the column is  $(k, i)$ -neutral.

If two columns  $\begin{smallmatrix} a_n \\ b_n \end{smallmatrix}$  and  $\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}$  are both  $(k, i)$ -positive or both  $(k, i)$ -negative we shall say that they have the same  $(k, i)$ -parity.

We now divide the Frobenius symbol into  $(k, i)$ -parity blocks. These are sets of contiguous columns maximally extended to the right:

$$\begin{array}{cccc} a_n & a_{n+1} & \cdots & a_j \\ b_n & b_{n+1} & \cdots & b_j \end{array}$$

where all the entries have either the same  $(k, i)$ -parity or are  $(k, i)$ -neutral.

For example, consider the Frobenius symbol

$$\begin{array}{cccccccccc} 15 & 14 & 12 & 8 & 7 & 6 & 5 & 4 & 1 & 0 \\ 15 & 13 & 10 & 9 & 7 & 6 & 4 & 2 & 1 & 0 \end{array}$$

The  $(3, 1)$ - parity blocks are

$$\begin{array}{c|c|c|c|c} 15 & 14 & 12 & 8 & 7 & 6 & 5 & 4 & 1 & 0 \\ 15 & 13 & 10 & 9 & 7 & 6 & 4 & 2 & 1 & 0 \end{array}$$

The  $(5, 2)$ -parity blocks are

$$\begin{array}{c|c|c} 15 & 14 & 12 & 8 & 7 & 6 & 5 & 4 & 1 & 0 \\ 15 & 13 & 10 & 9 & 7 & 6 & 4 & 2 & 1 & 0 \end{array}$$

We shall call the first non-neutral column in each parity block, the *anchor* of the block.

We shall say that a  $(k, i)$ -parity block is neutral if all columns in it are neutral. Owing to the maximality condition, this can only occur if all the columns in the Frobenius symbol are  $(k, i)$ -neutral. In all other cases we shall say that a  $(k, i)$ -parity block is positive if it contains no  $(k, i)$ -negative columns and  $(k, i)$ -negative if it contains no  $(k, i)$ -positive columns.

We shall say a Frobenius symbol is  $(k, i)$ -singular if (1) there are no overlined entries, or (2) the one overlined entry on the top row occurs in the anchor of a  $(k, i)$ -positive block, or (3) the one overlined entry on the bottom row occurs in an anchor of a  $(k, i)$ -negative block, and (4) if there is one overlined entry in each row, then they occur in adjacent  $(k, i)$ -parity blocks.

Hence, the following are all  $(5, 2)$ -singular

$$\begin{array}{c|c|c} 15 & 14 & 12 & 8 & 7 & 6 & 5 & 4 & 1 & 0 \\ 15 & 13 & 10 & 9 & 7 & 6 & 4 & 2 & 1 & 0 \\ 15 & 14 & \overline{12} & 8 & 7 & 6 & 5 & 4 & 1 & 0 \\ 15 & 13 & 10 & 9 & 7 & 6 & 4 & 2 & 1 & 0 \end{array}$$

15	14	12		8	7	6	5		4	1	0
15	13	10		$\overline{9}$	7	6	4		2	1	0
15	14	12		8	7	6	5		$\overline{4}$	1	0
15	13	10		9	7	6	4		2	1	0
15	14	$\overline{12}$		8	7	6	5		4	1	0
15	13	10		$\overline{9}$	7	6	4		2	1	0
15	14	12		8	7	6	5		$\overline{4}$	1	0
15	13	10		$\overline{9}$	7	6	4		2	1	0

### 3 Proof of Theorem 1.

With  $1 \leq i < \frac{K}{2}$ , we define  $\overline{Q}_{K,i}(n)$  to be the number of  $(K, i)$ -singular Frobenius symbols related to  $n$ .

**Theorem 1.**

$$\overline{Q}_{K,i}(n) \equiv \overline{C}_{K,i}(n)$$

where  $\overline{C}_{K,i}(n)$  is given in Section 1, is the number of overpartitions of  $n$  in which there are no multiples of  $K$  and only parts  $\equiv \pm i \pmod{K}$  may be overlined.

For example when  $K = 3, i = 1, n = 5$ , the 16  $(3, 1)$ -singular Frobenius symbols for 5 are

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} \overline{4} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ \overline{4} \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} \overline{3} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ \overline{3} \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ \overline{2} \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} \overline{2} & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & \overline{0} \end{pmatrix}, \begin{pmatrix} \overline{2} & 0 \\ 1 & \overline{0} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \overline{2} & 0 \end{pmatrix}$$

Note that the reason  $\begin{pmatrix} 1 & 0 \\ 2 & \overline{0} \end{pmatrix}$  is not listed is because  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  is not the anchor of its  $(3, 1)$ -parity block.

The 16 partitions enumerated by  $\overline{A}_{3,1}(5)$  are

$$\begin{aligned} &5, \overline{5}, 4 \ 1, \overline{4} \ 1, 4 \ \overline{1}, \overline{4} \ \overline{1}, 2 \ 2 \ 1, \overline{2} \ 2 \ 1, 2 \ 2 \ \overline{1}, \overline{2} \ 2 \ \overline{1}, 2 \ 1 \ 1 \ 1, \\ &\overline{2} \ 1 \ 1 \ 1, 2 \ \overline{1} \ 1 \ 1, \overline{2} \ \overline{1} \ 1 \ 1, 1 \ 1 \ 1 \ 1 \ 1, \overline{1} \ 1 \ 1 \ 1 \ 1. \end{aligned}$$

The necessary results to prove Theorem 1 are essentially facts first proved in [1] (cf [2, Ch 9]) that have lain without application prior to the study of overpartitions.

In order to utilize these facts we need to rewrite that work as follows. The work there is all valid provided  $2k + 1$  is an integer  $\geq 3$ . In particular, we shall take

$$k = \frac{K - 1}{2},$$

so that  $K$  is any positive integer  $\geq 3$ .

We shall now rewrite the first five definitions occurring in Section 9.3 of [9] in the language of Frobenius symbols.

**Definition 1.** *If  $\pi$  is a Frobenius symbol with only  $(K, i)$ -neutral columns, then  $\pi$  is said to have 0  $(K, i)$ -positive oscillation and 0  $(K, i)$ -negative oscillation.*

**Definition 2.** *Suppose  $\pi$  is a Frobenius symbol with  $r$   $(K, i)$ -parity blocks (and not all columns are  $(K, i)$ -neutral). If the first  $(K, i)$ -parity block is  $(K, i)$ -positive, we say that  $\pi$  has  $(K, i)$ -positive oscillation  $r$  and  $(K, i)$ -negative oscillation  $r - 1$ . If, on the other hand, the first  $(K, i)$ -parity block is  $(K, i)$ -negative, we say that  $\pi$  has  $(K, i)$ -negative oscillation  $r$  and  $(K, i)$ -positive oscillation  $r - 1$ .*

**Definition 3.** *Let  $p_{K,i}(a, b; \mu; N)$  (resp.  $m_{K,i}(a, b; \mu; N)$ ) denote the number of Frobenius symbols for  $N$  with the top entry of the first column  $< a$ , the top entry of the second column  $< b$ , and with  $(K, i)$ -positive (resp.  $(K, i)$ -negative) oscillation at least  $\mu$ .*

For these partition functions we have the associated generating functions:

$$P_{K,i}(a, b; \mu; q) = \sum_{N \geq 0} p_{K,i}(a, b; \mu; N) q^N, \quad (3.1)$$

$$M_{K,i}(a, b; \mu; q) = \sum_{N \geq 0} m_{K,i}(a, b; \mu; N) q^N. \quad (3.2)$$

$$P_{K,i}(\mu; q) = \lim_{a \rightarrow \infty} P_{K,i}(a, a; \mu; q) \quad (3.3)$$

$$M_{K,i}(\mu; q) = \lim_{a \rightarrow \infty} M_{K,i}(a, a; \mu; q). \quad (3.4)$$

Then Theorem 9.10 of [2, p. 152] may be restated as follows:

$$M_{K,i}(2\mu; q) = \frac{q^{2K\mu^2 + (K-2i)\mu}}{(q; q)_\infty} \quad \text{for } \mu \geq 0, \quad (3.5)$$

$$M_{K,i}(2\mu - 1; q) = \frac{q^{(2\mu-1)(K\mu-K+i)}}{(q; q)_\infty} \quad \text{for } \mu > 0, \quad (3.6)$$

$$P_{K,i}(2\mu; q) = \frac{q^{2K\mu^2 - (K-2i)\mu}}{(q; q)_\infty} \quad \text{for } \mu \geq 0, \quad (3.7)$$

$$P_{K,i}(2\mu - 1; q) = \frac{q^{(2\mu-1)(K\mu-i)}}{(q; q)_\infty} \quad \text{for } \mu > 0. \quad (3.8)$$

The proof of Theorem 1 now concludes in a way parallel to the proof of Theorem 9.12 of [2, p. 154]. The difference is that now we must start with the following observation:

$$\sum_{n=0}^{\infty} \overline{Q}_{K,i}(n) q^n = P_{K,i}(0; q) + \sum_{\mu=1}^{\infty} M_{K,i}(\mu; q) + \sum_{\mu=1}^{\infty} P_{K,i}(\mu; q). \quad (3.9)$$

To see this we consider three cases. First, if all columns are  $(K, i)$ -neutral, then there can be no overlined entries in such a Frobenius symbol. Consequently it is counted once by  $\overline{Q}_{K,i}(n)$  and on the right exactly once by  $P_{K,i}(0; q)$ .

Now suppose the first  $(K, i)$ -parity block is  $(K, i)$ -positive and there are  $\mu$  blocks. How many singular Frobenius symbols can arise from this given Frobenius symbol? There can be 0 overlines (counted by  $P_{K,i}(0; q)$ ), or one overline on each of the  $\mu$  anchors (counted once by each of  $P_{K,i}(1; q)$ ,  $P_{K,i}(2; q)$ ,  $\dots$ ,  $P_{K,i}(\mu; q)$ ) or there can be overlines in both rows on successive anchors (counted once by each of  $M_{K,i}(1; q)$ ,  $M_{K,i}(2; q)$ ,  $\dots$ ,  $M_{K,i}(\mu - 1; q)$ ). Now since the  $(K, i)$ -positive oscillation is  $\mu$ , the  $(K, i)$ -negative oscillation is  $\mu - 1$ . Consequently we see that the number of  $(K, i)$ -singular overpartitions that can be produced from the given Frobenius symbol is exactly what is counted by the right-hand side of (3.9).

Finally suppose the first  $(K, i)$ -parity block is  $(K, i)$ -negative. Then the preceding paragraph can be repeated with roles of  $(K, i)$ -positive and  $(K, i)$ -negative reversed.



In each case, the right hand side counts exactly the singular-overpartition for  $n$ , thus establishing (3.9).

One now substitutes the expression from (3.5)-(3.8) into the right hand side of (3.9). After simplification (entirely like that done in the proof of Theorem 9.12 in [2, p. 154] with the “ $(-1)^n$ ” removed) we see that

$$\begin{aligned} \sum_{n=0}^{\infty} Q_{K,i}(n)q^n &= \frac{\sum_{n=-\infty}^{\infty} q^{Kn(n-1)/2+in}}{(q; q)_{\infty}} \\ &= \frac{(q^K, -q^i, -q^{K-i}; q^K)_{\infty}}{(q; q)_{\infty}} \\ &= \sum_{n=0}^{\infty} \bar{C}_{K,i}(n)q^n, \end{aligned}$$

as desired.

## 4 Proof of Theorem 2

The first step is to find the generating function for  $\bar{Q}_{3,1}(3n)$ . Now by Theorem 1,

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{Q}_{3,1}(n)q^n &= \frac{(q^3, -q, -q^2; q^3)_{\infty}}{(q; q)_{\infty}} \\ &= \frac{(q^3; q^3)_{\infty}(-q; q)_{\infty}}{(-q^3; q^3)_{\infty}(q; q)_{\infty}} \\ &= \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2}}{\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}} \quad (\text{by [2, p. 23]}). \end{aligned}$$

Using Ramanujan’s notation [5, p. 17]

$$\begin{aligned} \phi(q) &:= \sum_{n=-\infty}^{\infty} q^{n^2} \\ &= \sum_{i=-1}^1 \sum_{m=-\infty}^{\infty} q^{(3m+i)^2} \\ &= \phi(q^9) + 2qW(q^3), \end{aligned}$$

where

$$W(q) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{Q}_{3,1}(n)q^n &= \frac{\phi(-q^3)}{\phi(-q)} \\ &= \frac{\phi(-q^3)}{\phi(-q^9) - 2qW(-q^3)} \\ &= \frac{\phi(-q^3) (\phi(-q^9) - 2q\omega W(-q^3)) (\phi(-q^9) - 2q\overline{\omega}W(-q^3))}{\phi(-q^9)^3 - 8q^3W(-q^3)^3} \\ &\quad \text{(where } \omega = e^{2\pi i/3}\text{)} \\ &= \frac{\phi(-q^3) (\phi(-q^9)^2 + 2q\phi(-q^9)W(-q^3) + 4q^2W(-q^3)^2)}{\phi(-q^9)^3 - 8q^3W(-q^3)^3}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{Q}_{3,1}(3n)q^n &= \frac{\phi(-q)\phi(-q^3)^2}{\phi(-q^3)^3 - 8qW(-q)^3} \\ &\equiv \frac{\phi(-q^3)^2 (\phi(-q^9) + qW(-q^3))}{\phi(-q^9) + qW(-q^3)} \pmod{3} \\ &\equiv \phi(-q^3)^2 \pmod{3}. \end{aligned} \tag{4.1}$$

Clearly we see that

$$\overline{Q}_{3,1}(9n+3) \equiv \overline{Q}_{3,1}(9n+6) \equiv 0 \pmod{3}$$

**Corollary 3.**

$$Q_{3,1}(9n) \equiv (-1)^n r_2(n),$$

where  $r_2(n)$  is the number of representations of  $n$  as the sum of two squares.

*Proof.* This follows directly from (4.1) and the fact that  $\phi(q)^2$  is the generating function for  $r_2(n)$ .  $\square$

As an example,  $Q_{3,1}(90) = 230021384 \equiv -1 \pmod{3}$ , and  $r_2(10) = 8 \equiv -1 \pmod{3}$ .

## 5 Related Identities

We can easily produce infinite series/infinite product identities for the generating function for  $\overline{Q}_{2k+1,k}(n)$ . Indeed, we begin with a result first given by Ramanujan [5, p. 99] and utilized by Bressoud [8] in his simple proof of the Rogers-Ramanujan identities. The result in question is

$$\sum_{n=0}^{\infty} \frac{(-z; q)_n (-q/z; q)_n q^{n^2}}{(q; q)_{2n}} = \frac{\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} z^n}{(q; q)_{\infty}}. \quad (5.1)$$

When  $z = -1$ , we retrieve the pentagonal number theorem [2, p. 11], and when  $z = 1$  we have (1.1).

The simplest proof of (5.1) (essentially that of Bressoud [8]) is to note that if

$$\beta_n = \frac{(-z; q)_n (-q/z; q)_n}{(q; q)_{2n}}, \quad (5.2)$$

and

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0 \\ z^n q^{\binom{n}{2}} + z^{-n} q^{\binom{n+1}{2}} & \text{if } n \neq 0 \end{cases} \quad (5.3)$$

Then (5.1) follows from the weak form of Bailey's Lemma with  $a = 1$ , namely

$$\sum_{n=0}^{\infty} q^{n^2} \beta_n = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} q^{n^2} \alpha_n.$$

If we put this Bailey pair in the  $k$ -fold iteration of the Bailey chain [4] (which is equivalent to what Bressoud did), we obtain Bressoud's main theorem [8, p. 238] with  $N \rightarrow \infty$ :

$$\sum_{s_1, s_2, \dots, s_k \geq 0} \frac{q^{s_1^2 + s_2^2 + \dots + s_k^2} (-xq; q)_{s_k} (-x^{-1}; q)_{s_k}}{(q; q)_{s_1 - s_2} (q; q)_{s_2 - s_3} \cdots (q; q)_{s_{k-1} - s_k} (q; q)_{2s_k}} = \frac{1}{(q; q)_{\infty}} \sum_{m=-\infty}^{\infty} x^m q^{((2k+1)m^3 + m)/2}$$

and if we replace  $x$  by  $q^i$ , we obtain a  $k$ -fold  $q$ -series representation of

$$\sum_{n \geq 0} \overline{Q}_{2k+1,i}(n) q^n,$$

which is the analog of the  $k$ -fold generalization of the Rogers-Ramanujan identities [2, p. 111].

## 6 Conclusion

Numerous questions suggest themselves.

*Question 1.* Is there a bijective proof of Theorem 1, especially the case  $K = 3$ ,  $i = 1$ ?

*Question 2.* Can one prove combinatorially  $\overline{Q}_{3,1}(n) = \overline{A}_3(n)$ , where the latter appears in the theorem of Lovejoy stated in the introduction?

*Question 3.* Is there a statistic analogous to the rank of a partition that would provide a combinatorial description of Theorem 2?

*Question 4.* Generalizations of Frobenius symbols have been studied extensively (cf. [3]). Are there corresponding fruitful generalizations of singular overpartitions?

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