Name: ID:

Quiz #1

MATH 697: Mathematical Statistics

7 November 2017, 9:00–10:00

Instructions

- This is a closed book exam.
- Answer all questions in the space provided below the questions. Extra paper will be provided if needed.
- There are a total of six questions. Each question is worth 10 marks. There are no optional questions.
- Show all your work. Be explicit about the distributions, assumptions and theorems you're using.
- Calculators and translation dictionaries are permitted.
- Discrete and continuous distributions are provided, as well as a table of standard normal probabilities.

Good Luck!

Problem 1. Let X be a uniform random variable on the interval $(-1, 1)$. Define the transformation $Y = X^2$.

(a) [**4 marks**] Compute $E(X)$, $V(X)$ and $E(X^4)$

(b) [**4 marks**] Find the CDF of Y. Be sure to include the range of Y.

(c) [**2 marks**] Find the PDF of Y. Is Y Uniform on the interval $(0, 1)$?

Problem 2. **[10 marks]** Let $U \sim Uniform(0,1)$. Define the transformation

$$X = \log\left(\frac{U}{1-U}\right)$$

(a) **[5 marks]** Find the CDF and PDF of X. Be sure to include the range of X.

(b) **[2 marks]** Compute $P(-2 < X < 2)$

(c) **[3 marks]** Find $E(X)$ without using calculus. *hint:* $1-U$ *has the same distribution as* U

Problem 3. The time that it takes a driver to react to the brake lights on a decelerating vehicle is critical in avoiding rear-end collisions. The article *Fast-Rise Brake Lamp as a Collision-Prevention Device* (Ergonomics, 1993: 391395) suggests that reaction time for an in-traffic response to a brake signal from standard brake lights can be modeled with a normal distribution having mean value 1.25 seconds and standard deviation of 0.46 seconds.

(a) [**3 marks**] What is the probability that reaction time is between 1.00 and 1.75 seconds? How would you solve this problem in `R` ? (write down the code)

(b) [**3 marks**] 2 seconds is viewed as a critically long-reaction time. What is the probability that actual reaction time will exceed this value? How would you solve this problem in `R` ? (write down the code)

(c) [**4 marks**] Find the reaction time X such that all reaction times are less than X with a probability of 0.99. How would you solve this problem in `R` ? (write down the code)

Problem 4. Let X and Y be continuous with joint PDF f given by

$$f_{X,Y}(x,y) = \begin{cases} 120x^3y & x \geqslant 0, y \geqslant 0, x+y \leqslant 1 \\ 0 & \text{else} \end{cases}$$

(a) [**3 marks**] Verify that f is indeed a valid joint PDF

(b) [**5 marks**] Find the marginal distributions of X and Y

(c) [**2 marks**] Find $E(X)$

Problem 5.

(a) [**4 marks**] Given that X has moment-generating function

$$M_X(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t}$$

find $P(|X| \leqslant 1)$

(b) [**5 marks**] A loss for a company has moment-generating function $M_X(t) = 0.16/(0.16t)$, for $t < 0.16$. An insurance policy pays a benefit equal to 70% of the loss. What is the moment-generating function of the benefit?

Problem 6. The joint distribution of X and Y is given in the following table

	X=0	X=1	X=3
Y=-1	0.11	0.03	0.00
Y=2.5	0.03	0.09	0.16
Y=3	0.15	0.15	0.06
Y=4.7	0.04	0.16	0.02

(a) [**5 marks**] What is the conditional distribution of Y given $X = 1$ denoted by $P(Y = y|X = 1)$.

(b) [**5 marks**] Compute $E(Y|X = 1)$

DISCRETE DISTRIBUTIONS

	RANGE	PARAMETERS	MASS FUNCTION	CDF	E(X)	var(X)	MGF
Bernoulli(θ)	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
Binomial(n,θ)	$\{0,\ldots,n\}$	$n \in \mathbb{N}, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
Poisson(λ)	$\{0,1,2,\ldots\}$	$\lambda \in \mathbb{R}^+$	$e^{-\lambda}\dfrac{\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
Geometric(θ)	$\{0,1,2,\ldots\}$	$\theta \in (0,1)$	$(1-\theta)^x\theta$		$\dfrac{1-\theta}{\theta}$	$\dfrac{1-\theta}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
NegBinom(r,θ)	$\{r,r+1,\ldots\}$	$r \in \mathbb{N}, \theta \in (0,1)$	$\binom{x-1}{r-1}\theta^r(1-\theta)^{x-r}$		$\dfrac{r}{\theta}$	$\dfrac{r(1-\theta)}{\theta^2}$	$\left\{\dfrac{\theta e^t}{1-e^t(1-\theta)}\right\}^r$
or	$\{0,1,2,\ldots\}$	$r \in \mathbb{N}, \theta \in (0,1)$	$\binom{r+x-1}{x}\theta^r(1-\theta)^x$		$\dfrac{r(1-\theta)}{\theta}$	$\dfrac{r(1-\theta)}{\theta^2}$	$\left\{\dfrac{\theta}{1-e^t(1-\theta)}\right\}^r$

For *continuous* distributions (see next page), define *Euler's gamma function*, for all $\alpha > 0$, by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,\mathrm{d}x.$$

Further note that the *location/scale* transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right), \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right), \qquad M_Y(t) = e^{\mu t}M_X(\sigma t), \qquad \mathrm{E}\,(Y) = \mu + \sigma\mathrm{E}\,(X)\,, \qquad \mathrm{var}\,(Y) = \sigma^2\mathrm{var}\,(X)\,.$$

CONTINUOUS DISTRIBUTIONS

		PARAM.	PDF	CDF	E(X)	var(X)	MGF
Uniform(α, β)	(α, β)	$\alpha < \beta \in \mathbb{R}$	$\dfrac{1}{\beta - \alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{\alpha+\beta}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t} - e^{\alpha t}}{t\,(\beta - \alpha)}$
Exponential(β)	$\mathbb{R}^+$	$\beta \in \mathbb{R}^+$	$\frac{1}{\beta}\, e^{-x/\beta}$	$1 - e^{-x/\beta}$	β	β^2	$(1-\beta t)^{-1}$
Gamma(α, β)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\dfrac{1}{\beta^\alpha \Gamma(\alpha)}\, x^{\alpha-1} e^{-x/\beta}$		$\alpha\beta$	$\alpha\beta^2$	$(1-\beta t)^{-\alpha}$
Weibull(α, β)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\alpha\beta x^{\alpha-1} e^{-\beta x^\alpha}$	$1 - e^{-\beta x^\alpha}$	$\dfrac{\Gamma\,(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\,(1+2/\alpha) - \Gamma\,(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
Normal(μ, σ^2)	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\mu t + \sigma^2 t^2/2}$
Student(ν)	$\mathbb{R}$	$\nu \in \mathbb{R}^+$	$\dfrac{(\pi\nu)^{-1/2}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left(1+\dfrac{x^2}{\nu}\right)^{(\nu+1)/2}}$		$0 \quad (\text{if } \nu > 1)$	$\dfrac{\nu}{\nu-2} \quad (\text{if } \nu > 2)$	
Pareto(θ, α)	(θ, ∞)	$\theta, \alpha \in \mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{x^{\alpha+1}}$	$1 - \left(\dfrac{\theta}{x}\right)^\alpha$	$\dfrac{\alpha\theta}{\alpha-1}$ (if $\alpha > 1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha > 2$)	
Beta(α, β)	$(0, 1)$	$\alpha, \beta \in \mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

Table of the Normal distribution

Entries in the table are the values of the cumulative distribution function Φ of the $\mathcal{N}(0,1)$ distribution, evaluated at z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998