Week 5: Continuous Random Variables and Probability Distributions

MATH697

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Introduction

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Definition 1 (Continuous Random Variable)

A random variable is continuous if

$$P(X = x) = 0, \quad \forall \quad x \in \mathbb{R}$$

Example

Example 2 (The Uniform Distribution)

Consider a random variable X whose distribution is the uniform distribution on $[0, 1]$:

$$P(a \leq X \leq b) = b - a, \quad 0 \leq a \leq b \leq 1$$

with $P(X < 0) = P(X > 1) = 0$. We write this as $X \sim \mathrm{Uniform}(0, 1)$.

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$$P\left(X \geq \frac{2}{3}\right) = P\left(\frac{2}{3} \leq X \leq 1\right) + P(X > 1) = \left(1 - \frac{2}{3}\right) + 0$$

Example cont

Example 3 (The Uniform Distribution)

$X \sim \mathrm{Uniform}(0, 1)$. Setting $a = b = x$ we see in particular that

$$P\left(x \leq X \leq x\right) = x - x = 0, \quad \forall \quad x \in \mathbb{R}$$

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Thus, the uniform distribution is an example of a continuous distribution.

Continous CDF and Probability Density Function (PDF)

Continous CDF

The probability distribution of a *continuous* random variable X is defined by the continuous **cumulative distribution function** or **c.d.f.**, F_X, specified by

$$F_X(x) = P[X \leq x] \qquad \text{for all } x \in \mathbb{X}$$

note: this is an identical definition to the discrete case.

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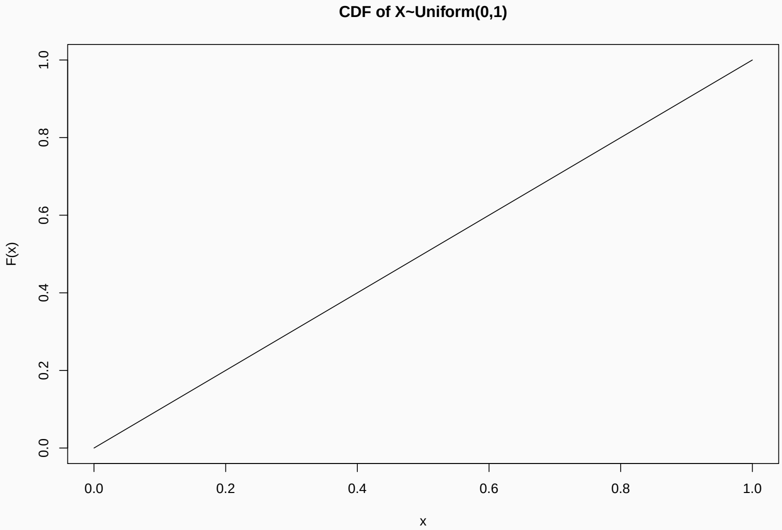
The continuous cdf F_X must exhibit the same properties as for the discrete cdf, except that

(iii) $\lim_{h \to 0} F_X(x+h) = F_X(x)$ [i.e. F_X is continuous]

Uniform(0,1) Distribution

```
curve(punif(x), ylab = "F(x)",  main = "CDF of X~Uniform(0,1)")
```



Continous PDF

Definition 4 (Continuous Random Variable)

Let X be a random variable with distribution function $F(x)$. If there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$F_X(x) = \int_{-\infty}^{x} f(t)dt, \quad x \in \mathbb{R}$$

then X is called a continous random variable with density function f.

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then X is called a continous random variable with density function f. By the fundamental theorem of calculus, we also have

$$F'(x) = f(x), \quad \forall \quad x$$

Continous PDF

Proposition 5 (Continuous PDF)

f has the following properties

1. $f(x) \geq 0$ *for all* $x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f(x) dx = 1$

Proof: *on board*

Proposition 6 (CDF from PDF)

Let X be a continuous random variable with density function f

1. *If $a < b$, then*

$$P(a < X \le b) = \int_a^b f(x)dx$$

   Notat that this is the area under the curve of f between a and b.

   More generally, we have

$$P(X \in A) = \int_A f(x)dx, \quad for\ any\ A \subset \mathbb{R}$$

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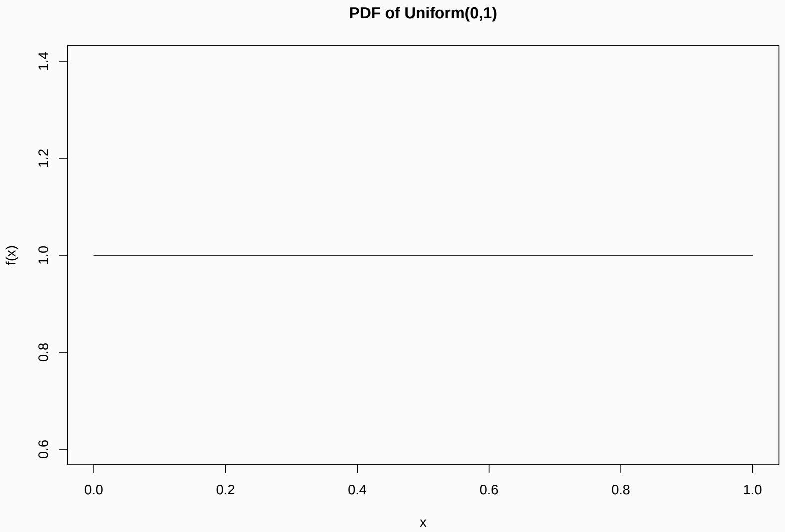
2. *$P(X = x) = 0$, for every $x \in \mathbb{R}$*

Proof: *on board*

Example

```r
# X ~ Uniform(0,1)
curve(dunif(x), xlab = "x", ylab = "f(x)", main = "PDF of Uniform(0,1)")
```



Continuous PDF

Remark: Because of part (2) on the previous slide, we can say that

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

Relationship between CDF and PDF

The **probability density function**, or **pdf**, f_X, is defined by

$$f_X(x) = \frac{d}{dx}\left\{F_X(x)\right\}$$

so that, by a fundamental calculus result,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

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In the continuous case, we calculate F_X from f_X by **integration**, and f_X from F_X by **differentiation**

Remarks about PDF and CDF

- We must use F_X to specify the probability distribution initially, although it is often easier to think of the shape of the distribution via the pdf f_X. Any function that satisfies the properties for a pdf can be used to construct a probability distribution.

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- Note that, for a continuous random variable

$$f_X(x) \neq P[X = x].$$

Special Continous Probability Distributions

1. Uniform Distribution

Definition 7 (The continous uniform distribution)

A model with **constant** probability density on a region,

$$f_X(x) = \frac{1}{b-a} \qquad a < x < b$$

the cumulative distribution function (cdf) is also straightforward

$$F_X(x) = \frac{x-a}{b-a} \qquad a < x < b$$

2. The Exponential Distribution

Definition 8 (The Exponential(λ) Distribution)

A **continuous** waiting-time model

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \in \mathbb{R}^+$$

The cdf for the exponential distribution can be calculated easily;

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x} \qquad x \geq 0.$$

and note that

$$P[\, X > x\,] = 1 - P[\, X \leq x\,] = 1 - F_X(x) = e^{-\lambda x}$$

- This is the parametrization used in R (see `?dexp`)

Memoryless Property of the Exponential

The exponential distribution can be used to model lifetimes as it shares the memoryless property of the geometric. If $X \sim Exponential(\lambda)$, then for $s > t \geq 0$

$$P(X > s | X > t) = P(X > s - t)$$

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- Tossing a fair coin is an example that is memoryless. Every time you toss the coin, you have a 50% chance of it coming up heads. It doesn't matter whether or not the last 5 times you tossed the coin it came up consistently tails; the probability of heads in the next throw is always going to be 0.5

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- Time until car failure is not memoryless.

$$P(7 \text{ years} < fail < 10 \text{ years}) \neq P(3 \text{ years} < fail < 6 \text{ years})$$

Memoryless Property of the Exponential (proof)

$$
\begin{aligned}
P(X > s | X > t) &= \frac{P(X > s \cap X > t)}{P(X > t)} \\
&= \frac{P(X > s)}{P(X > t)} \\
&= \frac{\int_s^\infty \lambda e^{-\lambda x} dx}{\int_t^\infty \lambda e^{-\lambda x} dx} \\
&= \frac{e^{-\lambda s}}{e^{-\lambda t}} \\
&= e^{-\lambda(s-t)} \\
&= 1 - F_X(s-t) \\
&= P(X > s - t)
\end{aligned}
$$

3. The Gamma Distribution

Gamma Function

The *gamma function* is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \qquad \alpha > 0 \tag{1}$$

It turns out that

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

and that

- $\Gamma(\alpha) = (\alpha - 1)! \rightarrow$ if α is a positive integer

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and that

- $\Gamma(\alpha) = (\alpha - 1)! \rightarrow$ if α is a positive integer
- $\Gamma(1/2) = \sqrt{\pi}$

Gamma Distribution

We can use the gamma function to define the density of the Gamma(α, β) distribution.

Definition 9 (The Gamma(α, β) Distribution)

$$f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \qquad 0 < x < \infty, \quad \alpha > 0, \quad \beta > 0, \tag{2}$$

where $\Gamma(\alpha)$ is defined in Equation (1) on the previous slide.

- A random variable X having density function f given by (2) is said to have the Gamma(α, β) distribution.

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- A random variable X having density function f given by (2) is said to have the $\text{Gamma}(\alpha, \beta)$ distribution.
- We write this as $X \sim \text{Gamma}(\alpha, \beta)$
- This is the parametrization used in `R` (see `?dgamma`)

Example 10 (The Gamma Distribution)

Verify that Equation (2) is really a density function.

Proof: on board

Gamma Distribution Parameters

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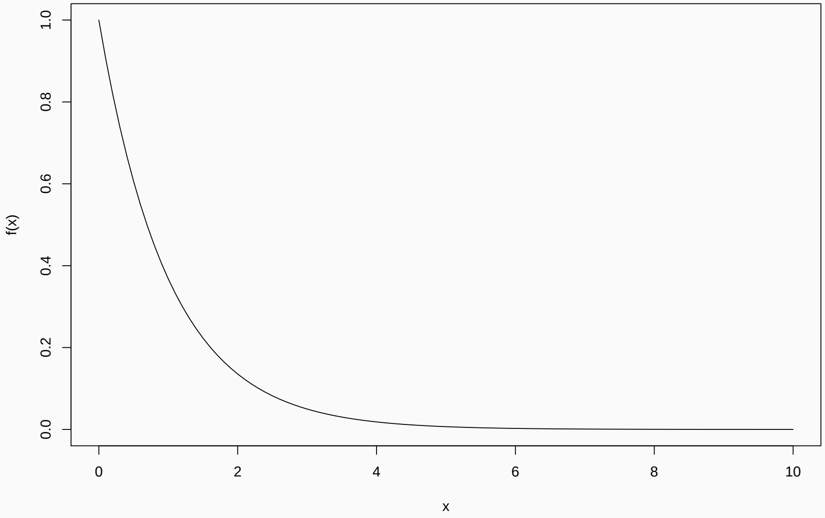
- The parameter β is called the scale parameter $\longrightarrow$ influences the spread of the distribution
- $\text{Gamma}(1, \beta) \equiv \text{Exponential}(\beta)$
- $\text{Gamma}(p/2, 2) \equiv \chi^2_{(p)}, p = \{0, 1, 2, 3, \ldots\}$

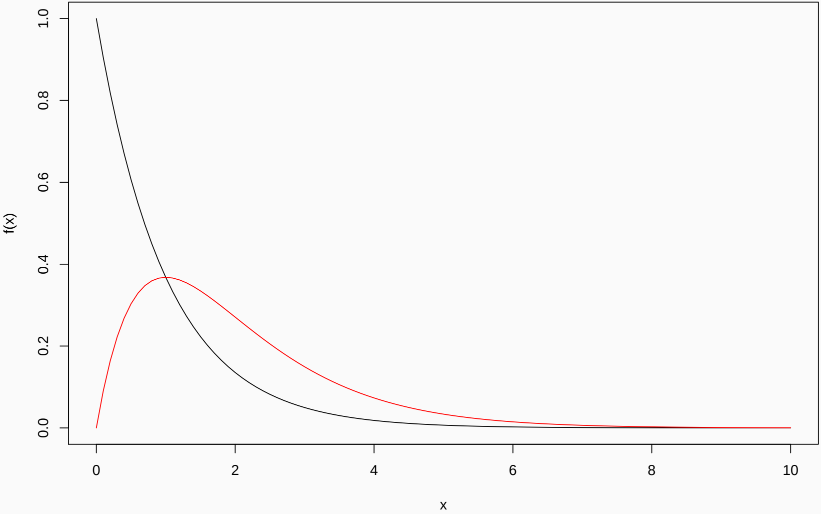
Gamma Distribution Parameters

```
curve(dgamma(x, shape = 1, scale = 1), from = 0, to = 10, ylab = "f(x)")
```



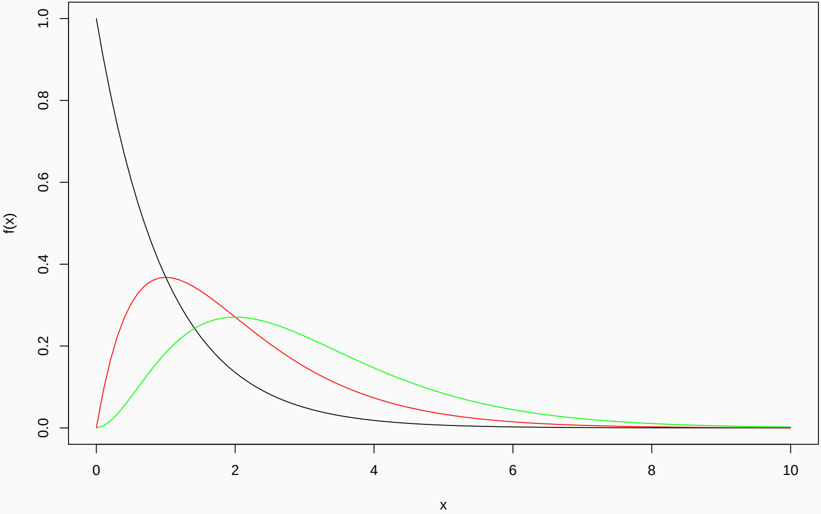
Gamma Distribution Parameters

```
curve(dgamma(x, shape = 2, scale = 1), add = TRUE, col = "red")
```



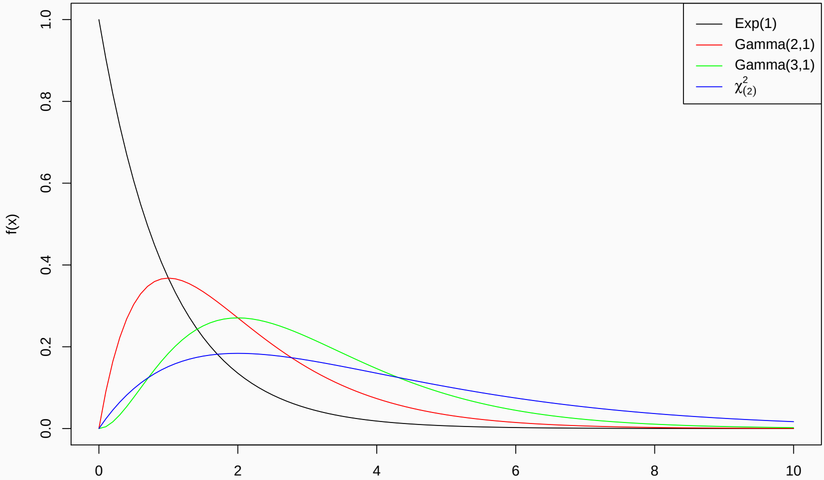
Gamma Distribution Parameters

```r
curve(dgamma(x, shape = 3, scale = 1), add = TRUE, col = "green")
```



Gamma Distribution Parameters

```
curve(dgamma(x, shape = 4/2, scale = 2), add = TRUE, col = "blue")
legend("topright",
       legend = c("Exp(1)", "Gamma(2,1)", "Gamma(3,1)",expression(chi[(2)]^2)),
       col = c("black","red","green","blue"), lty = 1)
```



Gamma Distribution (alternative definition)

Some books and software packages replace β with $1/\beta$:

Definition 11 (The Gamma(α, β) Distribution)

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad x \in \mathbb{R}^+$$

Relation between Exponential and Gamma Distribution

The Gamma distribution is another **continuous** waiting-time model. It can be shown the sum of i.i.d. Exponential random variables has a Gamma distribution, that is, if $X_1, X_2, \ldots, X_n$ are independent and identically distributed $Exponential(\lambda)$ random variables, then

$$X = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$$