Implicit Finite-Horizon Approximation and Efficient Optimal Algorithms for Stochastic Shortest Path

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Abstract

We introduce a generic template for developing regret minimization algorithms in the Stochastic Shortest Path (SSP) model, which achieves minimax optimal regret as long as certain properties are ensured. The key of our analysis is a new technique called implicit finite-horizon approximation, which approximates the SSP model by a finite-horizon counterpart *only in the analysis* without explicit implementation. Using this template, we develop two new algorithms: the first one is model-free (the first in the literature to our knowledge) and minimax optimal under strictly positive costs; the second one is model-based and minimax optimal even with zero-cost state-action pairs, matching the best existing result from (Tarbouriech et al., 2021b). Importantly, both algorithms admit highly sparse updates, making them computationally more efficient than all existing algorithms. Moreover, both can be made completely parameter-free.

1. Introduction

We study regret minimization in the Stochastic Shortest Path (SSP) model, where an agent aims to reach a goal state with minimum cost in a stochastic environment. Specifically, the problem proceeds in K episodes. In each episode, the learner starts at a fixed initial state, sequentially takes action, suffers some cost, and transits to the next state, until reaching a predefined goal state. The performance of the learner is measured by her regret, which is the difference between her total costs and that of the best policy.

(Tarbouriech et al., 2020a) develop the first regret minimization algorithm for SSP with a regret bound of $\tilde{\mathcal{O}}(D^{3/2}S\sqrt{AK/c_{\min}})$, where D is the diameter, S is the number of states, A is the number of actions, and $c_{\min}$ is the minimum cost among all state-action pairs. (Cohen et al., 2020) improve over their results and give a near optimal regret bound of $\tilde{\mathcal{O}}(B_\star S\sqrt{AK})$, where $B_\star \le D$ is the largest expected cost of the optimal policy starting from any state. Even more recently, (Cohen et al., 2021) achieve minimax regret of $\tilde{\mathcal{O}}(B_\star\sqrt{SAK})$ through a finite-horizon reduction technique, and concurrently (Tarbouriech et al., 2021b) also propose minimax optimal and parameter-free algorithms. Notably, all existing algorithms are model-based with space complexity $\Omega(S^2A)$. Moreover, they all update the learner's policy through full-planning (a term taken from (Efroni et al., 2019)), incurring a relatively high time complexity.

In this work, we further advance the state-of-the-art by proposing a generic template for regret minimization algorithms in SSP (Algorithm 1). By instantiating our template differently, we make the following two key algorithmic contributions:

- In Section 4, we develop the *first model-free* SSP algorithm called LCB-ADVANTAGE-SSP (Algorithm 2). Similar to most model-free reinforcement learning algorithms, LCB-ADVANTAGE-SSP enjoys a space complexity of $\tilde{\mathcal{O}}(SA)$. It achieves a regret bound of $\tilde{\mathcal{O}}(B_\star\sqrt{SAK} + B_\star^5S^2A/c_{\min}^4)$, which is minimax optimal when $c_{\min} > 0$. Moreover, it can be made parameter-free without worsening the regret bound.
- In Section 5, we develop another simple model-based algorithm called SVI-SSP (Algorithm 4), which achieves minimax regret $\tilde{\mathcal{O}}(B_\star\sqrt{SAK} + B_\star S^2A)$ even when $c_{\min} = 0$, matching the best existing result by Tarbouriech et al. (2021b). Notably, SVI-SSP is computationally much more efficient since it updates each state-action pair only logarithmically many times, and each update only performs *one-step planning* (again, a term taken from (Efroni et al., 2019)) as opposed to full-planning (such as value iteration or extended value iteration); SVI-SSP can also be made parameter-free following the idea of (Tarbouriech et al., 2021b).

Techniques Our main technical contribution is a new analysis framework called *implicit finite-horizon approximation* (Section 3), which is the key to analyze algorithms developed from our template. The high level idea is to ap-

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proximate an SSP instance by a finite-horizon counterpart. However, the approximation *only happens in the analysis*, a key difference compared to (Chen et al., 2021; Chen and Luo, 2021; Cohen et al., 2021) that explicitly implement such an approximation in their algorithms. As a result, our method not only avoids blowing up the time and space complexity by a factor of the horizon, but also allow one to derive a horizon-free regret bound.

In order to achieve the minimax optimal regret, our model-free algorithm LCB-ADVANTAGE-SSP uses a key variance reduction idea via a reference-advantage decomposition by (Zhang et al., 2020b). On the other hand, for our model-based algorithm SVI-SSP, we adopt a special Bernstein-style bonus term and bound the learner's total variance via recursion, taking inspiration from (Tarbouriech et al., 2021b; Zhang et al., 2020a).

Empirical Evaluation We support our theoretical findings with experiments in Appendix F.

Related Work For a detailed comparison of existing results for the same problem, we refer the readers to (Tarbouriech et al., 2021b, Table 1). There are also several works (Rosenberg and Mansour, 2020; Chen et al., 2021; Chen and Luo, 2021) that consider the even more challenging SSP setting where the cost function is decided by an adversary and can change over time. Apart from regret minimization, (Tarbouriech et al., 2021a) study the sample complexity of SSP with a generative model; (Lim and Auer, 2012) and (Tarbouriech et al., 2020b) investigate exploration problems involving multiple goal states (multi-goal SSP).

The special case of SSP with a fixed horizon has been studied extensively, for both stochastic costs (e.g., (Azar et al., 2017; Jin et al., 2018; Efroni et al., 2019; Zanette and Brunskill, 2019; Zhang et al., 2020a)) and adversarial costs (e.g., (Neu et al., 2012; Zimin and Neu, 2013; Rosenberg and Mansour, 2019; Jin et al., 2020)). Importantly, recent works (Wang et al., 2020; Zhang et al., 2020a) find that when the cost for each episode is at most a constant, it is in fact possible to obtain a regret bound with only logarithmic dependency on the horizon. (Tarbouriech et al., 2021b) generalize this concept to SSP and define horizon-free regret as a bound with only logarithmic dependence on the expected hitting time of the optimal policy starting from any state (which is bounded by $B_\star/c_{\min}$). They also propose the first algorithm with horizon-free regret for SSP, which is important for arguing minimax optimality even when $c_{\min} = 0$. Notably, our model-based algorithm SVI-SSP also achieves horizon-free regret (but the model-free one does not).

2. Preliminaries

An SSP instance is defined by a Markov Decision Process (MDP) $M = (\mathcal{S}, \mathcal{A}, s_{\text{init}}, g, c, P)$, where $\mathcal{S}$ is the state space, $\mathcal{A}$ is the action space, $s_{\text{init}} \in \mathcal{S}$ is the initial state, $g \notin \mathcal{S}$ is the goal state, $c : \mathcal{S}\times\mathcal{A} \to [c_{\min}, 1]$ is the cost function with some $c_{\min} \geq 0$, and $P : \mathcal{S}\times\mathcal{A} \to \Delta_{\mathcal{S}^+}$ with $\mathcal{S}^+ = \mathcal{S}\cup\{g\}$ is the transition function. Here, $\Delta_{\mathcal{S}^+}$ is the simplex over $\mathcal{S}^+$, and we use $P_{s,a} \in \Delta_{\mathcal{S}^+}$ to denote the distribution over $\mathcal{S}^+$ that the transition follows if one takes action a at state s. We assume that the transition P is unknown to the learner, while all other parameters are known.[1]

The learning process goes as follows: the learner interacts with the environment for K episodes. In the k-th episode, the learner starts in initial state s_{init}, sequentially takes an action, suffers a cost, and transits to the next state until reaching the goal state g. More formally, at the i-th step of the k-th episode, the learner observes the current state s_i^k (with $s_1^k = s_{\text{init}}$), takes action a_i^k, suffers a cost $c(s_i^k, a_i^k)$, and transits to the next state $s_{i+1}^k \sim P_{s_i^k, a_i^k}$. An episode ends when the current state is g, and we define the length of episode k as I_k, such that $s_{I_k+1}^k = g$.

Learning Objective At a high level, the learner's goal is to reach the goal with a small total cost. To this end, we focus on *proper policies* — a (stationary and deterministic) policy $\pi : \mathcal{S} \to \mathcal{A}$ is a mapping that assigns an action $\pi(s)$ to each state $s \in \mathcal{S}$, and it is proper if the goal is reached with probability 1 when following π (that is, taking action $\pi(s)$ whenever in state s). Given a proper policy π, one can define the cost-to-go function $V^\pi : \mathcal{S} \to [0, \infty)$ as $V^\pi(s) = \mathbb{E}\left[\sum_{i=1}^{I} c(s_i, \pi(s_i)) \middle| P, s_1 = s\right]$, where the expectation is with respect to the randomness of $s_{i+1} \sim P_{s_i, \pi(s_i)}$ and the number of steps I before reaching g. The optimal proper policy $\pi^\star$ is then defined as a policy such that $V^{\pi^\star}(s) = \min_{\pi\in\Pi} V^\pi(s)$ for all $s \in \mathcal{S}$, where Π is the set of all proper policies assumed to be nonempty. The formal objective of the learner is then to minimize her regret against $\pi^\star$, the difference between her total cost and that of the optimal proper policy, defined as

$$R_K = \sum_{k=1}^{K}\sum_{i=1}^{I_k} c(s_i^k, a_i^k) - K \cdot V^\star(s_{\text{init}}),$$

where we use $V^\star$ as a shorthand for $V^{\pi^\star}$. The minimax optimal regret is known to be $\tilde{\mathcal{O}}(B_\star\sqrt{SAK})$, where $B_\star = \max_{s\in\mathcal{S}} V^\star(s)$, and $S = |\mathcal{S}^+|$ and $A = |\mathcal{A}|$ are the numbers of states (including the goal state) and actions respectively (Cohen et al., 2020).

Bellman Optimality Equation For a proper policy π, the corresponding action-value function $Q^\pi : \mathcal{S} \times \mathcal{A} \to [0, \infty)$ is defined as $Q^\pi(s, a) = c(s, a) + \mathbb{E}_{s'\sim P_{s,a}}[V^\pi(s')]$. Similarly, we use $Q^\star$ as a shorthand for $Q^{\pi^\star}$. When $c_{\min} > 0$,

[1] In particular, the cost function is also known to the learner, but just as in many prior works, our results generalize directly to the case where the learner only obtains stochastic samples of the cost function.

Algorithm 1 A General Algorithmic Template for SSP

Initialize: $t \leftarrow 0$, $s_1 \leftarrow s_{\text{init}}$, $Q(s,a) \leftarrow c(s,a)$ for all $(s,a) \in \mathcal{S} \times \mathcal{A}$.
for $k = 1, \ldots, K$ **do**
  repeat
1   Increment time step $t \xleftarrow{+} 1$.
2   Take action $a_t = \operatorname{argmin}_a Q(s_t, a)$, suffer cost $c(s_t, a_t)$, transit to and observe s'_t.
3   Update Q (so that it satisfies Property 1 and Property 2).
4   **if** $s'_t \neq g$ **then** $s_{t+1} \leftarrow s'_t$; **else** $s_{t+1} \leftarrow s_{\text{init}}$, **break**.
Record $T \leftarrow t$ (that is, the total number of steps).

it is known that $\pi^\star$ satisfies the Bellman optimality equation: $V^\star(s) = \min_{a \in \mathcal{A}} Q^\star(s,a)$ for all $s \in \mathcal{S}$ (Bertsekas and Tsitsiklis, 1991). When $c_{\min} = 0$, one can first solve a modified SSP instance with the cost function redefined as $c_\epsilon(s,a) = \max\{c(s,a), \epsilon\}$ for some $\epsilon > 0$ so that $c_{\min} = \epsilon > 0$, and the regret in this modified SSP is similar to that in the original SSP up to an additive term of order $\mathcal{O}(\epsilon K)$ (Tarbouriech et al., 2020a). Therefore, throughout the paper we assume that $c_{\min} > 0$ and the Bellman optimality equation holds, unless explicitly stated otherwise.

Other Notations We use $C_K = \sum_{k=1}^{K} \sum_{i=1}^{I_k} c(s_i^k, a_i^k)$ in the analysis to denote the total costs suffered by the learner over K episodes. For a function $X : \mathcal{S}^+ \to \mathbb{R}$ and a distribution $P \in \Delta_{\mathcal{S}^+}$, denote by $PX = \mathbb{E}_{S \sim P}[X(S)]$ the expectation where S is drawn from P. For a scalar x, define $(x)_+ = \max\{x, 0\}$, and denote by $\lceil x \rceil_2 = 2^{\lceil \log_2 x \rceil}$ and $\lfloor x \rfloor_2 = 2^{\lfloor \log_2 x \rfloor}$ the closest power of two upper and lower bounding x respectively. For an integer m, $[m]$ denotes the set $\{1, \ldots, m\}$. In pseudocode, $x \xleftarrow{+} y$ is a shorthand for the increment operation $x \leftarrow x + y$.

3. Implicit Finite-Horizon Approximation

In this section, we introduce our main analytical technique, that is, implicitly approximating the SSP problem with a finite-horizon counterpart. We start with a general template of our algorithms shown in Algorithm 1. For notational convenience, we concatenate state-action trajectories of all episodes as one single sequence (s_t, a_t) for $t = 1, 2, \ldots, T$, where $s_t \in \mathcal{S}$ is one of the non-goal state and $a_t \in \mathcal{A}$ is the action taken at s_t by the learner. Note that the goal state g is never included in this sequence, and we also use the notation $s'_t \in \mathcal{S}^+$ to denote the after-state following (s_t, a_t), so that s_{t+1} is simply s'_t unless $s'_t = g$ (in which case s_{t+1} is reset to the initial state s_{init}); see Line 4.

The template follows a rather standard idea for many reinforcement learning algorithms: maintain an (optimistic) estimate Q of the optimal action-value function $Q^\star$, and act greedily by taking the action with the smallest estimate: $a_t = \operatorname{argmin}_a Q(s_t, a)$; see Line 2. The key of the analysis is often to bound the estimation error $Q^\star(s_t, a_t) - Q(s_t, a_t)$, which is relatively straightforward in a discounted setting (where the discount factor controls the growth of the error) or a finite-horizon setting (where the error vanishes after a fixed number of steps), but becomes highly non-trivial for SSP due to the lack of similar structures.

We propose to approximate the original SSP instance M with a finite-horizon counterpart $\widetilde{M}$ implicitly (that is, only in the analysis). We defer the formal definition of $\widetilde{M}$ to Appendix A, which is similar to those in (Chen et al., 2021; Cohen et al., 2021) and corresponds to interacting with the original SSP for H steps (for some integer H) and then teleporting to the goal. All we need in the analysis are the optimal value function $V_h^\star$ and optimal action-value function $Q_h^\star$ of $\widetilde{M}$ for each step $h \in [H]$, which can be defined recursively without resorting to the definition of $\widetilde{M}$:

$$Q_h^\star(s,a) = c(s,a) + P_{s,a} V_{h-1}^\star, V_h^\star(s) = \min_a Q_h^\star(s,a),$$

with $Q_0^\star(s,a) = 0$ for all (s,a).[2] Intuitively, $Q_H^\star$ approximates $Q^\star$ well when H is large enough. This is formally summarized in the lemma below (see Appendix A).

Lemma 1. *For any value of H, $Q_H^\star(s,a) \leq Q^\star(s,a)$ holds for all (s,a). For any $\delta \in (0,1)$, if $H \geq \frac{4B_\star}{c_{\min}} \ln(2/\delta) + 1$, then $Q^\star(s,a) \leq Q_H^\star(s,a) + B_\star \delta$ holds for all (s,a).*

In the remaining discussion, we fix a particular value of H. To carry out the regret analysis, we now specify two general requirements of the estimate Q. Let Q_t be the value of Q at the beginning of time step t. Then Q_t needs to satisfy:

Property 1 (Optimism)**.** *With high probability, $Q_t(s,a) \leq Q^\star(s,a)$ holds for all (s,a) and $t \geq 1$.*

Property 2 (Recursion)**.** *There exists a "bonus overhead" $\xi_H > 0$ and an absolute constant $d > 0$ such that the following holds with high probability:*

$$\sum_{t=1}^{T} (\mathring{Q}(s_t, a_t) - Q_t(s_t, a_t))_+ \leq \xi_H + \left(1 + \frac{d}{H}\right) \sum_{t=1}^{T} (\mathring{V}(s_t) - Q_t(s_t, a_t))_+,$$

for $\mathring{Q} = Q_h^\star$ and $\mathring{V} = V_{h-1}^\star$ ($h = 1, \ldots, H$) as well as $\mathring{Q} = Q^\star$ and $\mathring{V} = V^\star$.

Property 1 is standard and can usually be ensured by using a certain "bonus" term derived from concentration equalities

[2] Note that our notation is perhaps unconventional compared to most works on finite-horizon MDPs, where $Q_h^\star$ usually refers to our $Q_{H-h}^\star$. We make this switch since we want to highlight the dependence on H for $Q_H^\star$.

in the update. These bonus terms on (s_t, a_t) accumulate into some bonus overhead in the final regret bound, which is exactly the role of ξ_H in Property 2. In both of our algorithms, ξ_H has a leading-order term $\tilde{\mathcal{O}}(\sqrt{B_\star SAC_K})$ and a lower-order term that increases in H.

Property 2 is a key property that provides a recursive form of the estimation error and allows us to connect it to the finite-horizon approximation. This is illustrated through the following two lemmas.

Lemma 2. *Property 2 implies* $\sum_{t=1}^{T}(Q^\star_H(s_t,a_t) - Q_t(s_t,a_t))_+ \leq \mathcal{O}(H\xi_H)$.

Proof. With $\mathring{Q} = Q^\star_H$ and $\mathring{V} = V^\star_{H-1}$, Property 2 implies

$$\begin{aligned}
&\sum_{t=1}^{T}(Q^\star_H(s_t,a_t) - Q_t(s_t,a_t))_+ \\
&\leq \xi_H + \left(1+\frac{d}{H}\right)\sum_{t=1}^{T}(V^\star_{H-1}(s_t) - Q_t(s_t,a_t))_+ \\
&\leq \xi_H + \left(1+\frac{d}{H}\right)\sum_{t=1}^{T}(Q^\star_{H-1}(s_t,a_t) - Q_t(s_t,a_t))_+,
\end{aligned}$$

where in the last step we use the optimality of $V^\star_{H-1}$ from Eq. (1). Repeatedly applying this argument, we eventually arrive at $\sum_{t=1}^{T}(Q^\star_H(s_t,a_t) - Q_t(s_t,a_t))_+ \leq H\left(1+\frac{d}{H}\right)^H \xi_H + \left(1+\frac{d}{H}\right)^H \sum_{t=1}^{T}(Q^\star_0(s_t,a_t) - Q_t(s_t,a_t))_+ = \mathcal{O}(H\xi_H)$, where the last step uses the facts $Q^\star_0(s_t,a_t) = 0$ and $\left(1+\frac{d}{H}\right)^H \leq e^d$ (an absolute constant). ☐

Lemma 3. *For any $\delta \in (0,1)$, if $H \geq \frac{4B_\star}{c_{\min}}\ln(2/\delta)+1$, then Property 1 and Property 2 together imply* $\sum_{t=1}^{T} Q^\star(s_t,a_t) - V^\star(s_t) = \mathcal{O}(\delta C_K + \xi_H)$.

Proof. Applying Property 2 with $\mathring{Q} = Q^\star$ and $\mathring{V} = V^\star$, we have $\sum_{t=1}^{T}(Q^\star(s_t,a_t) - Q_t(s_t,a_t))_+ \leq \xi_H + \left(1+\frac{d}{H}\right)\sum_{t=1}^{T}(V^\star(s_t) - Q_t(s_t,a_t))_+$. Now note that by Property 1, the Bellman optimality equation $V^\star(s_t) = \min_a Q^\star(s_t,a)$, and the fact $Q_t(s_t,a_t) = \min_a Q_t(s_t,a)$ (by the definition of a_t), the arguments within the clipping operation $(\cdot)_+$ are all non-negative and thus the clipping can be removed. Rearranging terms then gives

$$\begin{aligned}
&\sum_{t=1}^{T} Q^\star(s_t,a_t) - V^\star(s_t) \\
&\leq \xi_H + \frac{d}{H}\sum_{t=1}^{T}(V^\star(s_t) - Q_t(s_t,a_t)) \\
&\leq \xi_H + \frac{d}{H}\sum_{t=1}^{T}(Q^\star(s_t,a_t) - Q_t(s_t,a_t)),
\end{aligned}$$

where the last inequality is by optimality of $V^\star$. It remains to bound the last term using the finite-horizon approximation $Q^\star_H$ as a proxy:

$$\begin{aligned}
&\sum_{t=1}^{T}(Q^\star(s_t,a_t) - Q_t(s_t,a_t)) \\
&= \sum_{t=1}^{T}(Q^\star(s_t,a_t) - Q^\star_H(s_t,a_t) + Q^\star_H(s_t,a_t) - Q_t(s_t,a_t)) \\
&= \mathcal{O}(TB_\star\delta + H\xi_H),
\end{aligned}$$

where the last step uses Lemma 1 and Lemma 2. Importantly, this term is finally scaled by d/H, which, together with the fact $\frac{TB_\star}{H} \leq c_{\min}T \leq C_K$, proves the claimed bound. ☐

Readers familiar with the literature might already recognize the term $\sum_{t=1}^{T} Q^\star(s_t,a_t) - V^\star(s_t)$ considered in Lemma 3, which is closely related to the regret. Indeed, with this lemma, we can conclude a regret bound for our generic algorithm.

Theorem 1. *For any $\delta \in (0,1)$, if $H \geq \frac{4B_\star}{c_{\min}}\ln(2/\delta)+1$, then Algorithm 1 ensures (with high probability)* $R_K = \tilde{\mathcal{O}}\left(\sqrt{B_\star C_K} + B_\star + \delta C_K + \xi_H\right)$.

Proof. We first decompose the regret as follows, which holds generally for any algorithm:

$$\begin{aligned}
R_K &= \sum_{k=1}^{K}\left(\sum_{i=1}^{I_k} c(s_i^k, a_i^k) - V^\star(s_1^k)\right) \\
&= \sum_{k=1}^{K}\sum_{i=1}^{I_k}\left(c(s_i^k,a_i^k) - V^\star(s_i^k) + V^\star(s_{i+1}^k)\right) \\
&= \sum_{t=1}^{T}(c(s_t,a_t) - V^\star(s_t) + V^\star(s'_t)) \\
&= \sum_{t=1}^{T}(V^\star(s'_t) - P_{s_t,a_t}V^\star) + \sum_{t=1}^{T}(Q^\star(s_t,a_t) - V^\star(s_t)).
\end{aligned}$$

The first term is the sum of a martingale difference sequence (since s'_t is drawn from P_{s_t,a_t}) and can be bounded by $\tilde{\mathcal{O}}\left(\sqrt{B_\star C_K} + B_\star\right)$ using a concentration inequality; see Lemma 33 and Lemma 4. The second term can be bounded using Lemma 3 directly, which finishes the proof. ☐

To get a sense of the regret bound in Theorem 1, first note that since $1/\delta$ only appears in a logarithmic term of the required lower bound of H, one can pick δ to be small enough so that the term δC_K is dominated by others. Moreover, if ξ_H is $\tilde{\mathcal{O}}(\sqrt{B_\star SAC_K})$ plus some lower-order term referred to as ρ_H, then by solving a quadratic of $\sqrt{C_K}$, the regret bound of Theorem 1 implies $R_K = \tilde{\mathcal{O}}(B_\star\sqrt{SAK} + \rho_H)$, which is minimax optimal (ignoring ρ_H)! Based on this analytical technique, in the following sections, we develop the first model-free SSP algorithm and an improved model-based SSP algorithm.

4. The First Model-free Algorithm: LCB-ADVANTAGE-SSP

In this section, we present a model-free algorithm (the first in the literature) called LCB-ADVANTAGE-SSP that falls into our generic template and satisfies the required properties. It is largely inspired by the state-of-the-art model-free algorithm UCB-ADVANTAGE (Zhang et al., 2020b) for the finite-horizon problem. The pseudocode is shown in Algorithm 2 in Appendix B along with a detailed description. Importantly, the space complexity of this algorithm is only $\mathcal{O}(SA)$ since we do not estimate the transition directly or conduct explicit finite-horizon reduction, and the time complexity is only $\mathcal{O}(1)$ in each step. In Appendix B, we show that Algorithm 2 indeed satisfies the two required properties.

Theorem 2. *Let $H = \lceil \frac{4B_\star}{c_{\min}} \ln(\frac{2}{\beta}) + 1 \rceil_2$ for $\beta = \frac{c_{\min}}{2B_\star^2 SAK}$ and $\theta^\star = \tilde{\mathcal{O}}(B_\star^2 H^3 SA / c_{\min}^2)$ be defined in Lemma 6, then Algorithm 2 satisfies Property 1 and Property 2 with $d = 3$ and $\xi_H = \tilde{\mathcal{O}}(\sqrt{B_\star SAC_K} + B_\star^2 H^3 S^2 A / c_{\min})$.*

As a direct corollary of Theorem 1, we arrive at the following regret guarantee.

Theorem 3. *With the same parameters as in Theorem 2, with probability at least $1-47\delta$, Algorithm 2 ensures $R_K = \tilde{\mathcal{O}}(B_\star\sqrt{SAK} + B_\star^5 S^2 A / c_{\min}^4)$.*

We make two remarks on our results. First, while Algorithm 2 requires setting parameters in terms of $B_\star$ to obtain the claimed regret bound, one can in fact achieve the exact same bound without knowing $B_\star$ by slightly changing the algorithm. Details are deferred to Section B.5.

Second, as mentioned in Section 2, when $c_{\min} = 0$, one can pretend that the cost function is $c_\epsilon(s,a) = \max\{c(s,a), \epsilon\}$ for some small ϵ, which introduces an additive regret term of order $\mathcal{O}(\epsilon K)$. By picking ϵ to be of order $K^{-1/5}$, our bound becomes $\tilde{\mathcal{O}}\left(K^{4/5}\right)$ ignoring other parameters.

5. An Optimal and Efficient Model-based Algorithm: SVI-SSP

In this section, we propose a simple model-based algorithm called SVI-SSP (Sparse Value Iteration for SSP), which not only achieves the minimax optimal regret even when $c_{\min} = 0$, matching the state-of-the-art by a recent work (Tarbouriech et al., 2021b), but also admits highly sparse updates, making it more efficient than all existing model-based algorithms. The pseudocode is shown in Algorithm 4 in Appendix B along with a detailed description.

Theorem 4. *If $B \geq B_\star$ and $H = \lceil \frac{4B}{c_{\min}} \ln(\frac{2}{\beta}) + 1 \rceil_2$ for $\beta = \frac{c_{\min}}{2B^2 SAK}$, then Algorithm 4 satisfies Property 1 and Property 2 with $d = 1$ and $\xi_H = \tilde{\mathcal{O}}(\sqrt{B_\star SAC_K} + BS^2A + \beta C_K)$, where the dependence on H in ξ_H is hidden in logarithmic terms.*

Again, as a direct corollary of Theorem 1, we arrive at the following regret guarantee.

Theorem 5. *With the same parameters as in Theorem 4, with probability at least $1-12\delta$, Algorithm 4 ensures $R_K = \tilde{\mathcal{O}}(B_\star\sqrt{SAK} + BS^2A)$.*

Setting $B = B_\star$, our bound becomes $\tilde{\mathcal{O}}(B_\star\sqrt{SAK} + B_\star S^2 A)$, which is minimax optimal even when $c_{\min} = 0$.[3] When $B_\star$ is unknown, we can use the same doubling trick from (Tarbouriech et al., 2021b) to obtain almost the same bound; see Section C.5 for details.[4]

Comparison with EB-SSP (Tarbouriech et al., 2021b) Our regret bounds match exactly the state-of-the-art by Tarbouriech et al. (2021b). Thanks to the sparse update, however, SVI-SSP has a much better time complexity. Specifically, for SVI-SSP, each (s,a) is updated at most $\tilde{\mathcal{O}}(H) = \tilde{\mathcal{O}}(B_\star / c_{\min})$ times (Lemma 14), and each update takes $\mathcal{O}(S)$ time, leading to total complexity $\tilde{\mathcal{O}}(B_\star S^2 A / c_{\min})$. On the other hand, for EB-SSP, although each (s,a) only causes $\tilde{\mathcal{O}}(1)$ updates, each update runs value iteration on all entries of Q until convergence, which takes $\tilde{\mathcal{O}}(B_\star^2 S^2 / c_{\min}^2)$ iterations (see their Appendix C) and leads to total complexity $\tilde{\mathcal{O}}(B_\star^2 S^5 A / c_{\min}^2)$, much larger than ours.

Comparison with (Cohen et al., 2021) Another recent work by Cohen et al. (2021) using explicit finite-horizon approximation also achieves minimax regret but requires the knowledge of some hitting time of the optimal policy. Without this knowledge, their bound has a large $1/c_{\min}^4$ dependence in the lower-order term just as our model-free algorithm. Our results in this section show that implicit finite-horizon approximation is much better than explicit approximation: the former does not necessarily introduce $\text{poly}(H)$ dependence even for the lower-order term, while the latter does unavoidably.

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[3]This is because the dependence on $1/c_{\min}$ is only logarithmic, and one can use the modified cost $c_\epsilon(s,a) = \max\{c(s,a), \epsilon\}$ with $\epsilon = 1/K$ in this case without introducing $\text{poly}(K)$ overhead to the regret

[4]We note that this doubling trick is in fact also applicable to Algorithm 2. However, the specific approach we propose for this algorithm in Section B.5 is better in the sense that it does not worsen the regret at all.

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Extra Notations in Appendix For conciseness, throughout the appendix, we use the following notational shorthands:

- $\mathbb{I}_s(s') = \mathbb{I}\{s = s'\}$;
- $P_t = P_{s_t,a_t}$;
- for a function $X : \mathcal{S}^+ \to \mathbb{R}$ and a distribution $P \in \Delta_{\mathcal{S}^+}$, denote by $PX^2 = \mathbb{E}_{S\sim P}[X(S)^2]$ and $\mathbb{V}(P,X) = \text{VAR}_{S\sim P}[X(S)]$, where S is drawn from P;
- for a function $f_t : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$, we often abuse the notation and use f_t to denote $f_t(s_t,a_t)$ when there is no confusion from the context; in fact, in Lemma 7 and Lemma 16, we also use f_t to denote $f_t(s,a)$ for a particular (s,a) pair;
- $\mathcal{V}_H = \{(Q^\star, V^\star)\} \cup \{(Q^\star_h, V^\star_{h-1})\}_{h=1}^H$.

Note that for any $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$, we have $\mathring{Q}(s,a) = c(s,a) + P_{s,a}\mathring{V}$, $\mathring{V}(s) \in [0, B_\star]$, $\mathring{V}(g) = 0$ and $\mathring{V}(s) \le \min_a \mathring{Q}(s,a)$. Throughout the paper, $\tilde{\mathcal{O}}(\cdot)$ also hides dependence on $\ln(1/\delta)$ where δ is some failure probability.

Assumptions For simplicity, we assume $1 \le B_\star \le \sqrt{K}$ (in general, a known constant $\alpha > 0$ such that $B_\star \le K^\alpha$ suffices). For failure probability δ, we always assume $\delta < e^{-1}$ so that $\ln\frac{1}{\delta} \ge 1$.

A. Omitted Detailts for Section 3

In this section, we provide omitted details and proofs for Section 3. We first introduce the class of finite horizon MDPs used in the approximation: given an SSP model $M = (\mathcal{S}, \mathcal{A}, s_{\text{init}}, g, c, P)$, we consider the costs of interacting with M for at most H steps and then directly teleporting to the goal state. Specifically, we define a finite-horizon SSP $\widetilde{M} = (\widetilde{\mathcal{S}}, \mathcal{A}, \widetilde{s}_{\text{init}}, g, \widetilde{c}, \widetilde{P})$ as follows:

- $\widetilde{\mathcal{S}} = \mathcal{S} \times [H]$, $\widetilde{s}_0 = (s_0, 1)$ and the goal state g remains the same;
- transition from (s,h) to (s',h') is only possible when $h' = h+1$, and the transition follows the original MDP: $\widetilde{P}((s',h+1)|(s,h),a) = P(s'|s,a)$ for $h \in [H-1]$ and $\widetilde{P}(g|(s,H),a) = 1$;
- cost function also follows the original MDP: $\widetilde{c}_k((s,h),a) = c_k(s,a)$.

We also define $Q^\star_0(s,a) = V^\star_0(s) = 0, Q^\star_h(s,a) = \widetilde{Q}^\star((s,H-h+1),a), V^\star_h(s) = \widetilde{V}^\star(s,H-h+1)$ for $h \in [H]$, where $\widetilde{Q}^\star$ and $\widetilde{V}^\star$ are optimal state-action and state value functions in $\widetilde{M}$. Then, it is straightforward to verify that $Q^\star_h, V^\star_h$ satisfy:

$$Q^\star_h(s,a) = c(s,a) + P_{s,a}V^\star_{h-1}, \qquad V^\star_h(s) = \min_a Q^\star_h(s,a). \tag{1}$$

Since M is equivalent to $\widetilde{M}$ with $H = \infty$, intuitively we should have $Q^\star(s,a) \approx Q^\star_H(s,a)$ for a sufficiently large H. The formal statement, shown in Lemma 1, is proven below:

Proof of Lemma 1. By definition $Q^\star_h(s,a) \le Q^\star(s,a)$ holds for all $(s,a) \in \mathcal{S} \times \mathcal{A}$ and $h \in [H]$, since $\widetilde{M}$ is a truncated version of M. Therefore, $V^\star_h(s) \le B_\star$ holds, and the expected hitting time (the number of steps needed to reach the goal) of the optimal policy in $\widetilde{M}$ starting from any (s,h) is upper bounded by $\frac{B_\star}{c_{\min}}$. By (Rosenberg and Mansour, 2020, Lemma 6), when $h \ge \frac{4B_\star}{c_{\min}}\ln\frac{2}{\delta}$, the probability of not reaching g in h steps is at most δ. Denote by $\widetilde{\pi}^\star_L$ the optimal policy of $\widetilde{M}$, and $\pi^\star_L$ a non-stationary policy in M which follows $\widetilde{\pi}^\star_L$ for the first H steps, and then follows $\pi^\star$ afterwards. We have for any $s \in \mathcal{S}$, $V^\star(s) - V^\star_{H-1}(s) \le V^{\pi^\star_L}(s) - V^{\widetilde{\pi}^\star_L}_{H-1}(s) \le B_\star\delta$, where we apply $H \ge \frac{4B_\star}{c_{\min}}\ln\frac{2}{\delta} + 1$, $V^\star(s) \le V^{\widetilde{\pi}^\star_L}(s)$ and $V^\star_{H-1}(s) = V^{\widetilde{\pi}^\star_L}_{H-1}(s)$. Finally, $Q^\star(s,a) - Q^\star_H(s,a) = P_{s,a}(V^\star - V^\star_{H-1}) \le B_\star\delta$. $\square$

The next lemma is used in the proof of Theorem 1, which shows that the sum of the variances of the optimal value function is of order $\tilde{\mathcal{O}}(B_\star C_K)$. It is also useful in bounding the overhead of Bernstein-style confidence interval (see Lemma 9 and (Cohen et al., 2020, Lemma 4.7) for example).

Lemma 4. *With probability at least* $1-\delta$, $\sum_{t=1}^T \mathbb{V}(P_{s_t,a_t}, V^\star) = \tilde{\mathcal{O}}\left(B_\star^2 + B_\star C_K\right)$.

Proof. Note that:

$$\begin{aligned}
\sum_{t=1}^{T}\mathbb{V}(P_{s_t,a_t},V^\star) &= \sum_{t=1}^{T}P_{s_t,a_t}(V^\star)^2-(P_{s_t,a_t}V^\star)^2\\
&=\sum_{k=1}^{K}\sum_{i=1}^{I_k}P_{s_i^k,a_i^k}(V^\star)^2-V^\star(s_i^k)^2+\sum_{k=1}^{K}\sum_{i=1}^{I_k}V^\star(s_i^k)^2-(P_{s_i^k,a_i^k}V^\star)^2\\
&\leq\sum_{k=1}^{K}\sum_{i=1}^{I_k}P_{s_i^k,a_i^k}(V^\star)^2-V^\star(s_{i+1}^k)^2+\sum_{k=1}^{K}\sum_{i=1}^{I_k}Q^\star(s_i^k,a_i^k)^2-(P_{s_i^k,a_i^k}V^\star)^2.\\
&\qquad (V^\star(s_{I_k+1}^k)=0 \text{ and } V^\star(s_i^k)\leq Q^\star(s_i^k,a_i^k))
\end{aligned}$$

For the first term, by Eq. (15) of Lemma 33 with $V^\star(s)\leq B_\star$ and Lemma 28 with $X=V^\star(S'),S'\sim P_{s_t,a_t}$, we have with probability at least $1-\delta$,

$$\begin{aligned}
\sum_{k=1}^{K}\sum_{i=1}^{I_k}P_{s_i^k,a_i^k}(V^\star)^2-V^\star(s_{i+1}^k)^2 &= \tilde{\mathcal{O}}\left(\sqrt{\sum_{t=1}^{T}\mathbb{V}(P_{s_t,a_t},(V^\star)^2)}+B_\star^2\right)\\
&=\tilde{\mathcal{O}}\left(B_\star\sqrt{\sum_{t=1}^{T}\mathbb{V}(P_{s_t,a_t},V^\star)}+B_\star^2\right).
\end{aligned}$$

For the second term, note that:

$$\begin{aligned}
\sum_{k=1}^{K}\sum_{i=1}^{I_k}Q^\star(s_i^k,a_i^k)^2-(P_{s_i^k,a_i^k}V^\star)^2 &= \sum_{k=1}^{K}\sum_{i=1}^{I_k}\left(Q^\star(s_i^k,a_i^k)-P_{s_i^k,a_i^k}V^\star\right)\left(Q^\star(s_i^k,a_i^k)+P_{s_i^k,a_i^k}V^\star\right)\\
&\leq\sum_{k=1}^{K}\sum_{i=1}^{I_k}3B_\star c(s_i^k,a_i^k)=3B_\star C_K.\\
&\qquad (Q^\star(s,a)\leq 2B_\star \text{ and } V^\star(s)\leq B_\star \text{ for any } (s,a)\in\mathcal{S}\times\mathcal{A})
\end{aligned}$$

Therefore, $\sum_{t=1}^{T}\mathbb{V}(P_{s_t,a_t},V^\star)=\tilde{\mathcal{O}}\left(B_\star\sqrt{\sum_{t=1}^{T}\mathbb{V}(P_{s_t,a_t},V^\star)}+B_\star^2+B_\star C_K\right)$. By Lemma 23 with $x=\sum_{t=1}^{T}\mathbb{V}(P_{s_t,a_t},V^\star)$, we have $\sum_{t=1}^{T}\mathbb{V}(P_{s_t,a_t},V^\star)=\tilde{\mathcal{O}}\left(B_\star^2+B_\star C_K\right)$. □

B. Omitted Details for Section 4

We first present details of our proposed model-free algorithm LCB-ADVANTAGE-SSP. It is largely inspired by the state-of-the-art model-free algorithm UCB-ADVANTAGE (Zhang et al., 2020b) for the finite-horizon problem. The pseudocode is shown in Algorithm 2, with only the lines instantiating the update rule of the Q estimates numbered. Importantly, the space complexity of this algorithm is only $\mathcal{O}(SA)$ since we do not estimate the transition directly or conduct explicit finite-horizon reduction, and the time complexity is only $\mathcal{O}(1)$ in each step.

Specifically, for each state-action pair (s,a), we divide the samples received when visiting (s,a) into consecutive stages of exponentially increasing length, and only update $Q(s,a)$ at the end of a stage. The number of samples e_j in stage j is defined through $e_1=H$ and $e_{j+1}=\lfloor(1+1/H)e_j\rfloor$ for some parameter H. Further define $\mathcal{L}^\star=\{E_j\}_{j\in\mathbb{N}^+}$ with $E_j=\sum_{i=1}^{j}e_i$, which contains all the indices indicating the end of some stage. As mentioned, the algorithm only updates $Q(s,a)$ when the total number of visits to (s,a) falls into the set $\mathcal{L}^\star$ (Line 4). The algorithm also maintains an estimate V for $V^\star$, which always satisfies $V(s)=\min_a Q(s,a)$ (Line 8), and importantly another reference value function V^{ref} whose role and update rule are to be discussed later.

In addition, some local and global accumulators are maintained in the algorithm. Local accumulators only store information related to the current stage. These include: $M(s,a)$, the number of visits to (s,a) within the current stage; $v(s,a)$, the

Algorithm 2 LCB-ADVANTAGE-SSP

Parameters: horizon H, threshold $\theta^\star$, and failure probability $\delta \in (0,1)$.

Define: $\mathcal{L}^\star = \{E_j\}_{j\in\mathbb{N}^+}$ where $E_j = \sum_{i=1}^{j} e_i$, $e_1 = H$ and $e_{j+1} = \lfloor (1+1/H)e_j \rfloor$.

Initialize: $t \leftarrow 0$, $s_1 \leftarrow s_{\text{init}}$, $B \leftarrow 1$, for all (s,a), $N(s,a) \leftarrow 0, M(s,a) \leftarrow 0$.

Initialize: for all (s,a), $Q(s,a) \leftarrow c(s,a)$, $V(s) \leftarrow \min_a c(s,a)$, $V^{\text{ref}}(s) \leftarrow V(s)$.

Initialize: for all (s,a), $\mu^{\text{ref}}(s,a) \leftarrow 0, \sigma^{\text{ref}}(s,a) \leftarrow 0, \mu(s,a) \leftarrow 0, \sigma(s,a) \leftarrow 0, v(s,a) \leftarrow 0$.

for $k = 1, \ldots, K$ **do**

  repeat

    Increment time step $t \overset{+}{\leftarrow} 1$.

    Take action $a_t = \operatorname{argmin}_a Q(s_t, a)$, suffer cost $c(s_t,a_t)$, transit to and observe s'_t.

1   Increment visitation counters: $n = N(s_t,a_t) \overset{+}{\leftarrow} 1, m = M(s_t,a_t) \overset{+}{\leftarrow} 1$.

2   Update global accumulators: $\mu^{\text{ref}}(s_t,a_t) \overset{+}{\leftarrow} V^{\text{ref}}(s'_t),\ \sigma^{\text{ref}}(s_t,a_t) \overset{+}{\leftarrow} V^{\text{ref}}(s'_t)^2$.

3   Update local accumulators: $v(s_t,a_t) \overset{+}{\leftarrow} V(s'_t),\ \mu(s_t,a_t) \overset{+}{\leftarrow} V(s'_t) - V^{\text{ref}}(s'_t),\ \sigma(s_t,a_t) \overset{+}{\leftarrow} (V(s'_t) - V^{\text{ref}}(s'_t))^2$.

4   **if** $n \in \mathcal{L}^\star$ **then**

5   Compute $\iota \leftarrow 256 \ln^6(4SAK^4t^5/\delta)$ and bonuses $b' \leftarrow 2\sqrt{\frac{B^2\iota}{m}}$ and

$$b \leftarrow \sqrt{\frac{\sigma^{\text{ref}}(s_t,a_t)/n - \left(\mu^{\text{ref}}(s_t,a_t)/n\right)^2}{n}\iota} + \sqrt{\frac{\sigma(s_t,a_t)/m - \left(\mu(s_t,a_t)/m\right)^2}{m}\iota} + \left(\frac{3B}{n} + \frac{3B}{m}\right)\iota.$$

6       $Q(s_t,a_t) \leftarrow \max\left\{c(s_t,a_t) + \frac{v(s_t,a_t)}{m} - b', Q(s_t,a_t)\right\}$.

7       $Q(s_t,a_t) \leftarrow \max\left\{c(s_t,a_t) + \frac{\mu^{\text{ref}}(s_t,a_t)}{n} + \frac{\mu(s_t,a_t)}{m} - b, Q(s_t,a_t)\right\}$.

8       $V(s_t) \leftarrow \min_a Q(s_t,a)$.

9   **if** $V(s_t) > B$ **then** $B \leftarrow 2V(s_t)$.

10   Reset local accumulators: $v(s_t,a_t) \leftarrow 0, \mu(s_t,a_t) \leftarrow 0, \sigma(s_t,a_t) \leftarrow 0, M(s_t,a_t) \leftarrow 0$.

11   **if** $\sum_a N(s_t,a)$ *is a power of two not larger than* $\theta^\star$ **then** $V^{\text{ref}}(s_t) \leftarrow V(s_t)$.

    if $s'_t \neq g$ **then** $s_{t+1} \leftarrow s'_t$; **else** $s_{t+1} \leftarrow s_{\text{init}}$, **break**.

cumulative value of $V(s')$ within the current stage, where s' represents the next state after each visit to (s,a); and finally $\mu(s,a)$ and $\sigma(s,a)$, the cumulative values of $V(s') - V^{\text{ref}}(s')$ and its square respectively within the current stage (Line 3). These local accumulators are reset to zero at the end of each stage (Line 10).

On the other hand, global accumulators store information related to all stages and are never reset. These include: $N(s,a)$, the number of visits to (s,a) from the beginning, as well as $\mu^{\text{ref}}(s,a)$ and $\sigma^{\text{ref}}(s,a)$, the cumulative value of $V^{\text{ref}}(s')$ and its square respectively from the beginning, where again s' represents the next state after each visit to (s,a) (Line 2).

We are now ready to describe the update rule of Q. The first update, Line 6, is intuitively based on the equality $Q^\star(s,a) = c(s,a) + P_{s,a}V^\star$ and uses $v(s,a)/M(s,a)$ as an estimate for $P_{s,a}V^\star$ together with a (negative) bonus b' derived from Azuma's inequality (Line 5). As mentioned, the bonus is necessary to ensure Property 1 (optimism) so that Q is always a lower confidence bound of $Q^\star$ (hence the name "LCB"). Note that this update only uses data from the current stage (roughly $1/H$ fraction of the entire data collected so far), which leads to an extra $\sqrt{H}$ factor in the regret.

To address this issue, Zhang et al. (2020b) introduce a variance reduction technique via a reference-advantage decomposition, which we borrow here leading to the second update rule in Line 7. This is intuitively based on the decomposition $P_{s,a}V^\star = P_{s,a}V^{\text{ref}} + P_{s,a}(V^\star - V^{\text{ref}})$, where $P_{s,a}V^{\text{ref}}$ is approximated by $\mu^{\text{ref}}(s,a)/N(s,a)$ and $P_{s,a}(V^\star - V^{\text{ref}})$ is approximated by $\mu(s,a)/M(s,a)$. In addition, a "variance-aware" bonus term b is applied, which is derived from a tighter Freedman's inequality (Line 5). The reference function V^{ref} is some snapshot of the past value of V, and is guaranteed to be $\mathcal{O}(c_{\min})$ close to $V^\star$ on a particular state as long as the number of visits to this state exceeds some threshold $\theta^\star = \tilde{\mathcal{O}}\left(B_\star^2 H^3 SA/c_{\min}^2\right)$ (Line 11). Overall, this second update rule not only removes the extra $\sqrt{H}$ factor as in (Zhang et al., 2020b), but also turns some terms of order $\tilde{\mathcal{O}}(\sqrt{T})$ into $\tilde{\mathcal{O}}(\sqrt{C_K})$ in our context, which is important for obtaining the optimal regret.

Despite the similarity, we emphasize several key differences between our algorithm and that of (Zhang et al., 2020b). First, (Zhang et al., 2020b) maintains a different Q estimate for each step of an episode (which is natural for a finite-horizon problem), while we only maintain one Q estimate (which is natural for SSP). Second, we update the reference function $V^{\text{ref}}(s)$ whenever the number of visits to s doubles (while still below the threshold $\theta^\star$; see Line 11), instead of only updating it once as in (Zhang et al., 2020b). We show in Lemma 6 that this helps reduce the sample complexity and leads to a smaller lower-order term in the regret. Third, since there is no apriori known upper bound on V (unlike the finite-horizon setting), we maintain an empirical upper bound B (in a doubling manner) such that $V(s) \leq B \leq B_\star$ (Line 9), which is further used in computing the bonus terms b and b'. This is important for eventually developing a parameter-free algorithm.

Finally, we point out that, just as in the finite-horizon case, the variance reduction technique is crucial for obtaining the minimax optimal regret. For example, if one instead uses an update rule similar to the (suboptimal) Q-learning algorithm of (Jin et al., 2018), while this still satisfies Property 2, the bonus overhead ξ_H would be $\sqrt{H}$ times larger, resulting in a suboptimal leading term in the regret.

Before we present the proof of Theorem 3 (Section B.3), we first quantify the sample complexity of the reference value function (Section B.1) and prove the two required properties (Section B.2).

Extra Notations Denote by $Q_t(s,a)$, $V_t(s)$, $V_t^{\text{ref}}(s)$, B_t, $N_t(s,a)$ the value of $Q(s,a)$, $V(s)$, $V^{\text{ref}}(s)$, B, $N(s,a)$ at the beginning of time step t. Define $N_t(s) = \sum_a N_t(s,a)$. Denote by $n_t(s,a), m_t(s,a), b_t(s,a), b'_t(s,a), \iota_t(s,a)$ the value of n, m, b, b', ι used in computing $Q_t(s,a)$. Note that, these are *not* necessarily their values at time step t. For example, $n_t(s,a)$ is the number of visits to (s,a) before the current stage (not before time t); $m_t(s,a)$ the number of visits to (s,a) in the last stage; and $b_t(s,a)$ and $b'_t(s,a)$ are the bonuses used in the last update of $Q_t(s,a)$ (and they are 0 when $n_t(s,a)=0$). Denote by $l_{t,i}(s,a)$ the i-th time step the agent visits (s,a) among those $n_t(s,a)$ steps before the current stage, and by $\check{l}_{t,i}(s,a)$ the i-th time step the agent visits (s,a) among those $m_t(s,a)$ steps within the last stage. With these notations, we have by the update rule of the algorithm:

$$Q_t(s,a) = \max\Bigg\{Q_{t-1}(s,a),\quad c(s,a) + \frac{1}{m_t}\sum_{i=1}^{m_t} V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) - b'_t, \\ c(s,a) + \frac{1}{n_t}\sum_{i=1}^{n_t} V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}}) + \frac{1}{m_t}\sum_{i=1}^{m_t}(V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) - V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})) - b_t\Bigg\}, \tag{2}$$

where m_t represents $m_t(s,a)$, $\check{l}_{t,i}$ represents $\check{l}_{t,i}(s,a)$, and similarly for n_t, $l_{t,i}$, b_t and b'_t.

We also define two empirical variances at time step t as:

$$\nu_t = \frac{1}{m_t}\sum_{i=1}^{m_t}(V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) - V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}))^2 - \left(\frac{1}{m_t}\sum_{i=1}^{m_t} V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) - V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})\right)^2$$

and

$$\nu_t^{\text{ref}} = \frac{1}{n_t}\sum_{i=1}^{n_t} V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})^2 - \left(\frac{1}{n_t}\sum_{i=1}^{n_t} V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})\right)^2.$$

Here, ν_t and ν_t^{ref} should be treated as a function of state-action pair (s,a), so that m_t, n_t, $\check{l}_{t,i}$, and $l_{t,i}$ in the formulas all represent $m_t(s,a)$, $n_t(s,a)$, $\check{l}_{t,i}(s,a)$, and $l_{t,i}(s,a)$. Except for Lemma 7, this input (s,a) is simply (s_t,a_t).

Further define $\varepsilon_t = \mathbb{I}\{n_t > 0\} = \mathbb{I}\{m_t > 0\}$, and $0/0$ to be 0 so that formula in the form $\frac{1}{n_t}\sum_{i=1}^{n_t} X_{l_{t,i}}$ is treated as 0 if $n_t = 0$ (similarly for m_t).

B.1. Sample Complexity for Reference Value Function

In this section, we assume $H = \lceil \frac{4B_\star}{c_{\min}} \ln(\frac{2}{\beta}) + 1 \rceil_2$ for some $\beta > 0$ (the form used in Theorem 2). We show that to obtain a reference value with precision $\rho \geq 2B_\star\beta$ at state s (that is, $|V^{\text{ref}}(s) - V^\star(s)| \leq \rho$), $\tilde{\mathcal{O}}\left(\frac{B_\star^2 H^3 SA}{\rho^2}\right)$ number of visits to state s is sufficient (Corollary 6). Moreover, the total costs appeared in regret for a reference value function with maximum precision ρ is $\tilde{\mathcal{O}}\left(\frac{B_\star^2 H^3 S^2 A}{\rho}\right)$ (Lemma 6).

Lemma 5. *With probability at least* $1-8\delta$, *Algorithm 2 ensures for any non-negative weights* $\{w_t\}_{t=1}^T$,

$$\sum_{t=1}^T w_t(V^\star(s_t)-V_t(s_t)) \le B_\star \|w\|_1 \beta + \tilde{\mathcal{O}}\left(H^2SAB_\star\|w\|_\infty + B_\star\sqrt{H^3SA\|w\|_\infty\|w\|_1}\right).$$

Proof. Define $w_t^{(0)}=w_t$ and $w_{t+1}^{(h+1)}=\sum_{t'=1}^T\sum_{i=1}^{m_{t'}}\frac{w_{t'}^{(h)}}{m_{t'}}\mathbb{I}\{t=\check{l}_{t',i}\}$. We first argue the following properties related to $w_t^{(h)}$ and vector $w^{(h)}=(w_1^{(h)},\ldots,w_T^{(h)})$. Denote by j_t the stage to which time step t belongs. When $t=\check{l}_{t',i}$, we have $m_{t'}=e_{j_t}$. Therefore,

$$\sum_{t'=1}^T\sum_{i=1}^{m_{t'}}\frac{1}{m_{t'}}\mathbb{I}\{t=\check{l}_{t',i}\}\le\frac{e_{j_t+1}}{e_{j_t}}\le 1+\frac{1}{H},$$

and thus, $\left\|w^{(h)}\right\|_\infty\le(1+\frac{1}{H})\left\|w^{(h-1)}\right\|_\infty\le\cdots\le(1+\frac{1}{H})^h\|w\|_\infty$. Moreover,

$$\left\|w^{(h+1)}\right\|_1=\sum_{t=1}^T\sum_{t'=1}^T\sum_{i=1}^{m_{t'}}\frac{w_{t'}^{(h)}}{m_{t'}}\mathbb{I}\{t=\check{l}_{t',i}\}=\sum_{t'=1}^T w_{t'}^{(h)}\sum_{i=1}^{m_{t'}}\sum_{t=1}^T\frac{\mathbb{I}\{t=\check{l}_{t',i}\}}{m_{t'}}\le\left\|w^{(h)}\right\|_1,$$

and thus $\left\|w^{(h)}\right\|_1\le\|w\|_1$ for any h. Also note that for any $\{X_t\}_t$ such that $X_t\ge 0$:

$$\sum_{t=1}^T\frac{w_t^{(h)}}{m_t}\sum_{i=1}^{m_t}X_{\check{l}_{t,i}}=\sum_{t'=1}^T\sum_{t=1}^T\frac{w_t^{(h)}}{m_t}\sum_{i=1}^{m_t}X_{t'}\mathbb{I}\{t'=\check{l}_{t,i}\}=\sum_{t'=1}^T w_{t'+1}^{(h+1)}X_{t'}. \tag{3}$$

Now we are ready to prove the lemma. First, we condition on Lemma 7, which happens with probability at least $1-7\delta$. Then for any $h\in\{0,\ldots,H-1\}$, $\mathring{Q}=Q_{H-h}$, $\mathring{V}=Q_{H-h-1}$ we have:

$$\sum_{t=1}^T w_t^{(h)}(\mathring{Q}(s_t,a_t)-Q_t(s_t,a_t))_+\le\sum_{t=1}^T w_t^{(h)}\left(P_t\mathring{V}-\frac{1}{m_t}\sum_{i=1}^{m_t}V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})\right)_+ + w_t^{(h)}b'_t$$

(by Eq. (2) and $\mathring{Q}(s,a)=c(s,a)+P_{s,a}\mathring{V}$)

$$\le\sum_{t=1}^T B_\star w_t^{(h)}\mathbb{I}\{m_t=0\}+\sum_{t=1}^T w_t^{(h)}\left(\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}\mathring{V}-\frac{1}{m_t}\sum_{i=1}^{m_t}V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})\right)_+ + w_t^{(h)}b'_t.$$

($P_t=P_{\check{l}_{t,i}}$ and $P_t\mathring{V}\le B_\star\mathbb{I}\{m_t=0\}+\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}\mathring{V}$)

Since $e_1=H$, we have $\sum_{t=1}^T w_t^{(h)}\mathbb{I}\{m_t=0\}\le SAH\left\|w^{(h)}\right\|_\infty$. Moreover, by Eq. (15) of Lemma 33 with $X_t=\mathring{V}(s'_t)$, we have with probability at least $1-\frac{\delta}{H}$: $\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}\mathring{V}\le\frac{1}{m_t}\sum_{i=1}^{m_t}\mathring{V}(s'_{\check{l}_{t,i}})+\tilde{\mathcal{O}}\left(\frac{B_\star\varepsilon_t}{\sqrt{m_t}}\right)$. Plugging these back to the previous inequality and using the definition of b'_t gives:

$$\begin{aligned}
&\sum_{t=1}^T w_t^{(h)}(\mathring{Q}(s_t,a_t)-Q_t(s_t,a_t))_+\\
&\le HSAB_\star\left\|w^{(h)}\right\|_\infty+\sum_{t=1}^T\frac{w_t^{(h)}}{m_t}\sum_{i=1}^{m_t}\left(\mathring{V}(s'_{\check{l}_{t,i}})-V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})\right)_+ +\tilde{\mathcal{O}}\left(\frac{B_\star w_t^{(h)}\varepsilon_t}{\sqrt{m_t}}\right)\\
&\le HSAB_\star\left\|w^{(h)}\right\|_\infty+\tilde{\mathcal{O}}\left(B_\star\sqrt{HSA\left\|w^{(h)}\right\|_\infty\|w\|_1}\right)+\sum_{t=1}^T w_{t+1}^{(h+1)}\left(\mathring{V}(s'_t)-V_t(s'_t)\right)_+ \quad \text{(Eq. (3) and Lemma 12)}\\
&\le\tilde{\mathcal{O}}\left(HSAB_\star\left\|w^{(h)}\right\|_\infty+B_\star\sqrt{HSA\left\|w^{(h)}\right\|_\infty\|w\|_1}\right)+\sum_{t=1}^T w_t^{(h+1)}(\mathring{Q}(s_t,a_t)-Q_t(s_t,a_t))_+,
\end{aligned}$$

where in the last inequality we apply:

$$\sum_{t=1}^{T} w_{t+1}^{(h+1)} \left(\mathring{V}(s'_t) - V_t(s'_t)\right)_+ \leq \sum_{t=1}^{T} w_{t+1}^{(h+1)}(\mathring{V}(s'_t) - V_{t+1}(s'_t))_+ + \tilde{\mathcal{O}}\left(\left\|w^{(h)}\right\|_\infty SB_\star\right) \qquad \text{(apply Lemma 26 on } \textstyle\sum_{t=1}^{T} V_{t+1}(s'_t) - V_t(s'_t))$$

$$\leq \sum_{t=1}^{T} w_t^{(h+1)}(\mathring{V}(s_t) - V_t(s_t))_+ + +\tilde{\mathcal{O}}\left(\left\|w^{(h)}\right\|_\infty SB_\star\right) \qquad ((\mathring{V}(s'_t) - V_{t+1}(s'_t))_+ \leq (\mathring{V}(s_{t+1}) - V_{t+1}(s_{t+1}))_+)$$

$$\leq \sum_{t=1}^{T} w_t^{(h+1)}(\mathring{Q}(s_t, a_t) - Q_t(s_t, a_t))_+ + \tilde{\mathcal{O}}\left(\left\|w^{(h)}\right\|_\infty SB_\star\right). \qquad (\mathring{V}(s_t) \leq \mathring{Q}(s_t, a_t) \text{ and } V_t(s_t) = Q_t(s_t, a_t))$$

By a union bound, the inequality above holds for $\mathring{Q} = Q_{H-h}, \mathring{V} = Q_{H-h-1}$ for all $h \in \{0, \ldots, H-1\}$ with probability at least $1-\delta$. Applying the inequality above recursively starting from $h = 0$, and by $Q_0^\star(s,a) - Q_t(s,a) \leq 0$, $(1+\frac{1}{H})^H \leq 3$:

$$\sum_{t=1}^{T} w_t(Q_H^\star(s_t, a_t) - Q_t(s_t, a_t))_+ = \tilde{\mathcal{O}}\left(H^2 SAB_\star \left\|w\right\|_\infty + B_\star\sqrt{H^3 SA \left\|w\right\|_\infty \left\|w\right\|_1}\right).$$

Therefore, by Lemma 1,

$$\sum_{t=1}^{T} w_t(V^\star(s_t) - V_t(s_t)) \leq \sum_{t=1}^{T} w_t(Q^\star(s_t, a_t) - Q_H^\star(s_t, a_t) + Q_H^\star(s_t, a_t) - Q_t(s_t, a_t))$$

$$\leq B_\star \left\|w\right\|_1 \beta + \tilde{\mathcal{O}}\left(H^2 SAB_\star \left\|w\right\|_\infty + B_\star\sqrt{H^3 SA \left\|w\right\|_\infty \left\|w\right\|_1}\right).$$

□

Corollary 6. *With probability at least $1-8\delta$, Algorithm 2 ensures for any $\rho \geq 2B_\star\beta$:*

$$\sum_{t=1}^{T} \mathbb{I}\left\{V^\star(s_t) - V_t(s_t) \geq \rho\right\} = \tilde{\mathcal{O}}\left(\frac{B_\star^2 H^3 SA}{\rho^2}\right) \triangleq U_\rho - 1,$$

and for any $s \in \mathcal{S}$, $N_t(s) \geq U_\rho$ implies $0 \leq V^\star(s) - V_t(s) \leq \rho$.

Proof. We can assume $\rho \leq B_\star$ since $\sum_{t=1}^{T} \mathbb{I}\{V^\star(s_t) - V_t(s_t) \geq \rho\} = 0$ when $\rho > B_\star$. By Lemma 5 with $w_t = \mathbb{I}\{V^\star(s_t) - V_t(s_t) \geq \rho\}$, $\rho w_t \leq w_t(V^\star(s_t) - V_t(s_t))$ and $\rho \geq 2B_\star\beta$, with probability at least $1-8\delta$:

$$\rho \left\|w\right\|_1 \leq \sum_{t=1}^{T} w_t(V^\star(s_t) - V_t(s_t)) \leq \frac{\rho}{2}\left\|w\right\|_1 + \tilde{\mathcal{O}}\left(H^2 SAB_\star + B_\star\sqrt{H^3 SA \left\|w\right\|_1}\right).$$

Therefore, by Lemma 23, $\left\|w\right\|_1 = \tilde{\mathcal{O}}\left(\frac{H^2 SAB_\star}{\rho} + \frac{B_\star^2 H^3 SA}{\rho^2}\right)$. We prove the second statement by contradiction: suppose $N_t(s) \geq U_\rho$ and $V^\star(s) - V_t(s) > \rho$. Then since V_t is non-decreasing in t, $N_t(s) \leq \left\|w\right\|_1$. Thus, $U_\rho \leq N_t(s) \leq \left\|w\right\|_1 < U_\rho$, a contradiction. □

Lemma 6. *Define $\beta_i = \frac{B_\star}{2^i}, \widetilde{N}_0 = 0, \widetilde{N}_i = U_{\beta_i}$ (defined in Corollary 6) for $i \geq 1$ and $q^\star = \inf\{i : \beta_i \leq c_{\min}\}$. Define $V^{\mathrm{REF}} = V_{T+1}^{\mathrm{ref}}, \theta^\star = \lceil \widetilde{N}_{q^\star} \rceil_2$, and B_t^{ref} such that:*

$$B_t^{\mathrm{ref}}(s) = \sum_{i=1}^{q^\star} \beta_{i-1}\mathbb{I}\{\lceil \widetilde{N}_{i-1} \rceil_2 \leq N_t(s) < \lceil \widetilde{N}_i \rceil_2\}.$$

Then with probability at least $1-8\delta$, $V^{\mathrm{REF}}(s) - V_t^{\mathrm{ref}}(s) \leq B_t^{\mathrm{ref}}(s)$, and

$$\sum_{t=1}^{T} V^{\mathrm{REF}}(s_t) - V_t^{\mathrm{ref}}(s_t) \leq \sum_{t=1}^{T} B_t^{\mathrm{ref}}(s_t) = \tilde{\mathcal{O}}\left(\frac{B_\star^2 H^3 S^2 A}{c_{\min}}\right) \triangleq C_{\mathrm{REF}},$$

$$\sum_{t=1}^{T} \left(V^{\mathrm{REF}}(s_t) - V_t^{\mathrm{ref}}(s_t)\right)^2 \leq \sum_{t=1}^{T} B_t^{\mathrm{ref}}(s_t)^2 = \tilde{\mathcal{O}}\left(B_\star^2 H^3 S^2 A\right) \triangleq C_{\mathrm{REF},\,2}.$$

Proof. We condition on Corollary 6, which happens with probability at least $1-8\delta$. By Corollary 6 with $\rho=\beta_i$ for each $i\in[q^\star]$, we have $V^{\text{REF}}(s)-V^{\text{ref}}_t(s)\leq B^{\text{ref}}_t(s)$. Moreover, $B^{\text{ref}}_t(s)^2=\sum_{i=1}^{q^\star}\beta_{i-1}^2\mathbb{I}\{\lceil\widetilde{N}_{i-1}\rceil_2\leq N_t(s)<\lceil\widetilde{N}_i\rceil_2\}$. Thus,

$$\sum_{t=1}^{T}B^{\text{ref}}_t(s_t)\leq\sum_s\sum_{i=1}^{q^\star}\beta_{i-1}\lceil\widetilde{N}_i\rceil_2=\tilde{\mathcal{O}}\left(\sum_s\sum_{i=1}^{q^\star}\frac{B_\star^2H^3SA}{\beta_i}\right)=\tilde{\mathcal{O}}\left(\frac{B_\star^2H^3S^2A}{\beta_{q^\star}}\right).$$

$$\sum_{t=1}^{T}B^{\text{ref}}_t(s_t)^2\leq\sum_s\sum_{i=1}^{q^\star}\beta_{i-1}^2\lceil\widetilde{N}_i\rceil_2=\tilde{\mathcal{O}}\left(\sum_s\sum_{i=1}^{q^\star}B_\star^2H^3SA\right)=\tilde{\mathcal{O}}\left(B_\star^2H^3S^2A\right).$$

□

B.2. Proofs of Required Properties

In this section, we prove Property 1 and Property 2 of Algorithm 2.

Lemma 7. *With probability at least $1-7\delta$, Algorithm 2 ensures $Q_t(s,a)\leq Q_{t+1}(s,a)\leq Q^\star(s,a)$ for any $(s,a)\in\mathcal{S}\times\mathcal{A},t\geq 1$.*

Proof. We fix a pair (s,a), and denote $n_t,m_t,l_{t,i},\check{l}_{t,i},b_t,b'_t,\iota_t$ as shorthands of the corresponding functions evaluated at (s,a). The first inequality is by the update rule of Q_t. Next, we prove $Q_t(s,a)\leq Q^\star(s,a)$ by induction on t. It is clearly true when when $t=1$. For the induction step, the statement is clearly true when $n_t=m_t=0$. When $n_t>0$, it suffices to consider two update rules, that is, the last two terms in the max operator of Eq. (2). For the second update rule, note that,

$$\begin{aligned}
&c(s,a)+\frac{1}{n_t}\sum_{i=1}^{n_t}V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})+\frac{1}{m_t}\sum_{i=1}^{m_t}\left(V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})-V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})\right)-b_t\\
&=c(s,a)+\frac{1}{n_t}\sum_{i=1}^{n_t}P_{s,a}V^{\text{ref}}_{l_{t,i}}+\frac{1}{m_t}\sum_{i=1}^{m_t}P_{s,a}\left(V_{\check{l}_{t,i}}-V^{\text{ref}}_{\check{l}_{t,i}}\right)\\
&\qquad+\underbrace{\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\mathbb{I}_{s'_{l_{t,i}}}-P_{s,a}\right)V^{\text{ref}}_{l_{t,i}}}_{\chi_1}+\underbrace{\frac{1}{m_t}\sum_{i=1}^{m_t}\left(\mathbb{I}_{s'_{\check{l}_{t,i}}}-P_{s,a}\right)\left(V_{\check{l}_{t,i}}-V^{\text{ref}}_{\check{l}_{t,i}}\right)}_{\chi_2}-b_t.
\end{aligned}$$

Define $C'_t=\lceil\ln(K^2n_t)\rceil^2\leq\min\{4\ln^2(K^2n_t),K^4n_t^2\}$. For χ_1, by Eq. (15) of Lemma 33 with $b=K$ (recall the assumption $B_\star\leq\sqrt{K}$) and $C\leq C'_t$, we have with probability at least $1-\frac{\delta}{SA}$:

$$|\chi_1|=\left|\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\mathbb{I}_{s'_{l_{t,i}}}-P_{s,a}\right)V^{\text{ref}}_{l_{t,i}}\right|\leq 4\ln^3\left(\frac{4SAK^4n_t^5}{\delta}\right)\left(\sqrt{\frac{8\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},V^{\text{ref}}_{l_{t,i}})}{n_t^2}}+\frac{5B_t}{n_t}\right),$$

Note that (recall that ν^{ref}_t represents $\nu^{\text{ref}}_t(s,a)$)

$$\frac{1}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},V^{\text{ref}}_{l_{t,i}})-\nu^{\text{ref}}_t=\chi_3+\chi_4+\chi_5,\tag{4}$$

where

$$\chi_3=\frac{1}{n_t}\sum_{i=1}^{n_t}\left(P_{s,a}(V^{\text{ref}}_{l_{t,i}})^2-V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})^2\right),\quad\chi_4=\left(\frac{1}{n_t}\sum_{i=1}^{n_t}V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})\right)^2-\left(\frac{1}{n_t}\sum_{i=1}^{n_t}P_{s,a}V^{\text{ref}}_{l_{t,i}}\right)^2,$$
$$\chi_5=\left(\frac{1}{n_t}\sum_{i=1}^{n_t}P_{s,a}V^{\text{ref}}_{l_{t,i}}\right)^2-\frac{1}{n_t}\sum_{i=1}^{n_t}(P_{s,a}V^{\text{ref}}_{l_{t,i}})^2.$$

By Eq. (15) of Lemma 33 with $b = K$ and $C \leq C'_t$, and Lemma 28 with $\left\|V^{\text{ref}}_{l_{t,i}}\right\|_\infty \leq B_t$, with probability at least $1 - \frac{2\delta}{SA}$,

$$
\begin{aligned}
|\chi_3| &\leq \frac{4\ln^3(4SAK^4n_t^5/\delta)}{n_t}\left(\sqrt{8\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},(V^{\text{ref}}_{l_{t,i}})^2)+5B_t^2}\right)\\
&\leq \frac{4\ln^3(4SAK^4n_t^5/\delta)}{n_t}\left(2B_t\sqrt{8\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},(V^{\text{ref}}_{l_{t,i}})^2)+5B_t^2}\right). \qquad (5)\\
|\chi_4| &\leq \left|\frac{1}{n_t}\sum_{i=1}^{n_t}V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})+\frac{1}{n_t}\sum_{i=1}^{n_t}P_{s,a}V^{\text{ref}}_{l_{t,i}}\right|\left|\frac{1}{n_t}\sum_{i=1}^{n_t}V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})-\frac{1}{n_t}\sum_{i=1}^{n_t}P_{s,a}V^{\text{ref}}_{l_{t,i}}\right|\\
&\leq 2B_t\cdot\frac{4\ln^3(4SAK^4n_t^5/\delta)}{n_t}\left(\sqrt{8\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},(V^{\text{ref}}_{l_{t,i}})^2)+5B_t}\right). \qquad (6)
\end{aligned}
$$

Moreover, $\chi_5 \leq 0$ by Cauchy-Schwarz inequality. Therefore,

$$
\frac{1}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},V^{\text{ref}}_{l_{t,i}})-\nu^{\text{ref}}_t \leq \frac{4B_t\ln^3(4SAK^4n_t^5/\delta)}{n_t}\left(4\sqrt{8\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},V^{\text{ref}}_{l_{t,i}})+15B_t}\right).
$$

Applying Lemma 23 with $x = \sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},V^{\text{ref}}_{l_{t,i}})$, we obtain:

$$
\frac{1}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{s,a},V^{\text{ref}}_{l_{t,i}}) \leq 2\nu^{\text{ref}}_t + \frac{4216B_t^2\ln^6\frac{4SAK^4n_t^5}{\delta}}{n_t}.
$$

Thus, $\left|\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\mathbb{I}_{s'_{l_{t,i}}}-P_{s,a}\right)V^{\text{ref}}_{l_{t,i}}\right| \leq \sqrt{\frac{\nu^{\text{ref}}_t}{n_t}\iota_t}+\frac{3B_t\iota_t}{n_t}$. By similar arguments, $|\chi_2| \leq \sqrt{\frac{\nu_t}{m_t}\iota_t}+\frac{3B_t\iota_t}{m_t}$ with probability at least $1-\frac{3\delta}{SA}$. Therefore, $|\chi_1|+|\chi_2|-b_t \leq 0$. By the non-decreasing property of V^{ref}_t and $V_{\check{l}_{t,i}}(s) \leq V^\star(s)$ for any $s \in \mathcal{S}^+$:

$$
c(s,a)+\frac{1}{n_t}\sum_{i=1}^{n_t}P_{s,a}V^{\text{ref}}_{l_{t,i}}+\frac{1}{m_t}\sum_{i=1}^{m_t}P_{s,a}\left(V_{\check{l}_{t,i}}-V^{\text{ref}}_{\check{l}_{t,i}}\right) \leq c(s,a)+P_{s,a}V^\star \leq Q^\star(s,a).
$$

For the first update rule, by Eq. (15) of Lemma 33 with $b = K$ and $C \leq C'_t$, with probability at least $1-\frac{\delta}{SA}$, $\frac{1}{m_t}\sum_{i=1}^{m_t}V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})-P_{\check{l}_{t,i}}V_{\check{l}_{t,i}} \leq 2\sqrt{\frac{B_t^2\iota_t}{m_t}}$. Therefore, $c(s,a)+\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}V_{\check{l}_{t,i}} \leq c(s,a)+P_{s,a}V^\star = Q^\star(s,a)$, and $Q_t(s,a) \leq Q^\star(s,a)$ for the fixed (s,a). By a union bound over $(s,a) \in \mathcal{S}\times\mathcal{A}$, we have $Q_t(s,a) \leq Q^\star(s,a)$ for any $(s,a) \in \mathcal{S}\times\mathcal{A}, t \geq 1$. □

Proof of Theorem 2. Property 1 is satisfied by Lemma 7. For Property 2, we conditioned on Lemma 7, Lemma 6, Lemma 8, and Lemma 9, which holds with probability at least $1-44\delta$. Then, for any $(\mathring{Q},\mathring{V}) \in \mathcal{V}_H$:

$$
\begin{aligned}
&\sum_{t=1}^{T}\mathring{Q}(s_t,a_t)-Q_t(s_t,a_t)\\
&\leq \sum_{t=1}^{T}P_t\mathring{V}-\frac{1}{n_t}\sum_{i=1}^{n_t}V^{\text{ref}}_{l_{t,i}}(s'_{l_{t,i}})-\frac{1}{m_t}\sum_{i=1}^{m_t}\left(V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})-V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})\right)+b_t \quad \text{(by Eq. (2) and } \mathring{Q}(s,a)=c(s,a)+P_{s,a}\mathring{V})\\
&\leq \sum_{t=1}^{T}B_\star\mathbb{I}\{m_t=0\}+\sum_{t=1}^{T}\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}\mathring{V}-\frac{1}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}V^{\text{ref}}_{l_{t,i}}-\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}\left(V_{\check{l}_{t,i}}-V^{\text{ref}}_{\check{l}_{t,i}}\right)+2b_t\\
&\qquad\qquad (P_t\mathring{V} \leq B_\star\mathbb{I}\{m_t=0\}+\tfrac{1}{m_t}\textstyle\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}\mathring{V} \text{ and } |\chi_1|+|\chi_2| \leq b_t \text{ from Lemma 7})\\
&\leq B_\star HSA+\sum_{t=1}^{T}\frac{1}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}\left(V^{\text{REF}}-V^{\text{ref}}_{l_{t,i}}\right)+\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}(\mathring{V}-V_{\check{l}_{t,i}})+2b_t.\\
&\qquad\qquad (\textstyle\sum_{t=1}^{T}\mathbb{I}\{m_t=0\} \leq SAH,\ P_t=P_{l_{t,i}}=P_{\check{l}_{t,i}}, \text{ and } V^{\text{ref}}_{\check{l}_{t,i}}(s) \leq V^{\text{REF}}(s) \text{ for any } s \in \mathcal{S})
\end{aligned}
$$

By Lemma 10 and Lemma 8,

$$\sum_{t=1}^{T}\frac{1}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}\left(V^{\text{REF}}-V^{\text{ref}}_{l_{t,i}}\right)=\tilde{\mathcal{O}}\left(\sum_{t=1}^{T}P_t(V^{\text{REF}}-V^{\text{ref}}_t)\right)=\tilde{\mathcal{O}}\left(C_{\text{REF}}\right).$$

Moreover, by Lemma 11, with probability at least $1-\frac{\delta}{H+1}$,

$$\frac{1}{m_t}\sum_{i=1}^{m_t}P_{\check{l}_{t,i}}(\mathring{V}-V_{\check{l}_{t,i}})\leq\left(1+\frac{1}{H}\right)^2\sum_{t=1}^{T}(\mathring{V}(s_t)-V_t(s_t))+\tilde{\mathcal{O}}\left(B_\star(H+S)\right).$$

Plug these back, we get:

$$\begin{aligned}\sum_{t=1}^{T}\mathring{Q}(s_t,a_t)-Q_t(s_t,a_t)&\leq\tilde{\mathcal{O}}\left(B_\star HSA+C_{\text{REF}}\right)+\left(1+\frac{1}{H}\right)^2\sum_{t=1}^{T}(\mathring{V}(s_t)-V_t(s_t))+2\sum_{t=1}^{T}b_t\\&\leq\left(1+\frac{3}{H}\right)\sum_{t=1}^{T}(\mathring{V}(s_t)-V_t(s_t))+\tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K}+\sqrt{SAHc_{\min}C_K}+\frac{B_\star^2H^3S^2A}{c_{\min}}\right).\end{aligned}$$

$((1+\frac{1}{H})^2\leq 1+\frac{3}{H}$, Lemma 9 and Lemma 6)

Taking a union bound over $(\mathring{Q},\mathring{V})\in\mathcal{V}_H$ and by $H=\tilde{\mathcal{O}}\left(\frac{B_\star}{c_{\min}}\right)$ proves the claim. □

B.3. Proof of Theorem 3

Proof. By Theorem 1 and Theorem 2 with probability at least $1-47\delta$ and $\beta=\frac{c_{\min}}{2B_\star^2SAK}$:

$$C_K-KV^\star(s_0)=R_K\leq\tilde{\mathcal{O}}\left(\beta C_K+\sqrt{B_\star SAC_K}+\frac{B_\star^2H^3S^2A}{c_{\min}}\right).$$

Then by $V^\star(s_0)\leq B_\star,\beta\leq\frac{1}{2}$ and Lemma 23, we have $C_K=\tilde{\mathcal{O}}\left(B_\star K\right)$. Substitute this back and by $\beta\leq\frac{c_{\min}}{B_\star K}, H=\tilde{\mathcal{O}}(B_\star/c_{\min})$, we get $R_K=\tilde{\mathcal{O}}\left(B_\star\sqrt{SAK}+\frac{B_\star^5S^2A}{c_{\min}^4}\right)$. □

B.4. Extra Lemmas for Section 4

In this section, we gives proofs of auxiliary lemmas used in Section 4. Lemma 8 quantifies the cost of using reference value function. Lemma 9 quantifies the cost of using the chosen variance-aware bonus terms b_t. Lemma 10, Lemma 11, and Lemma 12 deal with the bias induced by the sparse update scheme.

Lemma 8. *With probability at least $1-9\delta$, $\sum_{t=1}^{T}P_t\left(V^{\text{REF}}-V^{\text{ref}}_t\right)\leq\sum_{t=1}^{T}P_tB^{\text{ref}}_t=\tilde{\mathcal{O}}\left(C_{\text{REF}}\right)$.*

Proof. By Lemma 6, Lemma 34, Lemma 26 and $B^{\text{ref}}_{t+1}(s'_t)\leq B^{\text{ref}}_{t+1}(s_{t+1})$ in each step:

$$\begin{aligned}\sum_{t=1}^{T}P_t\left(V^{\text{REF}}-V^{\text{ref}}_t\right)&\leq\sum_{t=1}^{T}P_tB^{\text{ref}}_t\leq 2\sum_{t=1}^{T}B^{\text{ref}}_t(s'_t)+\tilde{\mathcal{O}}\left(B_\star\right)\\&=\tilde{\mathcal{O}}\left(\sum_{t=1}^{T}B^{\text{ref}}_t(s_t)+SB_\star\right)=\tilde{\mathcal{O}}\left(C_{\text{REF}}\right).\end{aligned}$$

□

Lemma 9. *With probability at least $1-20\delta$,*

$$\sum_{t=1}^{T}b_t=\tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K}+B_\star H^2S^{\frac{3}{2}}A+\sqrt{SAHc_{\min}C_K}\right).$$

Proof. We condition on Lemma 6, which holds with probability at least $1-8\delta$. By Eq. (7) and Eq. (8) of Lemma 12,

$$\sum_{t=1}^T b_t \leq \sum_{t=1}^T \sqrt{\frac{\nu_t^{\text{ref}}\varepsilon_t}{n_t}\iota_t} + \sqrt{\frac{\nu_t\varepsilon_t}{m_t}\iota_t} + B_\star\sum_t\left(\frac{3\varepsilon_t}{n_t}+\frac{3\varepsilon_t}{m_t}\right)\iota_t = \tilde{\mathcal{O}}\left(\sum_{t=1}^T\sqrt{\frac{\nu_t^{\text{ref}}\varepsilon_t}{n_t}} + \sqrt{\frac{\nu_t\varepsilon_t}{m_t}} + B_\star HSA\right).$$

Note that by Eq. (4), Eq. (5) and Eq. (6), when $n_t > 0$, with probability at least $1-2\delta$,

$$\begin{aligned}
&\nu_t^{\text{ref}} - \frac{1}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}}) \leq |\chi_3| + |\chi_4| - \chi_5 \\
&\leq \tilde{\mathcal{O}}\left(\frac{B_t}{n_t}\sqrt{\sum_{i=1}^{n_t}\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}})} + \frac{B_t^2}{n_t}\right) + \frac{1}{n_t}\sum_{i=1}^{n_t}(P_{l_{t,i}}V_{l_{t,i}}^{\text{ref}})^2 - \left(\frac{1}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}V_{l_{t,i}}^{\text{ref}}\right)^2 \\
&\overset{\text{(i)}}{=} \tilde{\mathcal{O}}\left(\frac{B_t}{n_t}\sqrt{\sum_{i=1}^{n_t}\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}})} + \frac{B_t^2}{n_t} + \frac{B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}}\right) \\
&\leq \frac{1}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}}) + \tilde{\mathcal{O}}\left(\frac{B_t^2}{n_t} + \frac{B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}}\right), \qquad \text{(AM-GM Inequality)}
\end{aligned}$$

where in (i) we apply:

$$\begin{aligned}
&\frac{1}{n_t}\sum_{i=1}^{n_t}(P_{l_{t,i}}V_{l_{t,i}}^{\text{ref}})^2 - \left(\frac{1}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}V_{l_{t,i}}^{\text{ref}}\right)^2 \leq (P_tV^{\text{REF}})^2 - \left(\frac{1}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}V_{l_{t,i}}^{\text{ref}}\right)^2 \qquad (V_{l_{t,i}}^{\text{ref}}(s) \leq V^{\text{REF}}(s) \text{ for any } s\in\mathcal{S}) \\
&\leq \frac{2B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}\left(V^{\text{REF}} - V_{l_{t,i}}^{\text{ref}}\right) \leq \frac{2B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}}. \qquad (\left\|V^{\text{REF}}\right\|_\infty \leq B_\star \text{ and Lemma 6})
\end{aligned}$$

Therefore, $\nu_t^{\text{ref}} - \frac{2}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}}) = \tilde{\mathcal{O}}\left(\frac{B_t^2}{n_t} + \frac{B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}}\right)$, and

$$\begin{aligned}
&\nu_t^{\text{ref}} - 2\mathbb{V}(P_t, V^\star) = \nu_t^{\text{ref}} - \frac{2}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}}) + \frac{2}{n_t}\sum_{i=1}^{n_t}(\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}}) - \mathbb{V}(P_{l_{t,i}}, V^\star)) \qquad (P_t = P_{l_{t,i}}) \\
&\overset{\text{(i)}}{\leq} \tilde{\mathcal{O}}\left(\frac{B_\star^2}{n_t} + \frac{B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}}\right) + \frac{4B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}\left(V^\star - V_{l_{t,i}}^{\text{ref}}\right) \\
&= \tilde{\mathcal{O}}\left(\frac{B_\star^2}{n_t} + \frac{B_\star}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}} + B_\star\beta_{q^\star}\right), \qquad (V^\star(s) - V_{l_{t,i}}^{\text{ref}}(s) \leq B_{l_{t,i}}^{\text{ref}}(s) + \beta_{q^\star}, \forall s)
\end{aligned}$$

where in (i) we apply the bound for $\nu_t^{\text{ref}} - \frac{2}{n_t}\sum_{i=1}^{n_t}\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}})$, $B_t \leq B_\star$ and

$$\mathbb{V}(P_{l_{t,i}}, V_{l_{t,i}}^{\text{ref}}) - \mathbb{V}(P_{l_{t,i}}, V^\star) \leq (P_{l_{t,i}}V^\star)^2 - (P_{l_{t,i}}V_{l_{t,i}}^{\text{ref}})^2 \leq 2B_\star P_{l_{t,i}}(V^\star - V_{l_{t,i}}^{\text{ref}}).$$

Plugging the inequality above back, we have with probability at least $1-10\delta$,

$$\begin{aligned}
&\sum_{t=1}^T\sqrt{\frac{\nu_t^{\text{ref}}}{n_t}} = \tilde{\mathcal{O}}\left(\sum_{t=1}^T\sqrt{\frac{\mathbb{V}(P_t,V^\star)}{n_t}} + \frac{B_\star}{n_t} + \frac{1}{n_t}\sqrt{B_\star\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}}} + \sqrt{\frac{B_\star\beta_{q^\star}}{n_t}}\right) \\
&= \tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star)} + B_\star SA + \sqrt{\sum_{t=1}^T\frac{B_\star}{n_t}}\sqrt{\sum_{t=1}^T\frac{1}{n_t}\sum_{i=1}^{n_t}P_{l_{t,i}}B_{l_{t,i}}^{\text{ref}}} + \sqrt{B_\star\beta_{q^\star}SAT}\right) \\
&\qquad\qquad \text{(Lemma 12 and Cauchy-Schwarz inequality)} \\
&= \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + B_\star SA + \sqrt{B_\star SAC_{\text{REF}}} + \sqrt{B_\star\beta_{q^\star}SAT}\right). \qquad \text{(Lemma 4, Lemma 12, Lemma 10 and Lemma 8)}
\end{aligned}$$

Moreover,

$$\sum_{t=1}^{T}\sqrt{\frac{\nu_t}{m_t}} \leq \sum_{t=1}^{T}\frac{\sqrt{\sum_{i=1}^{m_t}(V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) - V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}))^2}}{m_t} \leq \sum_{t=1}^{T}\frac{\sqrt{\sum_{i=1}^{m_t}(V^\star(s'_{\check{l}_{t,i}}) - V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}))^2}}{m_t}$$

$$= \tilde{\mathcal{O}}\left(\sum_{t=1}^{T}\frac{\sqrt{\sum_{i=1}^{m_t}B^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})^2}}{m_t} + \frac{\sqrt{\sum_{i=1}^{m_t}\beta_{q^\star}^2}}{m_t}\right)$$

$(V^\star(s'_{\check{l}_{t,i}}) - V^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) \leq B^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) + \beta_{q^\star}$, $(a+b)^2 \leq 2a^2 + 2b^2$, and $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$)

$$= \tilde{\mathcal{O}}\left(\sqrt{\sum_{t=1}^{T}\frac{1}{m_t}}\sqrt{\sum_{t=1}^{T}\frac{1}{m_t}\sum_{i=1}^{m_t}B^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})^2} + \sum_{t=1}^{T}\sqrt{\frac{\beta_{q^\star}^2}{m_t}}\right). \qquad \text{(Cauchy-Schwarz inequality)}$$

Note that by Lemma 10, Lemma 26, $B^{\text{ref}}_{t+1}(s'_t) \leq B^{\text{ref}}_{t+1}(s_{t+1})$ and Lemma 6:

$$\sum_{t=1}^{T}\frac{1}{m_t}\sum_{i=1}^{m_t}B^{\text{ref}}_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}})^2 \leq \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}B^{\text{ref}}_t(s'_t)^2$$

$$= \tilde{\mathcal{O}}\left(\sum_{t=1}^{T}B^{\text{ref}}_{t+1}(s'_t) + SB_\star^2\right) = \tilde{\mathcal{O}}\left(\sum_{t=1}^{T}B^{\text{ref}}_t(s_t) + SB_\star^2\right) = \tilde{\mathcal{O}}\left(C_{\text{REF, 2}}\right).$$

Plug this back to the last inequality and by Lemma 12, we have:

$$\sum_{t=1}^{T}\sqrt{\frac{\nu_t}{m_t}} = \tilde{\mathcal{O}}\left(\sqrt{SAHC_{\text{REF, 2}}} + \sqrt{SAH\beta_{q^\star}^2 T}\right).$$

Put everything together, and by $\beta_{q^\star} = \mathcal{O}\left(c_{\min}\right), \beta_{q^\star}T = \mathcal{O}\left(c_{\min}T\right) = \mathcal{O}\left(C_K\right)$:

$$\sum_{t=1}^{T}b_t = \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + \sqrt{B_\star SAC_{\text{REF}}} + \sqrt{SAHC_{\text{REF, 2}}} + \sqrt{SAHc_{\min}C_K} + B_\star HSA\right)$$

$$= \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + B_\star H^2 S^{\frac{3}{2}}A + \sqrt{SAHc_{\min}C_K}\right).$$

$(H = \Omega\left(\frac{B_\star}{c_{\min}}\right)$ and definition of $C_{\text{REF}}, C_{\text{REF, 2}}$ (Lemma 6))

□

Lemma 10 (bias of the update scheme)**.** *Assuming $X_t \geq 0$, we have:*

$$\sum_{t=1}^{T}\frac{1}{m_t}\sum_{i=1}^{m_t}X_{\check{l}_{t,i}} \leq \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}X_t, \qquad \sum_{t=1}^{T}\frac{1}{n_t}\sum_{i=1}^{n_t}X_{l_{t,i}} = \mathcal{O}\left(\ln(T)\sum_{t=1}^{T}X_t\right).$$

Proof. For the first inequality, denote by j_t the stage to which time step t belongs. When $t' = \check{l}_{t,i}$, we have $m_t = e_{j_{t'}}$. Therefore, $\sum_{t=1}^{T}\sum_{i=1}^{m_t}\frac{1}{m_t}\mathbb{I}\{t' = \check{l}_{t,i}\} \leq \frac{e_{j_{t'}+1}}{e_{j_{t'}}} \leq 1 + \frac{1}{H}$, and

$$\sum_{t=1}^{T}\frac{1}{m_t}\sum_{i=1}^{m_t}X_{\check{l}_{t,i}} = \sum_{t=1}^{T}\frac{1}{m_t}\sum_{i=1}^{m_t}\sum_{t'=1}^{T}X_{t'}\mathbb{I}\{t' = \check{l}_{t,i}\} = \sum_{t'=1}^{T}X_{t'}\sum_{t=1}^{T}\sum_{i=1}^{m_t}\frac{\mathbb{I}\{t' = \check{l}_{t,i}\}}{m_t} \leq \left(1+\frac{1}{H}\right)\sum_{t'=1}^{T}X_{t'}.$$

For the second inequality:

$$\sum_{t=1}^{T}\frac{1}{n_t}\sum_{i=1}^{n_t}X_{l_{t,i}} = \sum_{t=1}^{T}\frac{1}{n_t}\sum_{i=1}^{n_t}\sum_{t'=1}^{T}X_{t'}\mathbb{I}\{t' = l_{t,i}\} = \sum_{t'=1}^{T}X_{t'}\sum_{t=1}^{T}\sum_{i=1}^{n_t}\frac{\mathbb{I}\{t' = l_{t,i}\}}{n_t}$$

$$\leq \sum_{t'=1}^{T}X_{t'}\sum_{z:t' \leq E_{z-1} \leq T}\frac{e_z}{E_{z-1}} = \mathcal{O}\left(\ln(T)\sum_{t'=1}^{T}X_{t'}\right).$$

□

Lemma 11. *Assuming $X_t : \mathcal{S}^+ \to [0, B]$ is monotonic in t (i.e., $X_t(s)$ is non-increasing or non-decreasing in t for any $s \in \mathcal{S}^+$) and $X_t(g) = 0$, with probability at least $1 - \delta$,*

$$\sum_{t=1}^{T} \frac{1}{m_t} \sum_{i=1}^{m_t} P_{\check{l}_{t,i}} X_{\check{l}_{t,i}} \leq \left(1 + \frac{1}{H}\right)^2 \sum_{t=1}^{T} X_t(s_t) + \tilde{\mathcal{O}}\left(B(H + S)\right).$$

Proof. By Lemma 10, Lemma 34 and Lemma 26, $X_{t+1}(s'_t) \leq X_{t+1}(s_{t+1})$ in each step,

$$\begin{aligned}\sum_{t=1}^{T} \frac{1}{m_t} \sum_{i=1}^{m_t} P_{\check{l}_{t,i}} X_{\check{l}_{t,i}} &\leq \left(1 + \frac{1}{H}\right) \sum_{t=1}^{T} P_t X_t \leq \left(1 + \frac{1}{H}\right)^2 \sum_{t=1}^{T} X_t(s'_t) + \tilde{\mathcal{O}}\left(BH\right) \\ &\leq \left(1 + \frac{1}{H}\right)^2 \sum_{t=1}^{T} X_t(s_t) + \tilde{\mathcal{O}}\left(B(H + S)\right).\end{aligned}$$

□

Lemma 12. *For any non-negative weights $\{w_t\}_t$, and $\alpha \in (0, 1)$, we have:*

$$\sum_{t=1}^{T} \frac{w_t \varepsilon_t}{n_t^\alpha} = \mathcal{O}\left(\left(\|w\|_\infty SA\right)^\alpha \|w\|_1^{1-\alpha}\right), \quad \sum_{t=1}^{T} \frac{w_t \varepsilon_t}{m_t^\alpha} = \mathcal{O}\left(\left(\|w\|_\infty HSA\right)^\alpha \|w\|_1^{1-\alpha} \ln \frac{\|w\|_\infty}{\|w\|_1}\right).$$

Moreover, when $w_t = v(s_t, a_t)$ for some v,

$$\sum_{t=1}^{T} \frac{w_t \varepsilon_t}{n_t^\alpha} = \tilde{\mathcal{O}}\left(\sum_{(s,a)} v(s, a) N_{T+1}(s, a)^{1-\alpha}\right), \quad \sum_{t=1}^{T} \frac{w_t \varepsilon_t}{m_t^\alpha} = \tilde{\mathcal{O}}\left(H^\alpha \sum_{(s,a)} v(s, a) N_{T+1}(s, a)^{1-\alpha}\right).$$

In case $w_t = 1$ for all t, it holds that:

$$\sum_{t=1}^{T} \frac{\varepsilon_t}{n_t^\alpha} = \tilde{\mathcal{O}}\left((SA)^\alpha T^{1-\alpha}\right), \quad \sum_{t=1}^{T} \frac{\varepsilon_t}{m_t^\alpha} = \tilde{\mathcal{O}}\left((SAH)^\alpha T^{1-\alpha}\right), \tag{7}$$

when $0 < \alpha < 1$, and

$$\sum_{t=1}^{T} \frac{\varepsilon_t}{n_t} = \mathcal{O}\left(SA \ln T\right), \quad \sum_{t=1}^{T} \frac{\varepsilon_t}{m_t} = \mathcal{O}\left(SAH \ln T\right), \tag{8}$$

when $\alpha = 1$.

Proof. Define $\mathfrak{n}(s, a, j) = \sum_{t:(s_t,a_t)=(s,a),n_t=E_j} w_t$, $\mathfrak{n}(s, a) = \sum_{j \geq 0} \mathfrak{n}(s, a, j)$. Then, $\sum_{(s,a)} \mathfrak{n}(s, a) = \|w\|_1$, $\mathfrak{n}(s, a, j) \leq \|w\|_\infty e_{j+1} \leq \left(1 + \frac{1}{H}\right) \|w\|_\infty e_j$. Moreover, by definitions of e_j and E_j,

$$\sum_{j \geq 1} \mathbb{I}\left\{\left(1 + \frac{1}{H}\right) \|w\|_\infty E_{j-1} \leq \mathfrak{n}(s, a)\right\} = \mathcal{O}\left(H \ln \frac{\|w\|_1}{\|w\|_\infty}\right). \tag{9}$$

$$\sum_{j \geq 1} e_j \mathbb{I}\left\{\left(1 + \frac{1}{H}\right) \|w\|_\infty E_{j-1} \leq \mathfrak{n}(s, a)\right\} = \mathcal{O}(\mathfrak{n}(s, a) / \|w\|_\infty). \tag{10}$$

Since $\frac{1}{E_j^\alpha}$ and $\frac{1}{e_j^\alpha}$ is decreasing, by "moving weights to earlier terms" (from $\mathfrak{n}(s,a,j)$ to $\mathfrak{n}(s,a,i)$ for $i<j$),

$$\begin{aligned}
\sum_{t=1}^{T}\frac{w_t\varepsilon_t}{n_t^\alpha} &= \sum_{(s,a)}\sum_{j\geq 1}\frac{\mathfrak{n}(s,a,j)}{E_j^\alpha} \leq \sum_{(s,a)}\sum_{j\geq 1}\left(1+\frac{1}{H}\right)\|w\|_\infty\frac{e_j\mathbb{I}\left\{\left(1+\frac{1}{H}\right)\|w\|_\infty E_{j-1}\leq \mathfrak{n}(s,a)\right\}}{E_j^\alpha}\\
&= \mathcal{O}\left(\sum_{(s,a)}\|w\|_\infty\left(\frac{\mathfrak{n}(s,a)}{\|w\|_\infty}\right)^{1-\alpha}\right) && \left(\textstyle\sum_{j=1}^{J}\frac{e_j}{E_j^\alpha}=\mathcal{O}\left(E_J^{1-\alpha}\right)\text{ and Eq. (10)}\right)\\
&= \mathcal{O}\left(\left(\|w\|_\infty SA\right)^\alpha\|w\|_1^{1-\alpha}\right), && \text{(Hölder's inequality)}
\end{aligned}$$

$$\begin{aligned}
\sum_{t=1}^{T}\frac{w_t\varepsilon_t}{m_t^\alpha} &= \sum_{(s,a)}\sum_{j\geq 1}\frac{\mathfrak{n}(s,a,j)}{e_j^\alpha} \leq \sum_{(s,a)}\sum_{j\geq 1}\left(1+\frac{1}{H}\right)\|w\|_\infty e_j^{1-\alpha}\mathbb{I}\left\{\left(1+\frac{1}{H}\right)\|w\|_\infty E_{j-1}\leq \mathfrak{n}(s,a)\right\}\\
&\leq \left(1+\frac{1}{H}\right)\|w\|_\infty\left(\sum_{(s,a)}\sum_{j\geq 1}\mathbb{I}\left\{\|w\|_\infty E_{j-1}\leq \mathfrak{n}(s,a)\right\}\right)^\alpha\left(\sum_{(s,a)}\frac{\mathfrak{n}(s,a)}{\|w\|_\infty}\right)^{1-\alpha}\\
& && \text{(Hölder's inequality and Eq. (10))}\\
&= \mathcal{O}\left(\left(\|w\|_\infty HSA\right)^\alpha\|w\|_1^{1-\alpha}\ln\frac{\|w\|_1}{\|w\|_\infty}\right). && \text{(Eq. (9))}
\end{aligned}$$

In case $w_t=1$ and $\alpha\in(0,1)$, we have $\|w\|_\infty=1$, $\|w\|_1=T$, and Eq. (7) is proved. When $w_t=v(s_t,a_t)$ for some v, $\mathfrak{n}(s,a,j)\leq v(s,a)e_{j+1}\mathbb{I}\{j\leq J_{s,a}\}$, where $J_{s,a}$ is such that $E_{J_{s,a}}=n_T(s,a)$. Thus,

$$\begin{aligned}
\sum_{t=1}^{T}\frac{w_t\varepsilon_t}{n_t^\alpha} &\leq \sum_{(s,a)}v(s,a)\sum_{j=1}^{J_{s,a}}\frac{e_{j+1}}{E_j^\alpha} = \mathcal{O}\left(\sum_{(s,a)}v(s,a)\sum_{j=1}^{J_{s,a}}\frac{e_j}{E_j^\alpha}\right) = \mathcal{O}\left(\sum_{(s,a)}v(s,a)N_{T+1}(s,a)^{1-\alpha}\right).\\
\sum_{t=1}^{T}\frac{w_t\varepsilon_t}{m_t^\alpha} &\leq \sum_{(s,a)}v(s,a)\sum_{j=1}^{J_{s,a}}\frac{e_{j+1}}{e_j^\alpha} = \mathcal{O}\left(\sum_{(s,a)}v(s,a)\sum_{j=1}^{J_{s,a}}e_j^{1-\alpha}\right)\\
&= \tilde{\mathcal{O}}\left(\sum_{(s,a)}v(s,a)J_{s,a}^\alpha\left(\sum_{j=1}^{J_{s,a}}e_j\right)^{1-\alpha}\right) = \tilde{\mathcal{O}}\left(H^\alpha\sum_{(s,a)}v(s,a)N_{T+1}(s,a)^{1-\alpha}\right).\\
& && \text{(Hölder's inequality and } J_{s,a}=\tilde{\mathcal{O}}\left(H\right)\text{ by how } e_j \text{ grows)}
\end{aligned}$$

In case $\alpha=1$, we have:

$$\begin{aligned}
\sum_{t=1}^{T}\frac{\varepsilon_t}{n_t} &\leq \sum_{(s,a)}\sum_{j:0<E_{j-1}\leq T}\frac{e_j}{E_{j-1}} = \mathcal{O}\left(SA\ln T\right).\\
\sum_{t=1}^{T}\frac{\varepsilon_t}{m_t} &\leq \sum_{(s,a)}\sum_{j:0<E_{j-1}\leq T}\left(1+\frac{1}{H}\right) = \mathcal{O}\left(SAH\ln T\right).
\end{aligned}$$

□

B.5. Parameter free algorithm

In this section, we present a parameter-free model-free algorithm (Algorithm 3) that achieves the same regret guarantee as Algorithm 2. The high level idea is to try logarithmically many different values of H and $\theta^\star$ simultaneously following a doubling scheme, each leading to a different update rule for Q and V^{ref}. Specifically, define $N_\beta=\lceil\log_2(1/\beta)\rceil$ with $\beta=\frac{c_{\min}}{2SAK^2}$ (recall we assume that $B_\star\leq\sqrt{K}$), $H_p=2^p$ for $p\in\mathcal{P}$ with $\mathcal{P}=[N_\beta]$, and $\mathcal{H}=\{H_p\}_{p\in\mathcal{P}}$. Define $\mathcal{R}=[8N_\beta]$. Here, $\mathcal{H}$ and $\{2^r\}_{r\in\mathcal{R}}$ constitute the search range of H and $\theta^\star$.

For each p,r, we maintain accumulators $\mu^{\text{ref}}_{p,r},\sigma^{\text{ref}}_{p,r},\mu_{p,r},\sigma_{p,r},v_p,m_p$ similar to $\mu^{\text{ref}},\sigma^{\text{ref}},\mu,\sigma,v,m$ in Algorithm 2 (Line 1 and Line 2). For each $(s,a)\in\mathcal{S}\times\mathcal{A}$ and $p\in\mathcal{P}$, we divide the samples received into consecutive stages, where the length

Algorithm 3 LCB-ADVANTAGE-SSP without knowledge of $B_\star$

Parameter: failure probability $\delta \in (0,1)$.
Define: $\mathcal{L}_p = \{E_{p,j}\}_{j\in\mathbb{N}^+}$ where $E_{p,j} = \sum_{i=1}^{j} e_{p,i}$, $e_{p,1} = H_p$ and $e_{p,j+1} = \lfloor (1+1/H_p)e_{p,j} \rfloor$.
Initialize: $t \leftarrow 0$, $s_1 \leftarrow s_{\text{init}}$, $B \leftarrow 1$, for all $(s,a), p \in \mathcal{P}$, $N(s,a) \leftarrow 0$, $M_p(s,a) \leftarrow 0$.
Initialize: for all $(s,a), r \in \mathcal{R}, Q(s,a) \leftarrow c(s,a), V(s) \leftarrow \min_a c(s,a), V^{\text{ref}}_r(s) \leftarrow V(s)$.
Initialize: for all $(s,a), p \in \mathcal{P}, r \in \mathcal{R}$, $\mu^{\text{ref}}_{p,r}(s,a) \leftarrow 0$, $\sigma^{\text{ref}}_{p,r}(s,a) \leftarrow 0$, $\mu_{p,r}(s,a) \leftarrow 0$, $\sigma_{p,r}(s,a) \leftarrow 0$, $v_p(s,a) \leftarrow 0$.
for $k = 1, \ldots, K$ **do**
  repeat
    Increment time step $t \xleftarrow{+} 1$.
    Take action $a_t = \operatorname{argmin}_a Q(s_t,a)$, suffer cost $c(s_t,a_t)$, transit to and observe s'_t.
    Increment visitation counters: $n = N(s_t,a_t) \xleftarrow{+} 1$.
    for $p \in \mathcal{P}$ **do**
      for $r \in \mathcal{R}$ **do**
1   Update reference value accumulators: $\mu^{\text{ref}}_{p,r}(s_t,a_t) \xleftarrow{+} V^{\text{ref}}_r(s'_t),\ \sigma^{\text{ref}}_{p,r}(s_t,a_t) \xleftarrow{+} V^{\text{ref}}_r(s'_t)^2,\ \mu_{p,r}(s_t,a_t) \xleftarrow{+} V(s'_t) - V^{\text{ref}}_r(s'_t),\ \sigma_{p,r}(s_t,a_t) \xleftarrow{+} (V(s'_t) - V^{\text{ref}}_r(s'_t))^2$.
2   Update accumulators: $v_p(s_t,a_t) \xleftarrow{+} V(s'_t),\ m_p = M_p(s_t,a_t) \xleftarrow{+} 1$.
3   **if** $n \in \mathcal{L}_p$ **then**
        Compute $\iota \leftarrow 256 \ln^6(4SAK^4t^5/\delta)$.
        for $r \in \mathcal{R}$ **do**
          $b_{p,r} \leftarrow \sqrt{\frac{\sigma^{\text{ref}}_{p,r}(s_t,a_t)/n - (\mu^{\text{ref}}_{p,r}(s_t,a_t)/n)^2}{n}\iota} + \sqrt{\frac{\sigma_{p,r}(s_t,a_t)/m_p - (\mu_{p,r}(s_t,a_t)/m_p)^2}{m_p}\iota} + \left(\frac{3B}{n} + \frac{3B}{m_p}\right)\iota$.
4          $Q(s_t,a_t) \leftarrow \max\left\{c(s_t,a_t) + \frac{\mu^{\text{ref}}_{p,r}(s_t,a_t)}{n} + \frac{\mu_{p,r}(s_t,a_t)}{m_p} - b_{p,r}, Q(s_t,a_t)\right\}$.
          Reset local accumulators: $\mu_{p,r}(s_t,a_t) \leftarrow 0,\ \sigma_{p,r}(s_t,a_t) \leftarrow 0$.
        Compute bonus $b'_p \leftarrow 2\sqrt{\frac{B^2\iota}{m_p}}$.
5        $Q(s_t,a_t) \leftarrow \max\left\{c(s_t,a_t) + \frac{v_p(s_t,a_t)}{m_p} - b'_p, Q(s_t,a_t)\right\}$.
        Reset local accumulators: $v_p(s_t,a_t) \leftarrow 0,\ M_p(s_t,a_t) \leftarrow 0$.
    $V(s_t) \leftarrow \min_a Q(s_t,a)$.
    if $V(s_t) > B$ **then** $B \leftarrow 2V(s_t)$.
6   **if** $\sum_a N(s_t,a) = 2^r$ *for some* $r \in \mathcal{R}$ **then** $V^{\text{ref}}_{r'}(s_t) \leftarrow V(s_t),\ \forall r' \geq r$.
    if $s'_t \neq g$ **then** $s_{t+1} \leftarrow s'_t$; **else** $s_{t+1} \leftarrow s_{\text{init}}$, **break**.

of the j-th stage is $e_{p,j}$ with $e_{p,1} = H_p, e_{p,j+1} = \lfloor (1 + \frac{1}{H_p})e_{p,j} \rfloor$. Also define the indices indicating the end of a stage for a given p as $\mathcal{L}_p = \{E_{p,j}\}_{j\in\mathbb{N}^+}$ with $E_{p,j} = \sum_{i=1}^{j} e_{p,i}$. We update $Q(s,a)$ only when the number of visits to (s,a) falls into $\mathcal{L}_p$ for some $p \in \mathcal{P}$ (Line 3), and there are two types of update rules similar to Algorithm 2 (Line 4 and Line 5). We also maintain $|\mathcal{R}|$ reference value functions, each with different final precision (Line 6). We show that the way we combine different update rules enable us to apply analysis of Algorithm 2 w.r.t any choice of $(p,r) \in \mathcal{P} \times \mathcal{R}$. Notably, we can proceed with $(p^\star, r^\star)$ with $H_{p^\star} = H, 2^{r^\star} = \theta^\star$, which gives us the same regret bound as Algorithm 2 without knowing $B_\star$.

Now we define notations only used in this section. When it is clear from the context, we ignore dependency on p, and define $n_t(s,a), m_t(s,a), l_{t,i}(s,a), \check{l}_{t,i}(s,a)$ similarly as before for a given p. Denote by $V^{\text{ref}}_{r,t}(s)$ the value of $V^{\text{ref}}_r(s)$ at the beginning of time step t, and by $b_{p,r,t}(s,a), b'_{p,t}(s,a)$ the value of $b_{p,r}(s,a), b'_p(s,a)$ in $Q_t(s,a)$. Also define:

$$\overline{Q}_{p,r,t}(s,a) = c(s,a) + \frac{1}{n_t}\sum_{i=1}^{n_t} V^{\text{ref}}_{r,l_{t,i}}(s'_{l_{t,i}}) + \frac{1}{m_t}\sum_{i=1}^{m_t}\left(V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) - V^{\text{ref}}_{r,\check{l}_{t,i}}(s'_{\check{l}_{t,i}})\right) - b_{p,r,t}.$$

$$\overline{Q}'_{p,t}(s,a) = c(s,a) + \frac{1}{m_t}\sum_{i=1}^{m_t} V_{\check{l}_{t,i}}(s'_{\check{l}_{t,i}}) - b'_{p,t}.$$

Algorithm 4 SVI-SSP

Parameters: horizon H, value function upper bound B, and failure probability $\delta \in (0, 1)$.
Define: $\mathcal{L} = \{E_j\}_{j\in\mathbb{N}^+}$, where $E_j = \sum_{i=1}^{j} e_i, e_j = \lfloor \widetilde{e}_j \rfloor$, and $\widetilde{e}_1 = 1, \widetilde{e}_{j+1} = \widetilde{e}_j + \frac{1}{H} e_j$.
Initialize: $t \leftarrow 0, s_1 \leftarrow s_{\text{init}}$.
Initialize: for all $(s, a, s'), n(s, a, s') \leftarrow 0, n(s, a) \leftarrow 0, Q(s, a) \leftarrow c(s, a), V(s) \leftarrow \min_a c(s, a)$.
for $k = 1, \ldots, K$ **do**
  repeat
    Increment time step $t \xleftarrow{+} 1$.
    Take action $a_t = \operatorname{argmin}_a Q(s_t, a)$, suffer cost $c(s_t, a_t)$, transit to and observe s'_t.
1   Update visitation counters: $n = n(s_t, a_t) \xleftarrow{+} 1, n(s_t, a_t, s'_t) \xleftarrow{+} 1$.
2   **if** $n \in \mathcal{L}$ **then**
3   Update empirical transition: $\bar{P}_{s_t,a_t}(s') \leftarrow \frac{n(s_t,a_t,s')}{n}$ for all s'.
4   Compute $\iota \leftarrow \ln \frac{2SAt}{\delta}$ and bonus $b \leftarrow \max\left\{7\sqrt{\frac{\mathbb{V}(\bar{P}_{s_t,a_t},V)\iota}{n}}, \frac{49B\iota}{n}\right\}$.
5      $Q(s_t, a_t) \leftarrow \max\{c(s_t, a_t) + \bar{P}_{s_t,a_t} V - b, Q(s_t, a_t)\}$.
6      $V(s_t) \leftarrow \operatorname{argmin}_a Q(s_t, a)$.
    if $s'_t \neq g$ **then** $s_{t+1} \leftarrow s'_t$; **else** $s_{t+1} \leftarrow s_{\text{init}}$, **break**.

Note that for any $(s, a) \in \mathcal{S} \times \mathcal{A}, t > 1$,

$$Q_t(s, a) = \max\left\{\max_{p,r} \overline{Q}_{p,r,t}(s, a), \max_{p} \overline{Q}'_{p,t}(s, a), Q_{t-1}(s, a)\right\}. \tag{11}$$

Next, we prove the key lemma in this section, which shows that Q_t is an optimistic estimator of $Q^\star$.

Lemma 13. *With probability at least $1 - 7 \cdot 8N_\beta^2\delta$, Algorithm 3 ensures $Q_t(s, a) \le Q_{t+1}(s, a) \le Q^\star(s, a)$ for any $(s, a) \in \mathcal{S} \times \mathcal{A}$.*

Proof. The first inequality is by the update rule of Q_t. Next, we prove $Q_t(s, a) \le Q^\star(s, a)$ by induction on t. It is clearly true when when $t = 1$. For the induction step, note that for any p, r, the proof of Lemma 7 still proceeds to conclude that $\overline{Q}_{p,r,t}(s, a) \le Q^\star(s, a)$ and $\overline{Q}'_{p,t}(s, a) \le Q^\star(s, a)$, where we substitute b_t with $b_{p,r,t}$, b'_t with $b'_{p,t}$, and V_t^{ref} with $V_{r,t}^{\text{ref}}$. Thus, by a union bound over $8N_\beta^2$ update rules and Eq. (11), the claim is proved. □

Theorem 7. *With probability at least $1 - 47 \cdot 8N_\beta^2\delta$, Algorithm 3 ensures $R_K = \tilde{\mathcal{O}}\left(B_\star\sqrt{SAK} + \frac{B_\star^5 S^2 A}{c_{\min}^4}\right)$.*

Proof. Define $V^{\text{ref}} = V_{r^\star}^{\text{ref}}, V^{\text{REF}} = V_{r^\star,T+1}^{\text{ref}}, b_t = b_{p^\star,r^\star,t}, b'_t = b'_{p^\star,t}, H = H_{p^\star}$, and n_t, m_t, $l_{t,i}$, $\check{l}_{t,i}$ are defined for $p^\star$. We have Lemma 10, Lemma 5, Corollary 6, Lemma 6, Lemma 8, Lemma 9 and Theorem 2 holds for Algorithm 3. Following the steps in the proof of Theorem 3 gives the desired result. □

C. Omitted Details for Section 5

We first present details of our proposed model-based algorithm SVI-SSP, which not only achieves the minimax optimal regret even when $c_{\min} = 0$, matching the state-of-the-art by a recent work (Tarbouriech et al., 2021b), but also admits highly sparse updates, making it more efficient than all existing model-based algorithms. The pseudocode is in Algorithm 4, again with only the lines instantiating the update rule for Q numbered.

Similar to Algorithm 2, SVI-SSP divides samples of each (s, a) into consecutive stages of (roughly) exponentially increasing length, and only update $Q(s, a)$ at the end of a stage (Line 2). However, the number of samples e_j in stage j is defined slightly differently through $e_j = \lfloor \widetilde{e}_j \rfloor, \widetilde{e}_1 = 1$, and $\widetilde{e}_{j+1} = \widetilde{e}_j + \frac{1}{H} e_j$ for some parameter H. In the long run, this is almost the same as the scheme used in Algorithm 2, but importantly, it forces more frequent updates at the beginning — for example, one can verify that $e_1 = \cdots = e_H = 1$, meaning that $Q(s, a)$ is updated every time (s, a) is visited for the first H visits. This slight difference turns out to be important to ensure that the lower-order term in the regret has no poly(H) dependence, as shown in Lemma 14 and further discussed in Remark 2.

The update rule for Q is very simple (Line 5). It is again based on the equality $Q^\star(s,a) = c(s,a) + P_{s,a}V^\star$, but this time uses $\bar{P}_{s,a}V - b$ as an approximation for $P_{s,a}V^\star$, where $\bar{P}_{s,a}$ is the empirical transition directly calculated from two counters $n(s,a)$ and $n(s,a,s')$ (number of visits to (s,a) and (s,a,s') respectively), V is such that $V(s) = \min_a Q(s,a)$, and b is a special bonus term (Line 4) adopted from (Tarbouriech et al., 2021b; Zhang et al., 2020a) which ensures that Q is an optimistic estimate of $Q^\star$ and also helps remove poly(H) dependence in the regret.

SVI-SSP exhibits a unique structure compared to existing algorithms. In each update, it modifies only one entry of Q (similarly to model-free algorithms), while other model-based algorithms such as (Tarbouriech et al., 2021b) perform value iteration for every entry of Q repeatedly until convergence (concrete time complexity comparisons to follow). We emphasize that our implicit finite-horizon analysis is indeed the key to enable us to derive a regret guarantee for such a sparse value iteration algorithm. Specifically, in Section C.1, we show that SVI-SSP satisfies the two required properties.

Extra Notations Denote by $Q_t(s,a), V_t(s)$ the value of $Q(s,a), V(s)$ at the beginning of time step t, $V_0(s) = 0$, and $b_t(s,a), n_t(s,a), \bar{P}_{t,s,a}(s'), \iota_t(s,a)$ the value of $b, n, \bar{P}_{s,a}(s'), \iota$ used in computing $Q_t(s,a)$ (note that $b_t(s,a) = 0$ if $n_t(s,a) = 0$). Denote by $l_t(s,a)$ the last time step the agent visits (s,a) among those $n_t(s,a)$ steps before the current stage, and $l_t(s,a) = t$ if the first visit to (s,a) is at time step t. Also define $\bar{P}_t = \bar{P}_{t,s_t,a_t}$ and $n_t^+(s,a) = \max\{1, n_t(s,a)\}$. With these notations, we have by the update rule of the algorithm:

$$Q_t(s,a) = \max\{Q_{t-1}(s,a), c(s,a) + \bar{P}_{t,s,a}V_{l_t} - b_t\}, \tag{12}$$

where b_t represents $b_t(s,a)$, and l_t represents $l_t(s,a)$ for notational convenience.

Before proving Theorem 5 (Section C.3), we first show some basic properties of our proposed update scheme (Section C.1), and proves the two required properties for Algorithm 4 (Section C.2).

C.1. Properties of Proposed Update Scheme

In this section, we prove that our proposed update scheme has the desired properties, that is, it suffers constant cost independent of H, while maintaining sparse update in the long run similar to the update scheme of Algorithm 2 (Lemma 14). We also quantify the bias induced by the sparse update compared to full-planning (that is, update every state-action pair at every time step) in Lemma 15.

Lemma 14. *The proposed update scheme satisfies the following:*

1. *For $\{X_t\}_{t\geq 0}$ such that $X_t \in [0, \mathring{B}]$ and $t < t', (s_t, a_t) = (s_{t'}, a_{t'}) \implies X_t \geq X_{t'}$, we have: $\sum_{t=1}^T X_{l_t} \leq \mathring{B}SA + (1 + \frac{1}{H})\sum_{t=1}^T X_t$.*
2. *Denote $i_h^\star = \inf\{i \geq \mathbb{N}^+ : e_i \geq h\}$ for $h \in \mathbb{N}^+$. Then $i_h^\star = \mathcal{O}(H\ln(h))$.*

Proof. For any given $n \in \mathbb{N}^+$, define y_n as the index of the end of last stage, that is, the largest element in $\mathcal{L}$ that is smaller than n (also define $y_1 = 1$). For the first property, we first prove by induction that for any $j \in \mathbb{N}^+$, there exists non-negative weights $\{w_{n,i}\}_{n,i}$ such that:

1. For all $n \leq E_j$, $\sum_{i=1}^{y_n} w_{n,i} = \mathbb{I}\{n > 1\}$, and $w_{n,i} = 0$ for $i > y_n$.
2. $\sum_{n=1}^{E_j} w_{n,i} \leq 1 + \frac{1}{H}$ for any $i \leq E_j$.
3. $\widetilde{e}_{j+1} + \sum_{n=1}^{E_j}\sum_{n'=1}^{E_j} w_{n,n'} = (1 + 1/H)E_j$.

To give some intuition, we can imagine a continuous process where we process index n at time step n. Indices are divided into consecutive stages, and there are e_j indices in the j-th stage. At index n we need to consume 1 unit of energy accumulated up to the last stage (that is, up to index y_n) and then contributes $(1 + \frac{1}{H})$ energy to the future stages. We can think of $\widetilde{e}_j$ as the available amount of energy at the beginning of stage j (accumulated from indices up to E_{j-1}), and e_j as the amount of energy consumed in stage j (one unit by each index in stage j). The assignment of energy consumption is represented by $\{w_{n,i}\}$, where $w_{n,i}$ is the amount of energy consumed by index n which is contributed by index i. The result we are going to prove by induction states that the process described above can proceed indefinitely.

The base case of $j=1$ is clearly true by $w_{1,i}=0$ for any $i\in\mathbb{N}^+$ and $\widetilde{e}_2=1+\frac{1}{H}$. For the induction step, by the third property, there are in total $(1+\frac{1}{H})E_j$ energy contributed by indices up to E_j, where $\widetilde{e}_{j+1}$ is the amount of energy available to use for stages starting from $j+1$, and $\sum_{n=1}^{E_j}\sum_{n'=1}^{E_j}w_{n,n'}$ is the amount of energy consumed by indices up to E_j (we use one of the possible assignments of $\{w_{n,i}\}_{n,i}$ for $n\leq E_j$ from the previous induction step). We can easily distribute e_{j+1} weights (from $\widetilde{e}_{j+1}$) to indices in stage $j+1$ so that $\sum_{i=1}^{y_n}w_{n,i}=1$ and $w_{n,i}=0$ for $i>y_n$ for all $E_j<n\leq E_{j+1}$ (note that $y_n=E_j$ in this range), and $\sum_{n=1}^{E_{j+1}}w_{n,i}\leq 1+\frac{1}{H}$ for any $i\leq E_{j+1}$. Moreover,

$$\begin{aligned}\widetilde{e}_{j+2}+\sum_{n=1}^{E_{j+1}}\sum_{n'=1}^{E_{j+1}}w_{n,n'}&=\widetilde{e}_{j+1}+\frac{1}{H}e_{j+1}+\sum_{n=1}^{E_j}\sum_{n'=1}^{E_j}w_{n,n'}+e_{j+1}\\&=\left(1+\frac{1}{H}\right)E_j+\left(1+\frac{1}{H}\right)e_{j+1}=\left(1+\frac{1}{H}\right)E_{j+1}.\end{aligned}$$

Thus, the induction step also holds. We are now ready to prove the first property. Denote by $t_i(s,a)$ the time step of the i-th visit to (s,a), and by $N(s,a)$ the total number of visits to (s,a) in K episodes. We have

$$\begin{aligned}\sum_{t=1}^{T}X_{l_t}&=\sum_{(s,a)}\sum_{n=1}^{N(s,a)}X_{t_{y_n}(s,a)}\leq\sum_{(s,a)}X_{t_1(s,a)}+\sum_{(s,a)}\sum_{n=2}^{N(s,a)}\sum_{i=1}^{y_n}w_{n,i}X_{t_i(s,a)}\\&\qquad\qquad(y_1=1,\ X_{t_i(s,a)}\text{ is non-increasing in }i,\text{ and }\{w_{n,i}\}_{n,i}\text{ is from the induction result})\\&\leq\mathring{B}SA+\sum_{(s,a)}\sum_{i=1}^{N(s,a)}X_{t_i(s,a)}\sum_{n=1}^{N(s,a)}w_{n,i}\leq\mathring{B}SA+\left(1+\frac{1}{H}\right)\sum_{(s,a)}\sum_{i=1}^{N(s,a)}X_{t_i(s,a)}\\&\qquad\qquad(X_{t_1(s,a)}\leq\mathring{B}\text{ and }\textstyle\sum_{n=1}^{N(s,a)}w_{n,i}\leq 1+\frac{1}{H})\\&=\mathring{B}SA+\left(1+\frac{1}{H}\right)\sum_{t=1}^{T}X_t.\end{aligned}$$

For the second property, note that $i_h^\star=\inf\{i\in\mathbb{N}^+:\widetilde{e}_i\geq h\}$ since h is an interger. Moreover,

$$\begin{aligned}&\widetilde{e}_{i+1}=\left(1+\frac{1}{H}\right)\widetilde{e}_i+\frac{1}{H}(e_i-\widetilde{e}_i)\geq\left(1+\frac{1}{H}\right)\widetilde{e}_i-\frac{1}{H}\implies\widetilde{e}_{i+1}-1\geq\left(1+\frac{1}{H}\right)(\widetilde{e}_i-1)\\&\implies\widetilde{e}_i\geq(\widetilde{e}_{i_2^\star}-1)\left(1+\frac{1}{H}\right)^{i-i_2^\star}+1\geq\left(1+\frac{1}{H}\right)^{i-i_2^\star}+1,\quad\forall i\geq i_2^\star.\end{aligned}$$

Therefore, $i_h^\star\leq\inf_i\{i\geq i_2^\star:(1+1/H)^{i-i_2^\star}+1\geq h\}=i_2^\star+\mathcal{O}(H\ln(h))$. Also, by inspecting e_i for small i we observe that $i_2^\star=\mathcal{O}(H)$, which implies that $i_h^\star=\mathcal{O}(H\ln(h))$. □

Remark 1. *Lemma 14 implies that there are at most $\mathcal{O}(\min\{SAH\ln T,ST\})$ updates in T steps.*

Remark 2. *Note that the update scheme in (Zhang et al., 2020b) (also used in Algorithm 2) induces a constant cost of order $\tilde{\mathcal{O}}(B_\star HSA)$, which ruins the horizon free regret. This is because their update scheme collects H samples before the first update. On the contrary, our update scheme updates frequently at the beginning, but has the same update frequency as that of (Zhang et al., 2020b) in the long run. This reduces the constant cost to $\tilde{\mathcal{O}}(B_\star SA)$ while maintaining the $\tilde{\mathcal{O}}(SAH)$ time complexity.*

The following lemma quantifies the dominating bias introduced by the sparse update.

Lemma 15 (bias of the update scheme)**.** *$\sum_{t=1}^{T}P_t(V_t-V_{l_t})\leq B_\star SA+\frac{1}{H}\sum_{t=1}^{T}P_t(V^\star-V_t)$ and $\sum_{t=1}^{T}\mathbb{V}(P_t,V_t-V_{l_t})\leq\tilde{\mathcal{O}}\left(B_\star^2SA\right)+\frac{B_\star}{H}\sum_{t=1}^{T}P_t(V^\star-V_t)$.*

Proof. For the first statement, we apply Lemma 14 and $P_t=P_{l_t}$ to obtain

$$\sum_{t=1}^{T}P_t(V_t-V_{l_t})=\sum_{t=1}^{T}P_{l_t}(V^\star-V_{l_t})-\sum_{t=1}^{T}P_t(V^\star-V_t)\leq B_\star SA+\frac{1}{H}\sum_{t=1}^{T}P_t(V^\star-V_t).$$

Similarly, for the second statement

$$\sum_{t=1}^{T}\mathbb{V}(P_t, V_t - V_{l_t}) \le \sum_{t=1}^{T} P_t(V_t - V_{l_t})^2 \le B_\star \sum_{t=1}^{T} P_t(V_t - V_{l_t})$$
$$\le B_\star^2 SA + \frac{B_\star}{H}\sum_{t=1}^{T} P_t(V^\star - V_t).$$

□

C.2. Proofs of Required Properties

In this section, we prove Property 1 (Lemma 16) and Property 2 of Algorithm 4, where Lemma 17 proves a preliminary form of Property 2.

Lemma 16. *With probability at least* $1-\delta$, $Q_t(s,a) \le Q_{t+1}(s,a) \le Q^\star(s,a)$, *for any* $(s,a) \in \mathcal{S}\times\mathcal{A}, t \ge 1$.

Proof. The first inequality is clearly true by the update rule. Next, we prove $Q_t(s,a) \le Q^\star(s,a)$. By Eq. (12), it is clearly true when $n_t(s,a) = 0$. When $n_t(s,a) > 0$, by Lemma 29: (here, l_t, ι_t is a shorthand of $l_t(s,a), \iota_t(s,a)$):

$$c(s,a) + \bar{P}_{t,s,a}V_{l_t} - b_t(s,a) = c(s,a) + f(\bar{P}_{t,s,a}, V_{l_t}, n_t(s,a), B, \iota_t)$$
$$\le c(s,a) + f(\bar{P}_{t,s,a}, V^\star, n_t(s,a), B, \iota_t)$$
$$= c(s,a) + \bar{P}_{t,s,a}V^\star - \max\left\{7\sqrt{\frac{\mathbb{V}(\bar{P}_{t,s,a}, V^\star)\iota_t}{n_t(s,a)}}, \frac{49B\iota_t}{n_t(s,a)}\right\}$$
$$\le Q^\star(s,a) + (\bar{P}_{t,s,a} - P_{s,a})V^\star - 3\sqrt{\frac{\mathbb{V}(\bar{P}_{t,s,a}, V^\star)\iota_t}{n_t(s,a)}} - \frac{24B\iota_t}{n_t(s,a)}$$

($Q^\star(s,a) = c(s,a) + P_{s,a}V^\star$ and $\max\{a,b\} \ge \frac{a+b}{2}$)

$$\le Q^\star(s,a) + (2\sqrt{2}-3)\sqrt{\frac{\mathbb{V}(\bar{P}_{t,s,a}, V^\star)\iota_t}{n_t(s,a)}} + (19-24)\frac{B\iota_t}{n_t(s,a)} \le Q^\star(s,a). \qquad \text{(Lemma 32)}$$

□

Lemma 17. *With probability at least* $1-9\delta$, *for all* $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$

$$\sum_{t=1}^{T}(\mathring{Q}(s_t,a_t) - Q_t(s_t,a_t))_+ \le \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}(\mathring{V}(s_t) - V_t(s_t))_+$$
$$+ \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + BS^2A + \sqrt{\frac{B_\star S^2 A}{H}\sum_{t=1}^{T} V^\star(s_t) - V_t(s_t)}\right).$$

Proof. Note that with probability at least $1-2\delta$, for all $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$,

$$\sum_{t=1}^{T}(\mathring{Q}(s_t,a_t) - Q_t(s_t,a_t))_+ \le \sum_{t=1}^{T}(P_t\mathring{V} - \bar{P}_tV_{l_t})_+ + b_t \qquad (\mathring{Q}(s_t,a_t) = c(s_t,a_t) + P_t\mathring{V} \text{ and Eq. (12)})$$
$$= \sum_{t=1}^{T}(P_t(\mathring{V} - V_{l_t}) + (P_t - \bar{P}_t)V^\star + (P_t - \bar{P}_t)(V_{l_t} - V^\star))_+ + b_t$$
$$\le \sum_{t=1}^{T} P_t(\mathring{V} - V_{l_t})_+ + \tilde{\mathcal{O}}\left(\sqrt{\frac{\mathbb{V}(P_t, V^\star)}{n_t^+}} + \sqrt{\frac{S\mathbb{V}(P_t, V^\star - V_{l_t})}{n_t^+}} + \frac{SB_\star}{n_t^+}\right) + b_t.$$

($(x+y)_+ \le (x)_+ + (y)_+$, Lemma 32, and Lemma 21)

Note that:

$$\sum_{t=1}^{T} P_t(\mathring{V}-V_{l_t})_+ \leq \left(1+\frac{1}{H}\right)\sum_{t=1}^{T} P_t(\mathring{V}-V_t)_+ + B_\star SA \qquad (P_{l_t}=P_t \text{ and Lemma 14})$$

$$= B_\star SA + \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}\left((\mathring{V}(s'_t)-V_t(s'_t))_+ + (P_t-\mathbb{I}_{s'_t})(\mathring{V}-V_t)_+\right)$$

$$\leq \mathcal{O}\left(B_\star SA\right) + \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}\left((\mathring{V}(s_t)-V_t(s_t))_+ + (P_t-\mathbb{I}_{s'_t})(\mathring{V}-V_t)_+\right).$$

(Lemma 26 and $(\mathring{V}(s'_t)-V_{t+1}(s'_t))_+ \leq (\mathring{V}(s_{t+1})-V_{t+1}(s_{t+1}))_+$)

Plug this back to the previous inequality, and by Cauchy-Schwarz inequality and Lemma 22:

$$\sum_{t=1}^{T}(\mathring{Q}(s_t,a_t)-Q_t(s_t,a_t))_+ \leq \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}\left((\mathring{V}(s_t)-V_t(s_t))_+ + (P_t-\mathbb{I}_{s'_t})(\mathring{V}-V_t)_+ + b_t\right)$$

$$+\tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star-V_{l_t})+B_\star S^2A}\right).$$

Next, we bound the term $\sum_{t=1}^{T}(P_t-\mathbb{I}_{s'_t})(\mathring{V}-V_t)_+$. We condition on Lemma 18, which holds with probability at least $1-\delta$. Then, for a given $(\mathring{Q},\mathring{V})\in\mathcal{V}_H$, by Lemma 20 with $X_t=(\mathring{V}-V_t)_+/B_\star$, we have with probability $1-\frac{\delta}{H+1}$:

$$B_\star F_T(0) = \sum_{t=1}^{T}(P_t-\mathbb{I}_{s'_t})(\mathring{V}-V_t)_+ \leq B_\star(\sqrt{3Y_T\zeta_T}+4\zeta_T) = \tilde{\mathcal{O}}\left(\sqrt{B_\star^2 Y_T}+B_\star\right)$$

$$= \tilde{\mathcal{O}}\left(\sqrt{B_\star^2\left(S+1+\sum_{t=1}^{T}(X_t(s_t)-P_tX_t)_+\right)}+B_\star\right)$$

$$= \tilde{\mathcal{O}}\left(\sqrt{B_\star^2 S + B_\star\sum_{t=1}^{T}(\mathring{V}(s_t)-V_t(s_t)-P_t(\mathring{V}-V_t))_+}+B_\star\right). \qquad ((x)_+-(y)_+\leq(x-y)_+)$$

$$\overset{\text{(i)}}{=} \tilde{\mathcal{O}}\left(\sum_{t=1}^{T} b_t + B_\star S\sqrt{A} + \sqrt{\frac{B_\star}{H}\sum_{t=1}^{T}P_t(V^\star-V_t)}\right)$$

$$+\tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star)}+\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star-V_{l_t})}\right),$$

where in (i) we apply:

$$\sqrt{B_\star\sum_{t=1}^{T}(\mathring{V}(s_t)-V_t(s_t)-P_t(\mathring{V}-V_t))_+} \leq \sqrt{B_\star\left(\sum_{t=1}^{T} b_t + \frac{P_t(V^\star-V_t)}{H}\right)}$$

$$+\tilde{\mathcal{O}}\left(\sqrt{B_\star\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star)}+\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star-V_{l_t})}\right)+B_\star S\sqrt{A}}\right)$$

(Lemma 18 and $\sqrt{x+y}\leq\sqrt{x}+\sqrt{y}$)

$$\leq \sum_{t=1}^{T} b_t + \sqrt{\frac{B_\star}{H}\sum_{t=1}^{T}P_t(V^\star-V_t)} + \tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star)}+\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star-V_{l_t})+B_\star S\sqrt{A}}\right).$$

(AM-GM inequality and $\sqrt{x+y}\leq\sqrt{x}+\sqrt{y}$)

Hence, by a union bound, the bound above for $\sum_{t=1}^{T}(P_t - \mathbb{I}_{s'_t})(\mathring{V} - V_t)_+$ holds for all $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$ with probability at least $1-\delta$, and with probability at least $1-4\delta$, for all $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$,

$$\begin{aligned}
\sum_{t=1}^{T}(\mathring{Q}(s_t,a_t) - Q_t(s_t,a_t))_+ &\leq \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}(\mathring{V}(s_t) - V_t(s_t))_+ + \tilde{\mathcal{O}}\left(B_\star S^2 A + \sum_{t=1}^{T} b_t\right)\\
&\quad + \tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t})} + \sqrt{\frac{B_\star}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right)\\
&\leq \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}(\mathring{V}(s_t) - V_t(s_t))_+ + \tilde{\mathcal{O}}\left(BS^2A + \sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)}\right)\\
&\quad + \tilde{\mathcal{O}}\left(\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t})} + \sqrt{\frac{B_\star SA}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right). && \text{(Lemma 19)}
\end{aligned}$$

Note that:

$$\begin{aligned}
&\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t})}\\
&= \tilde{\mathcal{O}}\left(\sqrt{B_\star S^2 A\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + B^2S^4A^2 + \frac{B_\star S^2 A}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right) && \text{(Lemma 19)}\\
&= \tilde{\mathcal{O}}\left(\sqrt{B_\star S^2 A\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)}} + BS^2A + \sqrt{\frac{B_\star S^2 A}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right) && (\sqrt{x+y} \leq \sqrt{x} + \sqrt{y})\\
&= \tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + BS^2A + \sqrt{\frac{B_\star S^2 A}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right). && \text{(AM-GM inequality)}
\end{aligned}$$

Plug this back to the previous inequality, and then by Lemma 4

$$\begin{aligned}
\sum_{t=1}^{T}(\mathring{Q}(s_t,a_t) - Q_t(s_t,a_t))_+ &\leq \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}(\mathring{V}(s_t) - V_t(s_t))_+\\
&\quad + \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + BS^2A + \sqrt{\frac{B_\star S^2 A}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right).
\end{aligned}$$

Finally, applying Lemma 34, Lemma 26 and $(V^\star - V_{t+1})(s'_t) \leq (V^\star - V_{t+1})(s_{t+1})$, the claim is proved by

$$\sum_{t=1}^{T}P_t(V^\star - V_t) \leq \tilde{\mathcal{O}}\left(B_\star\right) + 2\sum_{t=1}^{T}V^\star(s'_t) - V_t(s'_t) \leq \tilde{\mathcal{O}}\left(SB_\star\right) + 2\sum_{t=1}^{T}V^\star(s_t) - V_t(s_t).$$

□

Proof of Theorem 4. Property 1 is proved in Lemma 16. For Property 2, by Lemma 17, it suffices to bound $\sum_{t=1}^{T}V^\star(s_t) - V_t(s_t)$. By Lemma 17, $V^\star_{h-1}(s_t) \leq Q^\star_h(s_t,a_t)$, and $V_t(s_t) = Q_t(s_t,a_t)$, we have with probability at least $1-9\delta$, for all

$\mathring{Q} = Q^\star_h, \mathring{V} = V^\star_{h-1}, h \in [H]$:

$$\sum_{t=1}^{T}(Q^\star_h(s_t,a_t) - Q_t(s_t,a_t))_+ \le \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}(Q^\star_{h-1}(s_t,a_t) - Q_t(s_t,a_t))_+ + \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + BS^2A + \sqrt{\frac{B_\star S^2 A}{H}\sum_{t=1}^{T} V^\star(s_t) - V_t(s_t)}\right), \quad \forall h \in [H].$$

Applying the inequality above recursively starting from $h = H$ and by $Q^\star_0(s,a) = 0, (1+\frac{1}{H})^H \le 3$ we have:

$$\sum_{t=1}^{T}(Q^\star_H(s_t,a_t) - Q_t(s_t,a_t))_+ = \tilde{\mathcal{O}}\left(H\left(\sqrt{B_\star SAC_K} + BS^2A\right) + \sqrt{B_\star HS^2A\sum_{t=1}^{T} V^\star(s_t) - V_t(s_t)}\right).$$

Then by Lemma 1 with $H = \lceil \frac{4B}{c_{\min}}\ln(\frac{2}{\beta}) + 1\rceil_2$:

$$\begin{aligned}\sum_{t=1}^{T} V^\star(s_t) - V_t(s_t) &\le \sum_{t=1}^{T} Q^\star(s_t,a_t) - Q^\star_H(s_t,a_t) + \sum_{t=1}^{T} Q^\star_H(s_t,a_t) - Q_t(s_t,a_t)\\ &\le B_\star\beta T + \tilde{\mathcal{O}}\left(H\left(\sqrt{B_\star SAC_K} + BS^2A\right) + \sqrt{BHS^2A\sum_{t=1}^{T} V^\star(s_t) - V_t(s_t)}\right).\end{aligned}$$

Solving a quadratic equation w.r.t $\sum_{t=1}^{T} V^\star(s_t) - V_t(s_t)$ (Lemma 23), we have:

$$\sum_{t=1}^{T} V^\star(s_t) - V_t(s_t) \le B_\star\beta T + \tilde{\mathcal{O}}\left(H\left(\sqrt{B_\star SAC_K} + BS^2A\right)\right).$$

Plug this back to the bound of Lemma 17 and by AM-GM inequality, we have for all $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$:

$$\begin{aligned}&\sum_{t=1}^{T}(\mathring{Q}(s_t,a_t) - Q_t(s_t,a_t))_+\\ &\le \left(1+\frac{1}{H}\right)\sum_{t=1}^{T}(\mathring{V}(s_t) - V_t(s_t))_+ + \frac{B_\star\beta T}{H} + \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + BS^2A\right).\end{aligned}$$

Moreover, by $H \ge \frac{B_\star}{c_{\min}}$, we have $\frac{B_\star\beta T}{H} \le \beta c_{\min} T \le \beta C_K$. Hence, Property 2 is satisfied with $d = 1, \xi_H = \beta C_K + \tilde{\mathcal{O}}(\sqrt{B_\star SAC_K} + BS^2A)$ with probability at least $1 - 9\delta$. □

C.3. Proof of Theorem 5

Proof. By Theorem 1 and Theorem 4, with probability at least $1 - 12\delta$:

$$C_K - KV^\star(s_0) = R_K \le \beta C_K + \tilde{\mathcal{O}}\left(\sqrt{B_\star SAC_K} + BS^2A\right).$$

Then by $V^\star(s_0) \le B_\star, \beta \le \frac{1}{2}$ and Lemma 23, we have $C_K = \tilde{\mathcal{O}}\left(B_\star K\right)$. Substituting this back and by $\beta \le \frac{c_{\min}}{B_\star K}, H = \tilde{\mathcal{O}}(B_\star/c_{\min})$, we get $R_K = \tilde{\mathcal{O}}\left(B_\star\sqrt{SAK} + BS^2A\right)$. □

C.4. Extra Lemmas for Section 5

In this section, we give full proofs of auxiliary lemmas used in Section 5. Notably, Lemma 18 and Lemma 19 bounds the additional terms appears in the recursion in Lemma 17. Lemma 20 gives recursion-based analysis on bounding the sum of martingale difference sequence, which is the key in obtaining horizon-free regret.

Lemma 18. *With probability at least $1-\delta$, we have for all $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$,*

$$\sum_{t=1}^{T}((\mathbb{I}_{s_t} - P_t)(\mathring{V} - V_t))_+ \leq \sum_{t=1}^{T} b_t + \frac{P_t(V^\star - V_t)}{H} + \tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t}) + B_\star S^2 A}\right).$$

Proof. With probability at least $1-\delta$, for all $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$,

$$\begin{aligned}
&\sum_{t=1}^{T}(\mathring{V}(s_t) - V_t(s_t) - P_t(\mathring{V} - V_t))_+ \leq \sum_{t=1}^{T}(\mathring{Q}(s_t, a_t) - P_t\mathring{V} + P_tV_t - V_t(s_t))_+ \\
&\leq \sum_{t=1}^{T}(c(s_t, a_t) + P_tV_{l_t} - V_t(s_t))_+ + P_t(V_t - V_{l_t}) \\
&\qquad (\mathring{Q}(s_t,a_t) = c(s_t,a_t) + P_t\mathring{V},\ (x+y)_+ \leq (x)_+ + (y)_+, \text{ and } V_t \text{ is increasing in } t) \\
&\leq B_\star SA + \sum_{t=1}^{T}((P_t - \bar{P}_t)V_{l_t})_+ + b_t + \frac{1}{H}P_t(V^\star - V_t) \qquad (V_t(s_t) = Q_t(s_t,a_t), \text{ Eq. (12), and Lemma 15}) \\
&= B_\star SA + \sum_{t=1}^{T}((P_t - \bar{P}_t)V^\star + (P_t - \bar{P}_t)(V_{l_t} - V^\star))_+ + b_t + \frac{1}{H}P_t(V^\star - V_t).
\end{aligned}$$

Now by Lemma 32 and Lemma 21, we have with probability at least $1-\delta$: $(P_t - \bar{P}_t)V^\star = \mathcal{O}\left(\sqrt{\frac{\mathbb{V}(P_t,V^\star)}{n_t^+}} + \frac{B_\star}{n_t}\right)$ and $(P_t - \bar{P}_t)(V_{l_t} - V^\star) = \tilde{\mathcal{O}}\left(\sqrt{\frac{S\mathbb{V}(P_t,V^\star - V_{l_t})}{n_t^+}} + \frac{SB_\star}{n_t^+}\right)$. Plug these back to the previous inequality, we have for all $(\mathring{Q}, \mathring{V}) \in \mathcal{V}_H$:

$$\begin{aligned}
&\sum_{t=1}^{T}(\mathring{V}(s_t) - V_t(s_t) - P_t(\mathring{V} - V_t))_+ \\
&\leq B_\star SA + \sum_{t=1}^{T}\tilde{\mathcal{O}}\left(\sqrt{\frac{\mathbb{V}(P_t,V^\star)}{n_t^+}} + \sqrt{\frac{S\mathbb{V}(P_t,V^\star - V_{l_t})}{n_t^+}} + \frac{SB_\star}{n_t^+}\right) + b_t + \frac{1}{H}P_t(V^\star - V_t) \\
&\leq \tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star - V_{l_t}) + B_\star S^2A}\right) + \sum_{t=1}^{T} b_t + \frac{P_t(V^\star - V_t)}{H}. \\
&\qquad (\text{Cauchy-Schwarz inequality and Lemma 22})
\end{aligned}$$

This completes the proof. □

Lemma 19. *With probability at least $1-3\delta$,*

$$\sum_{t=1}^{T} b_t = \tilde{\mathcal{O}}\left(BS^{3/2}A + \sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star)} + \sqrt{\frac{B_\star SA}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right).$$

$$\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t}) = \tilde{\mathcal{O}}\left(B_\star\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t,V^\star) + B^2S^2A + \frac{B_\star}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)}\right).$$

Proof. First note that:

$$\sum_{t=1}^{T} b_t \overset{\text{(i)}}{=} \tilde{\mathcal{O}}\left(BSA + \sum_{t=1}^{T}\sqrt{\frac{\mathbb{V}(\bar{P}_t, V_{l_t})}{n_t^+}}\right) \overset{\text{(ii)}}{=} \tilde{\mathcal{O}}\left(BSA + \sum_{t=1}^{T}\sqrt{\frac{\mathbb{V}(P_t, V_{l_t})}{n_t^+}} + \frac{B_\star\sqrt{S}}{n_t^+}\right).$$

where in (i) we apply $\max\{a,b\} \leq a+b$ and Lemma 22, and in (ii) we have with probability at least $1-\delta$,

$$\begin{aligned}
&\mathbb{V}(\bar{P}_t, V_{l_t}) = \bar{P}_t(V_{l_t} - \bar{P}_t V_{l_t})^2 \leq \bar{P}_t(V_{l_t} - P_t V_{l_t})^2 && \left(\tfrac{\sum_i p_i x_i}{\sum_i p_i} = \operatorname{argmin}_z \textstyle\sum_i p_i(x_i - z)^2\right)\\
&= \mathbb{V}(P_t, V_{l_t}) + (P_t - \bar{P}_t)(V_{l_t} - P_t V_{l_t})^2\\
&\leq \mathbb{V}(P_t, V_{l_t}) + \tilde{\mathcal{O}}\left(\sum_{s'}\left(\sqrt{\frac{P_t(s')}{n_t^+}} + \frac{1}{n_t^+}\right)(V_{l_t}(s') - P_t V_{l_t})^2\right) && \text{(Lemma 32)}\\
&\leq \mathbb{V}(P_t, V_{l_t}) + \tilde{\mathcal{O}}\left(B_\star\sqrt{\frac{S\mathbb{V}(P_t, V_{l_t})}{n_t^+}} + \frac{SB_\star^2}{n_t^+}\right) = \tilde{\mathcal{O}}\left(\mathbb{V}(P_t, V_{l_t}) + \frac{SB_\star^2}{n_t^+}\right).
\end{aligned}$$

(Cauchy-Schwarz inequality and AM-GM inequality)

Thus, by Lemma 27, Cauchy-Schwarz inequality, and Lemma 22, we have:

$$\begin{aligned}
\sum_{t=1}^{T} b_t &= \tilde{\mathcal{O}}\left(BS^{3/2}A + \sum_{t=1}^{T}\sqrt{\frac{\mathbb{V}(P_t, V^\star)}{n_t^+}} + \sum_{t=1}^{T}\sqrt{\frac{\mathbb{V}(P_t, V^\star - V_{l_t})}{n_t^+}}\right)\\
&= \tilde{\mathcal{O}}\left(BS^{3/2}A + \sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + \sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t})}\right). \qquad (13)
\end{aligned}$$

Applying Lemma 20 with $X_t(s) = (V^\star(s) - V_t(s))/B_\star$, we have with probability at least $1-\delta$,

$$\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_t) = B_\star^2 G_T(0) \leq 3B_\star^2 Y_T + 9B_\star^2\zeta_T \leq 3B_\star\sum_{t=1}^{T}((\mathbb{I}_{s_t} - P_t)(V^\star - V_t))_+ + \tilde{\mathcal{O}}\left(SB_\star^2\right).$$

By Lemma 18 and Eq. (13), with probability at least $1-\delta$,

$$\begin{aligned}
&\sum_{t=1}^{T}((\mathbb{I}_{s_t} - P_t)(V^\star - V_t))_+ \leq \sum_{t=1}^{T} b_t + \frac{1}{H}P_t(V^\star - V_t)\\
&\qquad + \tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t})} + B_\star S^2 A\right).\\
&= \tilde{\mathcal{O}}\left(BS^2A + \sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t})} + \frac{1}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)\right)\\
&\overset{\text{(i)}}{=} \tilde{\mathcal{O}}\left(BS^2A + \sqrt{SA\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_t)} + \frac{1}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)\right),
\end{aligned}$$

where in (i) we apply

$$\begin{aligned}
&\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_{l_t})} = \tilde{\mathcal{O}}\left(\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_t)} + \sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V_t - V_{l_t})}\right)\\
&\qquad\qquad (\textsc{Var}[X+Y] \leq 2\textsc{Var}[X] + 2\textsc{Var}[Y] \text{ and } \sqrt{x+y} \leq \sqrt{x} + \sqrt{y})\\
&= \tilde{\mathcal{O}}\left(\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_t)} + \sqrt{S^2A\left(B_\star^2 SA + \frac{B_\star}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)\right)}\right) \qquad \text{(Lemma 15)}\\
&= \tilde{\mathcal{O}}\left(\sqrt{S^2A\sum_{t=1}^{T}\mathbb{V}(P_t, V^\star - V_t)} + B_\star S^2 A + \frac{1}{H}\sum_{t=1}^{T}P_t(V^\star - V_t)\right).
\end{aligned}$$

($\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ and AM-GM Inequality)

Plug the bound on $\sum_{t=1}^T((\mathbb{I}_{s_t}-P_t)(V^\star-V_t))_+$ back, we have

$$
\begin{aligned}
&\sum_{t=1}^T \mathbb{V}(P_t, V^\star - V_t)\\
&=\tilde{\mathcal{O}}\left(B^2S^2A + B_\star\sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star)} + B_\star\sqrt{S^2A\sum_{t=1}^T\mathbb{V}(P_t,V^\star-V_t)} + \frac{B_\star}{H}\sum_{t=1}^T P_t(V^\star-V_t)\right).
\end{aligned}
$$

Solving a quadratic inequality w.r.t $\sum_{t=1}^T\mathbb{V}(P_t,V^\star-V_t)$ (Lemma 23), we obtain

$$
\sum_{t=1}^T \mathbb{V}(P_t, V^\star - V_t) = \tilde{\mathcal{O}}\left(B^2S^2A + B_\star\sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star)} + \frac{B_\star}{H}\sum_{t=1}^T P_t(V^\star-V_t)\right),
$$

and by $\textsc{Var}[X+Y]\leq 2\textsc{Var}[X]+2\textsc{Var}[Y]$ and Lemma 15,

$$
\begin{aligned}
\sum_{t=1}^T \mathbb{V}(P_t, V^\star - V_{l_t}) &= \tilde{\mathcal{O}}\left(\sum_{t=1}^T \mathbb{V}(P_t,V^\star-V_t)+\mathbb{V}(P_t,V_t-V_{l_t})\right)\\
&= \tilde{\mathcal{O}}\left(B_\star\sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star)} + B^2S^2A + \frac{B_\star}{H}\sum_{t=1}^T P_t(V^\star-V_t)\right).
\end{aligned}
$$

Moreover, by $\sqrt{x+y}\leq\sqrt{x}+\sqrt{y}$ and AM-GM inequality:

$$
\begin{aligned}
&\sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star-V_{l_t})}\\
&=\tilde{\mathcal{O}}\left(\sqrt{B_\star SA\sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star)}} + BS^{3/2}A + \sqrt{\frac{B_\star SA}{H}\sum_{t=1}^T P_t(V^\star-V_t)}\right)\\
&=\tilde{\mathcal{O}}\left(\sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star)} + BS^{3/2}A + \sqrt{\frac{B_\star SA}{H}\sum_{t=1}^T P_t(V^\star-V_t)}\right).
\end{aligned}
$$

Plug this back to Eq. (13):

$$
\sum_{t=1}^T b_t = \tilde{\mathcal{O}}\left(BS^{3/2}A + \sqrt{SA\sum_{t=1}^T\mathbb{V}(P_t,V^\star)} + \sqrt{\frac{B_\star SA}{H}\sum_{t=1}^T P_t(V^\star-V_t)}\right).
$$

$\square$

Lemma 20. *Suppose $X_t:\mathcal{S}^+\rightarrow[0,1]$ is monotonic in t (that is, $X_t(s)$ is non-decreasing or non-increasing in t for all $s\in\mathcal{S}^+$), and $X_t(g)=0$. Define:*

$$
F_n(d)=\sum_{t=1}^n P_tX_t^{2^d}-(X_t(s'_t))^{2^d},\quad G_n(d)=\sum_{t=1}^n\mathbb{V}(P_t,X_t^{2^d}).
$$

Then with probability at least $1-\delta$, for all $n\in\mathbb{N}^+$ simultaneously, $G_n(0)\leq 3Y_n+9\zeta_n$, $F_n(0)\leq\sqrt{3Y_n\zeta_n}+4\zeta_n$, where $Y_n=S+1+\sum_{t=1}^n(X_t(s_t)-P_tX_t)_+$, $\zeta_n=32\ln^3\frac{4n^4}{\delta}$.

Proof. Note that:

$$
\begin{aligned}
G_n(d) &= \sum_{t=1}^n P_t X_t^{2^{d+1}} - (P_t X_t^{2^d})^2 \leq \sum_{t=1}^n P_t X_t^{2^{d+1}} - (P_t X_t)^{2^{d+1}} && (x^p \text{ is convex for } p > 1)\\
&= \sum_{t=1}^n P_t X_t^{2^{d+1}} - X_t(s'_t)^{2^{d+1}} + \sum_{t=1}^n X_t(s'_t)^{2^{d+1}} - X_t(s_t)^{2^{d+1}} + \sum_{t=1}^n X_t(s_t)^{2^{d+1}} - (P_t X_t)^{2^{d+1}}\\
&\overset{\text{(i)}}{\leq} F_n(d+1) + S + 1 + 2^{d+1}(X_t(s_t) - P_t X_t)_+ \leq F(d+1) + 2^{d+1} Y_n,
\end{aligned}
$$

where in (i) we apply Lemma 24 and,

$$
\begin{aligned}
&\sum_{t=1}^n X_t(s'_t)^{2^{d+1}} - X_t(s_t)^{2^{d+1}} = \sum_{t=1}^n X_t(s'_t)^{2^{d+1}} - X_{t+1}(s'_t)^{2^{d+1}} + \sum_{t=1}^n X_{t+1}(s'_t)^{2^{d+1}} - X_t(s_t)^{2^{d+1}}\\
&\leq S + \sum_{t=1}^n X_{t+1}(s_{t+1})^{2^{d+1}} - X_t(s_t)^{2^{d+1}} = S + X_{n+1}(s_{n+1})^{2^{d+1}} - X_1(s_1)^{2^{d+1}} \leq S + 1.\\
&&& (\text{Lemma 26 and } X_{t+1}(s'_t) \leq X_{t+1}(s_{t+1}))
\end{aligned}
$$

For a fixed d, n, by Eq. (14) of Lemma 33, with probability $1 - \frac{\delta}{2n^2\lceil \log_2 n + 1\rceil}$,

$$
F_n(d) \leq \sqrt{G_n(d)\zeta_n} + \zeta_n \leq \sqrt{(F_n(d+1) + 2^{d+1}Y_n)\zeta_n} + \zeta_n.
$$

Taking a union bound on $d = 0, \ldots, \lceil \log_2 n \rceil$, and by Lemma 25 with $\lambda_1 = n, \lambda_2 = \sqrt{\zeta_n}, \lambda_3 = Y_n, \lambda_4 = \zeta_n$, we have:

$$
F_n(1) \leq \max\{(\sqrt{\zeta_n} + \sqrt{2\zeta_n})^2, \sqrt{8Y_n\zeta_n} + \zeta_n\} \leq \max\{6\zeta_n, \sqrt{8Y_n\zeta_n} + \zeta_n\}.
$$

Therefore, $G_n(0) \leq F_n(1) + 2Y_n \leq \max\{6\zeta_n, Y_n + 9\zeta_n\} + 2Y_n \leq 3Y_n + 9\zeta_n$, and $F_n(0) \leq \sqrt{G_n(0)\zeta_n} + \zeta_n \leq \sqrt{3Y_n\zeta_n} + 4\zeta_n$. Taking a union bound over $n \in \mathbb{N}^+$ proves the claim. $\square$

Lemma 21. *Given $X_t : \mathcal{S}^+ \to \mathbb{R}$ with $\|X_t\|_\infty \leq B$, with probability at least $1-\delta$, it holds that for all $t \geq 1$ simultaneously:* $(P_t - \bar{P}_t)X_t = \tilde{\mathcal{O}}\left(\sqrt{\frac{S\mathbb{V}(P_t, X_t)}{n_t^+}} + \frac{SB}{n_t^+}\right)$.

Proof. For a fixed $(s, a) \in \mathcal{S} \times \mathcal{A}$, by Lemma 32, with probability $1 - \frac{\delta}{SA}$, for any $t \geq 1$ such that $(s_t, a_t) = (s, a)$:

$$
\begin{aligned}
(P_t - \bar{P}_t)X_t &= \sum_{s'} (P_t(s') - \bar{P}_t(s'))(X_t(s') - P_t X_t) && \left(\textstyle\sum_{s'} P_t(s') - \bar{P}_t(s') = 0\right)\\
&= \tilde{\mathcal{O}}\left(\sum_{s'}\left(\sqrt{\frac{P_t(s')}{n_t^+}} + \frac{1}{n_t^+}\right)|X_t(s') - P_t X_t|\right) = \tilde{\mathcal{O}}\left(\sqrt{\frac{S\mathbb{V}(P_t, X_t)}{n_t^+}} + \frac{SB}{n_t^+}\right).
\end{aligned}
$$

Taking a union bound over $(s, a) \in \mathcal{S} \times \mathcal{A}$, the statement is proved. $\square$

Lemma 22. $\sum_{t=1}^T \frac{1}{n_t^+} = \mathcal{O}(SA \ln T)$.

Proof. Define $J_{s,a}$ such that $E_{J_{s,a}} = n_T(s, a)$. It is easy to see that $e_{j+1}/e_j \leq 2$. Then,

$$
\sum_{t=1}^T \frac{1}{n_t^+} \leq SA + \sum_{(s,a)} \sum_{j=1}^{J_{s,a}} \frac{e_{j+1}}{E_j} \leq SA + 2\sum_{(s,a)} \sum_{j=1}^{J_{s,a}} \frac{e_j}{E_j} = \mathcal{O}\left(SA \ln T\right).
$$

$\square$

Algorithm 5 SVI-SSP without knowledge of $B_\star$

Parameters: failure probability $\delta \in (0,1)$.
Define: $\mathcal{L} = \{E_j\}_{j\in\mathbb{N}^+}$, where $E_j = \sum_{i=1}^j e_i, e_j = \lfloor \widetilde{e}_j \rfloor$, and $\widetilde{e}_1 = 1, \widetilde{e}_{j+1} = \widetilde{e}_j + \frac{1}{H}e_j$.
Initialize: $B \leftarrow \frac{\sqrt{K}}{S^{3/2}A^{1/2}}, H \leftarrow \lceil \frac{4B}{c_{\min}} \ln \frac{4B^2SAK}{c_{\min}} \rceil_2, C \leftarrow 0, t \leftarrow 0, s_1 \leftarrow s_{\text{init}}$.
Initialize: for all (s,a,s'), $n(s,a,s') \leftarrow 0$, $n(s,a) \leftarrow 0$, $Q(s,a) \leftarrow c(s,a)$, $V(s) \leftarrow \min_a c(s,a)$.
for $k = 1,\ldots,K$ **do**
  repeat
    Increment time step $t \overset{+}{\leftarrow} 1$.
    Take action $a_t = \text{argmin}_a Q(s_t,a)$, suffer cost $c(s_t,a_t)$, transit to and observe s'_t.
    Update visitation counters: $n = n(s_t,a_t) \overset{+}{\leftarrow} 1, n(s_t,a_t,s'_t) \overset{+}{\leftarrow} 1$.
    Update cost accumulator $C \overset{+}{\leftarrow} c(s_t,a_t)$.
    if $n \in \mathcal{L}$ **then**
      Update empirical transition: $\bar{P}_{s_t,a_t}(s') \leftarrow \frac{n(s_t,a_t,s')}{n}$ for all s'.
      Compute $\iota \leftarrow \ln\frac{2SAt}{\delta}$ and bonus $b \leftarrow \max\left\{7\sqrt{\frac{\mathbb{V}(\bar{P}_{s_t,a_t},V)\iota}{n}}, \frac{49B\iota}{n}\right\}$.
      $Q(s_t,a_t) \leftarrow \max\{c(s_t,a_t) + \bar{P}_{s_t,a_t}V - b, Q(s_t,a_t)\}$.
      $V(s_t) \leftarrow \text{argmin}_a Q(s_t,a)$.
    if $\|V\|_\infty > B$ *or* $C > KB + x(B\sqrt{SAK} + BS^2A)$ **then**
      $B \leftarrow 2B, H \leftarrow \lceil \frac{4B}{c_{\min}} \ln \frac{4B^2SAK}{c_{\min}} \rceil_2, C \leftarrow 0$, and update x.
      $n(s,a,s') \leftarrow 0, n(s,a) \leftarrow 0, Q(s,a) \leftarrow c(s,a), V(s) \leftarrow \min_a c(s,a)$ for all (s,a,s').
    if $s'_t \neq g$ **then** $s_{t+1} \leftarrow s'_t$; **else** $s_{t+1} \leftarrow s_{\text{init}}$, **break**.

C.5. Parameter-free Algorithm

Following (Tarbouriech et al., 2021b), we divide the learning process into epochs indexed by ϕ. We maintain value function upper bound B initialized with $\frac{\sqrt{K}}{S^{3/2}A^{1/2}}$ and cost accumulator C recording the total costs suffered in current epoch. In epoch ϕ, we execute an instance of Algorithm 4 with value function upper bound B. Moreover, we start a new epoch whenever:

1. $\|V\|_\infty > B$.
2. $C > KB + x(B\sqrt{SAK} + BS^2A)$.

Here, x is a large enough constant determined by Theorem 5, so that when $B \geq B_\star$, we have with probability at least $1-12\delta$:

$$C - V^\star(s^\phi_{\text{init}}) - (K-1)V^\star(s_0) \leq x(B_\star\sqrt{SAK} + BS^2A),$$

where s^ϕ_{init} is the initial state of epoch ϕ (note that Theorem 5 still holds when the initial state is changing over episodes). Moreover, we double the value of B whenever a new epoch starts. We summarize ideas above in Algorithm 5.

Theorem 8. *With probability at least $1-12\delta$, Algorithm 5 ensures $R_K = \tilde{\mathcal{O}}(B_\star\sqrt{SAK} + B_\star^3S^3A)$.*

Proof. Denote by B_ϕ the value of B in epoch ϕ, and by C_ϕ the value of C at the end of epoch ϕ. Define $\phi^\star = \inf_\phi\{B_\phi \geq B_\star\}$. Clearly $B_\phi \leq \max\{2B_\star, \sqrt{K}/S^{3/2}A^{1/2}\}$ for $\phi \leq \phi^\star$. By Theorem 5, with probability at least $1-12\delta$, there is at most $\phi^\star$ epochs since the condition of starting a new epoch will never be triggered in epoch $\phi^\star$, and the regret in epoch $\phi^\star$ is properly bounded:

$$C_{\phi^\star} - V^\star(s^{\phi^\star}_{\text{init}}) - (K-1)V^\star(s_0) = \tilde{\mathcal{O}}\left(B_\star\sqrt{SAK} + B_{\phi^\star}S^2A\right) = \tilde{\mathcal{O}}\left(B_\star\sqrt{SAK} + B_\star S^2A\right).$$

Conditioned on the event that there are at most $\phi^\star$ epochs, we partition the regret into two parts: the total costs suffered before epoch $\phi^\star$, and the regret starting from epoch $\phi^\star$. It suffices to bound the total costs before epoch $\phi^\star$ assuming

$K \le B_\star^2 S^3 A$ (otherwise $\phi^\star = 1$). By the update scheme of B, we have at most $\lceil \log_2 B_\star \rceil + 1$ epochs before epoch $\phi^\star$. Moreover, by the second condition of starting a new epoch, the accumulated cost in epoch $\phi < \phi^\star$ is bounded by:

$$C_\phi \le KB_\phi + \tilde{\mathcal{O}}\left(B_\phi\sqrt{SAK} + B_\phi S^2 A\right) = \tilde{\mathcal{O}}\left(B_\star^3 S^3 A\right).$$

Combining these two parts, we get:

$$\begin{aligned}
R_K &= \sum_{\phi=1}^{\phi^\star - 1} C_\phi + \left(C_{\phi^\star} - V^\star(s_{\text{init}}^{\phi^\star}) - (K-1)V^\star(s_0)\right) + \left(V^\star(s_{\text{init}}^{\phi^\star}) - V^\star(s_0)\right) \\
&= \tilde{\mathcal{O}}\left(B_\star\sqrt{SAK} + B_\star^3 S^3 A\right),
\end{aligned}$$

where we assume $C_{\phi^\star} = 0$ and $s_{\text{init}}^{\phi^\star} = s_0$ if there are less than $\phi^\star$ epochs. □

D. Auxiliary Lemmas

Lemma 23. $x \le a\sqrt{x} + b \implies x \le (a + \sqrt{b})^2 \le 2a^2 + 2b$.

Proof. This is simply by solving a quadratic inequality with variable $\sqrt{x}$ (by completing the square, for example) and the AM-GM inequality: $2a\sqrt{b} \le a^2 + b$. □

Lemma 24. *For any $a, b \in [0, 1]$ and $k \in \mathbb{N}^+$, we have: $a^k - b^k \le k(a-b)_+$.*

Proof. $a^k - b^k = (a-b)(\sum_{i=1}^k a^{i-1}b^{k-i}) \le (a-b)_+ \cdot \sum_{i=1}^k 1 = k(a-b)_+$. □

Lemma 25. *(Zhang et al., 2020a, Lemma 11) Let $\lambda_1, \lambda_2, \lambda_4 \ge 0, \lambda_3 \ge 1$ and $i' = \log_2(\lambda_1)$. Let $a_1, a_2, \ldots, a_{i'}$ be non-negative reals such that $a_i \le \lambda_1$ and $a_i \le \lambda_2\sqrt{a_{i+1} + 2^{i+1}\lambda_3} + \lambda_4$ for any $1 \le i \le i'$. Then, $a_1 \le \max\{(\lambda_2 + \sqrt{\lambda_2^2 + \lambda_4})^2, \lambda_2\sqrt{8\lambda_3} + \lambda_4\}$.*

Lemma 26. *Assume $v_t : \mathcal{S}^+ \to [0, B]$ is monotonic in t (i.e., $v_t(s)$ is non-increasing or non-decreasing in t for any $s \in \mathcal{S}^+$). Then, for any state sequence $\{s_t\}_{t=1}^n, n \in \mathbb{N}^+$, we have: $|\sum_{t=1}^n v_{t+1}(s_t) - v_t(s_t)| \le SB$.*

Proof.

$$\begin{aligned}
\left|\sum_{t=1}^n v_{t+1}(s_t) - v_t(s_t)\right| &\le \sum_{s\in\mathcal{S}^+}\left|\sum_{t=1}^n (v_{t+1}(s) - v_t(s))\mathbb{I}\{s_t = s\}\right| \\
&\le \sum_{s\in\mathcal{S}^+}\left|\sum_{t=1}^n v_{t+1}(s) - v_t(s)\right| \le \sum_{s\in\mathcal{S}^+}|v_{n+1}(s) - v_1(s)| \le SB. \qquad (v_t(s) \text{ is monotonic in } t)
\end{aligned}$$

□

Lemma 27. *(Cohen et al., 2021, Lemma C.3) For any two random variables X, Y with $\textsc{Var}[X] < \infty, \textsc{Var}[Y] < \infty$. We have: $\sqrt{\textsc{Var}[X]} - \sqrt{\textsc{Var}[Y]} \le \sqrt{\textsc{Var}[X-Y]}$.*

Lemma 28. *For any two random variables X, Y, we have:*

$$\textsc{Var}[XY] \le 2\textsc{Var}[X]\left\|Y\right\|_\infty^2 + 2(\mathbb{E}[X])^2\textsc{Var}[Y].$$

Consequently, $\left\|X\right\|_\infty \le C$ implies $\textsc{Var}[X^2] \le 4C^2\textsc{Var}[X]$.

Proof. First note that for any two random variables U, V, we have $\textsc{Var}[U+V] \le 2\textsc{Var}[U] + 2\textsc{Var}[V]$. Now let $U = (X - \mathbb{E}[X])Y$ and $V = \mathbb{E}[X]Y$, we have:

$$\begin{aligned}
\textsc{Var}[XY] &\le 2\textsc{Var}[(X - \mathbb{E}[X])Y] + 2\textsc{Var}[\mathbb{E}[X]Y] \le 2\mathbb{E}[(X - \mathbb{E}[X])^2 Y^2] + 2(\mathbb{E}[X])^2\textsc{Var}[Y] \\
&\le 2\textsc{Var}[X]\left\|Y\right\|_\infty^2 + 2(\mathbb{E}[X])^2\textsc{Var}[Y].
\end{aligned}$$

□

Lemma 29. *(Tarbouriech et al., 2021b, Lemma 14) Define $\Upsilon = \{v \in [0, B]^{\mathcal{S}^+} : v(g) = 0\}$. Let $f : \Delta_{\mathcal{S}^+} \times \Upsilon \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with $f(p, v, n, B, \iota) = pv - \max\left\{c_1\sqrt{\frac{\mathbb{V}(p,v)\iota}{n}}, c_2\frac{B\iota}{n}\right\}$, with $c_1 = 7$ and $c_2 = 49$. Then f satisfies for all $p \in \Delta_{\mathcal{S}^+}, v \in \Upsilon$ and $n, \iota > 0$,*

1. *$f(p, v, n, B, \iota)$ is non-decreasing in $v(s)$, that is,*

$$\forall v, v' \in \Upsilon, v(s) \le v'(s), \forall s \in \mathcal{S}^+ \implies f(p, v, n, B, \iota) \le f(p, v', n, B, \iota);$$

2. *$f(p, v, n, B, \iota) \le pv - \frac{c_1}{2}\sqrt{\frac{\mathbb{V}(p,v)\iota}{n}} - \frac{c_2}{2}\frac{B\iota}{n} \le pv - 3\sqrt{\frac{\mathbb{V}(p,v)\iota}{n}} - 24\frac{B\iota}{n}$.*

Lemma 30. *(Jaksch et al., 2010, Lemma 19), (Cohen et al., 2020, Lemma B.18) For any sequence of numbers $z_1, \ldots, z_n$ with $0 \le z_t \le Z_{t-1} = \max\{1, \sum_{i=1}^{t-1} z_i\}$:*

$$\sum_{t=1}^{n} \frac{z_t}{Z_{t-1}} \le 2\ln Z_n, \quad \sum_{t=1}^{n} \frac{z_t}{\sqrt{Z_{t-1}}} \le 3\sqrt{Z_n}.$$

E. Concentration Inequalities

Lemma 31. *(Cohen et al., 2020, Theorem D.1) Let $\{X_t\}_t$ be a martingale difference sequence such that $|X_t| \le B$. Then with probability at least $1 - \delta$,*

$$\left|\sum_{t=1}^{n} X_t\right| \le B\sqrt{n\ln\frac{2n}{\delta}}, \quad \forall n \ge 1.$$

Lemma 32. *Let $\{X_t\}_t$ be a sequence of i.i.d random variables with mean μ, variance σ^2, and $0 \le X_t \le B$. Then with probability at least $1 - \delta$, the following holds for all $n \ge 1$ simultaneously:*

$$\left|\sum_{t=1}^{n}(X_t - \mu)\right| \le 2\sqrt{2\sigma^2 n\ln\frac{2n}{\delta}} + 2B\ln\frac{2n}{\delta}.$$
$$\left|\sum_{t=1}^{n}(X_t - \mu)\right| \le 2\sqrt{2\hat{\sigma}_n^2 n\ln\frac{2n}{\delta}} + 19B\ln\frac{2n}{\delta}.$$

where $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{t=1}^{n} X_t^2 - (\frac{1}{n}\sum_{t=1}^{n} X_t)^2$.

Proof. For a fixed n, the first inequality holds with probability at least $1 - \frac{\delta}{4n^2}$ by Freedman's inequality. Then by (Efroni et al., 2021, Lemma 19), with probability at least $1 - \frac{\delta}{4n^2}$, $|\sigma - \hat{\sigma}_n| \le \sqrt{\frac{36B^2\ln(2n/\delta)}{n^+}}$. Therefore, $\sqrt{n}\sigma = \sqrt{n}\hat{\sigma}_n + \sqrt{n}(\sigma - \hat{\sigma}_n) \le \sqrt{n}\hat{\sigma}_n + 6B\sqrt{\ln(2n/\delta)}$. Plugging this back to the first inequality gives the second inequality. □

Lemma 33. *(Strengthened Freedman's inequality) Let $X_{1:\infty}$ be a martingale difference sequence with respect to a filtration $\{\mathcal{F}_t\}_t$ such that $\mathbb{E}[X_t|\mathcal{F}_{t-1}] = 0$. Suppose $B_t \in [1, b]$ for a fixed constant b, $B_t \in \mathcal{F}_{t-1}$ and $X_t \le B_t$ almost surely. Then for a given n, with probability at least $1 - \delta$:*

$$\left|\sum_{t=1}^{n} X_t\right| \le C\left(\sqrt{8V_{1,n}\ln(2C/\delta)} + 5B_{1,n}\ln(2C/\delta)\right), \tag{14}$$

and with probability at least $1 - \delta$ we have for all $1 \le l \le n$ simultaneously

$$\left|\sum_{t=l}^{l+n-1} X_t\right| \le C\left(\sqrt{8V_{l,n}\ln(4Cn^3/\delta)} + 5B_{l,n}\ln\left(4Cn^3/\delta\right)\right) \le 8CB_{l,n}\sqrt{n}\ln(4Cn^3/\delta), \tag{15}$$

where $V_{l,n} = \sum_{t=l}^{l+n-1}\mathbb{E}[X_t^2|\mathcal{F}_{t-1}]$, $B_{l,n} = \max_{l\le t<l+n} B_t$, and $C = \lceil\ln(b)\rceil\lceil\ln(nb^2)\rceil$.

Proof. Eq. (14) is simply from applying (Lee et al., 2020, Theorem 2.2) to $\{X_t\}_t$ and $\{-X_t\}_t$. Fix some $l, n \geq 1$. Eq. (15) holds with probability at least $1 - \frac{\delta}{2n^3}$ by Eq. (14). By a union bound (first sum over l, then sum over n), the statement is proved. $\square$

Lemma 34. *Given $\alpha \geq 1$ and a martingale sequence $\{X_t\}_t$ such that $X_t \in \mathcal{F}_t, 0 \leq X_t \leq B$, with probability at least $1-\delta$:*

$$\sum_{t=1}^{n} \mathbb{E}[X_t|\mathcal{F}_{t-1}] \leq \left(1 + \frac{1}{\alpha}\right) \sum_{t=1}^{n} X_t + 8B\alpha \ln \frac{2n}{\delta}, \quad \forall n \geq 1.$$

Proof. Define $Y_t = \mathbb{E}[X_t|\mathcal{F}_{t-1}] - X_t$. For a given n, by Freedman's inequality, with probability at least $1 - \frac{\delta}{2n^2}$:

$$\sum_{t=1}^{n} Y_t \leq \eta \sum_{t=1}^{n} \mathbb{E}[(X_t - \mathbb{E}[X_t|\mathcal{F}_{t-1}])^2|\mathcal{F}_{t-1}] + \frac{2\ln(2n/\delta)}{\eta} \leq B\eta \mathbb{E}[X_t|\mathcal{F}_{t-1}] + \frac{2\ln(2n/\delta)}{\eta},$$

for some $\eta < \frac{1}{B}$. Reorganizng terms, we get when $\eta = \frac{1}{2B\alpha} < \frac{1}{B}$ (note that $B\eta \leq \frac{1}{2}$):

$$\begin{aligned}\sum_{t=1}^{n} \mathbb{E}[X_t|\mathcal{F}_{t-1}] &\leq \frac{1}{1-B\eta}\left(\sum_{t=1}^{n} X_t + \frac{2\ln(2n/\delta)}{\eta}\right) \leq (1+2B\eta)\sum_{t=1}^{n} X_t + \frac{4\ln(2n/\delta)}{\eta} \\ &\leq \left(1+\frac{1}{\alpha}\right)\sum_{t=1}^{n} X_t + 8B\alpha \ln\frac{2n}{\delta}. \qquad \left(\tfrac{1}{1-x} \leq 1+2x \text{ when } x \in [0, \tfrac{1}{2}]\right)\end{aligned}$$

By a union bound over n, we obtain the desired bound. $\square$

F. Experiments

In this section, we benchmark known SSP algorithms empirically. We consider two environments, RandomMDP and GridWorld. In RandomMDP, there are 5 states and 2 actions, and both transition and cost function are chosen uniformly at random. In GridWorld, there are 12 states (including the goal state) and 4 actions (LEFT, RIGHT, UP, DOWN) forming a 3×4 grid. The agent starts at the upper left corner of the grid, and the goal state is at the lower right corner of the grid. Taking each action initiate an attempt to moves one step towards the indicated direction with probability 0.85, and moves randomly towards the other three directions with probability 0.15. The movement attempt fails if the agent tries to move out of the grid, and in this case the agent stays at the same position. The cost is 1 for each state-action pair.

We implement two model-free algorithms: Q-learning with ϵ-greedy exploration (Yu and Bertsekas, 2013) and LCB-ADVANTAGE-SSP, and five model-based algorithms: UC-SSP (Tarbouriech et al., 2020a)[5], Bernstein-SSP (Cohen et al., 2020), ULCVI (Cohen et al., 2021), EB-SSP (Tarbouriech et al., 2021b), and SVI-SSP. For each algorithm, we optimize hyper-parameters for the best possible results. Moreover, instead of incorporating the logarithmic terms from confidence intervals suggested by the theory, we treat it as a hyper-parameter ι and search its best value. The hyper-parameters used in the experiments are shown in Table 2. All experiments are performed in Google Cloud Platform on a compute engine with machine type "e2-medium".

The plot of accumulated regret is shown in Figure 1. Q-learning with ϵ-greedy exploration suffers linear regret, indicating that naive ϵ-greedy exploration is inefficient. UC-SSP and SVI-SSP show competitive results in both environments. SVI-SSP also consistently outperforms EB-SSP, both of which are minimax-optimal and horizon-free.

In Table 1, we also show the time spent in updates (policy, accumulators, etc) in the whole learning process for each algorithm. Our model-based algorithm SVI-SSP spends least time in updates among all algorithms, confirming our theoretical arguments. ULCVI and UC-SSP spend most time in updates, which is reasonable since these two algorithms computes a new policy in each episode, instead of exponentially sparse updates.

[5]we implement a variant of UC-SSP with a fixed pivot horizon for a much better empirical performance, where $\gamma_{k,j} = 10^{-6}$ always (see their Algorithm 2 for the definition of $\gamma_{k,j}$)

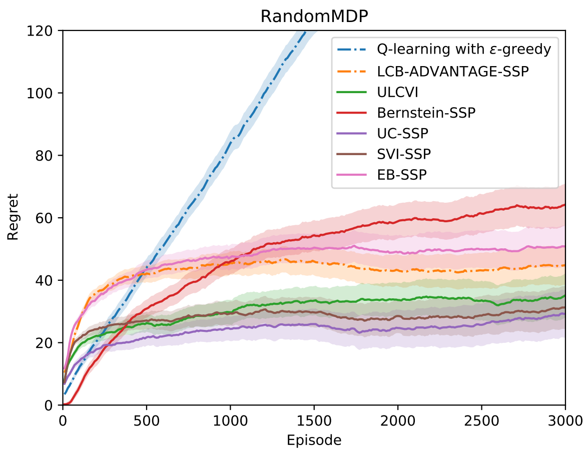

Figure 1. Accumulated regret of each algorithm on RandomMDP (left) and GridWorld (right) in 3000 episodes. Each plot is an average of 500 repeated runs, and the shaded area is 95% confidence interval. Dotted lines represent model-free algorithms and solid lines represent model-based algorithms.

Table 1. Average time (in seconds) spent in updates in 3000 episodes for each algorithm. Our model-based algorithm SVI-SSP is the most efficient algorithm.

	RandomMDP	GridWorld
Q-learning with ϵ-greedy	0.3385	0.3773
LCB-ADVANTAGE-SSP	0.3517	0.3982
UC-SSP	14.4472	8.6886
Bernstein-SSP	0.2918	0.4656
ULCVI	15.7128	22.8062
EB-SSP	0.2319	0.4619
SVI-SSP	**0.1207**	**0.1419**

Table 2. Hyper-parameters used in the experiments. We search the best parameters for each algorithm.

	Algorithm	Parameters
RandomMDP	Q-learning with ϵ-greedy	$\epsilon = 0.05$
	LCB-ADVANTAGE-SSP	$H = 5, \iota = 0.05, \theta^\star = 4096$
	UC-SSP	$\iota = 1.0$
	Bernstein-SSP	$\iota = 2.0$
	ULCVI	$H = 80, \iota = 2.0$
	EB-SSP	$\iota = 0.05$
	SVI-SSP	$H = 15, \iota = 0.05$
GridWorld	Q-learning with ϵ-greedy	$\epsilon = 0.05$
	LCB-ADVANTAGE-SSP	$H = 5, \iota = 0.1, \theta^\star = 4096$
	UC-SSP	$\iota = 0.5$
	Bernstein-SSP	$\iota = 0.5$
	ULCVI	$H = 100, \iota = 1.0$
	EB-SSP	$\iota = 0.01$
	SVI-SSP	$H = 10, \iota = 0.01$