Q1 A gas station operates two pumps, each of which can pump up to 10,000 gallons of gas in a month. The total amount of gas pumped at the station in a month is a random variable Y, measured in 10,000 gallons, with a probability density function given by

$$f(y) = \begin{cases} y, & 0 \leq y < 1, \\ 2 - y, & 1 \leq y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the cumulative distribution function F of Y denoted by $F_Y(y)$.

(b) Find the probability that the station will pump between 8500 and 11,500 gallons in a particular month.

(c) What is the expected monthly revenue if the station sells the gas at the price of 2.10$ per gallon?

Q2 In each of the following cases, identify the standard distribution that is most likely to be suitable for the random phenomenon at hand. Justify your choice, and specify which parameters are known and which parameters are unknown from the context.

(a) The size of a home-owner's financial loss due to the fire damage in Fort McMurray.

(b) The number of bags filled with coffee beans that needs to be examined before 15 bags containing spoiled beans are found.

(c) The number of bags filled with coffee beans that contains spoiled beans out of the 15 bags examined.

(d) The actual speed of a vehicle at a specific location on a highway.

Q3 Define the random variable X with PDF given by

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

for $\alpha, \beta > 0$.

(a) Derive closed-form expressions for $\mathrm{E}(X)$ and $V(X)$.
recall: for any $\alpha > 0$, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

(b) Compute the moment generating function $M(t)$ of the Poisson distribution with rate parameter $\lambda > 0$. Use $M(t)$ to show that the mean and variance of a Poisson random variable are equal.

Q4 Suppose $X \sim Unif(a, b)$.

(a) Derive $E(X)$ and $V(X)$

(b) Derive the CDF of X, given by $F_X(x)$

(c) Derive an expression for $P(c \leq X < d)$, where $a < c < d < b$

(d) Derive the Moment Generating Function (MGF) of X

(e) Using the MGF, verify the results in (a)

Q5 Suppose $Y \sim Gamma(\alpha, \beta)$, with $\alpha = \nu/2$ and $\beta = 2$.

(a) Derive $E(Y)$ and $V(Y)$

(b) Derive the CDF of Y, given by $F_Y(y)$

(c) Derive the Moment Generating Function (MGF) of Y

(d) Using the MGF, verify the results in (a)

Q6 Suppose $Z \sim N(\mu, \sigma^2)$.

(a) Derive $E(Z)$ and $V(Z)$

(b) Derive the CDF of Z, given by $F_Z(z)$

(c) Derive the Moment Generating Function (MGF) of Z

(d) Using the MGF, verify the results in (a)

Q7 Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

for $\alpha > 0$, $\beta > 0$.

(a) Derive $E(X)$ and $V(X)$

(b) Derive the CDF of X, given by $F_X(x)$

Q8 Determine the PDF of the following transformed random variables Y (be sure to include the support of Y in your final answer):

(a) $Y = \frac{\beta X}{\alpha(1-X)}$ where the PDF of X is given by

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

for $\alpha, \beta > 0$.

(b) $Y = \frac{\max(X,6)-1}{\max(X,6)}$ where $X \sim Geometric(p)$, $R_X = \{0, 1, 2, \ldots\}$, and $\max(X, 6)$ is the maximum of the random variable X and the number 6.

(c) $Y = X^2$, if X is uniformly distributed on [-1, 3]

(d) $Y = \sqrt{Z}$ where $Z = X^3$ and $X \sim Uniform[2, 7]$ (*hint: first obtain the density of* Z*, then the density of* Y)

(e) $Y = e^X$ where $X \sim N(0, \sigma^2)$

(f) $Y = Z^2$ where $Z \sim N(0, 1)$. Compare the PDF of Y to the one in **Q5**. What do you notice?

Q9 A new treatment for a disease is being tested, to see whether it is better than the standard treatment. The existing treatment is effective on 50% of patients. It is believed initially that there is a 2/3 chance that the new treatment is effective on 60% of patients, and a 1/3 chance that the new treatment is effective on 50% of patients. In a pilot study, the new treatment is given to 20 random patients, and is effective for 15 of them.

(a) Given this information, what is the probability that the new treatment is better than the standard treatment?

(b) A second study is done later, giving the new treatment to 20 new random patients. Given the results of the first study, what is the PMF for how many of the new patients the new treatment is effective on? (Letting p be the answer to (a), your answer can be left in terms of p.)

Q10 Consider the following simplified scenario based on Who Wants to Be a Millionaire?, a game show in which the contestant answers multiple-choice questions that have 4 choices per question. The contestant has answered 9 questions correctly already, and is now being shown the 10th question. He has no idea what the right answers are to the 10th or 11th questions are. He has one lifeline available, which he can apply on any question, and which narrows the number of choices from 4 down to 2. The contestant has the following options available. Find the expected value of each of these options. Which option has the highest expected value? Which option has the lowest variance?

(a) Walk away with \$16,000.

(b) Apply his lifeline to the 10th question, and then answer it. If he gets it wrong, he will leave with $1,000. If he gets it right, he moves on to the 11th question. He then leaves with $32,000 if he gets the 11th question wrong, and $64,000 if he gets the 11th question right.

(c) Same as the previous option, except not using his lifeline on the 10th question, and instead applying it to the 11th question (if he gets the 10th question right).

Q11 Let X be a valid PMF given by

$$P(X = k) = \frac{cp^k}{k}, \qquad \text{for } k = 1, 2, 3, \ldots$$

where p is a parameter $0 < p < 1$ and c is a normalising constant.

(a) Determine the normalising constant c

(b) Find the mean and variance of X.

Q12 In class we discussed the memoryless property for the exponential distribution which is a continous distribution. Equivalently, a discrete distribution for the random variable X has the memoryless property if

$$P(X \geq j + k \mid X \geq j) = P(X \geq k)$$

for all integers $j, k > 0$

(a) Assume X has the memoryless property with CDF $F_X(x)$ and PMF $f_X(x)$. Find an expression for $P(X \geq j + k)$ in terms of $F_X(j), F_X(k), f_X(k), f_X(j)$

(b) Name a discrete distribution which has the memoryless property. Justify your answer in a way similar to the exponential distribution we showed in class, i.e., with formulas.

Q13 Suppose that X has density

$$f_X(x) = \frac{1}{\sigma}\left(1 + \frac{\xi x}{\sigma}\right)^{-(1/\xi+1)}$$

where $x > 0$, $\xi \in (0, 1)$ and $\sigma > 0$

(a) Compute $P(X \geq x)$ and compute the upper bound on this quantity given by Markov's Inequality

(b) Compute an upper bound on $P(X \geq x)$ using Chebychev's inequality when $\xi < 1/2$

(c) Compare the two bounds graphically to one another as well as to the true probability when $\sigma = 1$ and $\xi \in \{1/10, 1/4, 0.45\}$. What do you observe? How could these bounds be improved when x is large?