Notes on Ito-Taylor Expansion

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Notes on SDE Calibration
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General Taylor Expansion

Let's review the method of application of a general Taylor expansion. There exists some $a(X_t)$ by which

$$\frac{d}{dt}X_t = a(X_t)$$

for a continuos-time variable X. Let a there be function $f(X_t)$, the evolution of which can be expressed by the chain rule as

$$\frac{df}{dt} = \frac{dX}{dt}\frac{\partial f}{\partial X} = a(X_t)\frac{\partial}{\partial X}f(X_t)$$

We can simply define this a linear operator $\mathcal{L}$ as

$$\mathcal{L} = a(X)\frac{\partial}{\partial X}$$

such that f can be written

$$f(X_t) = f(X_0) + \int_0^t \mathcal{L}f(X_\tau)d\tau$$

which happens to look very physical. This process can iterate such that f has an expanded form

$$\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t \mathcal{L}\left(f(X_0) + \int_0^{\tau_1} \mathcal{L}f(X_{\tau_2})d\tau_2\right)d\tau_1 \\
&= f(X_0) + \mathcal{L}f(X_0)\int_0^t d\tau_1 + \int_0^t \int_0^{\tau_1} \mathcal{L}^2 f(X_{\tau_2})d\tau_2 d\tau_1 \\
&= f(X_0) + \mathcal{L}f(X_0)(t-t_0) + \int_0^t \int_0^{\tau_1} \mathcal{L}^2 f(X_{\tau_2})d\tau_2 d\tau_1
\end{aligned}$$

Further iteration reveals

$$\begin{aligned}
\mathcal{L}^2 f(X_{\tau_2}) &= \mathcal{L}^2\left(f(X_0) + \int_0^{\tau_2} \mathcal{L}f(X_{\tau_3})d\tau_3\right) \\
&= \mathcal{L}^2 f(X_0) + \mathcal{L}^2\int_0^{\tau_2} \mathcal{L}f(X_{\tau_3})d\tau_3 \\
&= \mathcal{L}^2 f(X_0) + \mathcal{O}(\mathcal{L}^3)
\end{aligned}$$

which can be used in the term

$$\begin{aligned}
\int_0^t \int_0^{\tau_1} \mathcal{L}^2 f(X_{\tau_2})d\tau_2 d\tau_1 &= \int_0^t \int_0^{\tau_1} \left(\mathcal{L}^2 f(X_0) + \mathcal{O}(\mathcal{L}^3)\right) d\tau_2 d\tau_1 \\
&= \mathcal{L}^2 f(X_0)\int_0^t \int_0^{\tau_1} d\tau_2 d\tau_1 + \mathcal{O}(\mathcal{L}^3) \\
&= \frac{1}{2}\mathcal{L}^2 f(X_0)(t-t_0)^2 + \mathcal{O}(\mathcal{L}^3)
\end{aligned}$$

to return the familiar Taylor expansion

$$f(X_t) = f(X_0) + \mathcal{L}f(X_0)(t-t_0) + \frac{1}{2}\mathcal{L}^2 f(X_0)(t-t_0)^2 + \mathcal{O}(\mathcal{L}^3)$$

Ito-Taylor Expansion

Now we want to similarly expand an SDE of the form

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

As we implied above, we want a simple autonomous case in which

$$a = a(X_t) \quad \text{and} \quad b = b(X_t)$$

such that neither depends on time explicitly. Ito's lemma states that

$$df(X_t) = \left(a\partial_X f(X_t) + \frac{1}{2}b^2\partial_X^2 f(X_t) \right) dt + b\partial_X f(X_t)dW_t$$

As before we define linear operators

$$\begin{aligned}
\mathcal{L}^0 &= a\partial_X + \frac{1}{2}b^2\partial_X^2 \\
\mathcal{L}^1 &= b\partial_X
\end{aligned}$$

such that Ito's lemma can be stated

$$df(X_t) = \mathcal{L}^0 f(X_t)dt + \mathcal{L}^1 f(X_t)dW_t$$

or

$$f(X_t) = f(X_0) + \int_0^t \mathcal{L}^0 f(X_s)ds + \int_0^t \mathcal{L}^1 f(X_s)dW_s$$