Lecture 10: Linear regression with R

- Linear regression
  - Simple linear regression
  - Multiple linear regression
- Analysis of variance (ANOVA)

10.1 Linear regression

10.1.1 Simple linear regression

I will use simulated data for this section. This will help you understand model assumptions.

For this lecture, try to understand the data generating process.

Some background information.

- I spent a few weeks shopping for a diamond ring.
- I browsed many diamonds with G or H color, VVS1 or VVS2 clarity, excellent cut and between 1.00 and 1.50 carats.
  - I found out that the prices of such diamonds were predictable.
  - My gut feeling was that there was an algorithm for pricing, which waw then rounded it to the nearest 100x.
- As a statistician, I was thinking of fitting a linear regression to explore the relationship between the 4C's and the price.
  - So that I could pick the one that was significantly cheaper than it should be according to my statistical model.
  - Therefore having the best price-quality ratio.
  - I was too young and naive.
- **The most important thing is carat!**
- But still the other 3C's and some other minor factors may affect the price by a smaller margin.
  - G color is more expensive, H color is less expensive.

The assumptions on the small effects are,

- Normal distribution.
  - This small deviation is **normally distributed with mean 0.**
- **Constant variance.**
  - **This small deviation is the same for all diamonds, regardless of carat.**

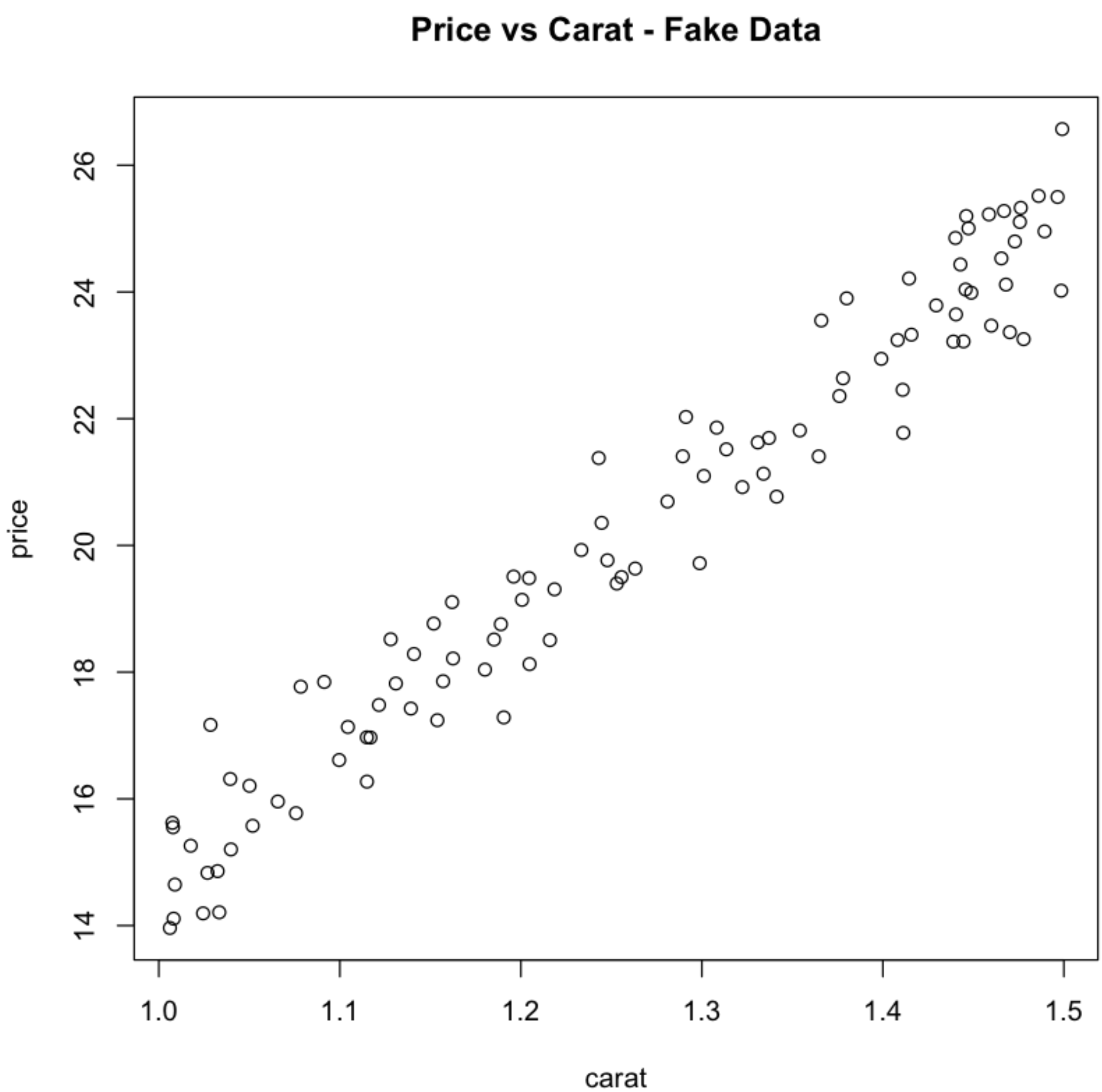
Therefore the data follows,

$$price_i = \beta_0 + \beta_1 \times carat_i + \epsilon_i$$

where $\epsilon_i \sim_{i.i.d} Normal(0, \sigma^2)$ **is the small deviation, which is sometimes called the noise.**

In [1]:

```
# Set seed for reproducibility
set.seed(613)

# 100 diamond rings ranging from 1 to 1.5 carats
n <- 100
carat <- runif(n = n, min = 1, max = 1.5)
color <- as.factor(sample(c("G", "H"), size = n, replace = TRUE)
)

# Normally distributed deviation from my perfect model caused by
other factors
error <- rnorm(n = n, mean = 0, sd = 0.5)

# On average:
# 1.00 carat: $10k;
# ................;
# 1.50 carat: $20k;

# Comparing to the average price.
# G color increases the price by 500
# H color decreases the price by 500
price <- -5 + 20 * carat + error + ifelse(color == "G", 0.5, -0.
5)

plot(x = carat, y = price, main = "Price vs Carat - Fake Data")
```


Price vs Carat - Fake Data
price
carat
14
16
18
20
22
24
26
1.0
1.1
1.2
1.3
1.4
1.5

The model is,

$$\widehat{price}_i = \hat{\beta}_0 + \hat{\beta}_1 \times carat_i$$

The goal is to de-noise and capture the association - find a straight line that best describes the association. How?

Consider the residual (e_i), which is the deviation of the observed data from the best fit line.

$$e_i = price_i - \widehat{price_i} = price_i - (\hat{\beta}_0 + \hat{\beta}_1 \times carat_i)$$

It makes sense to find a line that minimizes the sum of the residuals squared $\sum_{i=1}^{n} e_i^2$.

Why squared?

The math is,

in EPIB 621.

There is closed-form solution to linear model coefficients - nice.

Fit a linear model of the data considering only carat

In [2]:

```r
fit1 <- lm(price~carat); fit1
```

```
Call:
lm(formula = price ~ carat)

Coefficients:
(Intercept)        carat
     -6.159       20.903
```

In [3]:

```r
# If our data is a data.frame, note the data argument.
diamond <- data.frame(Price = price, Carat = carat, Color = colo
r)
head(diamond)
aggregate(Price~Color, data = diamond, mean)
table(diamond$Color)
```

A data.frame: 6 × 3

Price	Carat	Color
<dbl>	<dbl>	<fct>
24.95654	1.489319	G
23.25550	1.477755	H
15.57460	1.051832	H
21.77606	1.411278	H
18.21557	1.162526	G
24.53011	1.465442	G

A data.frame: 2 ×
2

Color	Price
<fct>	<dbl>
G	21.54011
H	19.09459

```
 G  H
48 52
```

In [4]:

```
fit1 <- lm(Price~Carat, data = diamond)
summary(fit1)
```

```
Call:
lm(formula = Price ~ Carat, data = diamond)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.56430 -0.56169 -0.02658  0.60028  1.82743 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -6.1591     0.6008  -10.25   <2e-16 ***
Carat        20.9027     0.4715   44.34   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 
0.1 ' ' 1

Residual standard error: 0.7509 on 98 degrees of fre
edom
Multiple R-squared:  0.9525,    Adjusted R-squared: 
0.952 
F-statistic:  1966 on 1 and 98 DF,  p-value: < 2.2e-
16
```

In [5]:

```
plot(x = carat, y = price, main = "Price vs Carat - Fake Data")
# Draw the best fit line from our linear model.
abline(fit1, col = "red")
```

In [6]:

```
# Explore the structure of fit1 and summary(fit1) to see how to
extract values.
# str(fit1)
# str(summary(fit1))
```

Interpretation of the coefficients along with confidence intervals of the estimated coefficients.

In [7]:

```
summary(fit1)$coefficients
```

A matrix: 2 × 4 of type dbl

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-6.159122	0.6007728	-10.25200	3.437439e-17
Carat	20.902675	0.4714502	44.33697	1.167533e-66

In [8]:

```
# Confidence intervals - default is 95%
confint(fit1, level = 0.95)
```

A matrix: 2 × 2 of type dbl

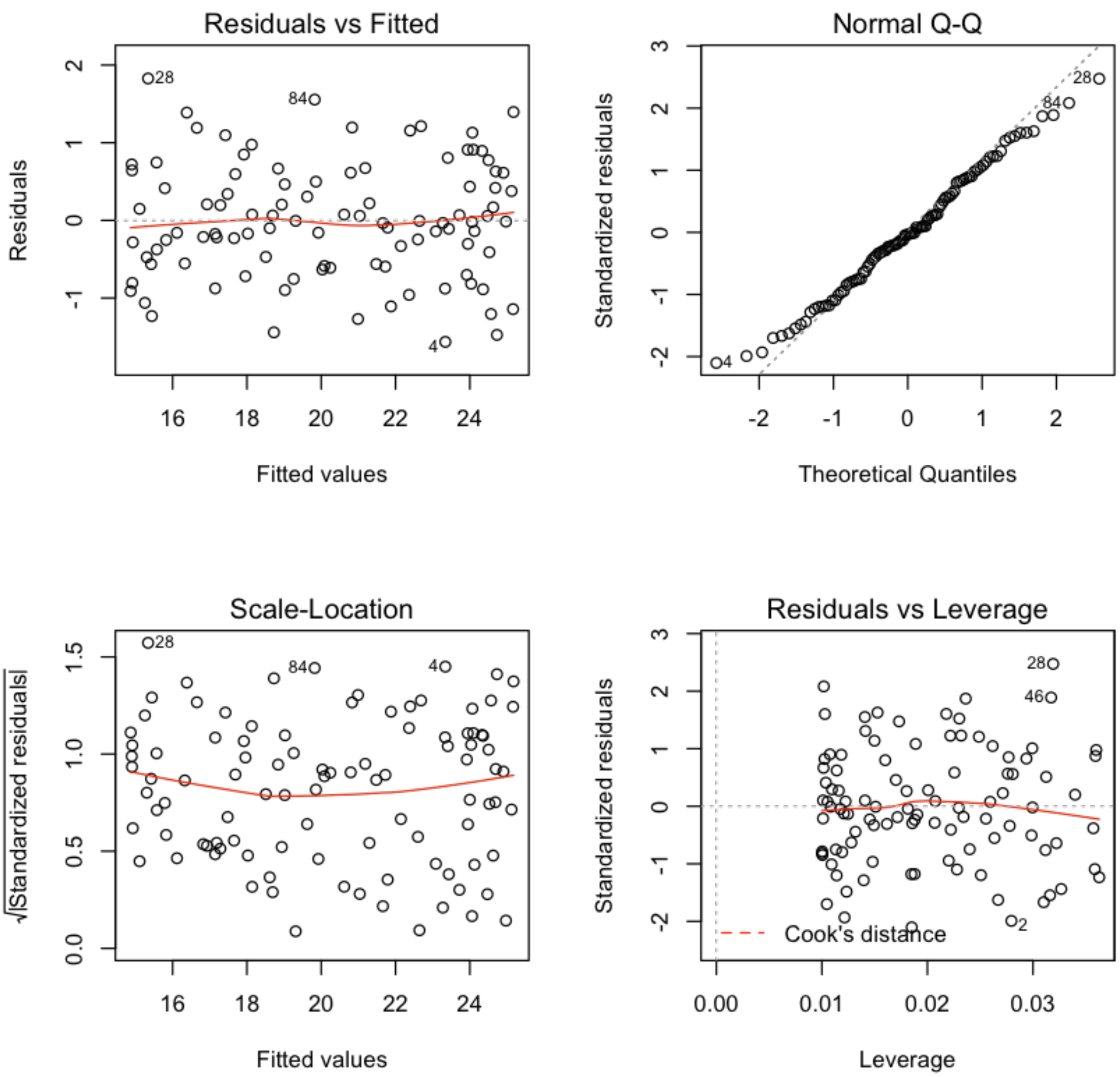
	2.5 %	97.5 %
(Intercept)	-7.351336	-4.966908
Carat	19.967097	21.838252

Check model fit and assumptions using plots

- First plot checks the constant variance assumption
- Second plot checks the normality assumption

In [9]:

```
par(mfrow = c(2,2))
plot(fit1)
```



10.1.2 Multiple linear regression

Simply more than one x's.

Fit a linear model considering both carat and color

In [10]:

```
fit2 <- lm(Price~Carat+Color, data = diamond)
summary(fit2)
```

```
Call:
lm(formula = Price ~ Carat + Color, data = diamond)

Residuals:
     Min       1Q   Median       3Q      Max
-0.94938 -0.42954 -0.02353  0.34489  1.09171

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -4.6503     0.4227  -11.00   <2e-16 ***
Carat        20.1772     0.3209   62.88   <2e-16 ***
ColorH       -1.1376     0.1023  -11.12   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'
0.1 ' ' 1

Residual standard error: 0.5004 on 97 degrees of fre
edom
Multiple R-squared:  0.9791,    Adjusted R-squared:
0.9787
F-statistic:  2275 on 2 and 97 DF,  p-value: < 2.2e-
16
```

In [11]:

```
# Adjust for all other variables in the dataset other than Price
.
# lm(Price~., data = diamond)

# Adjust for interaction terms - effect measure modification EPI
B 603.
# lm(Price~ Carat + Color + Carat:Color, data = diamond)
# Equivalent to
# lm(Price~ Carat * Color, data = diamond)
```

Prediction using the regression model

Predict the average price of an 1.22 carat diamond with G color - confidence interval.

In [12]:

```
predict(fit2, newdata = data.frame(Carat=1.22, Color="G"), inter
val = "confidence")
```

A matrix: 1 × 3 of type dbl

fit	lwr	upr
19.96585	19.81413	20.11758

Predict the price of an 1.22 carat diamond with G color - prediction interval.

In [13]:

```
predict(fit2, newdata = data.frame(Carat=1.22, Color="G"), inter
val = "prediction")
```

A matrix: 1 × 3 of type dbl

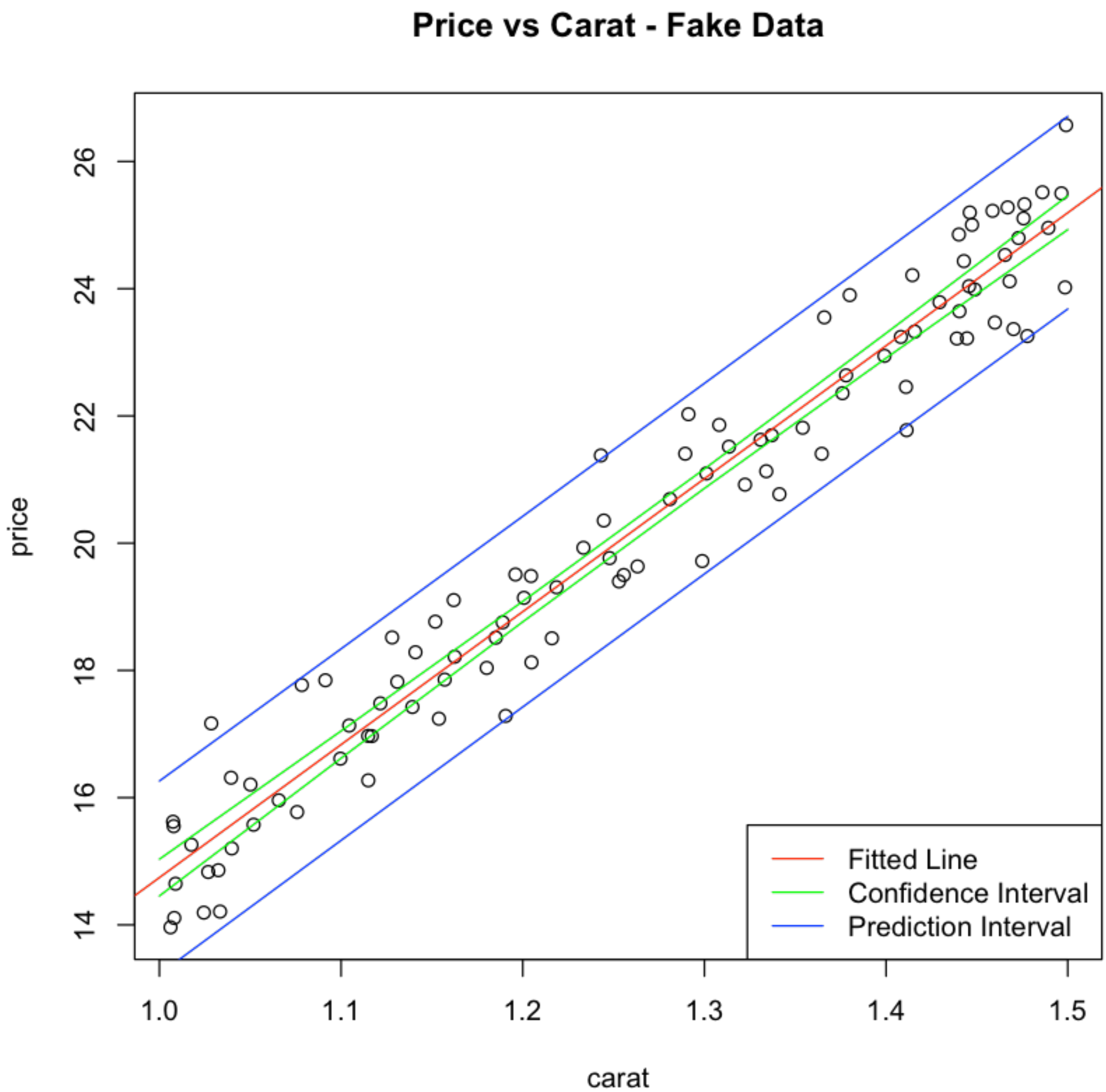
fit	lwr	upr
19.96585	18.9611	20.97061

Draw confidence interval and prediction interval for simple linear regression (fit1)

Hard to visualize when there are more than one x's.

In [14]:

```
CI <- predict(fit1, newdata = data.frame(Carat=seq(1,1.5,0.1)),
interval = "confidence")
PI <- predict(fit1, newdata = data.frame(Carat=seq(1,1.5,0.1)),
interval = "prediction")
plot(x = carat, y = price, main = "Price vs Carat - Fake Data")
# Draw the best fit line from our linear model.
abline(fit1, col = "red")
lines(x = seq(1,1.5,0.1), y = CI[,2], col = "green")
lines(x = seq(1,1.5,0.1), y = CI[,3], col = "green")

lines(x = seq(1,1.5,0.1), y = PI[,2], col = "blue")
lines(x = seq(1,1.5,0.1), y = PI[,3], col = "blue")

legend("bottomright", col = c("red", "green", "blue"), lty = c(1
,1,1),
       legend = c("Fitted Line", "Confidence Interval", "Predict
ion Interval"))
```



Programming-wise, simple linear regression and multiple linear regressions are the same.

10.2 Analysis of Variance

10.2.1 ANOVA without regression

Recall that we could use t-test to test whether two treatment effects are significantly different.

In [15]:

```
price.G <- subset(x = diamond, Color == "G")$Price
price.H <- subset(x = diamond, Color == "H")$Price
mu.G <- mean(price.G)
mu.H <- mean(price.H)
mu.tot <- mean(diamond$Price)
data.frame(G=mu.G, H=mu.H, Total=mu.tot)
```

A data.frame: 1 × 3

G	H	Total
<dbl>	<dbl>	<dbl>
21.54011	19.09459	20.26844

In [16]:

```
t.test(price.G, price.H)
```

```
        Welch Two Sample t-test

data:  price.G and price.H
t = 3.8049, df = 97.895, p-value = 0.0002472
alternative hypothesis: true difference in means is
not equal to 0
95 percent confidence interval:
 1.170011 3.721021
sample estimates:
mean of x mean of y
 21.54011  19.09459
```

This is a direct comparison of the two means.

- **G color is significantly more expensive than H color.**

We can also use the variance to test whether two groups are different. How?

- **Decompose the total variation in the dataset into within group variation and between group variation.**
- **Think of the test as: if too much of the total variation comes from the between group variation, we believe that the two groups are different.**

In [17]:

```
myanova <- aov(Price~Color, data = diamond)
summary(myanova)
```

```
            Df Sum Sq Mean Sq F value   Pr(>F)    
Color        1  149.3  149.27   14.42 0.000254 ***
Residuals   98 1014.4   10.35                     
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'
0.1 ' ' 1
```

The null hypothesis that the two groups are equal is rejected.

In [18]:

```
# Reproduce the test result from scratch
# ss means sum of squares
ss.within = sum((price.G-mu.G)^2) + sum((price.H-mu.H)^2)
ss.between = (mu.G-mu.tot)^2 * 48 + (mu.H-mu.tot)^2 * 52
ss.total = sum((price-mu.tot)^2)
data.frame(Within = ss.within, Between = ss.between, Total = ss.
total)
```

A data.frame: 1 × 3

Within	Between	Total
<dbl>	<dbl>	<dbl>
1014.435	149.2745	1163.709

10.2.2 ANOVA in regression

- We saw that the calculations involve within group variance calculation for each group.
- We had 2 groups. What if we had more?
- What if we had many more? e.g. carat in the first dataset - a continuous variable.

In [19]:

```
head(diamond$Carat, 20)
```

1.48931883147452 1.4777549788123 1.05183249094989 1.41127772070467 1.16252580692526 1.46544234571047 1.28101023298223 1.44567985611502 1.09136897884309 1.21604923321865 1.30815492232796 1.18015707843006 1.20065634301864 1.11689501185901 1.29879072017502 1.35407601017505 1.11498201265931 1.13925164530519 1.01760521158576 1.03983685944695

The logic: fit a linear model, decompose the total variation into model variation (between group) and residual variation (within group).

In [20]:

```
anova(fit1)
```

A anova: 2 × 5

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
	<int>	<dbl>	<dbl>	<dbl>	<dbl>
Carat	1	1108.44923	1108.4492293	1965.767	1.167533e-66
Residuals	98	55.25986	0.5638761	NA	NA

In [21]:

```
anova(fit2)
```

A anova: 3 × 5

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
	<int>	<dbl>	<dbl>	<dbl>	<dbl>
Carat	1	1108.44923	1108.4492293	4426.0268	9.597565e-83
Color	1	30.96729	30.9672870	123.6521	5.228797e-19
Residuals	97	24.29257	0.2504389	NA	NA

Compare two regression models.

- **Our fit1 is a reduced version of fit2**
- **Test whether the deletion of "Color" significantly affect model fit.**
    - **Rejected: use full model.**

In [22]:

```r
print(anova(fit1, fit2))
```

```
Analysis of Variance Table

Model 1: Price ~ Carat
Model 2: Price ~ Carat + Color
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)    
1     98 55.260                                  
2     97 24.293  1    30.967 123.65 < 2.2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’
0.1 ‘ ’ 1
```

Assignment 2