Bootstrap Confidence Intervals

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September 26, 2018

Review of Confidence Intervals

Sampling Distribution

Definition 1 (Sampling Distribution)

- *The sampling distribution of a statistic is the distribution of values taken by the statistic in* ***all possible samples of the same size*** *from the same population.*
- *The standard deviation of a sampling distribution is called a* ***standard error***

Sampling Distributions

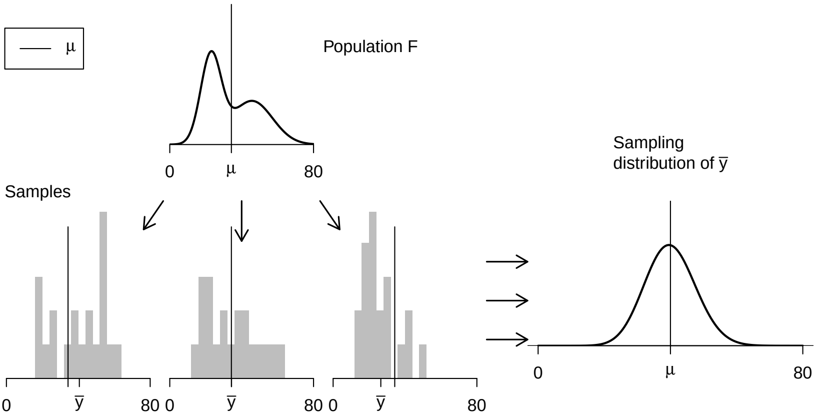

Fig.: Ideal world. Sampling distributions are obtained by drawing repeated samples from the population, computing the statistic of interest for each, and collecting (an infinite number of) those statistics as the sampling distribution

Sampling Distribution

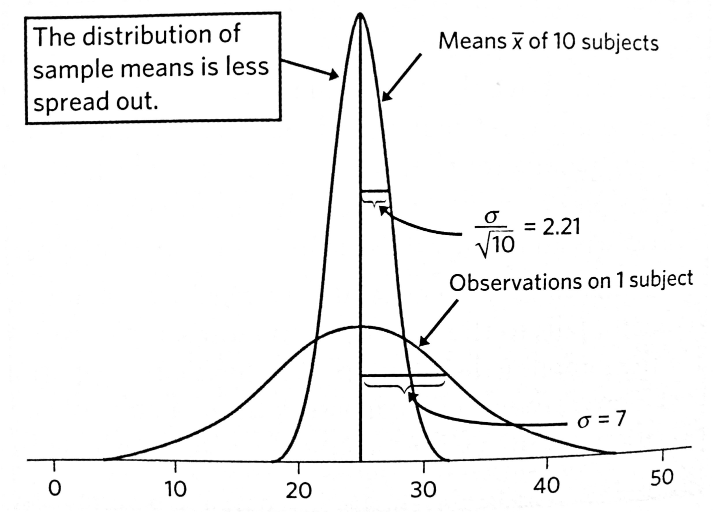

Fig.: Averages are less variable than individual observations

Traditional way to calculate CIs

How to construct a CI for the population mean?

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$$P(-z^* < Z < z^*) = \mathcal{C} = 1 - \alpha \tag{1}$$

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  - That is, we want the value z^* that gives a *lower tail probability* of $(1-\mathcal{C})/2 = \alpha/2$.
  - Often this value is denoted $z^* = z_{\alpha/2}$; thus we have

$$P(Z < -z_{\alpha/2}) = \alpha/2,$$

    and

$$P(Z > z_{\alpha/2}) = \alpha/2.$$

Traditional way to calculate CIs

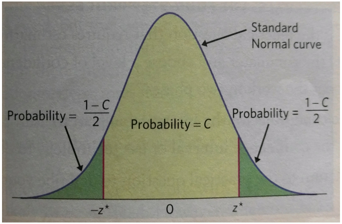

Fig.: The critical value $z^\star$ is the number that catches central probability $\mathcal{C}$ under a standard normal $\mathcal{N}(0,1)$ curve between $-z^\star$ and $z^\star$

We can use this probability statement about the standardized version of the sample mean $(\bar{y}-\mu)/\sigma/\sqrt{n}$, to place bounds on where we think the true mean lies by examining the probability that $\bar{y}$ is within $z^\star \cdot \frac{\sigma}{\sqrt{n}}$ of μ

$$
\begin{aligned}
\mathcal{C} &= P\left(-z^\star \leq \frac{\bar{y}-\mu}{\sigma/\sqrt{n}} \leq z^\star\right) \\
&= P\left(-z^\star \frac{\sigma}{\sqrt{n}} \leq \bar{y}-\mu \leq +z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(-\bar{y}-z^\star \frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\bar{y}+z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\bar{y}+z^\star \frac{\sigma}{\sqrt{n}} \geq \mu \geq \bar{y}-z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\bar{y}-z^\star \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{y}+z^\star \frac{\sigma}{\sqrt{n}}\right) \\
&= 1-\alpha
\end{aligned}
$$

We call the interval $\left(\bar{y}-z^\star \frac{\sigma}{\sqrt{n}}, \bar{y}+z^\star \frac{\sigma}{\sqrt{n}}\right)$ a **(1-α)100% confidence interval** for μ.

Confidence intervals for depths of the ocean

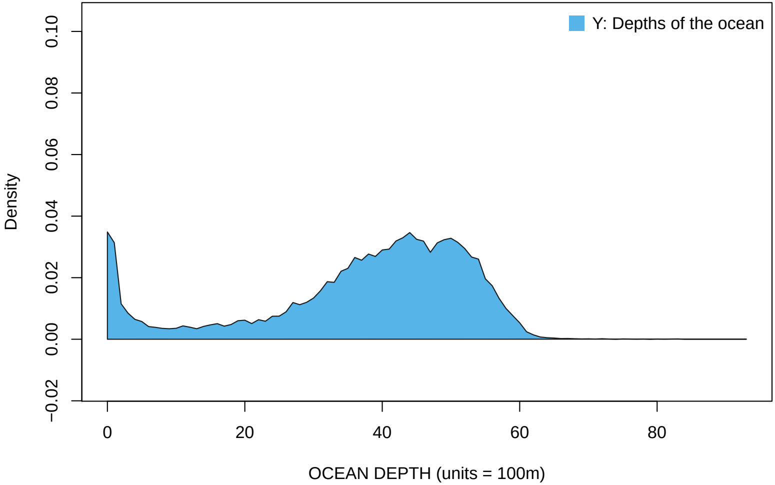

Fig.: The original data distribution of sampled depths of the ocean. Note that it has multiple modes and not Normal looking.

The CLT is 'kicking in' at $n = 16$

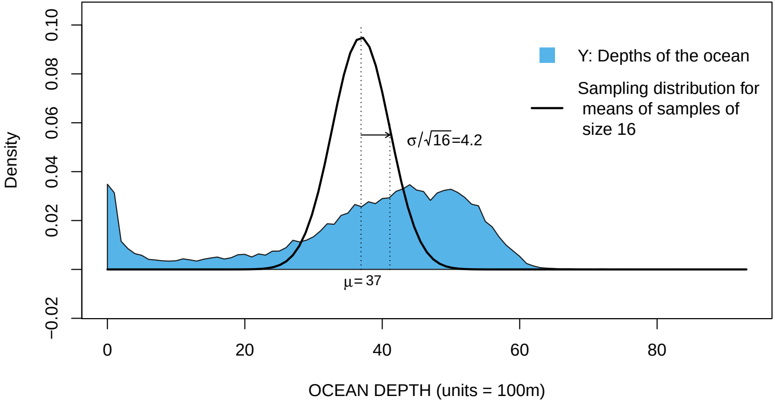

Fig.: The sampling distribution for the mean depth of the ocean with samples of size $n = 16$, looks normal (centered at $\mu = 37$ and SD equal to $\sigma/\sqrt{16}$)

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```
mosaic::xqnorm(p = c(0.025, 0.975))
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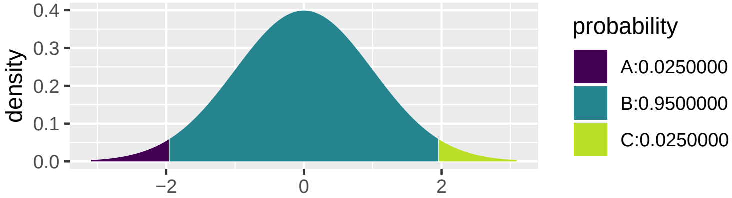


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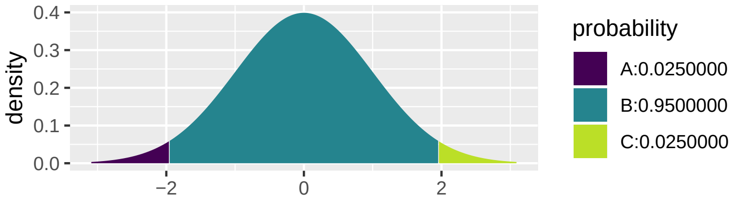


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3. 95% CI for μ: $(37 - 1.96 \cdot 4.2, 37 + 1.96 \cdot 4.2) = [29, 45]$

Alternative way of calculating CI with CLT: `qnorm`

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- There is an alternative, **yet equivalent**, way to calculate the CI without standardizing $\bar{y}$, and without using the $\pm$ formula
- This is accomplished using `qnorm`
- Note: we **still need the CLT regardless** of whether we use the $\pm$ formula or `qnorm`

68% Confidence interval using `qnorm`

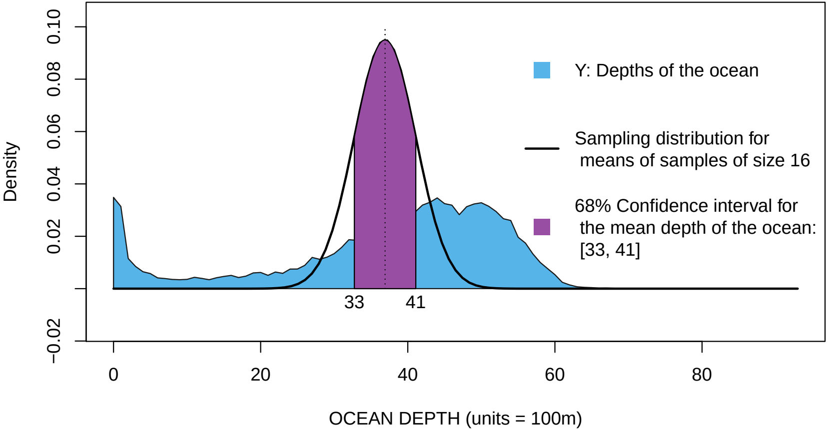

Fig.: 68% Confidence interval calculated using
`qnorm(p = c(0.16,0.84), mean = 37, sd = 4.2)`

95% Confidence interval using `qnorm`

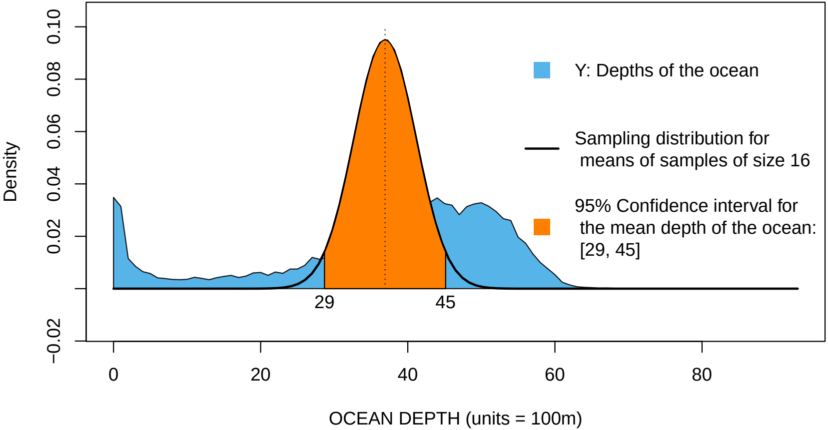

Fig.: 95% Confidence interval calculated using
`qnorm(p = c(0.025,0.975), mean = 37, sd = 4.2)`

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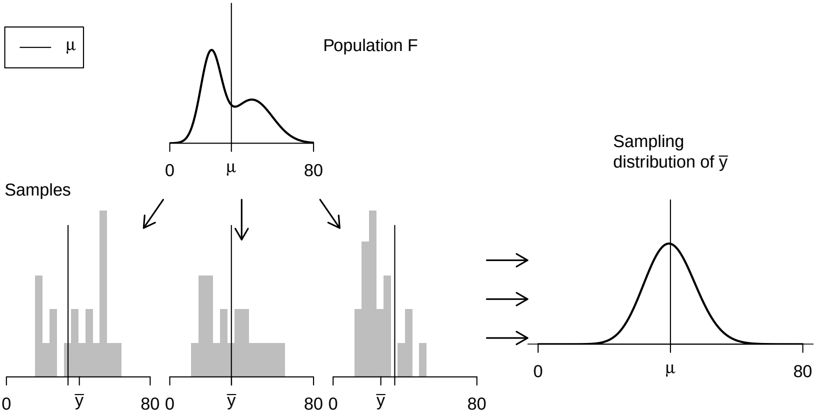
A: Bootstrap

The Bootstrap

Ideal world: known sampling distribution



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Reality: use the bootstrap distribution instead

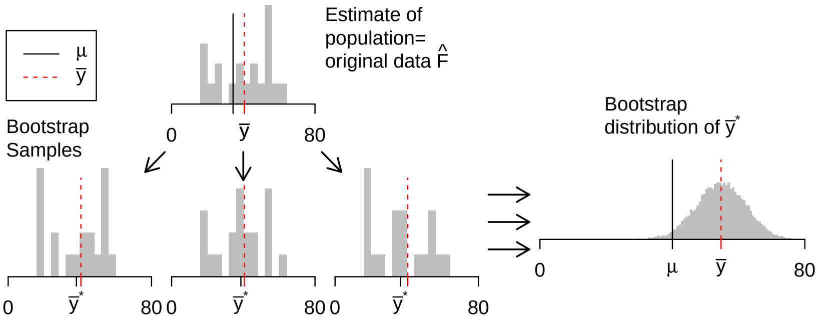


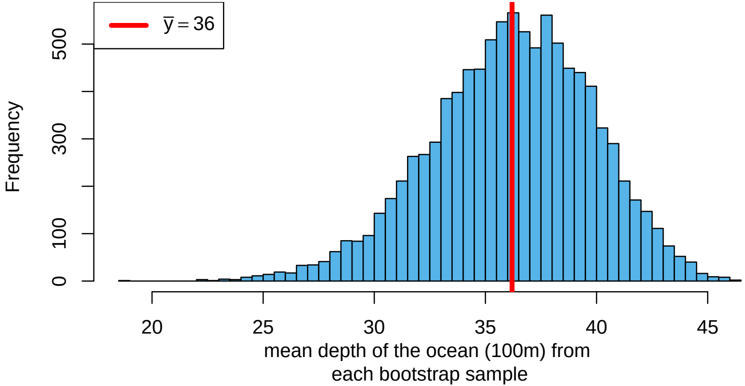
Fig.: Bootstrap world. The bootstrap distribution is obtained by drawing repeated samples from an estimate of the population, computing the statistic of interest for each, and collecting those statistics. The distribution is centered at the observed statistic ($\bar{y}$), not the parameter (μ).

Main idea: simulate your own sampling distribution

```
mean(~ alt, data = depths.n.20)
```

```
## [1] 36.2
```

```
library(mosaic)
s_dist <- do(10000) * mean( ~ alt, data = resample(depths.n.20))
```



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