STAT 153, FALL 2015
HOMEWORK 1 SOLUTIONS

Problem 1

(a) A solution in `R` is provided in Figure 1. We conclude that if $\delta = 0$ the process is a random walk without drift. When $\delta \neq 0$ there is a linear drift with slope equal to δ and as we increase δ, the drift begins to overwhelm the noise.

(b) We use induction. By definition $X_0 = W_0$ and for X_{t+1} we have:

$$\begin{aligned}
X_{t+1} &= \delta + X_t + W_{t+1} \\
&= \delta + (\delta t + \sum_{i \leq t} W_i) + W_{t+1} \\
&= \delta(t+1) + \sum_{i \leq t+1} W_i
\end{aligned}$$

The penlutimate equality follows from the inductive assumption.

(c) We have

$$\begin{aligned}
\mu_X(t) &= \mathbb{E}\left[\delta t + \sum_{i \leq t} W_i\right] \\
&= \mathbb{E}[\delta t] + \sum_{i \leq t} \mathbb{E}[W_i] \\
&= \delta t
\end{aligned}$$

$$\begin{aligned}
R_X(t, s) &= \operatorname{cov}(X_t, X_s) \\
&= \operatorname{cov}(\delta t + \sum_{i \leq t} W_i, \delta s + \sum_{i \leq s} W_i) \\
&= \operatorname{cov}\left(\sum_{i \leq t} W_i, \sum_{i \leq s} W_i\right) \\
&= \sigma^2 \min(t, s)
\end{aligned}$$

We conclude the process is not stationary because its mean function is not independent of t and its autocovariance function is not a function of $|s - t|$ alone (either one of these is sufficent).

```
X <- function (delta = 0, sd, steps) {
  c(0, cumsum(delta + rnorm(steps, 0, sd)))
}

plot_x <- function(delta) {
  print(matplot(replicate(10, X(delta, 1, 20)), type = 'l',
        main = bquote(delta == .(delta)),
        xlab = 't', ylab = expression('X'[t])))
}
par(mfrow = c(2,2))
sapply(c(0, 1, 0.2, 5), plot_x)
```

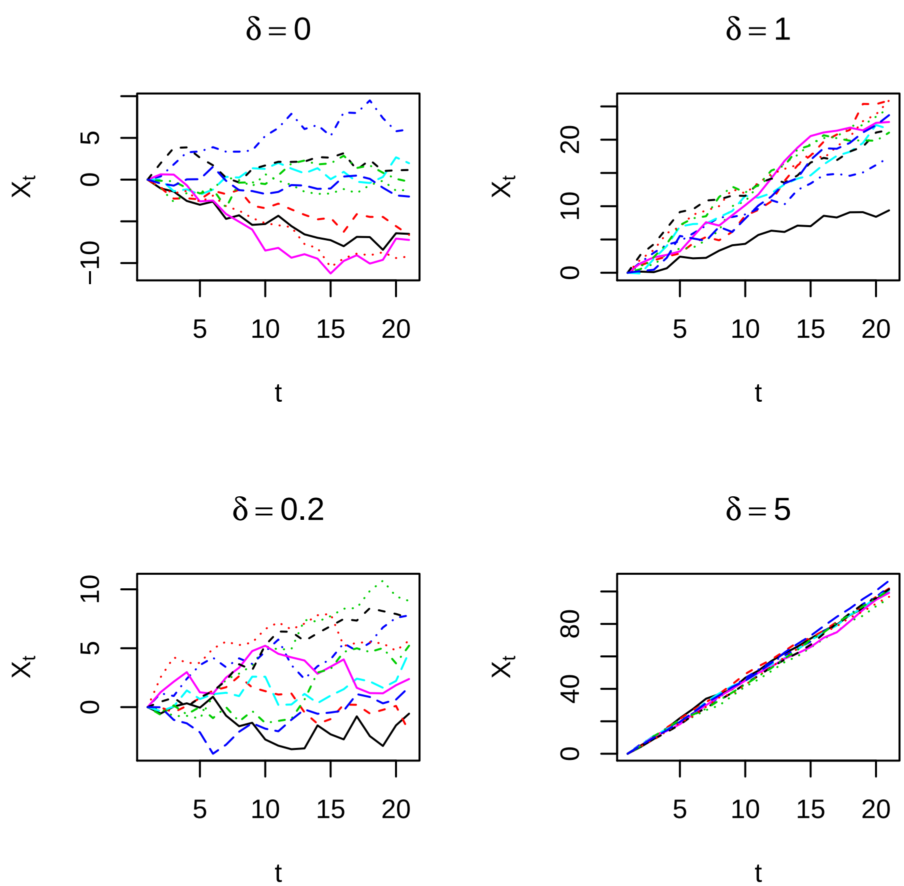


FIGURE 1. Code and Figures for Problem 1(a)

(d) The most natural transformation is $Y_t = X_t - X_{t-1}$. We then have:

$$\begin{aligned} Y_t &= (\delta t + \sum_{i \leq t} W_i) - (\delta(t+1) + \sum_{i \leq t+1} W_i) \\ &= \delta + W_t \end{aligned}$$

This is a white noise process with $\mu_Y(t) = \delta$ and $R_Y(h) = \sigma^2 \mathbb{I}(h = 0)$. Differencing is a common strategy for removing linear trends from time series.

Problem 2

(a) This model is appropriate for processes that are periodic.

(b)

$$\begin{aligned}
\mu_X(t) &= \mathbb{E}[U_1 sin(2\pi\omega_1 t) + U_2 cos(2\pi\omega_1 t)] \\
&= \mathbb{E}[U_1] sin(2\pi\omega_1 t) + \mathbb{E}[U_2] cos(2\pi\omega_1 t) \\
&= 0 \\
R_X(t,s) &= \operatorname{cov}(X_t, X_s) \\
&= \mathbb{E}(X_t X_s) \\
&= \mathbb{E}[\{U_1 sin(2\pi\omega_1 t) + U_2 cos(2\pi\omega_1 t)\}\{U_1 sin(2\pi\omega_1 s) + U_2 cos(2\pi\omega_1 s)\}] \\
&= \mathbb{E}[U_1^2]\sin(2\pi\omega_1 t)\sin(2\pi\omega_1 s) + \mathbb{E}[U_1 U_2]\sin(2\pi\omega_1 t)\cos(2\pi\omega_1 s) + \\
&\quad \mathbb{E}[U_2 U_1]\cos(2\pi\omega_1 s)\sin(2\pi\omega_1 t)] + \mathbb{E}[U_2^2]\cos(2\pi\omega_1 t)\cos(2\pi\omega_1 s) \\
&= \sigma^2 \sin(2\pi\omega_1 t)\sin(2\pi\omega_1 s) + \sigma^2\cos(2\pi\omega_1 t)\cos(2\pi\omega_1 s) \\
&= \sigma^2\cos(2\pi\omega_1(t-s)) \\
&= \sigma^2\cos(2\pi\omega_1|t-s|) \\
R_X(0) &= \sigma^2\cos(0) = \sigma^2 < \infty
\end{aligned}$$

So we conclude that X_t is weakly stationary because it has constant mean, covariance function depending only on $|t-s|$ and finite variance.

(c)

$$\begin{aligned}
\mu_Y(t) &= \mathbb{E}(Y_t) = \mathbb{E}(X_t + \tilde{X}_t) = \mathbb{E}(X_t) + \mathbb{E}(\tilde{X}_t) = 0 \\
R_X(t,s) &= \operatorname{cov}(Y_t, Y_s) = \operatorname{cov}(X_t + \tilde{X}_t, X_s + \tilde{X}_s) \\
&= \operatorname{cov}(X_t, X_s) + \operatorname{cov}(\tilde{X}_t, \tilde{X}_s) \\
&= \sigma^2\cos(2\pi\omega_1|t-s|) + \sigma^2\cos(2\pi\omega_2|t-s|) \\
&= \sigma^2(\cos(2\pi\omega_1|t-s|) + \cos(2\pi\omega_2|t-s|)) \\
R_Y(0) &= 2\sigma^2 < \infty
\end{aligned}$$

So we conclude that Y_t is also weakly stationary.

(d) It is not necessarily true that if X_t and Y_t are weakly stationary then $Z_t = X_t + Y_t$ is also weakly stationary. Let $X_t = W_t$ where W_t is white noise with variance σ^2 and define

$$Y_t = \begin{cases} W_{t+2} & \text{if } t = 1 \bmod 4 \\ W_{t-2} & \text{if } t = 3 \bmod 4 \\ W_t & \text{otherwise} \end{cases}$$

This means the first values of Y_t are $W_3, W_2, W_1, W_4, W_7, W_6 \ldots$.

We then have

$$R_Z(t,t) = \begin{cases} 2\sigma^2 & \text{if } t = 1 \mod 2 \\ 4\sigma^2 & \text{otherwise} \end{cases}$$

Problem 3

(a) The key fact that we use here is that a multivariate Gaussian distribution is completely determined by its mean vector and covariance matrix. Let X be a weakly stationary Gaussian process with $\mu_X(t) - \mu$ and $R_X(t,s) = f(|t-s|)$, then

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \sim \mathcal{N}\left(\underbrace{[\mu, \ldots, \mu]}_{\text{k copies}}, \begin{bmatrix} f(0) & f(|t_2 - t_1|) & \ldots & f(|t_k - t_1|) \\ f(|t_1 - t_2|) & f(0) & \ldots & f(|t_k - t_2|) \\ \vdots & \ddots & \ddots & \vdots \\ f(|t_1 - t_k|) & \cdots & \cdots & f(0) \end{bmatrix}\right)$$

When we evaluate the mean and covariance matrix for $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h})$ we find that they are identical to those for $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$ and hence the process is strictly stationary.

(b) There are many possibilites here. One simple case is

$$Y_t = \begin{cases} W_t & \text{if } t \text{ is odd} \\ V_t & \text{if } t \text{ is even} \end{cases}$$

Where W_t is iid white noise with marginal distribution uniform on $[-\sqrt{3}, \sqrt{3}]$ and V_t is iid white noise with $P(V_t = -1) = P(V_t = 1) = 1/2$. Incidentally, V_t follows what is known as the Rademacher distribution.

(c) Let $h(x)$ be a continuous transformation and define $h^{-1}((-\infty, c])$ to be the pre-image of the set $[-\infty, c]$: ie it is the collection of all points in $\mathbb{R}$ which h maps to a point in the interval $(\infty, c]$. Then we have:

$$\begin{aligned} \mathbb{P}(Y_{t_1} < c_1, \ldots, Y_{t_k} < c_k) &= \mathbb{P}(h(X_{t_1}) < c_1, \ldots, h(X_{t_k}) < c_k) \\ &= \mathbb{P}(X_{t_1} \in h^{-1}((-\infty, c_1]), \ldots, X_{t_k} \in h^{-1}((-\infty, c_k])) \\ &= \mathbb{P}(X_{t_1+h} \in h^{-1}((-\infty, c_1]), \ldots, X_{t_k+h} \in h^{-1}((-\infty, c_k])) \\ &= \mathbb{P}(h(X_{t_1+h}) < c_1, \ldots, h(X_{t_k+h}) < c_k) \\ &= \mathbb{P}(Y_{t_1+h} < c_1, \ldots, Y_{t_k+h} < c_k) \end{aligned}$$

The third equality follows because X_t is strictly stationary.

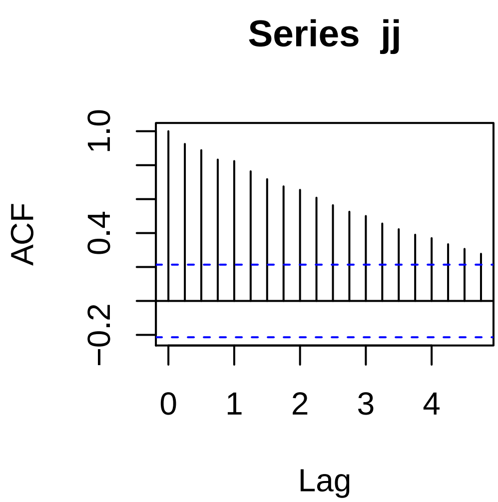
(d) The answer is no. Let Y_t be the process defined in part (b) and suppose $Z_t = h(X_t) = X_t^2$ then it is straightforward to verify that:

$$R_Z(0) = \begin{cases} \frac{4}{5} & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even} \end{cases}$$

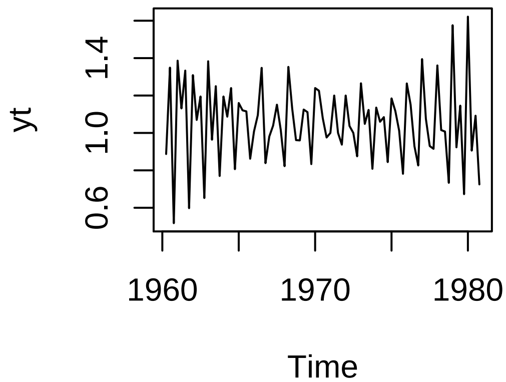
Problem 4

(a) There are many approaches here. If we look at contiguous subseries we can see that the mean for the subseries increases monotonically. We can also examine the ACF and observe that it appears to decay roughly linearly, suggestive of a trend.

```
library(astsa)
tapply(jj, rep(1:12, each = 7), mean)
##          1         2         3         4          5          6
##  0.6928571 0.7414286 1.0128571 1.4371429  1.8214286  2.5714286
##          7         8         9        10         11         12
##  4.0628571 5.4000000 6.8142857 8.2414286 11.0442856 13.7571429
acf(jj)
```

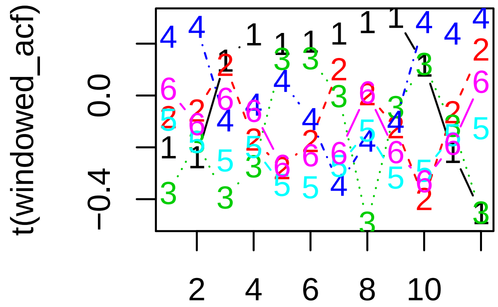


(b)
```
yt <- jj/lag(jj,-1)
plot(yt)
```



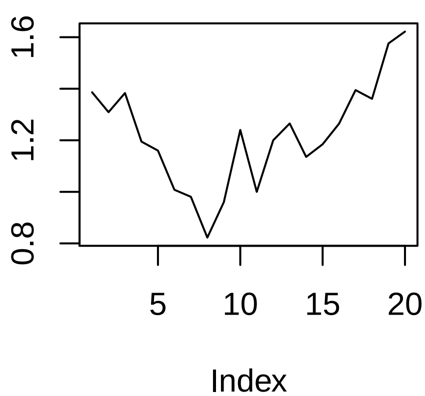
(c) We will be open minded in grading this problem, but the best approach may be an approach similar to that in 4(a): estimate a separate autocovariance function for disjoint, contiguous subseries and see if they are similar.

```
my_acovf <- function(h, ts) {
  1/length(ts)*
    sum((ts - mean(ts))*(lag(ts, h) - mean(ts)))
  }
my_acorf <- function(h, ts){
  ts <- as.ts(ts)
  my_acovf(h,ts)/my_acovf(0, ts)
}
my_acorf_vec <- function(ts) {
  sapply(1:(length(ts)-1), my_acorf, ts = ts)
}
windowed_acf <- simplify2array(tapply(jj, rep(1:12, each = 7), my_acorf_vec))
matplot(t(windowed_acf), type = "b")
```



The figure is a bit difficult to interpret but it seems that different windows have markedly different sample ACFs, so we conclude the series is probably not stationary.

(d)
```
plot(yt[4*(1:(length(yt)/4))], type = "l", ylab = "")
```



The series now looks relatively stationary but it is difficult to provide a conclusive answer because there are comparatively few data points.