Introduction to Quantile Regression and Applications

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Outline

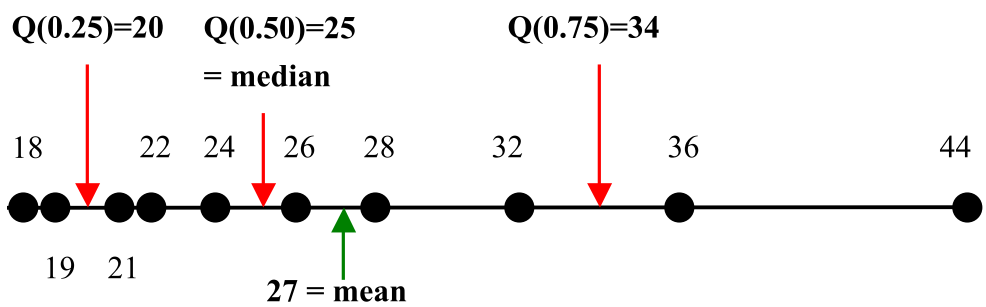
I. Mean, Quantiles, and Distribution
II. Quantile Regression
III. Fundamentals of Quantile Estimation
IV. Computation of Quantile Regression: R-Program
V. Conclusion Remarks

I. Mean, Quantiles, and Distribution

1. Distribution of a Random Variable
   - Example: Starting Salary of 10 Employees

     Y = 18, 19, 21, 22, 24, 26, 28, 32, 36, 44 (in thousands of NT$)

     ❖ Mean = 27



❖ Quartiles:

First Quartile (25 Percentile) = 20

Second Quartile (50 Percentile) = 25 = Median

Third Quartile (75 Percentile) = 34

❖ Distribution Skewness

Symmetric (No Skewness):

Median = Mean

Median – First Quartile = Third Quartile – Median

Positive Skewness (Skew to the Right)

Median < Mean

Median – First Quartile < Third Quartile – Median

Negative Skewness (Skew to the Left)

Median > Mean

Median – First Quartile > Third Quartile – Median

- Quantiles of order τ: $Q(\tau)$, $0 \le \tau \le 1$

❖ $Q(0.20) = 19$; $Q(0.25) = 20$; $Q(0.50) = 25$

$Q(0.75) = 34$; $Q(0.90) = 36$; $Q(1.00) = ?$

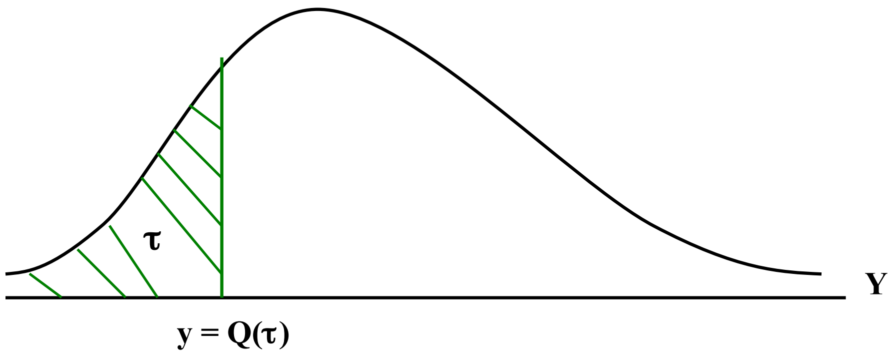
❖ Quantile as a Ranking:

$Q(\tau) < Q(\lambda)$ if $\tau < \lambda$

2. Quantile and Distribution Function

- Distribution Function

$$F(y) = P(Y \le y)$$



- Quantiles

❖ Quantiles of order τ:

$$F(y) = P(Y \le y) = \tau \quad \text{implies} \quad y = F^{-1}(\tau) = Q(\tau)$$

Examples:

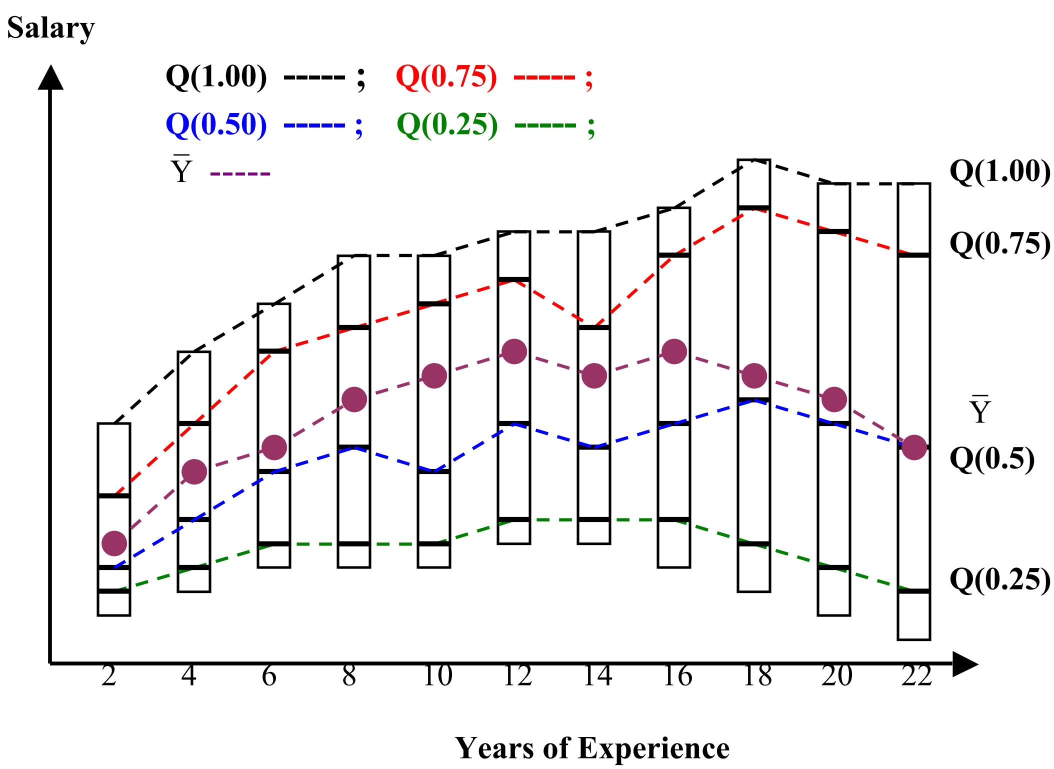
$$25 = F^{-1}(0.5) = Q(0.5)$$

$$34 = F^{-1}(0.75) = Q(0.75)$$

II. Quantile Regression

1. Classical Regression versus Quantile Regression
   - Example:

     y = salary; x = years of experience



   - Classical Regression

     $$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

     ❖ Conditional Mean Regression

     $$E(y_i \mid x_i) = \beta_0 + \beta_1 x_i \equiv \mu_i$$

   - Qunatile Regression of order $\tau \in [0, 1]$

     $$y_i = \beta_0^\tau + \beta_1^\tau x_i + \varepsilon_i$$

     ❖ Conditional Quantile Regression

     $$Q(\tau \mid x_i) = \beta_0^\tau + \beta_1^\tau x_i$$

2. Example: Quantile Regressions of Wage Determination in Taiwan
   - Data: 29,133 plant workers in 1996

     wage = hourly wage; experience = months on current job
   - Regressions:

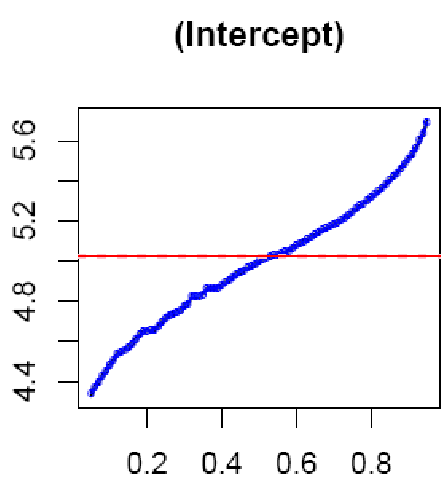
- Conditional Mean Regression

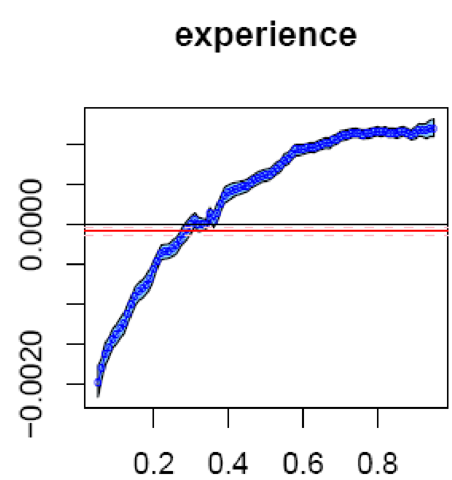
  $\ln(\text{wage}) = \beta_0 + \beta_1 (\text{experience}) + \varepsilon$

- Qunatile Regression

  $\ln(\text{wage}) = \beta_0^\tau + \beta_1^\tau (\text{experience}) + \varepsilon$

τ	β_0^τ	β_1^τ
0.1	4.47987	-0.00137
0.2	4.65334	-0.00056
0.3	4.77952	0.00000
0.4	4.88014	0.00040
0.5	4.99839	0.00062
0.6	5.07736	0.00094
0.7	5.18110	0.00111
0.8	5.31419	0.00117
0.9	5.50766	0.00115
0.95	5.69128	0.00121
Conditional Mean	5.02400	-0.000086





III. Fundamentals of Quantile Regression Estimation

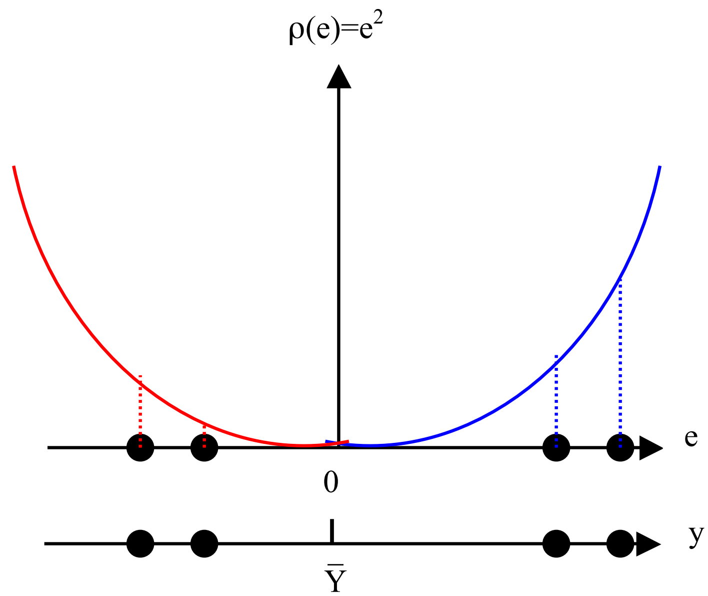
1. Estimation of Mean and Quantile

- Sample Observations: $y = \left(y_1, y_2, \ldots, y_n\right)$
- Population Mean and Population Quantile

  $E\left(y_i\right) = \mu;\ \ Q\left(\tau\right) = F^{-1}(\tau)$
- Least-square Estimation of Population Mean
  - ❖ Model: $y_i = \mu + \varepsilon_i = \bar{Y} + e_i$

    Error: $e_i = y_i - \bar{Y}$
  - ❖ Loss Function: $\rho\left(e_i\right) = e_i^2 = \left(y_i - \bar{Y}\right)^2$
  - ❖ Least-square method:

$$\min_{\bar{Y}} \sum_{i=1}^{n} \rho\left(e_i\right) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - \bar{Y}\right)^2; \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$



- Quantile Estimation of $Q\left(\tau\right)$
  - ❖ Model: $y_i = Q\left(\tau\right) + \varepsilon_i = \hat{Q} + e_i$

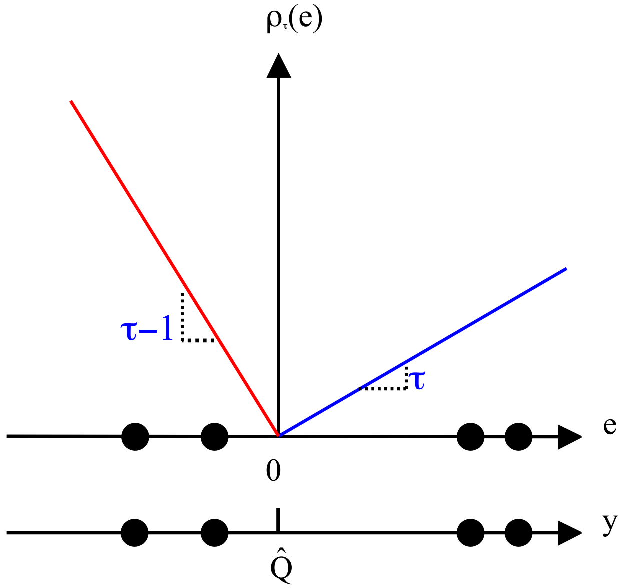
    Error: $e_i = y_i - \hat{Q}$

    Example: $\tau = 0.25$ implies $100\,\tau$ % of $e_i = y_i - \hat{Q} \le 0$

    $100\,(1-\tau)$ % of $e_i = y_i - \hat{Q} \ge 0$

❖ Loss Function:

$$\rho_\tau(e_i) = \tau \times e_i = \tau \times \left(y_i - \hat{Q}\right) \quad \text{if } e_i \ge 0$$
$$= (\tau - 1) \times e_i = (\tau - 1) \times \left(y_i - \hat{Q}\right) \quad \text{if } e_i \le 0$$
$$\rho_\tau(e_i) = \left(\tau - I(e_i < 0)\right) \times e_i = \left(\tau - I\left(y_i - \hat{Q} < 0\right)\right) \times \left(y_i - \hat{Q}\right)$$



❖ Quantile Estimation Method

$$\min_{\hat{Q}} \sum_{i=1}^{n} \rho_\tau(e_i) = \sum_{i=1}^{n} \left(\tau - I\left(y_i - \hat{Q} < 0\right)\right) \times \left(y_i - \hat{Q}\right)$$

Note: (1) For any i, there is the constraint

$$\text{If } \left(y_i - \hat{Q}\right) \begin{cases} \ge \\ \le \end{cases} 0, \text{ then } \rho_\tau(e_i) = \begin{cases} \tau \times \left(y_i - \hat{Q}\right) \\ (1 - \tau) \times \left(\hat{Q} - y_i\right) \end{cases}$$

i.e, for any i,

$$\hat{Q} + \left(y_i - \hat{Q}\right) - \left(\hat{Q} - y_i\right) = y_i$$

(2) Let $u_i = \left(y_i - \hat{Q}\right) \ge 0$, $v_i = \left(\hat{Q} - y_i\right) \ge 0$ be two "slack" variables,

then for any i,

$$\rho_\tau(e_i) = \tau \times u_i + (1 - \tau) \times v_i$$

subject to $\hat{Q} + u_i - v_i = y_i$

❖ Linear Programming Estimation of Quantiles

$$\min_{\hat{Q},u,v} \sum_{i=1}^{n} \left(\tau \times u_i + (1-\tau) \times v_i\right)$$

$$\text{subject to } \hat{Q} + u_i - v_i = y_i, \quad i = 1, 2, \ldots, n$$

$$u_i, v_i \geq 0$$

2. Estimation of Classical Regression and Quantile Regression
   - Model: $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
   - Classical Regression

     ❖ Conditional Mean: $E(y_i \mid x_i) = \beta_0 + \beta_1 x_i$

     ❖ Least-square Method: $\min_{\hat{\beta}_0, \hat{\beta}_1} \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2$

   - Quantile Regression:

     ❖ Conditional Quantile: $Q(\tau \mid x_i) = \beta_0^{\tau} + \beta_1^{\tau} x_i$

     ❖ Quantile Method: $\min_{\hat{\beta}_0^{\tau}, \hat{\beta}_1^{\tau}} \sum_{i=1}^{n} \rho_{\tau}(e_i) = \sum_{i=1}^{n} \left(\tau - I(e_i < 0)\right) \times (e_i)$

$$e_i = y_i - \hat{\beta}_0^{\tau} - \hat{\beta}_1^{\tau} x_i$$

IV. Computation of Quantile Regression: R-Program

- Free Download of **R-Program** at
  http://www.r-project.org
- Example on Wage Determination in Taiwan
  - ❖ Observations: 29,133 plant workers in 1996
  - ❖ Variables: wage = hourly wage; lnwage = ln(wage); gender = 1 male, 0 female;
    age = year of age; married = 1 married, 0 unmarried;
    education = year of education;
    experience = months of working experience on current job
- Use of R-Program
  - ❖ Data file: wage.dat

wage	lnwage	gender	age	married	education	experience
270.83	5.60	1	33	0	9	2
175	5.16	1	36	1	12	125
116.07	4.75	0	28	1	12	26
...	…	..	..	..	…	….

  - ❖ Program file:

```
> library(quantreg)
## change directory
{
dat=read.table("wage.dat",header=T)
attach(dat)
ls <- lm(lnwage~gender+married+education+experience)
summary(ls)
qt <- summary(rq(lnwage~gender+married+education+experience,
                  tau=5:95/100))
postscript("wage.ps", horizontal = FALSE, width = 6.5, height = 3.5)
plot(qt, nrow =1, ncol = 2)
dev.off()
}
```

  - ❖ Outputs:

(1) Least-square outputs:

Coefficients:

(Intercept)	gender	married	education	experience
4.078e+00	3.074e-01	2.242e-01	5.660e-02	-2.925e-05

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.078e+00	1.099e-02	370.912	<2e-16 ***
gender	3.074e-01	6.071e-03	50.641	<2e-16 ***
married	2.242e-01	6.344e-03	35.340	<2e-16 ***
education	5.660e-02	7.833e-04	72.260	<2e-16 ***
experience	-2.925e-05	2.814e-05	-1.039	0.299

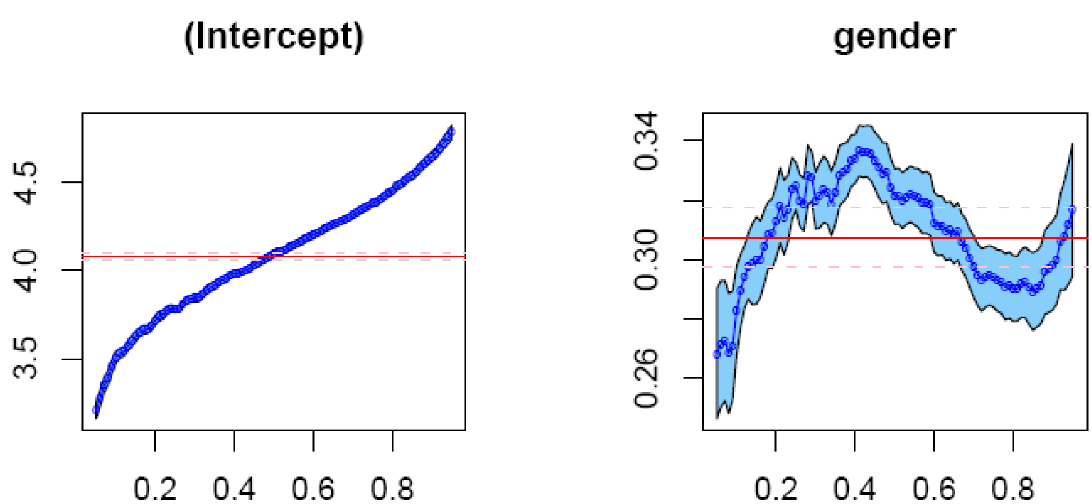
(2) Quantile Outputs:

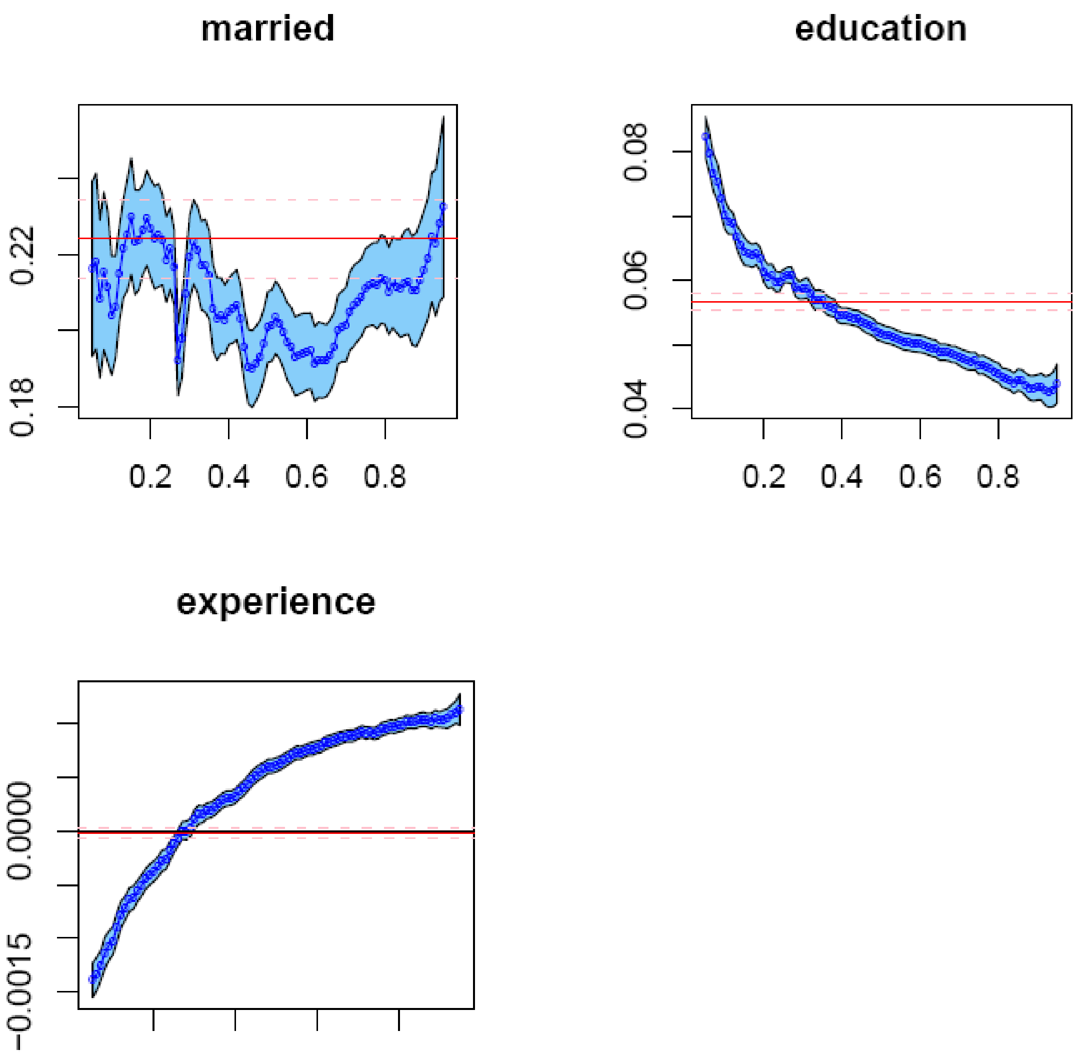
tau: [1] 0.05

Coefficients:

	Value	Std. Error	t value	Pr(>|t|)
(Intercept)	3.20542	0.02845	112.66064	0.00000
gender	0.26821	0.01336	20.08050	0.00000
married	0.21626	0.01395	15.49835	0.00000
education	0.08225	0.00195	42.17080	0.00000
experience	-0.00139	0.00010	-14.33464	0.00000

(3) Graphs of $y_i = \beta\tau$




married
0.18
0.22
0.2
0.4
0.6
0.8
education
0.04
0.06
0.08
0.2
0.4
0.6
0.8
experience
-0.0015
0.0000
0.2
0.4
0.6
0.8

V. Conclusion Remarks on Quantile Regression

1. Regression:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

- Homocedastic Case: $\varepsilon_i \sim \text{iid}$

$$Q_Y(\tau \mid x_i) = \beta_0 + \beta_1 x_i + Q_\varepsilon(\tau) = \left(\beta_0 + Q_\varepsilon(\tau)\right) + \beta_1 x_i$$
$$= \beta_0^\tau + \beta_1 x_i$$

Intercept varies with τ, but not slope.

- Heteroscedastic Case: $\varepsilon_i = h(x_i) w_i = x_i \ w_i$

$$Q_Y(\tau \mid x_i) = \beta_0 + \beta_1 x_i + x_i Q_w(\tau) = \beta_0 + \left(\beta_1 + Q_w(\tau)\right) x_i$$
$$= \beta_0 + \beta_1^\tau x_i$$

Slope varies with τ, but not intercept.

2. Interpreting Quantile Regression Coefficients:

- Classical Regression:
  - ❖ $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i; \ \ E(y_i \mid x_i) = \beta_0 + \beta_1 x_i$

$$\frac{\partial E(y_i \mid x_i)}{\partial x_i} = \beta_1$$

  - ❖ Transformation: $\ln(y_i) = \beta_0 + \beta_1 x_i + \varepsilon_i; \ \ E(\ln(y_i) \mid x_i) = \beta_0 + \beta_1 x_i;$

$$\frac{\partial E(\ln(y_i) \mid x_i)}{\partial x_i} = \beta_1, \ \text{but} \ \frac{\partial E(y_i \mid x_i)}{\partial x_i} \neq e^{\beta_1}$$

- Quantile Regression:
  - ❖ $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i; \ \ Q_y(\tau \mid x_i) = \beta_0^\tau + \beta_1^\tau x_i$

$$\frac{\partial Q_y(\tau \mid x_i)}{\partial x_i} = \beta_1^\tau$$

  - ❖ Transformation: $\ln(y_i) = \beta_0 + \beta_1 x_i + \varepsilon_i; \ \ Q_{\ln(y)}(\tau \mid x_i) = \beta_0^\tau + \beta_1^\tau x_i$

$$\frac{\partial Q_{\ln(y)}(\tau \mid x_i)}{\partial x_i} = \beta_1^\tau \ \text{ and } \ \frac{\partial Q_y(\tau \mid x_i)}{\partial x_i} = e^{\beta_1^\tau}$$