# Bootstrap Confidence Intervals

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# Review of Confidence Intervals

# Sampling Distribution

## Definition 1 (Sampling Distribution)

- *The sampling distribution of a statistic is the distribution of values taken by the statistic in* ***all possible samples of the same size*** *from the same population.*
- *The standard deviation of a sampling distribution is called a* ***standard error***

# Sampling Distributions

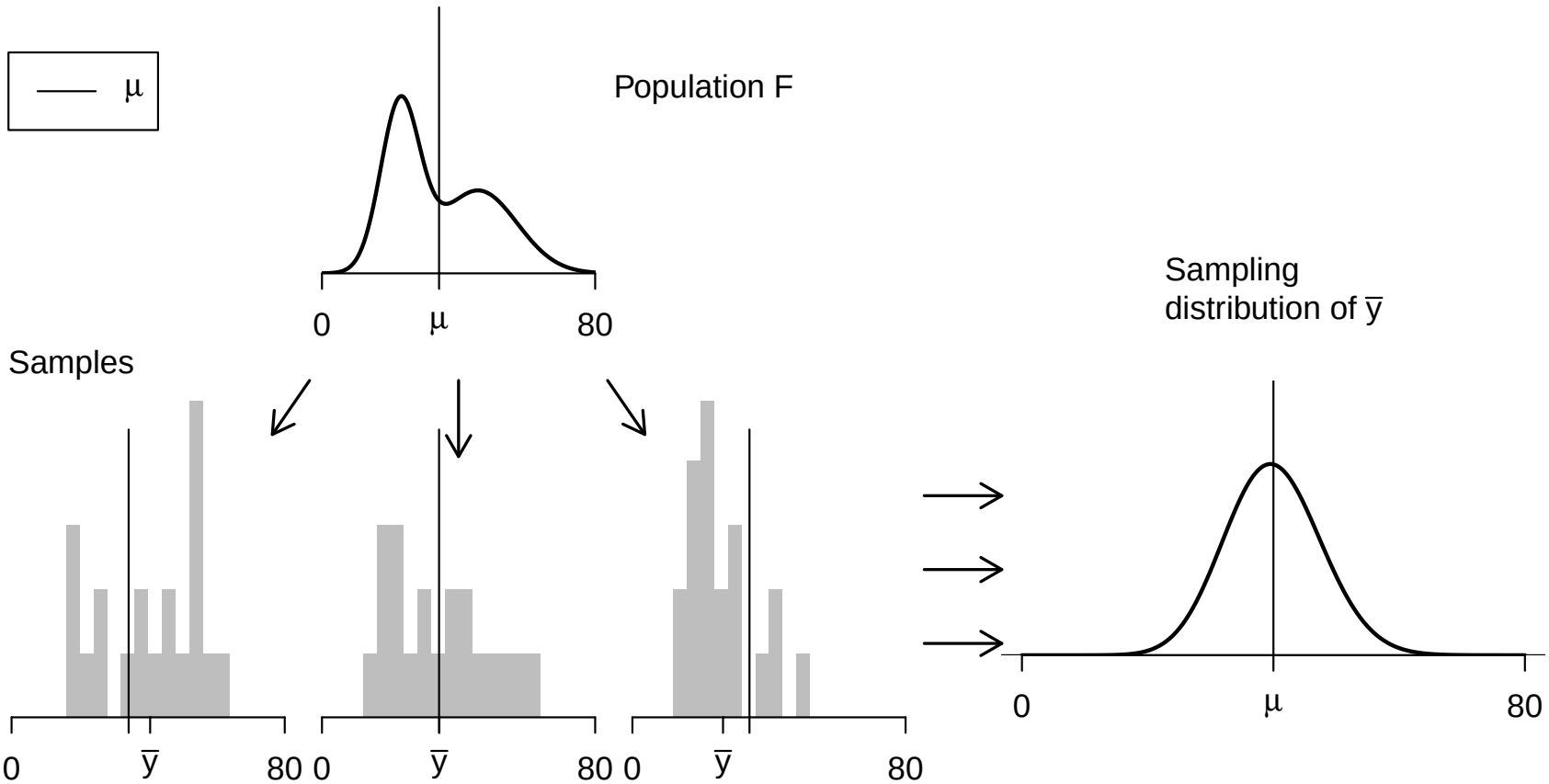


Fig.: Ideal world. Sampling distributions are obtained by drawing repeated samples from the population, computing the statistic of interest for each, and collecting (an infinite number of) those statistics as the sampling distribution

# Sampling Distribution

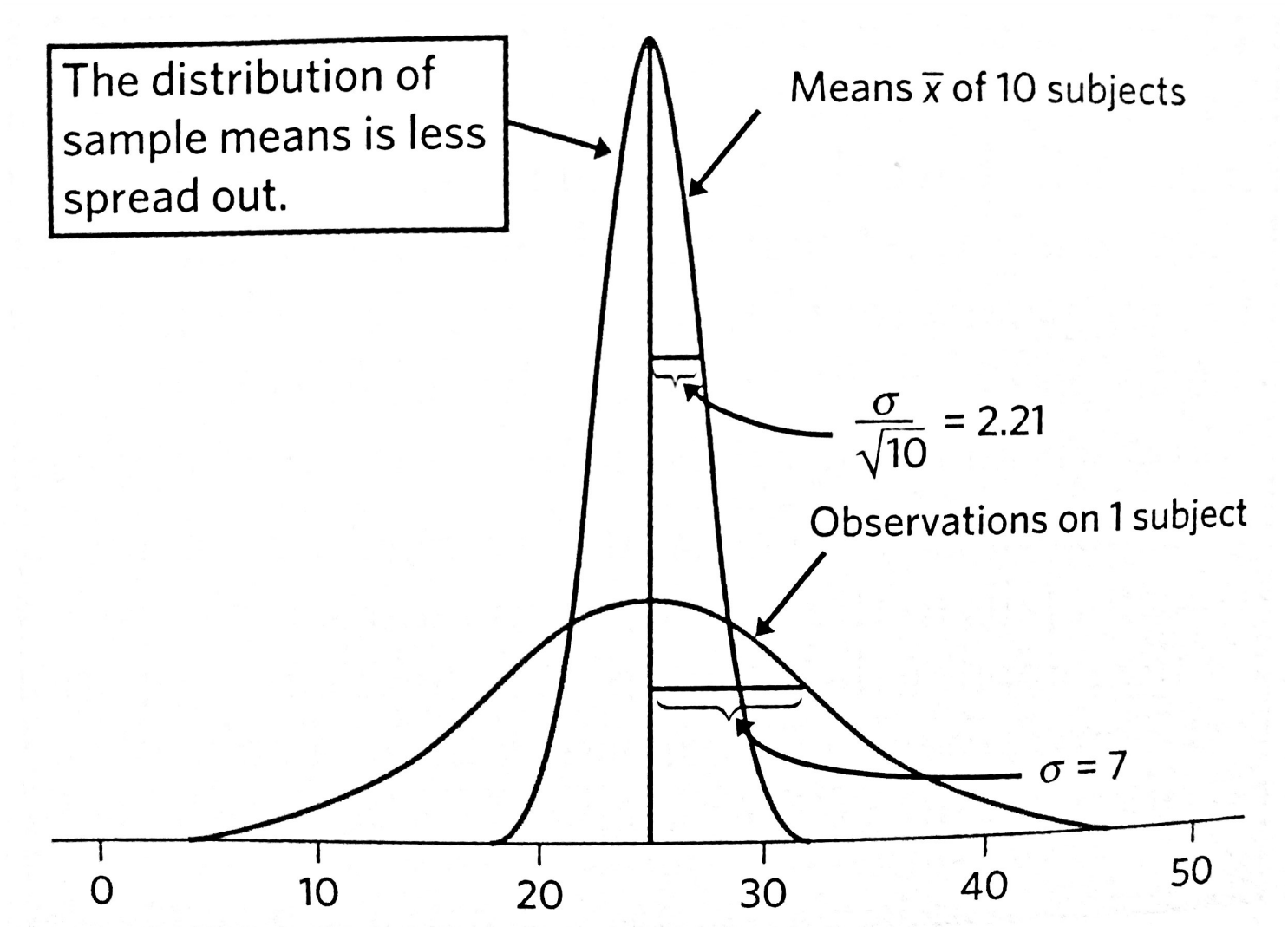


Fig.: Averages are less variable than individual observations

# Traditional way to calculate CIs

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- To find a CI with confidence level $\mathcal{C} = 1 - \alpha$, we must calculate the critical value $z^*$ such that

$$P(-z^* < Z < z^*) = \mathcal{C} = 1 - \alpha \tag{1}$$

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  - That is, we want the value $z^*$ that gives a *lower tail probability* of $(1-\mathcal{C})/2 = \alpha/2$.
  - Often this value is denoted $z^* = z_{\alpha/2}$; thus we have

$$P(Z < -z_{\alpha/2}) = \alpha/2,$$

and

$$P(Z > z_{\alpha/2}) = \alpha/2.$$

# Traditional way to calculate CIs

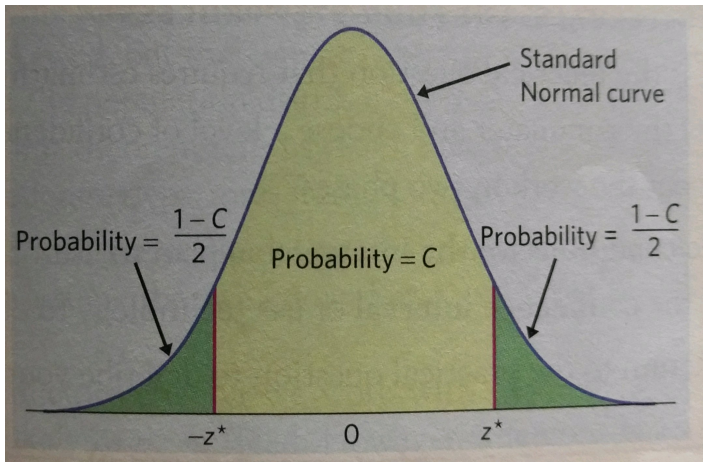


Fig.: The critical value $z^\star$ is the number that catches central probability $\mathcal{C}$ under a standard normal $\mathcal{N}(0,1)$ curve between $-z^\star$ and $z^\star$

We can use this probability statement about the standardized version of the sample mean $(\bar{y}-\mu)/\sigma/\sqrt{n}$, to place bounds on where we think the true mean lies by examining the probability that $\bar{y}$ is within $z^\star \cdot \frac{\sigma}{\sqrt{n}}$ of $\mu$

$$
\begin{aligned}
\mathcal{C} &= P\left(-z^\star \leq \frac{\bar{y}-\mu}{\sigma/\sqrt{n}} \leq z^\star\right) \\
&= P\left(-z^\star \frac{\sigma}{\sqrt{n}} \leq \bar{y}-\mu \leq +z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(-\bar{y}-z^\star \frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\bar{y}+z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\bar{y}+z^\star \frac{\sigma}{\sqrt{n}} \geq \mu \geq \bar{y}-z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\bar{y}-z^\star \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{y}+z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= 1-\alpha
\end{aligned}
$$

We call the interval $\left(\bar{y}-z^\star \frac{\sigma}{\sqrt{n}}, \bar{y}+z^\star \frac{\sigma}{\sqrt{n}}\right)$ a **(1-$\alpha$)100% confidence interval** for $\mu$.

# Confidence intervals for depths of the ocean

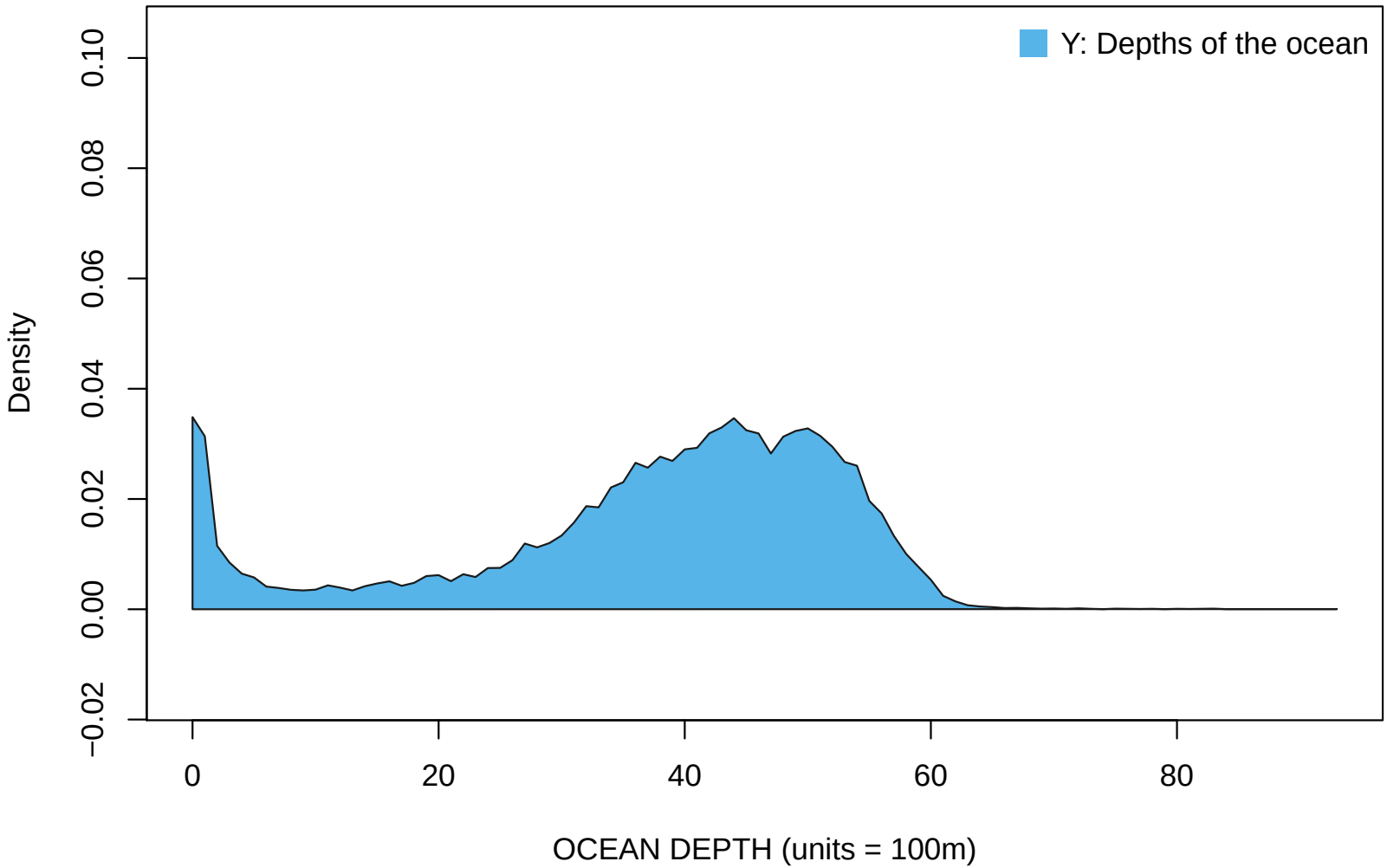


Fig.: The original data distribution of sampled depths of the ocean. Note that it has multiple modes and not Normal looking.

# The CLT is 'kicking in' at $n = 16$

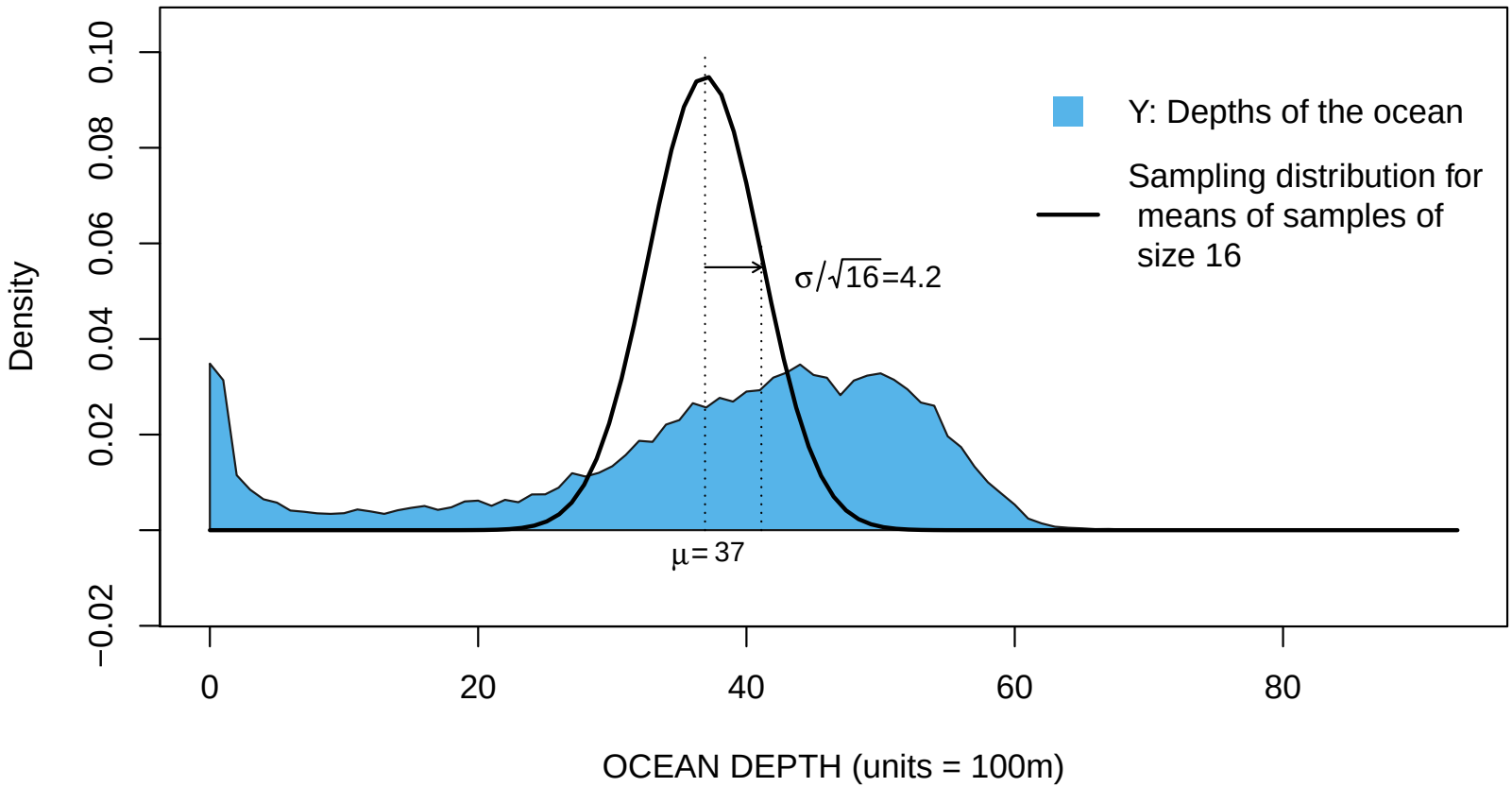


Fig.: The sampling distribution for the mean depth of the ocean with samples of size $n = 16$, looks normal (centered at $\mu = 37$ and SD equal to $\sigma/\sqrt{16}$)

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We want to construct a $\mathcal{C} = 95\%$ confidence interval for the mean.
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$P(Z < -z^\star) = P(Z > z^\star) = \alpha/2 = 0.025$ is given by

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mosaic::xqnorm(p = c(0.025, 0.975))
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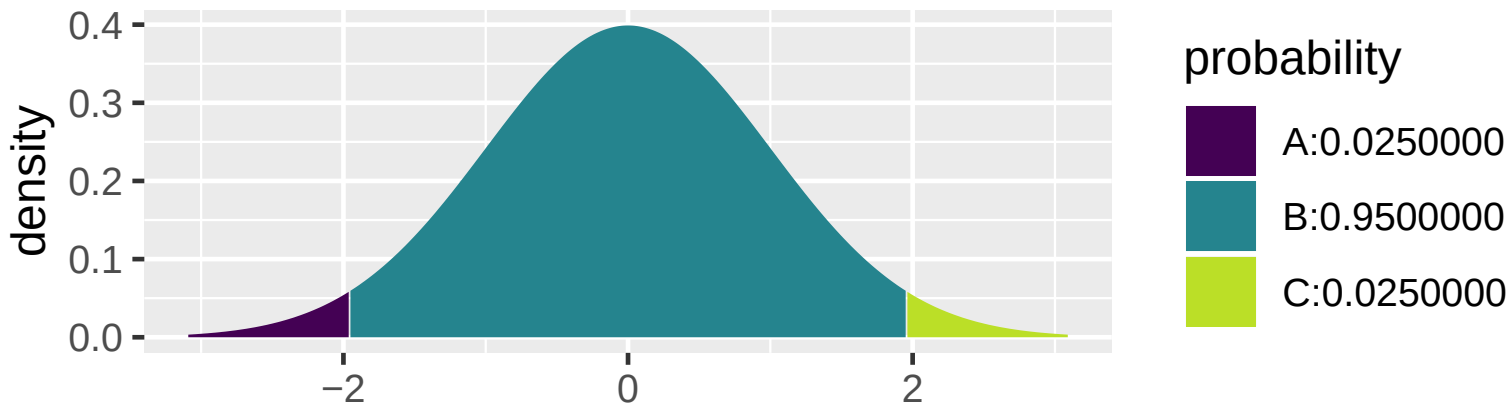


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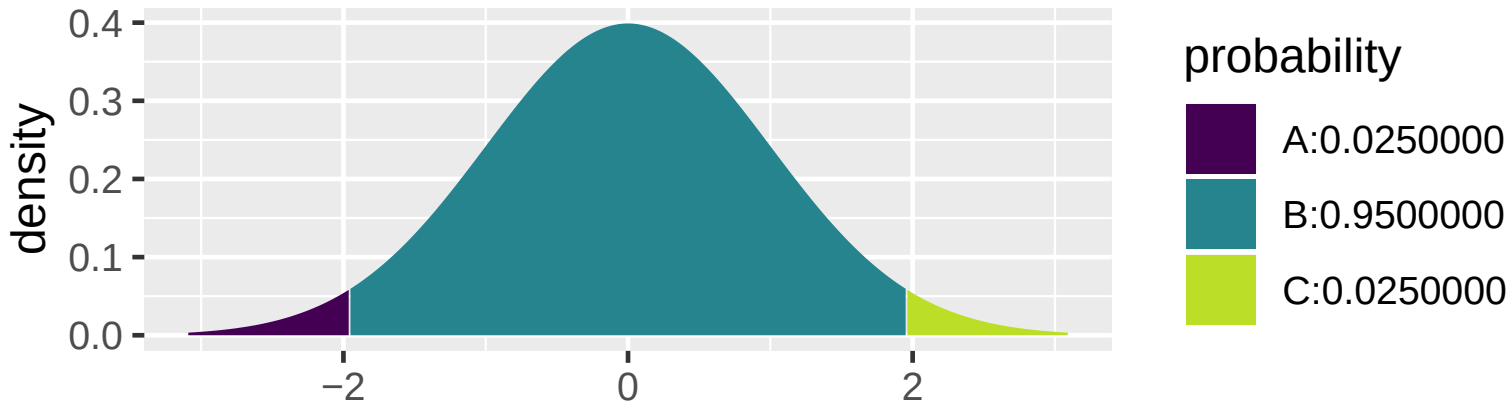


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3. 95% CI for $\mu$: $(37 - 1.96 \cdot 4.2, 37 + 1.96 \cdot 4.2) = [29, 45]$

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- There is an alternative, **yet equivalent**, way to calculate the CI without standardizing $\bar{y}$, and without using the $\pm$ formula
- This is accomplished using `qnorm`
- Note: we **still need the CLT regardless** of whether we use the $\pm$ formula or `qnorm`

# 68% Confidence interval using `qnorm`

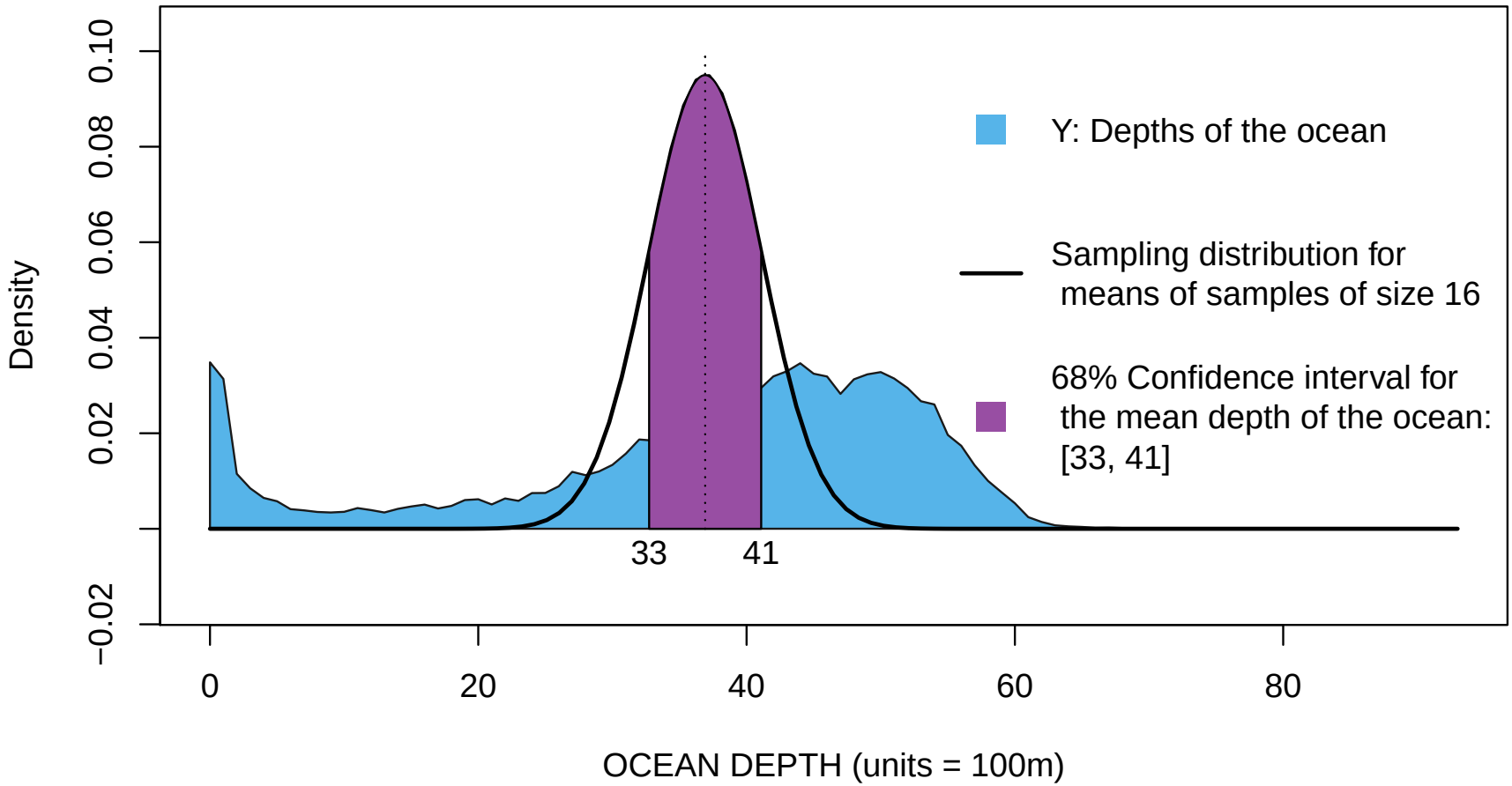


Fig.: 68% Confidence interval calculated using

```
qnorm(p = c(0.16,0.84), mean = 37, sd = 4.2)
```

# 95% Confidence interval using `qnorm`

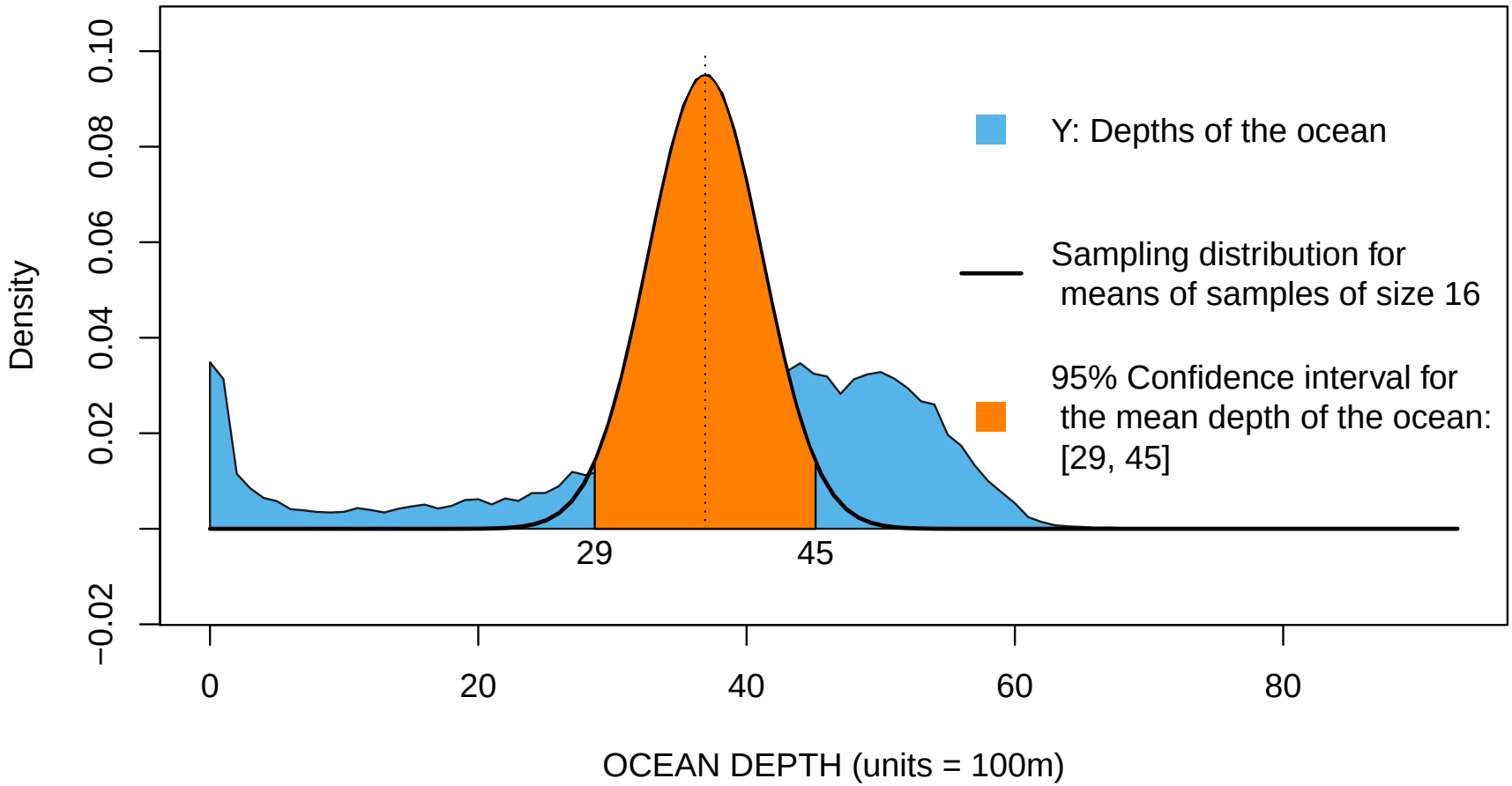


Fig.: 95% Confidence interval calculated using

```
qnorm(p = c(0.025,0.975), mean = 37, sd = 4.2)
```

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A: Bootstrap

# The Bootstrap

# Ideal world: known sampling distribution

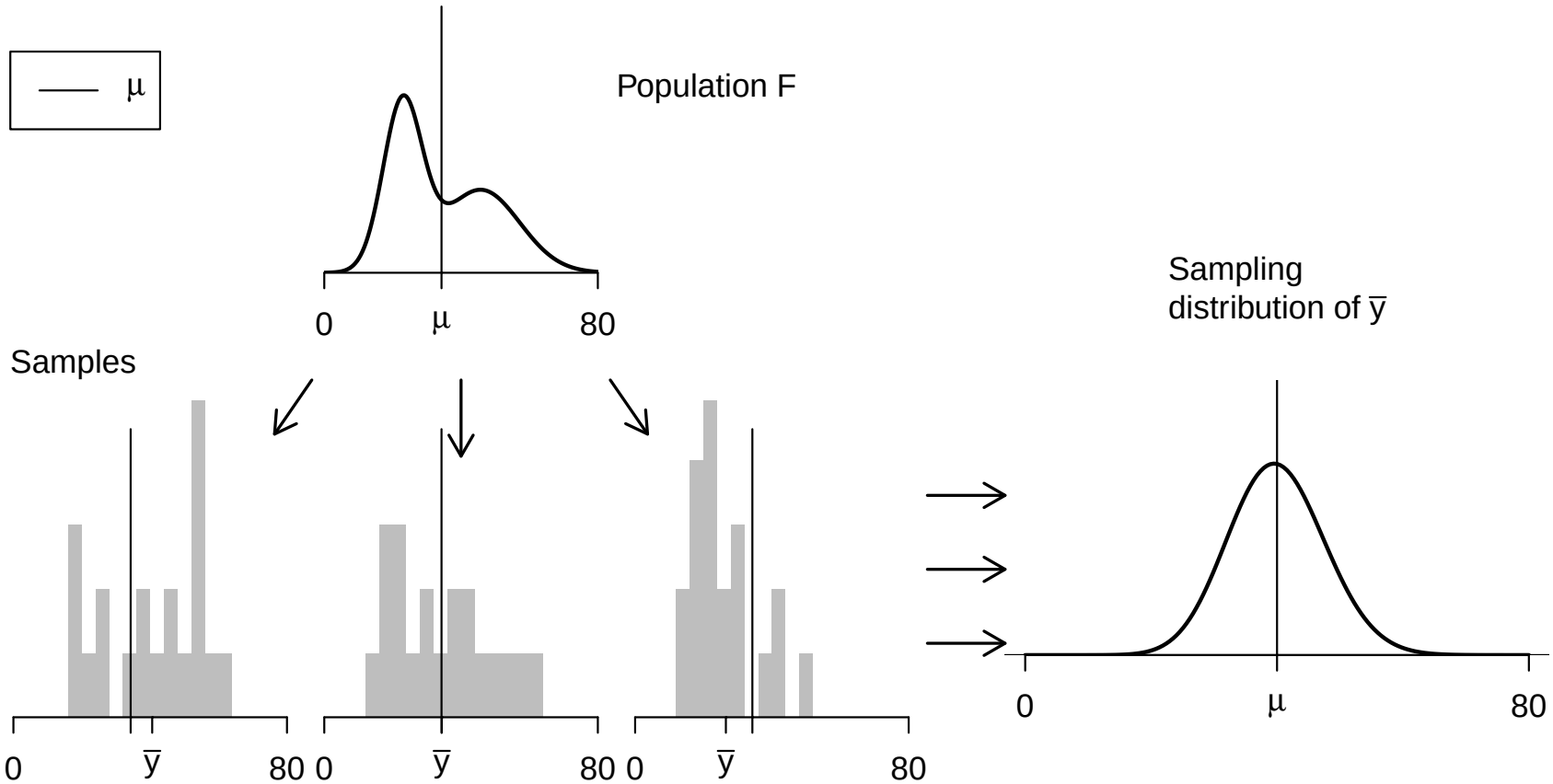



Fig.: Ideal world. Sampling distributions are obtained by drawing repeated samples from the population, computing the statistic of interest for each, and collecting (an infinite number of) those statistics as the sampling distribution

# Reality: use the bootstrap distribution instead

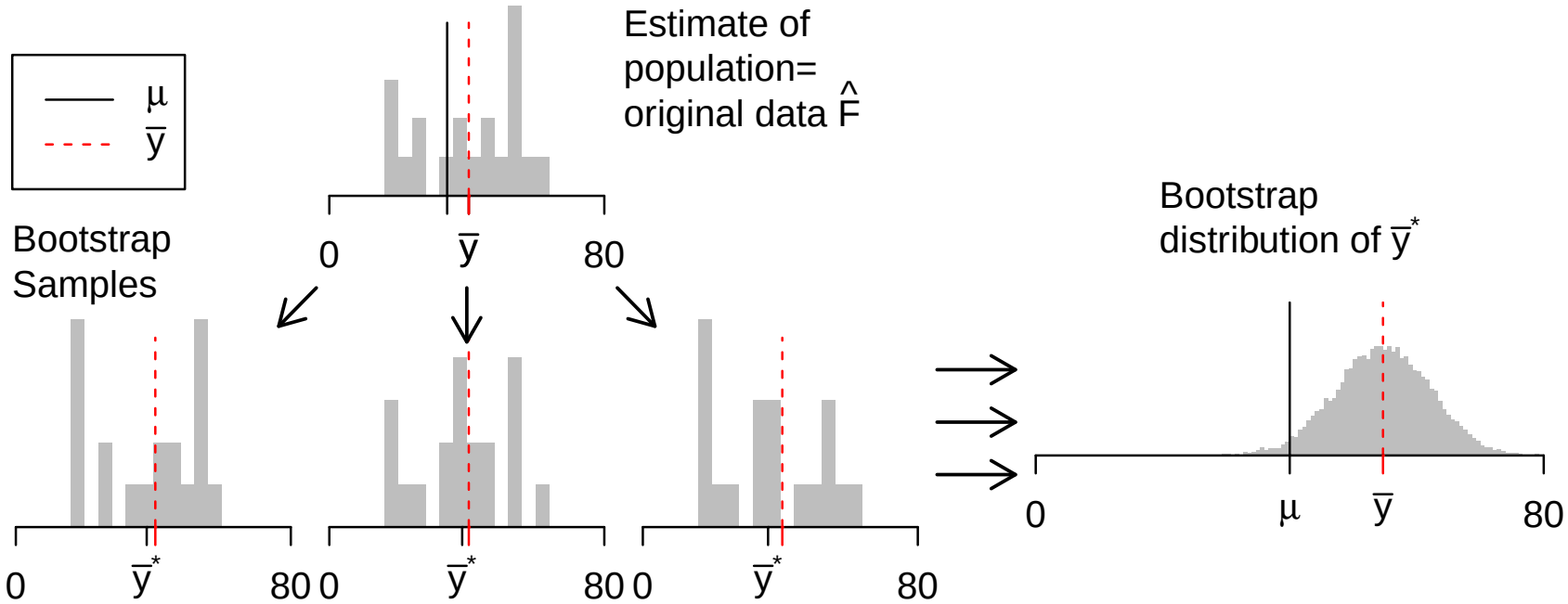
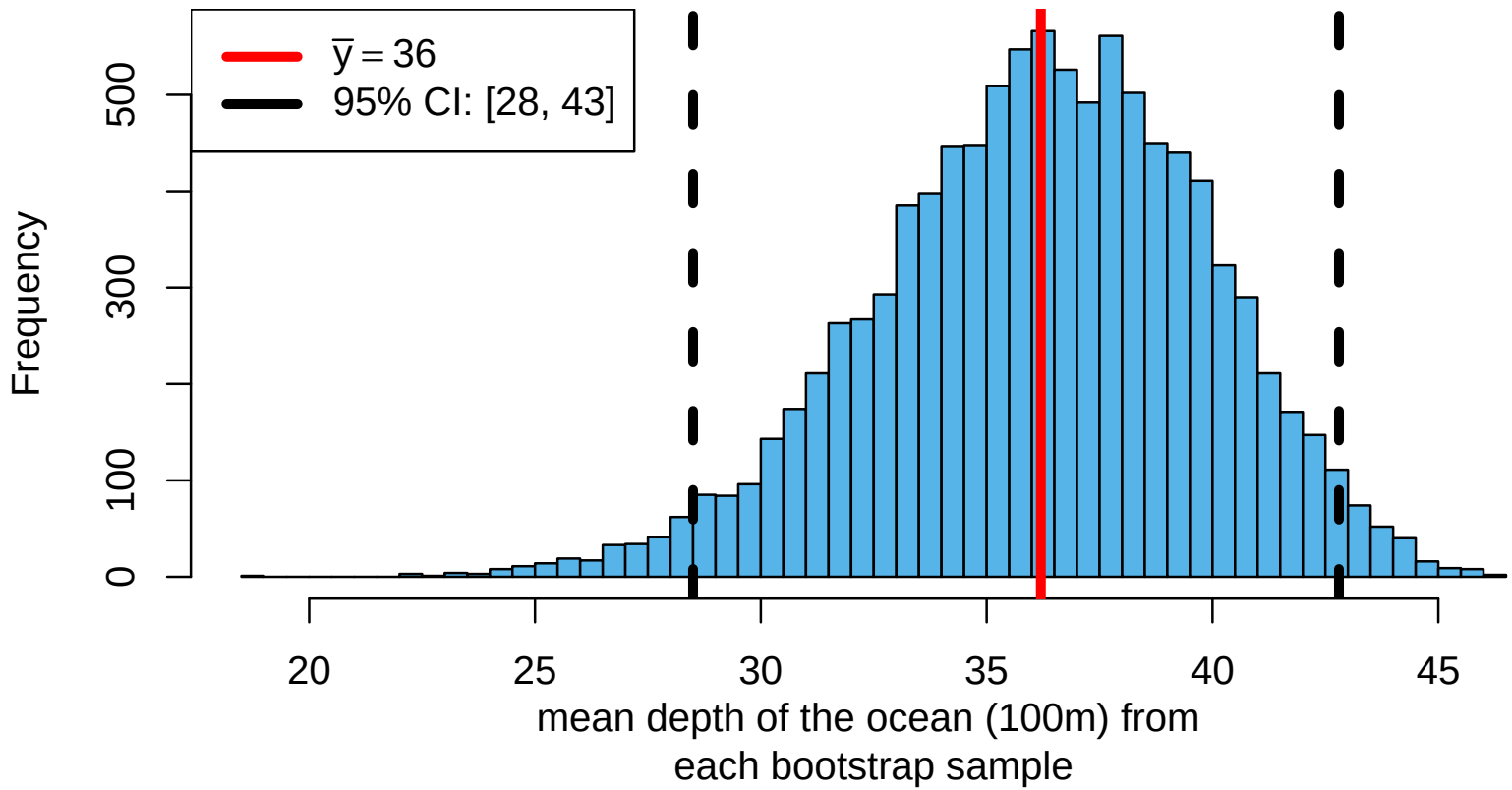



Fig.: Bootstrap world. The bootstrap distribution is obtained by drawing repeated samples from an estimate of the population, computing the statistic of interest for each, and collecting those statistics. The distribution is centered at the observed statistic ($\bar{y}$), not the parameter ($\mu$).

# Main idea: simulate your own sampling distribution

```
library(mosaic)
s_dist <- do(10000) * mean( ~ alt, data = resample(depths.n.20))
CI_95 <- quantile(~ mean, data = s_dist, probs = c(0.025, 0.975))
```



# Bootstrap code 1

```
# function for sampling ocean depths
source("https://github.com/sahirbhatnagar/EPIB607/raw/
master/exercises/water/automate_water_task.R")

# from the in-class exercise
index.n.20 <- c(2106,2107,2108,2109,2110,2111,2112,
2113,2114,2115,2116,2117,2118,2119,
2120,2121,2122,2123,2124,2125)

# get depths of ocean sample n=20
depths.n.20 <- automate_water_task(index = index.n.20,
student_id = 260194225, type = "depth")

# change to 100m units
depths.n.20$alt = round(depths.n.20$alt/100,0)

library(mosaic)

# calculate mean depth for your sample
mean_depth <- mean(~ alt, data = depths.n.20)
```

# Bootstrap code 2

```
# 10000 bootstrap samples
s_dist <- do(10000) * mean( ~ alt, data = resample(depths.n.20))

# 95% CI
CI_95 <- quantile(~ mean, data = s_dist, probs = c(0.025, 0.975))

# plot sampling distribution
hist(s_dist$mean, breaks = 50, col = "#56B4E9",
main="",
xlab = "mean depth of the ocean (100m) from each bootstrap sample")

# draw red line at the sample mean
abline(v = mean_depth, lty =1, col = "red", lwd = 4)

# draw black dotted lines at 95% CI
abline(v = CI_95[1], lty =2, col = "black", lwd = 4)
abline(v = CI_95[2], lty =2, col = "black", lwd = 4)

# include legend
library(latex2exp)
legend("topleft",
legend = c(TeX("$\\bar{y} = 36$"),
sprintf("95%% CI: [%.f, %.f]",CI_95[1], CI_95[2])),
lty = c(1,1),
col = c("red","black"), lwd = 4)
```