On the pseudo-periodicity of the integer hull of parametric convex polygons

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Outline

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Motivation

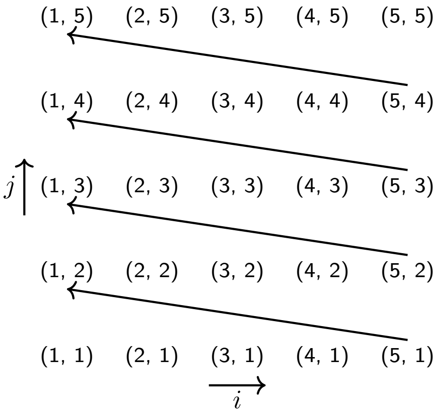
Data dependence analysis for arrays:

for $i = 1$ to 5 do

  for $j = i$ to 5 do

    $A[i,\ j+1] = A[5,\ j]$

Can we parallel the two for loops?



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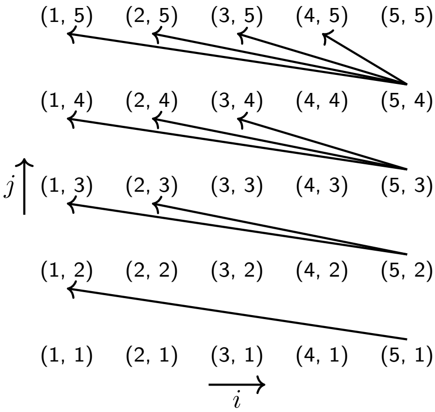

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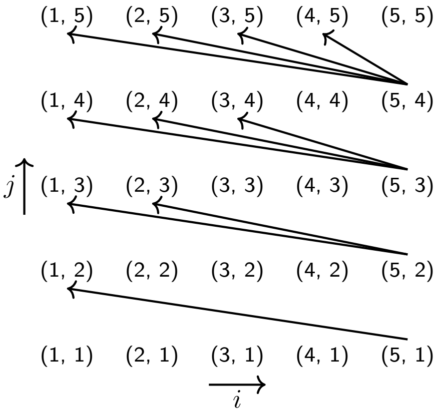
$$i', j', i, j \in \mathbb{Z}$$
$$1 \leq i \leq j \leq 5$$
$$1 \leq i' \leq j' \leq 5$$
$$i = 5$$
$$j + 1 = j'$$



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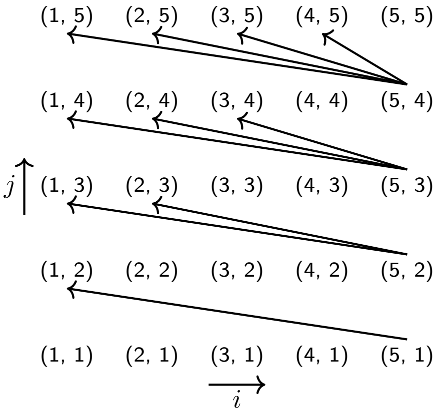
⇒ no solution!!
⇒ no dependence exists!!
⇒ can be parallelized



Simple example

```
for(i = 0; i ≤ n; i ++)
  for(j = i; j ≤ n; j ++)
    A[i][j]...
```

$$\begin{cases} 0 \leq i \leq n \\ i \leq j \leq n \end{cases}$$

- Loop counters could only be integer
- This leads to the problem of finding the integer polyhedral set of the iteration space
- Often time the space is parametric (ex. with variable n)

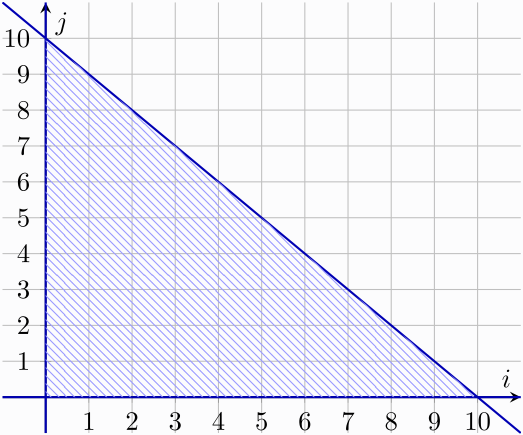
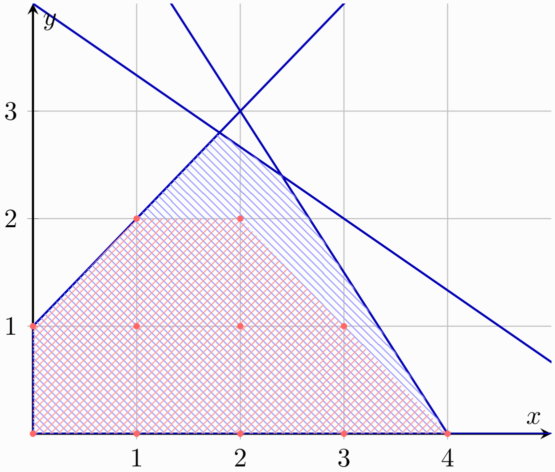


Figure: Iteration space when $n = 10$

Simple example

$$\begin{cases} 0 & \leq & x \\ 0 & \leq & y \\ 3\,x + 2\,y & \leq & 12 \\ 2\,x + 3\,y & \leq & 12 \\ -x + y & \leq & 1 \end{cases}$$

$$\begin{cases} 0 & \leq & x \\ 0 & \leq & y \\ y & \leq & 2 \\ x + y & \leq & 4 \\ -x + y & \leq & 1 \end{cases}$$

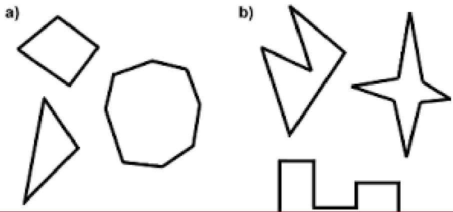


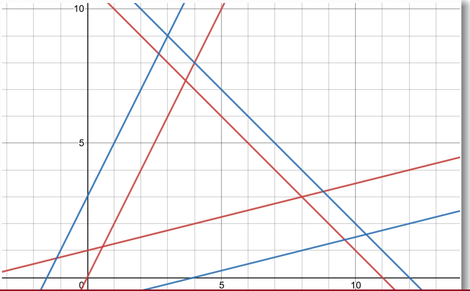
Figure

Outline

Convex Polyhedron and Integer Hull

- A subset $P \subseteq \mathbb{Q}^n$ is called a convex polyhedron (or simply a polyhedron) if
$$P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$$
holds, for a matrix $A \in \mathbb{Q}^{m \times n}$ and a vector $\mathbf{b} \in \mathbb{Q}^m$, where n, m are positive integers.
- We are interested in computing P_I the integer hull of P that is the smallest convex polyhedron containing all the integer points of P.
- Parametric polyhedron, particularly we define $P(\mathbf{b}) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ where $\mathbf{b}$ is unknown at compile time, either the whole vector is unknown or some element of the vector is unknown, we can represent the later case by $P(b_i)$.





Integer Hull Algorithms

Computing P_I

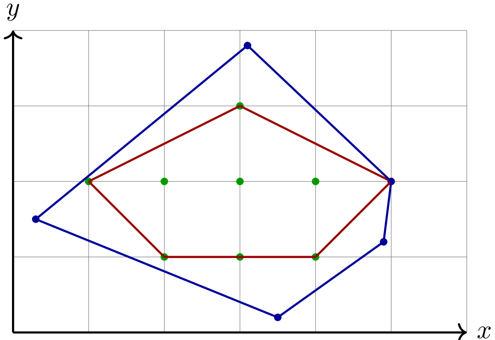
- The cutting plane method, originally introduced by Gomory to solve integer linear programs.
  Chvátal and Schrijver developed a procedure to compute P_I based on this method.
- The branch and bound method, introduced by Land and Doig in the early 1960s. This method recursively divides P into sub-polyhedra, then the vertices of the integer hull of each part of the partition are computed.

Counting lattice points in P

- Pick's theorem: a formula for the area of a simple polygon P with integer vertices, in terms of the number of integer points within P and on its boundary.
- In the 1990s, Barvinok created an algorithm for counting the integer points inside a polyhedron, which runs in polynomial time, for a fixed dimension of the ambient space. (First implemented in `LattE` in 2004)
- Yanagisawa from IBM Tokyo gave a simpler approach for lattice point counting, which divides a polygon into *right-angle triangles* and calculates the number of lattice points within each such triangle.

Vertices of Integer Hull

- Convex hull of all integer points inside a polyhedral set is its **integer hull**
- Vertices or Facets are enough to describe the integer hull
- $P = P_I$ if and only if every vertex of P is integer point.
- The convex hull of all the vertices of P_I is P_I itself.
- One way to represent an integer hull is to give all its vertices
- Our problem becomes:
  Given a polyhedron P, find all the vertices V of P_I.



Vertices of Integer Hull

Number of the vertices of P_I

- The earliest study by Cook, Hartmann, Kannan and McDiarmid, shows that the number of vertices of P_I is related to the *size* of the coefficients of the inequalities that describe P.
- $x = p/q$ is a rational number, p and q are coprime
- $size(x) = 1 + \lceil (log(|p| + 1)) \rceil + \lceil (log(|q| + 1)) \rceil$
- The size of a linear inequality with coefficient vector $\vec{a} = (a_1, \ldots, a_n)$ is
$$size(\vec{a}) = n + size(a_1) + \ldots + size(a_n)$$
- For a polyhedron $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ where matrix $A \in \mathbb{Q}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{Q}^m$, φ is the maximum size of the m inequalities, then the number of vertices of P_I is at most $2m^n(6n^2\varphi)^{n-1}$.

“Periodicity” of P_I

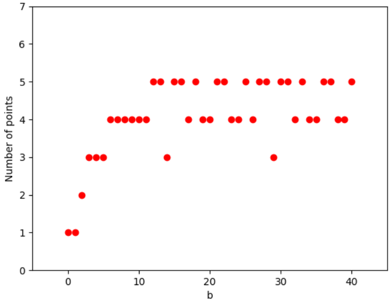
- In 2004, Meister presents a new method for computing the integer hull of a parameterized rational polyhedron. The author introduces a concept of periodic polyhedron (with facets given by equalities depending on periodic numbers). Hence, the word “periodic” means that the polyhedron can be defined in a periodic manner which is different from our perspective.
- For each integer $n \geq 1$, Eugème Ehrhart defined the *dilation* of the polyhedron P by n as the polyhedron $nP = \{nq \in \mathbb{Q}^d \mid q \in P\}$. Ehrhart studied the number of lattice points in nP, that is:
$$i(P, n) = \#(nP \cap \mathbb{Z}^d) = \#\{q \in P \mid nq \in Z^d\}.$$
He proved that there exists an integer $N > 0$ and polynomials $f_0, f_1, \ldots, f_{N-1}$ such that $i(P, n) = f_i(n)$ if $n \equiv i \mod N$. The quantity $i(P, n)$ is called the ***Ehrhart quasi-polynomial*** of P, in the dilation variable n.

Outline

Periodic behavior of the integer hulls of a fix-shaped triangle

Consider the triangle defined by the facets

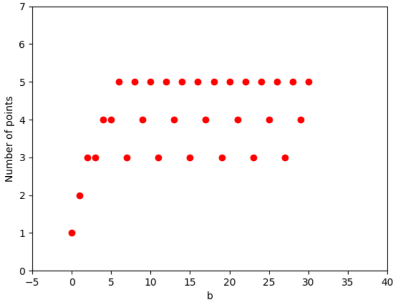
$$\left\{ \begin{array}{lcl} 0 & \leq & 2\,x - y \\ 0 & \leq & y \\ -b & \leq & -3\,x - y \end{array} \right. \quad (1)$$



This example has a period of 15

Consider the triangle defined by the facets

$$\left\{ \begin{array}{lcl} 0 & \leq & 2\,x - y \\ 0 & \leq & y \\ -b & \leq & -2\,x - y \end{array} \right. \quad (2)$$



This example has a period of 4

Visulization of the above example

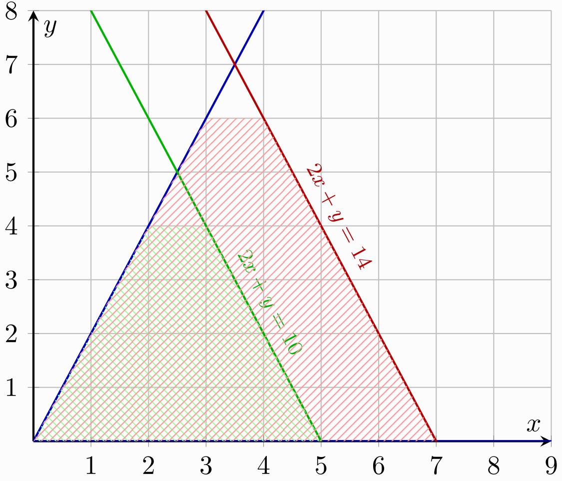


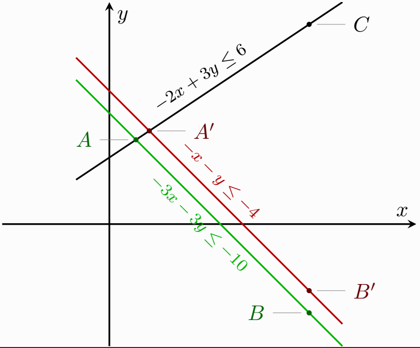
Figure: the shapes of the integer hulls are similar

Angular sector and its integer hull

Definition

An angular sector in an affine plane is defined by the intersection of two half-planes whose boundaries intersect in a single point, called the vertex of that angular sector.

- The integer hull of sector BAC is the same as that of sector $B'A'C$
- The vertices of the integer hull of sector $B'A'C'$ are the same as that of $\triangle B'A'C$ if B' and C are the closest integer points to A' on the facets



Periodicity of the integer hull of an angular sector

Theorem

Let us consider a parametric angular sector $S(b_i)$ defined by

$$\begin{cases} a_1\,x + c_1\,y & \leq & b_1 \\ a_2\,x + c_2\,y & \leq & b_2 \end{cases}$$

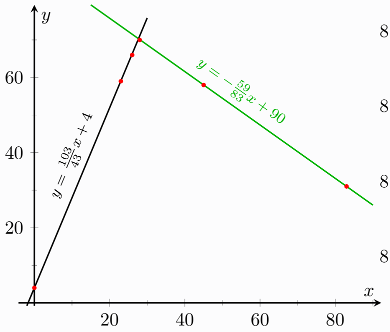
where $\gcd(a_i, b_i, c_i) = 1$ *for* $i \in \{1, 2\}$ *and* a_i, b_i, c_i *are all integers,* $b_i \in \{b_1, b_2\}$*. Let* $S_I(b_i)$ *be the integer hull of* $S(b_i)$*. Then, there exists an integer* T *and a vector* $\vec{u}$ *such that* $S_I(b_i + T)$ *is the translation of* $S_I(b_i)$ *by* $\vec{u}$*.*

$$T = \frac{1}{g_2}\,|a_2\,c_1 - a_1\,c_2| \quad or \quad \frac{1}{g_1}\,|a_2\,c_1 - a_1\,c_2|$$

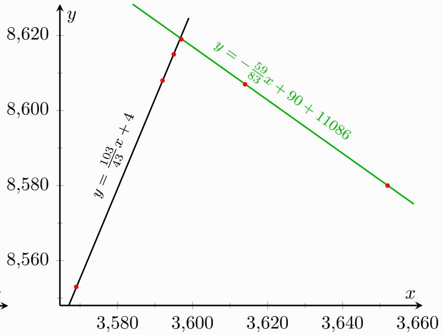
$$\vec{u} = (\frac{c_2\,T}{a_2\,c_1 - a_1\,c_2}, \frac{a_2\,T}{a_2\,c_1 - a_1\,c_2}) \quad or \quad (\frac{c_1\,T}{a_1\,c_2 - a_2\,c_1}, \frac{a_1\,T}{a_1\,c_2 - a_2\,c_1})$$

for $b_i = b_1$ *or* $b_i = b_2$ *respectively, where* $g_i = \gcd(a_i, c_i)$*. Note that* $a_2\,c_1 - a_1\,c_2 \neq 0$ *holds.*

Periodicity of the integer hull of an 2D-angular sector



(a)



(b)

Figure: A more complicated example. The red dots are the vertices of the integer hull of the sector.

Some observations

- Let P be a polygon with n vertices defined by
$$a_i\, x + c_i\, y \leq b_i$$
where $i \in \{1, \ldots, n\}$ and $S_i, i = 1, \ldots, n$ be the angular sectors define by
$$\left\{ \begin{array}{rcl} a_i\, x + c_i\, y & \leq & b_i \\ a_{i+1}\, x + c_{i+1}\, y & \leq & b_{i+1} \end{array} \right.$$
where $n + 1 = 1$. Then we have $P = \bigcap_{i=1}^{n} S_i$.
- Let P_I and S_{iI} be the integer hulls of P and S_i, respectively. Then, we have
$$P_I = \bigcap_{i=1}^{n} S_{iI}$$
.
- Let V, V_i be the vertex sets of P_I, S_{iI}, respectively. Then, we have $V = \bigcup_{i=1}^{n} V_i$ and the pairwise intersections of the V_i's are all empty, if b_i are large enough. (The precise bound for b_i is given in our paper)

Theorem

Let $P(b)$ be a parametric polygon given by

$$a_i\,x + c_i\,y \leq b_i \tag{3}$$

where $i \in \{1, \ldots, n\}$ and the parameter $b \in \{b_1, \ldots, b_n\}$ and $P_I(b)$ be the integer hull of $P(b)$. Specifically, $a_i\,x + c_i\,y \leq b_i$ and $a_{i+1}\,x + c_{i+1}\,y \leq b_{i+1}$ define an angular sector S_i of P, we have $i + 1 = 1$ if $i = n$, There exists an integer T and n vectors $\vec{v_1}, \ldots, \vec{v_n}$, such that for $|b|$ large enough $P_I(b + T)$ can be obtained from $P_I(b)$ as following.

As is defined above, denoting S_i the sectors of $P(b)$ and by S_{iI}, their respective integer hulls. then

$$P_I(b + T) = \bigcap f_{v_i}(S_{iI})$$

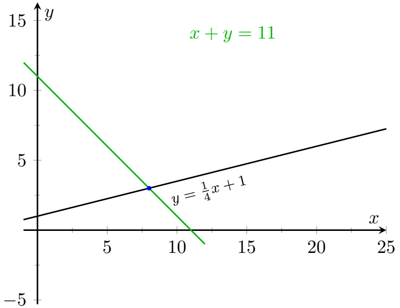
where f_{v_i} is the translation by v_i.

Triangle $P(b)$ defined by

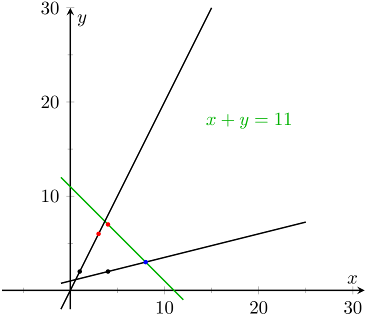
$$\left\{\begin{array}{rcl} -x+4\,y & \geq & 4 \\ 2\,x-y & \geq & 0 \\ -x-y & \geq & b \end{array}\right. \tag{4}$$

First, we look at the angular sector $S(b)$ given by $-x+4\,y \geq 4$ and $-x-y \geq b$. $S_I(b-5\,n)$ is a transformation of $S_I(b-5\,(n-1))$ by $\overrightarrow{(4,1)}$ for any $n \geq 1$.

When $|b| \geq 11$, the integer hull of $P(b-15n)$ is a translation of $P(b-15(n-1))$ by $\overrightarrow{(0,0)}, \overrightarrow{(12,3)}, \overrightarrow{(5,10)}$ for $n \geq 1$.



(a) Integer hull of an angular sector



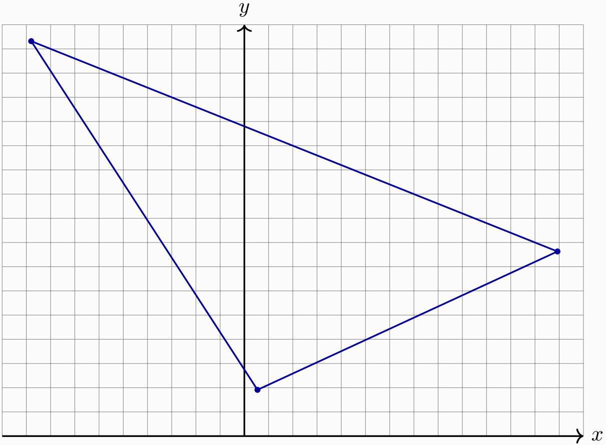
(b) Integer hull of a triangle

Computing the integer hull of polygons

Basic idea: Computing the integer hull of each angular sector that form the polygon

An example with vertices: $(-44/5, 408/25), (349/27, 206/27), (85/57, 109/57)$

$$\begin{cases} 2x + 5y & \leq & 64 \\ 7x + 5y & \geq & 20 \\ 3x - 6y & \leq & -7 \end{cases}$$



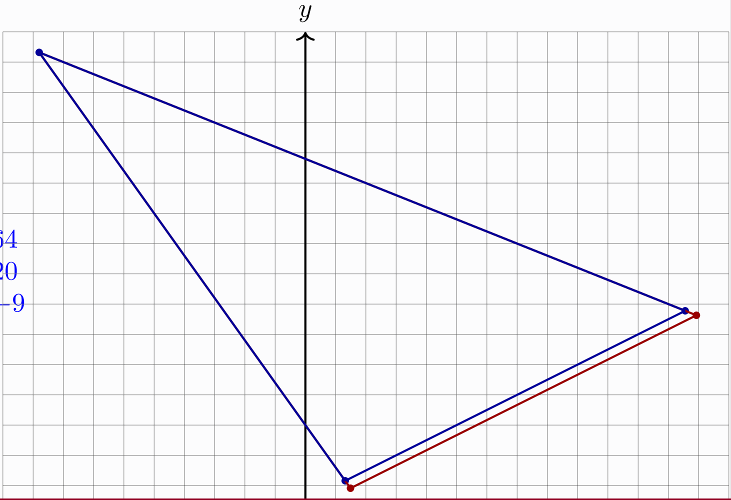
Example (1/4)

Replace the facets that could not have integer point

Vertices: $(-44/5, 408/25)$,~~$(349/27, 206/27)$,$(85/57, 109/57)$,~~
$(113/9, 70/9)$,$(25/19, 41/19)$

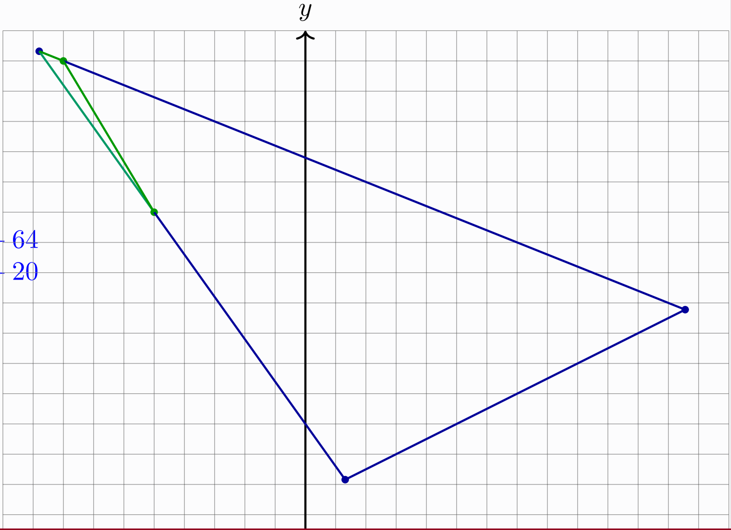
$\cancel{3x - 6y \leq -7}$

$$\begin{cases} 2x + 5y & \leq & 64 \\ 7x + 5y & \geq & 20 \\ 3x - 6y & \leq & -9 \end{cases}$$



Vertices: $(-44/5, 408/25), (113/9, 70/9), (25/19, 41/19)$
Find the triangle with vertices: $(-8, 16), (-44/5, 408/25), (-5, 11)$

$$\begin{cases} 5y & \leq & -2x+64 \\ 5y & \geq & -7x+20 \\ 2y & \geq & x+3 \end{cases}$$

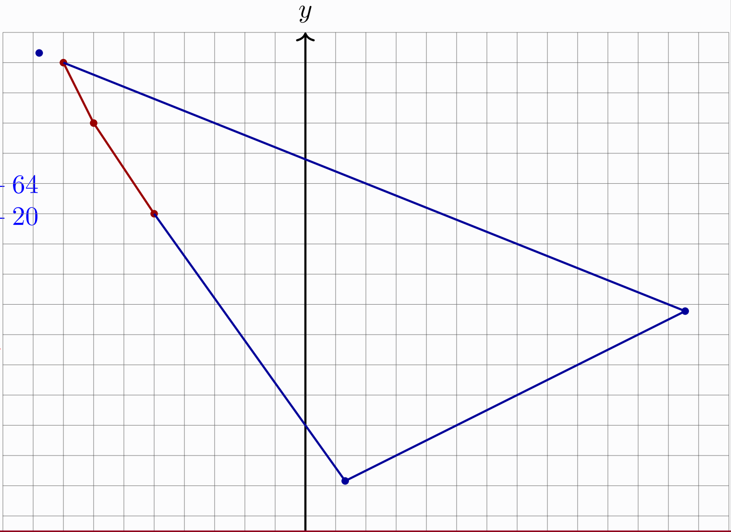


Compute the integer hull of the triangle

Vertices: $(-8, 16), (-7, 14), (-5, 11), (113/9, 70/9), (25/19, 41/19)$

$$\begin{cases} 5y & \le & -2x + 64 \\ 5y & \ge & -7x + 20 \\ 2y & \ge & x + 3 \end{cases}$$

$$\begin{cases} y & \ge & -2x \\ 2y & \ge & 3x + 7 \end{cases}$$

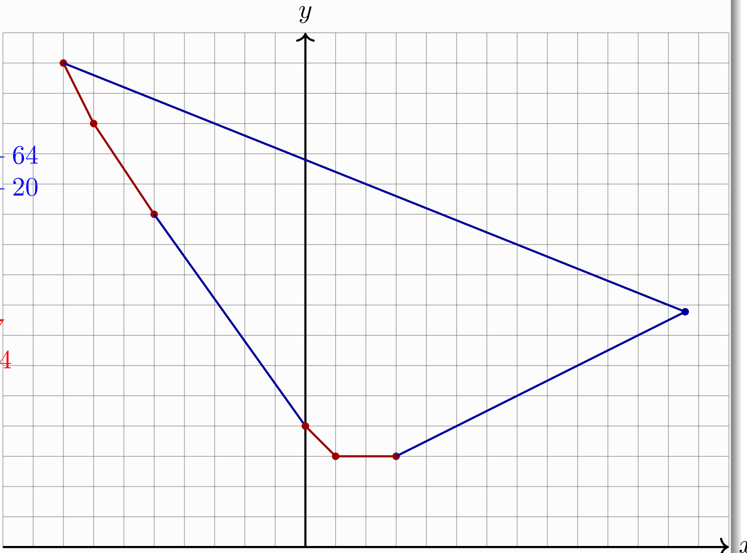


Example (3/4)

Vertices: $(-8,16),(-7,14),(-5,11),(0,4),(1,3),(3,3),(25/19,41/19)$

$$\begin{cases} 5y & \leq & -2x+64 \\ 5y & \geq & -7x+20 \\ 2y & \geq & x+3 \end{cases}$$

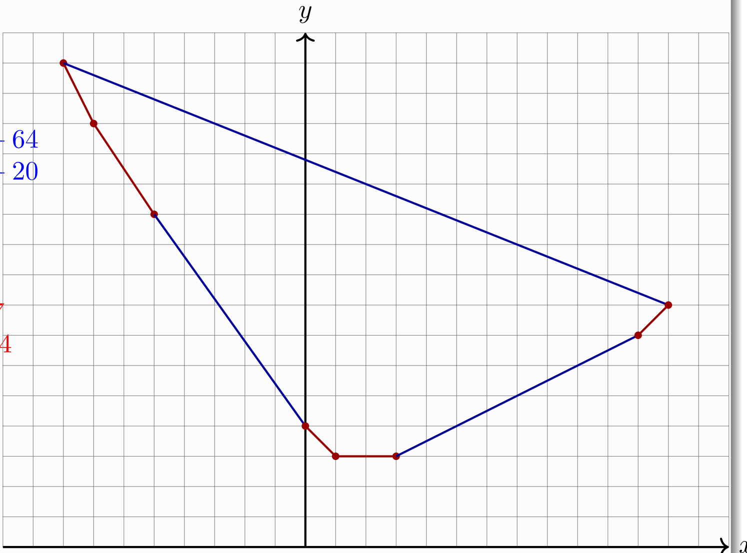
$$\begin{cases} y & \geq & -2x \\ 2y & \geq & 3x+7 \\ y & \geq & -x+4 \\ y & \geq & 3 \end{cases}$$



Vertices: $(-8,16),(-7,14),(-5,11),(0,4),(1,3),(3,3),(11,7),(12,8)$

$$\begin{cases} 5y & \leq & -2x+64 \\ 5y & \geq & -7x+20 \\ 2y & \geq & x+3 \end{cases}$$

$$\begin{cases} y & \geq & -2x \\ 2y & \geq & 3x+7 \\ y & \geq & -x+4 \\ y & \geq & 3 \\ y & \geq & x-4 \end{cases}$$



Application

Parametric integer hull

If the given P is a parametric polyhedron where b_i is unknown, we propose the following steps to compute the vertices of P_I:

- compute the length of a cycle T and the transformation vectors.
- compute the integer hull of every non-parametric polyhedron in one cycle.
- when the values of the parameters are available, using the corresponding solution from the previous step and the vectors from step 2 to compute the integer hull of the P with the given parameters.

Note that we can finish the first two steps "off-line", once the parameters are given the only computation that needs to be done is the translations which could be done in linear time. This method is both time and space efficient if the cycle T is short.

On-going work

What's next?

- Expand the theoretical results to higher dimensions
- Implement the non-parametric IntegerHull algorithm for arbitrary dimension
- Implement the parametric IntegerHull procedure

Thank You!

Your Questions?