Name: ID:

Quiz #3

MATH 697: Mathematical Statistics

28 November 2017, 9:00–10:00

Instructions

- This is a closed book exam.
- Answer all questions in the space provided below the questions. Extra paper will be provided if needed.
- There are a total of five questions. Each question is worth 10 marks. There are no optional questions.
- Show all your work. Be explicit about the distributions, assumptions and theorems you're using.
- Calculators and translation dictionaries are permitted.
- Discrete and continuous distributions are provided

Good Luck!

Problem 1. Suppose that X and Y are independent, each with distribution $Exponential(\lambda)$. Consider the following transformation: $U = X + Y$ and $V = X/(X + Y)$.

(a) [**7 marks**] What is the joint density of U and V?

(b) [**3 marks**] Are U and V independent? What are the distributions of U and V?

Problem 2. [10 marks] Let $X_i|P_i \sim Bernoulli(P_i)$ for $i = 1, \ldots, n$ and $P_i \sim beta(\alpha, \beta)$, where the X_i's are independent and the PDF of P_i is given by

$$f(p_i) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p_i^{\alpha-1}(1 - p_i)^{\beta-1}$$

A random variable of interest is $Y = \sum_{i=1}^{n} X_i$. You are also given that $E(P_i) = \frac{\alpha}{\alpha+\beta}$ and that $V(P_i) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$. What is $E(Y)$ and $V(Y)$ (*5 marks* each)?

Problem 3. Let X and Y be continuous random variables with joint PDF

$$f_{X,Y}(x,y) = 3x, \quad 0 \leqslant y \leqslant x \leqslant 1$$

Compute the covariance $Cov(X,Y)$ and correlation $Corr(X,Y)$.

Problem 4. Let X_1, X_2, X_3 denote a random sample of size 3 from a $Gamma(\alpha = 7, \beta = 5)$ distribution. Let

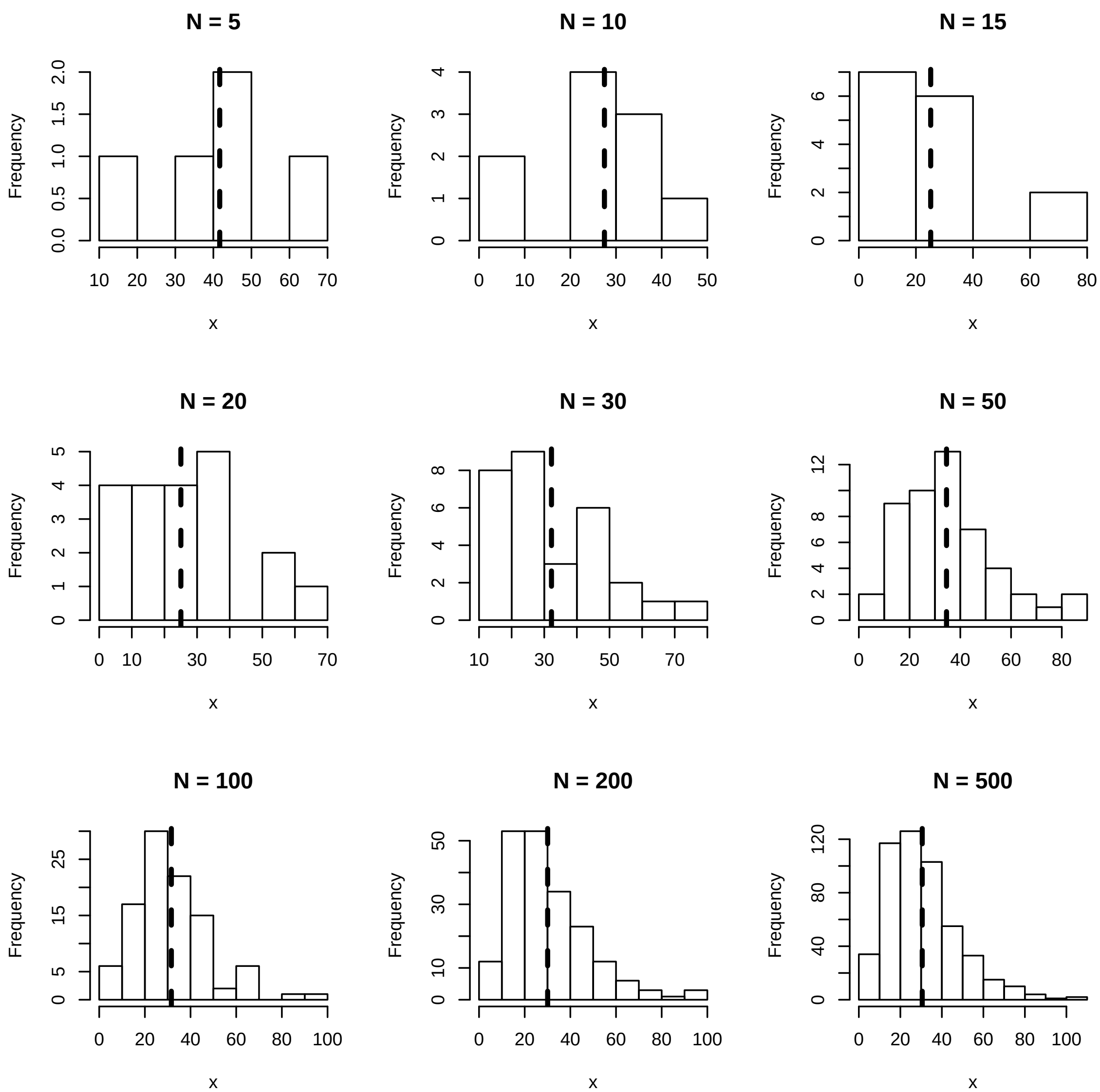
$$Y = X_1 + X_2 + X_3$$

(a) [**5 marks**] What is the distribution of Y?

(b) [**5 marks**] What is the distribution of the sample mean $\bar{Y}$?

Problem 5.

(a) [**5 marks**] The figure below contains nine histograms of random samples from a $Gamma(3, 10)$ distribution with varying sample size N. The dotted line represents the sample mean $\bar{X}_n$. What do you notice as the sample size increases? What is this phenomenon called?



(b) [**5 marks**] A certain brand of MP3 player comes in three configurations:

memory	2 GB	4 GB	8 GB
x (cost)	80	100	120
$p(x)$	0.20	0.30	0.50

With $\mu = 106, \sigma^2 = 244$. Suppose only two MP3 players are sold today: X_1 and X_2 representing the cost of the 1st and 2nd player, respectively. The table below lists possible (x_1, x_2) pairs, the probability of each assuming independence and the resulting $\bar{x}$ and s^2 values (when $n = 2$, $s^2 = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2$)

x_1	x_2	$p(x_1, x_2)$	$\bar{x}$	s^2
80	80	(.2)(.2) = .04	80	0
80	100	(.2)(.3) = .06	90	200
80	120	(.2)(.5) = .10	100	800
100	80	(.3)(.2) = .06	90	200
100	100	(.3)(.3) = .09	100	0
100	120	(.3)(.5) = .15	110	200
120	80	(.5)(.2) = .10	100	800
120	100	(.5)(.3) = .15	110	200
120	120	(.5)(.5) = .25	120	0

1. What is a statistic? Which quantities in the above table are statistics?
2. What is a sampling distribution ?
3. Compute $P(\bar{X} = 110)$ and $P(S^2 = 200)$

End of exam.

DISCRETE DISTRIBUTIONS							
	RANGE	PARAMETERS	MASS FUNCTION	CDF	E(X)	var(X)	MGF
Bernoulli(θ)	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
Binomial(n,θ)	$\{0,\ldots,n\}$	$n \in \mathbb{N}, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
Poisson(λ)	$\{0,1,2,\ldots\}$	$\lambda \in \mathbb{R}^+$	$e^{-\lambda}\dfrac{\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
Geometric(θ)	$\{0,1,2,\ldots\}$	$\theta \in (0,1)$	$(1-\theta)^x\theta$		$\dfrac{1-\theta}{\theta}$	$\dfrac{1-\theta}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
NegBinom(r,θ)	$\{r,r+1,\ldots\}$	$r \in \mathbb{N}, \theta \in (0,1)$	$\binom{x-1}{r-1}\theta^r(1-\theta)^{x-r}$		$\dfrac{r}{\theta}$	$\dfrac{r(1-\theta)}{\theta^2}$	$\left\{\dfrac{\theta e^t}{1-e^t(1-\theta)}\right\}^r$
or	$\{0,1,2,\ldots\}$	$r \in \mathbb{N}, \theta \in (0,1)$	$\binom{r+x-1}{x}\theta^r(1-\theta)^x$		$\dfrac{r(1-\theta)}{\theta}$	$\dfrac{r(1-\theta)}{\theta^2}$	$\left\{\dfrac{\theta}{1-e^t(1-\theta)}\right\}^r$

For *continuous* distributions (see next page), define *Euler's gamma function*, for all $\alpha > 0$, by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,\mathrm{d}x.$$

Further note that the *location/scale* transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right), \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right), \qquad M_Y(t) = e^{\mu t}M_X(\sigma t), \qquad \mathrm{E}\,(Y) = \mu + \sigma\mathrm{E}\,(X)\,, \qquad \mathrm{var}\,(Y) = \sigma^2\mathrm{var}\,(X)\,.$$

CONTINUOUS DISTRIBUTIONS

		PARAM.	PDF	CDF	E(X)	var(X)	MGF
Uniform(α, β)	(α, β)	$\alpha < \beta \in \mathbb{R}$	$\frac{1}{\beta - \alpha}$	$\frac{x - \alpha}{\beta - \alpha}$	$\frac{\alpha + \beta}{2}$	$\frac{(\beta - \alpha)^2}{12}$	$\frac{e^{\beta t} - e^{\alpha t}}{t\,(\beta - \alpha)}$
Exponential(β)	$\mathbb{R}^+$	$\beta \in \mathbb{R}^+$	$\frac{1}{\beta}\, e^{-x/\beta}$	$1 - e^{-x/\beta}$	β	β^2	$(1 - \beta t)^{-1}$
Gamma(α, β)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{1}{\beta^\alpha \Gamma(\alpha)}\, x^{\alpha - 1} e^{-x/\beta}$		$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Weibull(α, β)	$\mathbb{R}^+$	$\alpha, \beta \in \mathbb{R}^+$	$\alpha\beta x^{\alpha - 1} e^{-\beta x^\alpha}$	$1 - e^{-\beta x^\alpha}$	$\frac{\Gamma\,(1 + 1/\alpha)}{\beta^{1/\alpha}}$	$\frac{\Gamma\,(1 + 2/\alpha) - \Gamma\,(1 + 1/\alpha)^2}{\beta^{2/\alpha}}$	
Normal(μ, σ^2)	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\mu t + \sigma^2 t^2/2}$
Student(ν)	$\mathbb{R}$	$\nu \in \mathbb{R}^+$	$\frac{(\pi\nu)^{-1/2}\Gamma\left(\frac{\nu + 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(1 + \frac{x^2}{\nu}\right)^{(\nu + 1)/2}}$		$0 \quad (\text{if } \nu > 1)$	$\frac{\nu}{\nu - 2} \quad (\text{if } \nu > 2)$	
Pareto(θ, α)	(θ, ∞)	$\theta, \alpha \in \mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{x^{\alpha + 1}}$	$1 - \left(\frac{\theta}{x}\right)^\alpha$	$\frac{\alpha\theta}{\alpha - 1}$ (if $\alpha > 1$)	$\frac{\alpha\theta^2}{(\alpha - 1)(\alpha - 2)}$ (if $\alpha > 2$)	
Beta(α, β)	$(0, 1)$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha - 1}(1 - x)^{\beta - 1}$		$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	