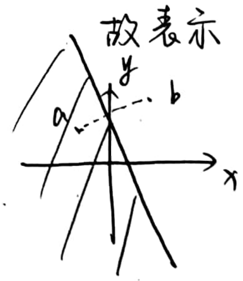
期中测验参考答案：

1. $\frac{|z-a|}{|z-b|}\le 1$. $|z-a|\le|z-b|$. $\Rightarrow |a|^2-|b|^2\le 2Re(z\bar{a})-2Re(z\bar{b})$.

故表示 a.b 线段中垂线分割 z 平面中包含 a 的半平面，包括边界.




注意：a.b 为复数且 $a\neq b$.

很多同学认为 $a,b\in R$. 得出 $x\le\frac{a+b}{2}$. ⊗.

2. (1) $\cos z=2$.

$$\frac{e^{iz}+e^{-iz}}{2}=2.$$

设 $t=e^{iz}$. $t+\frac{1}{t}=4$.

$\therefore t=2\pm\sqrt{3}$

$$\therefore z=\frac{1}{i}Ln(t)=\frac{1}{i}Ln(2\pm\sqrt{3})$$

$$=-i(\ln(2\pm\sqrt{3})+2k\pi i)$$

$$=2k\pi-i\ln(2\pm\sqrt{3})$$

(2) $i^{\sqrt{2}+i\sqrt{3}}$

$$e^{(\sqrt{2}+i\sqrt{3})Ln i}$$

$$=e^{(\sqrt{2}+i\sqrt{3})(\ln 1+i(\frac{\pi}{2}+2k\pi))}$$

$$=e^{(\sqrt{2}+i\sqrt{3})i(\frac{\pi}{2}+2k\pi)}$$

$$=e^{i\sqrt{2}(\frac{\pi}{2}+2k\pi)-\sqrt{3}(\frac{\pi}{2}+2k\pi)}$$

3. $0<\alpha<1$. (1) $\lim\limits_{\varepsilon\to 0}\int_{|z|=\varepsilon}\frac{z^{\alpha-1}}{z+1}dz$. (2) $\lim\limits_{R\to\infty}\int_{|z|=R}\frac{z^{\alpha-1}}{z+1}dz$.

(1) $f(z)=\frac{z^{\alpha-1}}{z+1}$. $z=\varepsilon e^{i\theta}$. 由长大不等式：$\left|\int_{|z|=\varepsilon}f(z)dz\right|\le\max\limits_{|z|=\varepsilon}|f(z)|\cdot l$

由于 $|f(z)|=\frac{\varepsilon^{\alpha-1}}{\sqrt{\varepsilon^2+1+2\varepsilon\cdot\cos\theta}}$ $\therefore\max\limits_{|z|=\varepsilon}|f(z)|\overset{\theta=\pi}{=\!=}\frac{\varepsilon^{\alpha-1}}{|\varepsilon-1|}$

$$\therefore\left|\int_{|z|=\varepsilon}f(z)dz\right|\le 2\pi\varepsilon\cdot\frac{\varepsilon^{\alpha-1}}{|\varepsilon-1|}=2\pi\frac{\varepsilon^{\alpha}}{|\varepsilon-1|}$$

$\therefore \varepsilon\to 0$ 时：$\left|\int_{|z|=\varepsilon}\frac{z^{\alpha-1}}{z+1}dz\right|\le 2\pi\frac{\varepsilon^{\alpha}}{|\varepsilon-1|}\to 0$.

$$\therefore\lim_{\varepsilon\to 0}\int_{|z|=\varepsilon}\frac{z^{\alpha-1}}{z+1}dz=0.$$

(2) 同(1). $\left|\int_{|z|=R} \frac{z^{\alpha-1}}{z+1} dz\right| \le 2\pi R \cdot \frac{R^{\alpha-1}}{|R-1|} = 2\pi \frac{R^{\alpha}}{|R-1|}$

$= 2\pi \frac{R^{\alpha-1}}{|1-\frac{1}{R}|} = 2\pi \frac{1}{|1-\frac{1}{R}|\cdot R^{1-\alpha}}$.

当 $R\to\infty$ 时: $\left|\int_{|z|=R} f(z) dz\right| \le 2\pi\cdot \frac{1}{|1-\frac{1}{R}|\cdot R^{1-\alpha}} \to 0$

$\therefore \lim_{R\to\infty} \int_{|z|=R} \frac{z^{\alpha-1}}{z+1} dz = 0$.

4. $I = \int_{|z-1|=2} \frac{z+1}{z^2(z-i)} dz$.

柯西积分公式: $I = 2\pi i \cdot \left( \left(\frac{z+1}{z-i}\right)'\Big|_{z=0} + \frac{z+1}{z^2}\Big|_{z=i} \right)$

$= 2\pi i \cdot (1+i-i-1) = 0$.

5. $v(x,y) = e^{x^2-y^2}\sin(2xy)$. $f(0)=0$. 求 $f(z)$.

① 验证调和: $\frac{\partial v}{\partial x} = [2x\sin(2xy) + 2y\cos(2xy)]\, e^{x^2-y^2}$

$\frac{\partial^2 v}{\partial x^2} = [2\sin(2xy) + 2x\cdot 2y\cos(2xy) + 2y\cdot 2y(-\sin(2xy))$
$+ 2x(2x\sin(2xy) + 2y\cos(2xy))]\, e^{x^2-y^2}$

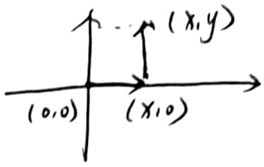
$\frac{\partial v}{\partial y} = [-2y\sin(2xy) + 2x\cos(2xy)]\, e^{x^2-y^2}$

$\frac{\partial^2 v}{\partial y^2} = [-2\sin(2xy) - 2y\cdot 2x\cos(2xy) + 2x\cdot 2x(-\sin(2xy))$
$-2y\cdot(-2y\cdot\sin(2xy) + 2x\cos(2xy))]\, e^{x^2-y^2}$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. 故 $v(x,y)$ 为平面调和.

② 求 $u(x,y)$.

$$u(x,y)=\int_{(0,0)}^{(x,y)}\frac{\partial v}{\partial y}dx-\frac{\partial v}{\partial x}dy+C$$




$$=\underbrace{\int_0^x 2x\cdot e^{x^2}dx}_{①}-\underbrace{\int_0^y[2x\sin(2xy)+2y\cos(2xy)]e^{x^2-y^2}dy}_{②}+C$$

对①: $\int_0^x 2xe^{x^2}dx=e^{x^2}\Big|_0^x=e^{x^2}-1$

对②: $\int_0^y[2x\sin(2xy)+2y\cos(2xy)]e^{x^2-y^2}dy$

$$=e^{x^2}\cdot\left[\int_0^y 2x\cdot\sin(2xy)\cdot e^{-y^2}dy+\int_0^y 2y\cdot\cos(2xy)\cdot e^{-y^2}dy\right]$$

$$=e^{x^2}\cdot\left[-\cos(2xy)\cdot e^{-y^2}\Big|_0^y+\int_0^y\cos(2xy)\cdot(-2y)e^{-y^2}dy+\int_0^y 2y\cdot\cos(2xy)\cdot e^{-y^2}dy\right]$$

相抵.

$$=e^{x^2}\cdot(-\cos(2xy)e^{-y^2}+1)$$

$$\therefore u(x,y)=e^{x^2}-1-e^{x^2}(-\cos(2xy)\cdot e^{-y^2}+1)+C$$

$$=e^{x^2-y^2}\cos(2xy)-1+C$$

$\because f(z)=u+iv\Rightarrow f(0)=0$. $\therefore u(0,0)+iv(0,0)=0$

$\therefore 1-1+C+i\cdot 0=0\Rightarrow C=0$

$$\therefore f(x,y)=e^{x^2-y^2}\cos(2xy)-1+i\cdot(e^{x^2-y^2}\sin(2xy))$$

③. 令 $x=z$. $y=0$. $f(z)=e^{z^2}-1$.

6. 附加题：(多种解法，在此展示用期中所学知识解的方法).

$0<r<1.\quad \int_0^{\pi} \ln(1-2r\cos\theta+r^2)\,d\theta$

设 $u(r,\theta)=\ln(1-2r\cos\theta+r^2)$.

极坐标形式的拉普拉斯算子 $\Delta = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial\theta^2}$

$\therefore \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$.

由于 $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\dfrac{-2r^2\cos\theta+4r-2\cos\theta}{r(1-2r\cos\theta+r^2)^2}$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac{2\cos\theta+2r^2\cos\theta-4r}{r(1-2r\cos\theta+r^2)^2}$$

$\therefore \Delta u=0.\quad \therefore u(r,\theta)$ 在全平面调和.

$$\therefore \frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\,d\theta=u(0,0)=\ln 1=0.$$

又$\because \int_0^{\pi}u(r,\theta)\,d\theta\overset{\tilde\theta=-\theta}{=}-\int_0^{-\pi}\ln(1-2r\cos\tilde\theta+r^2)\,d\tilde\theta=\int_{-\pi}^{0}\ln(1-2r\cos\tilde\theta+r^2)\,d\tilde\theta.$

又$\because \int_{-\pi}^{0}\ln(1-2r\cos\theta+r^2)\,d\theta\overset{\tilde\theta=\theta+2\pi}{=}\int_{\pi}^{2\pi}\ln(1-2r\cos\tilde\theta+r^2)\,d\tilde\theta$

$$\therefore \int_0^{\pi}u(r,\theta)\,d\theta=\int_{\pi}^{2\pi}u(r,\theta)\,d\theta.$$

$$\therefore \int_0^{2\pi}u(r,\theta)\,d\theta=2\int_0^{\pi}u(r,\theta)\,d\theta=0.$$

$\therefore I=0.$