Week 11: Convergence in Distribution and Central Limit Theorem

MATH697

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Convergence in Distribution

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Definition 1 (Convergence in Distribution)

Let $X_1, X_2, \ldots$ be a sequence of random variables. Then we say that the **sequence converges in distribution** to a random variable X if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

at all points x where $F_X(x)$ is continuous and we write $X_n \xrightarrow{D} X$

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- The importance of this, is that often the distribution of X_n is difficult to work with, while that of X is much simpler
- With X_n converging in distribution to X, however, we can approximate the distribution of X_n by that of X.

Convergence in Probability Implies in Distribution

Theorem 2 (Convergence in Probability Implies in Convergence in Distribution)

If the sequence of random variables $X_1, X_2, \ldots$ converges to a random variable X, the sequence also converges in distribution to X. That is if $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{D} X$

Example 3 (Poisson Approximation to the Binomial)

Suppose that $X_n \sim Binomial(n, \lambda/n)$ and $X \sim Poisson(\lambda)$. We have previously seen that as $n \longrightarrow \infty$

$$P(X_n = j) = \binom{n}{j}\left(\frac{\lambda}{n}\right)^j \left(1 - \frac{\lambda}{n}\right)^{n-j} \to e^{-\lambda}\frac{\lambda^j}{j!}$$

This implies that $F_{X_n}(x) \longrightarrow F_X(x)$ and thus $X_n \xrightarrow{D} X$

Central Limit Theorem

Introduction

- Let $X_1, X_2, \ldots,$ be iid with mean μ and finite variance σ^2. The law of large numbers says that as $n \to \infty$, $\bar{X}_n$ converges to the constant μ (with probability 1).

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- But what is its distribution along the way to becoming a constant?
- This is addressed by the central limit theorem (CLT)

Central Limit Theorem (CLT)

- The CLT states that for large n, the distribution of $\bar{X}_n$ (after standardization) approaches a standard Normal distribution.
- By standardization, we mean that we subtract μ, the expected value of $\bar{X}_n$, and divide by $\sigma/\sqrt{n}$, the standard deviation of $\bar{X}_n$.

Theorem 4 (CLT)

As $n \to \infty$

$$\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \xrightarrow{D} \mathcal{N}(0,1)$$

. From this we can also say that for large n

$$\bar{X}_n \sim \mathcal{N}(\mu, \sigma^2/n)$$

Proof:on board

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- Starting from virtually no assumptions (other than independence and finite variances), **we end up with normality**

Running Proportion of Heads Revisited

Before, we used the law of large numbers to conclude that $\bar{X}_n \to 1/2$ as $n \to \infty$. Now, using the central limit theorem, we can say more: $E(\bar{X}_n) = 1/2$ and $Var(\bar{X}_n) = 1/(4n)$, so for large n,

$$\bar{X}_n \sim \mathcal{N}\left(\frac{1}{2}, \frac{1}{4n}\right)$$

Poisson Convergence to Normal

Let $Y \sim Pois(n)$. Using MGFs it can be shown that Y can be expressed as a sum of n iid Poisson(1) random variables. Therefore by CLT, for large n,

$$Y \sim \mathcal{N}(n, n)$$

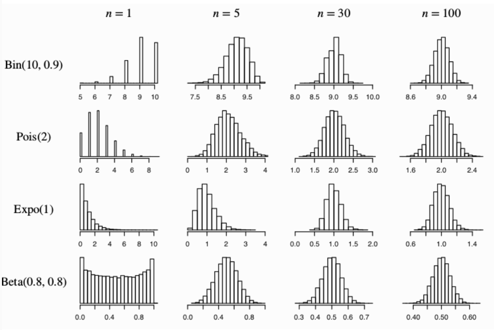
Binomial Convergence to Normal

Let $Y \sim Binomial(n, p)$. Using MGFs it can be shown that Y can be expressed as a sum of n iid Bernoulli(p) random variables. Therefore by CLT, for large n,

$$Y \sim \mathcal{N}(np, np(1-p))$$

CLT Visual

Histograms of the distribution of $\bar{X}_n$ for different starting distributions of the X_j (indicated by the rows) and increasing values of n (indicated by the columns). Each histogram is based on 10,000 simulated values of $\bar{X}_n$. Regardless of the starting distribution of the X_j, the distribution of $\bar{X}_n$ approaches a Normal distribution as n grows.



CLT Exercise

One way to visualize the CLT for a distribution of interest is to plot the distribution of $\overline{X}_n$ for various values of n. To do this, we first have to generate iid $X_1, \ldots, X_n$ a bunch of times from our distribution of interest. For example, suppose that our distribution of interest is $Unif(0, 1)$, and we are interested in the distribution of $\overline{X}_{12}$, i.e., we set $n = 12$. In the following code, we create a matrix of iid standard Uniforms. The matrix has 12 columns, corresponding to X_1 through X_{12}. Each row of the matrix is a different realization of X_1 through X_{12}.

```
nsim <- 10^4
n <- 12
x <- matrix(runif(n*nsim), nrow=nsim, ncol=n)
xbar <- rowMeans(x)
hist(xbar)
# change runif for rexp. what do you notice?
```

Session Info

```
devtools::session_info()
```

```
## setting  value
## version  R version 3.4.1 (2017-06-30)
## system   x86_64, linux-gnu
## ui       X11
## language en_US
## collate  en_US.UTF-8
## tz       Canada/Eastern
## date     2017-11-14
##
## package     * version    date       source
## abind         1.4-5      2016-07-21 cran (@1.4-5)
## arm           1.9-3      2016-11-27 cran (@1.9-3)
## assertthat    0.2.0      2017-04-11 CRAN (R 3.4.1)
## backports     1.1.0      2017-05-22 cran (@1.1.0)
## base        * 3.4.1      2017-07-08 local
## bindr         0.1        2016-11-13 CRAN (R 3.4.1)
## bindrcpp      0.2        2017-06-17 CRAN (R 3.4.1)
## blme          1.0-4      2015-06-14 cran (@1.0-4)
## broom         0.4.2      2017-02-13 CRAN (R 3.4.1)
```