Diffusion and Strategic Interaction on Social Networks

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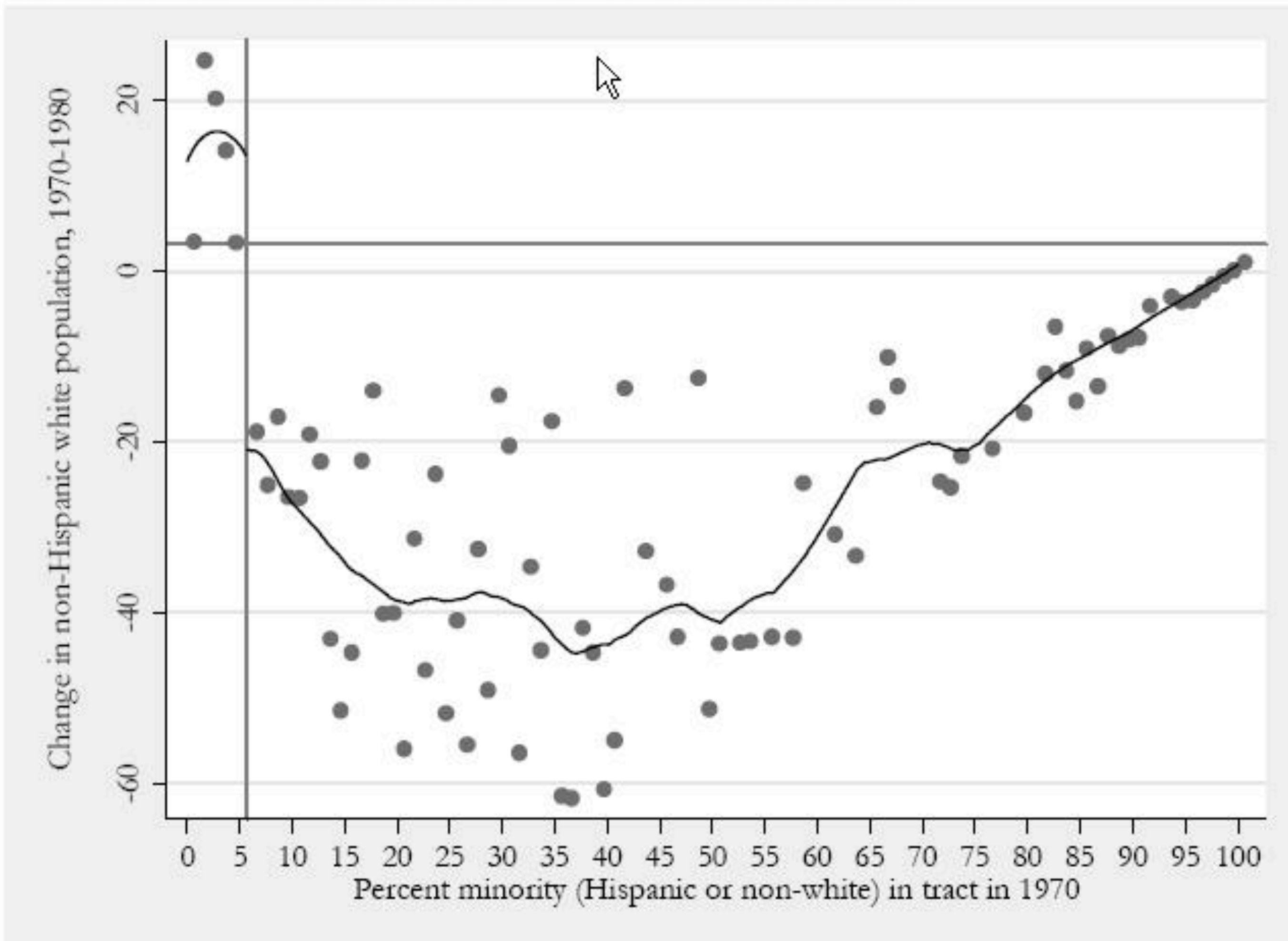
  - How does network structure impact behavior and welfare?
  - How does relative location in a network impact behavior and welfare?
- How does behavior propagate through networks (important for marketing, epidemiology, etc.)?

Choices without Networks: Some History

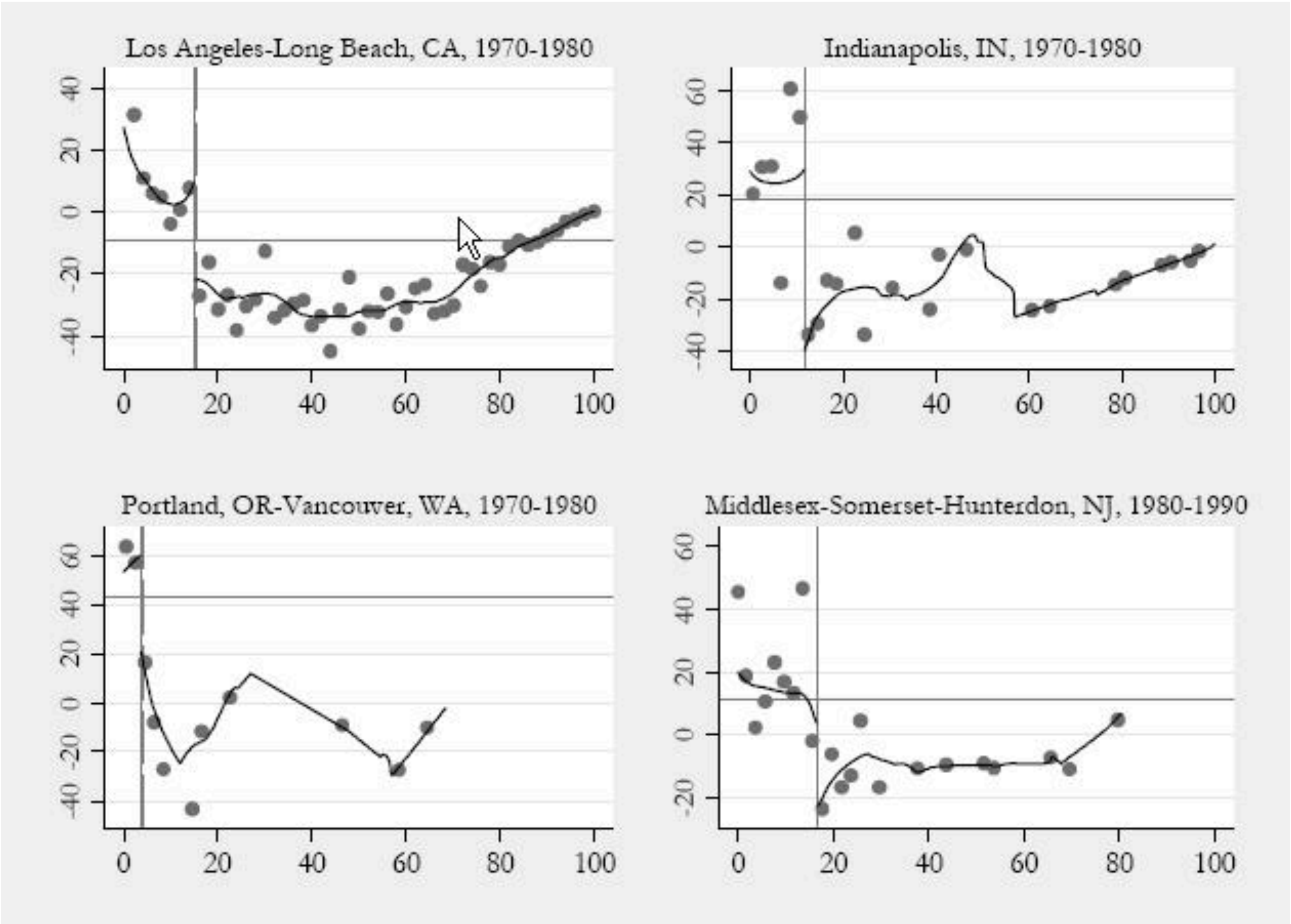
- Morton Grodzins (1957) coined the term *tipping point* when explaining white flight
- **Tipping Point:** A time in which a large number of individuals rapidly and dramatically change behavior

Card, Mas, and Rothstein (2008)

Figure 1: Neighborhood change in Chicago, 1970-1980

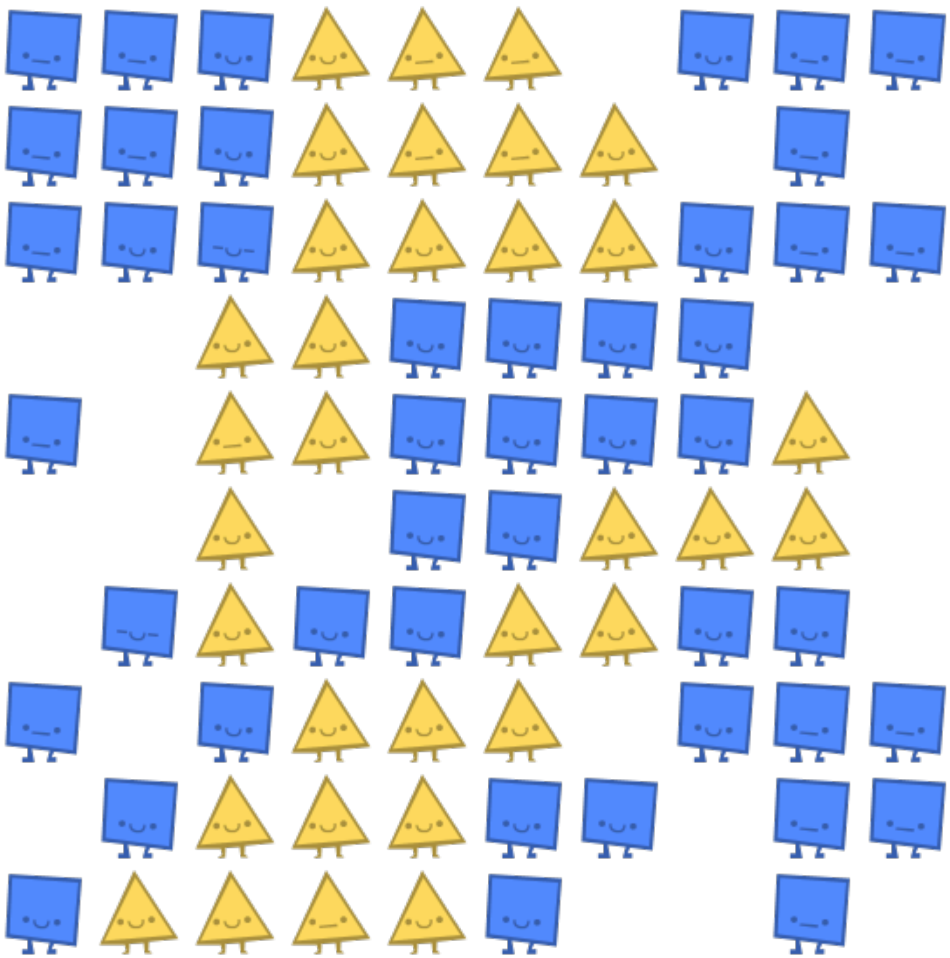


In Other Cities as Well



Schelling (1969, 1971)

- "A general theory of tipping"



Granovetter (1978): Threshold Models of Collective Behavior

- N agents, all connected
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- When agent *i* faces a profile $(x_1, x_2, \ldots, x_n)$,

$$u_i(x_1, \ldots, x_N) = \left[\frac{\sum_{j \neq i} x_j}{N-1} - c_i \right] x_i$$

- c_i is cost of choosing 1, distributed according to a continuous F over [0,1]

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  (could be privately or commonly known)

  (analogous to random thresholds)

Granovetter (1978)

- Assume at each stage agents best respond to previous period’s action distribution

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- Suppose at period t a fraction x^t choose 1

- At period $t+1$, adopt $\leftrightarrow$

$$\frac{Nx^t - x_i^t}{N-1} \geq c_i$$

- For N large,

$$\frac{Nx^t - x_i^t}{N-1} \approx x^t$$

Equilibrium in Granovetter

- Approximated transition formula:

$$x^{t+1} \approx F(x^t)$$

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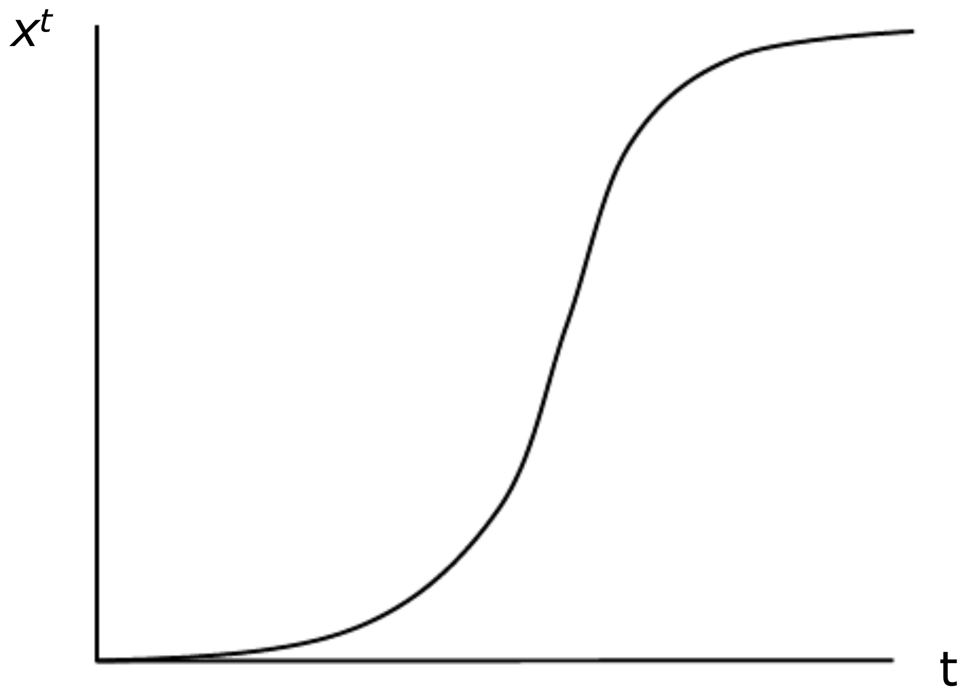
- The shape of F determines which equilibria are tipping points (more formal soon)
- [Contrast with Bass' F(t)]

S-shape Adoption in Granovetter (1978)

- Level of change:

$$\Delta(x^t) = F(x^t) - x^t$$



S-shape Adoption in Granovetter (1978)

- Level of change:

$$\Delta(x^t) = F(x^t) - x^t$$

- Assume *F* differentiable
- The derivative of *F(x)-x* is *F′(x)-1*
- S-shape → *F′(x) > 1* up to some point and *F′(x) < 1* afterwards

Introducing Networks

- So far, no networks
- How does network architecture affect diffusion?
- How does location within a network affect diffusion (recall the adoption of Tetracycline…)?

Example - Experimentation

Knowing the Network Structure

- Suppose you gain 1 if anyone experiments, 0 otherwise, but experimentation is costly (grains, software, etc.)

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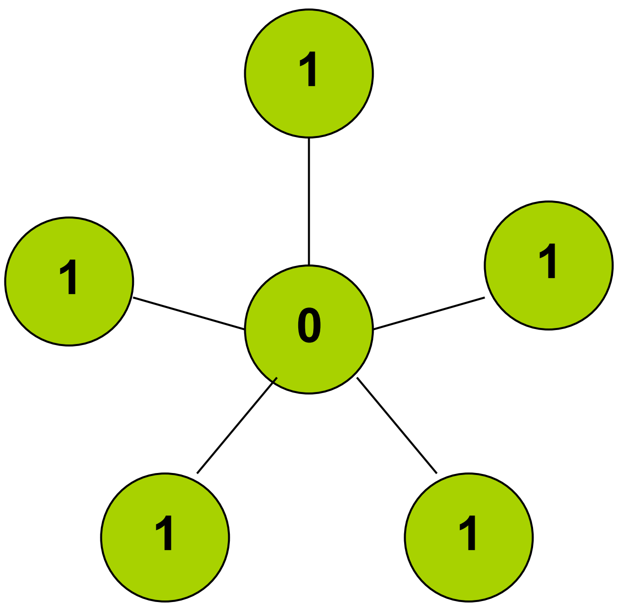
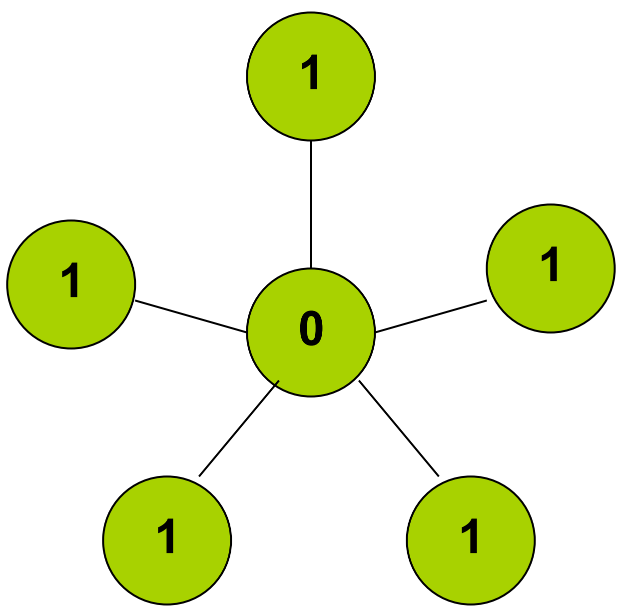
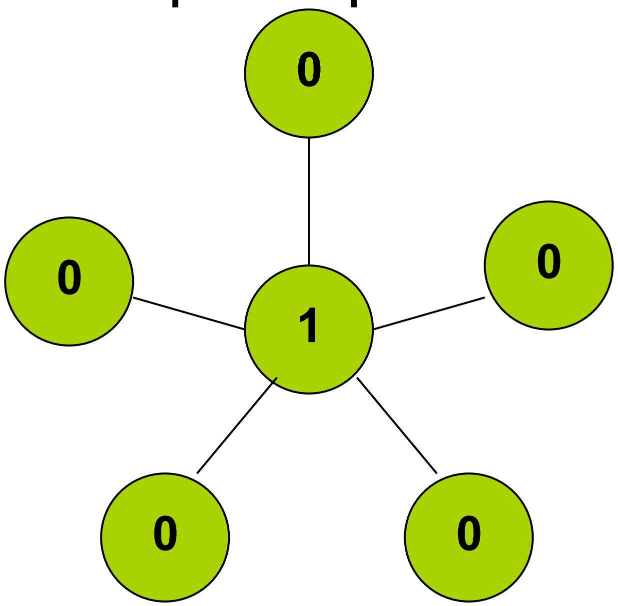


Example - Experimentation

- Suppose you gain 1 if anyone experiments, 0 otherwise, but experimentation is costly (grains, software, etc.) EXPERIMENTATION – 1
  NO EXPERIMENTATION - 0

- Network structure known – multiple equilibria:





Example – Experimentation (2)

Not knowing the structure

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- Probability p of a link between any two agents (Poisson..)

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- Probability that a neighbor experiments independent of own degree (number of neighbors)
  - → Higher degree less willing to choose 1
  - → Threshold equilibrium: low degrees experiment, high degrees do not.

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- Probability that a neighbor experiments independent of own degree (number of neighbors)
  - → Higher degree less willing to choose 1
  - → Threshold equilibrium: low degrees experiment, high degrees do not.
- Strong dependence on p
  - $p=0 \rightarrow$ all choose 1,
  - $p=1 \rightarrow$ all choose 1 with probability $1-c^{1/(n-1)}$

General Messages

- **Information Matters**

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  - Monotonicity with respect to degrees
    - Regarding behavior (complementarities…)
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- Information Matters
- Location Matters
  - Monotonicity with respect to degrees
    - Regarding behavior (complementarities…)
    - Regarding expected benefits (externalities…)
- **Network Structure Matters**
  - Adding links affects behavior monotonically (complementarities…)
  - Increasing heterogeneity?

Challenge

- Complexity of networks
- Tractable way to study behavior outside of simple (regular structures)?

Focus on key characteristics:

- Degree Distribution
- How connected is the network?
  - average degree, FOSD shifts
- How are links distributed across agents?
  - variance, skewness, etc.

Network Games – Payoff Structure

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Network Games – Payoff Structure

- Actions in {0,1}
- **Action 0**: Normalize payoff to 0
- **Action 1**: payoffs depend on number of neighbors choosing 1
- $\mathbf{v(d,x) - c_i}$ payoff from choosing 1 if degree is d and a fraction x of neighbors choose 1
- c_i distributed according to H, with no atoms

Examples (payoff: $v(d,x)-c$)

- **Average Action**: $v(d,x)=v(d)x= x$

  (classic coordination games, choice of technology)

- **Total Number**: $v(d,x)=v(d)x=dx$

  (learn a new language, need partners to use new good or technology, need to hear about it to learn)

- **Critical Mass**: $v(d,x)=0$ for x up to some M/d and $v(d,x)=1$ above M/d

  (uprising, voting, …)

- **Decreasing**: $v(d,x)$ declining in d

  (information aggregation, lower degree correlated with leaning towards adoption)

Network Games – Information

- (today) Incomplete information
  - know only own degree and assume others' types are governed by degree distribution
  - presume no correlation in degree
  - Bayesian equilibrium – as function of degree

Incomplete Information

- g drawn from some set of networks G such that (assuming large population):
  - degrees of neighbors are independent
  - Probability of any node having degree d is p(d)
  - probability of given neighbor having degree d is **P(d)=dp(d)/E(d)**

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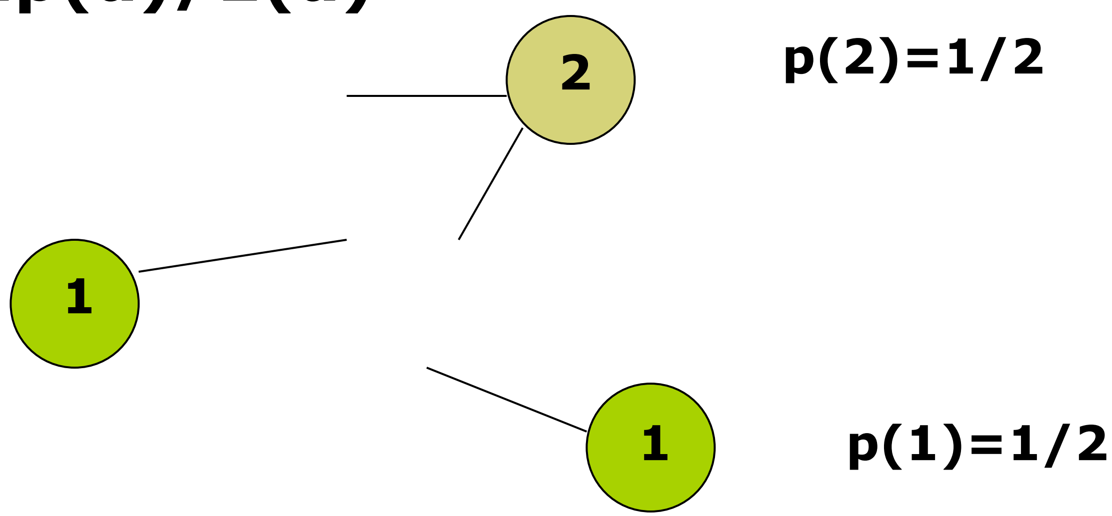

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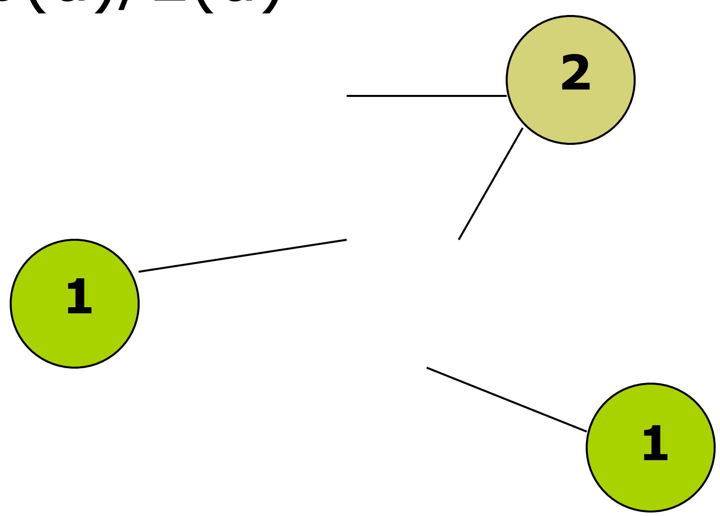

Probability of hitting **2** is twice as high as that of hitting **1** → **P(2)=2/3.**

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- type of i is ($d_i(g)$, c_i); space of types T_i

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- type of i is ($d_i(g)$, c_i); space of types T_i

- strategy: $\sigma_i: T_i \to \Delta(X)$

Equilibrium as a fixed point:

- Adopt if and only if $v(d,x) - c_i \geq 0$

- $H(v(d,x))$ is the percent of degree d types adopting action 1 if x is fraction of random neighbors adopting

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$$\mathbf{x = \varphi(x) = \Sigma\, P(d)\, H(v(d,x))}$$

$$\mathbf{= \Sigma\, d\, p(d)\, H(v(d,x)) \,/\, E[d]}$$

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- **v continuous in x → fixed point exists**

Monotone Behavior

Observation 1:

In a game of incomplete information, every symmetric equilibrium is monotone

- Non-decreasing in degree if v(d,x) is increasing in d
- Non-increasing in degree if v(d,x) is decreasing in d

Monotone Behavior

Intuition

- Symmetric equilibrium – a random neighbor has probability x of choosing 1, probability 1-x of choosing 0

Monotone Behavior

Intuition

- Symmetric equilibrium – a random neighbor has probability x of choosing 1, probability 1-x of choosing 0
- Consider agent of degree d+1
  - v(d,x) non-decreasing → payoff from 1 is $v(d+1,x) \geq v(d,x)$
  - v(d,x) non-increasing → payoff from 1 is $v(d+1,x) \leq v(d,x)$

Diffusion

$$\mathbf{x = \varphi(x) = \Sigma\, P(d)\, H(v(d,x))}$$

- start with some x^0
- let $x^1 = \varphi(\, x^0)$, $x^t = \varphi(x^{t-1})$, ...

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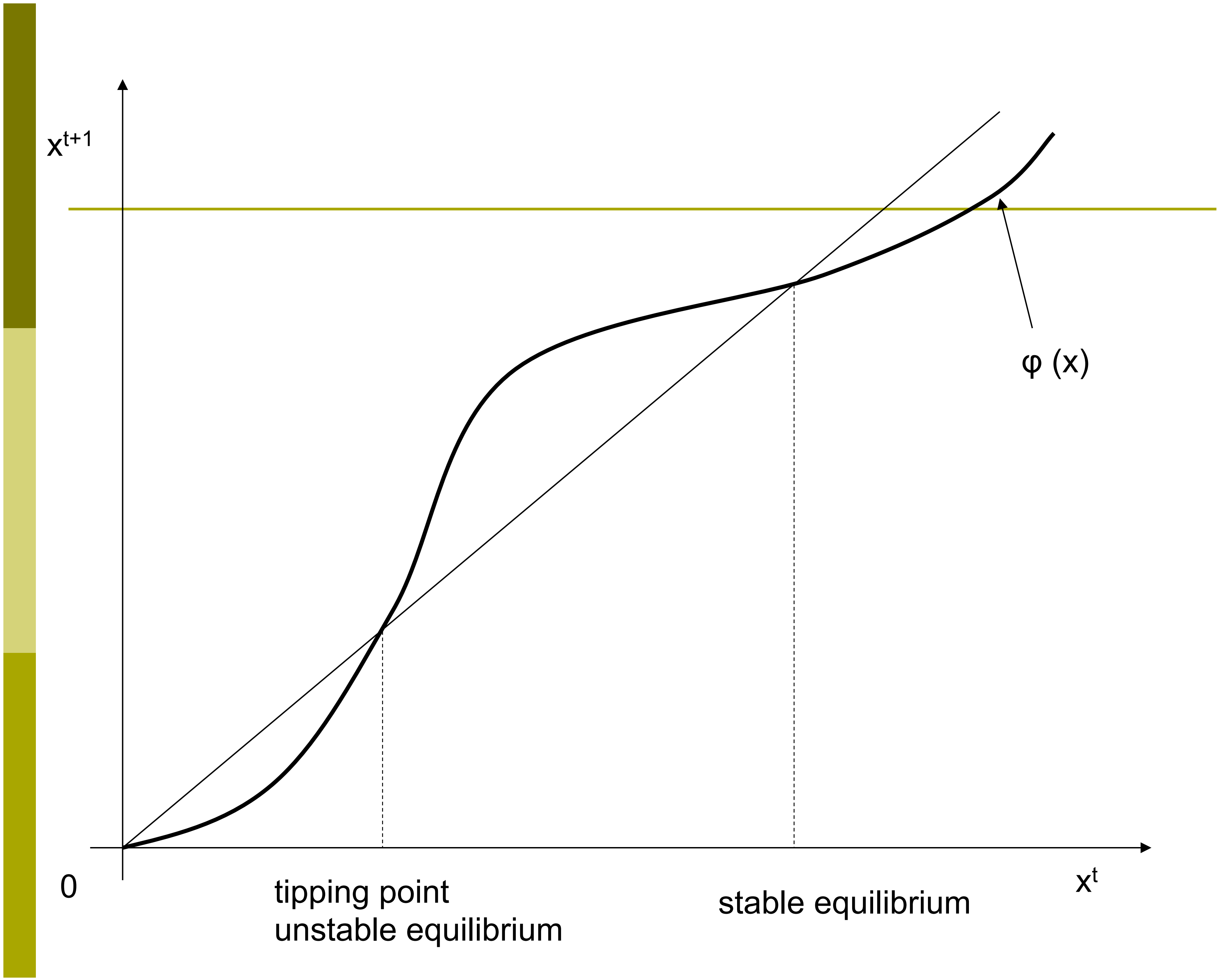
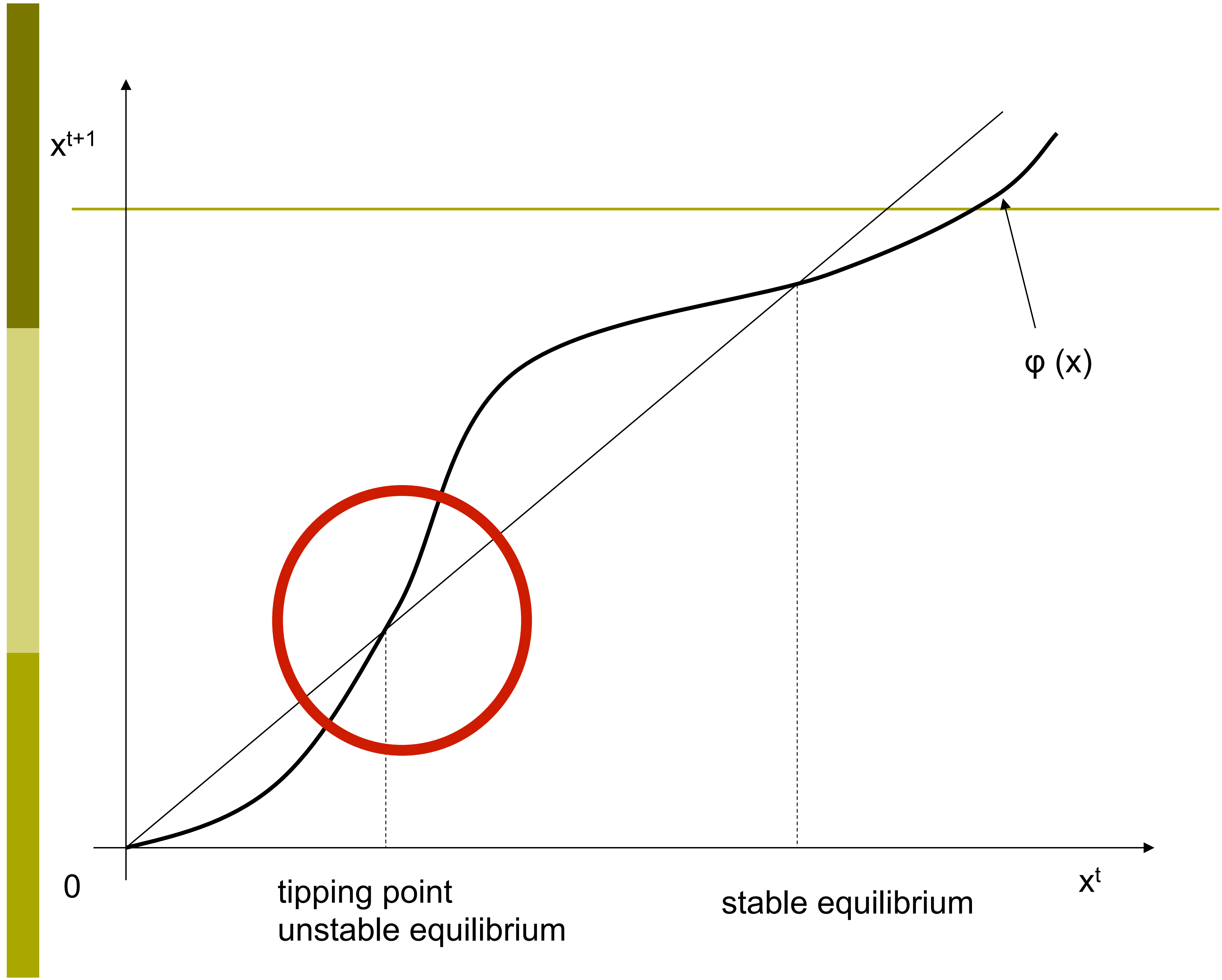
- start with some x^0
- let $x^1 = \varphi(x^0)$, $x^t = \varphi(x^{t-1})$, …
- **Interpretations**
  - examining equilibrium set with incomplete information
    - Stable equilibria are converged to from above and below
  - looking at diffusion: best response dynamics on "large, well-mixed" social network (mean-field approximation)



x^{t+1}
φ (x)
0
x^t
tipping point
unstable equilibrium
stable equilibrium



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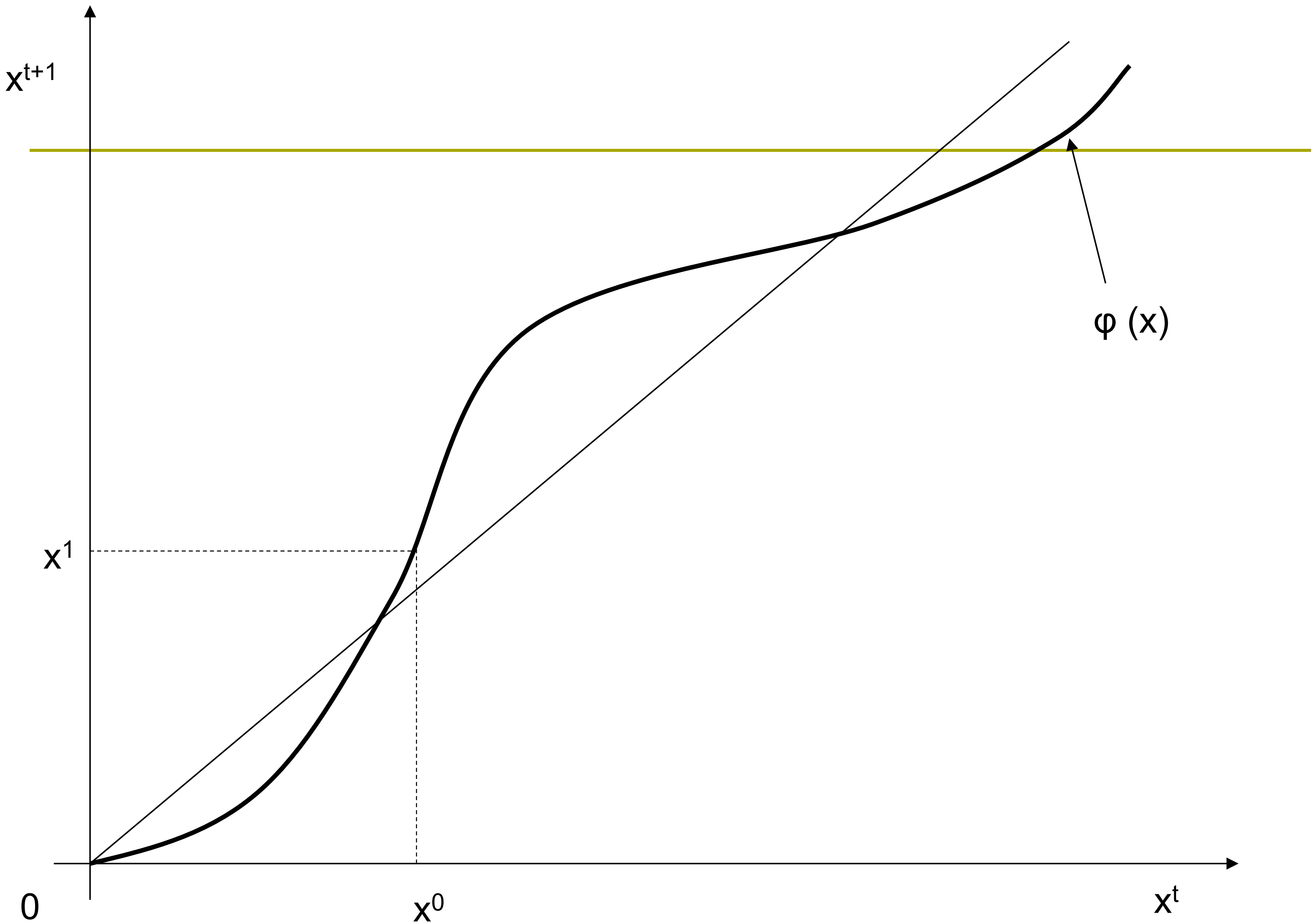
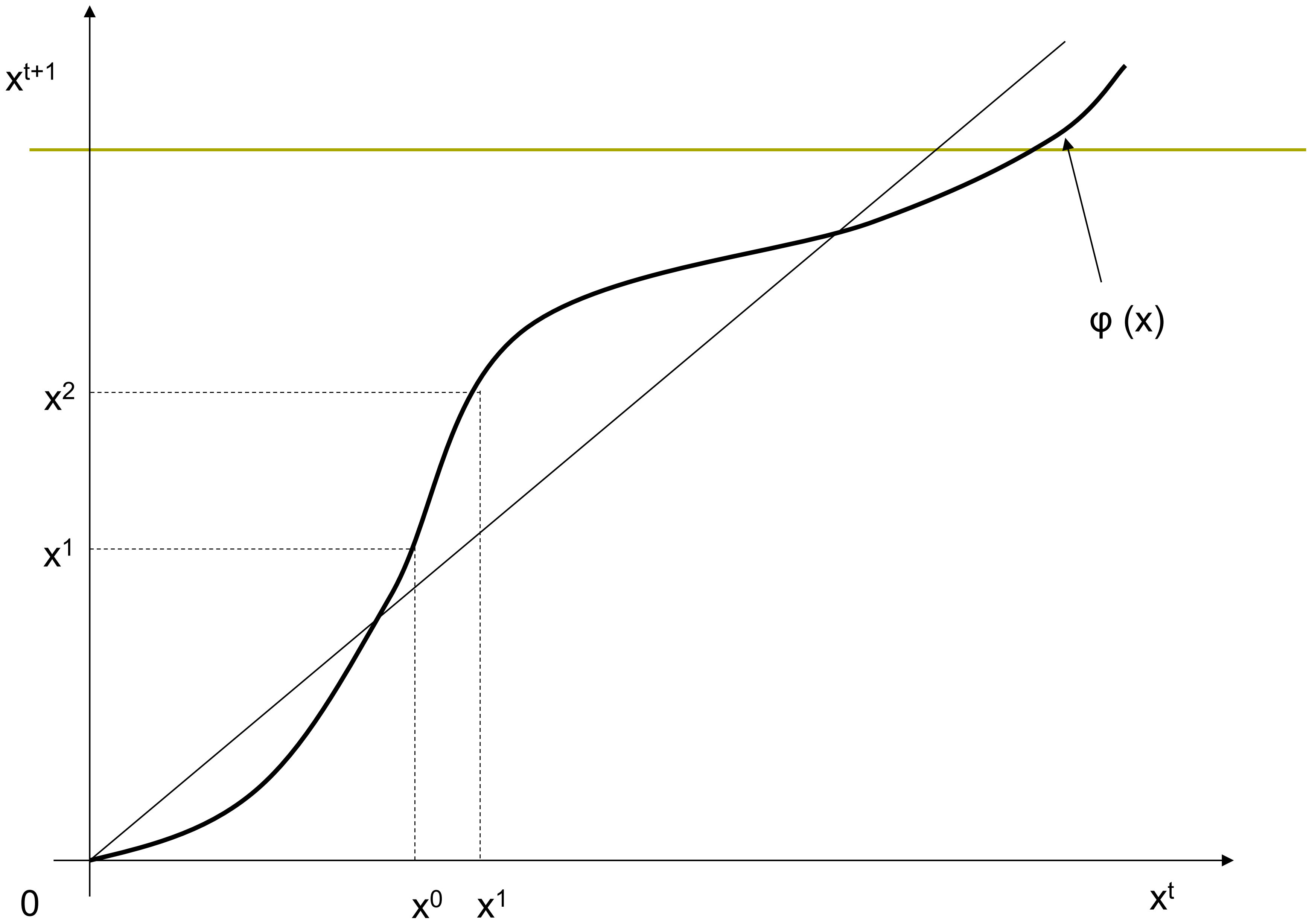


x^t+1
x^1
0
x^0
x^t
φ (x)



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x^2
x^1
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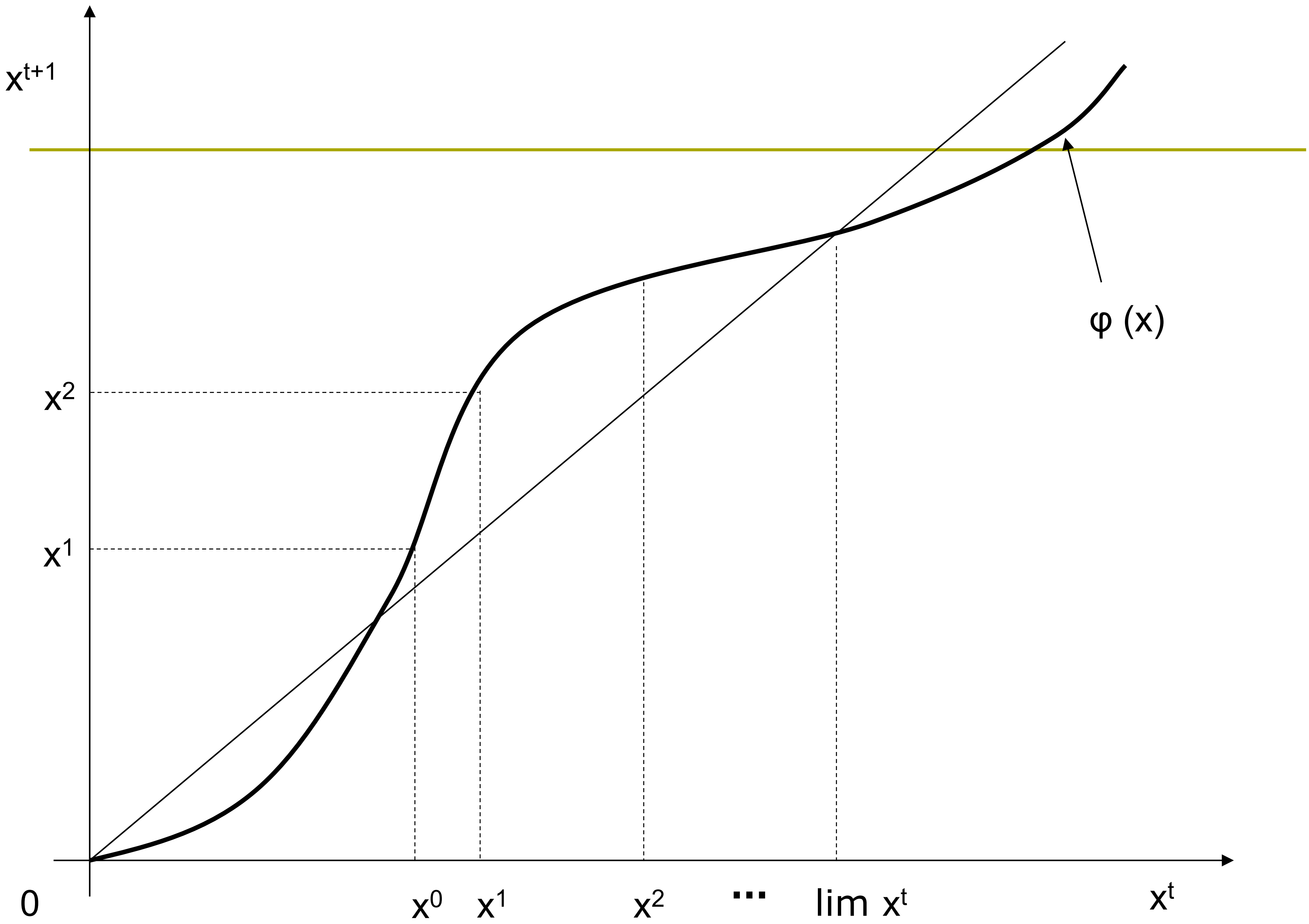
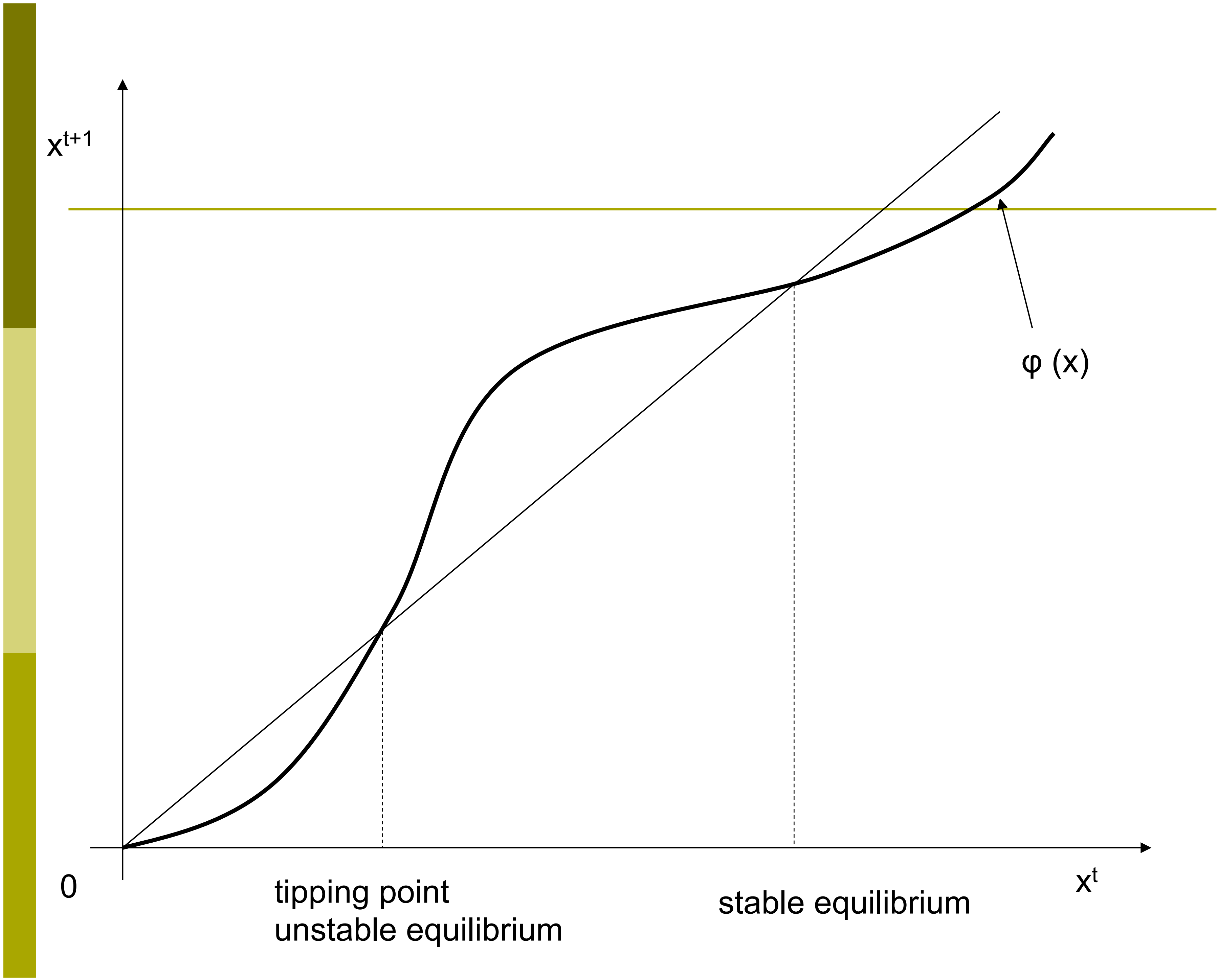
x^0
x^1
x^t
φ (x)



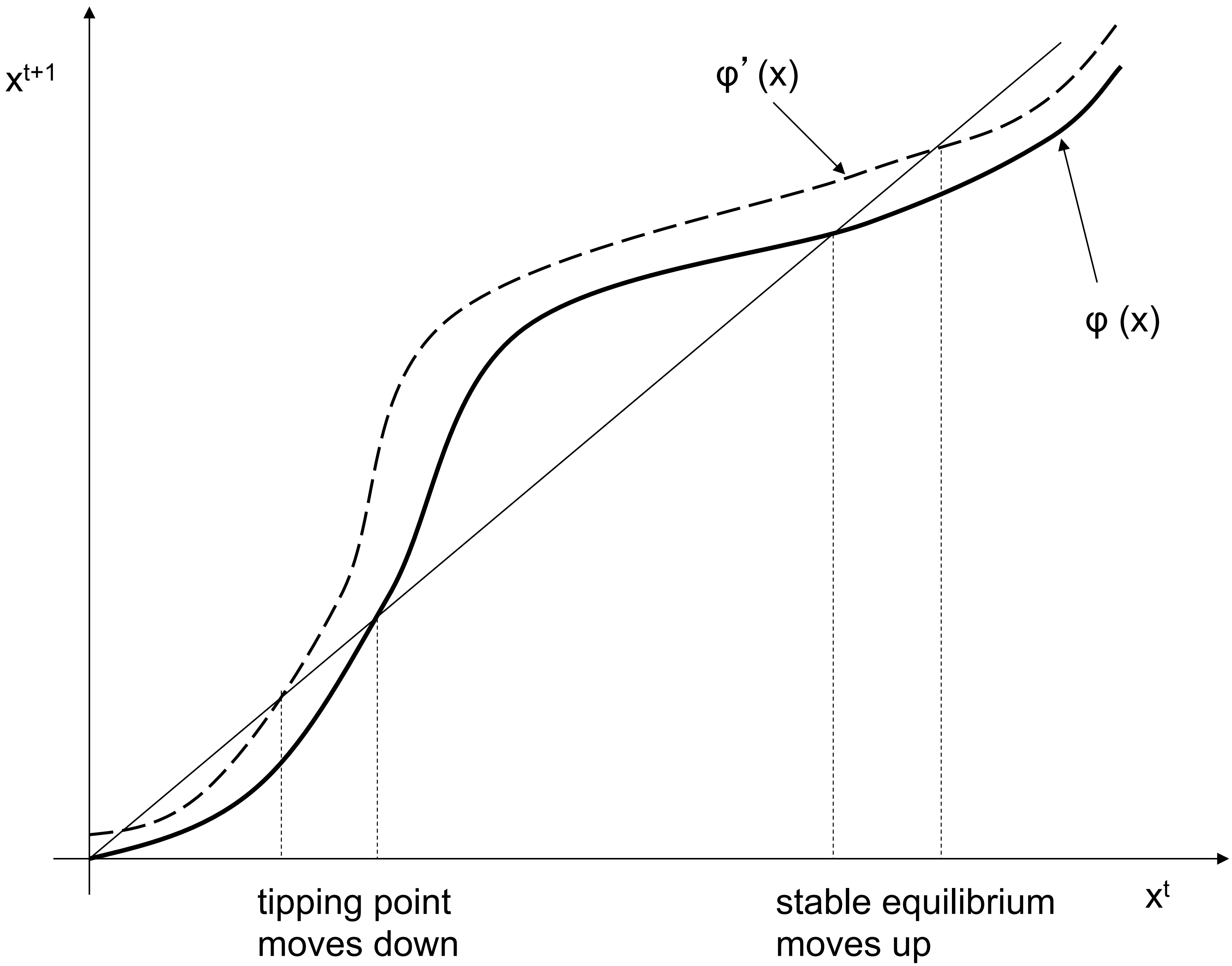
x^{t+1}
x^2
x^1
0
x^0
x^1
x^2
...
$\lim x^t$
x^t
$\varphi(x)$



x^{t+1}
φ (x)
0
x^t
tipping point
unstable equilibrium
stable equilibrium

How can we relate structure (network or payoff) to diffusion?

- Concentrate on “regular” environments – no tangencies
- Tipping and stable points alternate
- Keep track of how φ shifts with changes



x^{t+1}
φ' (x)
φ (x)
x^t
tipping point
moves down
stable equilibrium
moves up

Adding Links

- Consider a FOSD shift in distribution P(d)
  - More weight on higher degrees
  - v(d,x) non-decreasing in d → Higher expectations of higher actions (Observation 1)
  - More likely to take higher action

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- If v(d,x) is non-decreasing in d, then this leads to a point-wise increase of

$$\varphi(x) = \sum P(d)\, H(v(d,x))$$

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- Does not translate to FOSD shifts in p(d)

Adding Links – Welfare

- Suppose v(d,x) non-decreasing in x
- → FOSD shift in P increases payoffs for all agents corresponding to stable equilibria
- → Higher welfare

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- Suppose v(d,x) non-decreasing in x
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- → Higher welfare
- In general, monotonicity with respect to x (externalities) is important for welfare

Raising Costs

- Raising of costs of adoption of action 1 (FOSD shift of H) lowers $\varphi(x)$ pointwise
  - raises tipping points, lowers stable equilibria

Increasing Variance of Degrees

- v(d,x) increasing convex in d, H convex
  - e.g., v(d,x)=dx, H uniform[0,C] (with high C)

- p' is MPS of p implies φ(x) is pointwise higher under p'

- Roughly, increasing variance leads to lower tipping points and higher stable equilibria

- Fixing means,

$$\boldsymbol{\varphi^{power}(x) \geq \varphi^{Poisson}(x) \geq \varphi^{regular}(x)}$$

Can we relate the payoff structure to equilibrium?

- Assume $v(d,x)=v(d)x$
- Vary $v(d)$
- If we can influence v, whom should we target to shift equilibrium?

Proposition: impact of v(d)

Consider changing v(d) by rearranging its ordering

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Proposition: impact of v(d)

Consider changing v(d) by rearranging its ordering.

If p(d)d increasing, then v(d) increasing raises φ(x) pointwise (raises stable equilibria, lowers unstable)
[e.g., p is uniform]

If p(d)d decreasing, then v(d) decreasing raises φ(x) pointwise
[e.g., p is power]

Optimal Targeting

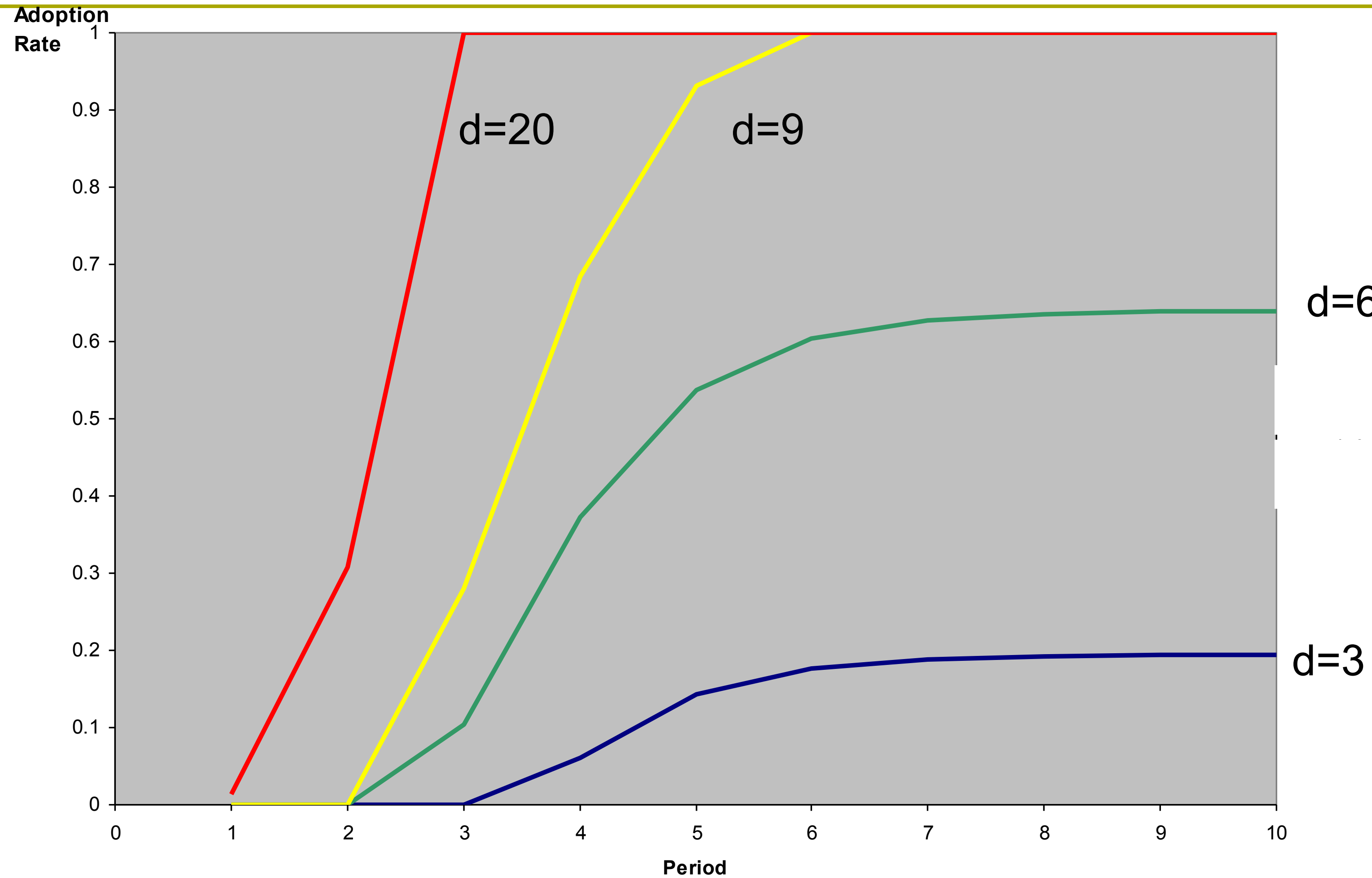
- Goes against idea of “targeting” high degree nodes
- Want the most probable neighbors to have the best incentives to adopt

Adoption Across Degrees

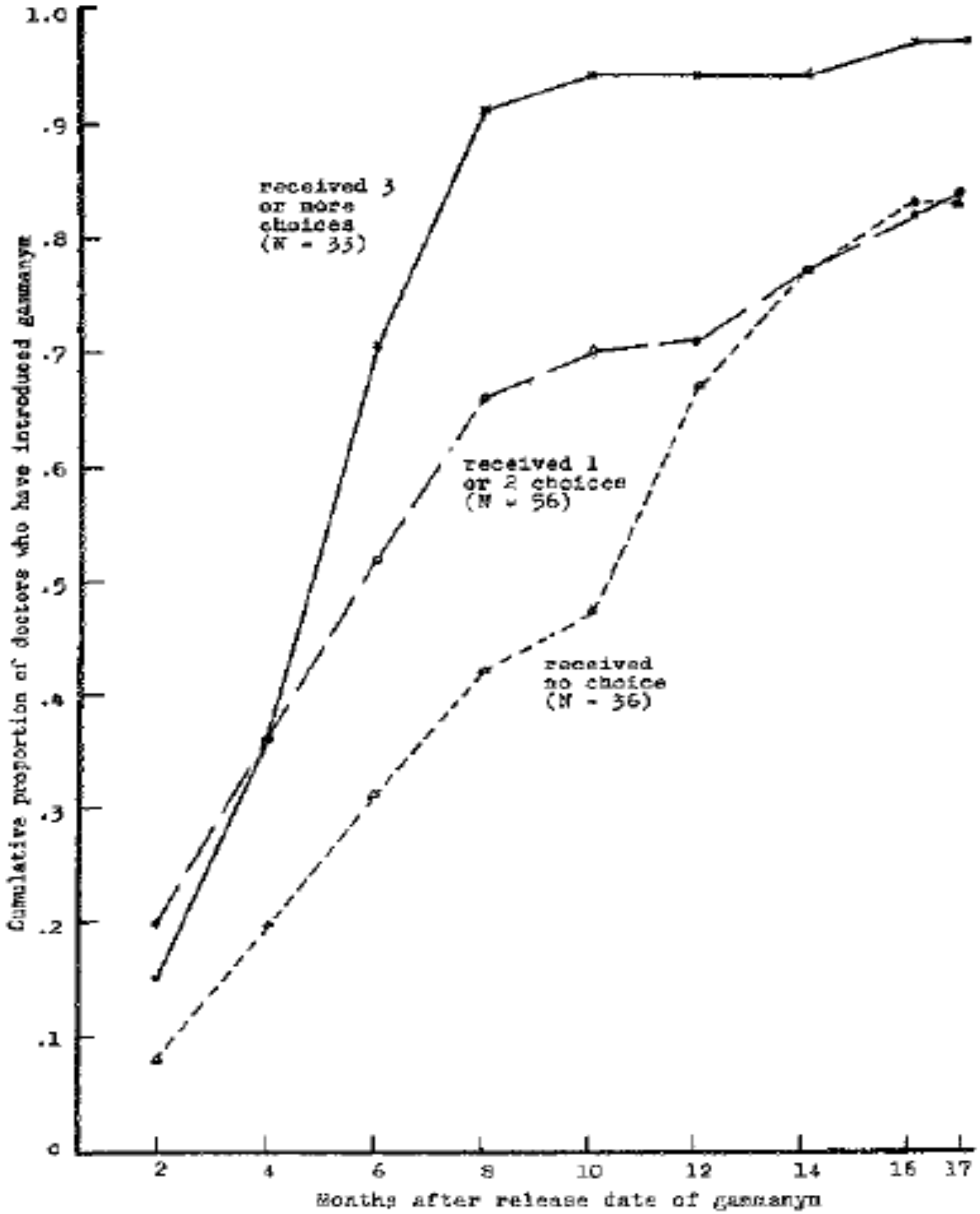
- If $v(d,x)$ is increasing in d, then higher d adopt in higher percentage for each x
  - adoption fraction is $H(v(d,x))$ which is increasing

- Patterns over time – depend on concavity of H

Diffusion Across Degrees



fraction adopting over time, power distribution exponent -2, initial seed x=.03, costs Uniform[1,5], v(d)=d



Tetracycline Adoption
(Coleman, Katz, and Menzel, 1966)

Summary:

- Location matters:
  - v(d,x) increasing in d
    - more connected adopt "earlier," at higher rate
    - have higher expected payoffs

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- Location matters:
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- Structure matters:
  - Lower tipping points, higher stable equilibria if:
    - lower costs (downward shift FOSD of H)
    - increase in connectedness (FOSD shift of P)
    - MPS of p if v, H (weakly) convex
    - match higher propensity v(d) to more prevalent degrees p(d)d (want *decreasing* v for power laws)
  - adoption speeds vary over time depending on curvature of the cost distribution

The End

Stability at 0

$\varphi(x)<x$ in a neighborhood around 0

(joint condition on H, $v(d,x)$, $P(d)$)

If H is continuous, and 0 is stable, then "generically": next unstable (first tipping point, where volume of adopters grows), next is stable, etc.

"Regular" environment: No tangencies

Speed of adoption over time

If $H(0)=0$ and H is C^2 and increasing

- If H is concave, then $\varphi(x)/x$ is decreasing
  - Convergence upward slows down, convergence downward speeds up
- If H is convex, then $\varphi(x)/x$ is increasing
  - Convergence upward speeds up, convergence downward slows down