Inference about a Population Mean (μ)

AAO unit 26; Baldi & Moore, Ch 17

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The t distribution

Inference for μ when σ is not known

Up until now, all of our calculations have relied on us knowing the value of the population standard deviation (σ). It is rare that this is the case.

We now consider methods of inference for when σ is unknown.

When σ is unknown, we must estimate it from the data using s, the sample standard deviation.

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- There is a different t distribution for each sample size. The degrees of freedom specify which distribution we use, and are determined by the denominator used in estimating s which is $(n-1)$.

A note about the conditions

- B&M stress that the **first** of their conditions as *very important*: *we can regard* our data as a simple random sample (SRS) from the population
- The **second**, observations from the population have a *<u>Normal</u>* distribution with unknown mean parameter μ and unknown standard deviation parameter σ less so
- *In practice*, inference procedures *can accommodate some deviations from the Normality condition* when the sample is large enough. (think CLT)

t distributions

The t distribution is symmetric, but has heavier tails than the Normal distribution.

As the degrees of freedom increase (i.e., as n increases), the t-distribution becomes more and more similar to a Normal distribution.

In fact, the quantiles/area under the curve are similar for $n \geq 30$:

Quantiles

Distribution	Cumulative Probability			
	0.005	0.010	0.025	0.050
Normal	-2.58	-2.33	-1.96	-1.64
$t(50)$	-2.68	-2.40	-2.01	-1.68
$t(30)$	-2.75	-2.46	-2.04	-1.70
$t(10)$	-3.17	-2.76	-2.23	-1.81

t procedures

We can calculate CIs and perform significance tests much as before (example coming up soon).

A significance test of a single sample mean using the t-statistic is called a one-sample t-test.

Collectively, the significance tests and confidence-interval based tests using the t distribution are called t procedures. (Tests using the Z statistic and the Normal distribution are a special case of these.)

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- Recall: Unless there is a very compelling reason (e.g. known/confirmed error in the recorded data), outliers should not be discarded.

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  - When n is large, s^2 is a good estimate of σ^2 (recall that s^2 is unbiased and, like most estimates, precision improves with increasing sample size)
  - CLT: $\bar{x}$ will be Normal when n is large, even if the population data are not

- …so how large is large enough?

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- When $n \geq 40$, t procedures can be used even if the data are highly skewed
  - Even when σ is not known, can use tests/CIs based on the Z statistic/Normal distribution rather than on the t statistic/t distribution. (Quantiles are very similar.)

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- If the data are very skewed, there are other options to consider (whether n is large or not):
    - Transformations (be careful with interpretation!)
    - Non-parametric models which do not assume any particular distribution of the data (we will study these in Part 5 of the course)

Confidence intervals for μ when σ^2 is unknown

When we must estimate the SE, the $(1-\alpha)100\%$ confidence interval now formed is:

$$\begin{aligned} \text{CI:} \quad & \text{estimate} \pm \text{margin of error} \\ & = \bar{x} \pm t_{\alpha/2} s/\sqrt{n} \end{aligned}$$

where $t_{\alpha/2}$ is the quantile of the $t(n-1)$ distribution curve such that $P(-t_{\alpha/2} < t_{n-1} < t_{\alpha/2}) = 1-\alpha$.

The CI is exactly correct when the distribution of the population data x is Normal, and approximately correct otherwise.

Recall that with highly skewed distributions and small samples, the CLT may not provide a good approximation to the distribution of $\bar{x}$.

Confidence intervals for μ when σ^2 is unknown

Exercise:
Compute a 95% and a 99% confidence interval for the mean Wechsler score of children at Lake Wobegon assuming variance is unknown.

Recall, for a 95% *Normal* CI, we use $z_{\alpha/2} = z_{0.025} = 1.96$, and for a 99% *Normal* CI, we have $z_{\alpha/2} = z_{0.005} = 2.58$. What quantile values $t_{\alpha/2}$ are used to construct CIs if we don't know the variance?

Confidence intervals for μ when σ^2 is unknown

Exercise:
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Components required to construct the CIs:

- $\bar{x} = 112.8$
- $s = \sqrt{160.4} = 12.7$
- $n = 9$
- for a 95% CI, we use $t_{\alpha/2} = t_{0.025} = 2.306$
- for a 99% CI, we use $t_{\alpha/2} = t_{0.005} = 3.355$

Inference about a population mean

(FREQUENTIST) INFERENCE for (PARAMETER) μ – the mean of an (effectively) infinite-size universe of Y values – based on the n values, $y_1, \ldots, y_n$ in an SRS from that universe/population

‘***Certain conditions apply***’ [1]

Point-estimate of μ : $\hat{\mu} = \bar{y}$

(Symmetric) Confidence Interval CI for μ : $\bar{y} \pm ME$,

where the Margin of Error (ME) is a

- z-multiple of SE [2], if n is ‘large’ AND

  the Y values in the universe have a ‘Normal’ (Gaussian’) distribution or, if not, n is large enough so that the Central Limit Theorem guarantees that the sampling distribution of possible $\bar{y}$’s of size n from this universe is well enough approximated by a Gaussian distribution
- t-multiple of SE if ‘small’ n AND

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t-distribution

When and why we use the t-distribution

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- the spread of the t distribution is greater than $\mathcal{N}(0, 1)$

Rejecting the Null ($H_0 : \mu = \mu_0$) when σ is known

$$\underbrace{z_{0.975}}_{\text{critical value}} = 1.96 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \to \frac{1.96\sigma}{\sqrt{n}} = \bar{x} - \mu_0$$

which means that to reject H_0 the difference between your sample mean and μ_0 needs to be greater than $\frac{1.96}{\sqrt{n}}$ standard deviations

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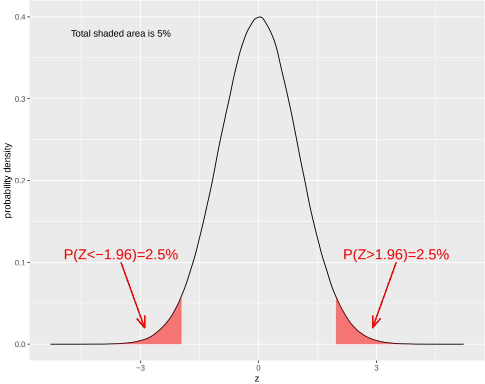

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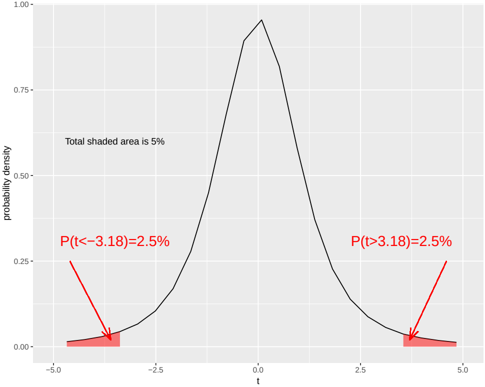

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- This is reflected in the fact that there is a different t distribution for each sample size
- As $n \to \infty$, sample variance S gets closer to σ
- As degrees of freedom increase, t distribution gets closer to Normal distribution

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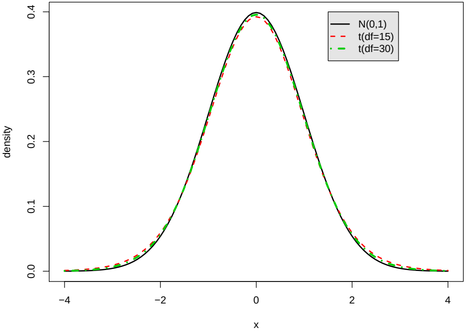
Sample size increases $\rightarrow$ degrees of freedom increase $\rightarrow$ t starts to look like $\mathcal{N}(0, 1)$



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df	$t_{0.975,df}$
3	3.1824463
5	2.5705818
30	2.0422725
50	2.0085591
100	1.9839715
250	1.9694984
500	1.9647198