Q1 The correlation between two random variables X and Y is given by $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$. Proove the following

(a) Shifting X and Y has no effect on their correlation, i.e., $Corr(X+a, Y+b) = Corr(X,Y)$ for any constants a and b

(b) Scaling X and Y has no effect on their correlation, i.e., $Corr(cX, Y) = Corr(X,Y)$ for any constant c

(c) Scaling and shifting both X and Y has no effect on their correlation, i.e., $Corr(cX+a, dY+b) = Corr(X,Y)$ for any constants a, b, c, d

(d) Starting from the definition of expectations, show that $Var(aX+bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$

(e) $Cov(X+Y, Z) = Cov(X,Z) + Cov(Y,Z)$

(f) $Cov(X+Y, Z+W) = Cov(X,Z) + Cov(X,W) + Cov(Y,Z) + Cov(Y,W)$

(g) Show that $E((X-EX)(Y-EY)) = E(XY) - E(X)E(Y)$

(h) Show that $Corr(X,Y)$ is bounded between -1 and 1

Q2 Let X and Y have joint PDF given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} y^{\beta-1} e^{-x} e^{-y} & 0 < x < \infty, 0 < y < \infty \\ 0 & else \end{cases}$$

Let $U = X + Y$ and $V = X/(X+Y)$.

(a) Find the joint PDF of (U,V)

(b) Find the marginal distributions of U and V.

(c) Are U and V independent?

Q3 Let X and Y be independent $Uniform(0,1)$ random variables. Define the transformations $U = \sqrt{-2\log(X)}\cos(2\pi Y)$ and $V = \sqrt{-2\log(X)}\sin(2\pi Y)$ (where log is the natural logarithm).

(a) Find the joint PDF of (U,V)

(b) Find the marginal distributions of U and V.

(c) Are U and V independent?

Q4 Let $X \sim Uniform(0,1)$ and $Y \sim Uniform(0,1/10)$, where X and Y are independent random variables. Consider the transformation $U = X$ and $V = X + Y$

(a) Find the joint PDF of (U,V)

(b) Find the $Cov(X,Y)$ and $Corr(X,Y)$