§4.5　傅里叶变换的性质

- 线性
- 奇偶性
- 对称性
- 尺度变换
- 时移特性
- 频移特性
- 卷积定理
- 时域微分和积分
- 频域微分和积分
- 相关定理

一. 线性性质(Linear Property)

If $f_1(t) \longleftrightarrow F_1(j\omega)$, $f_2(t) \longleftrightarrow F_2(j\omega)$

then

$$\boxed{[a\, f_1(t) + b\, f_2(t)] \longleftrightarrow [a\, F_1(j\omega) + b\, F_2(j\omega)]}$$

Proof: $\mathscr{F}[a\, f_1(t) + b\, f_2(t)]$

$$= \int_{-\infty}^{\infty} [af_1(t) + bf_2(t)] \mathrm{e}^{-j\omega t} \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} \mathrm{a}\, f_1(t) \mathrm{e}^{-j\omega t} \mathrm{d}t + \int_{-\infty}^{\infty} \mathrm{b}\, f_1(t) \mathrm{e}^{-j\omega t} \mathrm{d}t$$

$$= [a\, F_1(j\omega) + b\, F_2(j\omega)]$$

Example

二. 奇偶虚实性(Parity)

If $f(t)$ is real function, and

$$f(t) \longleftrightarrow F(\mathrm{j}\omega)=|F(\mathrm{j}\omega)|e^{\mathrm{j}\varphi(\omega)} = R(\omega)+\mathrm{j}X(\omega)$$

$$|F(j\omega)| = \sqrt{R^2(\omega)+X^2(\omega)} \qquad \varphi(\omega) = \arctan\left(\frac{X(\omega)}{R(\omega)}\right)$$

then

- $R(\omega)= R(-\omega)$, $X(\omega)= -X(-\omega)$,
  $|F(\mathrm{j}\omega)|= |F(-\mathrm{j}\omega)|$, $\varphi(\omega)= -\varphi(-\omega)$,
- $f(-t) \longleftrightarrow F(-\mathrm{j}\omega) = F^*(\mathrm{j}\omega)$
- If $f(t)= f(-t)$ then $X(\omega)=0$, $F(\mathrm{j}\omega) = R(\omega)$
  If $f(t)= -f(-t)$ then $R(\omega)=0$, $F(\mathrm{j}\omega) = \mathrm{j}X(\omega)$

Proof

三、对称性(Symmetrical Property)

If $f(t) \longleftrightarrow F(\mathrm{j}\omega)$ **then** $\boxed{F(\mathrm{j}t) \longleftrightarrow 2\pi f(-\omega)}$

Proof:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \qquad (1)$$

in (1) $t \rightarrow \omega$, $\omega \rightarrow t$ **then**

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{j}t)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}t \qquad (2)$$

in (2) $\omega \rightarrow -\omega$ **then**

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{j}t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$$

$\therefore$ $F(\mathrm{j}t) \longleftrightarrow 2\pi f(-\omega)$ **end**

Example

练习

$$f_1(t) = \frac{\sin t}{t} \leftrightarrow ?$$

$$f_2(t) = t + \frac{1}{t} \leftrightarrow ?$$

四、尺度变换性质(Scaling Transform Property)

If $f(t) \longleftrightarrow F(j\omega)$ then $\boxed{f(at) \longleftrightarrow \frac{1}{|a|} F\left(j\frac{\omega}{a}\right)}$

where "a" is a nonzero real constant.

Proof

Also,letting $a = -1$, $\boxed{f(-t) \longleftrightarrow F(-j\omega)}$

Example-1

意义

五、时移特性(Timeshifting Property)

If $f(t) \longleftrightarrow F(j\omega)$ **then** $\boxed{f(t-t_0) \longleftrightarrow e^{-j\omega t_0} F(j\omega)}$

where "t_0" is real constant.

Example 1

Example 2

Example 3

Proof: $\mathscr{F}[f(t-t_0)]$

$$= \int_{-\infty}^{\infty} f(t-t_0) e^{-j\omega t} dt$$

$$\overset{t-t_0=\tau}{=} \int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} d\tau \; e^{-j\omega t_0}$$

$$= e^{-j\omega t_0} F(j\omega)$$

六、频移性质(Frequency Shifting Property)

If $f(t) \longleftrightarrow F(j\omega)$ then

$$\boxed{e^{j\omega_0 t} f(t) \longleftrightarrow F[j(\omega - \omega_0)]}$$

where "ω_0" is real constant.

Proof:

$$\mathscr{F}[e^{j\omega_0 t} f(t)] = \int_{-\infty}^{\infty} e^{j\omega_0 t} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_0)t} dt$$

$$= F[j(\omega - \omega_0)]$$

end

For example 1

$$f(t) = e^{j3t} \longleftrightarrow F(j\omega) = ?$$

Ans: $1 \longleftrightarrow 2\pi\delta(\omega)$

$$e^{j3t} \times 1 \longleftrightarrow 2\pi\delta(\omega - 3)$$

Example 2

七、卷积性质(Convolution Property)

Convolution in time domain:

If $f_1(t) \longleftrightarrow F_1(\mathrm{j}\omega)$, $f_2(t) \longleftrightarrow F_2(\mathrm{j}\omega)$

Then $f_1(t)*f_2(t) \longleftrightarrow F_1(\mathrm{j}\omega)F_2(\mathrm{j}\omega)$

Proof

Convolution in frequency domain:

If $f_1(t) \longleftrightarrow F_1(\mathrm{j}\omega)$, $f_2(t) \longleftrightarrow F_2(\mathrm{j}\omega)$

Then $f_1(t)\, f_2(t) \longleftrightarrow \dfrac{1}{2\pi} F_1(\mathrm{j}\omega)*F_2(\mathrm{j}\omega)$

Example

八、时域的微分和积分

(Differentiation and Integration in time domain)

If $f(t) \longleftrightarrow F(j\omega)$ then

$$f^{(n)}(t) \longleftrightarrow (j\omega)^n F(j\omega)$$

$$\int_{-\infty}^{t} f(x)\,dx \longleftrightarrow \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega} \qquad F(0) = F(j\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} f(t)\,dt$$

Proof:

$$f^{(n)}(t) = \delta^{(n)}(t) * f(t) \longleftrightarrow (j\omega)^n F(j\omega)$$

$$f^{(-1)}(t) = \varepsilon(t) * f(t) \longleftrightarrow [\pi\delta(\omega) + \frac{1}{j\omega}]F(j\omega) = \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}$$

Example 1 Example 2

已知$f'(t) \longleftrightarrow F_1(j\omega)$

$f(t) \longleftrightarrow F(j\omega) = ?$

九、频域的微分和积分

(Differentiation and Integration in frequency domain)

If $f(t) \longleftrightarrow F(j\omega)$ then

$$(-jt)^n f(t) \longleftrightarrow F^{(n)}(j\omega)$$

$$\pi f(0)\delta(t) + \frac{1}{-jt} f(t) \longleftrightarrow \int_{-\infty}^{\omega} F(jx)\,\mathrm{d}x$$

where $f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\,\mathrm{d}\omega$

Example 1

Example 2

十、相关定理(Correlation theorem)

If

$$f_1(t) \longleftrightarrow F_1(j\omega),\quad f_2(t) \longleftrightarrow F_2(j\omega),\quad f(t) \longleftrightarrow F(j\omega)$$

then

$$\mathcal{F}[R_{12}(\tau)] = F_1(j\omega)\,F_2^*(j\omega)$$

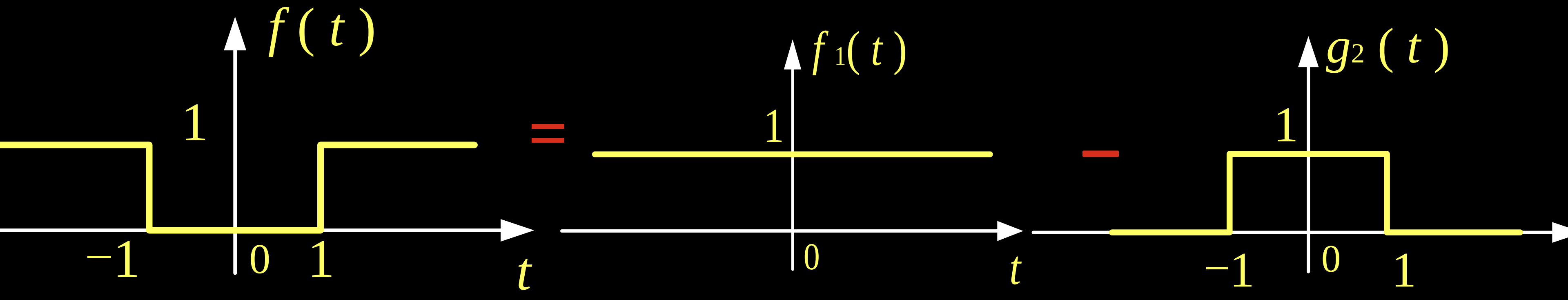
$$\mathcal{F}[R_{21}(\tau)] = F_1^*(j\omega)\,F_2(j\omega)$$

$$\mathcal{F}[R(\tau)] = |F(j\omega)|^2$$

Proof

线性性质例

For example $F(j\omega) = ?$



Ans: $f(t) = f_1(t) - g_2(t)$

$$f_1(t) = 1 \longleftrightarrow 2\pi\delta(\omega)$$

$$g_2(t) \longleftrightarrow 2Sa(\omega)$$

$$\therefore F(j\omega) = 2\pi\delta(\omega) - 2Sa(\omega)$$

对称性举例

For example

$$f(t)=\frac{1}{1+t^2} \longleftrightarrow F(\mathrm{j}\omega)=?$$

Ans: $\mathrm{e}^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$

if α **=1,** $\mathrm{e}^{-|t|} \longleftrightarrow \frac{2}{1+\omega^2}$

$\therefore$ $\frac{2}{1+t^2} \longleftrightarrow 2\pi\,\mathrm{e}^{-|\omega|}$ $\qquad \frac{1}{1+t^2} \longleftrightarrow \pi\,\mathrm{e}^{-|\omega|}$

尺度变换例1

For example 1

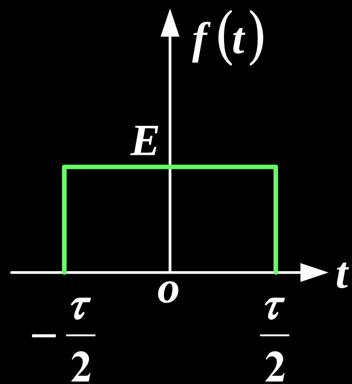
$$f(t) = \frac{1}{jt-1} \longleftrightarrow F(j\omega) = ?$$

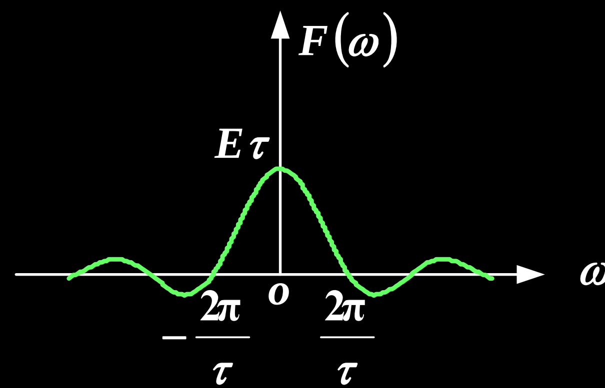
Ans: $$e^{-t}\varepsilon(t) \longleftrightarrow \frac{1}{j\omega+1}$$

Using symmetry, $$\frac{1}{jt+1} \longleftrightarrow 2\pi e^{\omega}\varepsilon(-\omega)$$

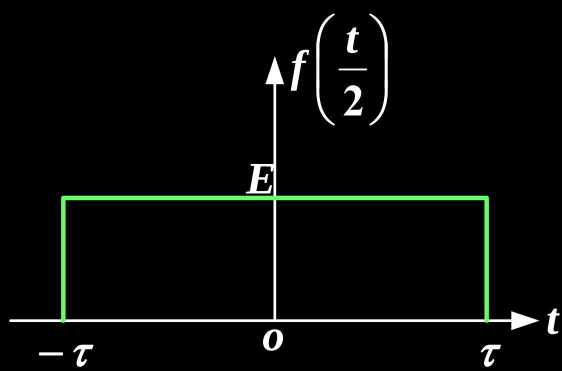
so that, $$-\frac{1}{-jt+1} \longleftrightarrow -2\pi e^{-\omega}\varepsilon(\omega)$$

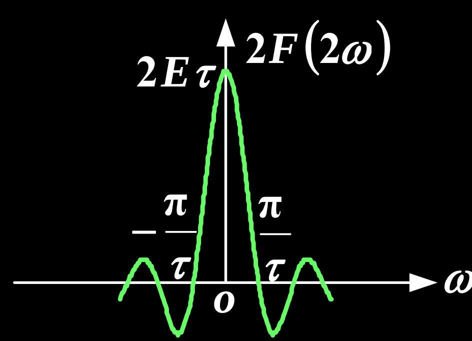
尺度变换意义





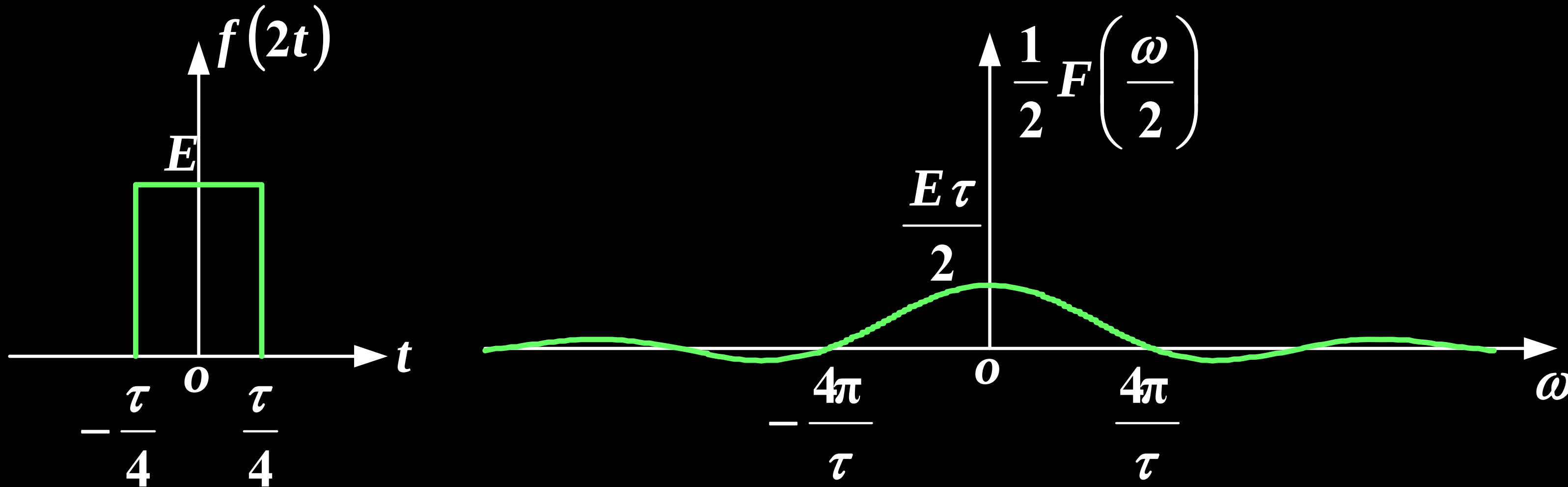
(1) $0<a<1$ 时域扩展，频带压缩。





脉冲持续时间增加 a 倍，变化慢了，信号在频域的频带压缩 a 倍。高频分量减少，幅度上升 a 倍。

(2) $a>1$ 时域压缩，频域扩展a倍。



持续时间短，变化快。信号在频域高频分量增加，频带展宽，各分量的幅度下降a倍。

(3) $a=-1$ 时域反转，频域也反转。

时移尺度举例

For example 2

Given that $f(t) \longleftrightarrow F(j\omega)$**, find** $f(at-b) \longleftrightarrow ?$

Ans: $f(t-b) \longleftrightarrow e^{-j\omega b} F(j\omega)$

$$f(at-b) \longleftrightarrow \frac{1}{|a|} e^{-j\frac{\omega}{a}b} F\left(j\frac{\omega}{a}\right)$$

or

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(j\frac{\omega}{a}\right)$$

$$f(at-b) = f\left[a(t-\frac{b}{a})\right] \longleftrightarrow \frac{1}{|a|} e^{-j\frac{\omega}{a}b} F\left(j\frac{\omega}{a}\right)$$

时移特性举例

For example $F(j\omega) = ?$

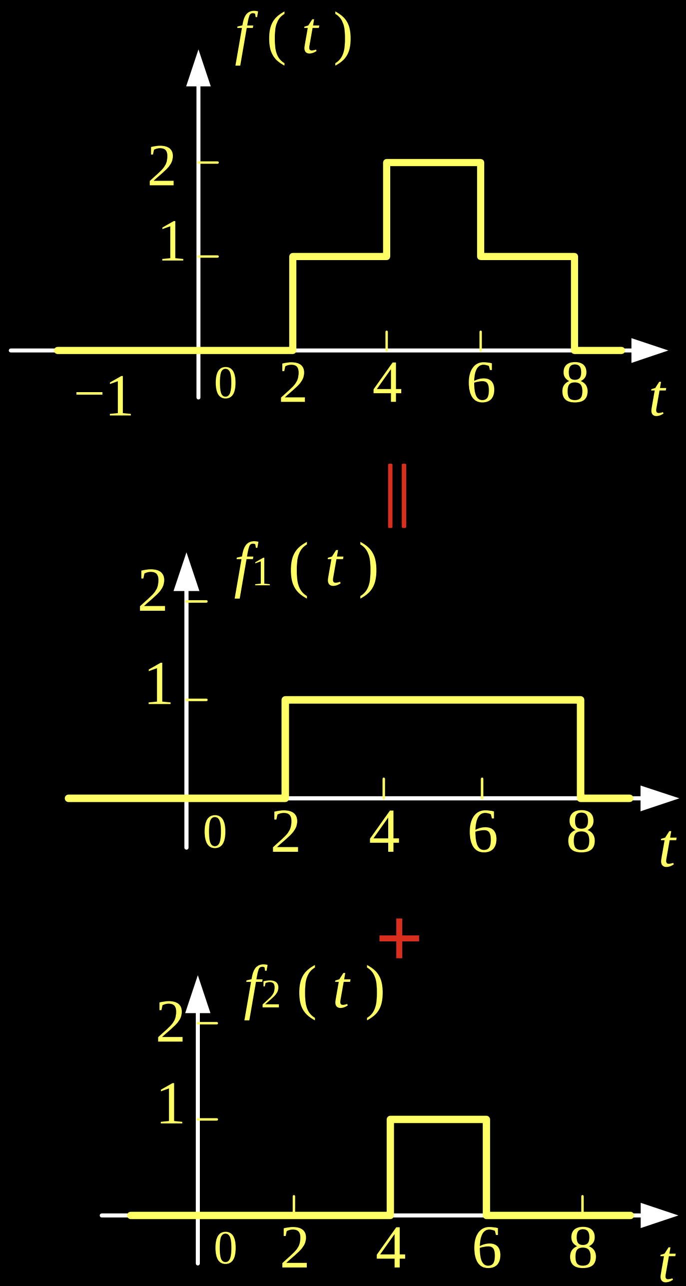
Ans: $f_1(t) = g_6(t-5)$,
$f_2(t) = g_2(t-5)$

$$g_6(t-5) \longleftrightarrow 6\,\mathrm{Sa}(3\omega)\,\mathrm{e}^{-j5\omega}$$

$$g_2(t-5) \longleftrightarrow 2\,\mathrm{Sa}(\omega)\,\mathrm{e}^{-j5\omega}$$

$\therefore F(j\omega) =$

$$[6\,\mathrm{Sa}(3\omega) + 2\,\mathrm{Sa}(\omega)]\,\mathrm{e}^{-j5\omega}$$



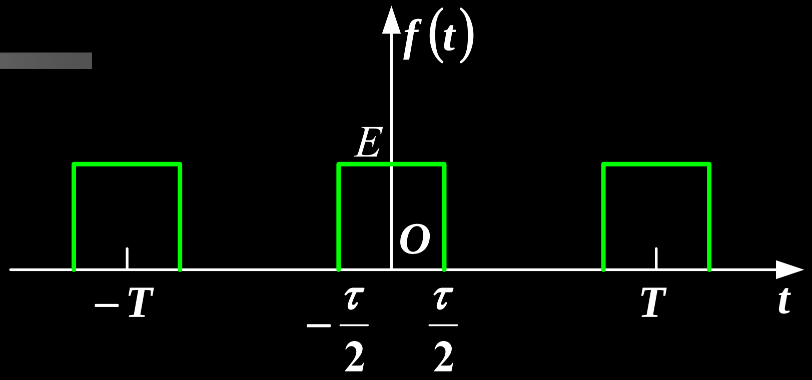
时移举例3

求图(a)所示三脉冲信号的频谱。

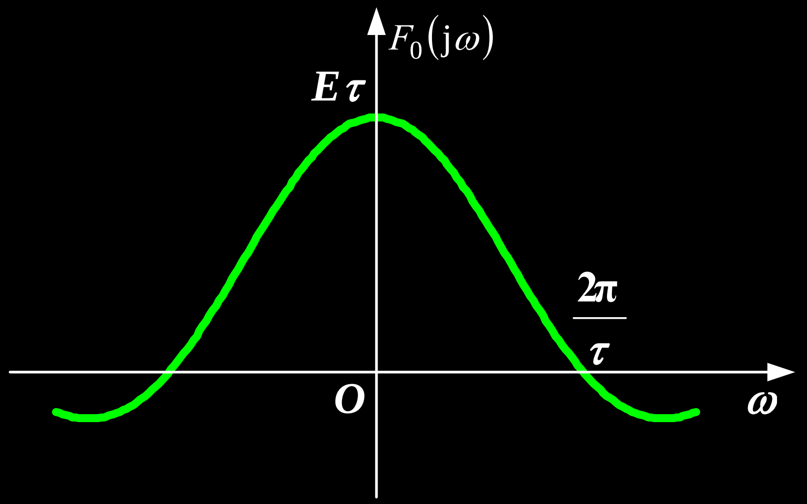
解：

令$f_0(t)$表示矩形单脉冲信号，其频谱函数 $F_0(\mathrm{j}\omega)$,

$$F_0(\mathrm{j}\omega)=E\tau\cdot\mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$$



(a) 三脉冲信号的波形



(b)

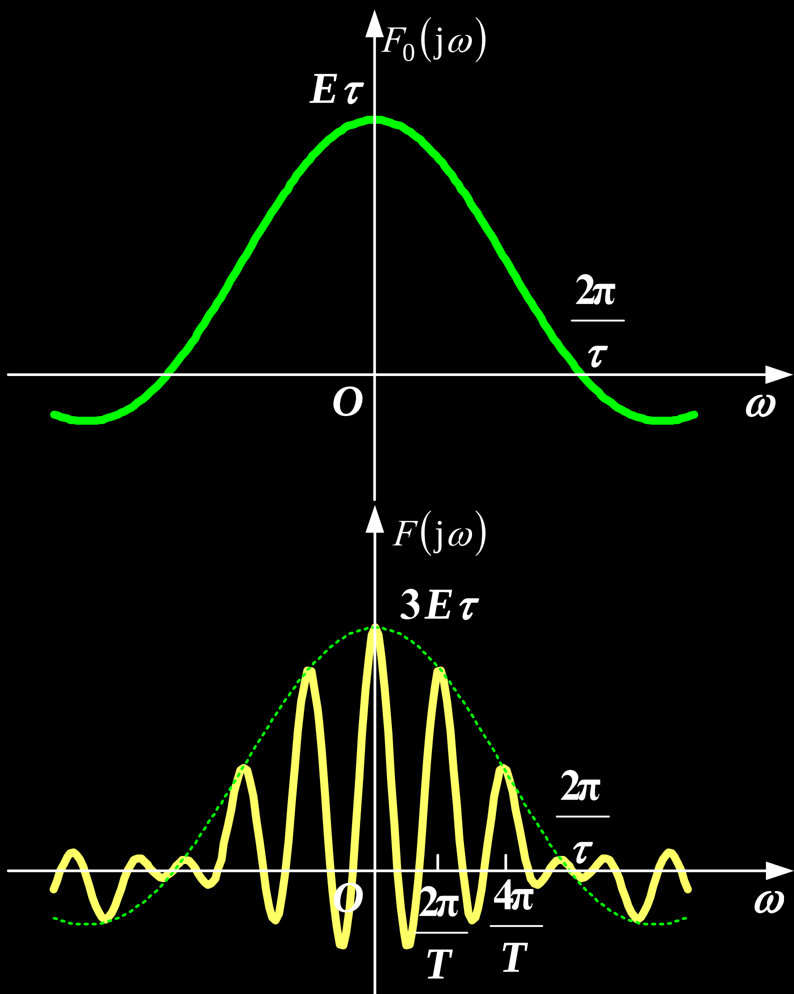
因为

$$f(t)=f_0(t)+f_0(t+T)+f_0(t-T)$$

由时移性质知三脉冲函数$f(t)$的频谱函数$F(\mathrm{j}\omega)$为：

$$F(\mathrm{j}\omega)=F_0(\mathrm{j}\omega)\left(1+\mathrm{e}^{\mathrm{j}\omega T}+\mathrm{e}^{-\mathrm{j}\omega T}\right)$$

$$=E\tau\cdot\mathbf{Sa}\left(\frac{\omega\tau}{2}\right)\left[1+2\cos(\omega T)\right]$$



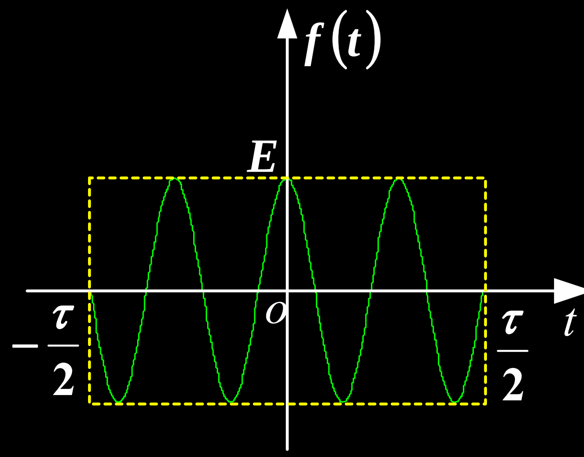
(c)

脉冲个数增多，频谱包络不变，带宽不变。

频移（调制）特性例

已知矩形调幅信号 $f(t)=Eg_\tau(t)\cos(\omega_0 t)$,

其中$g_\tau(t)$为矩形脉冲，脉宽为 τ ，试求其频谱函数。



(a)矩形调幅信号的波形

解：已知矩形脉冲$g_\tau(t)$的频谱$G_\tau(\mathrm{j}\omega)$为

$$G_\tau(\mathrm{j}\omega)=\tau\cdot\mathbf{Sa}\left(\frac{\omega\tau}{2}\right)$$

因为

$$f(t)=\frac{1}{2}Eg_\tau(t)\left(\mathrm{e}^{\mathrm{j}\omega_0 t}+\mathrm{e}^{-\mathrm{j}\omega_0 t}\right)$$

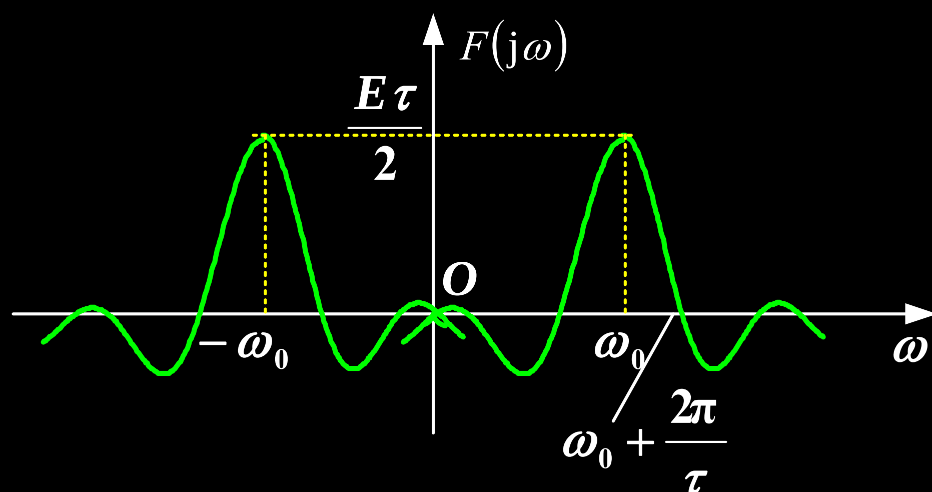
根据频移性质，$f(t)$频谱$F(\mathrm{j}\omega)$为

$$F(\mathrm{j}\omega)=\frac{1}{2}EG_\tau[\mathrm{j}(\omega-\omega_0)]+\frac{1}{2}EG_\tau[\mathrm{j}(\omega+\omega_0)]$$

$$F(\mathrm{j}\omega)=\frac{1}{2}EG_{\tau}[\mathrm{j}(\omega-\omega_0)]+\frac{1}{2}EG_{\tau}[\mathrm{j}(\omega+\omega_0)]$$

$$=\frac{E\tau}{2}\mathrm{Sa}\left[\frac{(\omega-\omega_0)\tau}{2}\right]+\frac{E\tau}{2}\mathrm{Sa}\left[\frac{(\omega+\omega_0)\tau}{2}\right]$$

将包络线的频谱一分为二，向左、右各平移 ω_0



(b)

频域微分积分特性例1

For example 1 **Determine $f(t) = t\,\varepsilon(t) \longleftrightarrow F(j\omega)$=?**

Ans:

$$\varepsilon(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$-\,jt\ \varepsilon(t) \longleftrightarrow \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

$$t\varepsilon(t) \longleftrightarrow j\pi\delta'(\omega) - \frac{1}{\omega^2}$$

频域微分积分特性例2

For example 2

Determine $$\int_{-\infty}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{d}\omega$$

Ans:

$$g_{2a}(t)\longleftrightarrow\frac{2\sin(a\omega)}{\omega}$$

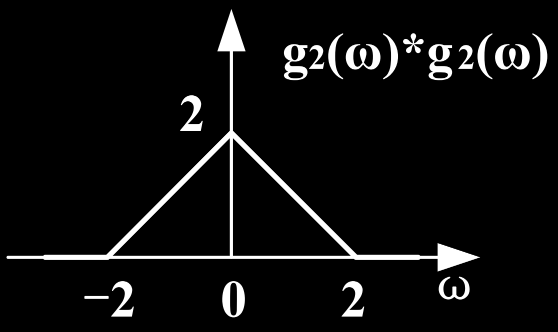
$$g_{2a}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin(a\omega)}{\omega}\mathrm{e}^{j\omega t}\mathrm{d}\omega=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

$$g_{2a}(0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{d}\omega \qquad \int_{0}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{d}\omega=\frac{\pi}{2}$$

卷积定理举例

For example

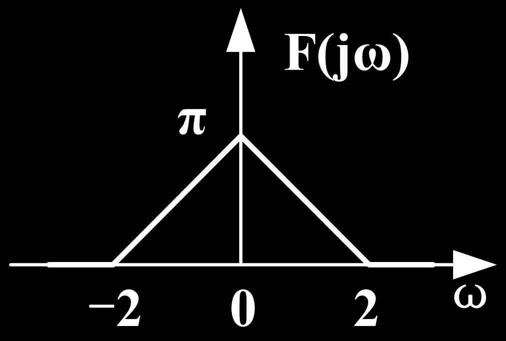
$$\left(\frac{\sin t}{t}\right)^2 \longleftrightarrow F(j\omega)=?$$



Ans:

$$g_2(t) \longleftrightarrow 2\,\mathrm{Sa}(\omega)$$

Using symmetry,



$$2\,\mathrm{Sa}(t) \longleftrightarrow 2\pi\ g_2(-\omega)$$

$$\mathrm{Sa}(t) \longleftrightarrow \pi\ g_2(\omega)$$

$$\left(\frac{\sin t}{t}\right)^2 \longleftrightarrow \frac{1}{2\pi}[\pi\ g_2(\omega)]*[\pi\ g_2(\omega)]=\frac{\pi}{2}g_2(\omega)*g_2(\omega)$$

时域微分特性例1

For example 1

f(t)= 1/t² ←→ ?

Ans: $\operatorname{sgn}(t) \longleftrightarrow \dfrac{2}{j\omega}$

$$\frac{2}{jt} \longleftrightarrow 2\pi \operatorname{sgn}(-\omega)$$

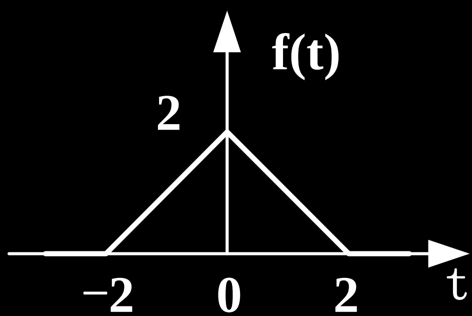
$$\frac{1}{t} \longleftrightarrow -j\pi \operatorname{sgn}(\omega)$$

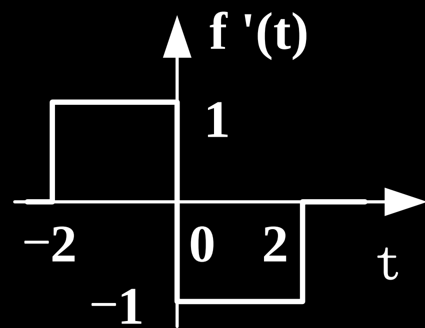
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\right) \longleftrightarrow -(j\omega)j\pi \operatorname{sgn}(\omega) = \pi\ \omega \operatorname{sgn}(\omega)$$

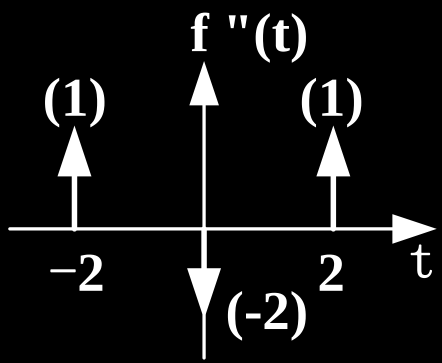
$$\frac{1}{t^2} \longleftrightarrow -\pi\ \omega \operatorname{sgn}(\omega) = -\pi\,|\,\omega\,|$$

时域微分积分特性例2

For example 2







Determine $f(t) \longleftrightarrow F(j\omega)$

Ans: $f''(t) = \delta(t+2) - 2\,\delta(t) + \delta(t-2)$

$$F_2(j\omega) = \mathrm{F}\,[f''(t)] = e^{j2\omega} - 2 + e^{-j2\omega} = 2\cos(2\omega) - 2$$

$$F(j\omega) = \frac{F_2(j\omega)}{(j\omega)^2} = \frac{2 - 2\cos(2\omega)}{\omega^2}$$

Notice: $d\,\varepsilon(t)/dt = \delta(t) \longleftrightarrow 1$

$\varepsilon(t) \leftarrow\!\times\!\rightarrow 1/(j\omega)$

Summary:

If $f^{(n)}(t) \longleftrightarrow F_n(j\omega)$, and

$$f(-\infty) + f(\infty) = 0$$

then

$$f(t) \longleftrightarrow F(j\omega) = F_n(j\omega) / (j\omega)^n$$

频域微分积分特性例1

For example 1 **Determine** $f(t) = t\,\varepsilon(t) \longleftrightarrow F(j\omega)=?$

Ans:

$$\varepsilon(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$-jt\ \varepsilon(t) \longleftrightarrow \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

$$t\varepsilon(t) \longleftrightarrow j\pi\delta'(\omega) - \frac{1}{\omega^2}$$

频域微分积分特性例2

For example 2

Determine $\int_{-\infty}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{d}\omega$

Ans:

$$g_{2a}(t)\longleftrightarrow\frac{2\sin(a\omega)}{\omega}$$

$$g_{2a}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin(a\omega)}{\omega}\mathrm{e}^{j\omega t}\mathrm{d}\omega=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

$$g_{2a}(0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{d}\omega \qquad \int_{0}^{\infty}\frac{\sin(a\omega)}{\omega}\mathrm{d}\omega=\frac{\pi}{2}$$