Time Series Analysis Workshop

Practical Overview of State Space Models

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September 5th 2017

Outline

Motivation

State space models

Spacing out

Extensions

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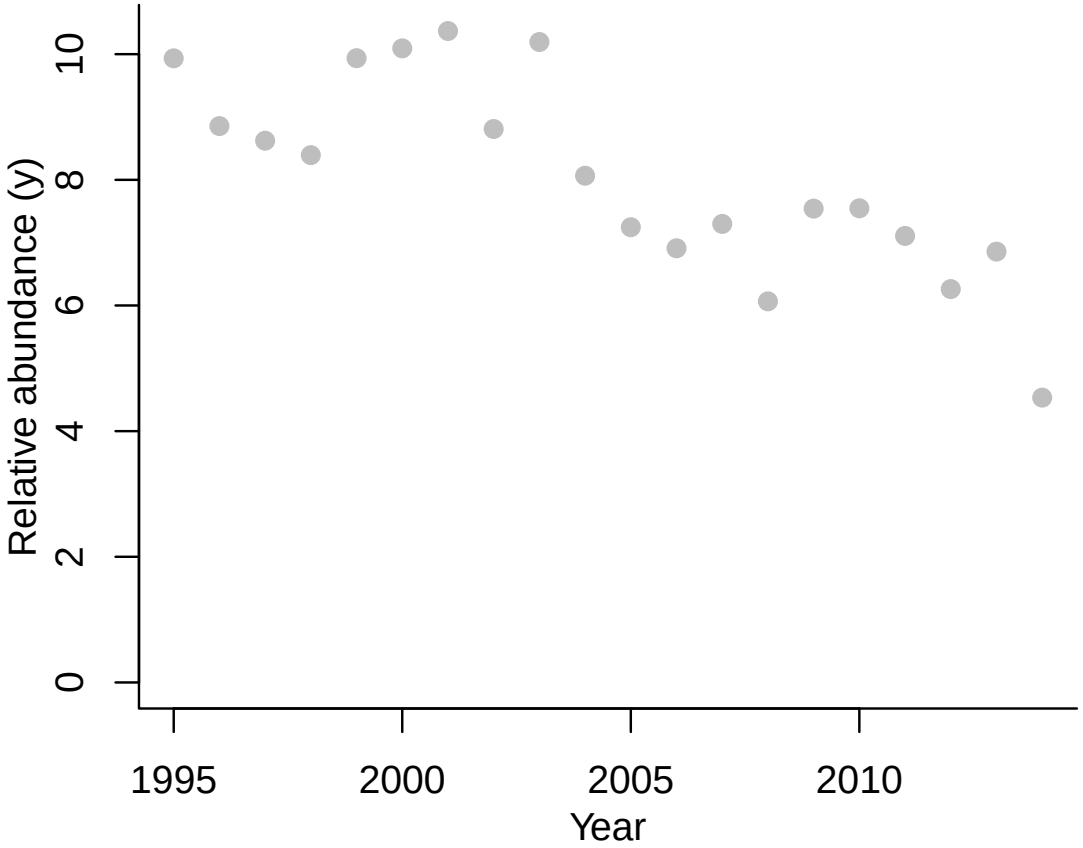
Extensions

Bread and butter

We often have data series that look like this



And are tasked with quantifying the trend

Measurement error only *traditional* option

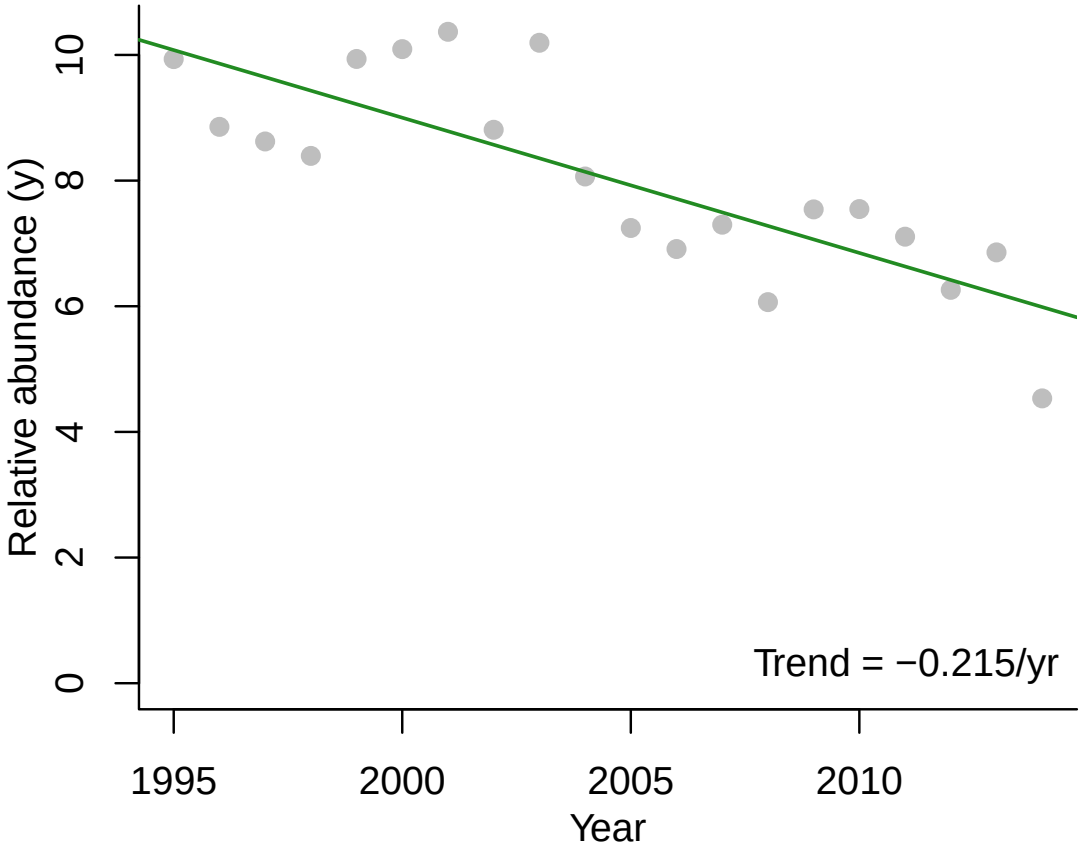
Regression likelihood

$$L(a, b, \sigma^2_{obs}|\mathbf{y}) = \prod_t \mathrm{N}(y_t - (a + bt), \sigma^2_{obs})$$

Maximize with respect to $\{a, b, \sigma^2_{obs}\}$ or let R's `lm` routine do it for us

Measurement error only *traditional* option

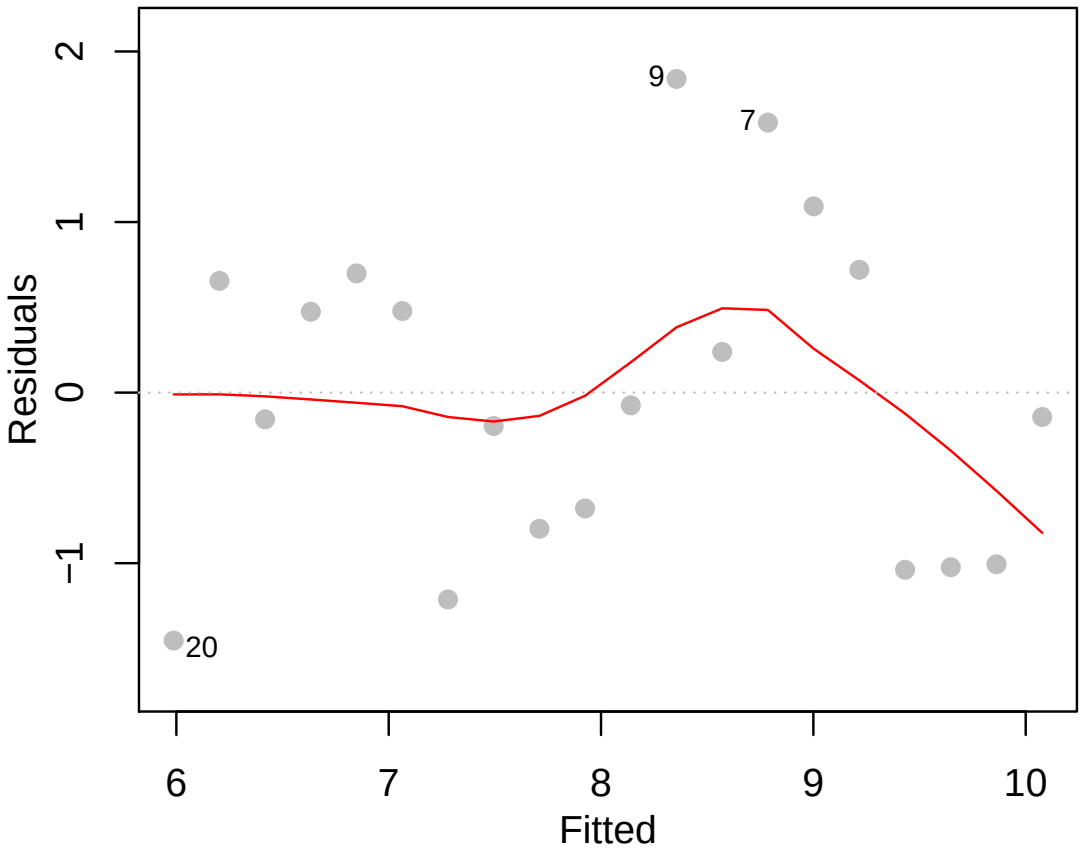
Treats all structure other than trend as observation error



Regression (GLM for other data types) approach
Shoots through entire series from initial conditions (intercept)

Measurement error only *traditional* option

Residual patterns exist for this fit



Indicating that this representation of the data might be off

Measurement error only *traditional* option

- Assumes the true model is often a smooth curve (or a surface if other variables included)
- All variability not captured by the line through time is assumed to be observation error
- No room for true systematic deviation from our understanding (as represented by the line)
- Could be a persistent environmental signal

Process error only option

Process error only likelihood

$$L(b, \sigma^2_{proc}|\mathbf{y}) = \prod_{t=2}^{T} \mathrm{N}(y_t - (y_{t-1} + b), \sigma^2_{proc})$$

Maximize with respect to $\{b, \sigma^2_{proc}\}$

Process error only option

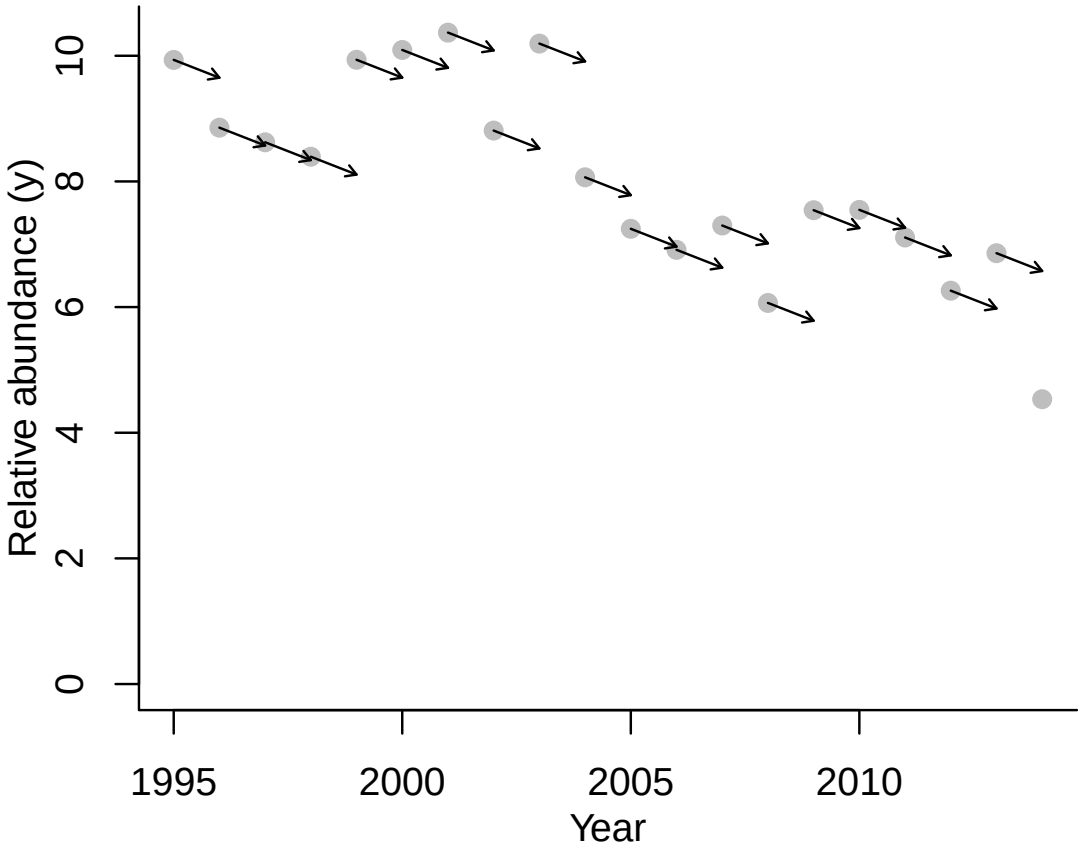
Need to roll your own in R

```
proc.only.like<-function(theta){
  delta<-theta[1]
  sigma<-exp(theta[2])
  pred.y<-y[1:(T-1)]+delta
  nll<--sum(dnorm(y[2:T], mean=pred.y, sigma,log=TRUE))
  return(nll)
}

proc.only.fit<-optim(par=c(1,1), fn=proc.only.like,
             control=list(trace=1))
```

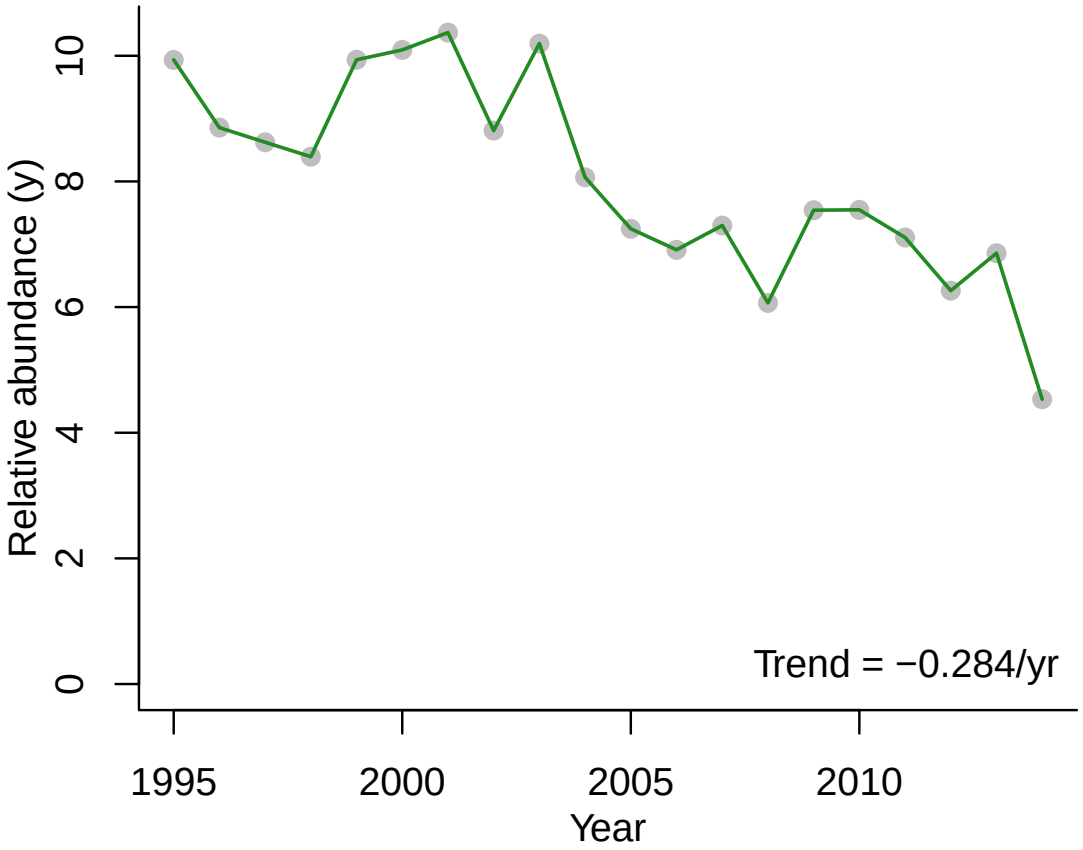
Process error only option

Steps through and predicts next observation from previous one



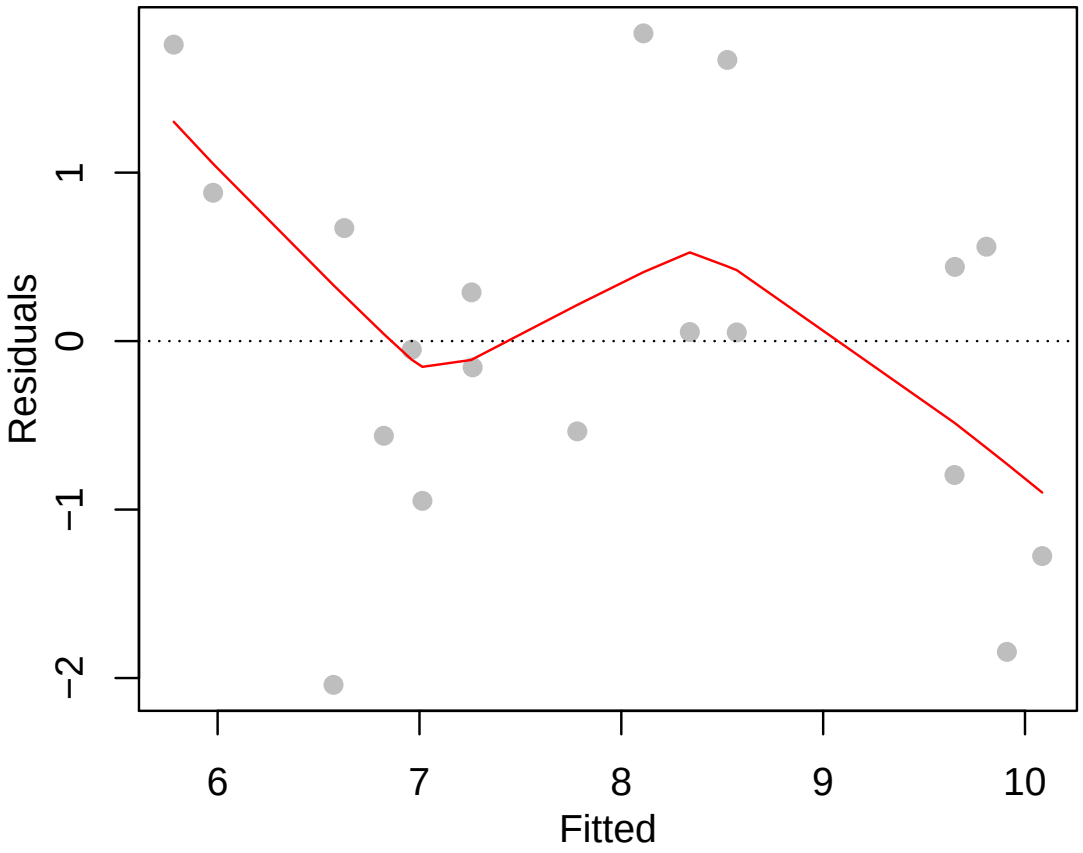
Process error only option

Treats all observations as perfectly known



Process error only option

Residual (one step ahead forecast errors) patterns exist here too



Process error only option

- Assumes the data are exact represtations of the true state of the system
- All variability is assumed to be process variability no captured by the one step ahead predictions
- No room for general observation error (think measuring anything to $\pm$ some tolerance)
- Some of the observations could be bogus but we treat them as perfectly known

Observation versus process error

- Until now it's been observation <u>or</u> process error assumptions
- Observation shoots through all observations
- Process treats each observation as known exactly
- They result in different trend estimates
- Neither seems satisfactory - is there a better approach?

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Goal of state space modelling

Is to explicitly account for the dual components of process and observation when making inference from time series

State space origins

- Control theory for space missions
- Process model from motion in space
- Observations from radar
- Need to produce best estimate of where it's at and where it's going (trajectory)
- Coupled process and observation

State space origins

- Kalman filter
- Optimal method for coupling process and observation
- Incorporates complex physical linear/linearized models
- Extracts signals from noisy data

Obama gets to meet Kálmán

Draper prize 2008: *For the development and dissemination of the optimal digital technique (known as the Kalman Filter) that is pervasively used to control a vast array of consumer, health, commercial and defense products.*

A state space model by any other name...

- Dynamic linear models (statistics)
- Bayesian forecasting models (statistics)
- Structural models (econometrics)
- Linear system models (engineering)
- Kalman filtering models (control engineering)
- Latent state models (ecology)

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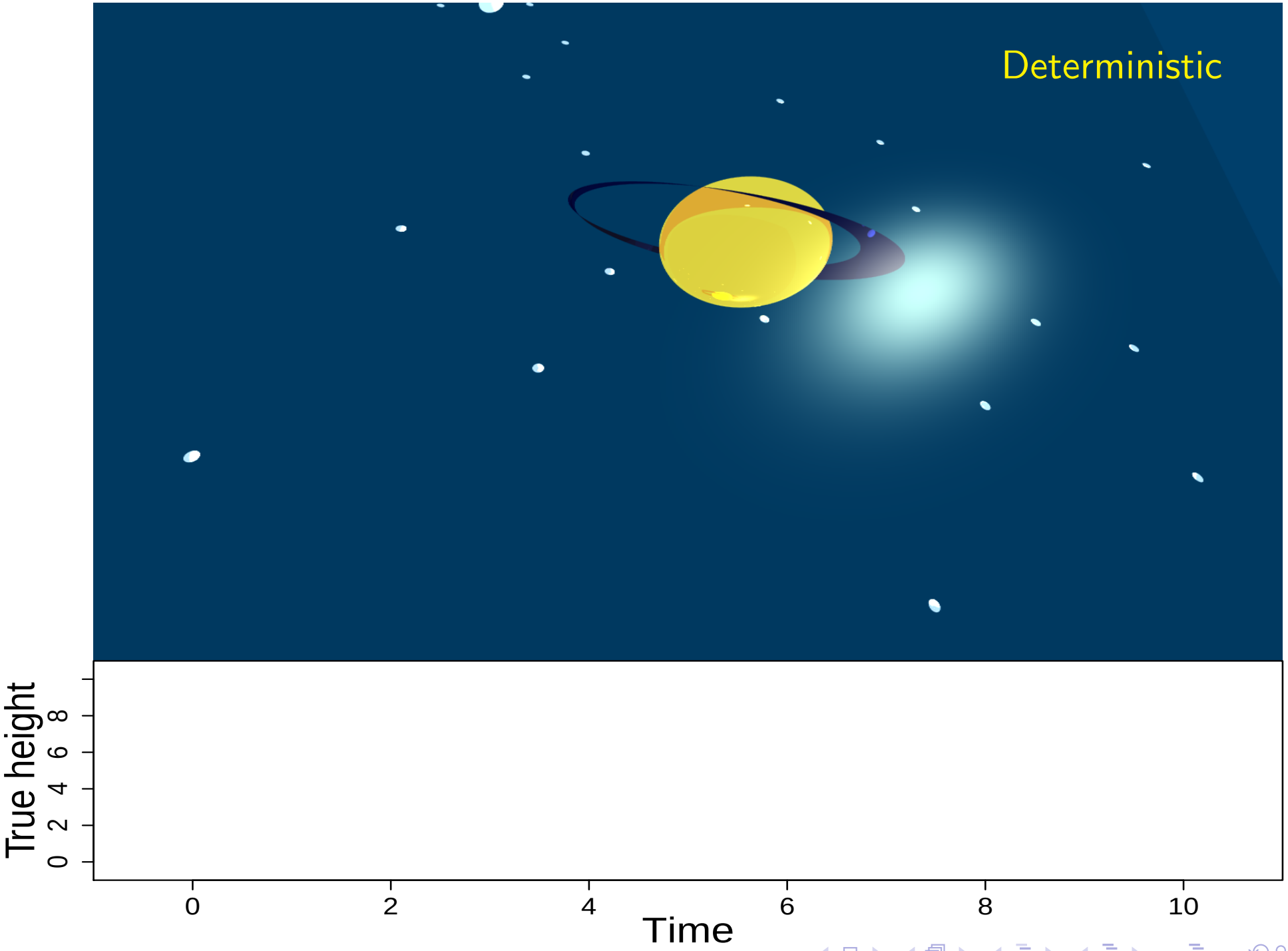
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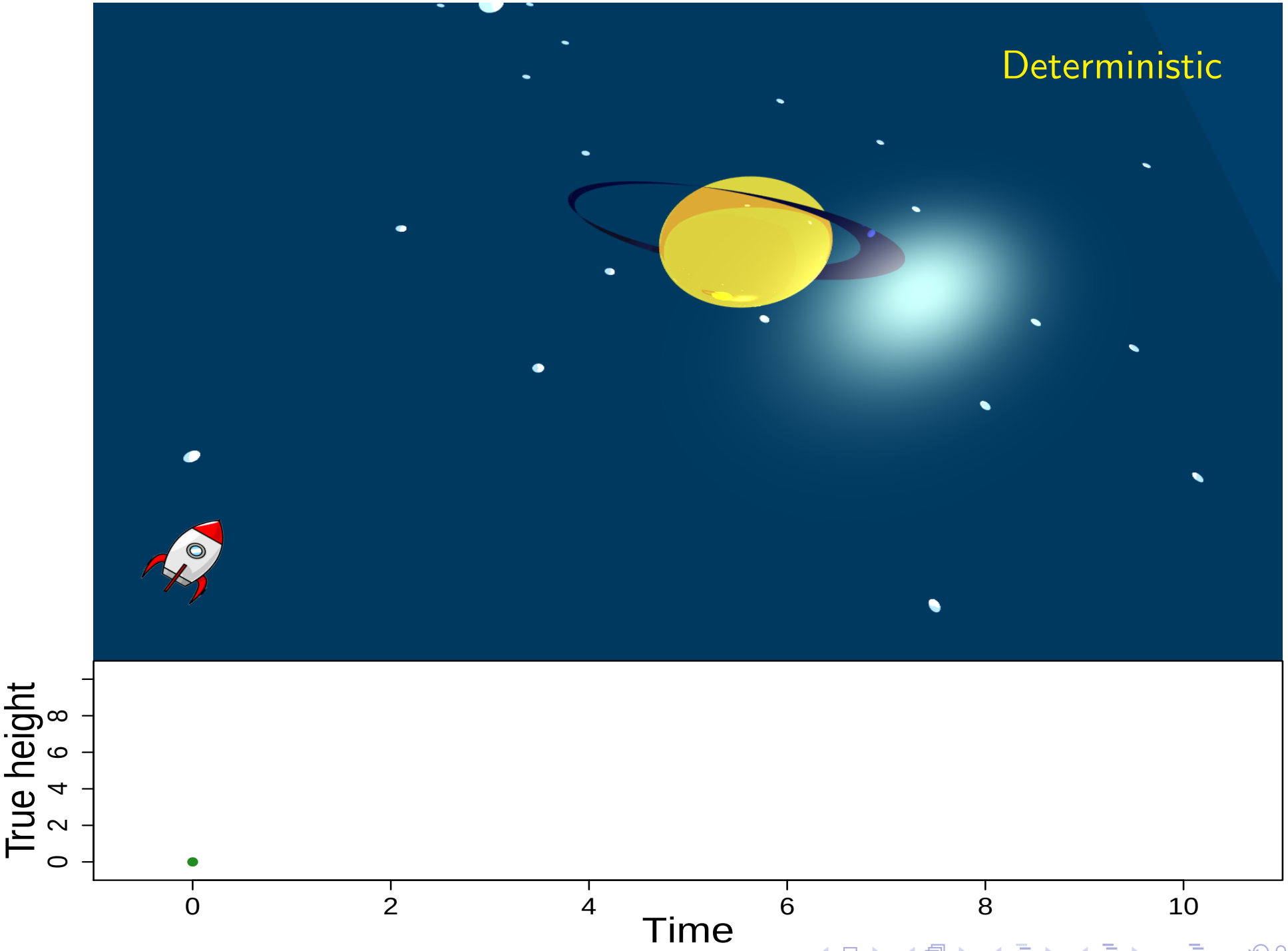
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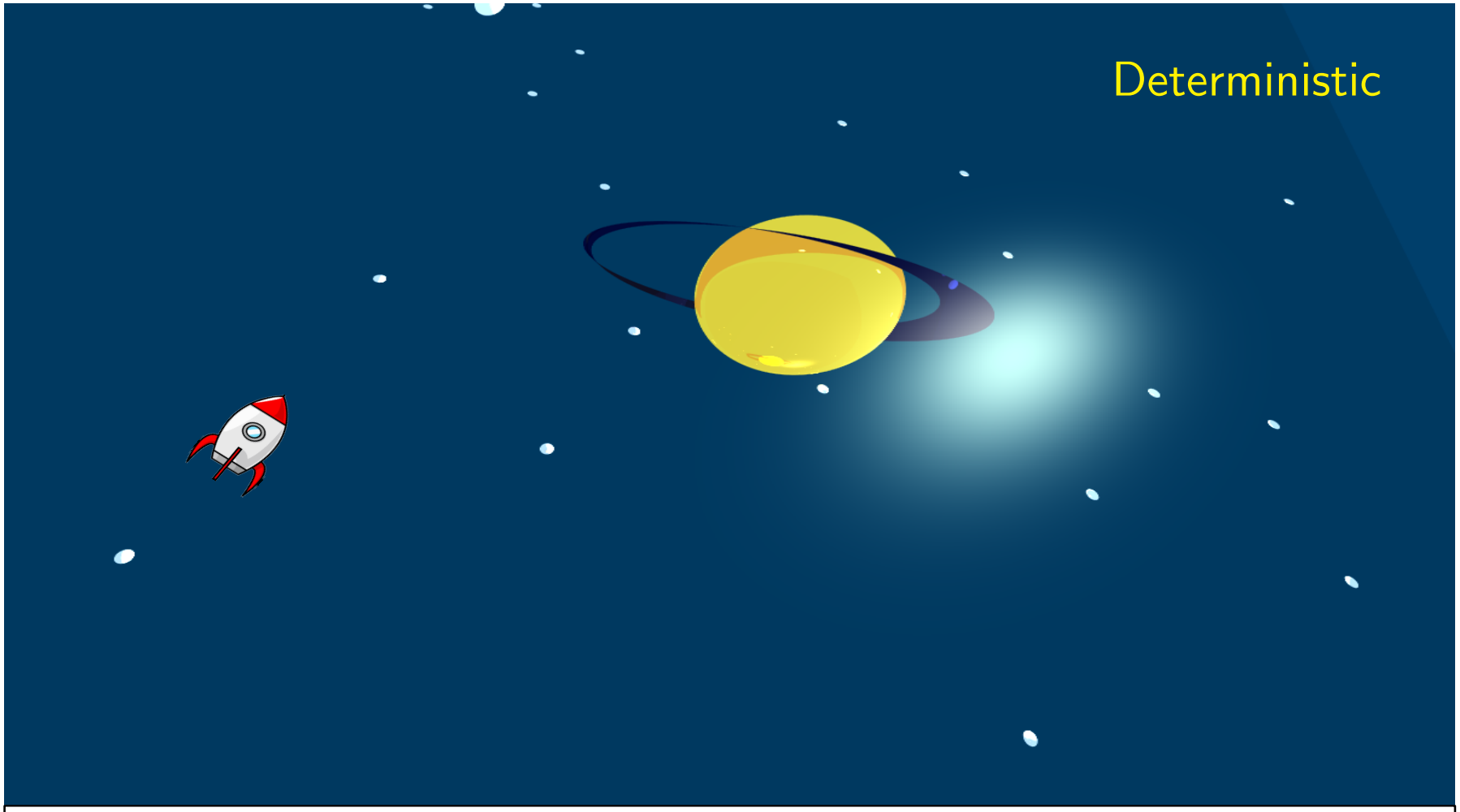
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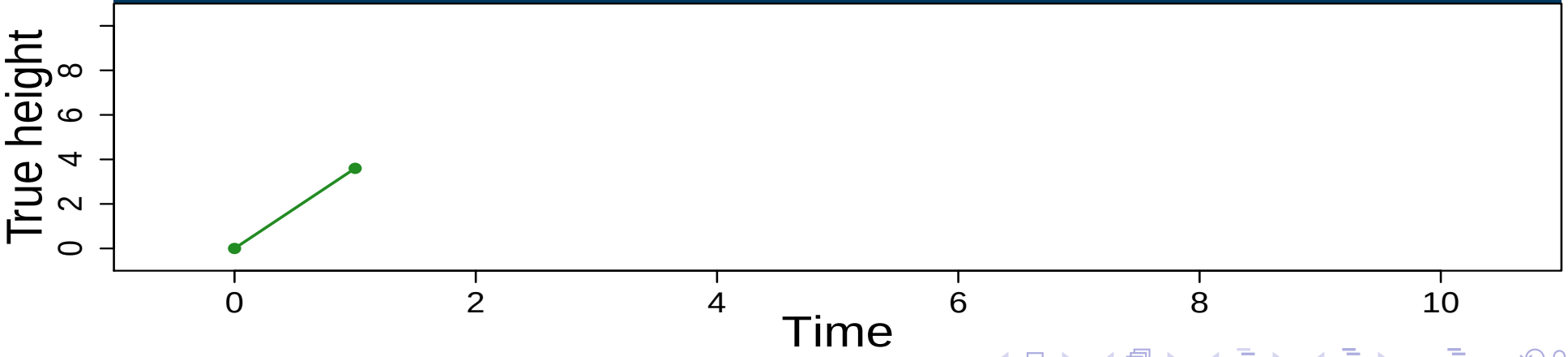
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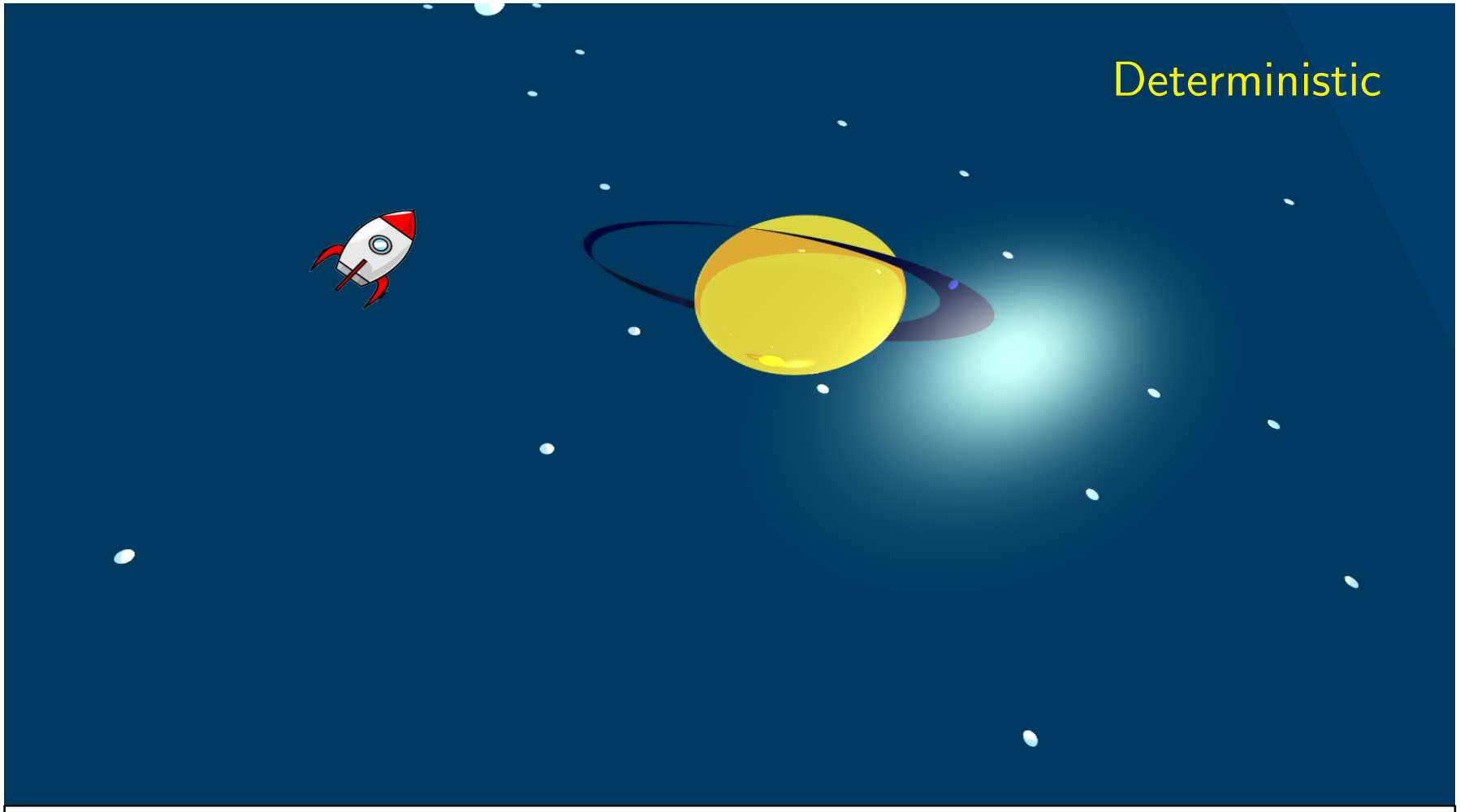
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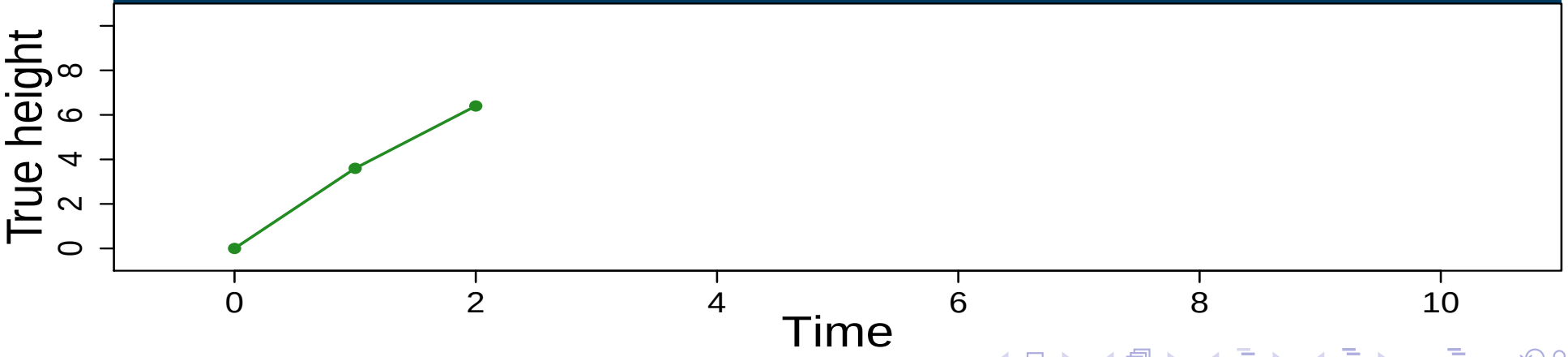
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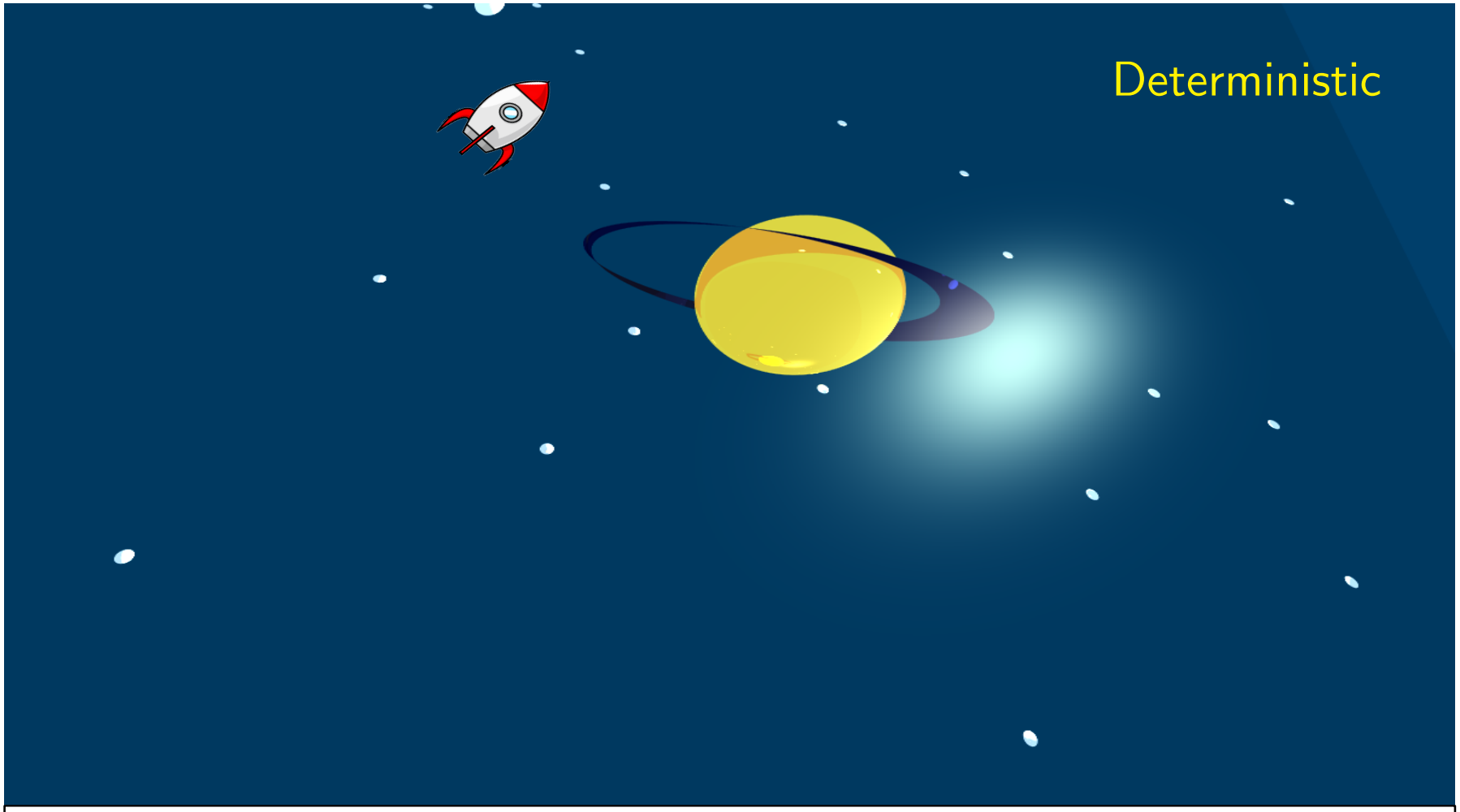
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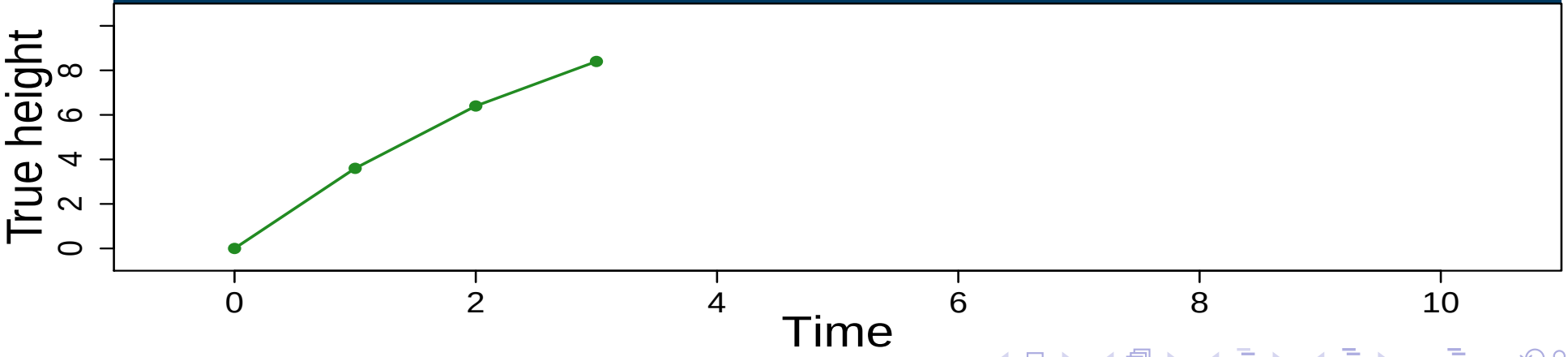
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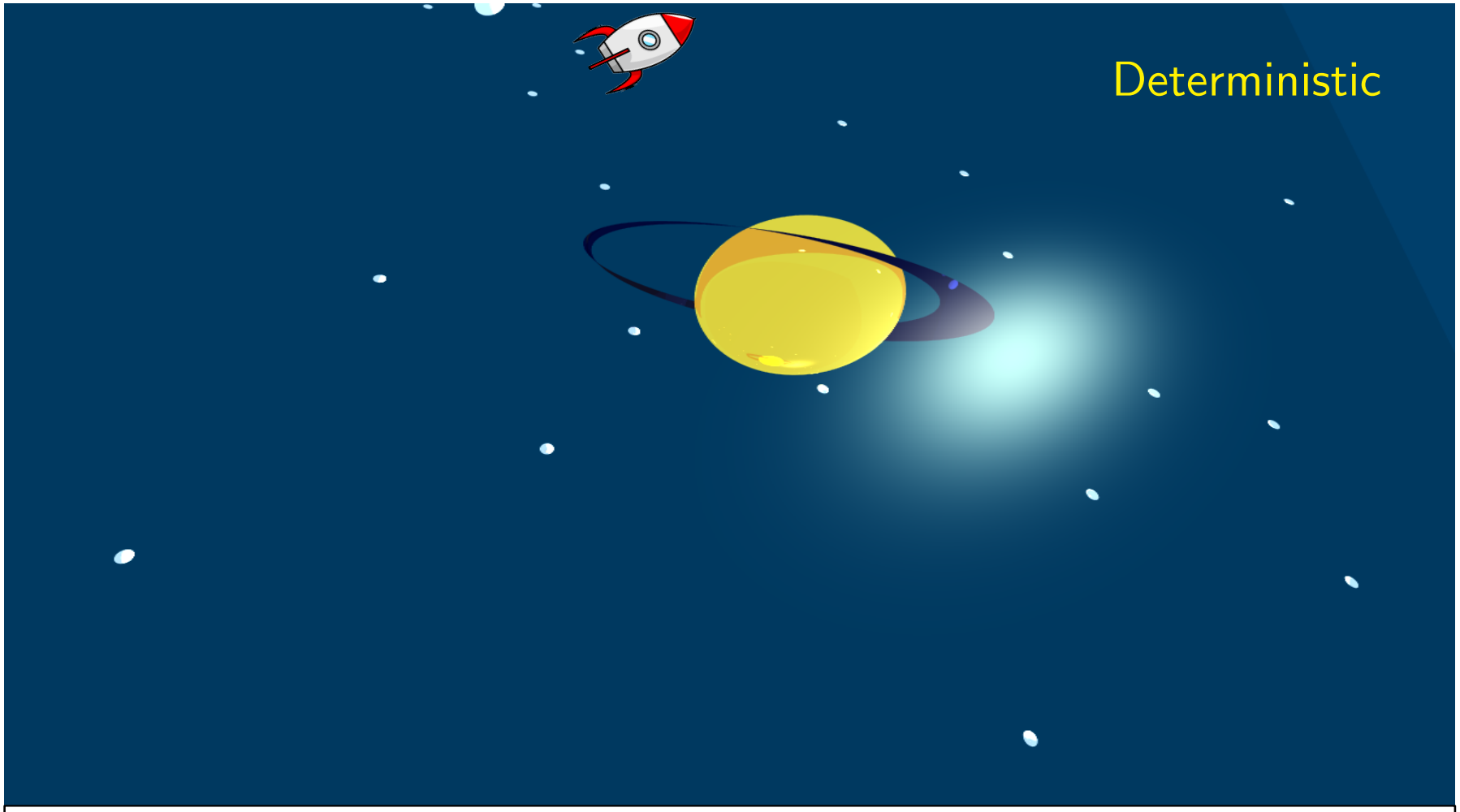
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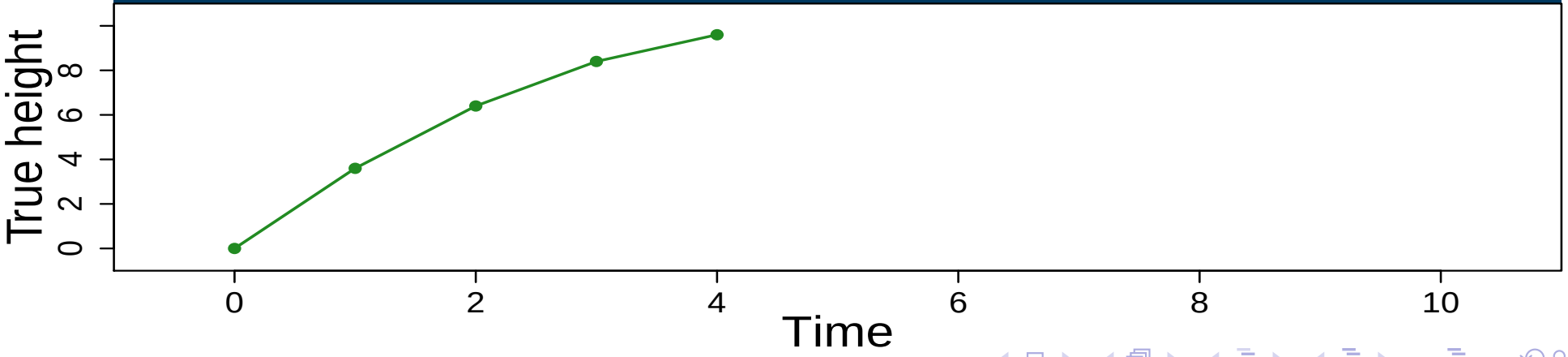
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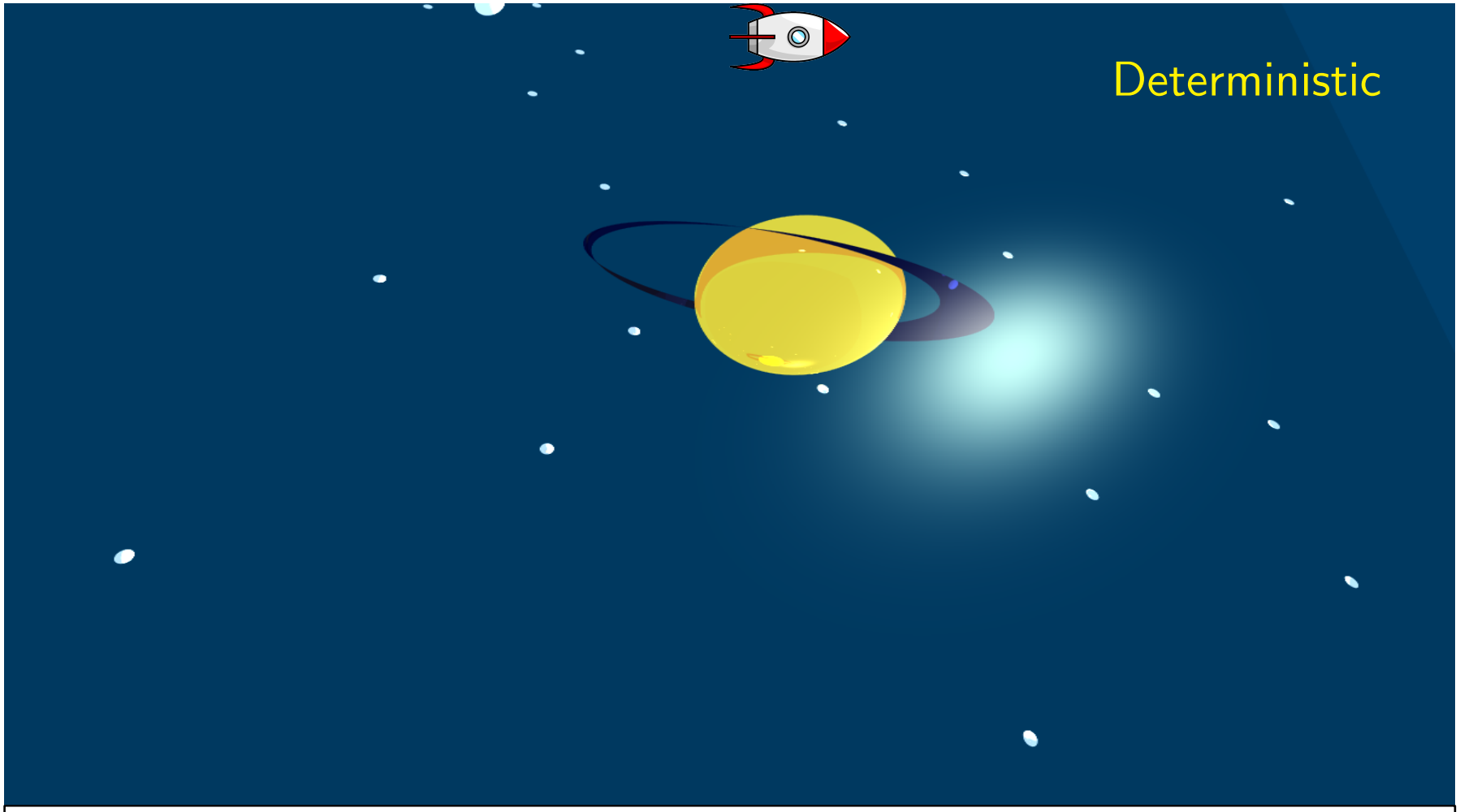
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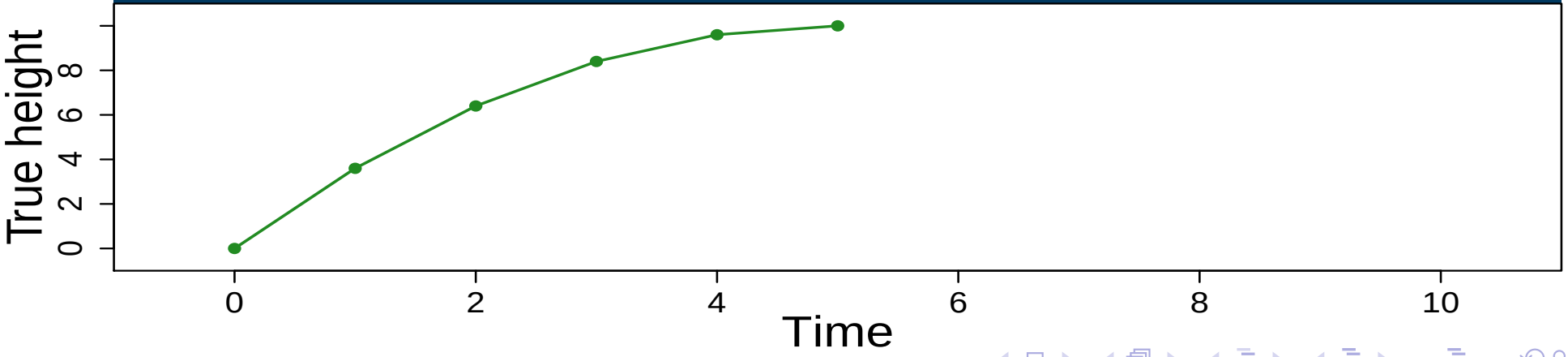
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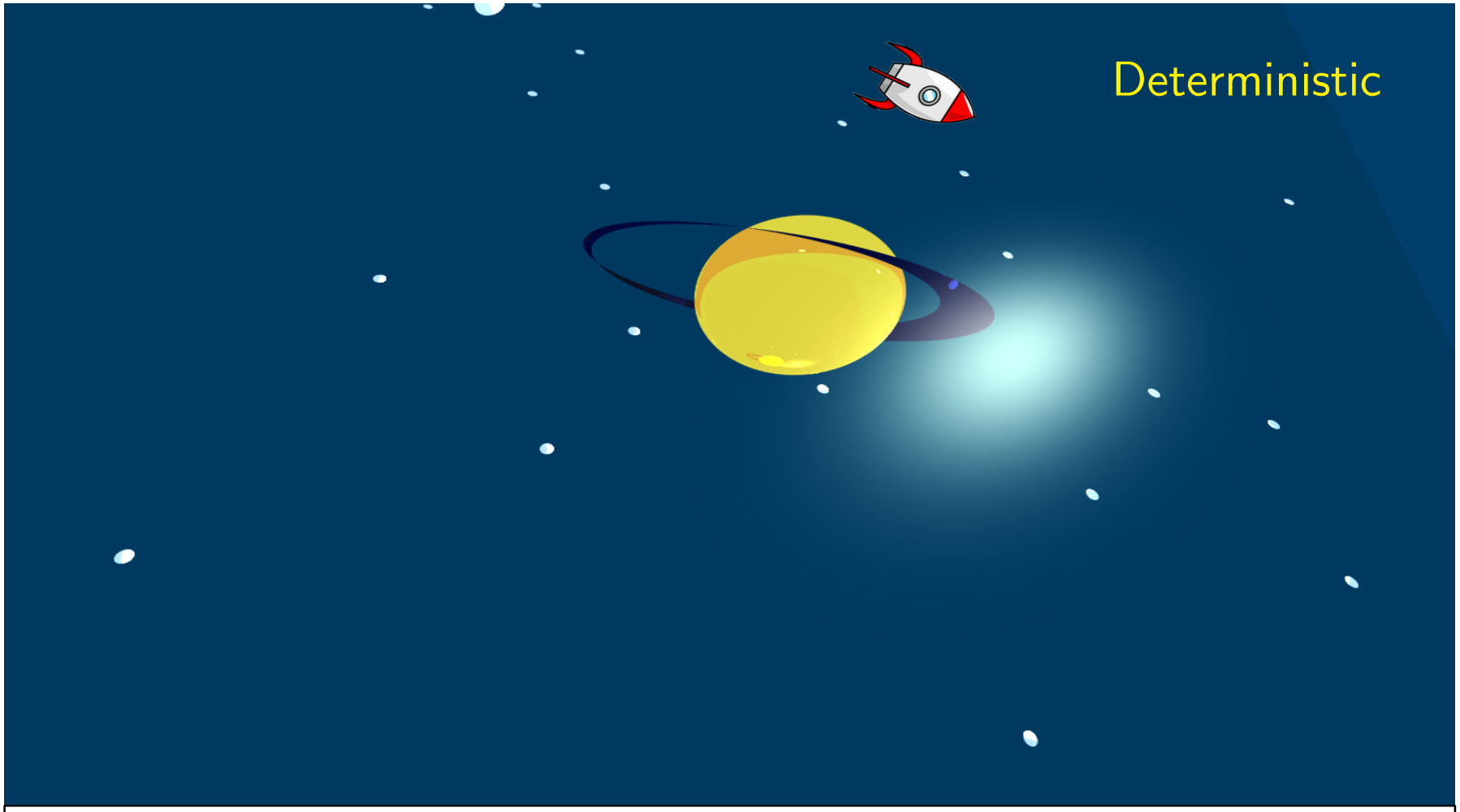
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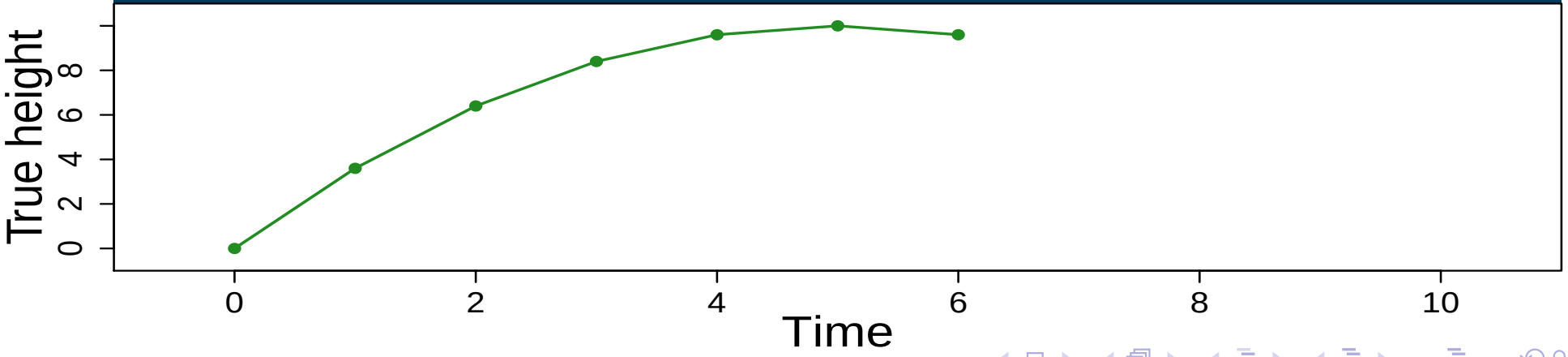
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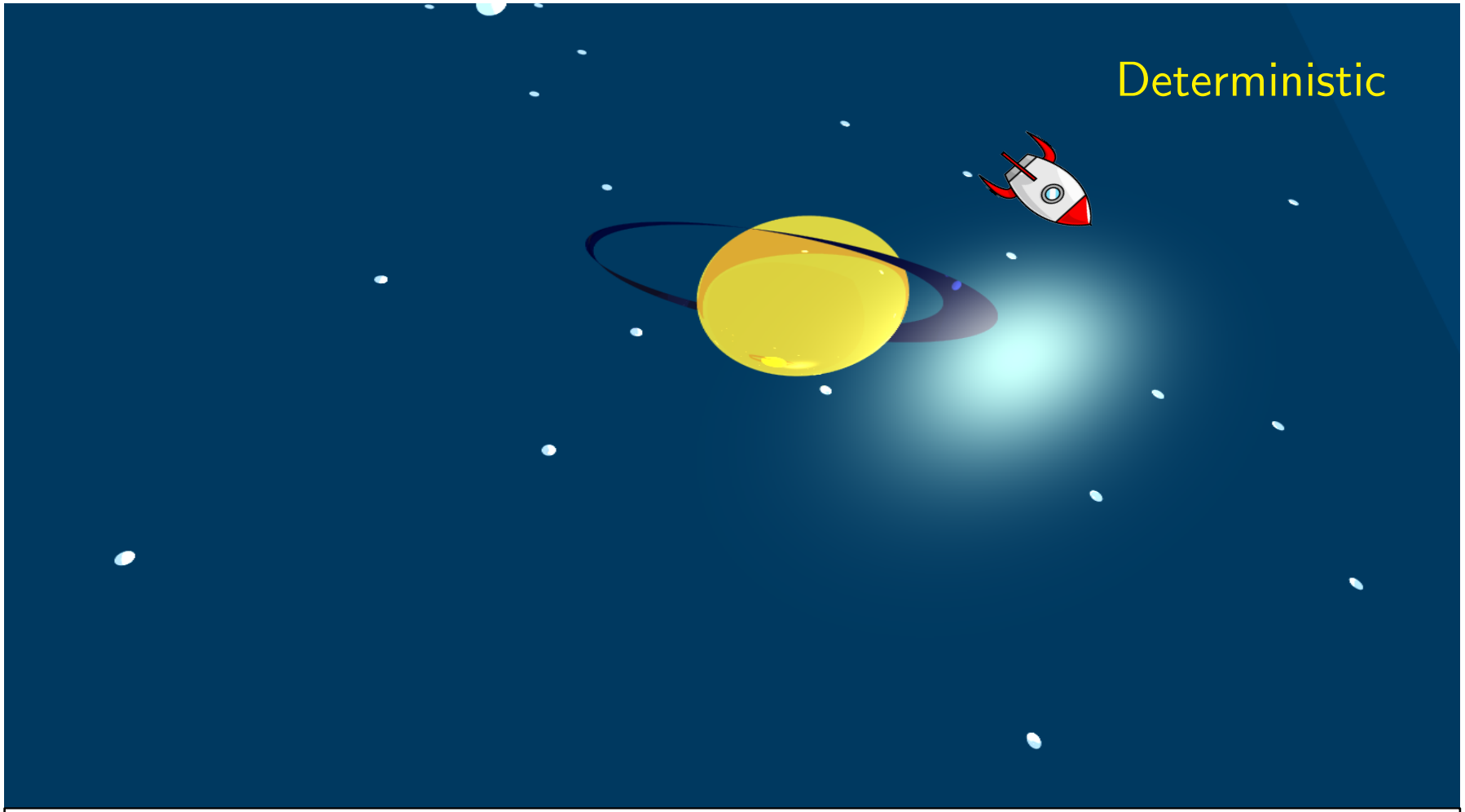
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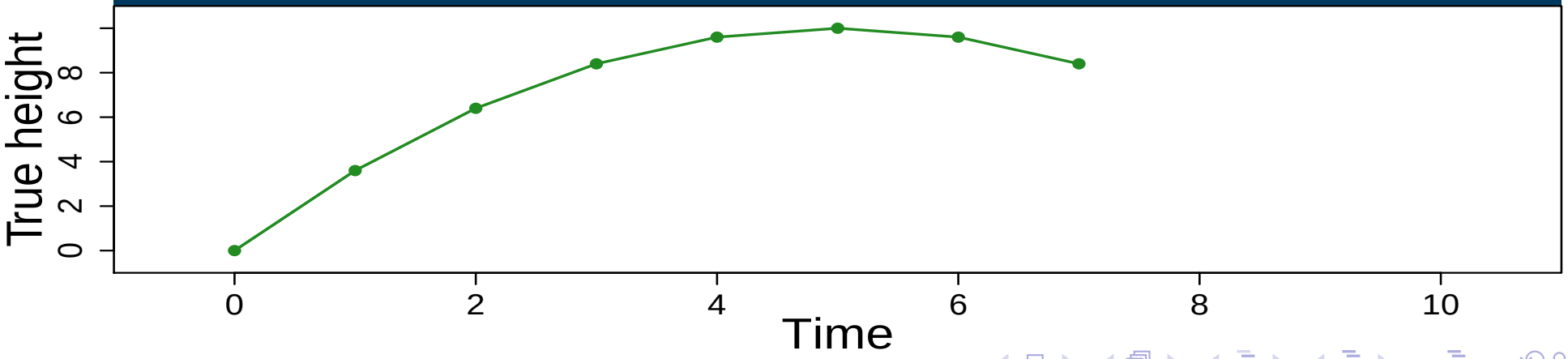
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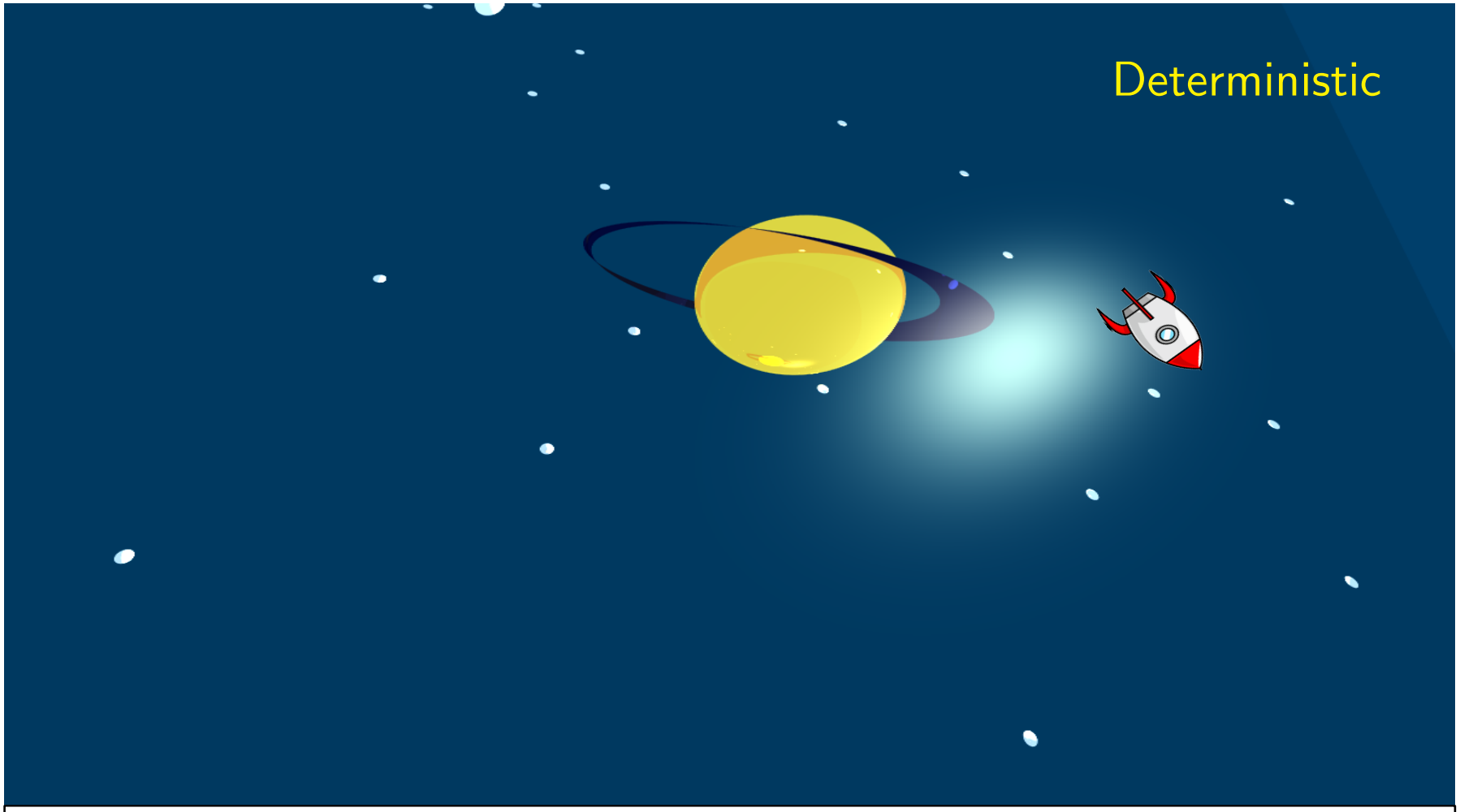
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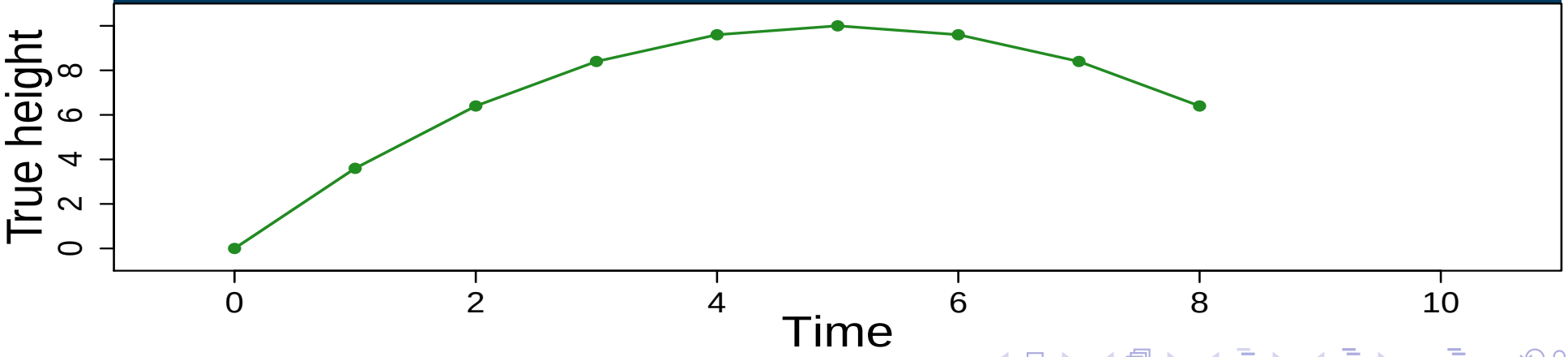
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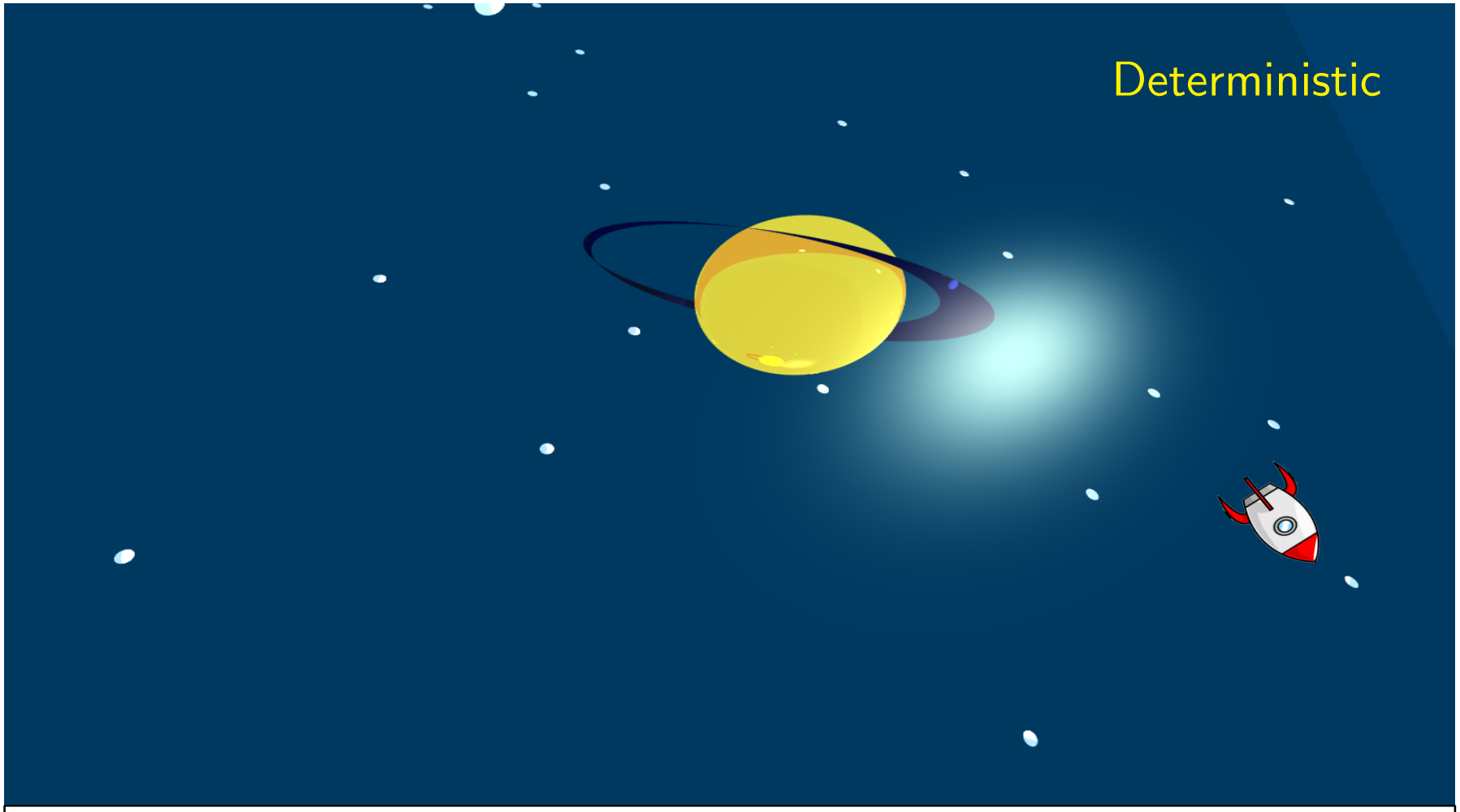
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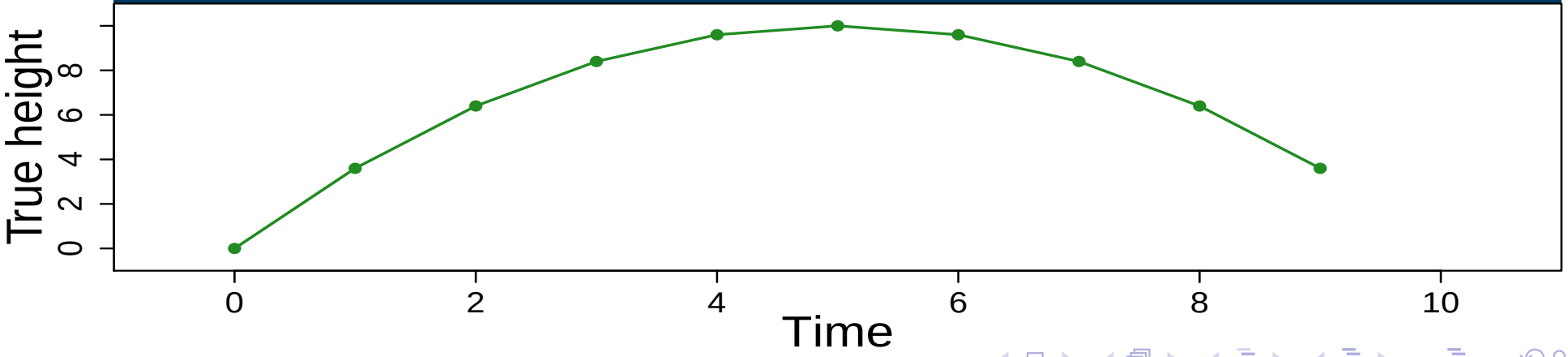
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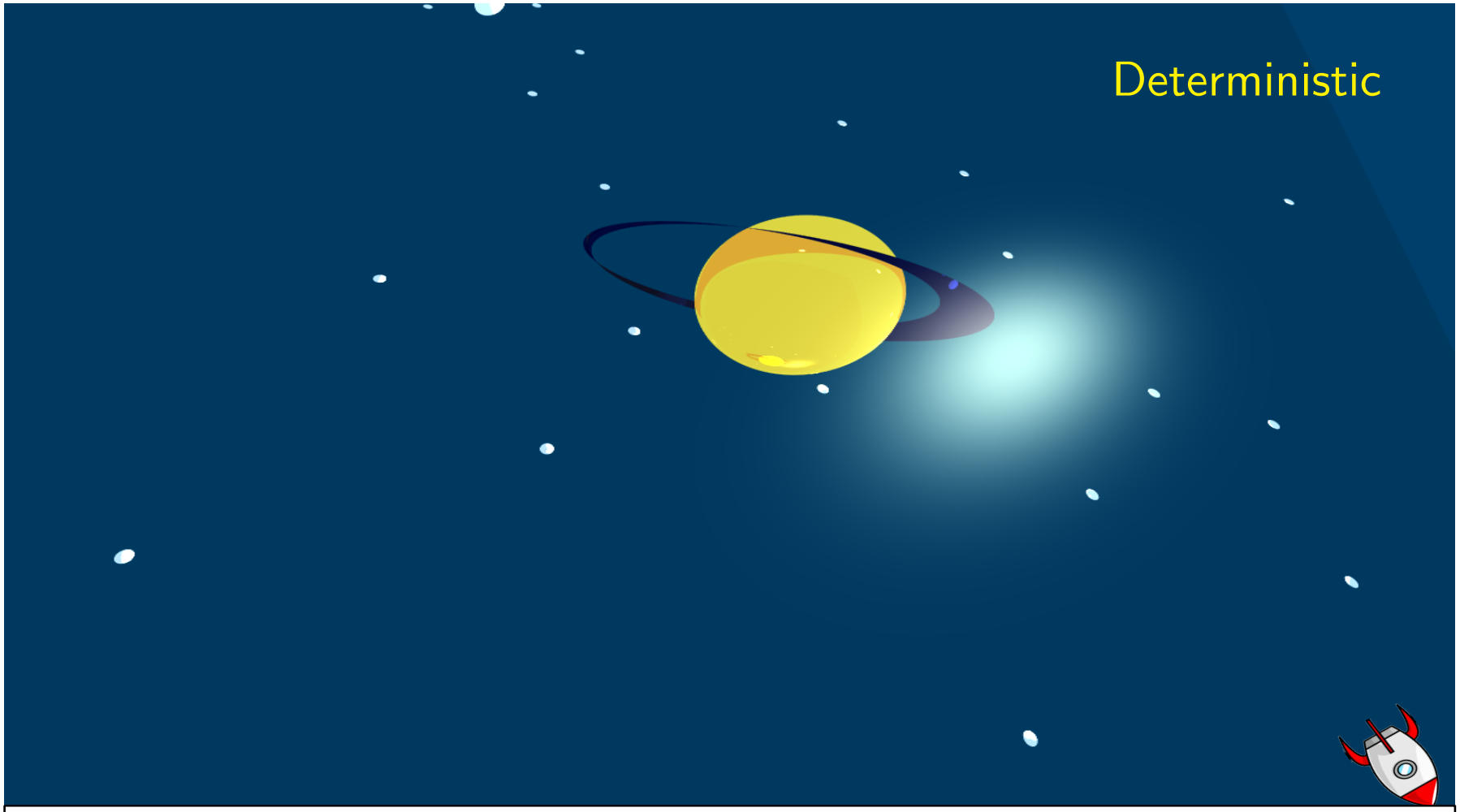
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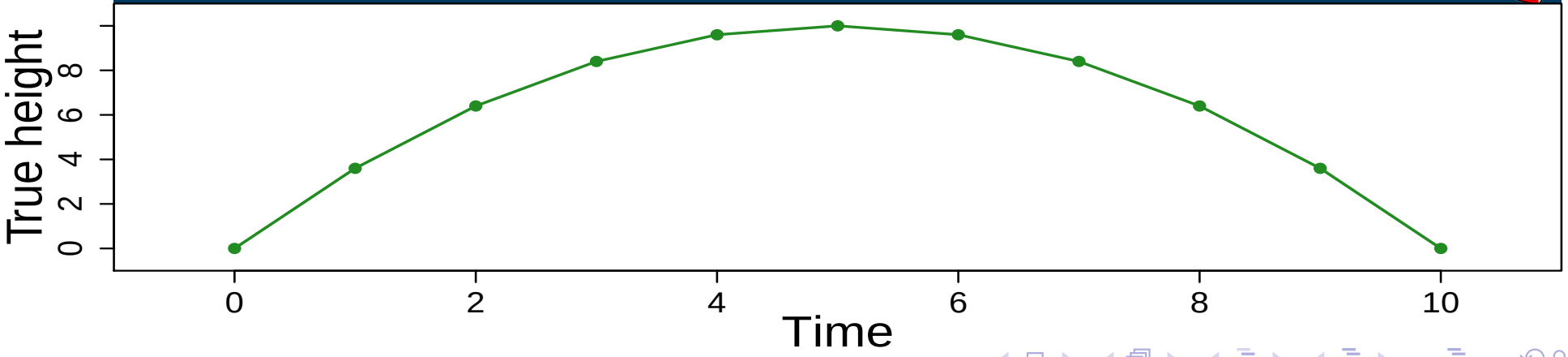
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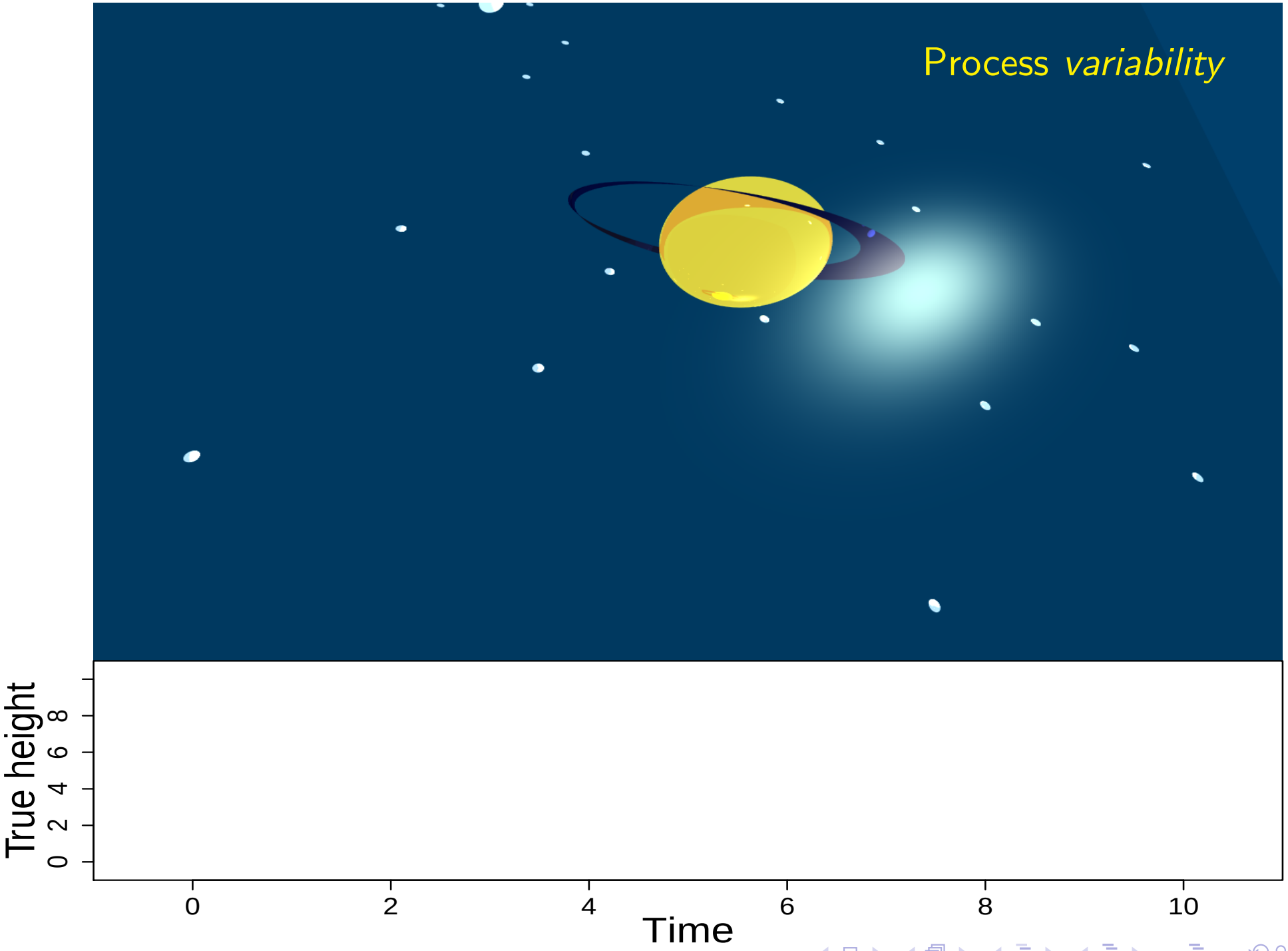
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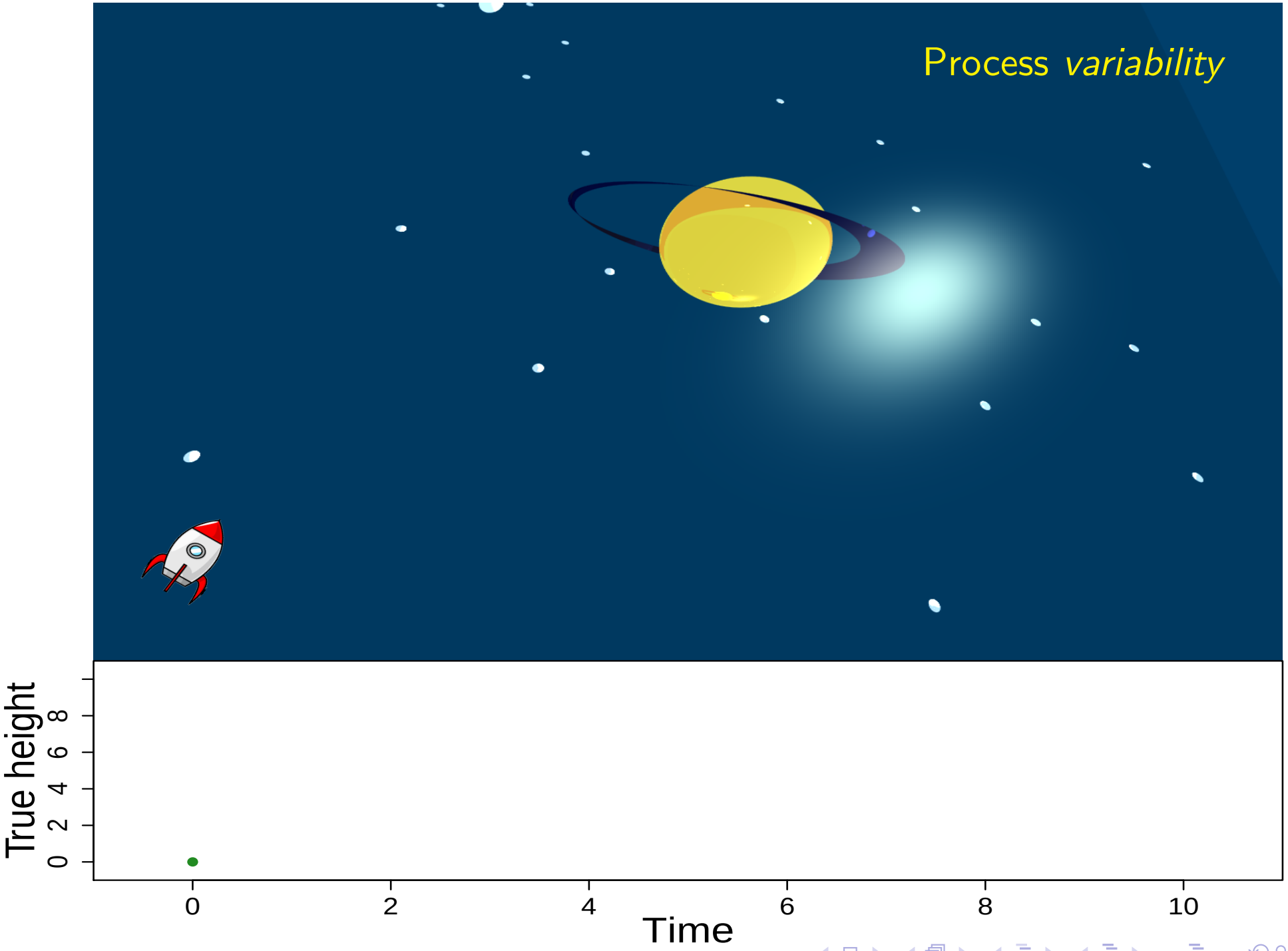
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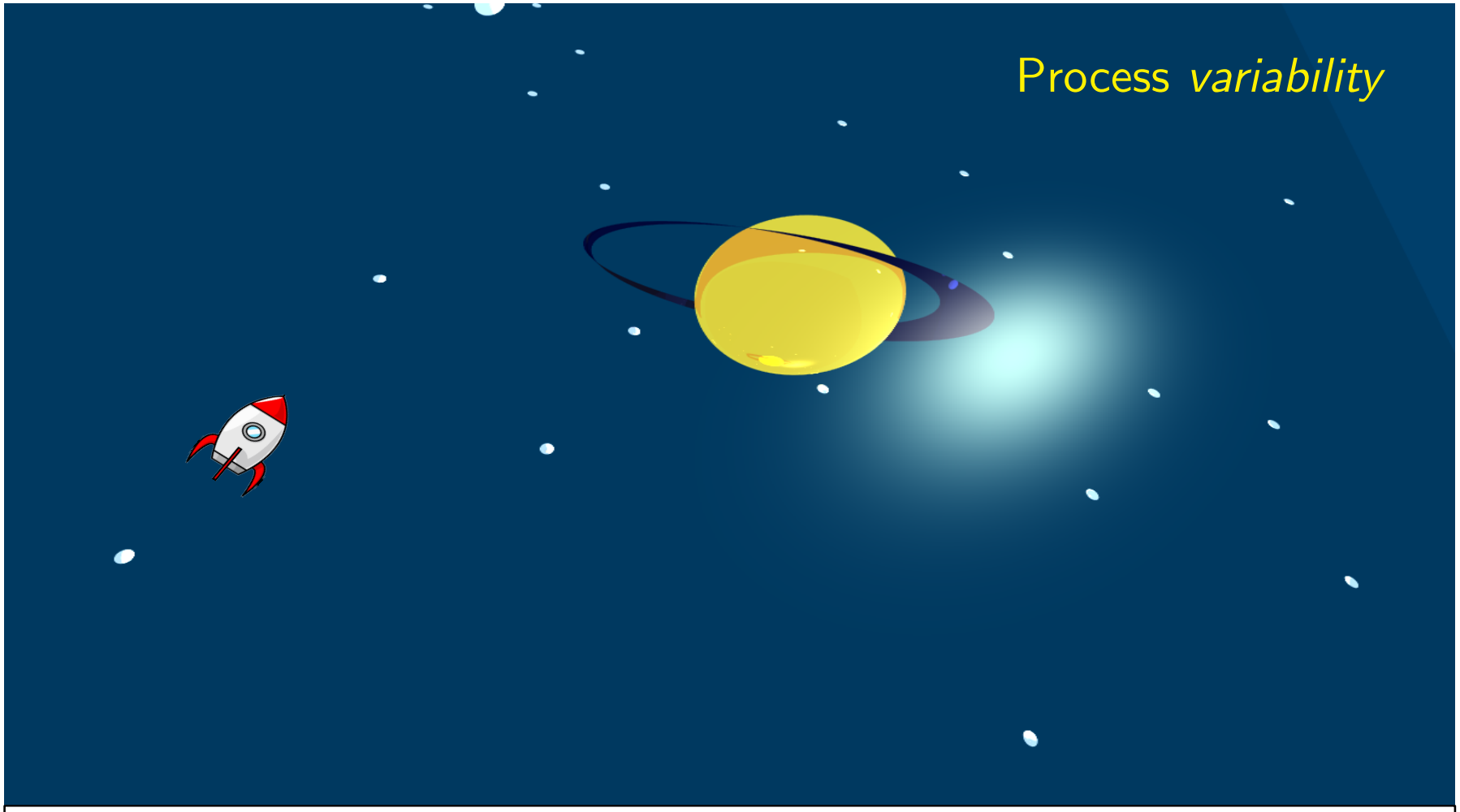
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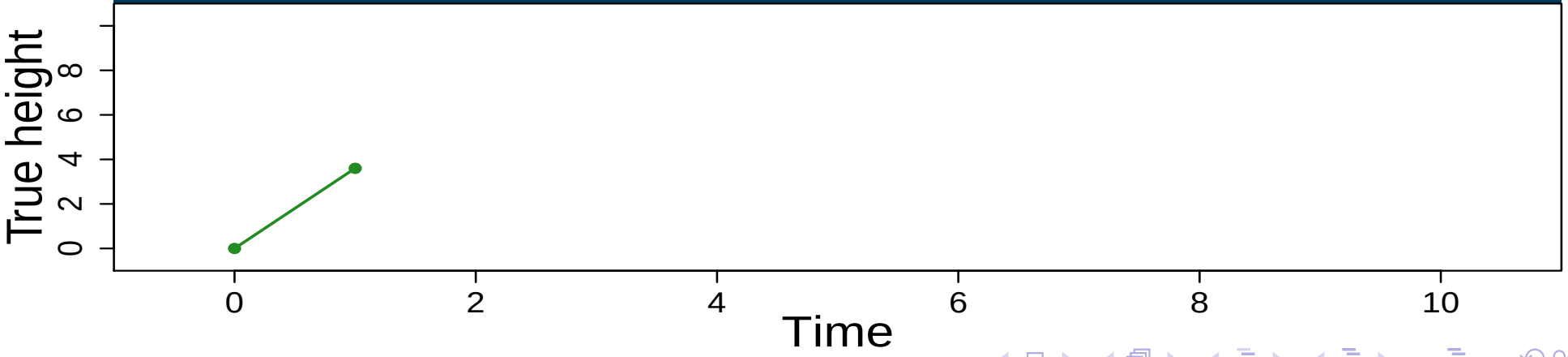
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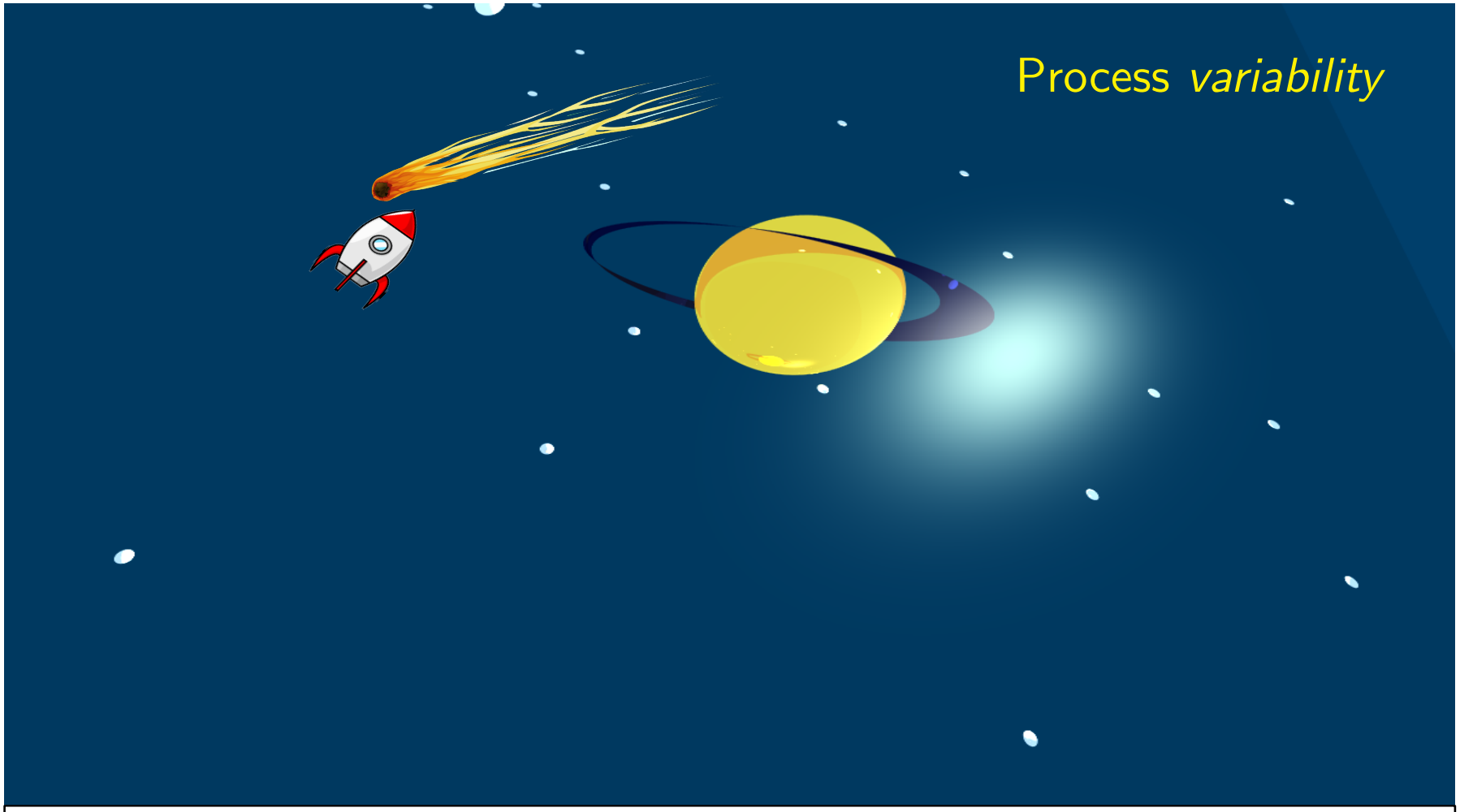
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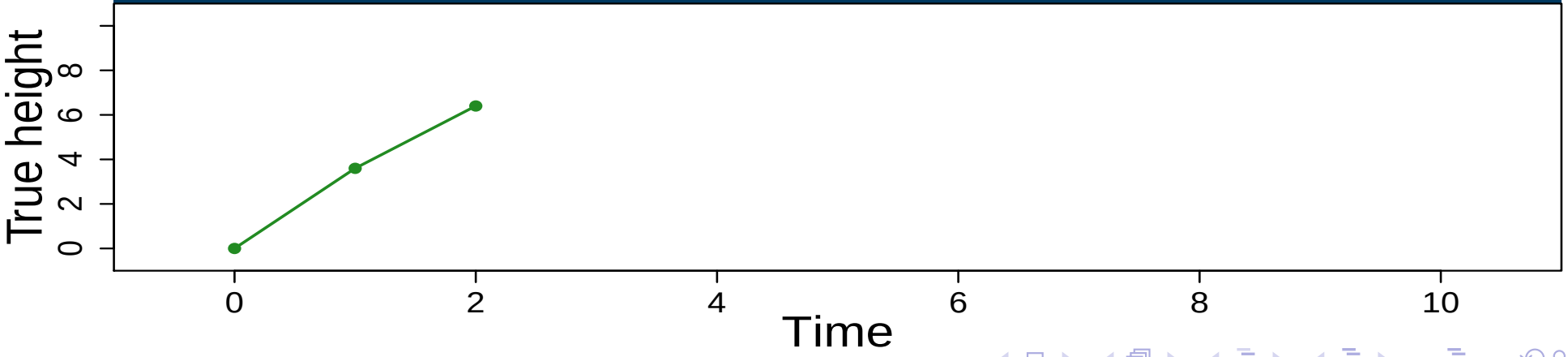
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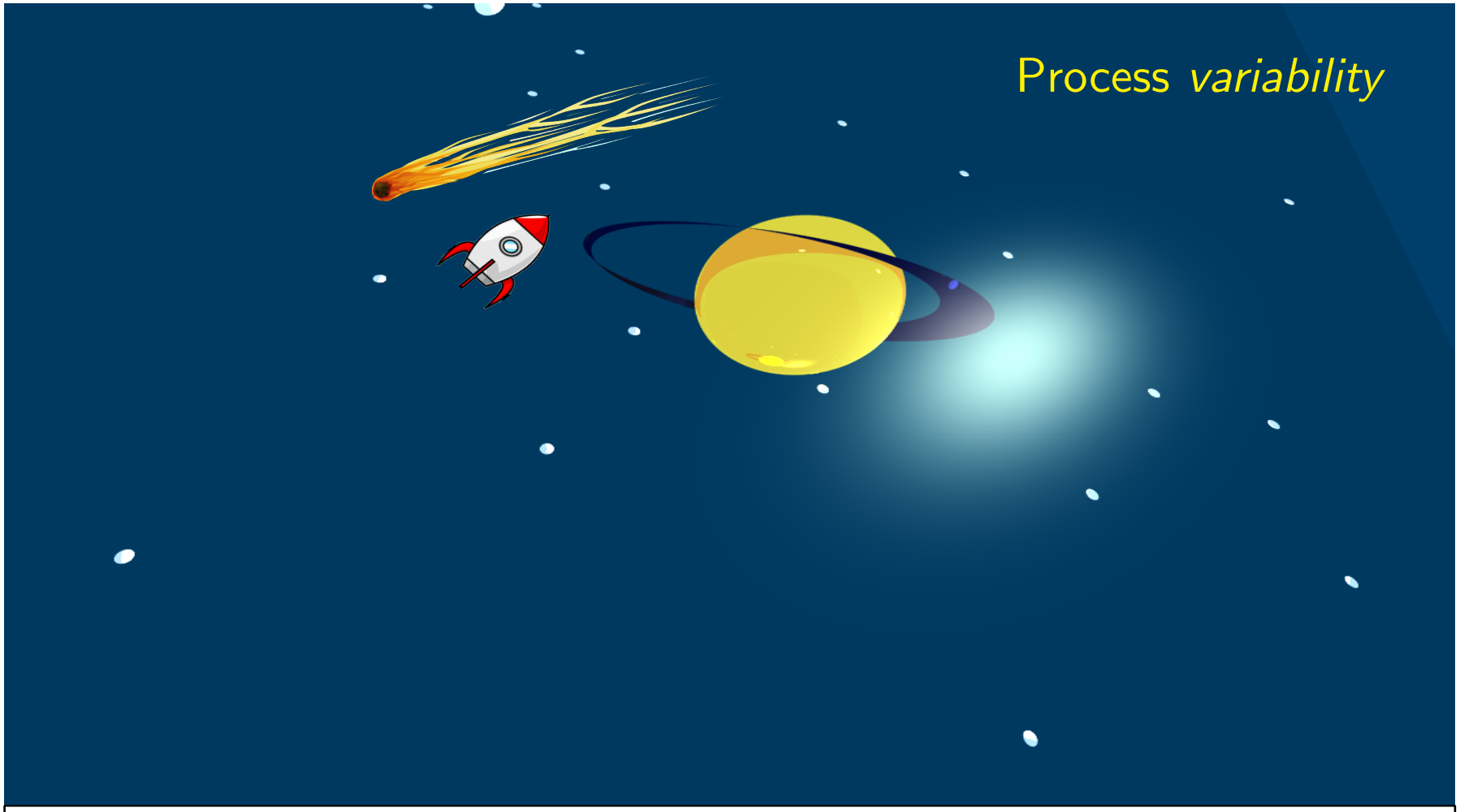
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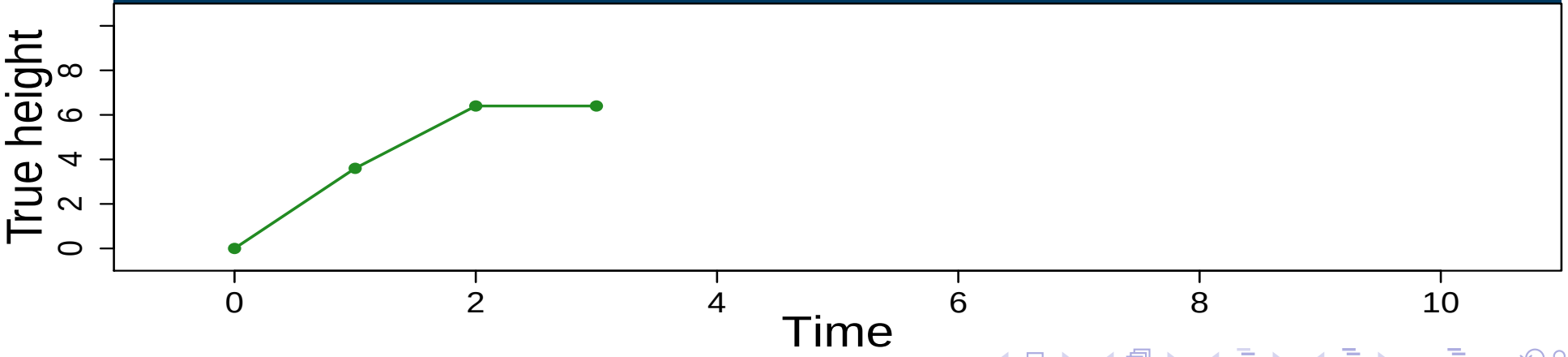
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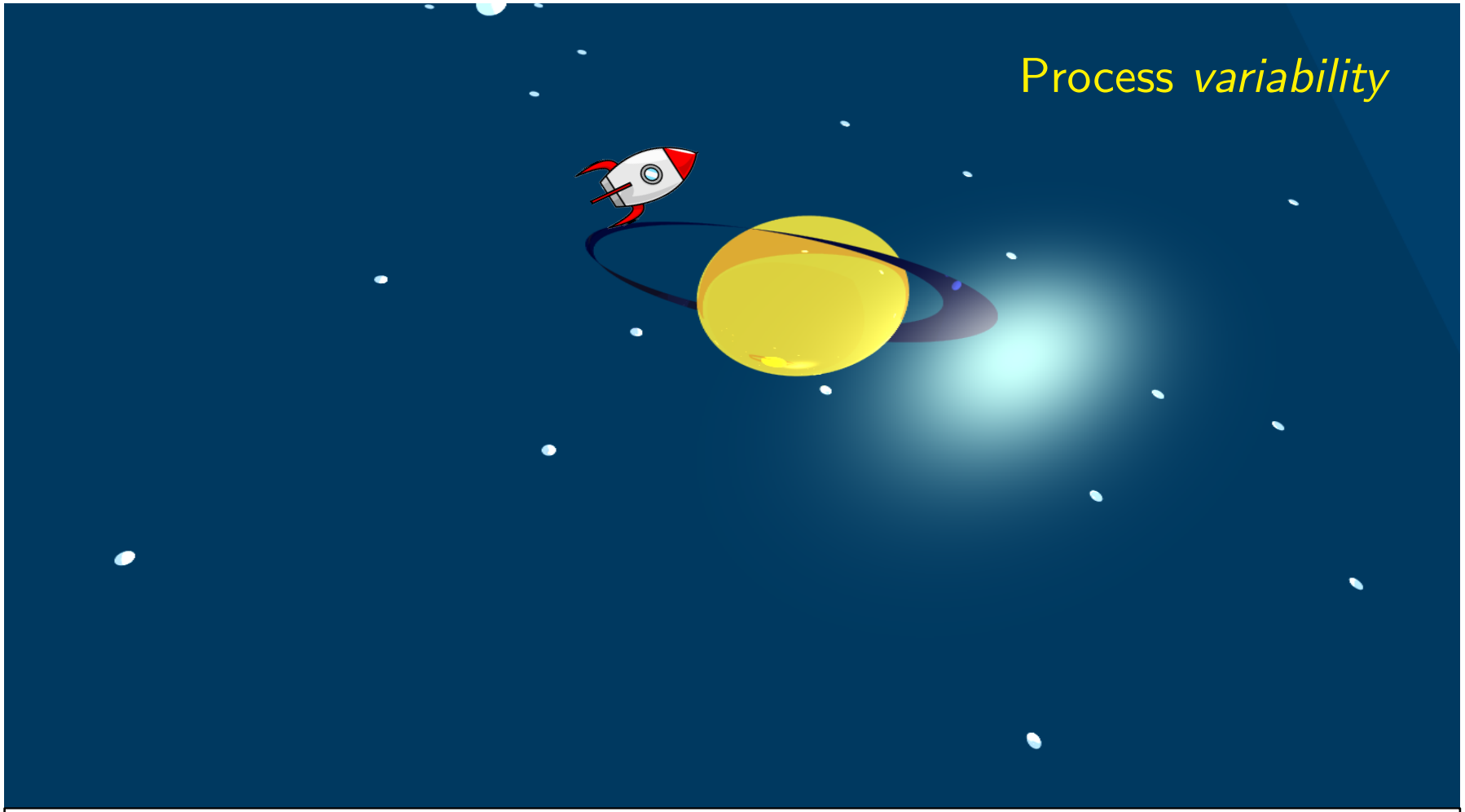
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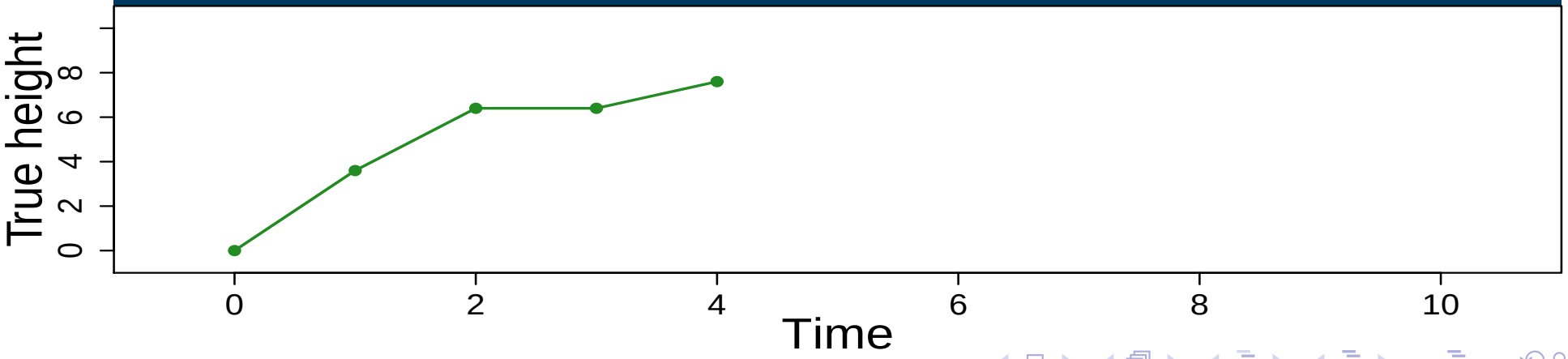
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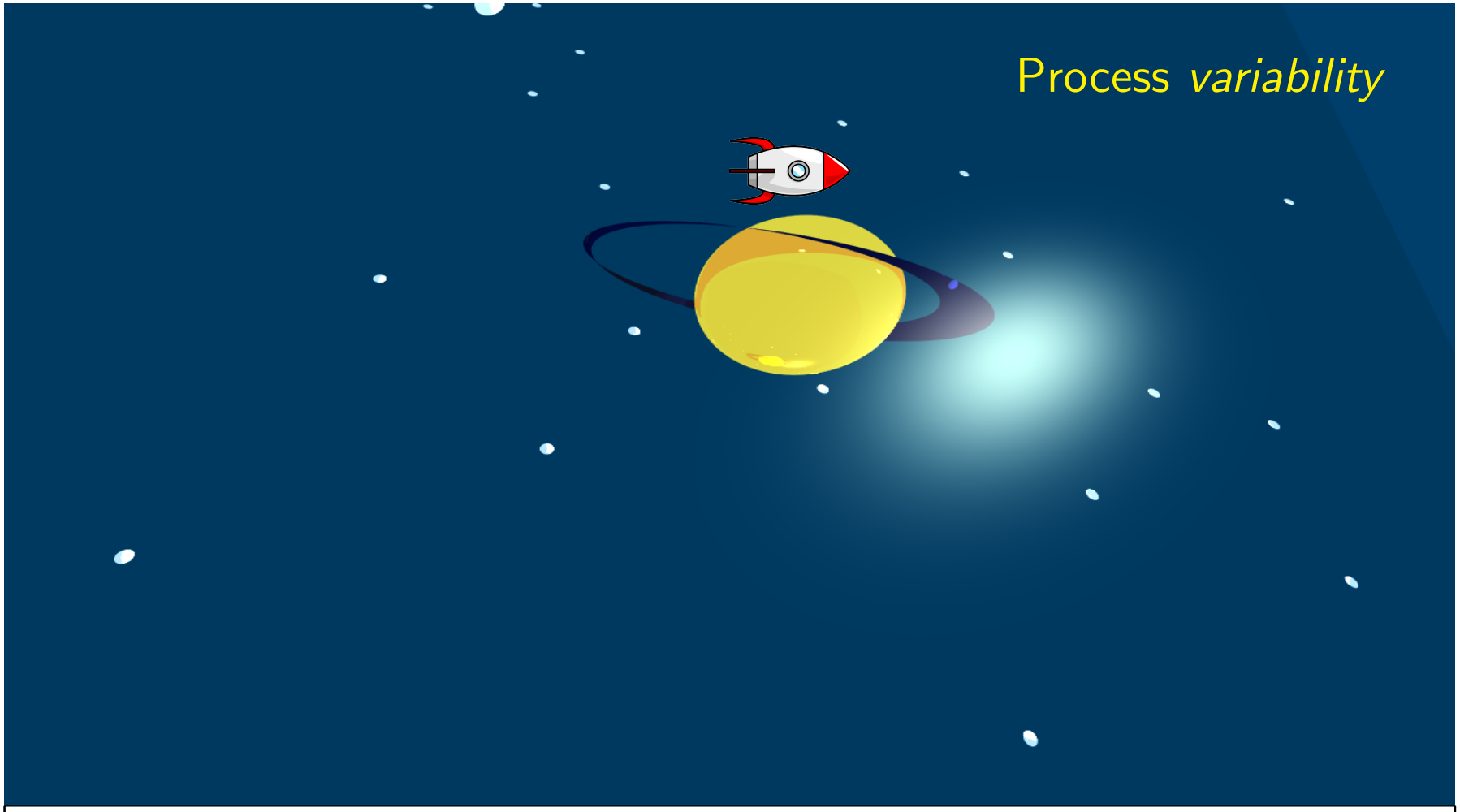
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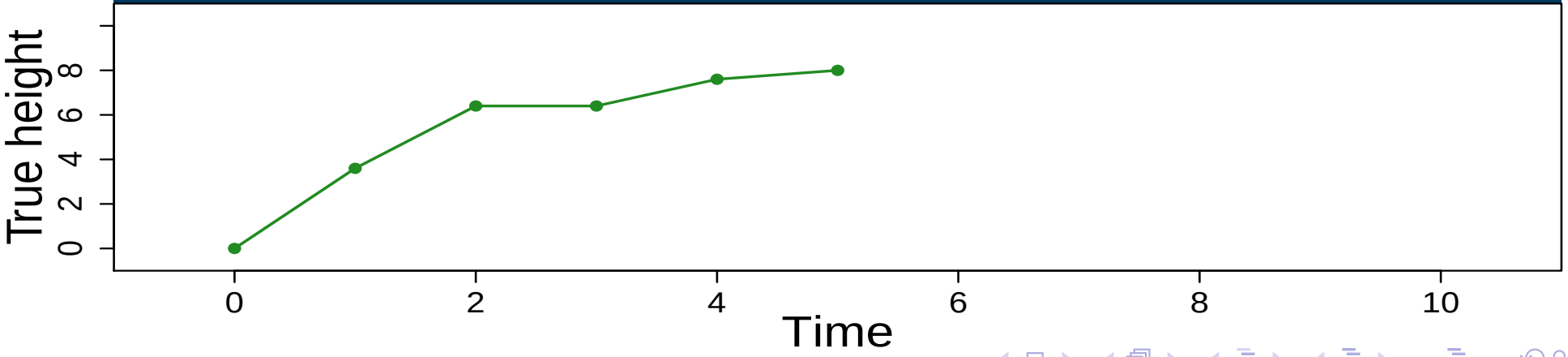
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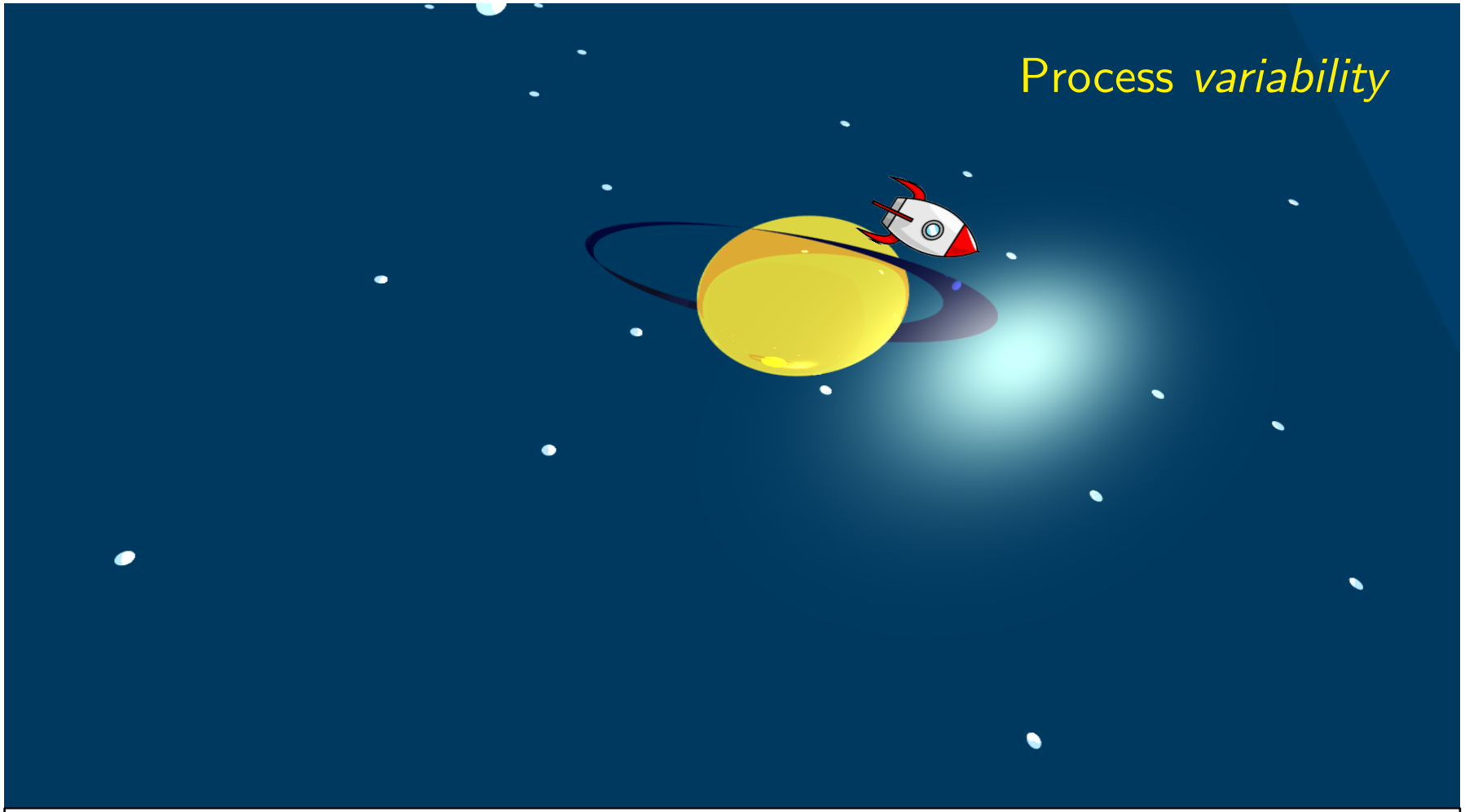
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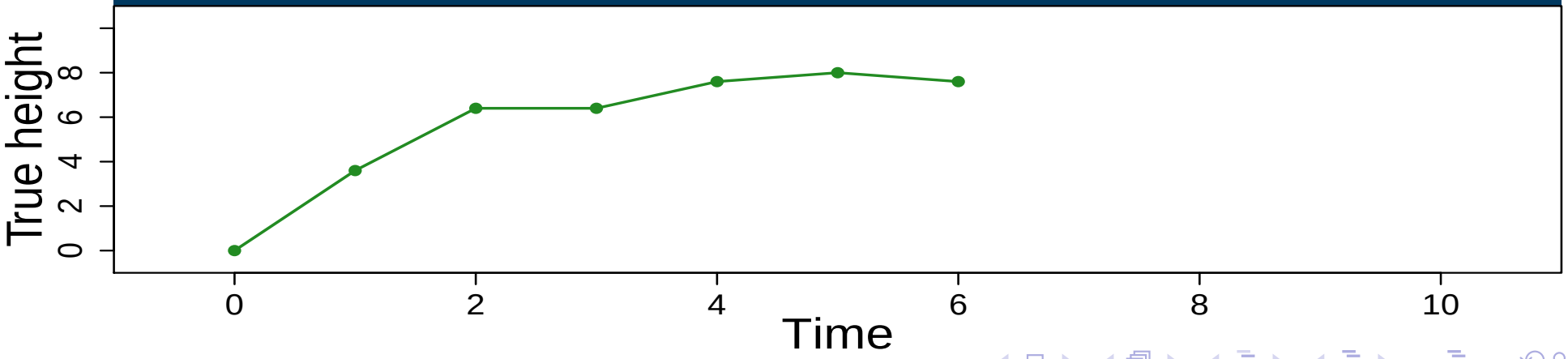
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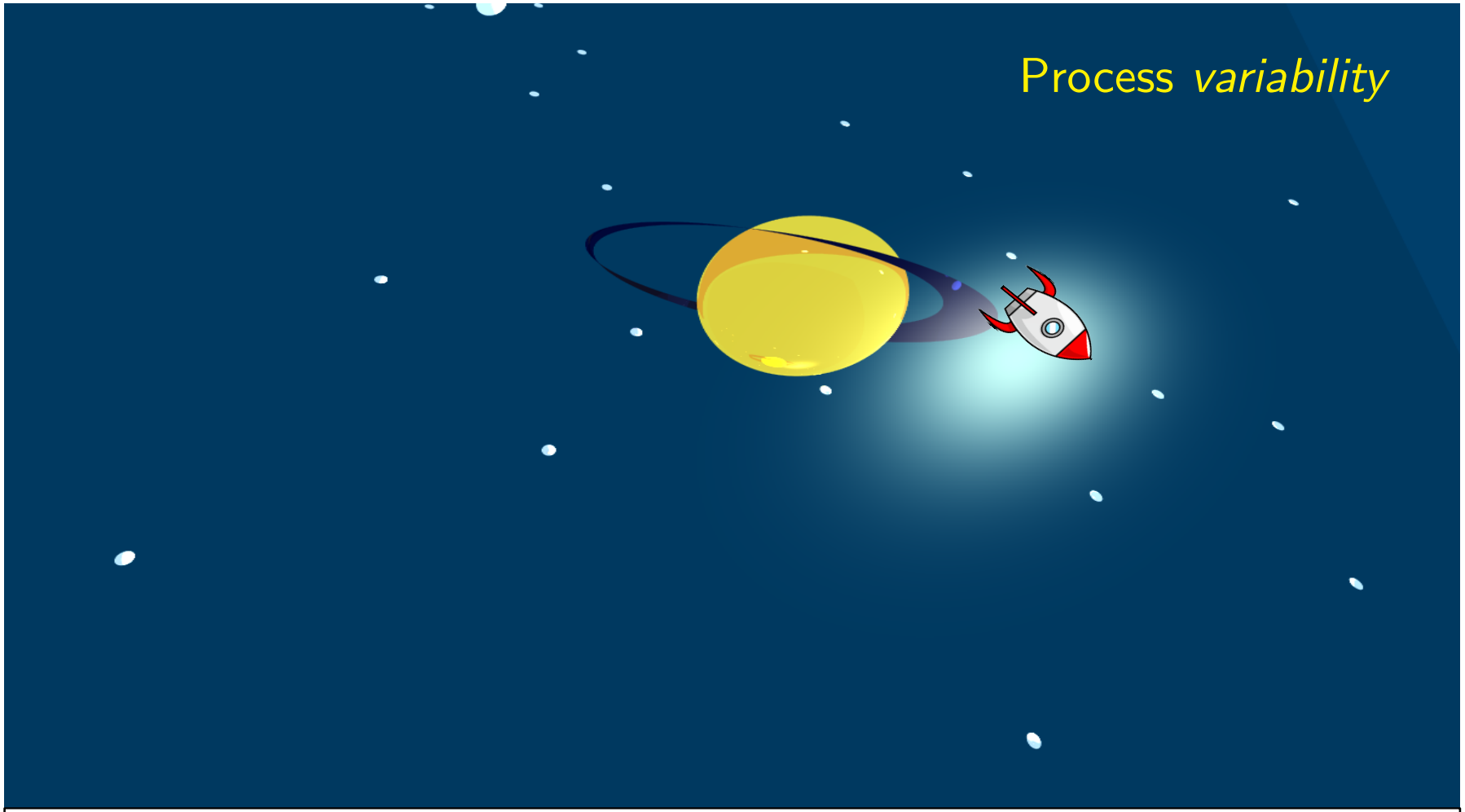
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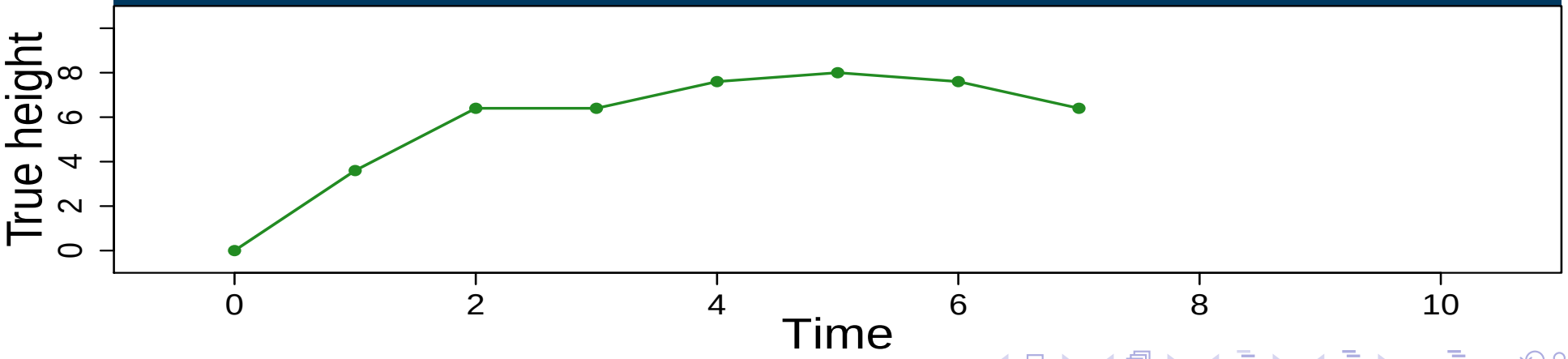
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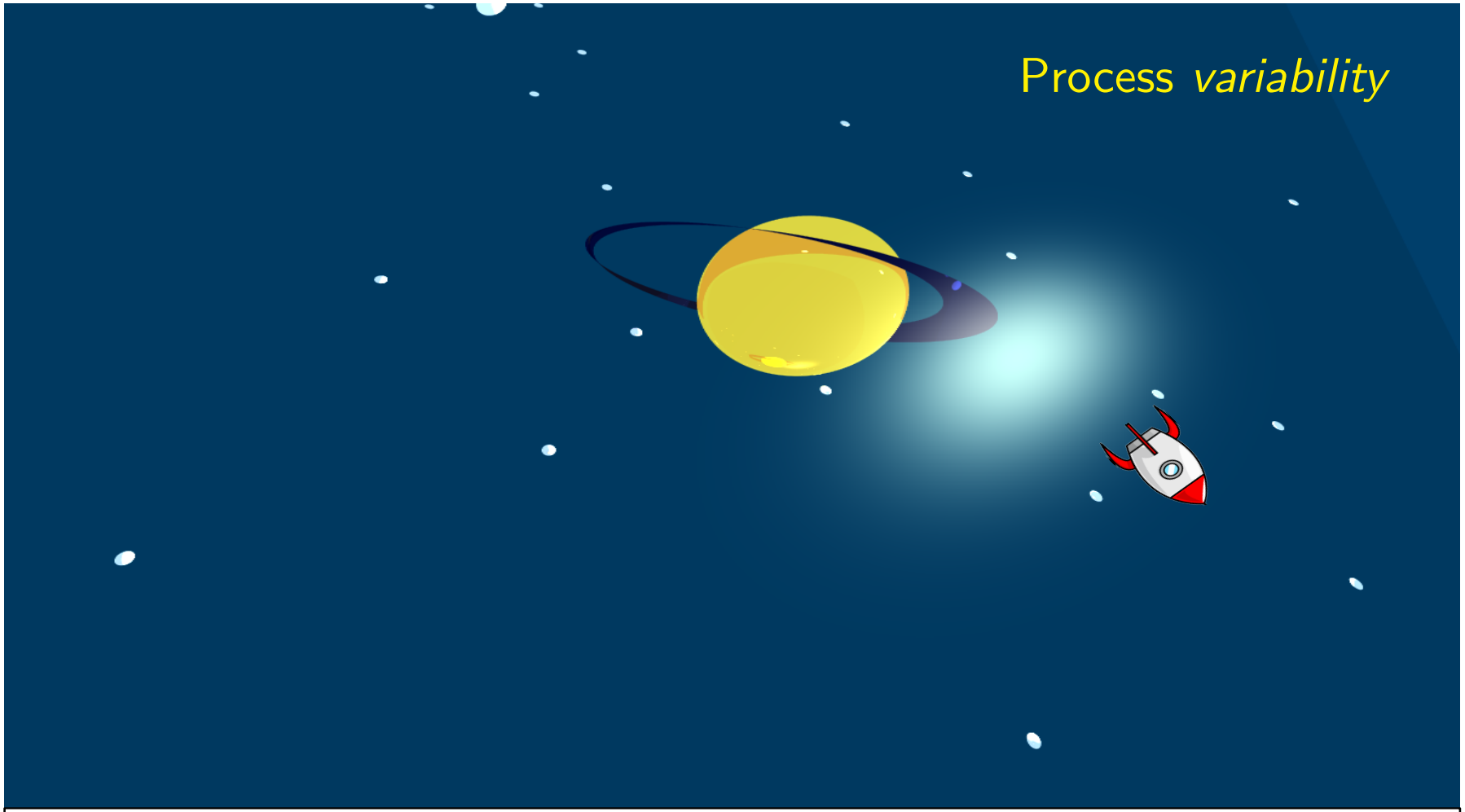
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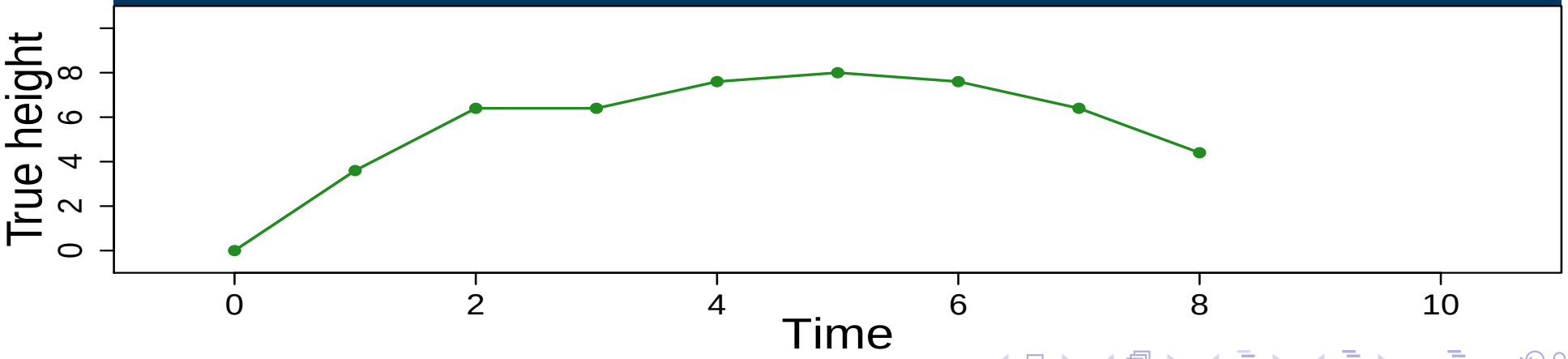
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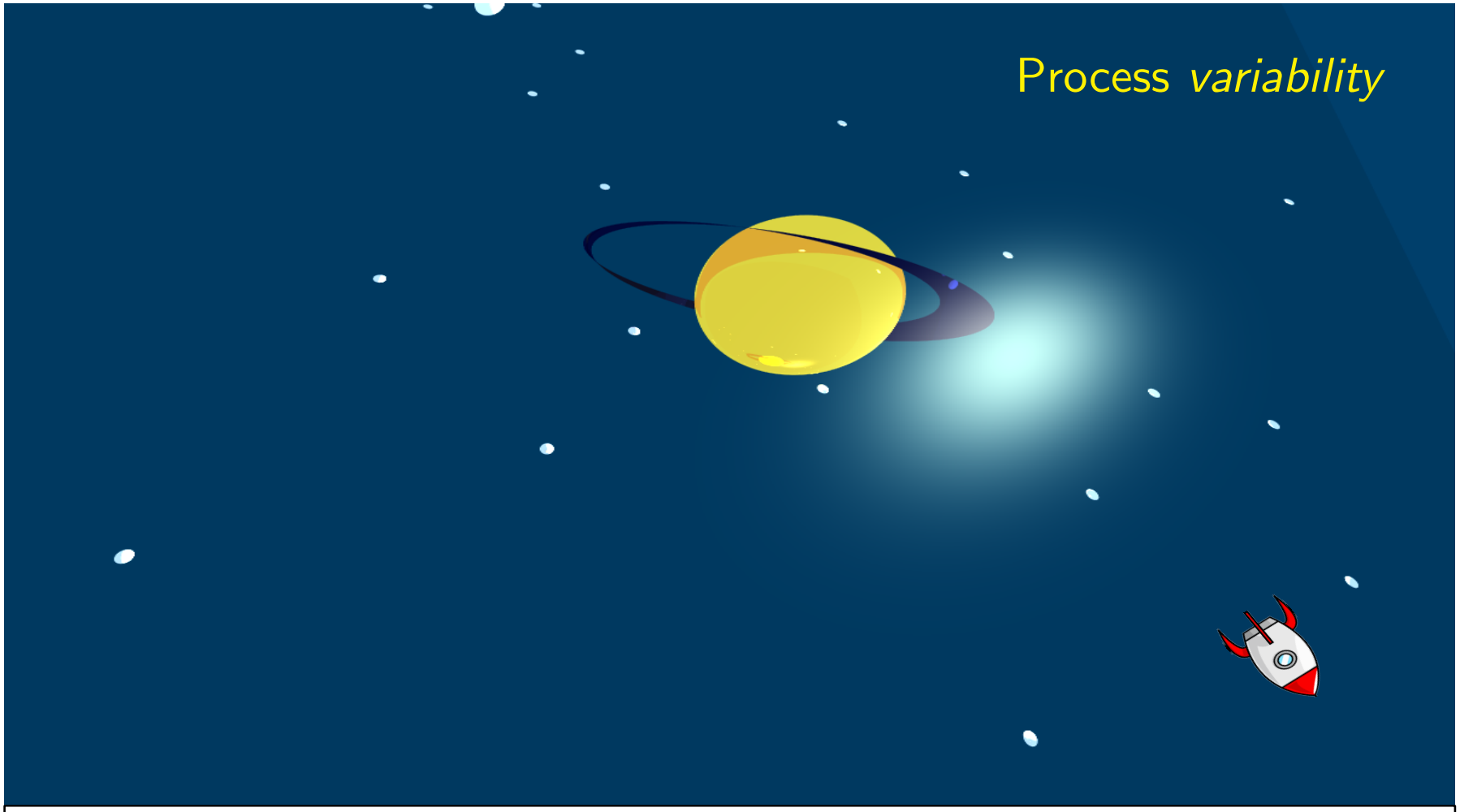
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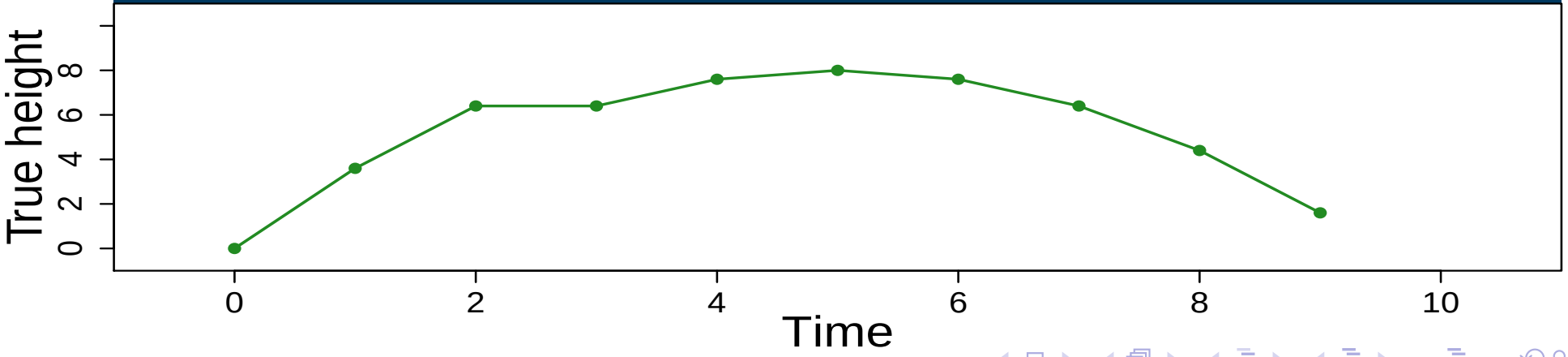
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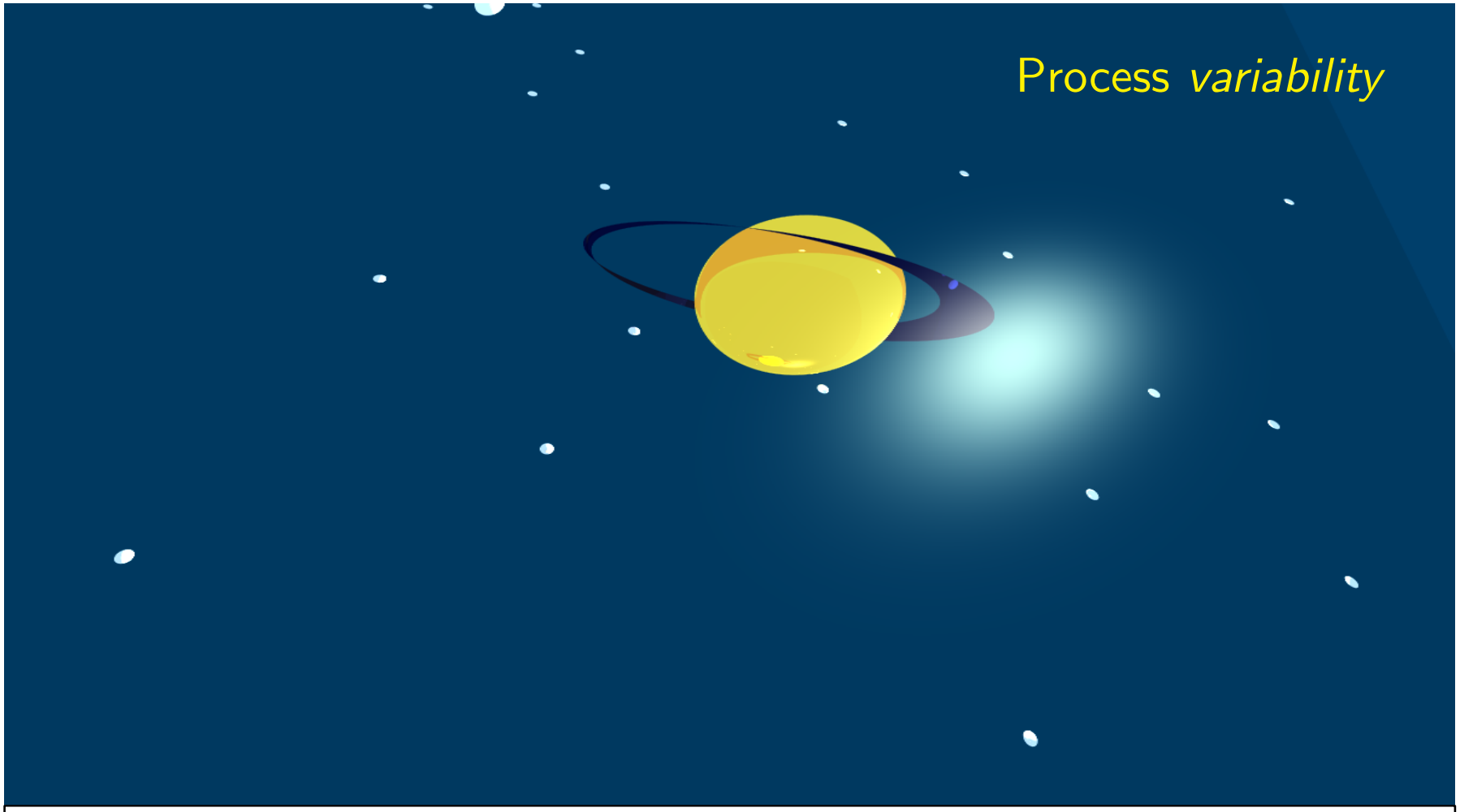
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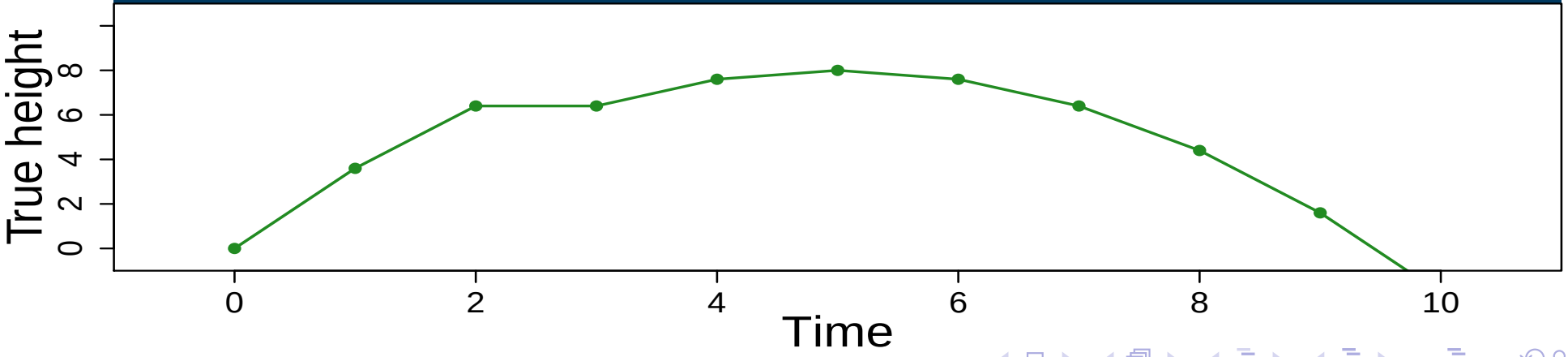
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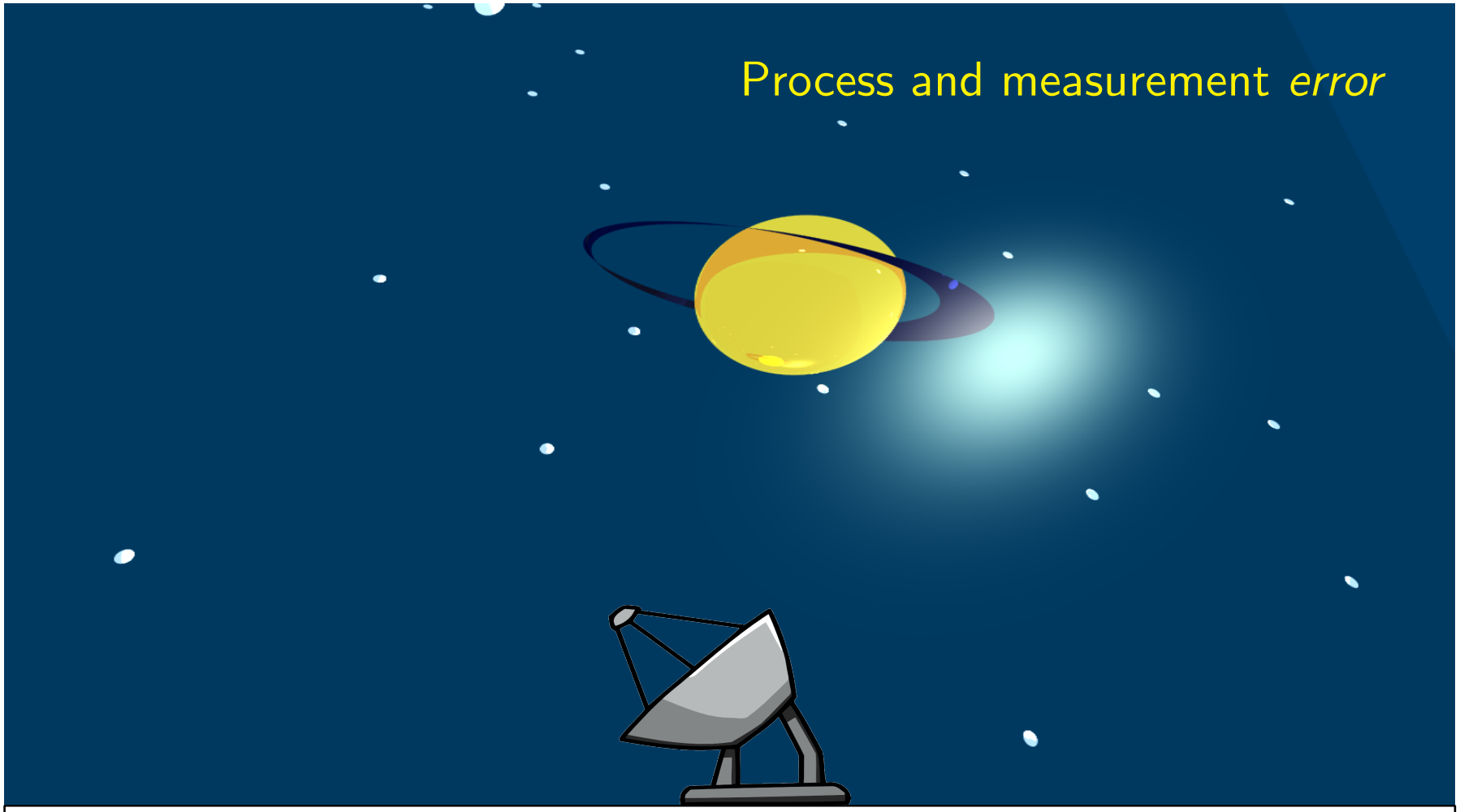
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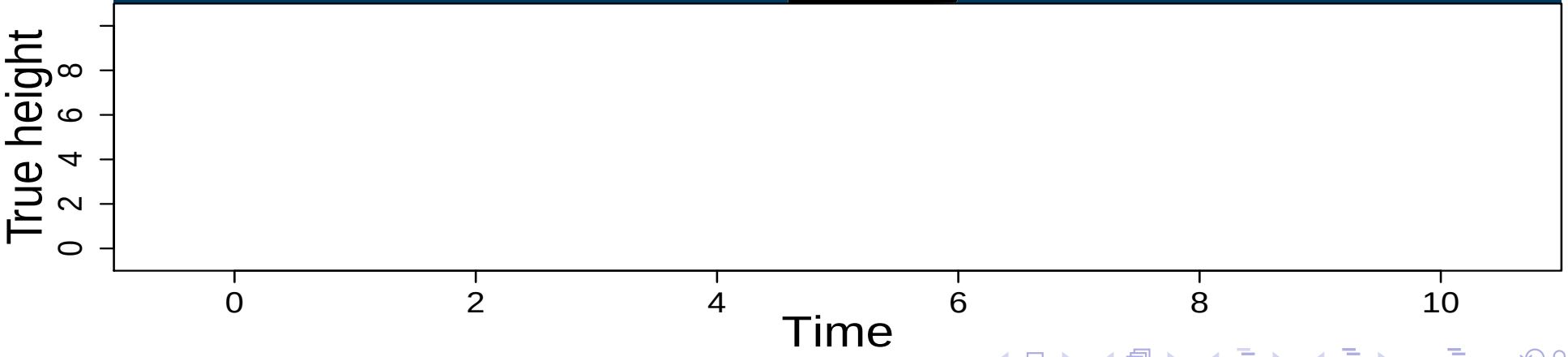
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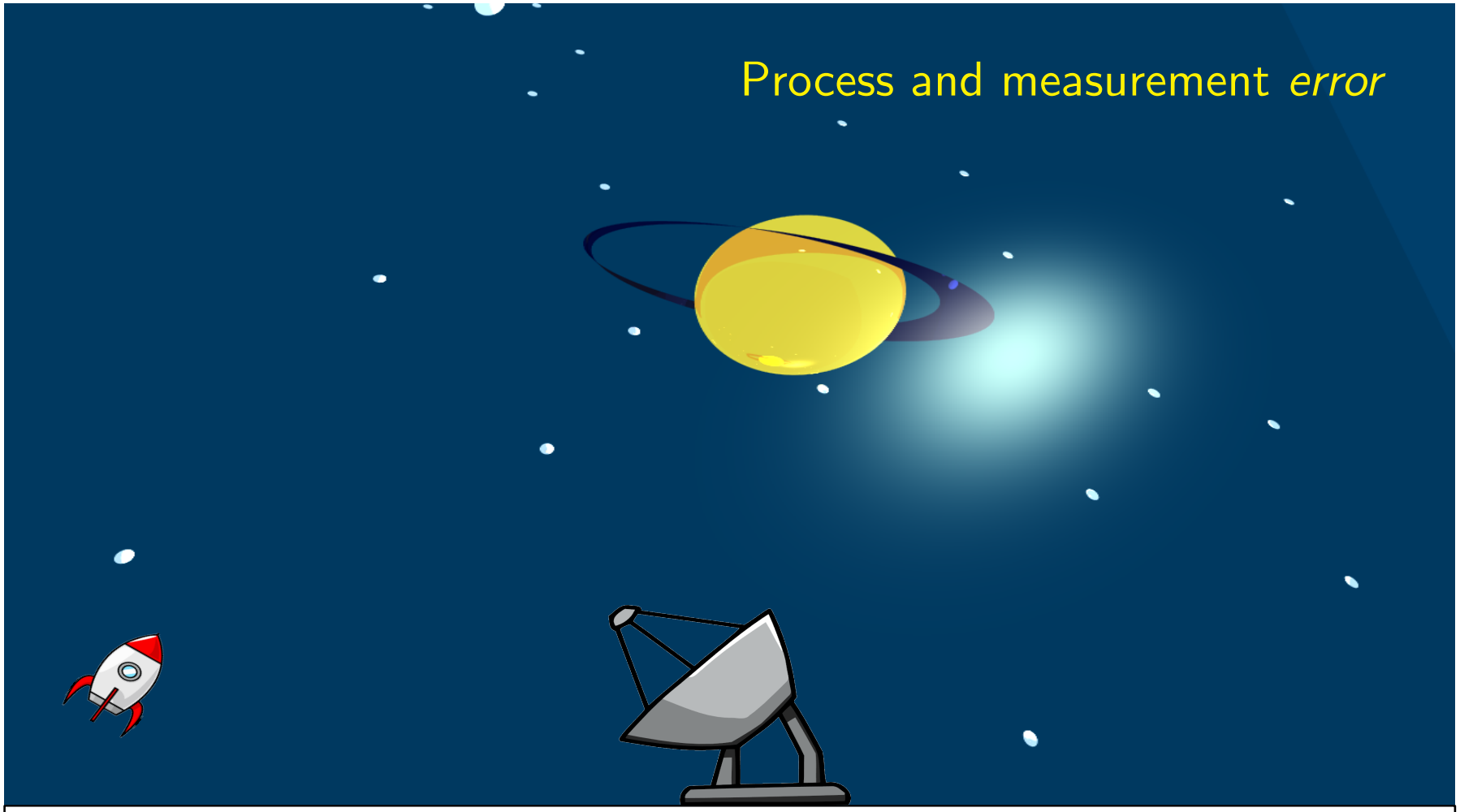
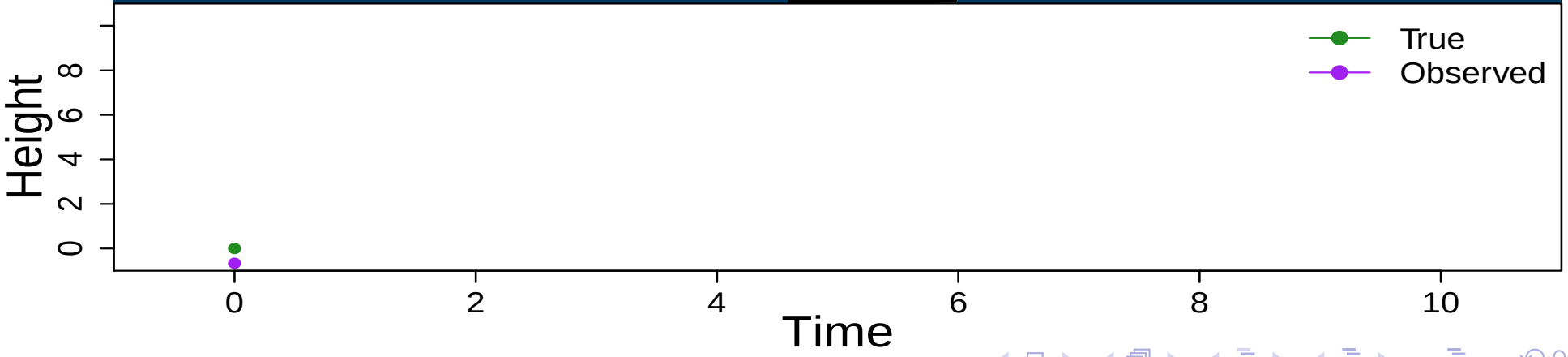
Process and measurement *error*





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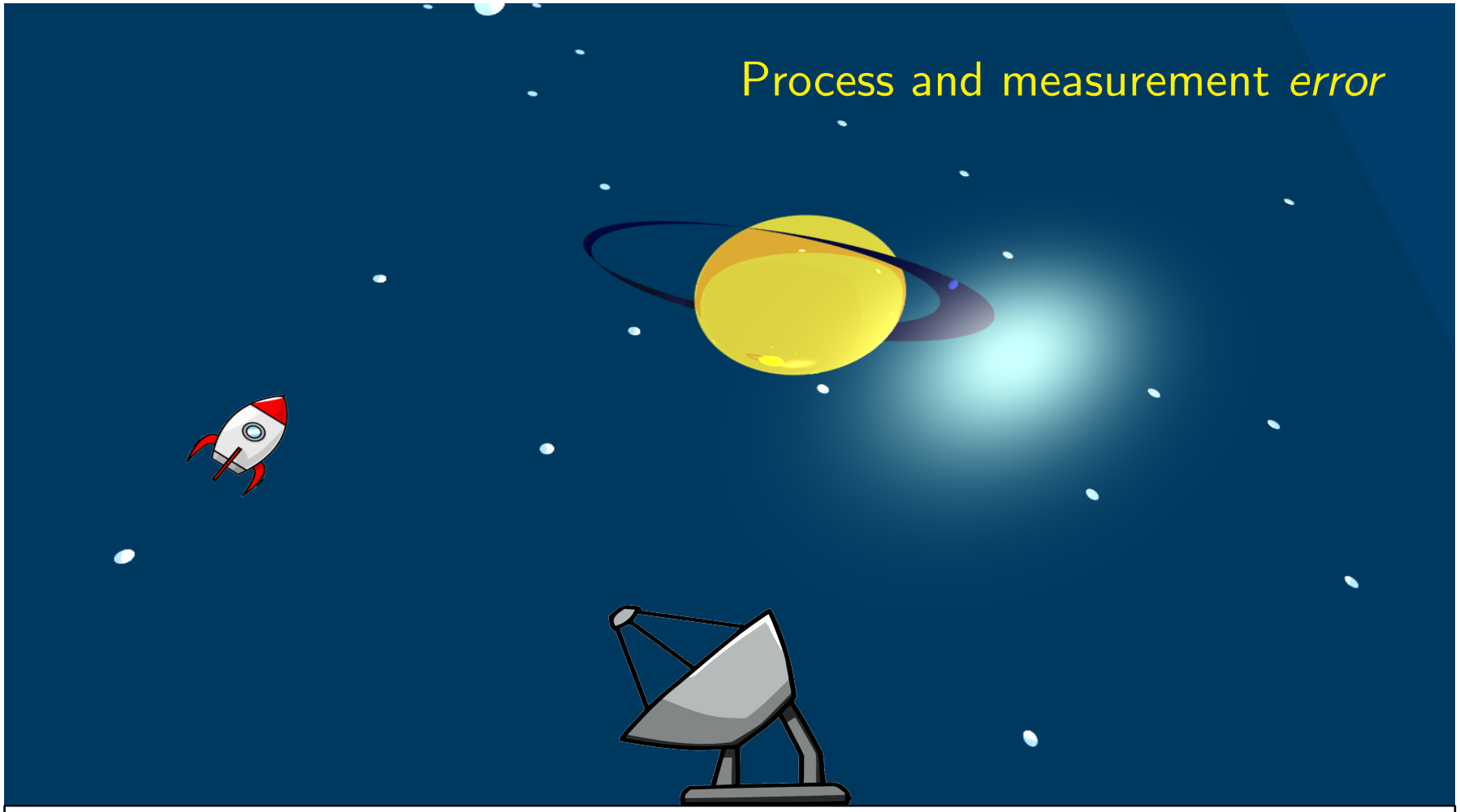

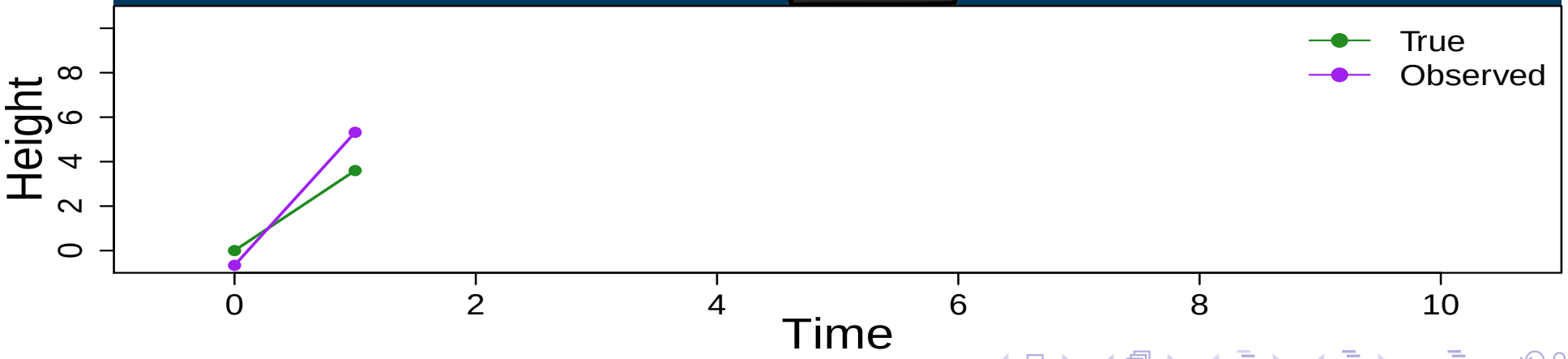

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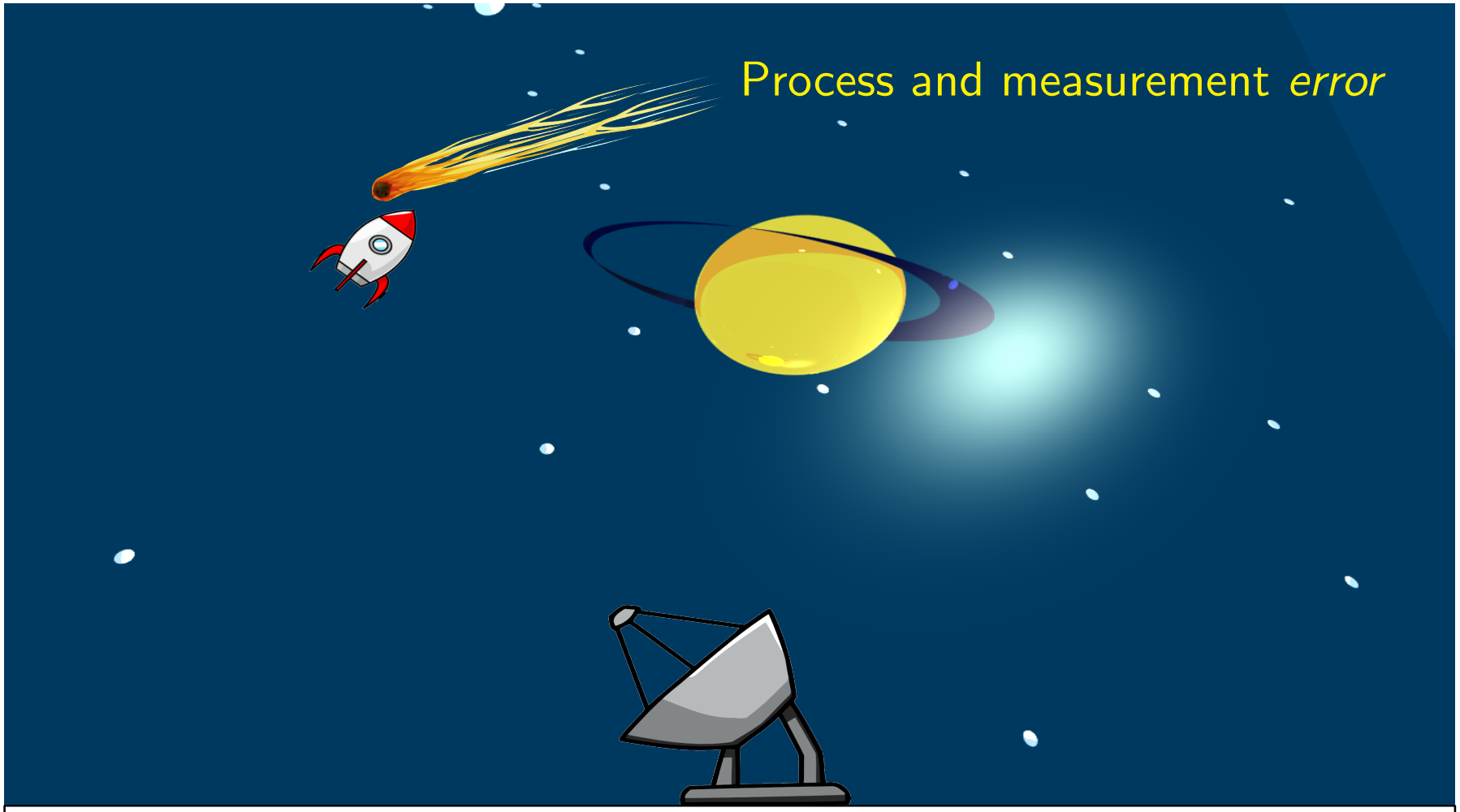

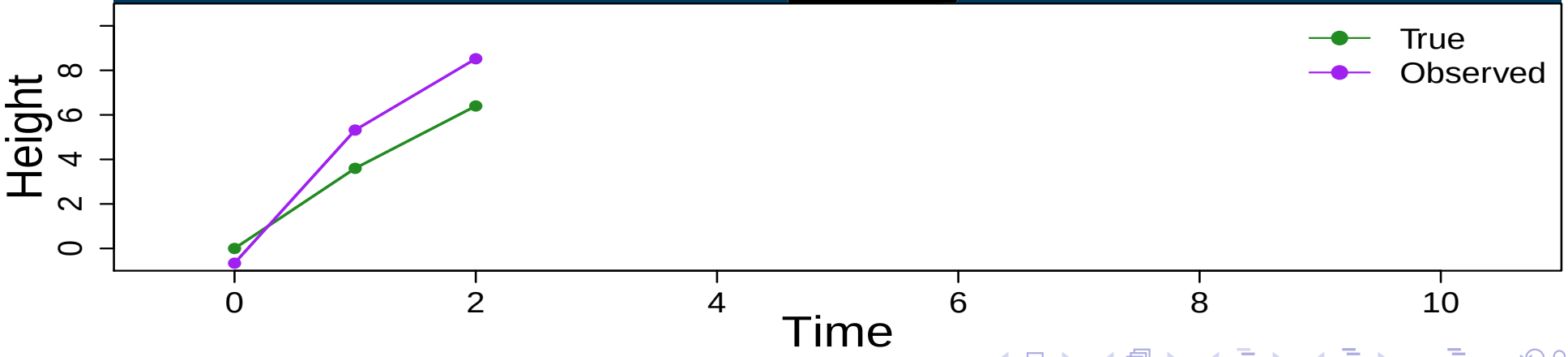

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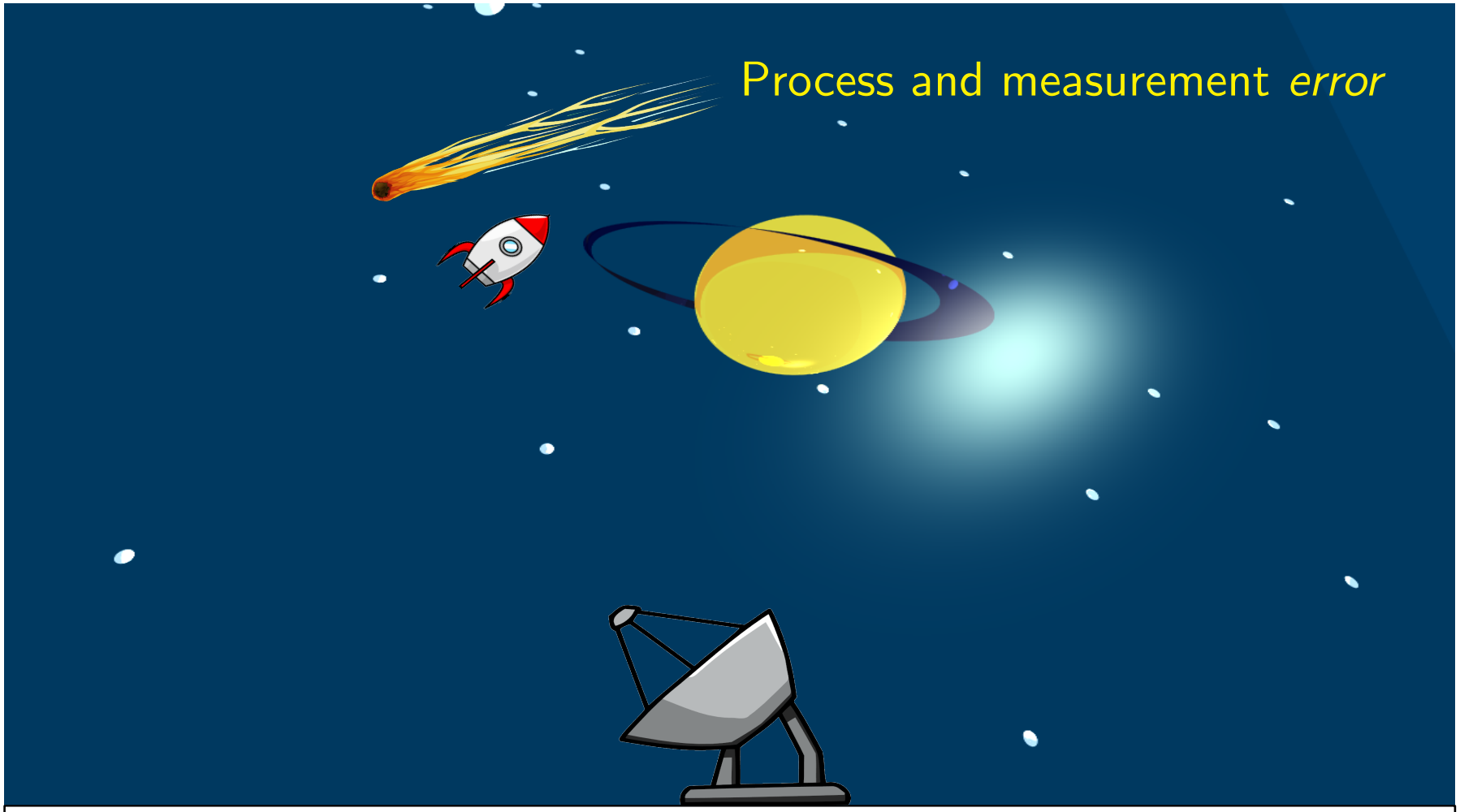

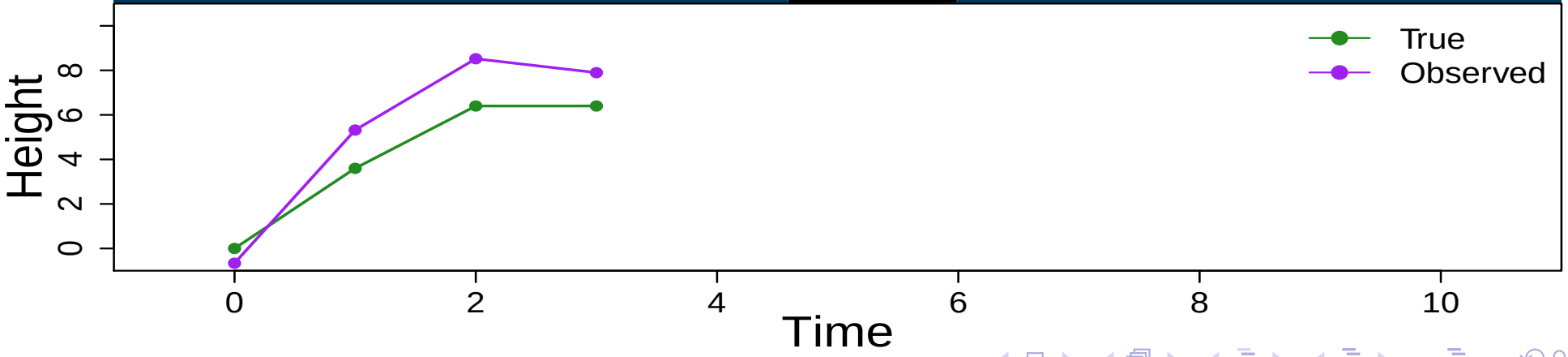

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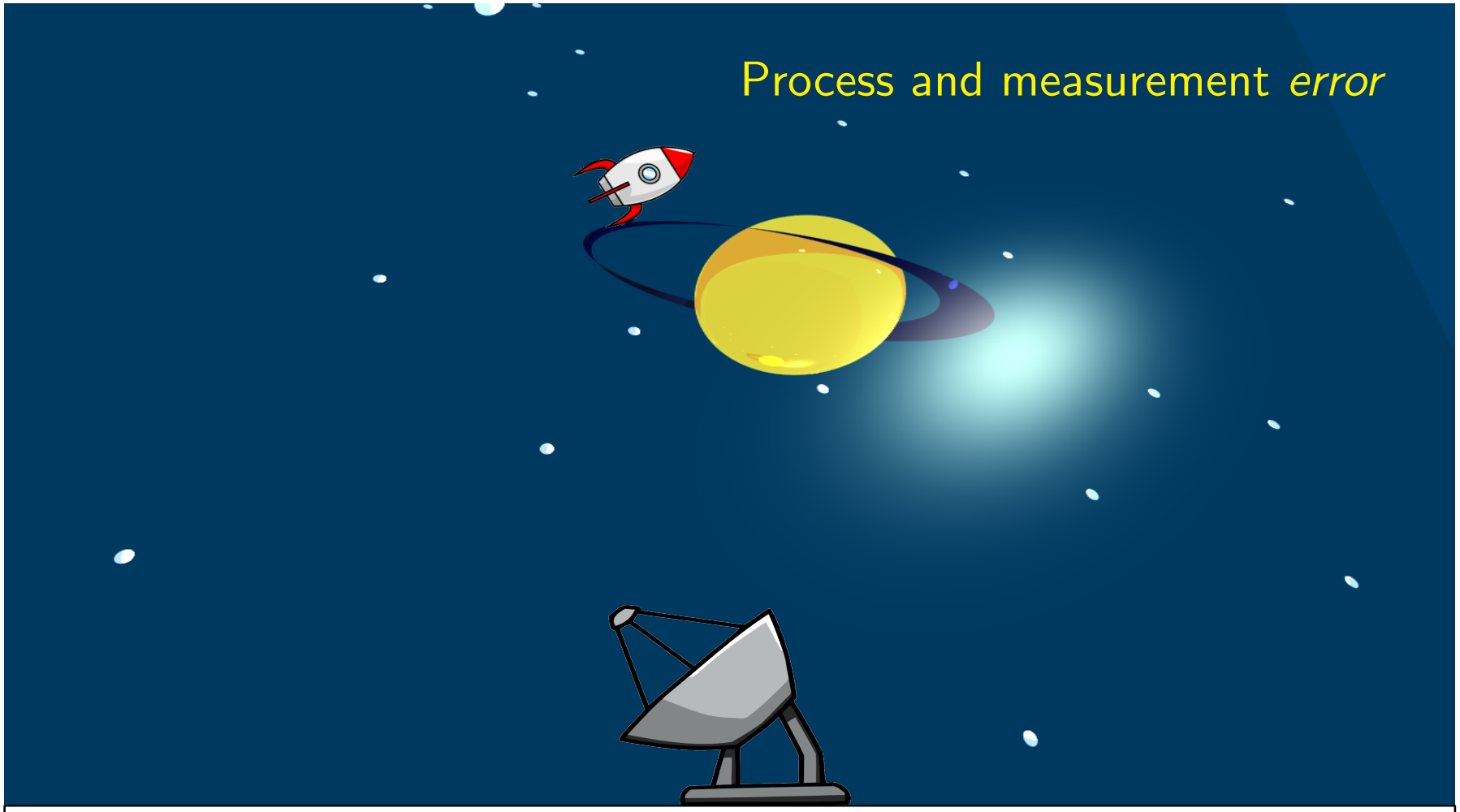

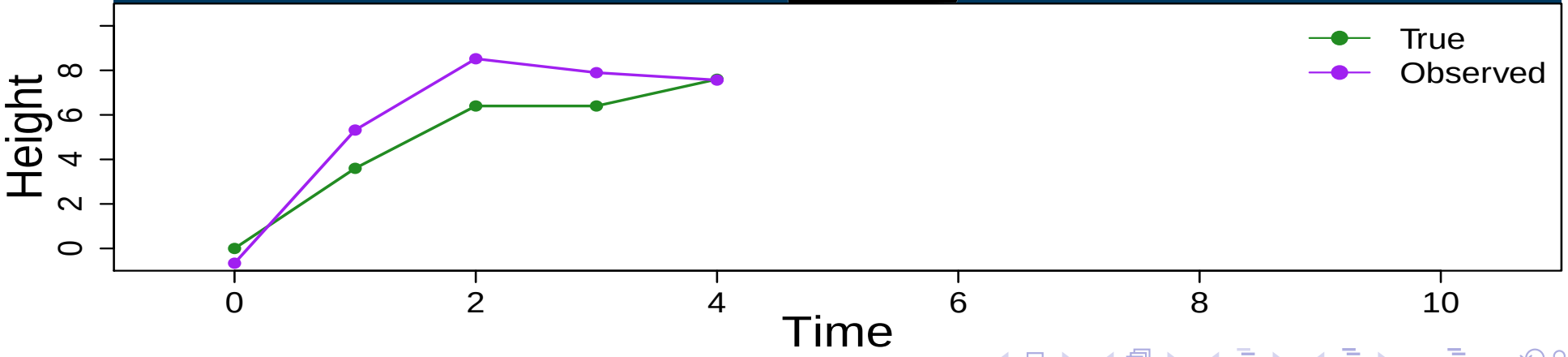

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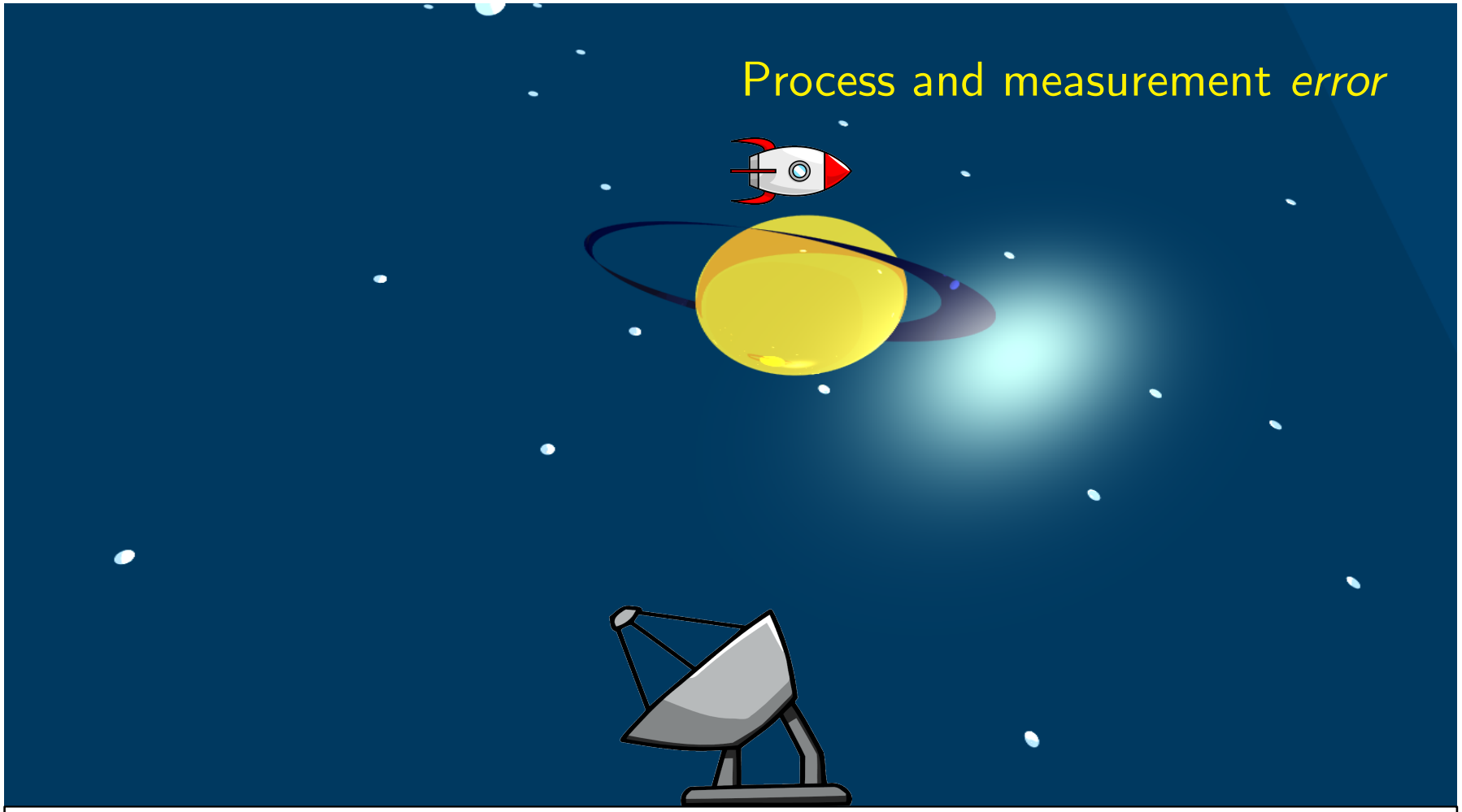

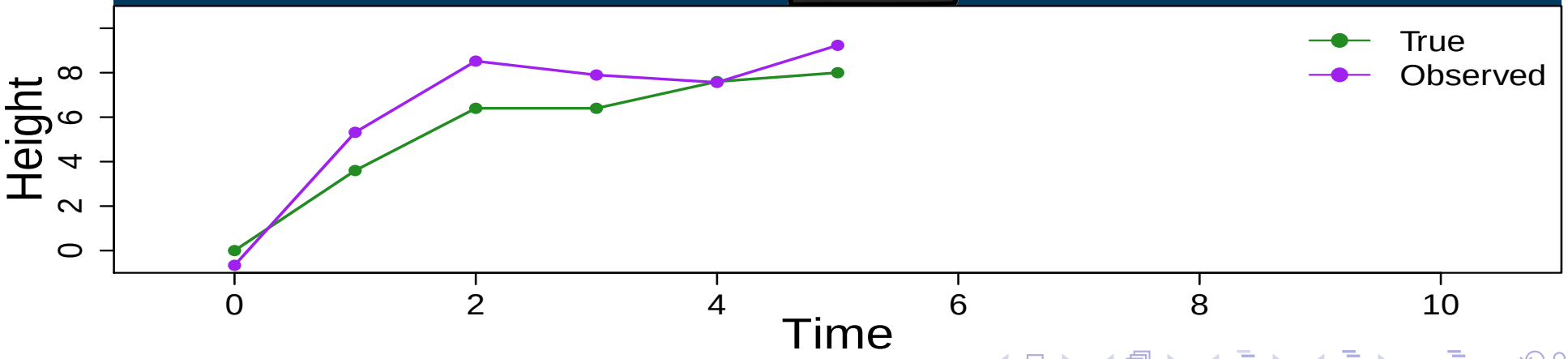

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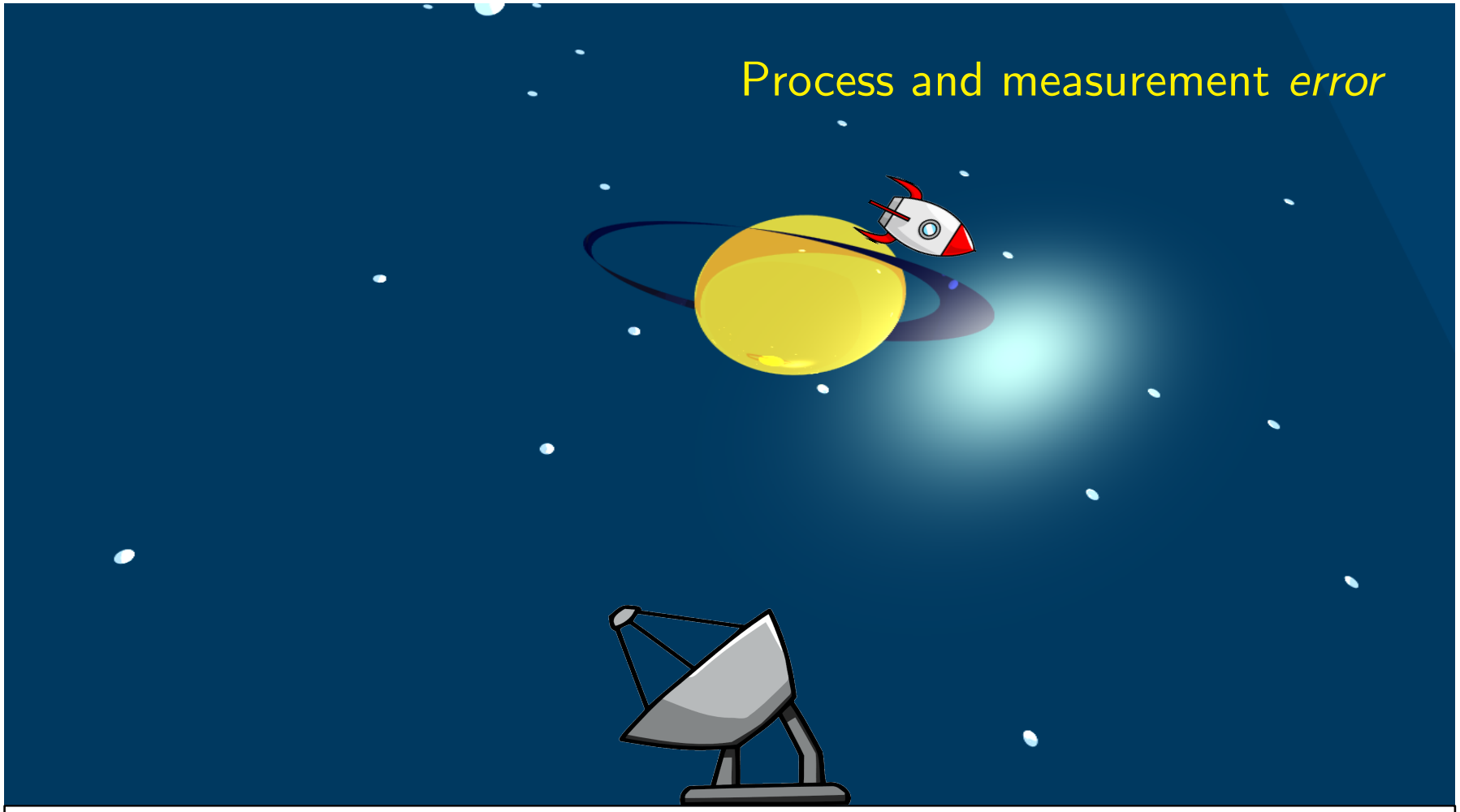

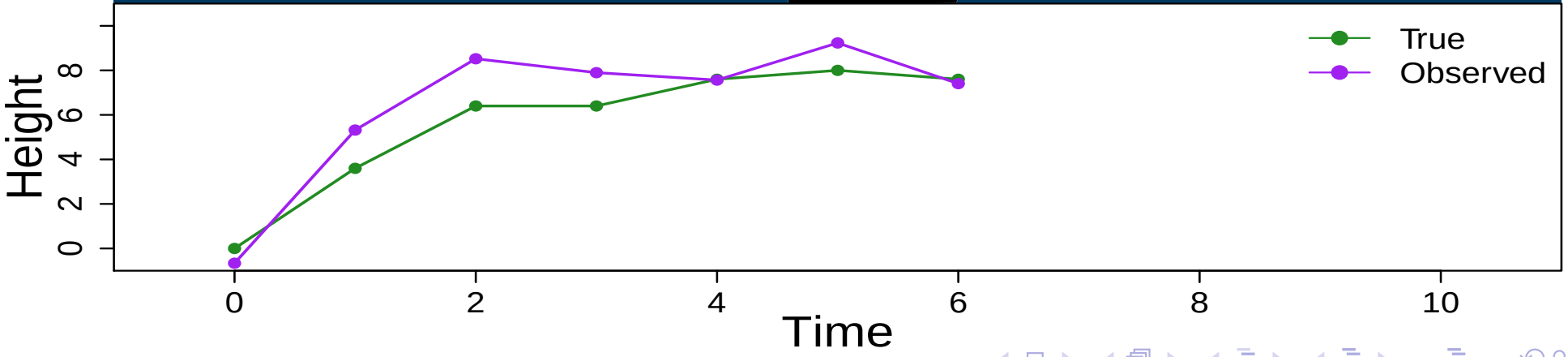

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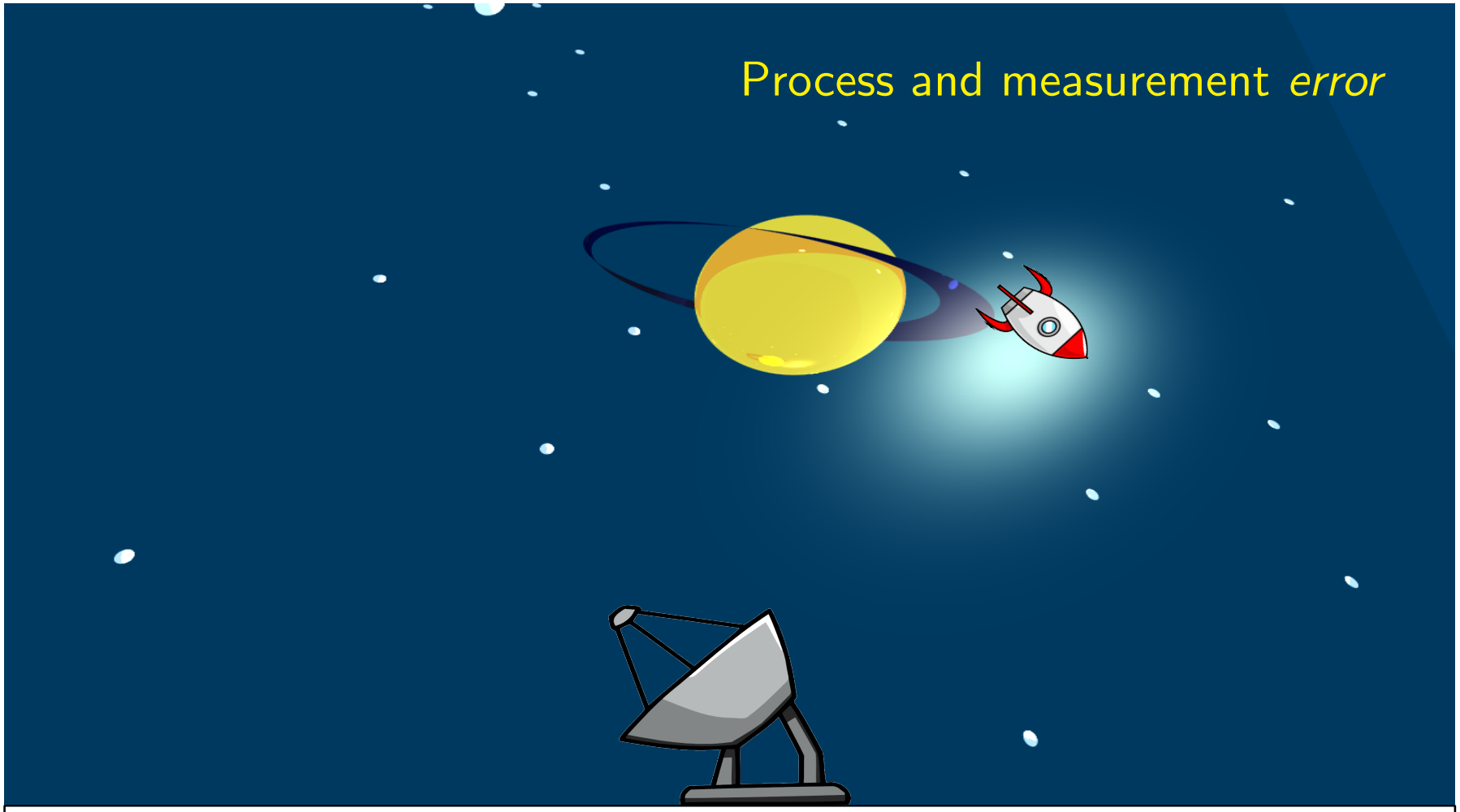

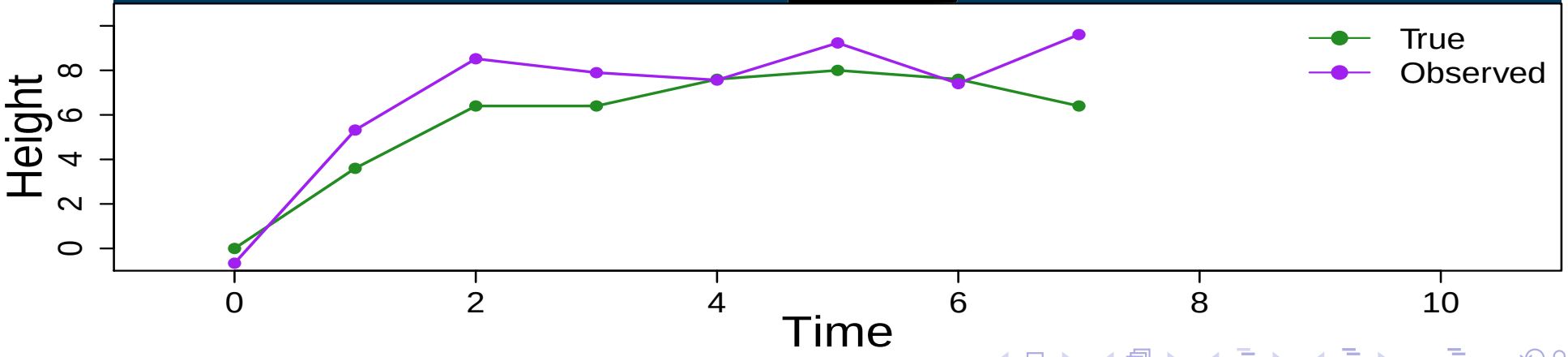

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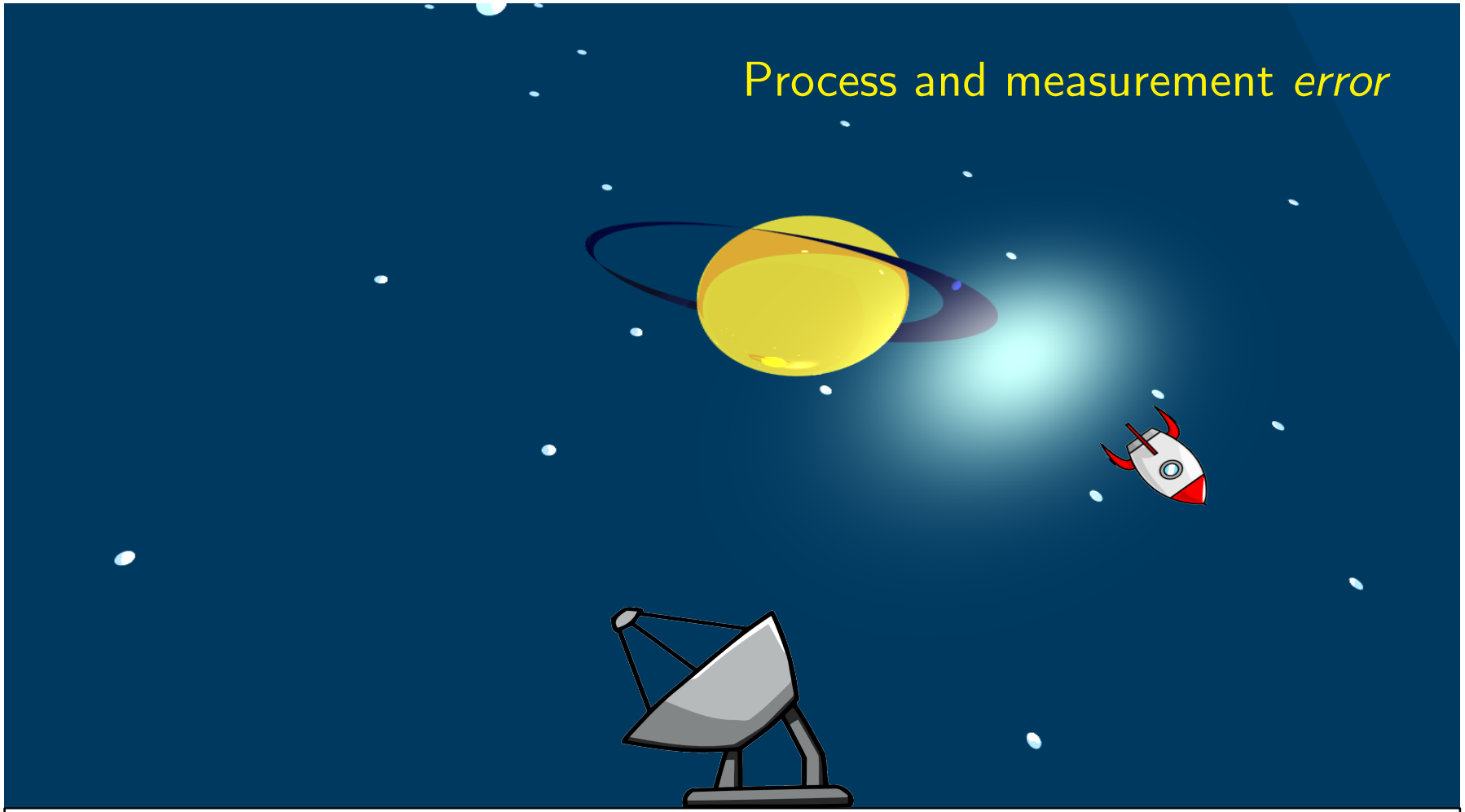

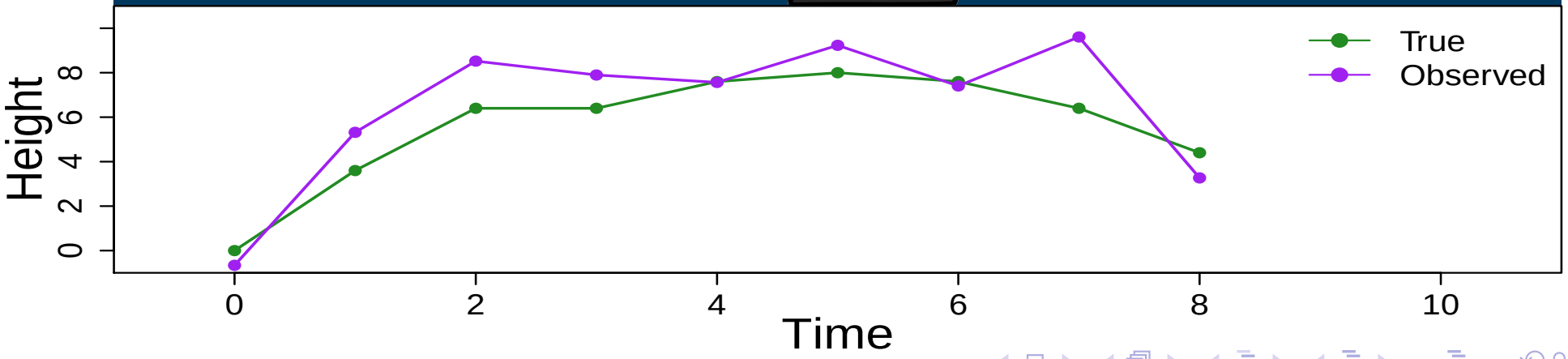

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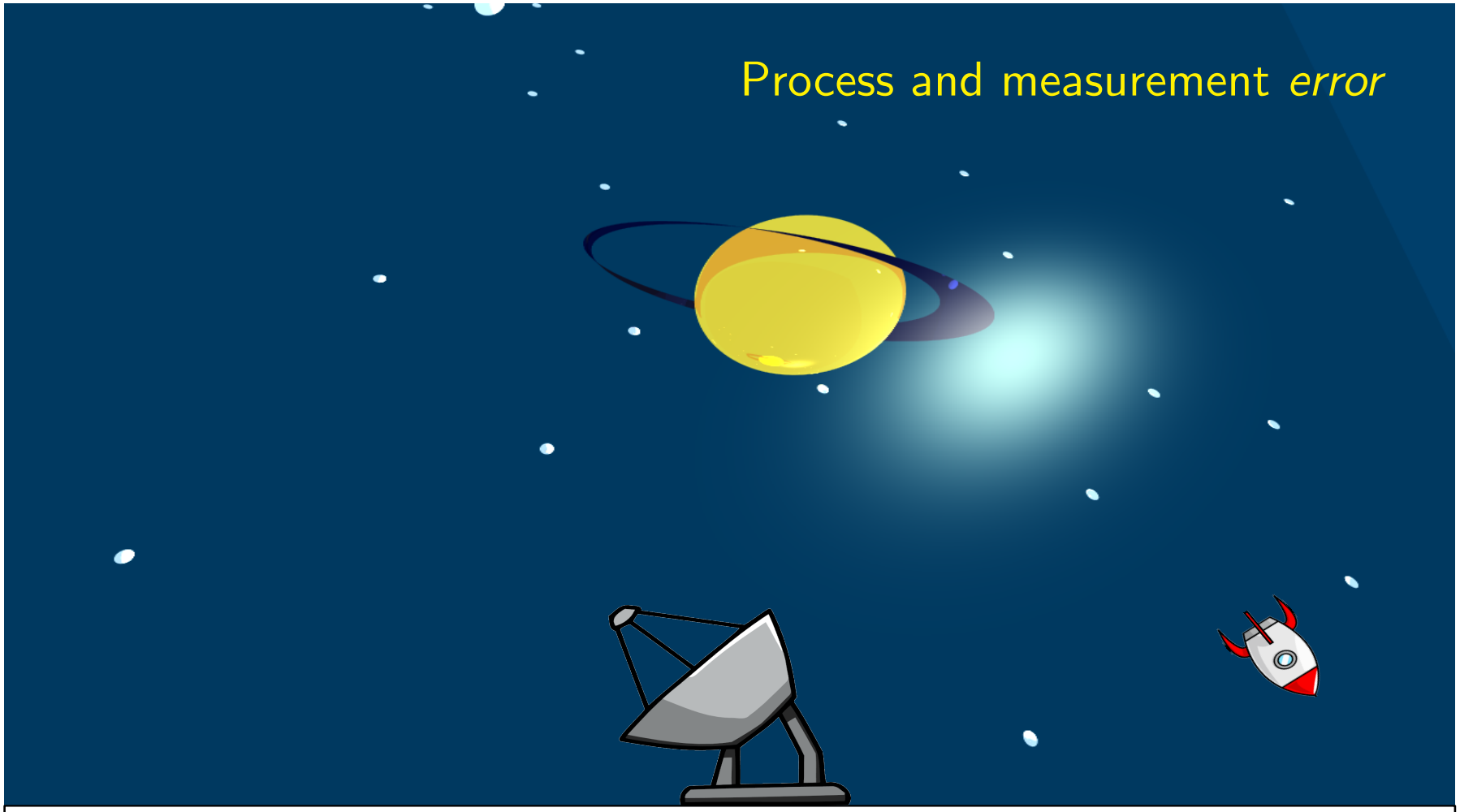

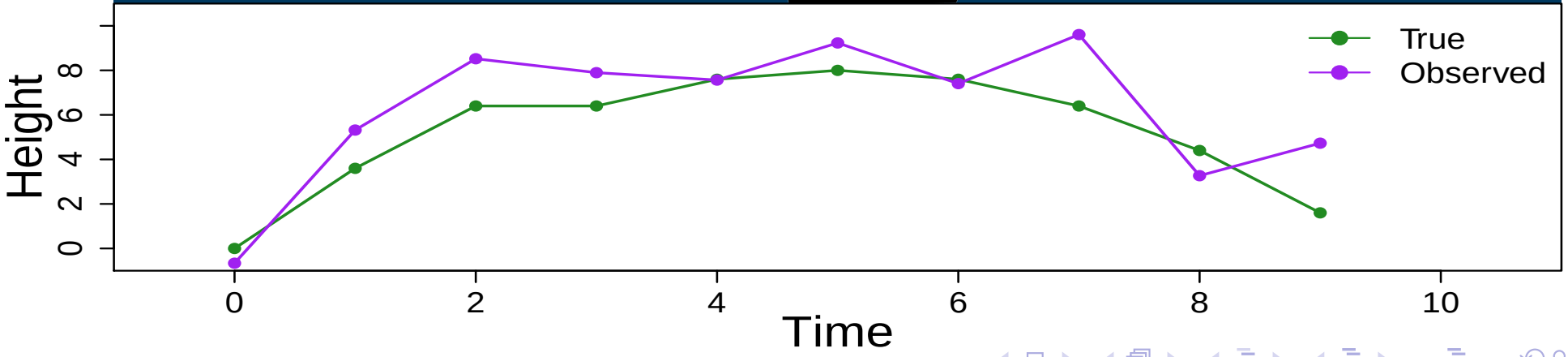

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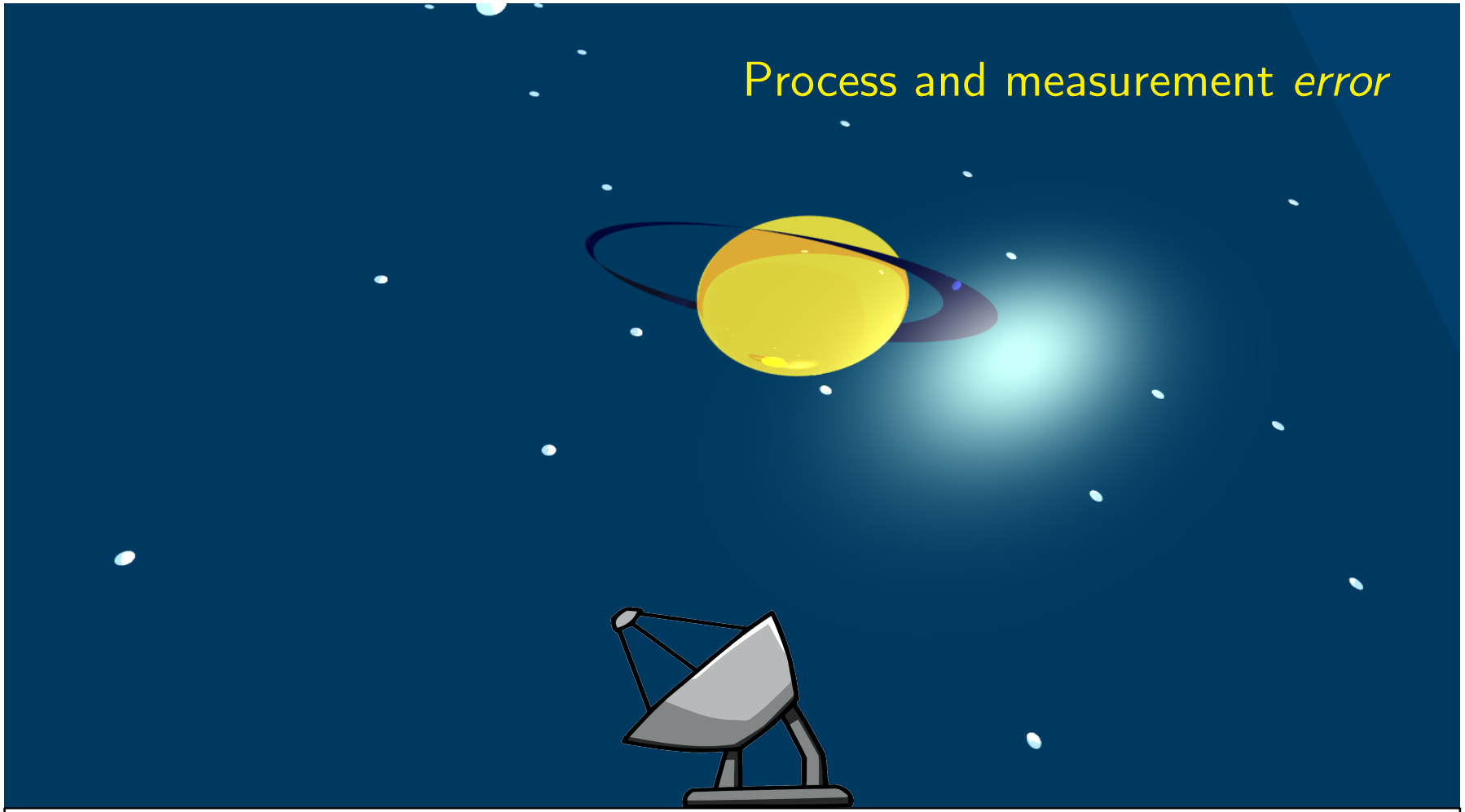

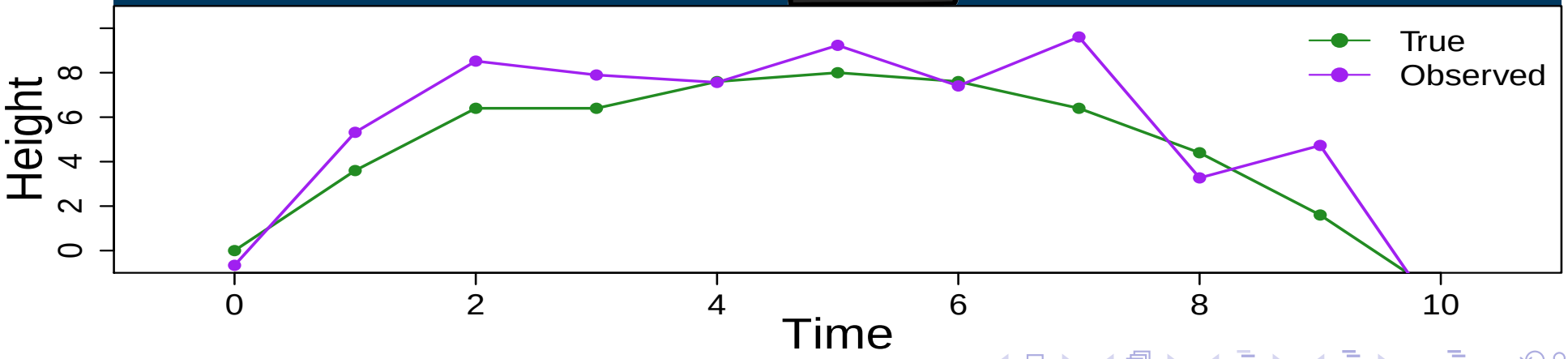

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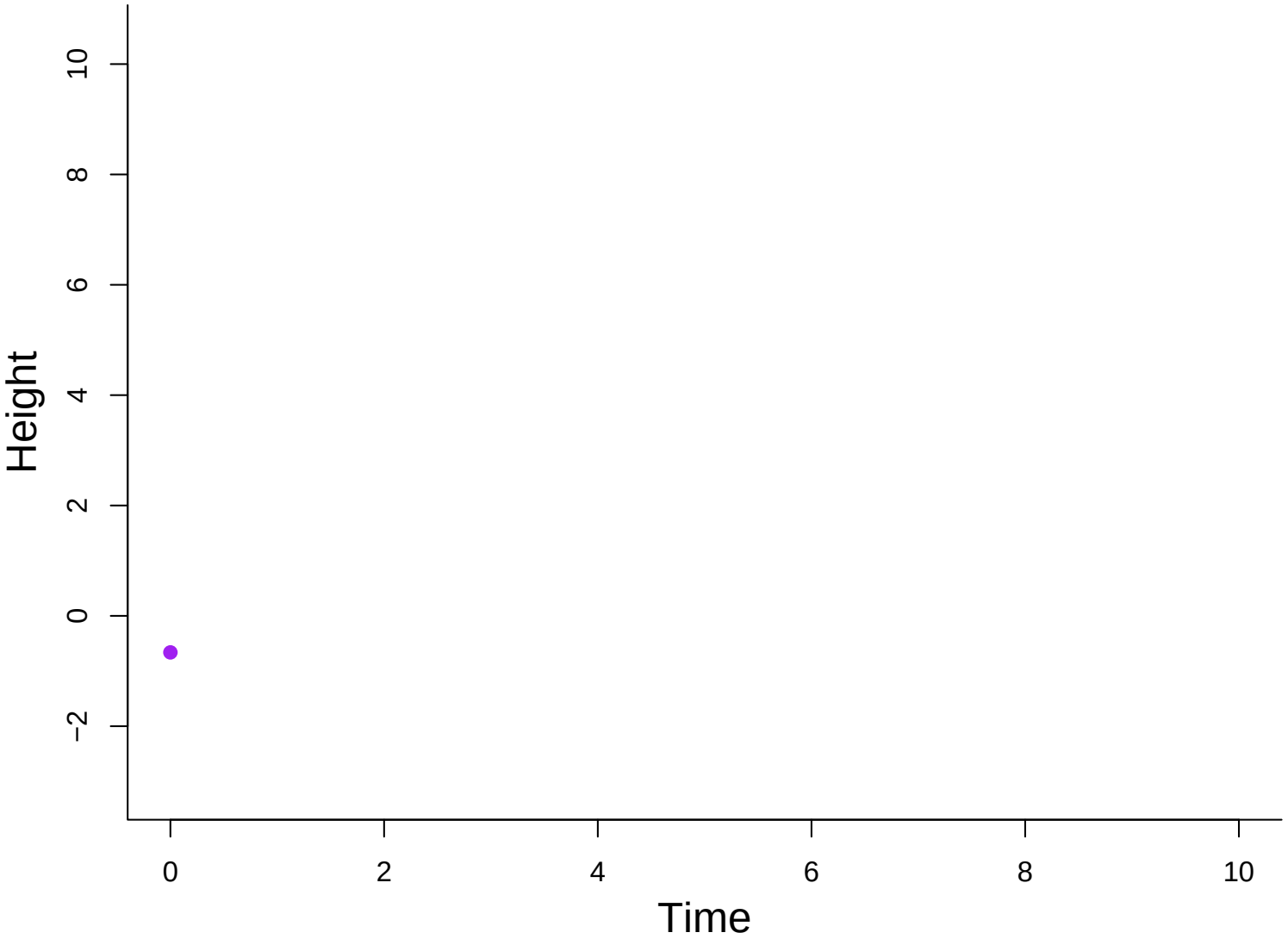

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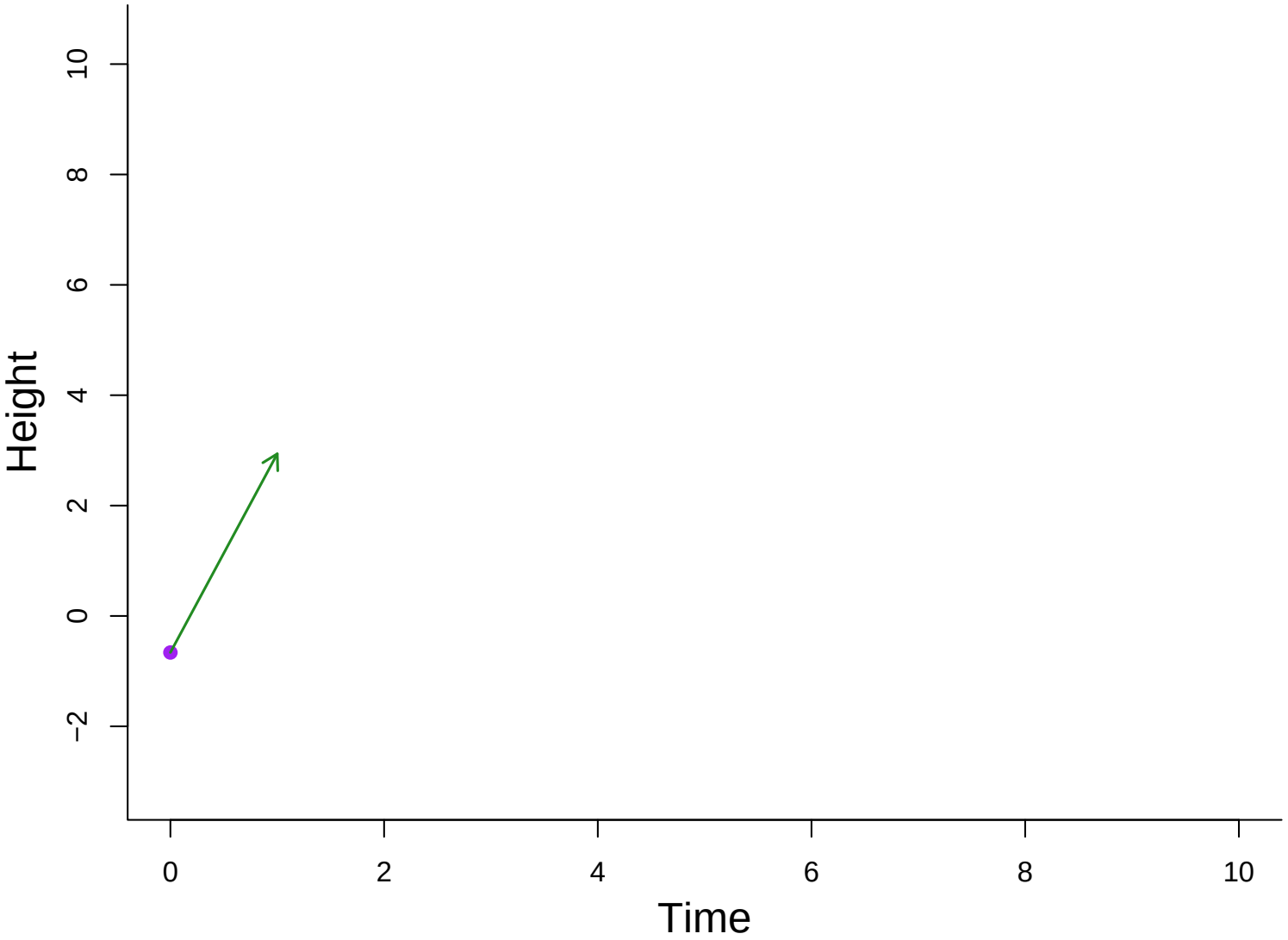

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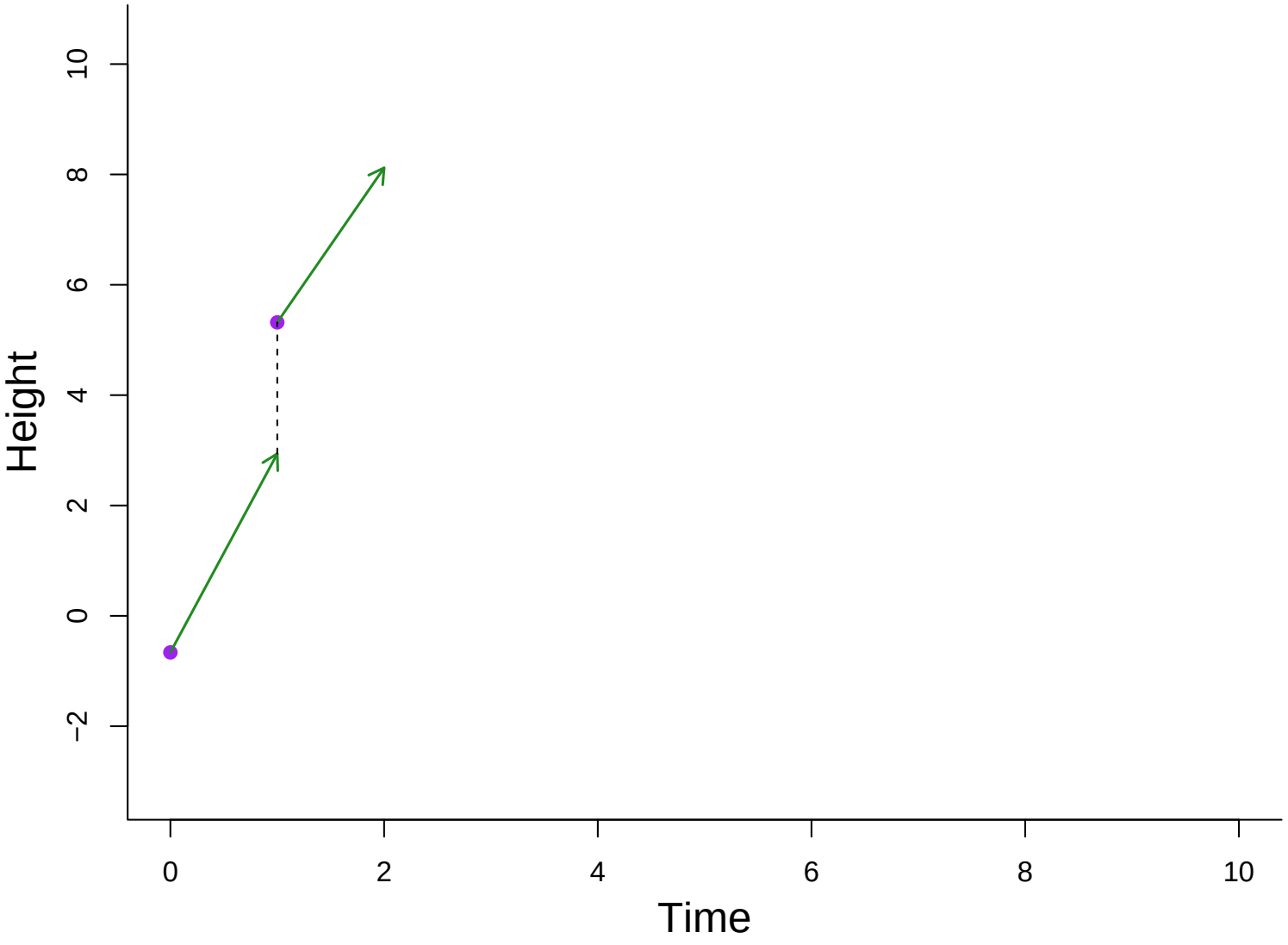

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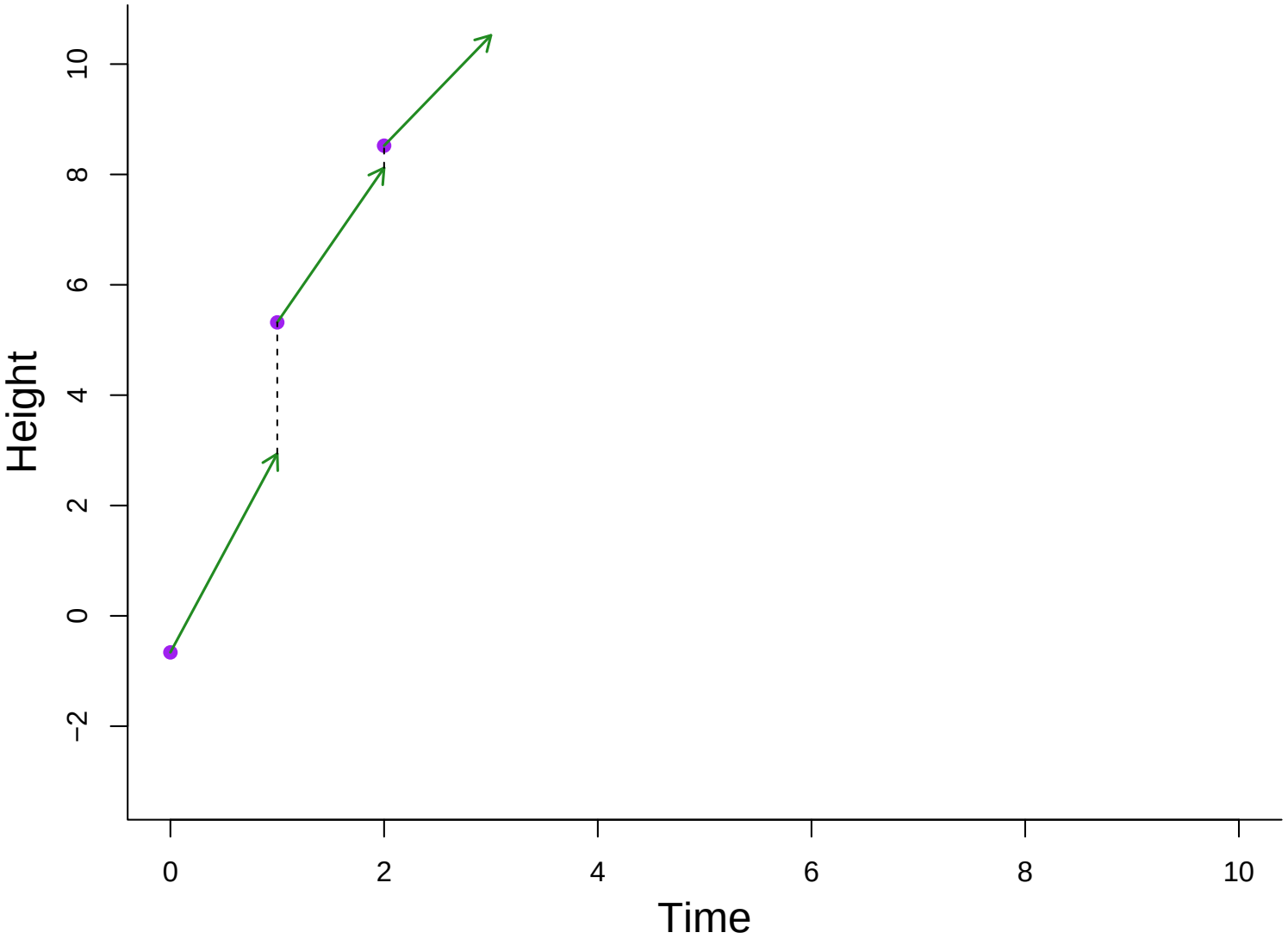

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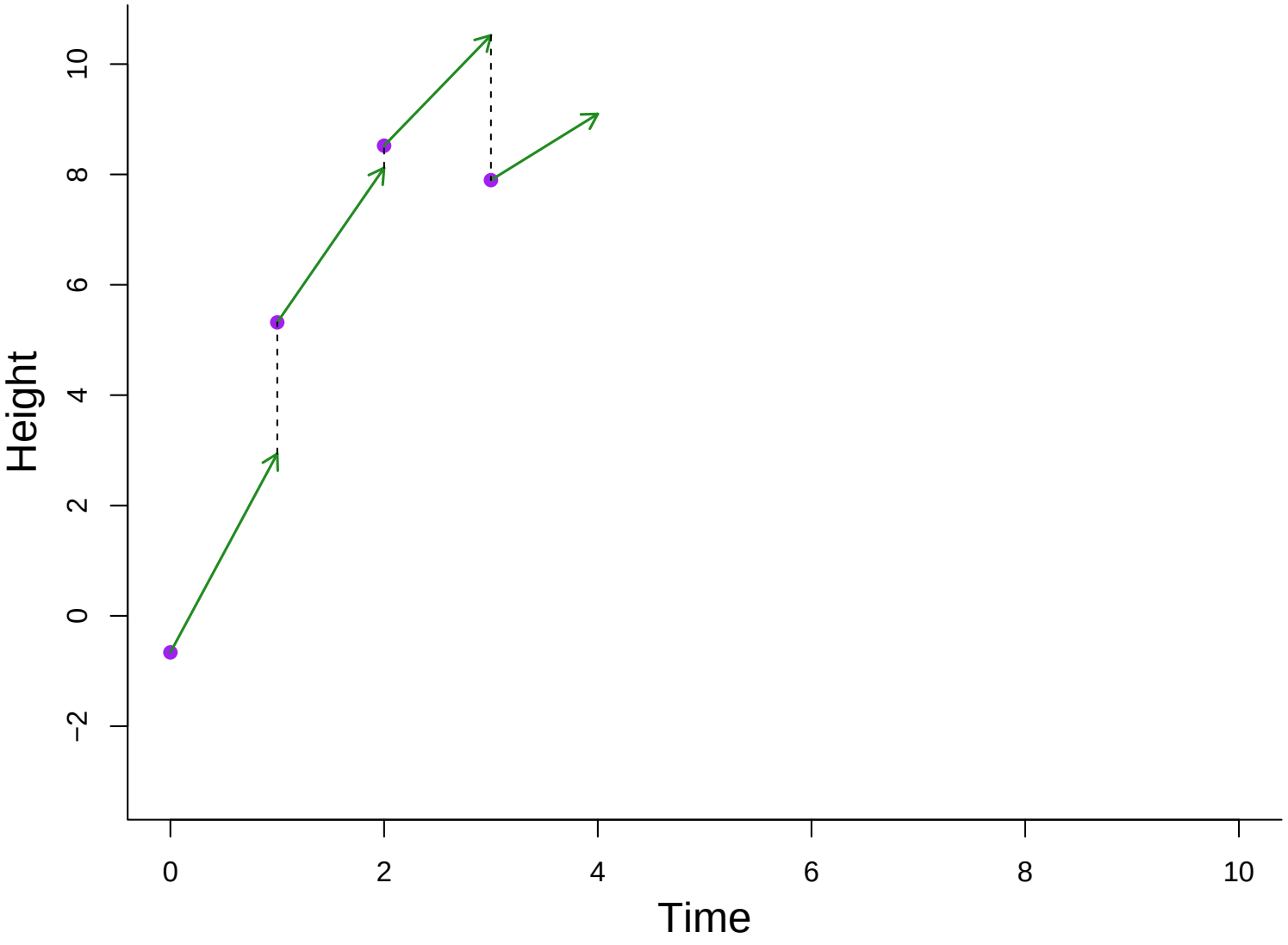

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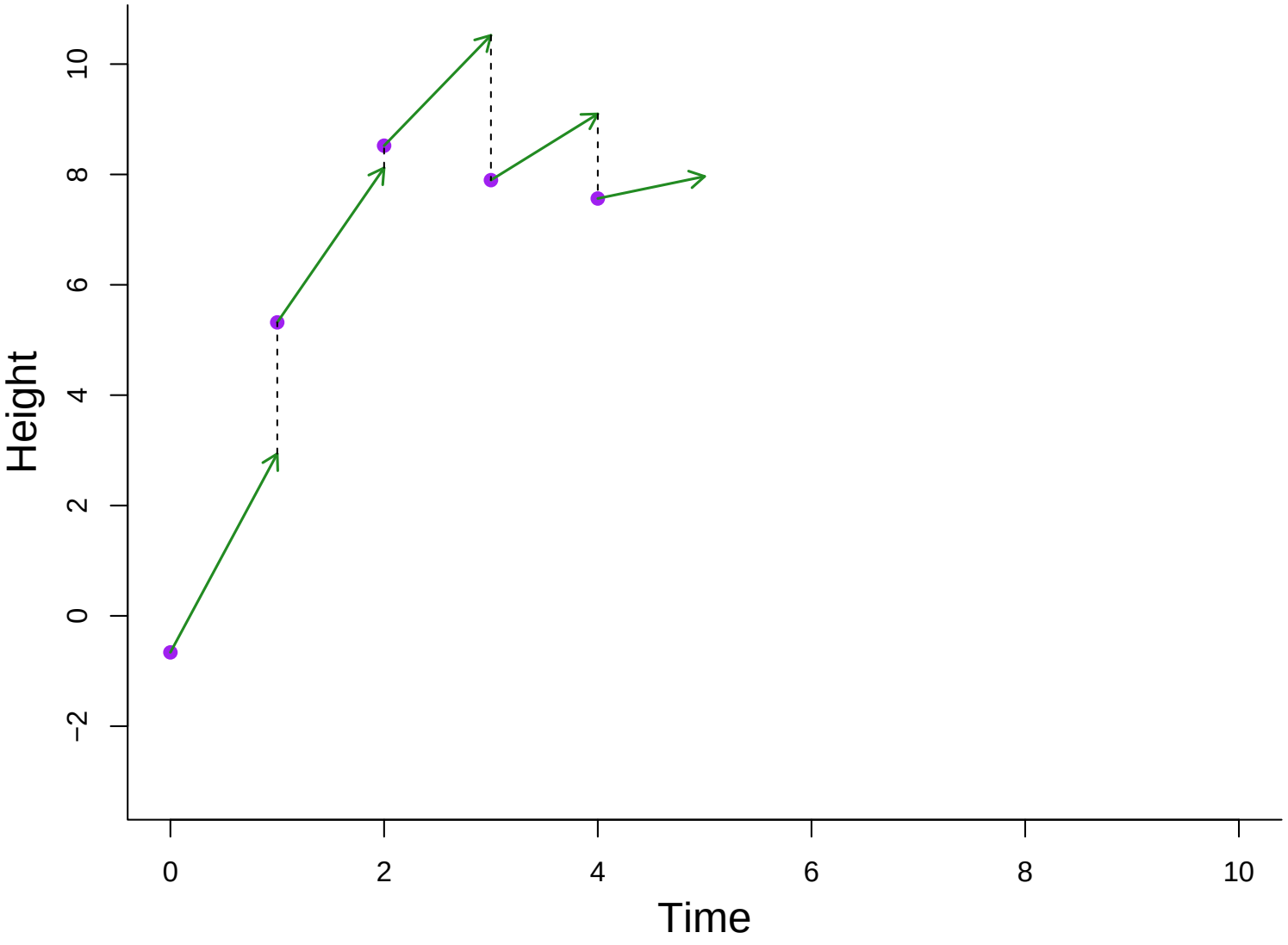

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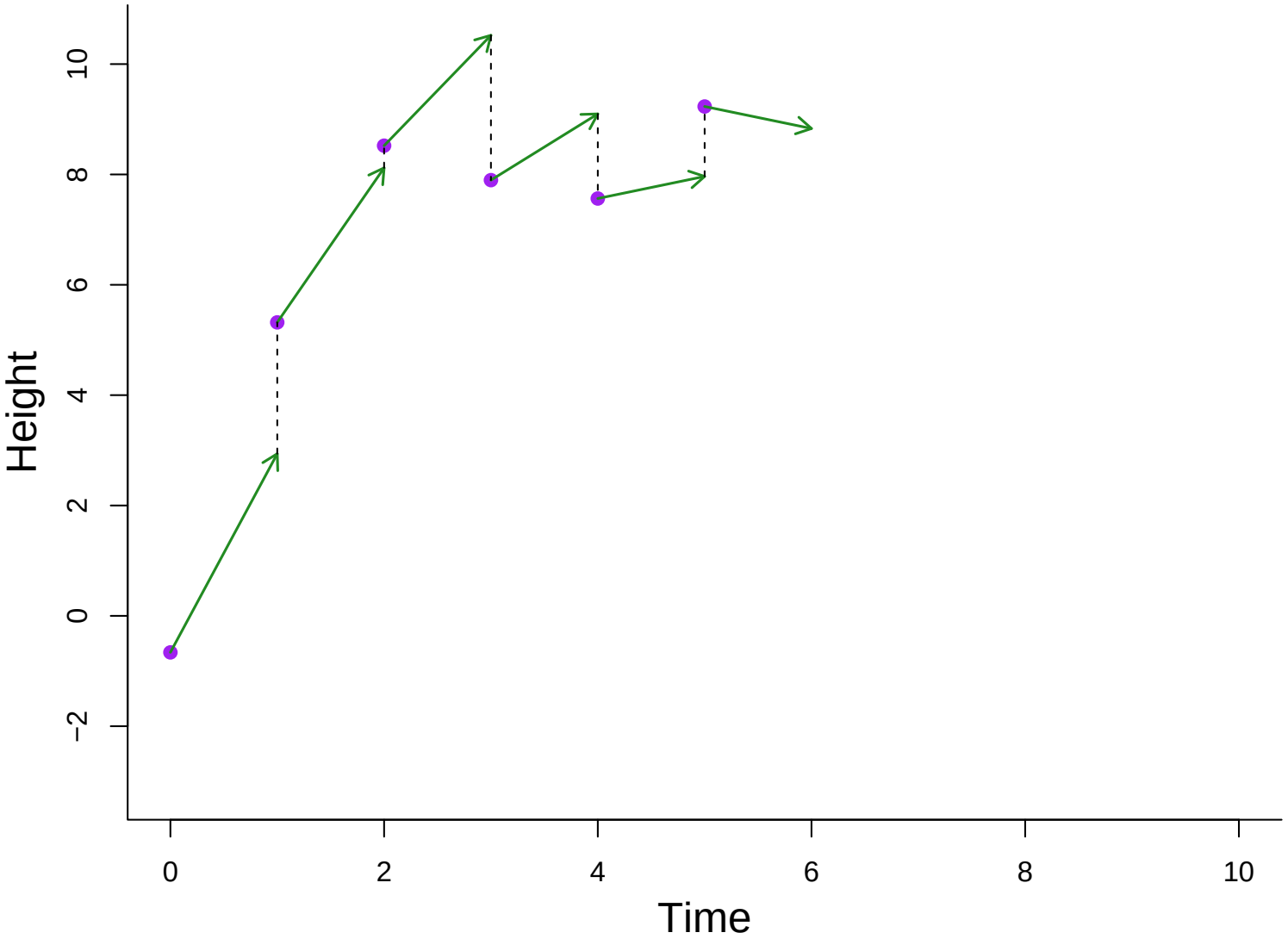

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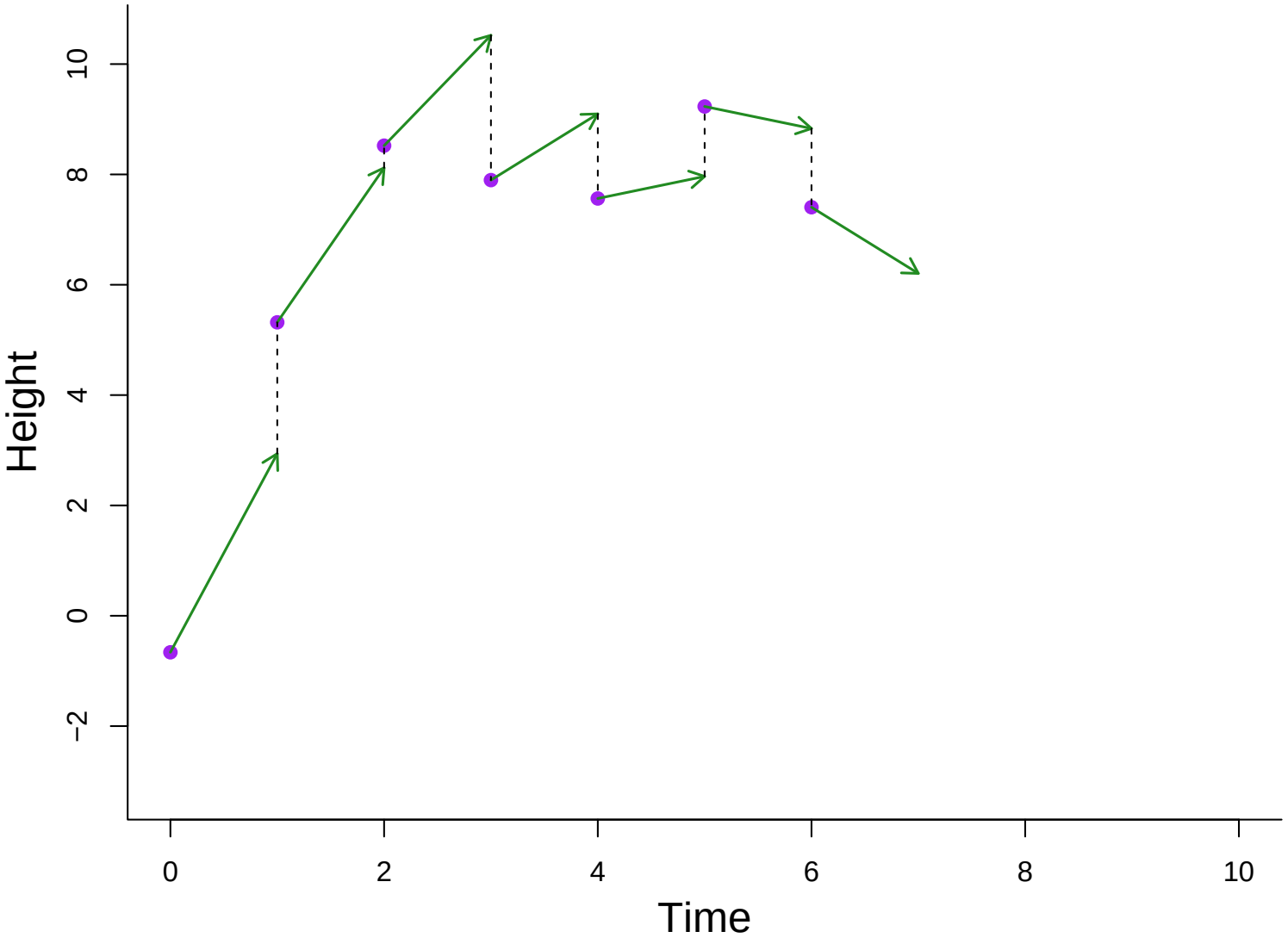

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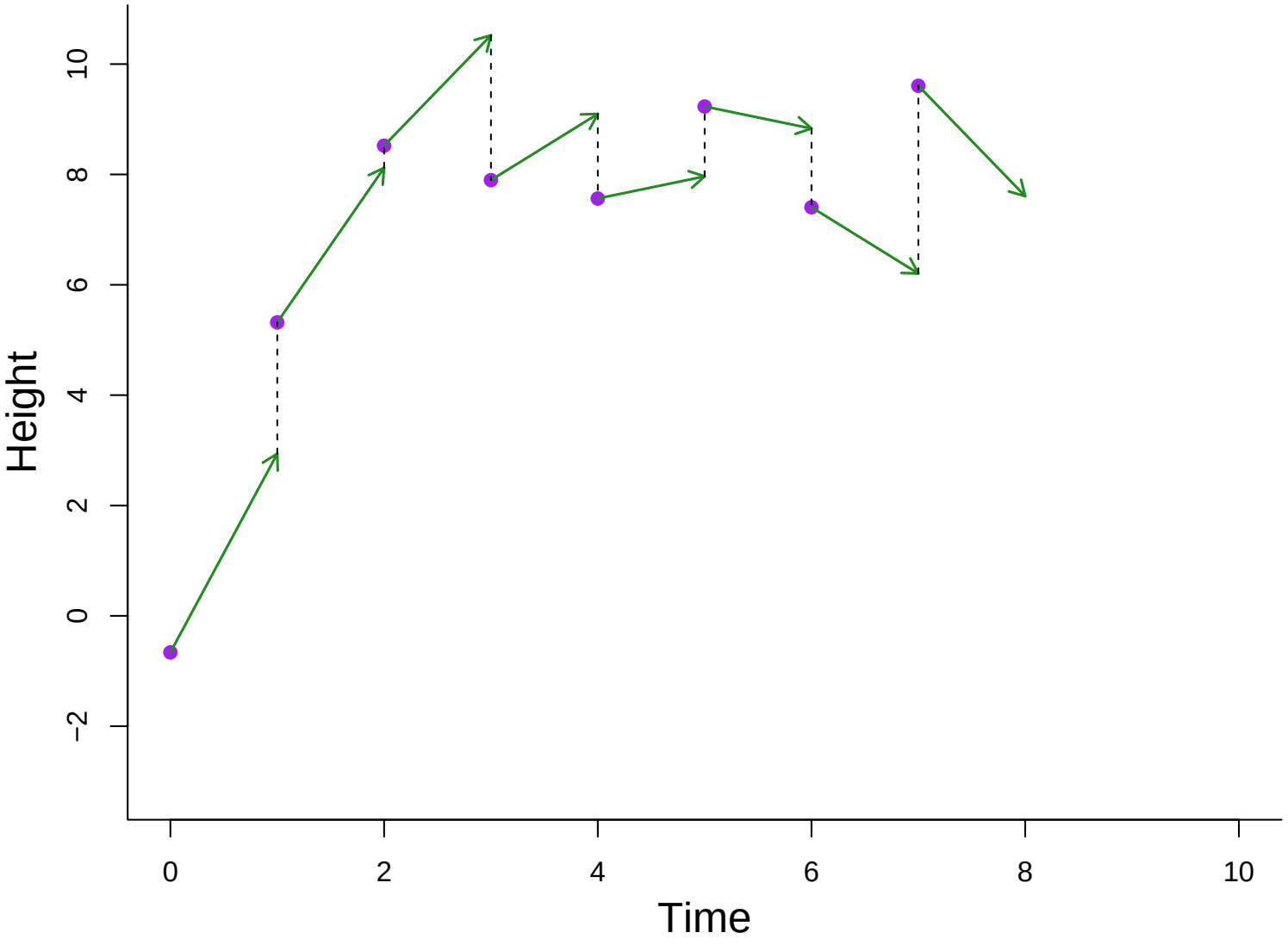

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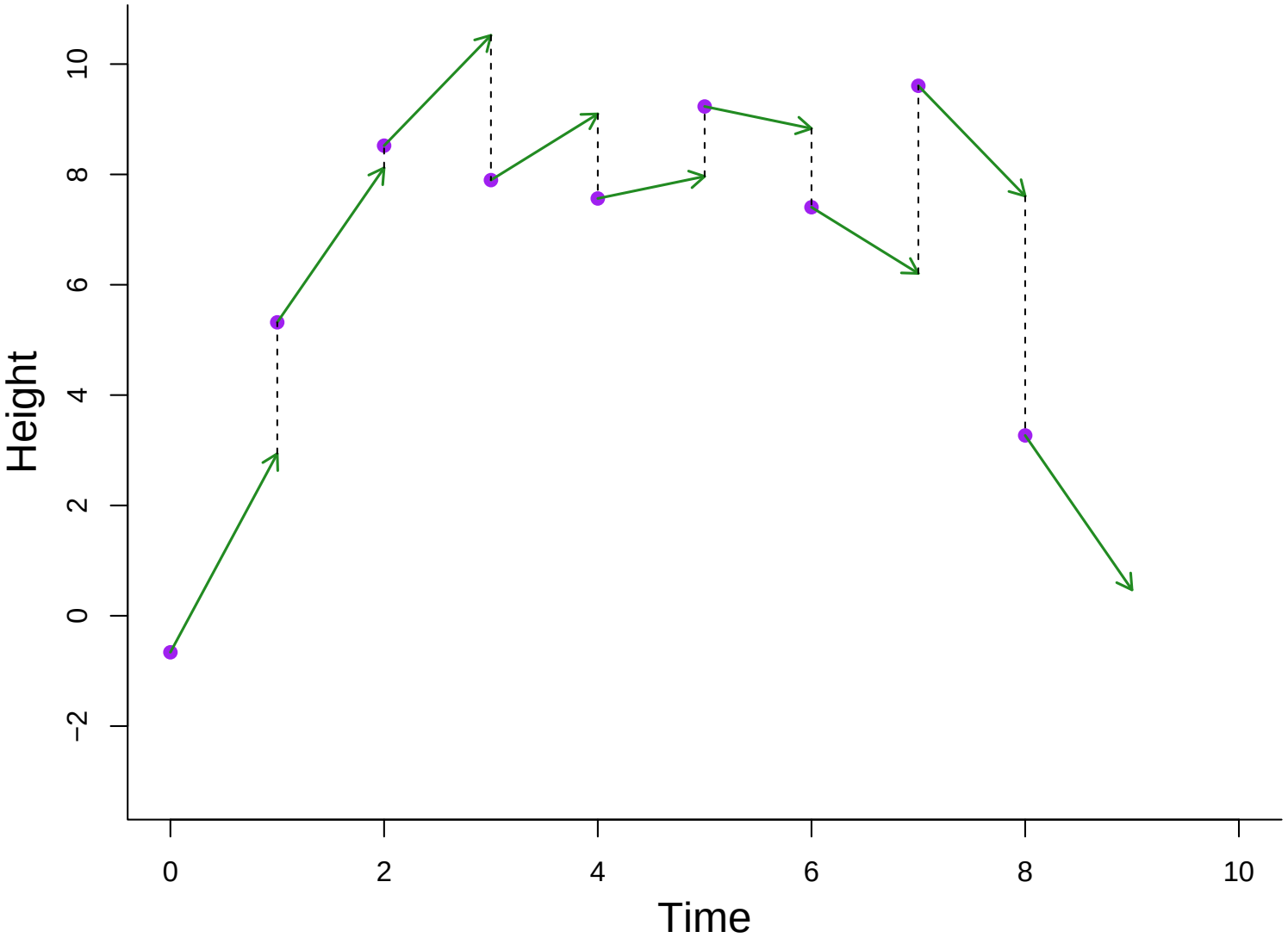

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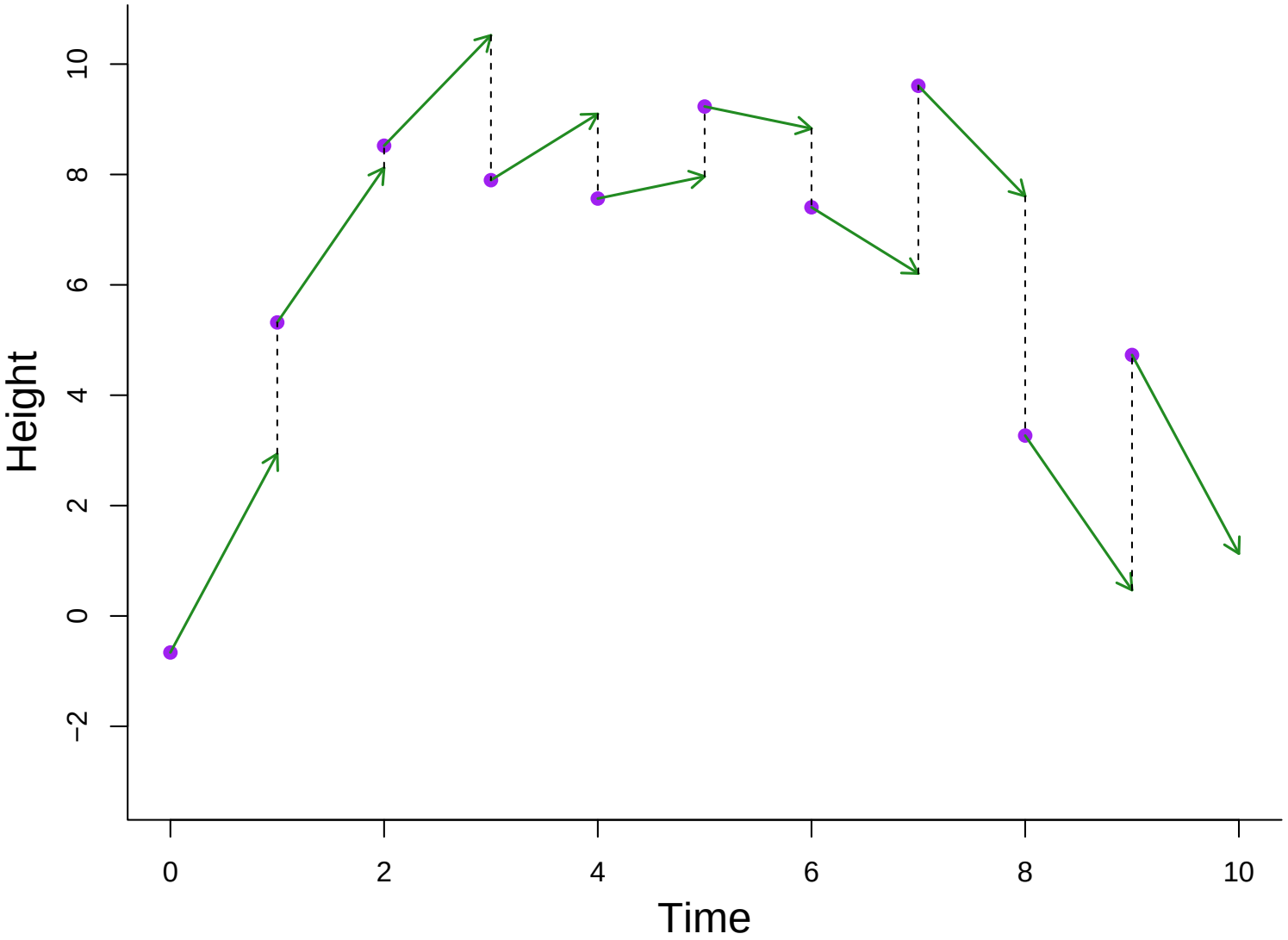

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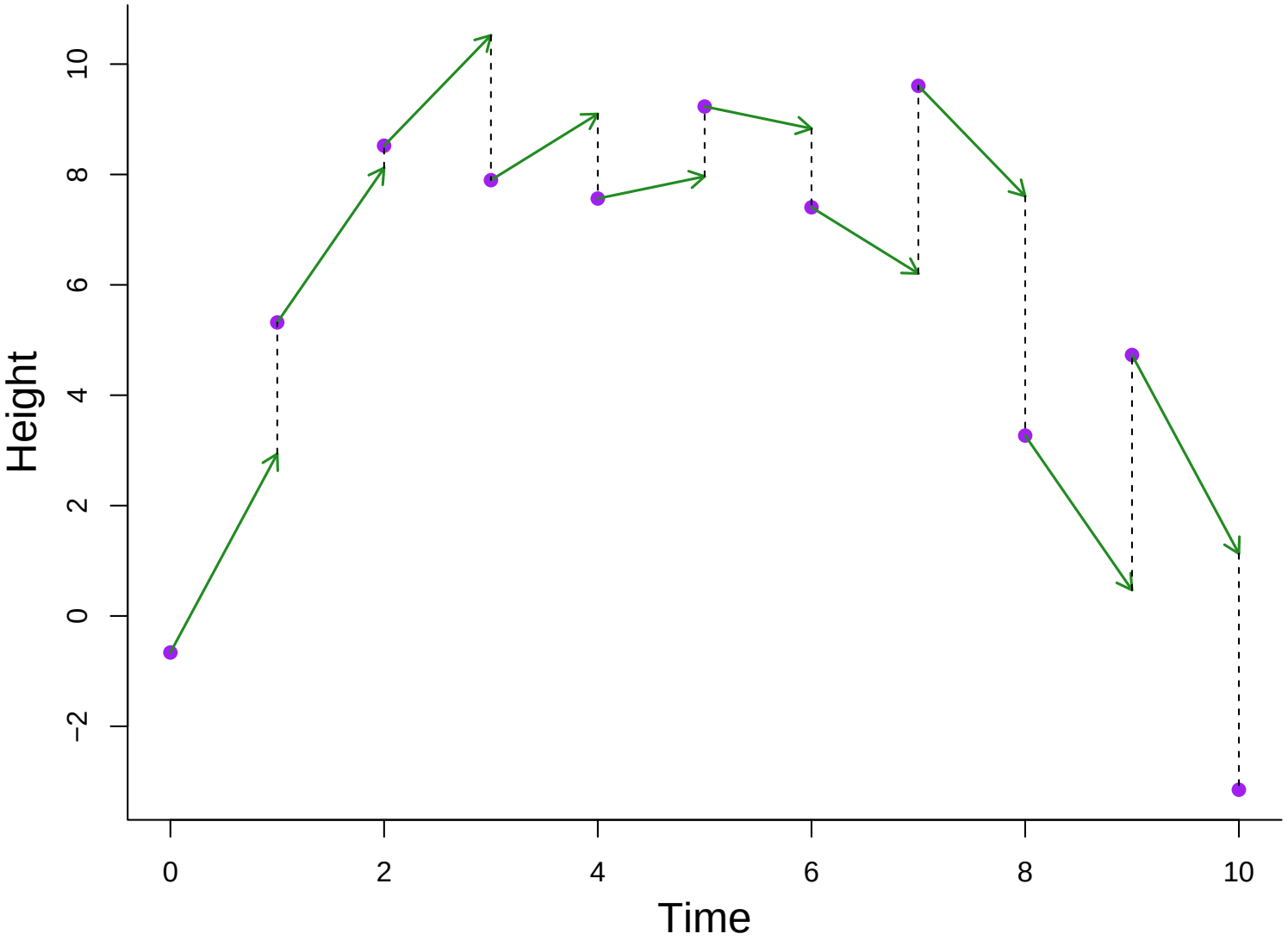
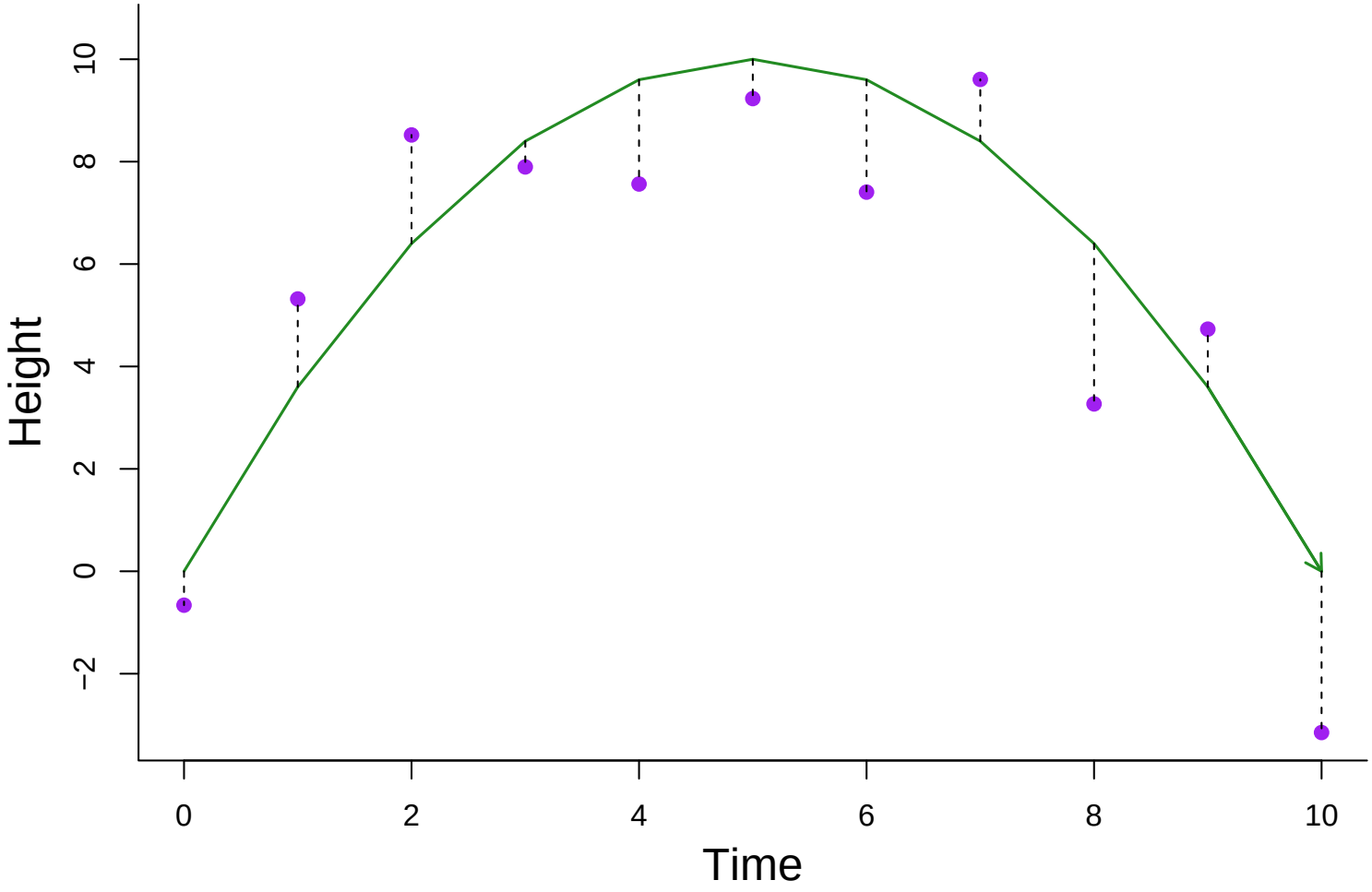
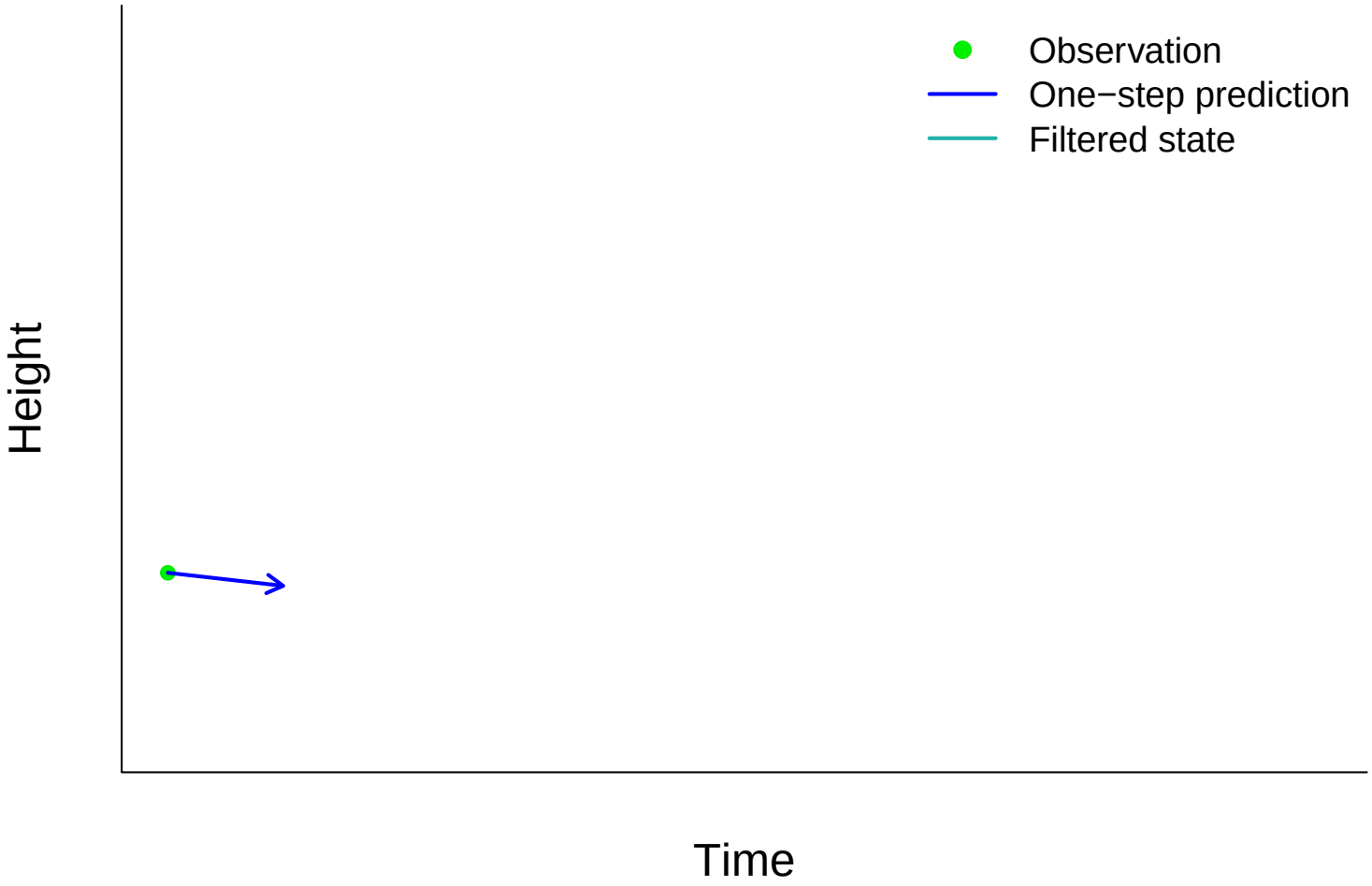


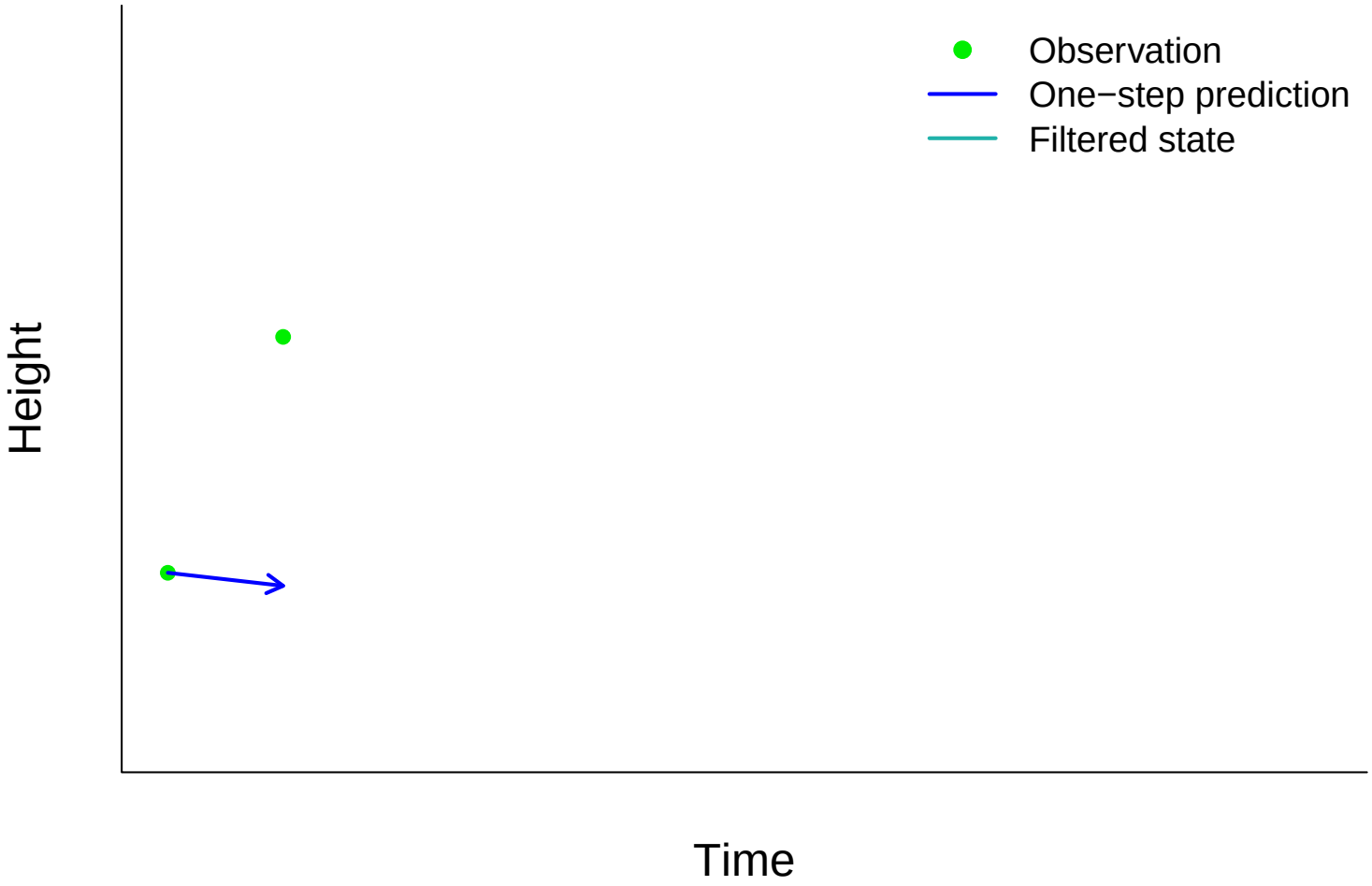
Measurement error only



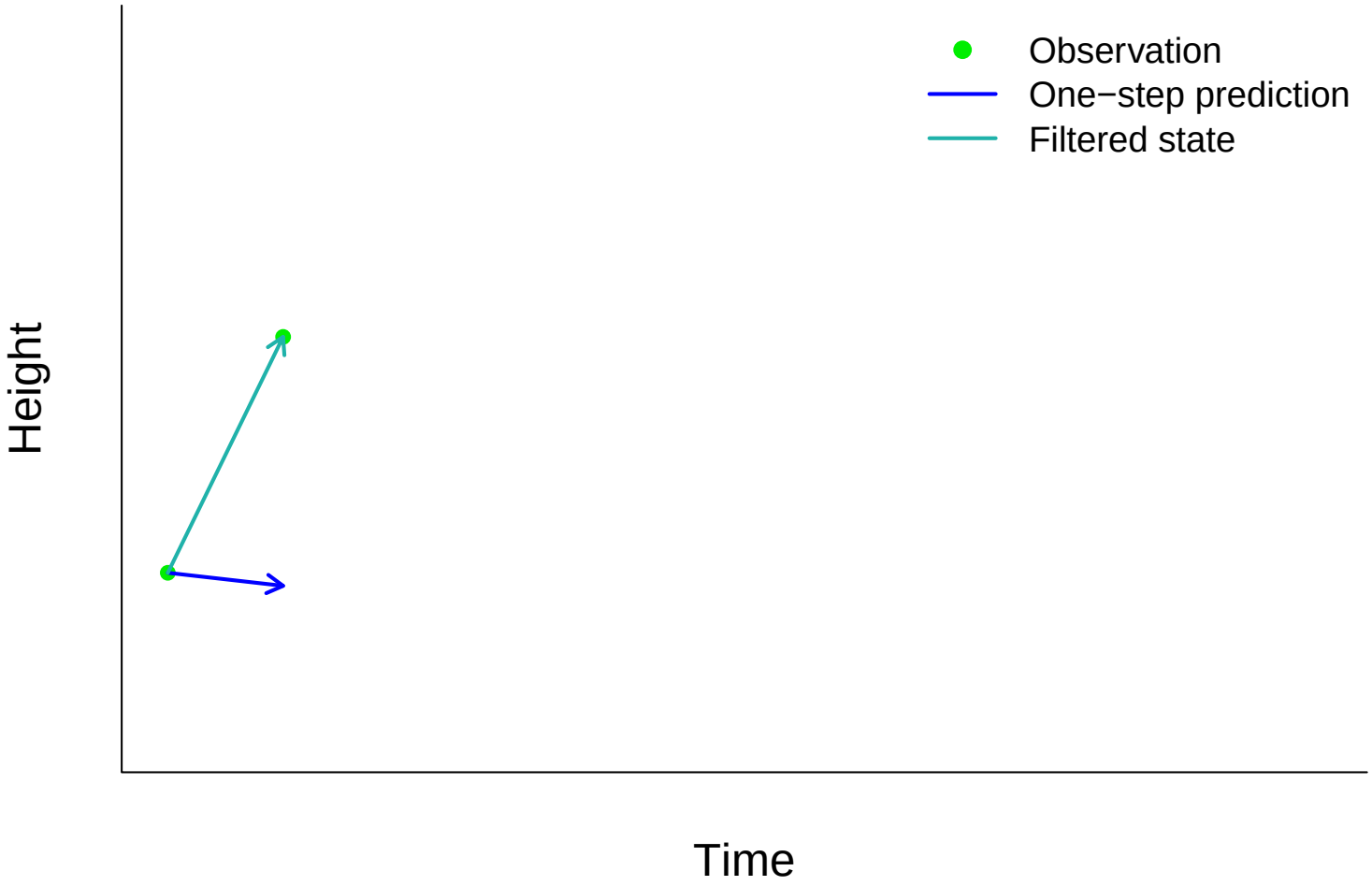
Both – Kalman filter



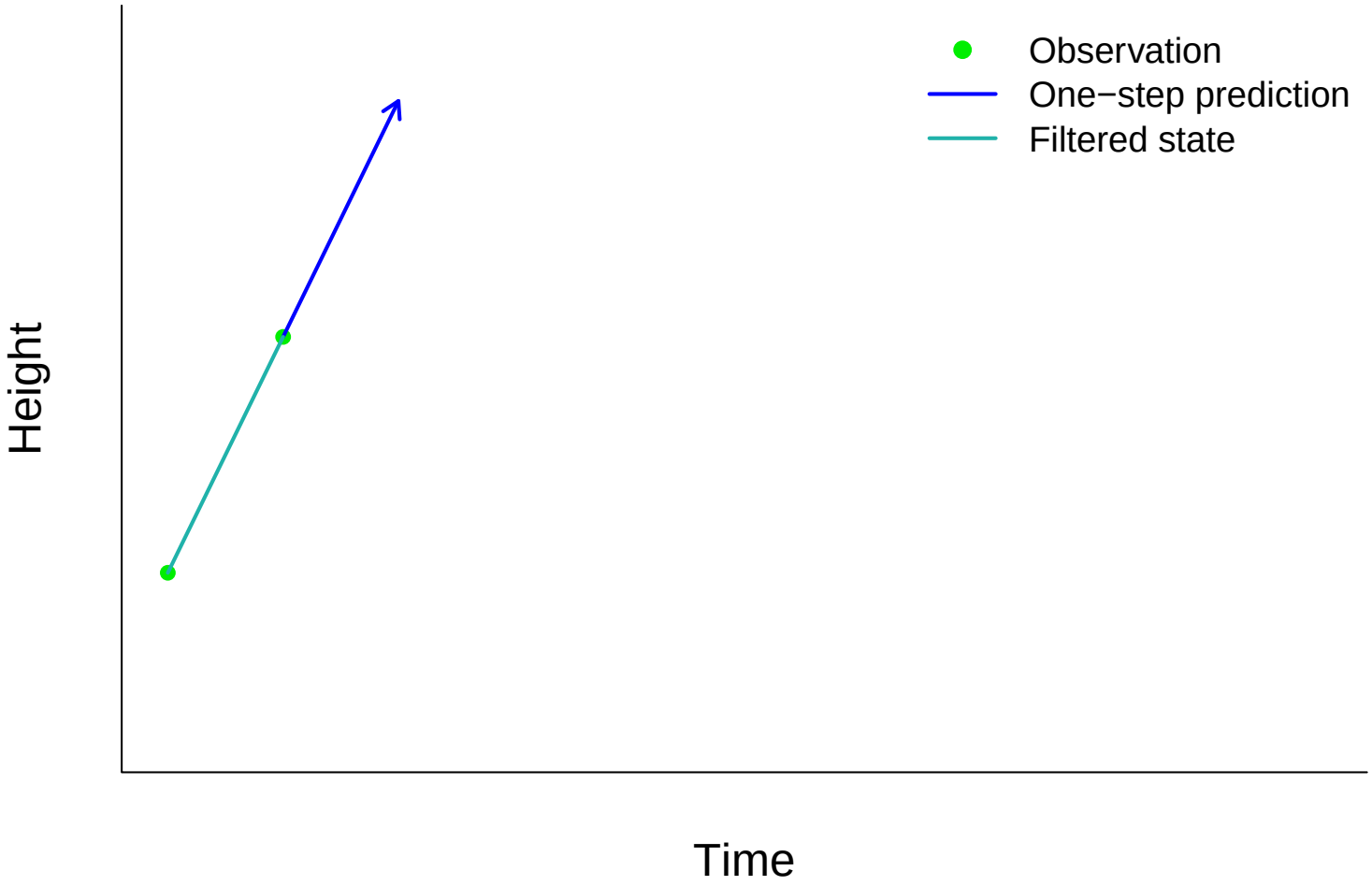
Both – Kalman filter



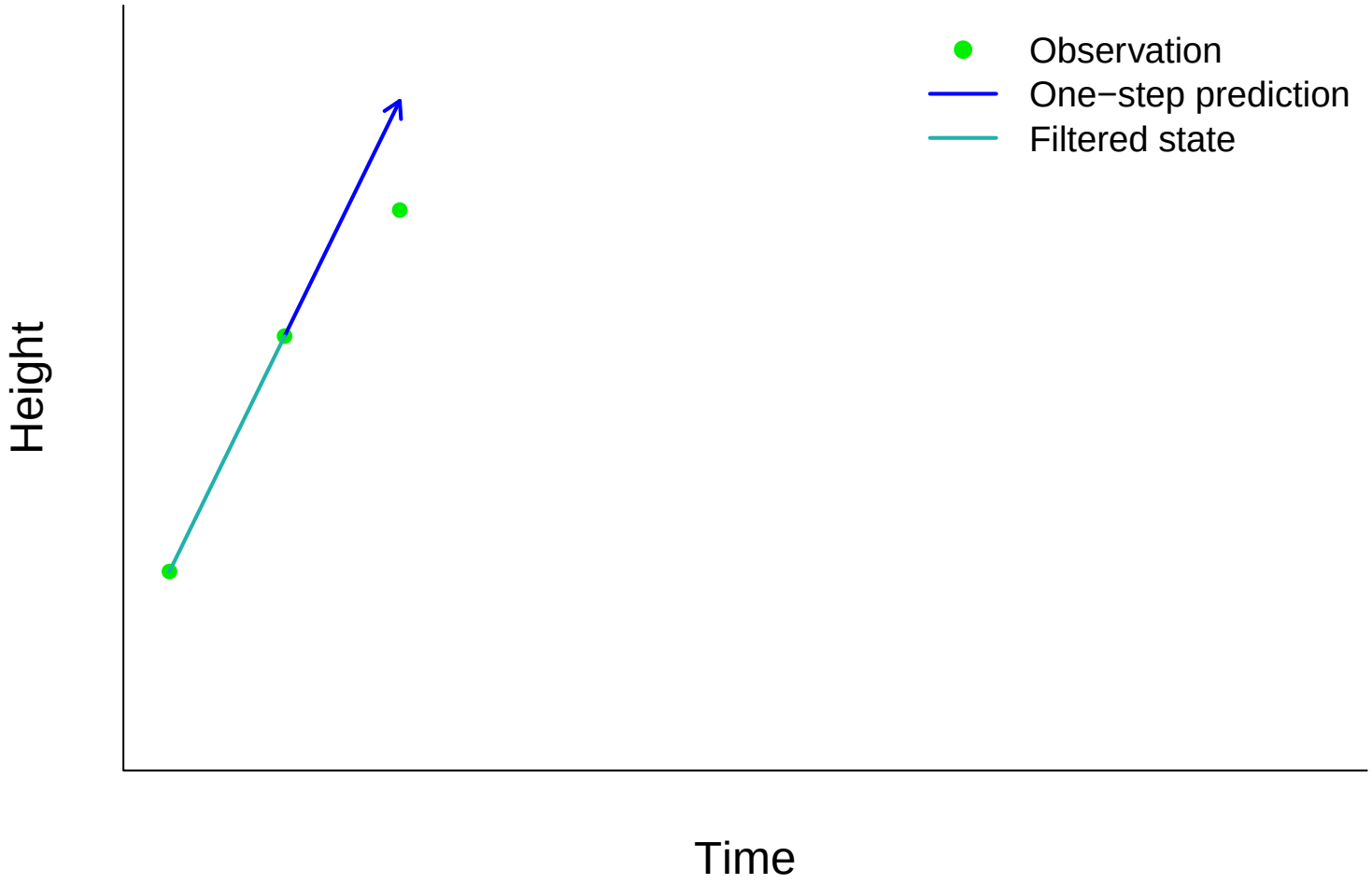
Both – Kalman filter



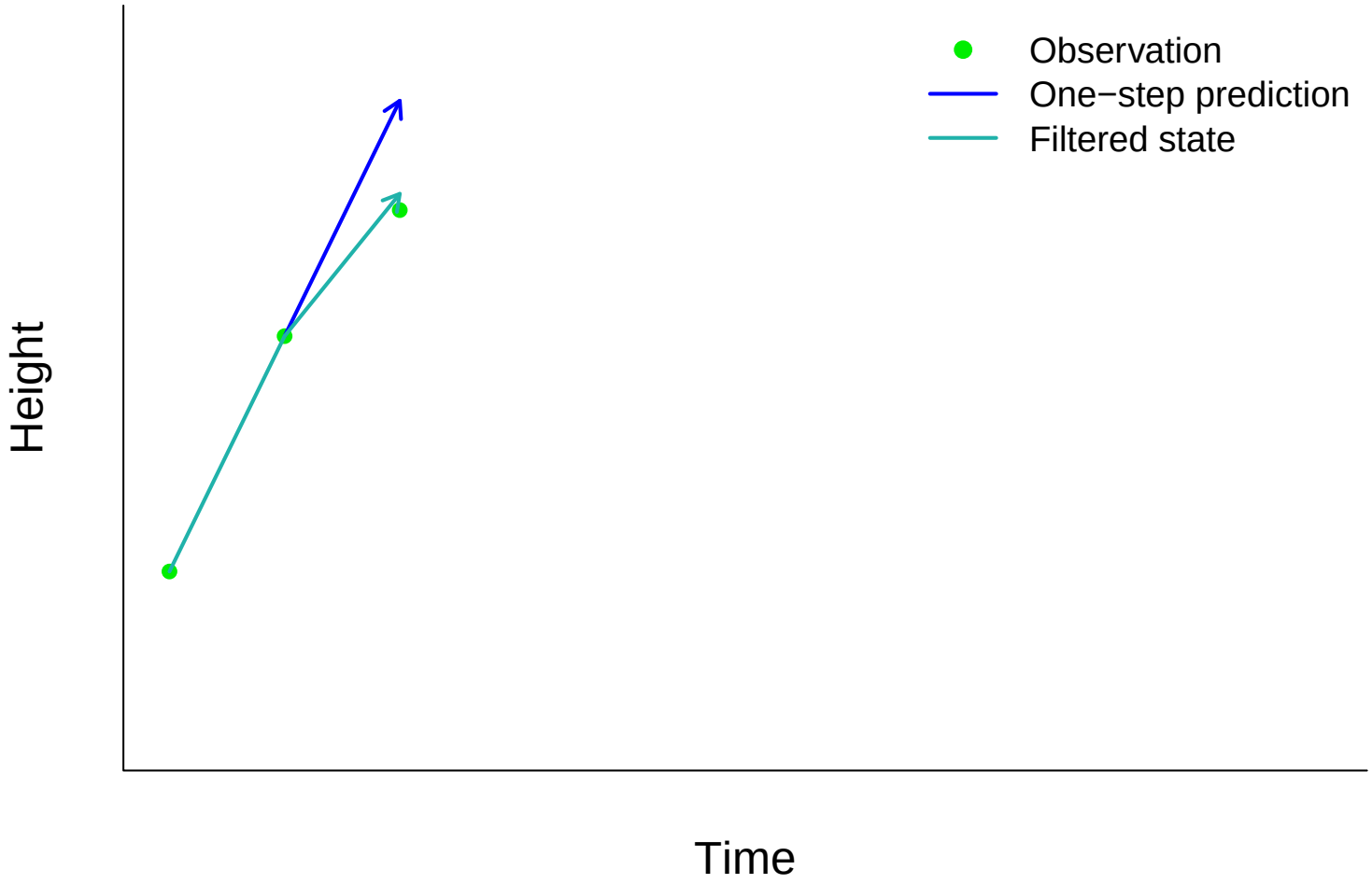
Both – Kalman filter



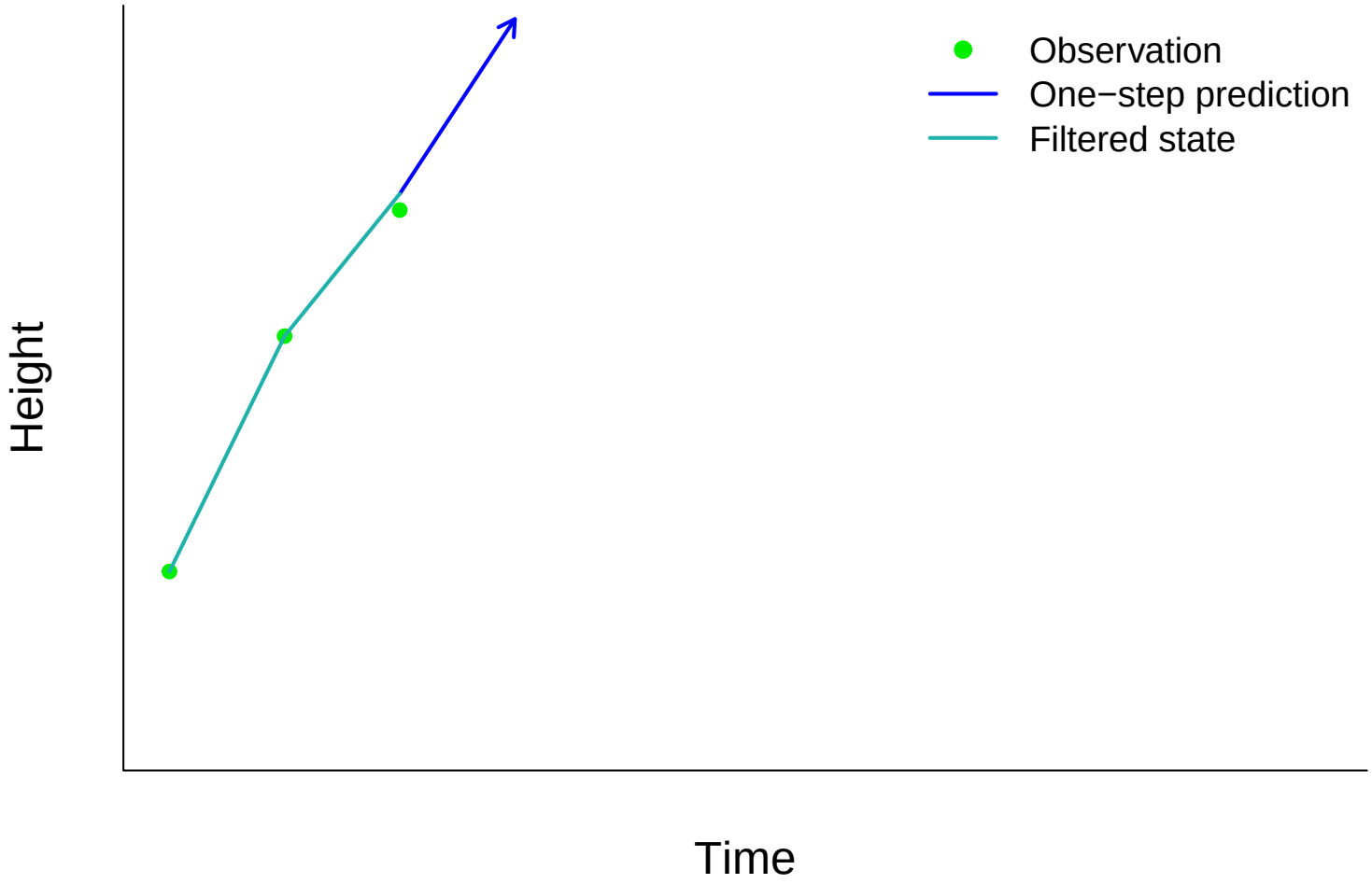
Both – Kalman filter



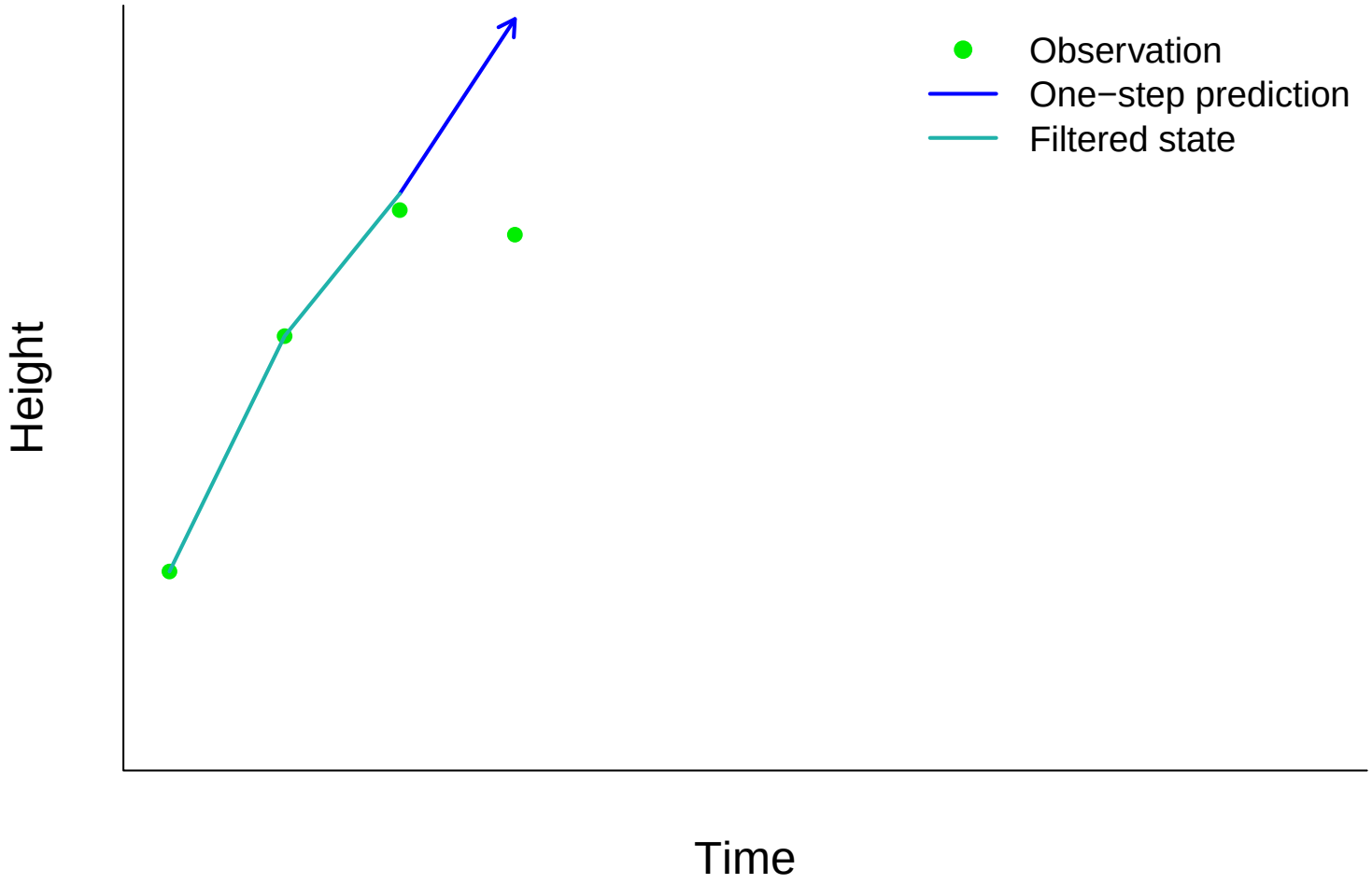
Both – Kalman filter



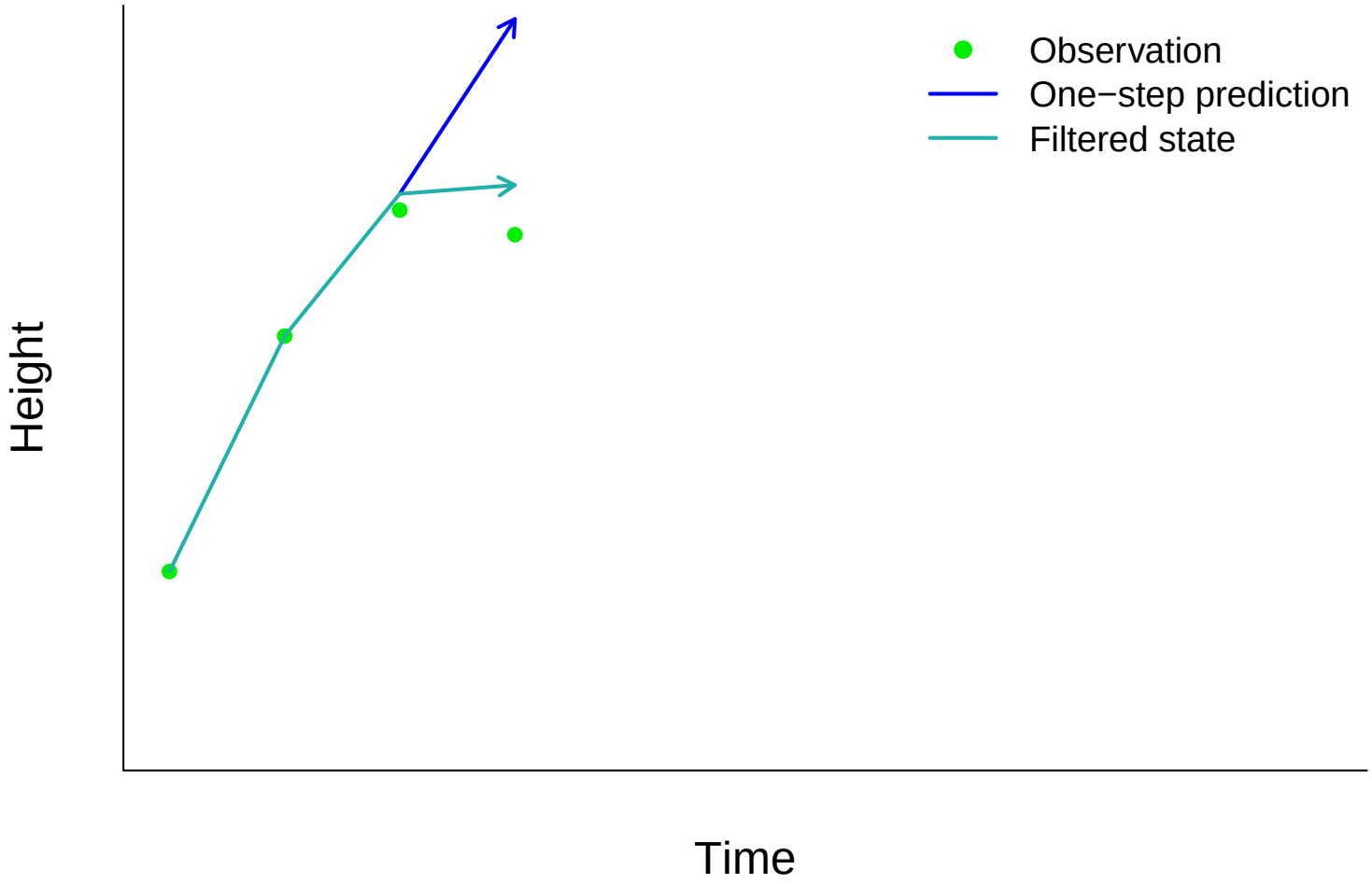
Both – Kalman filter



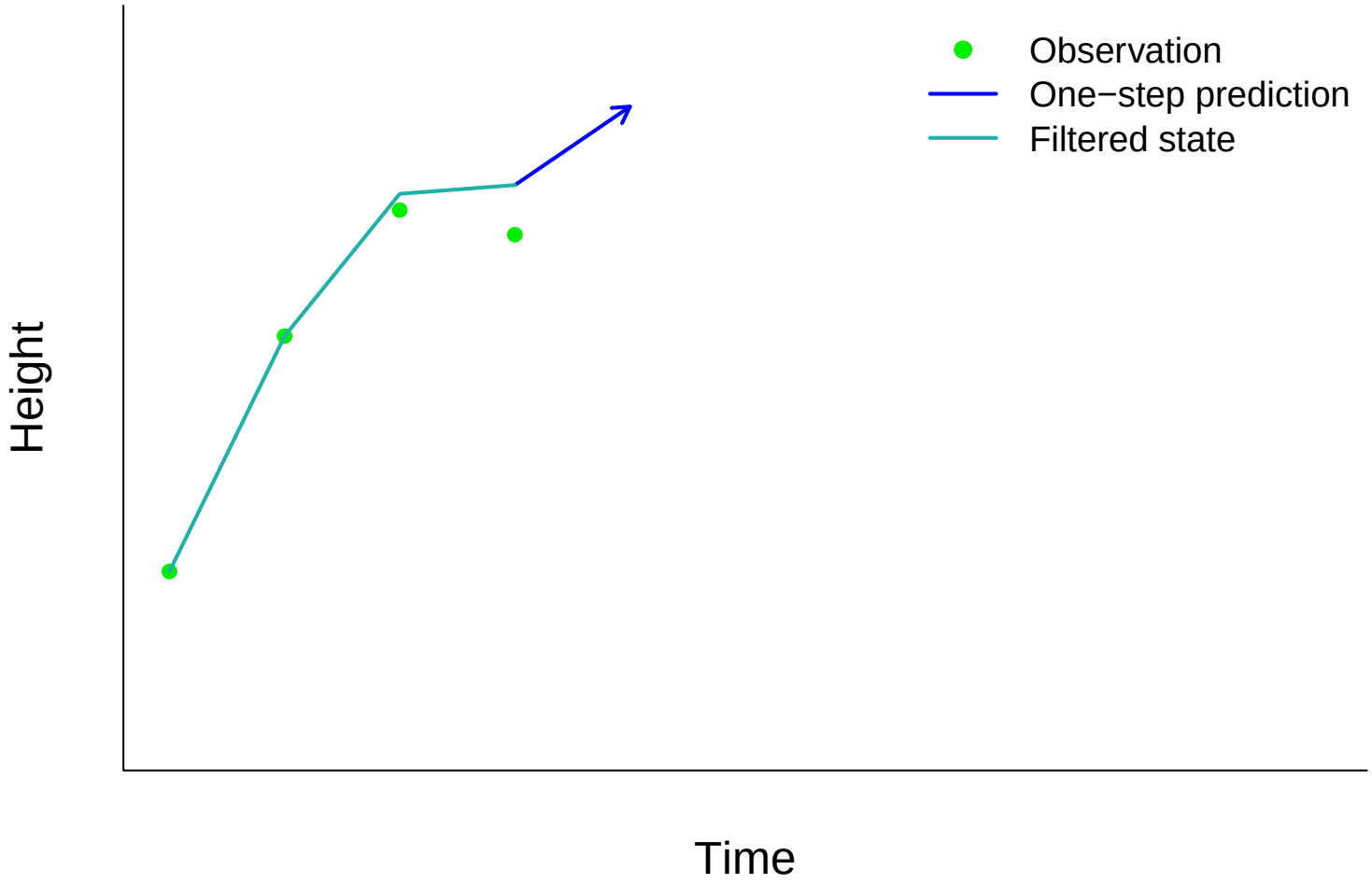
Both – Kalman filter



Both – Kalman filter



Both – Kalman filter



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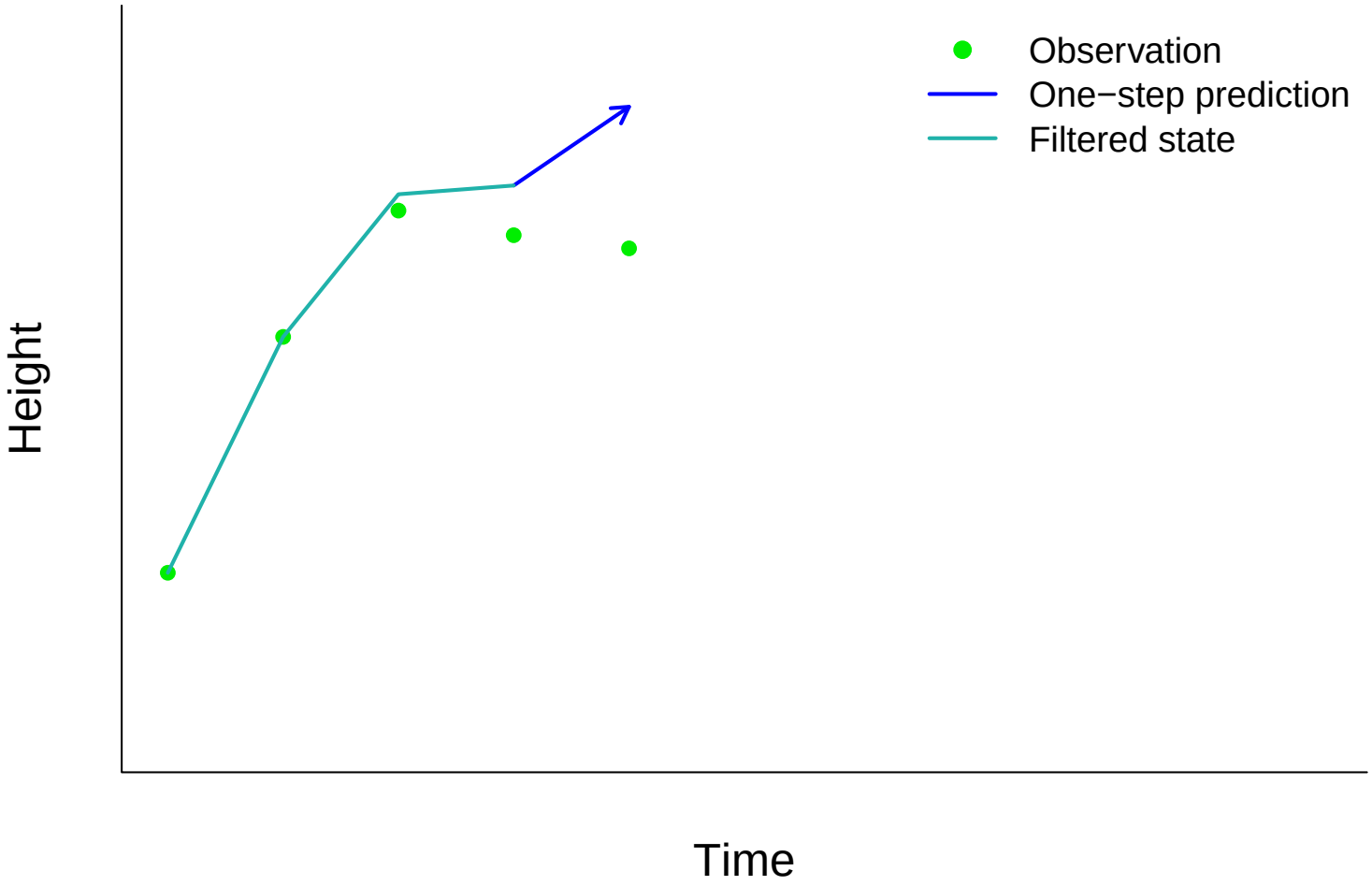

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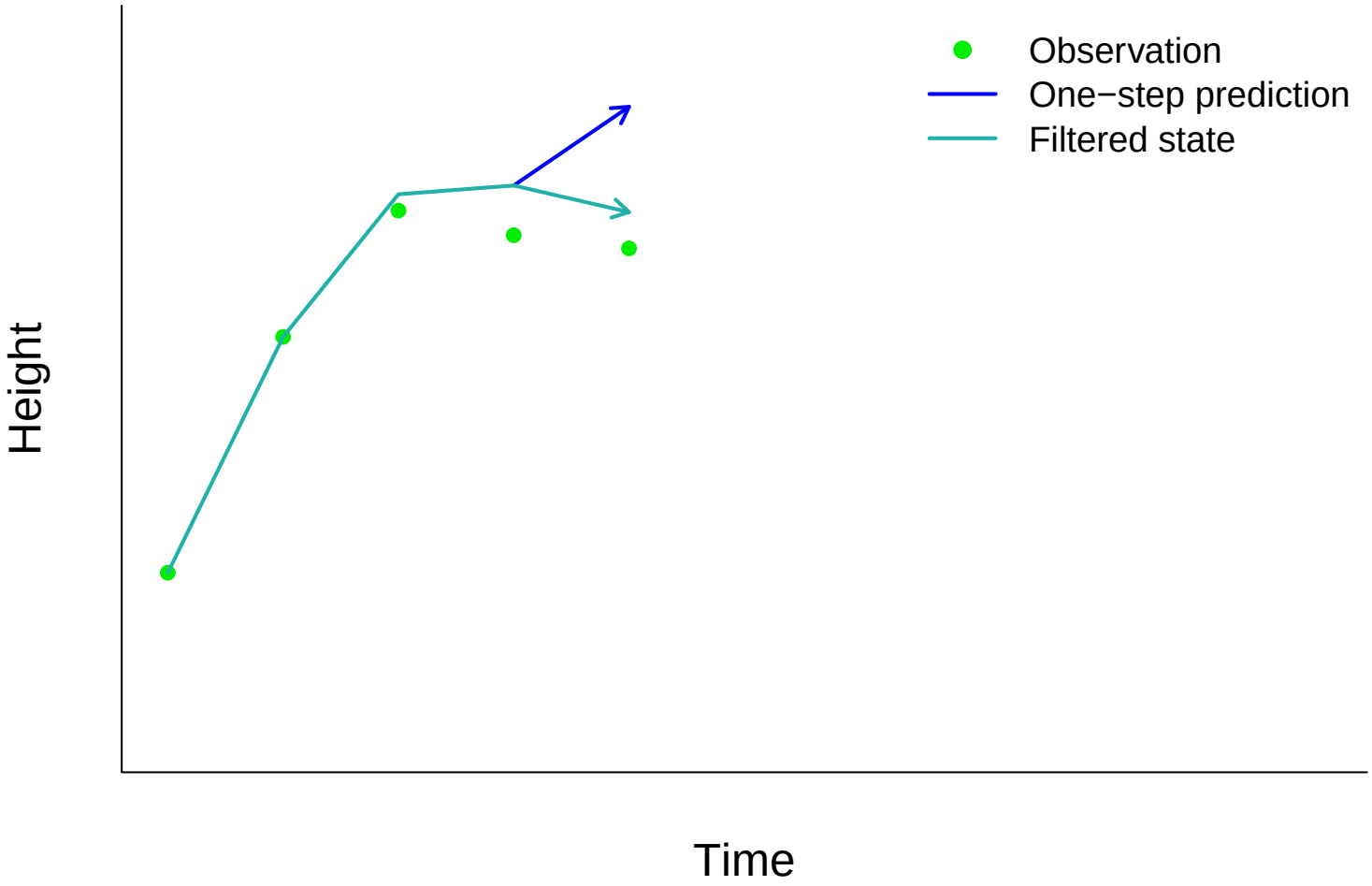
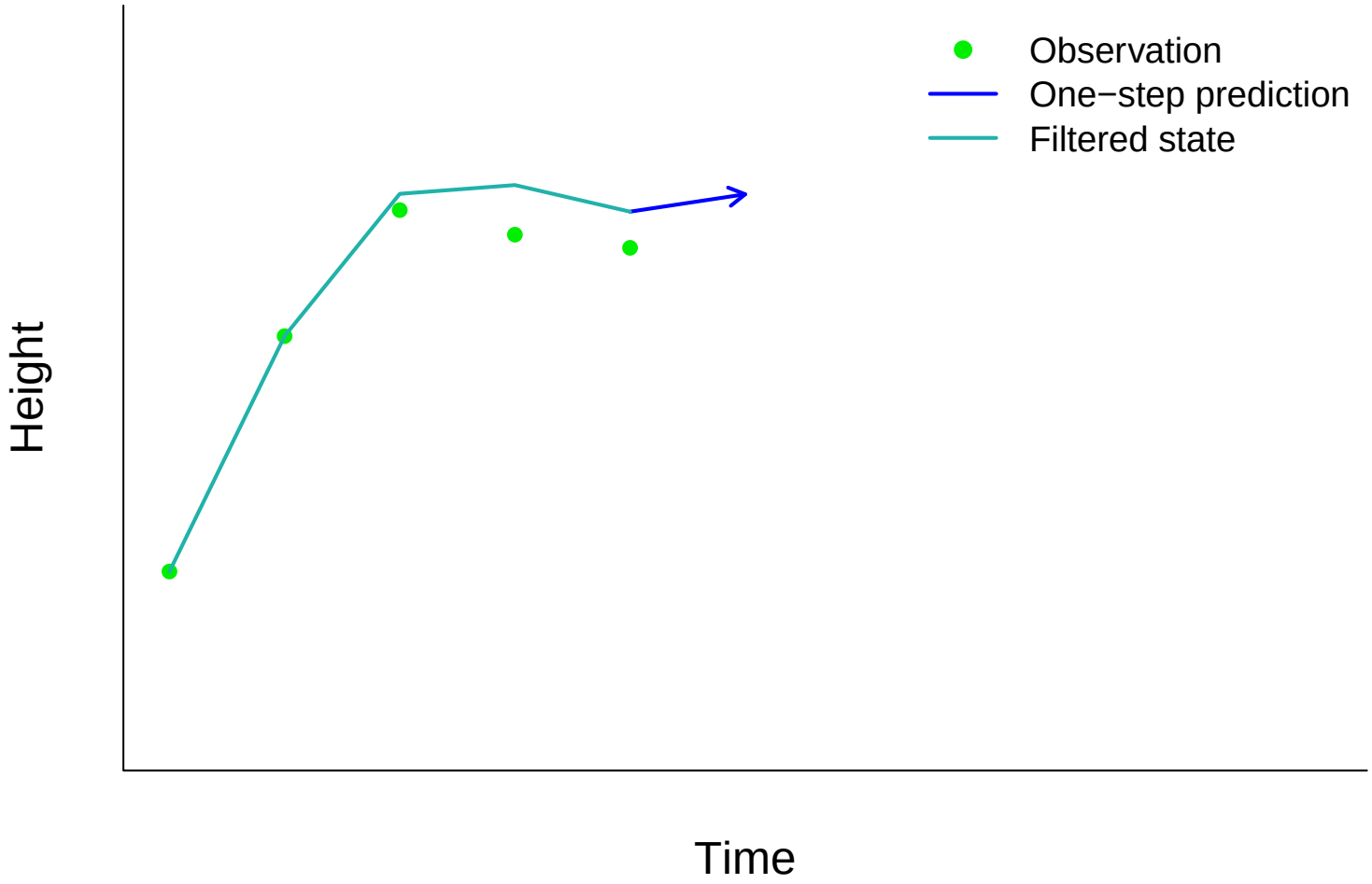


Both – Kalman filter



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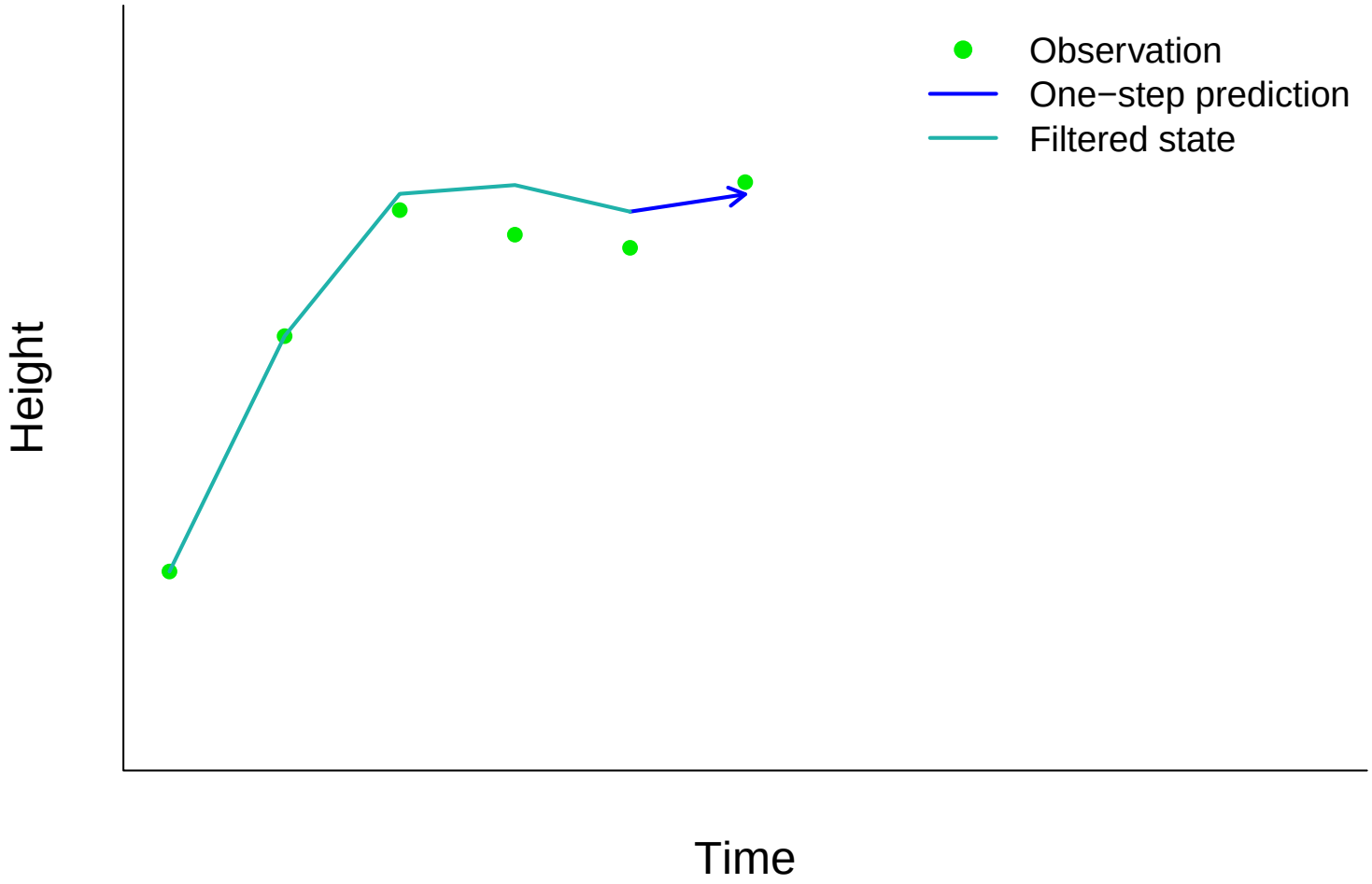

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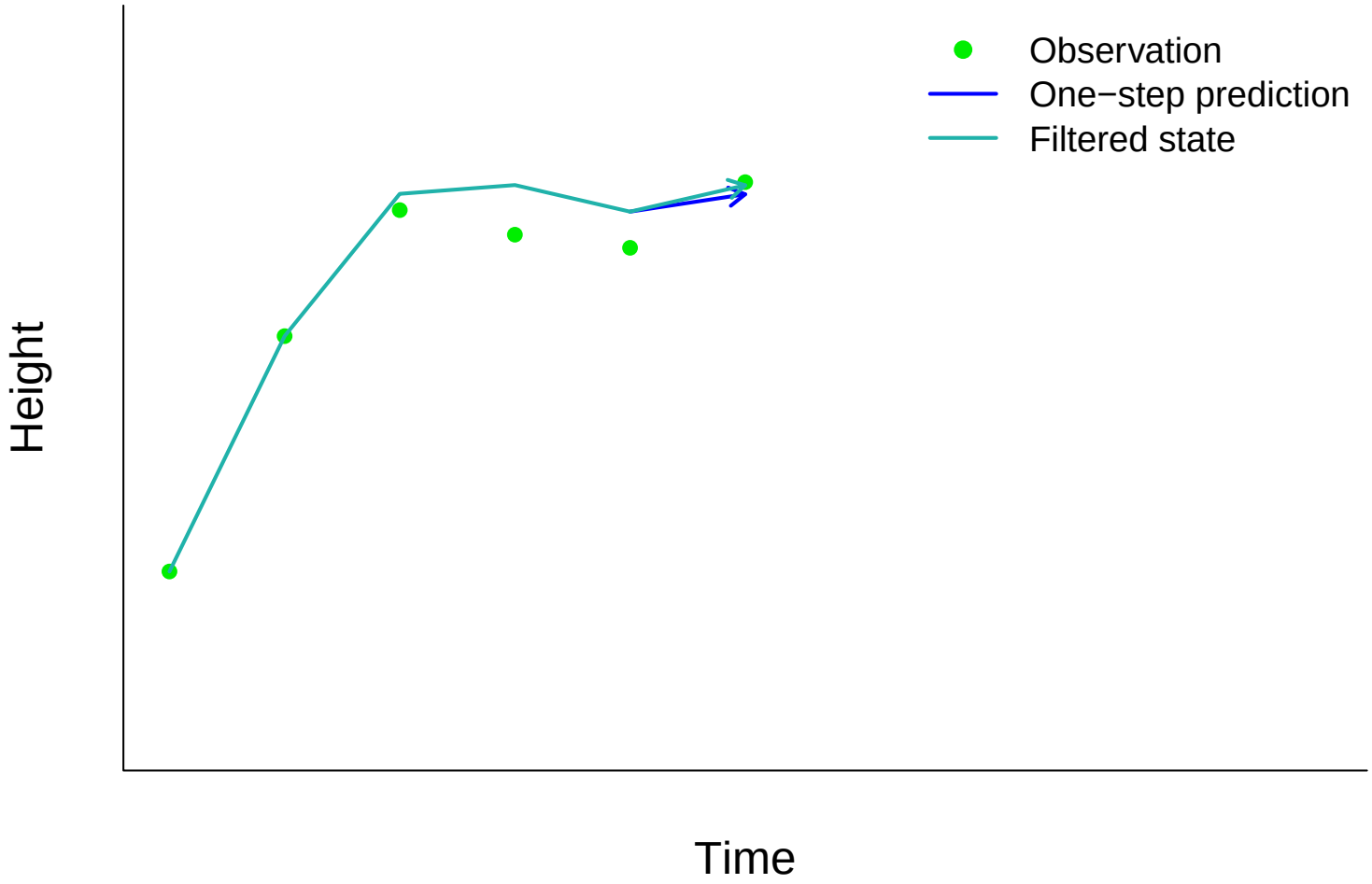
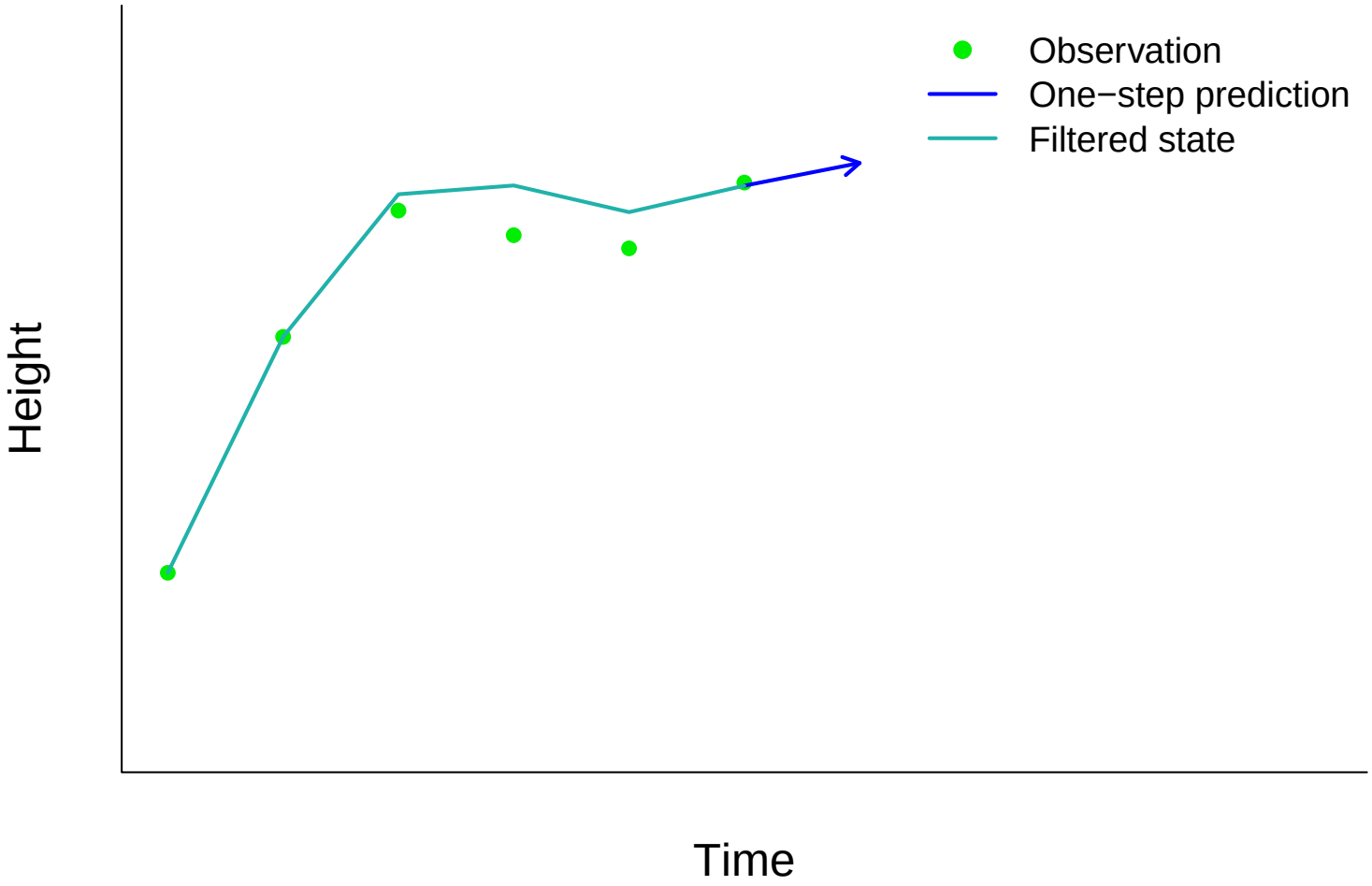
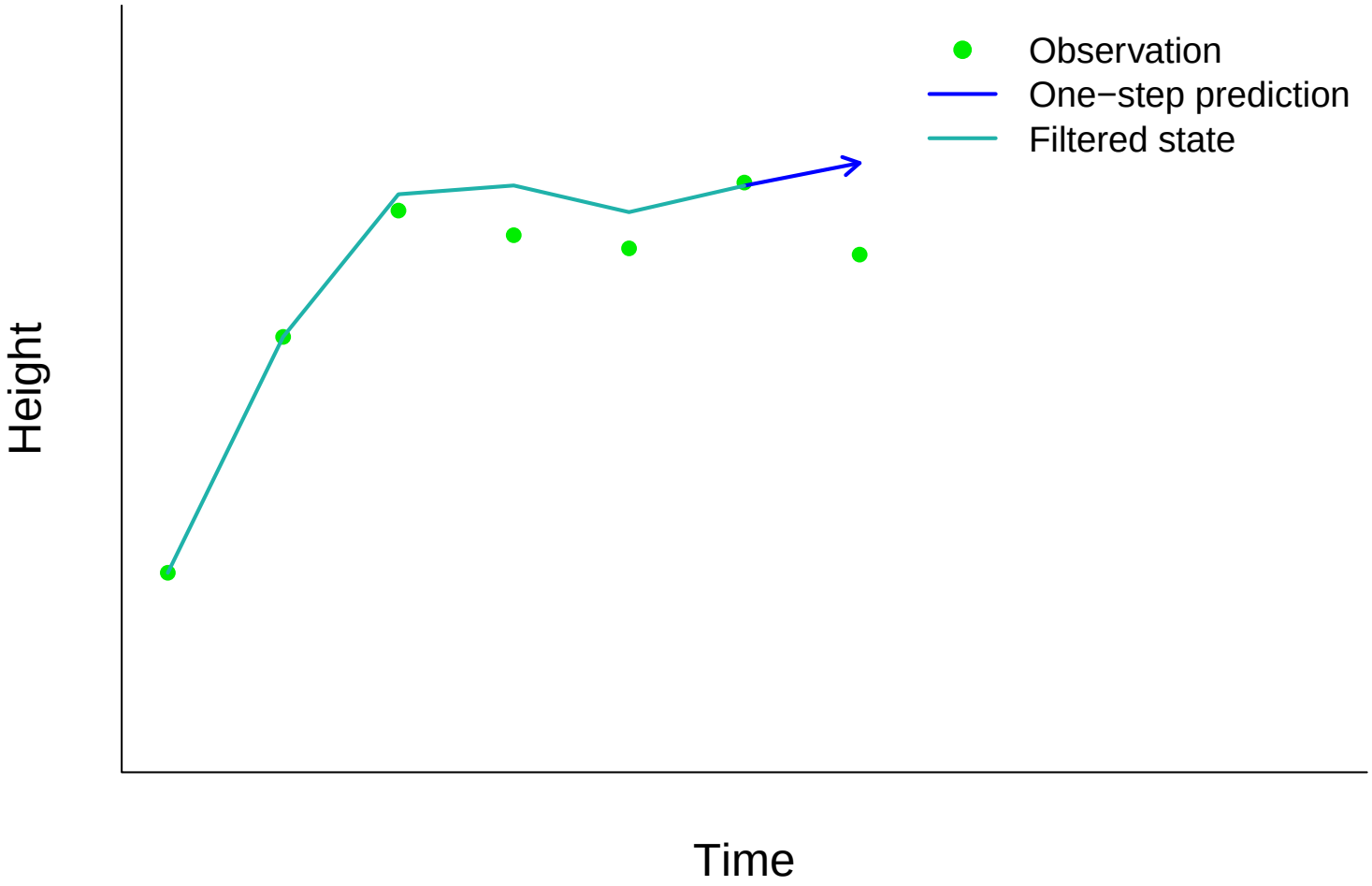


Both – Kalman filter



Both – Kalman filter



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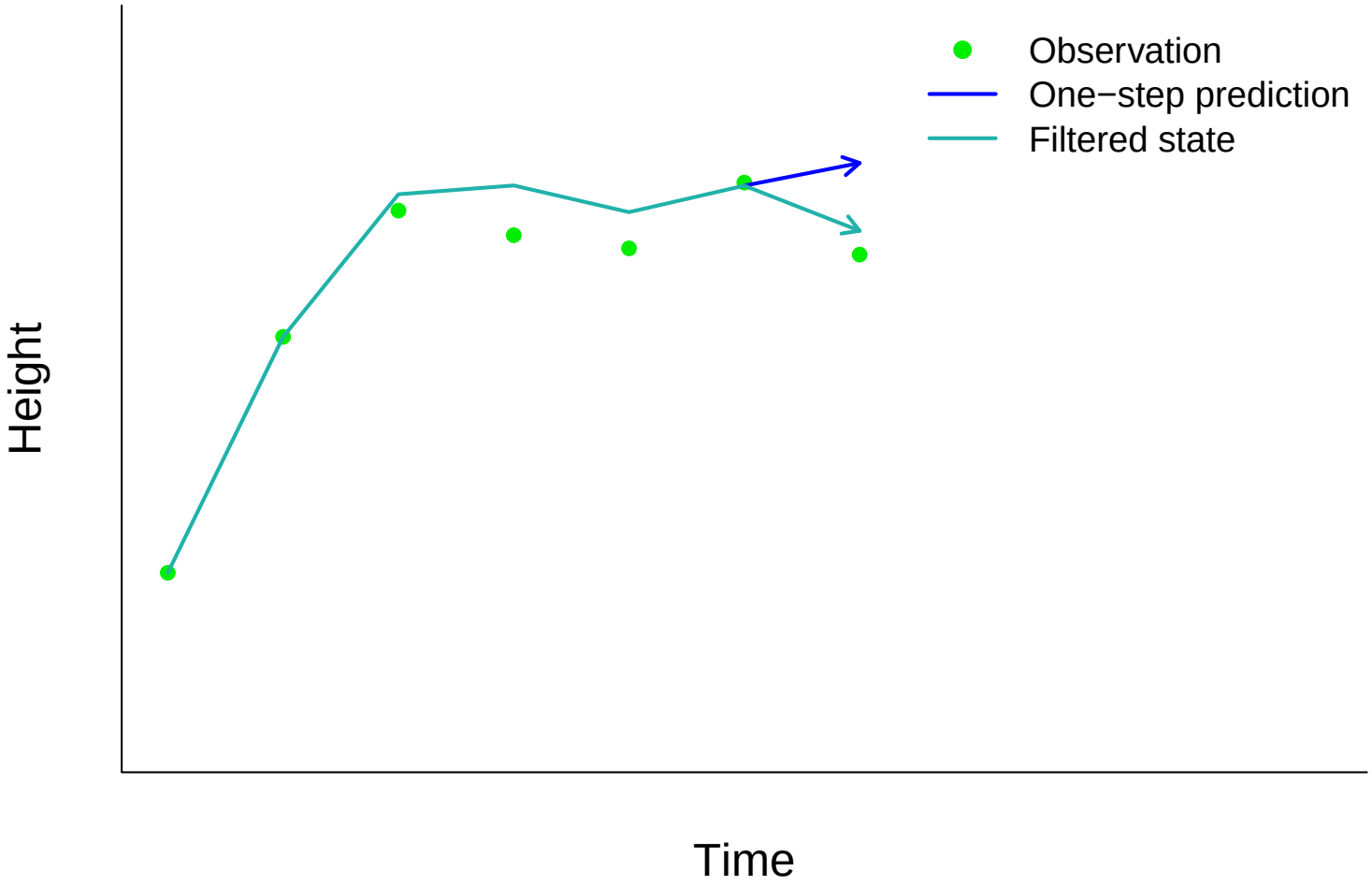

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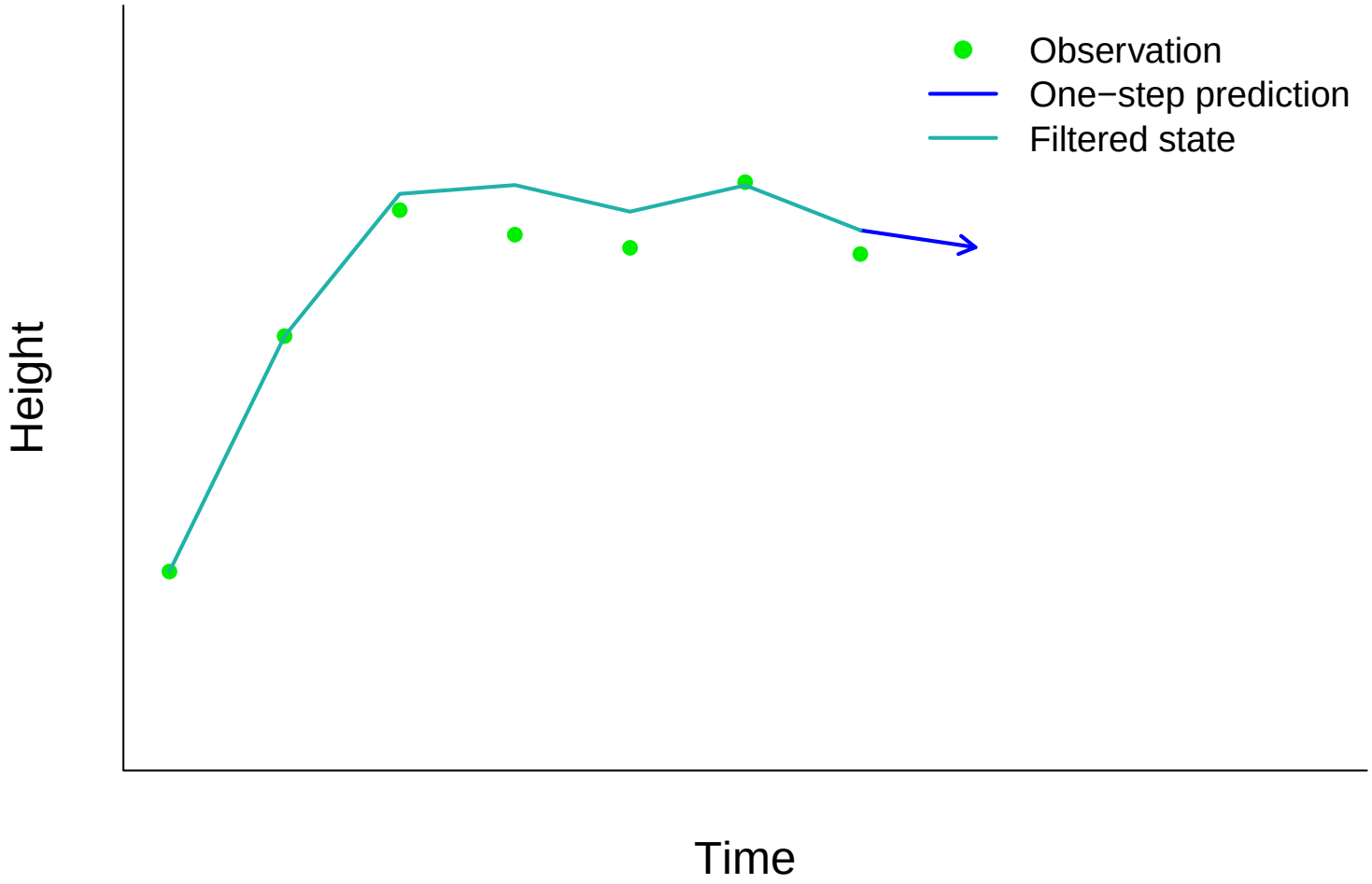

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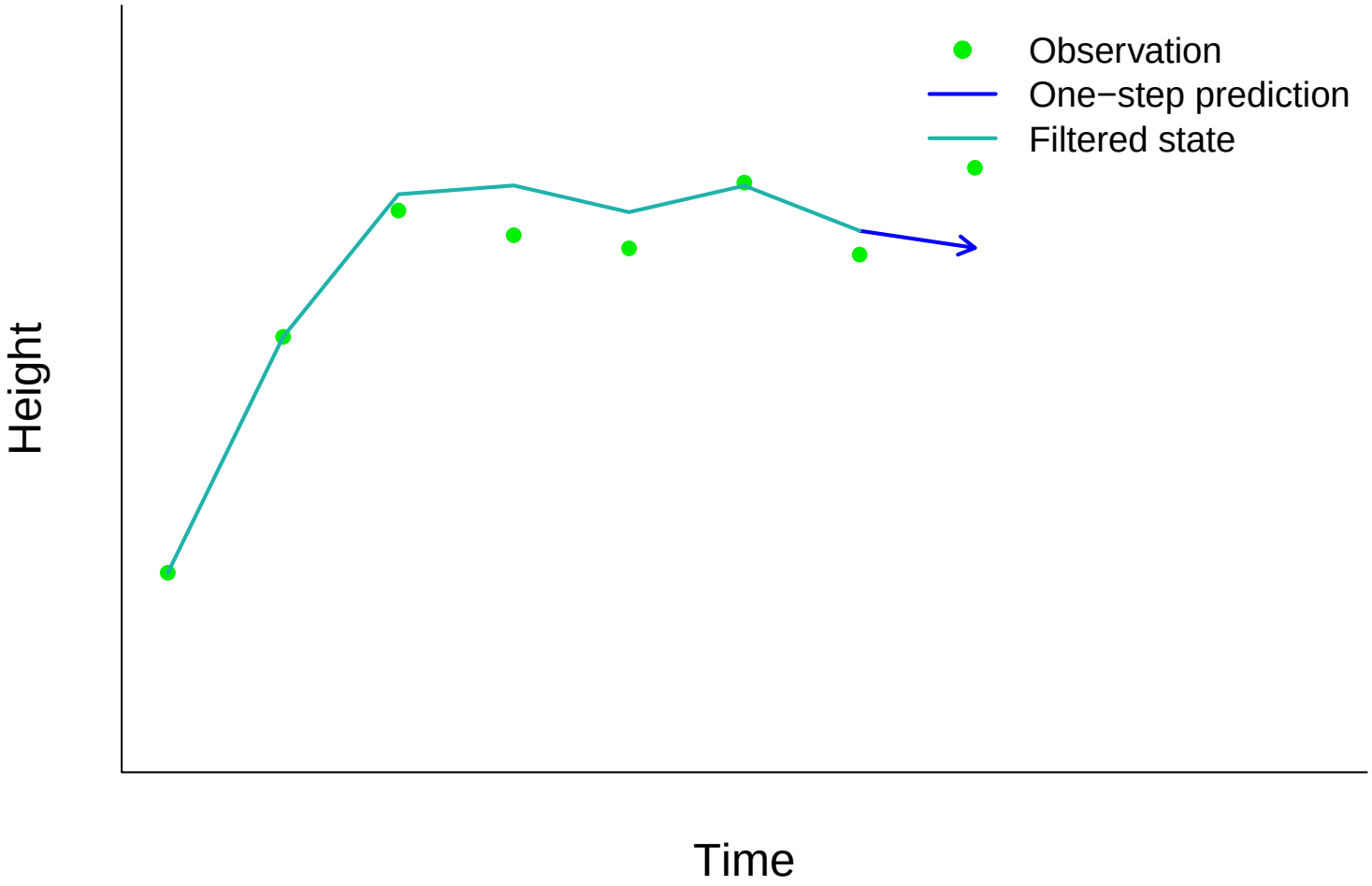

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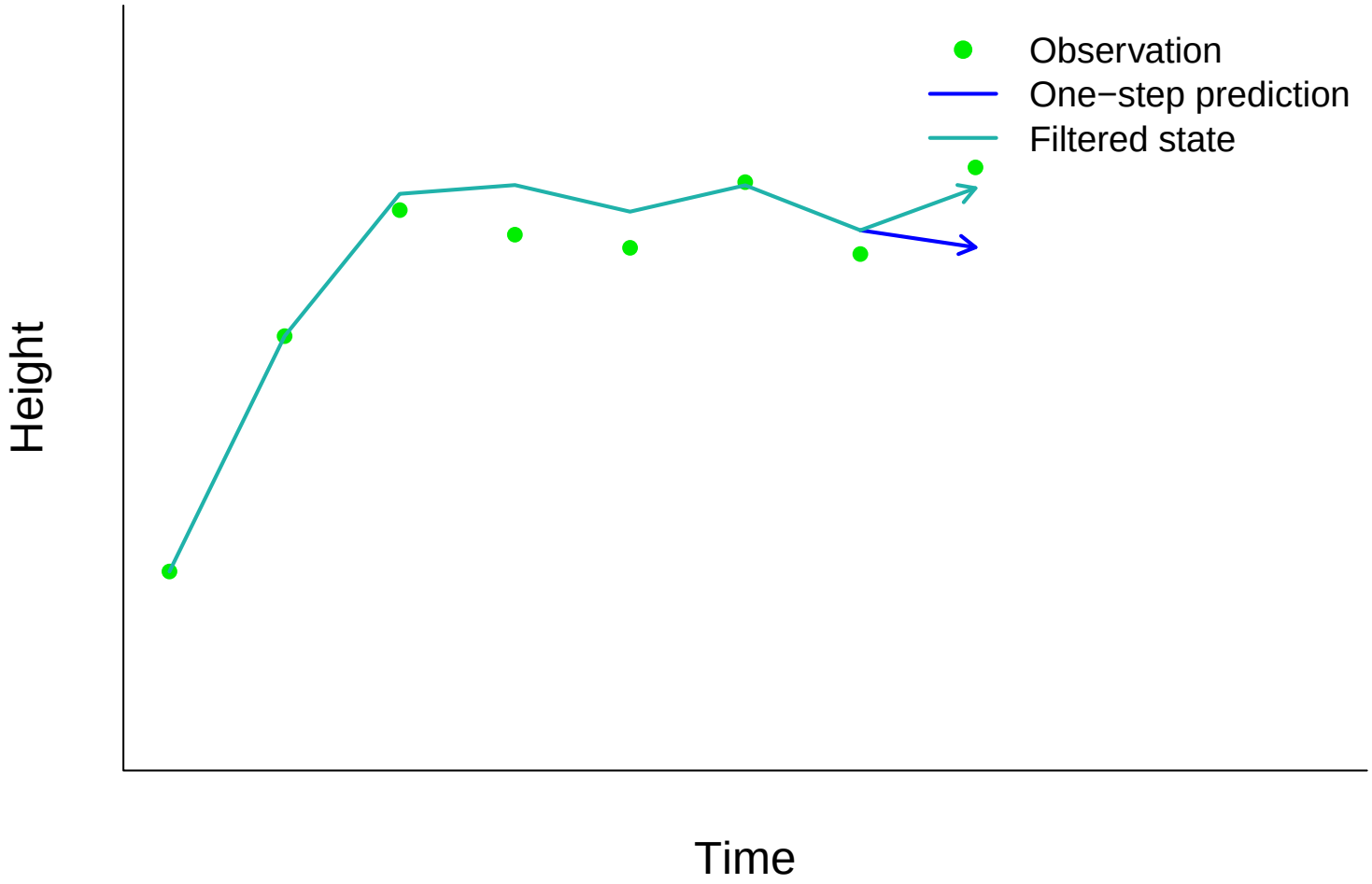

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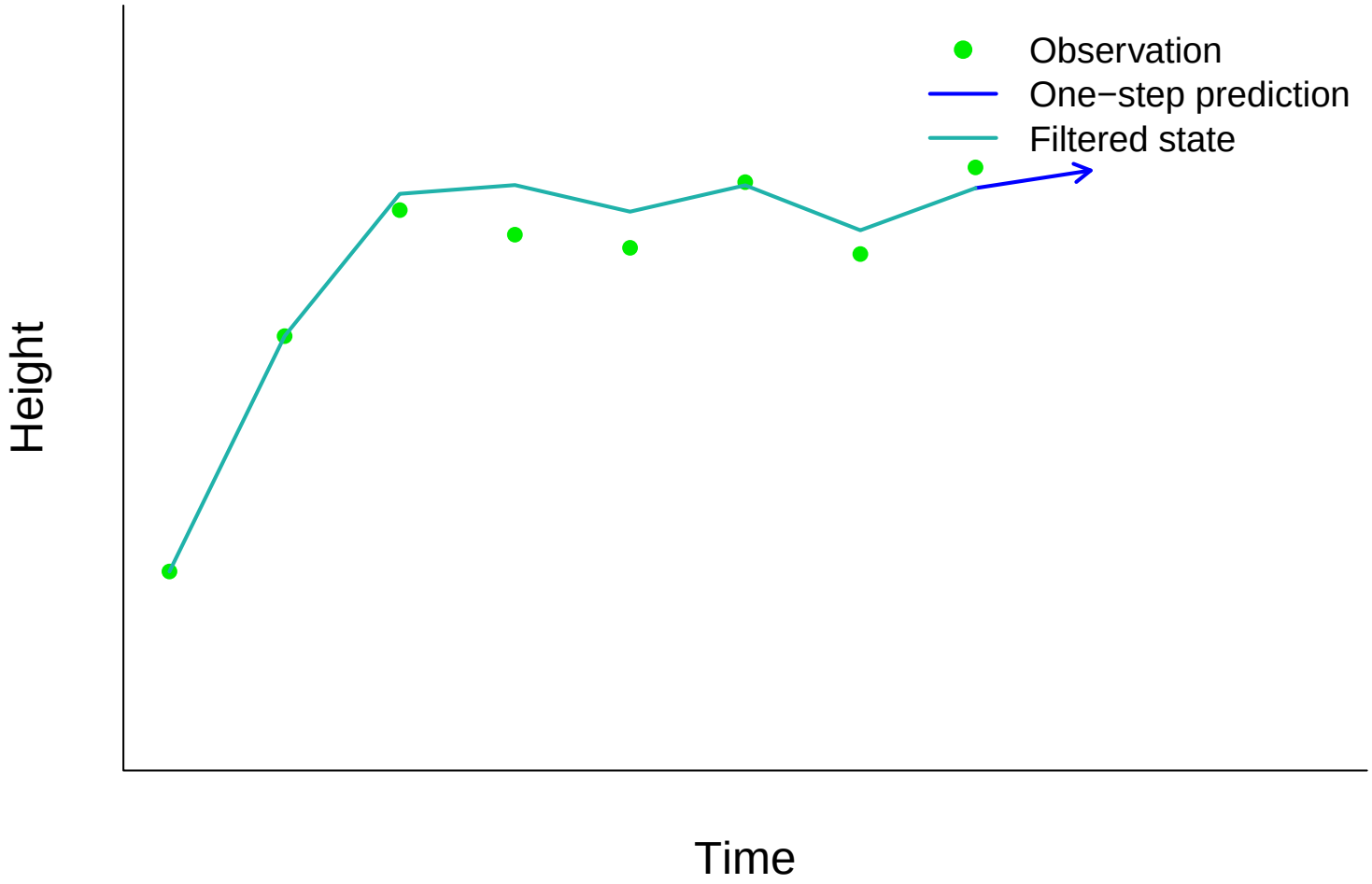

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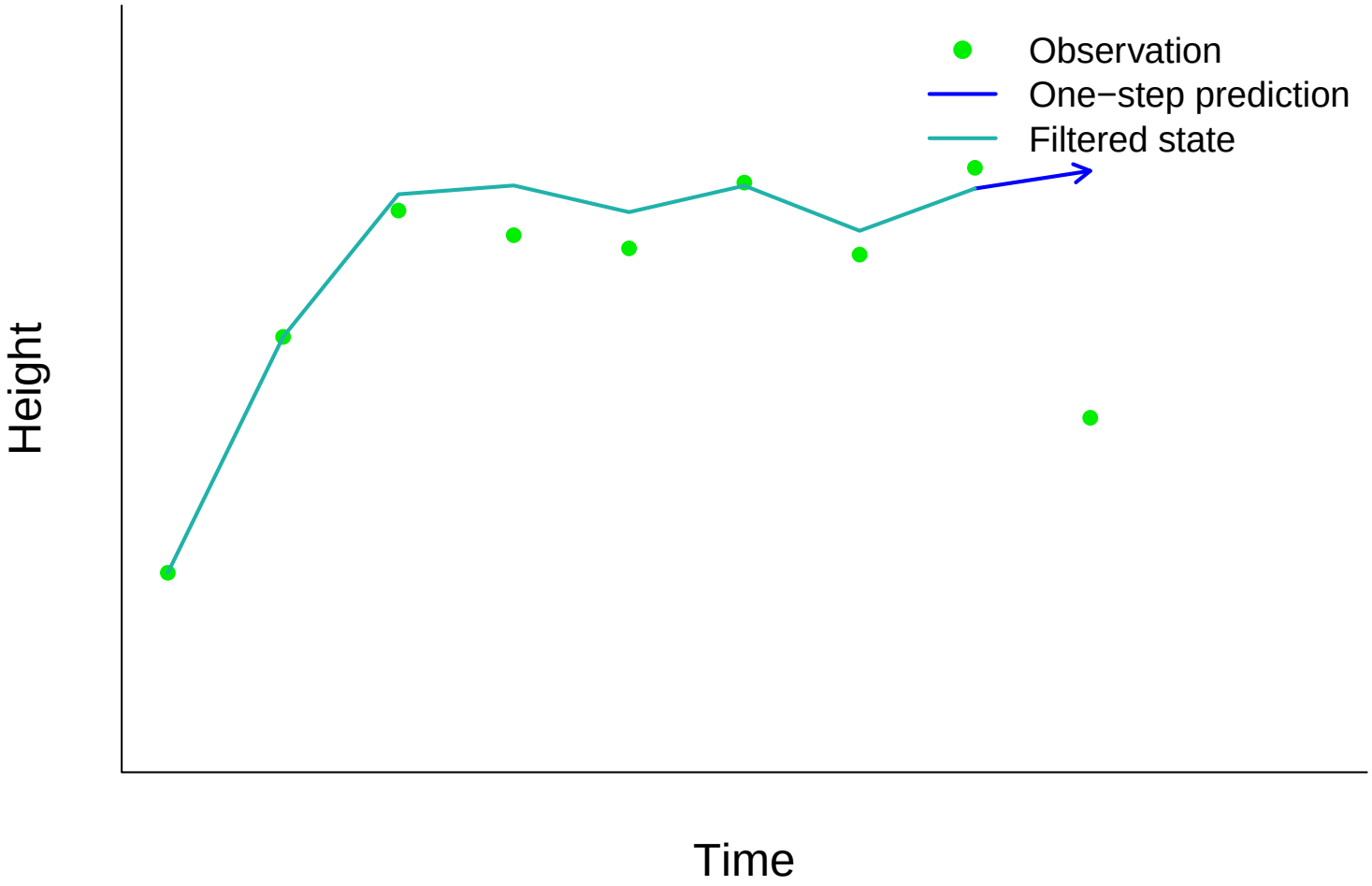

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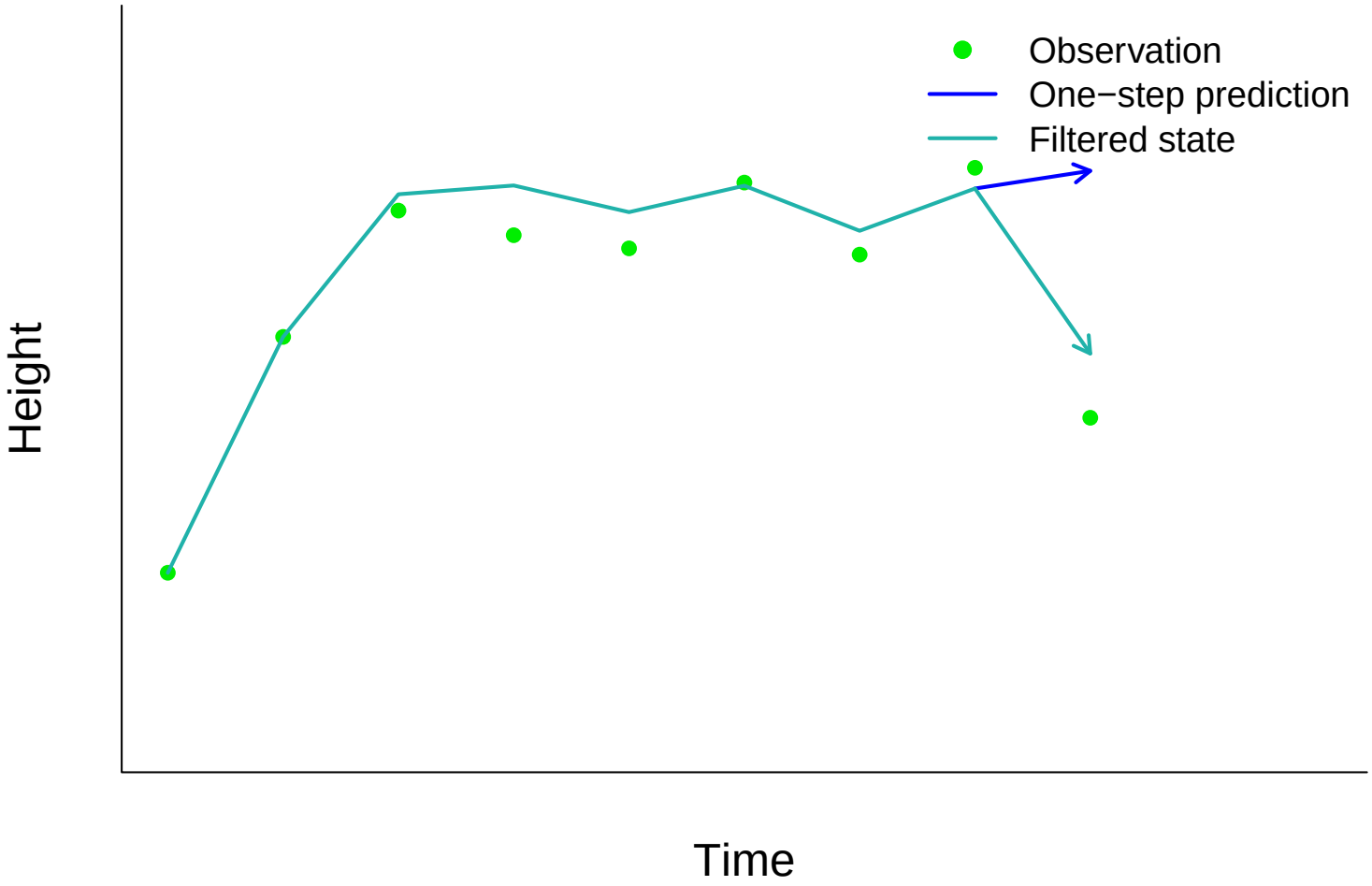

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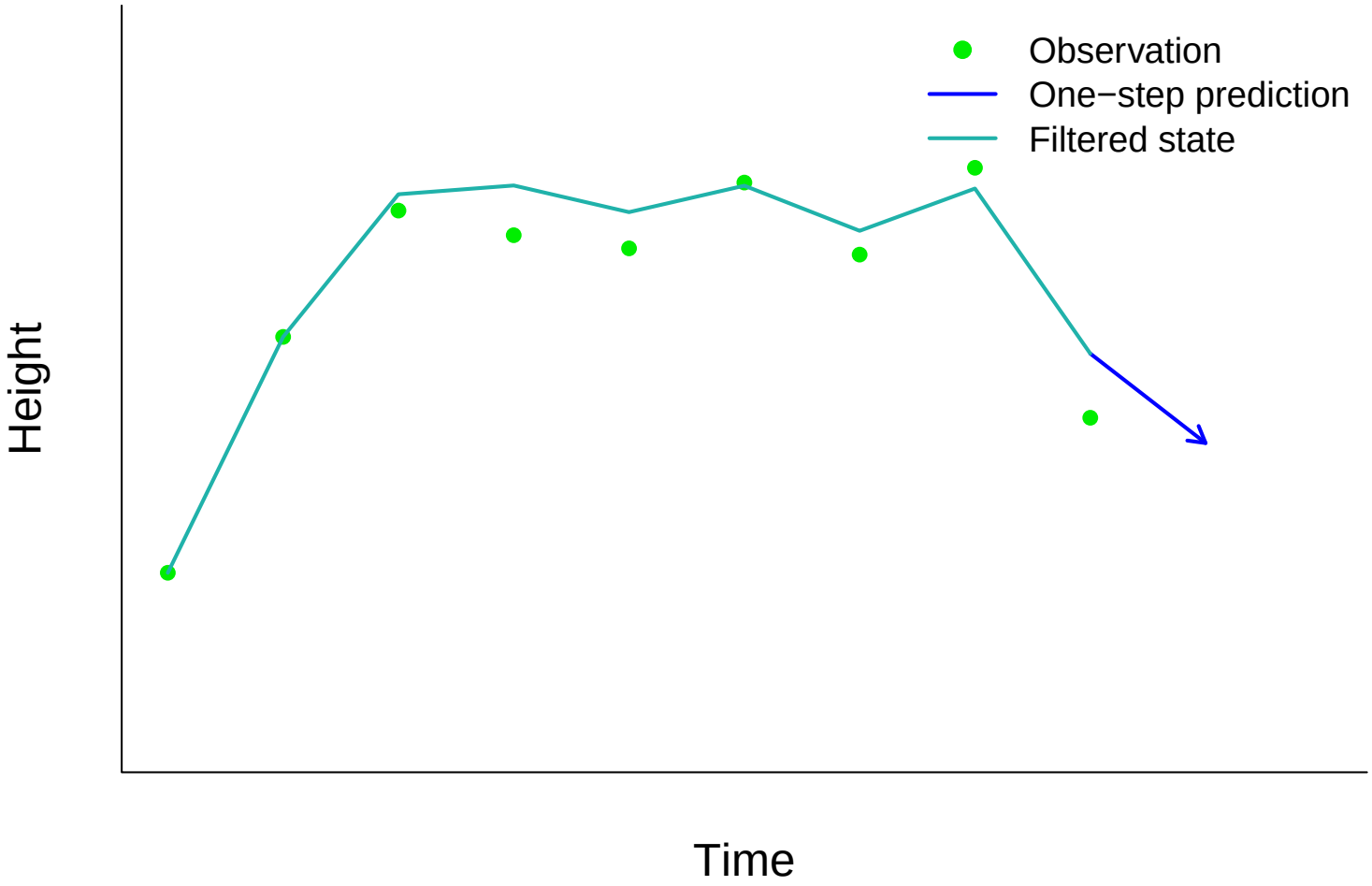

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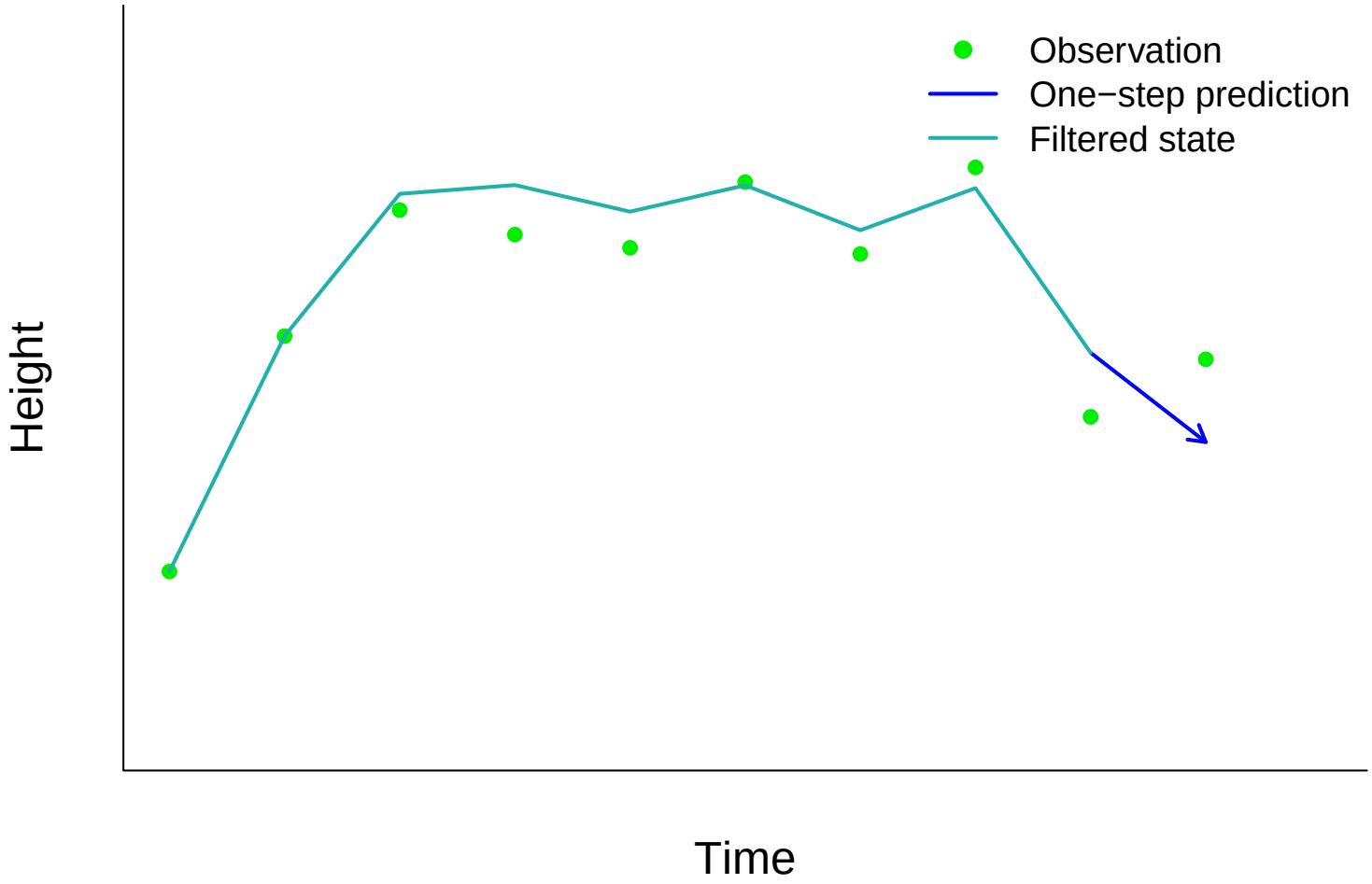

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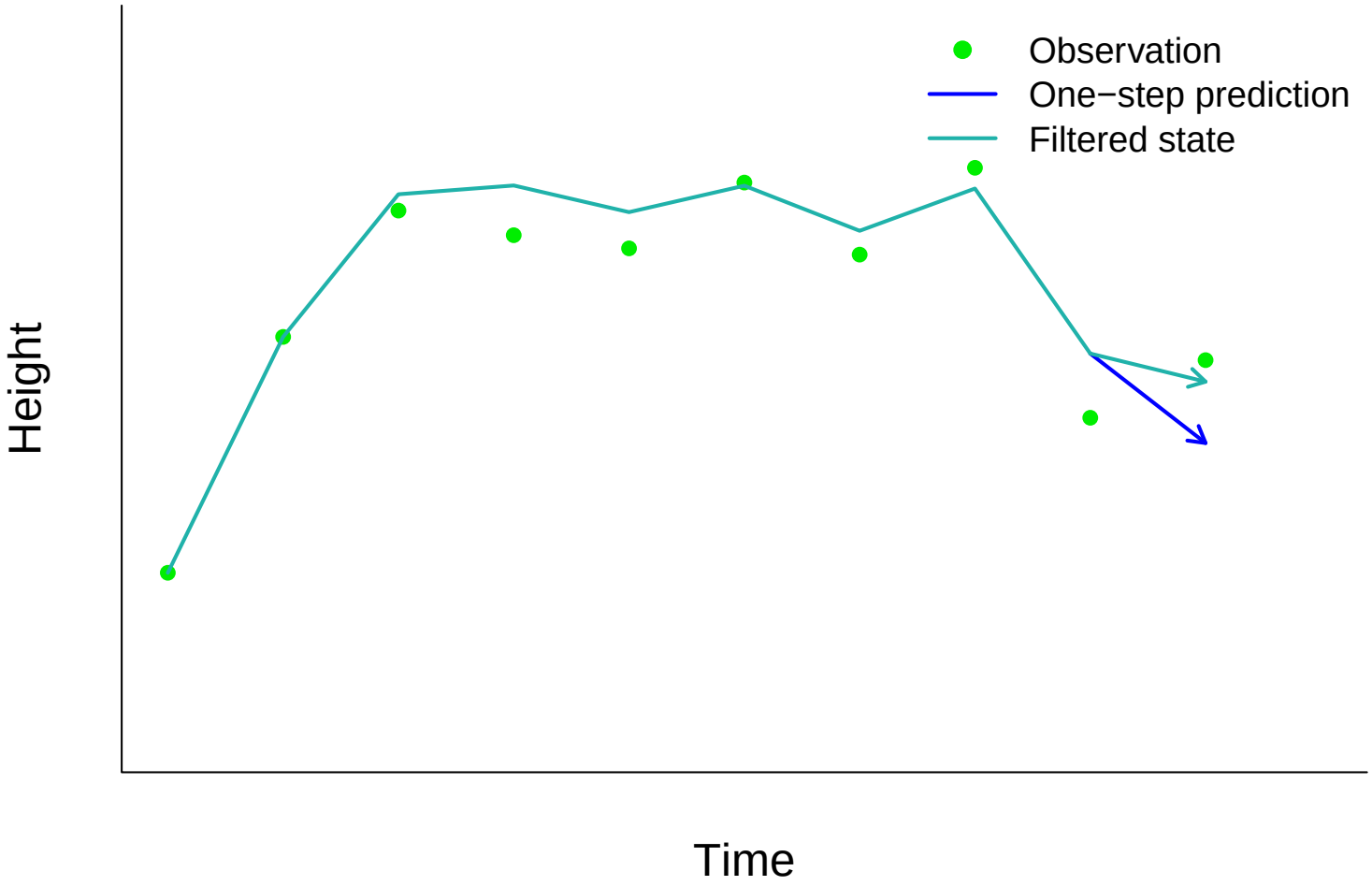

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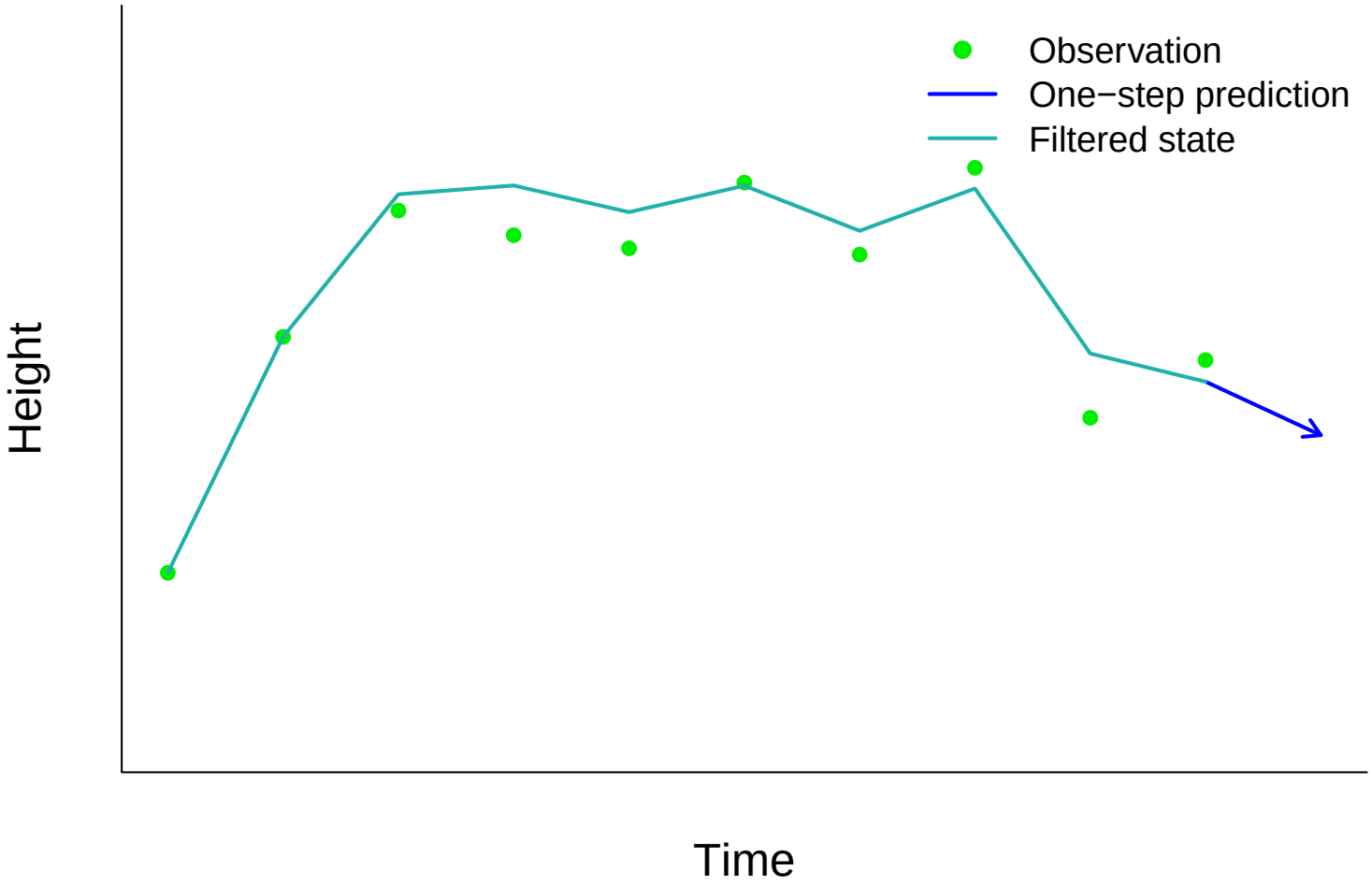

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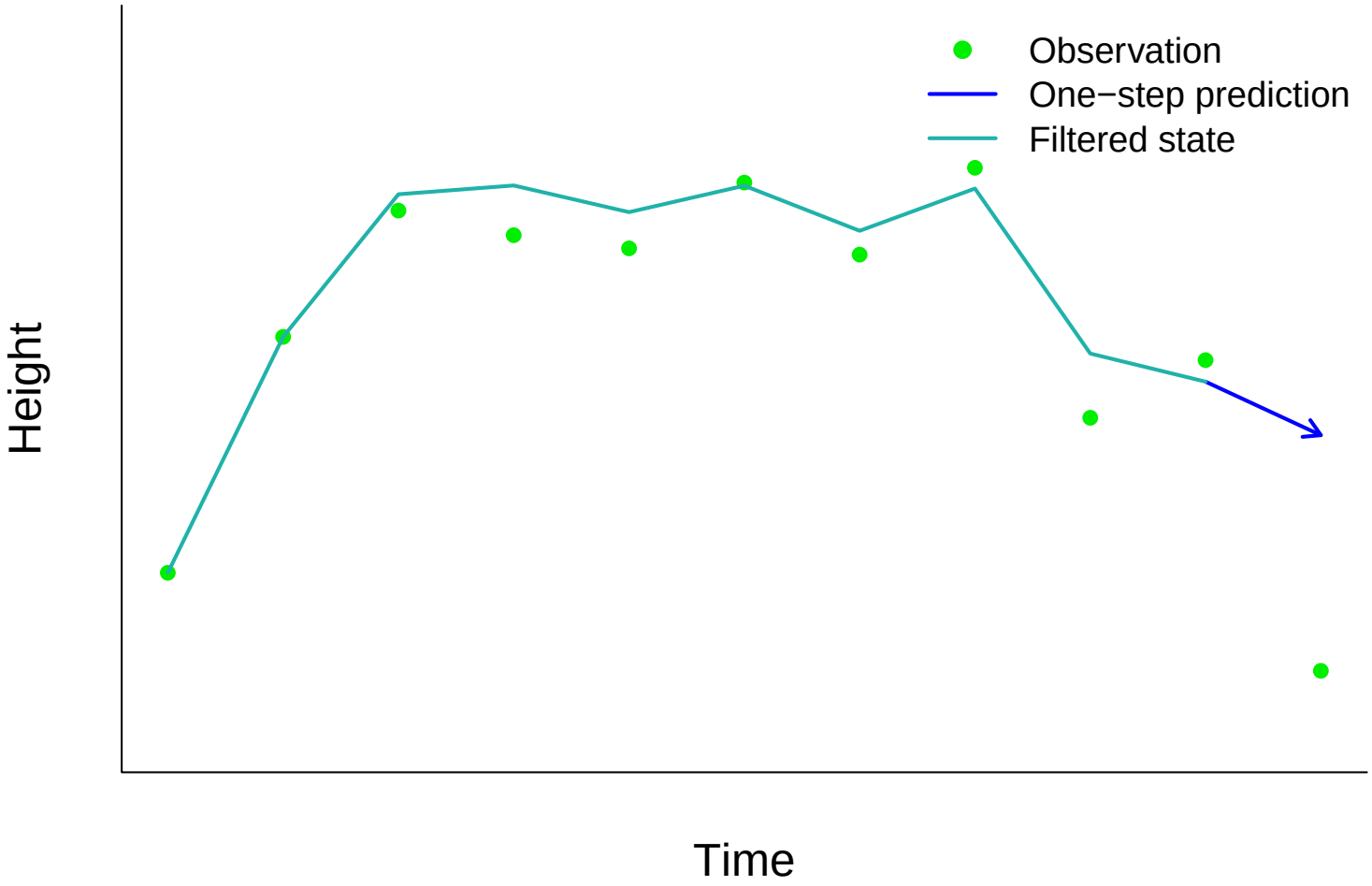

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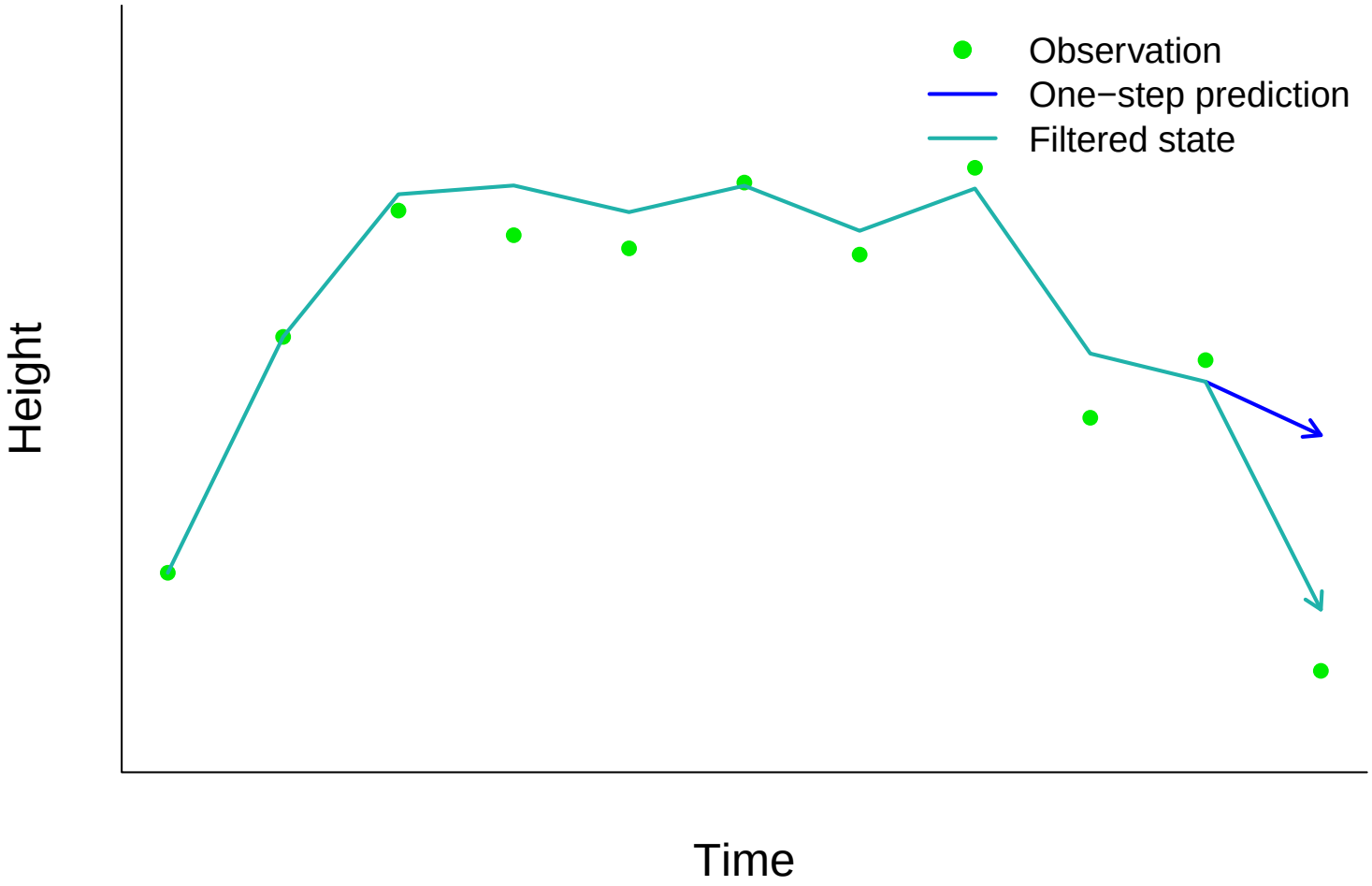


First state space model

Local level model

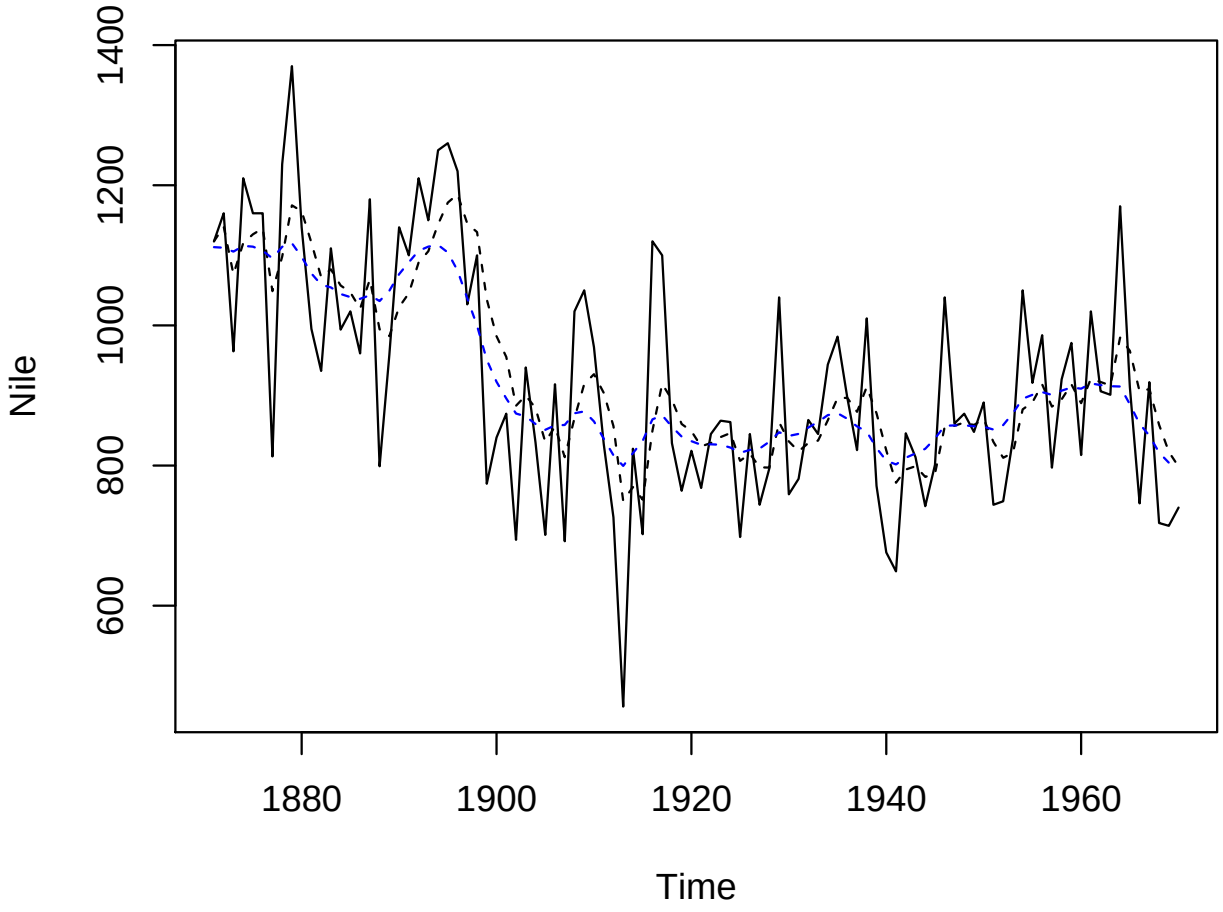
$$\begin{aligned} \text{Process:} \quad & x_t = x_{t-1} + \eta_t, \quad \eta_t \sim N(0, \sigma^2_{proc}) \\ \text{Observation:} \quad & y_t = x_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_{obs}) \end{aligned}$$

Can set up a Kalman filter to estimate the states [1]
Maximum likelihood uses the one step ahead forecast errors and their covariance matrix.

[1] See: Harvey (1989). *Forecasting, Structural Time Series Models and the Kalman Filter.*

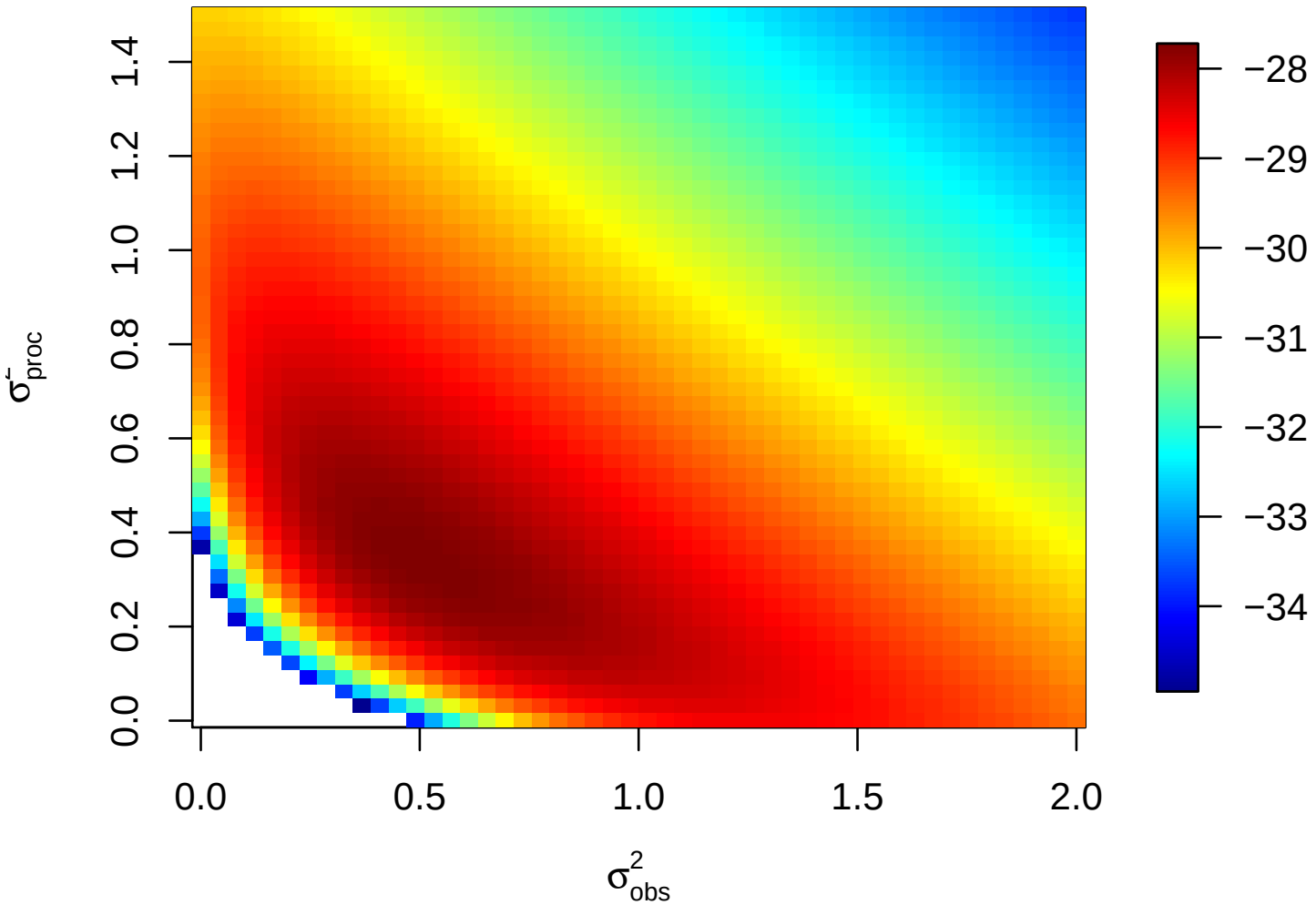
First state space model

Aswan dam flow data fit with a local level model



First state space model

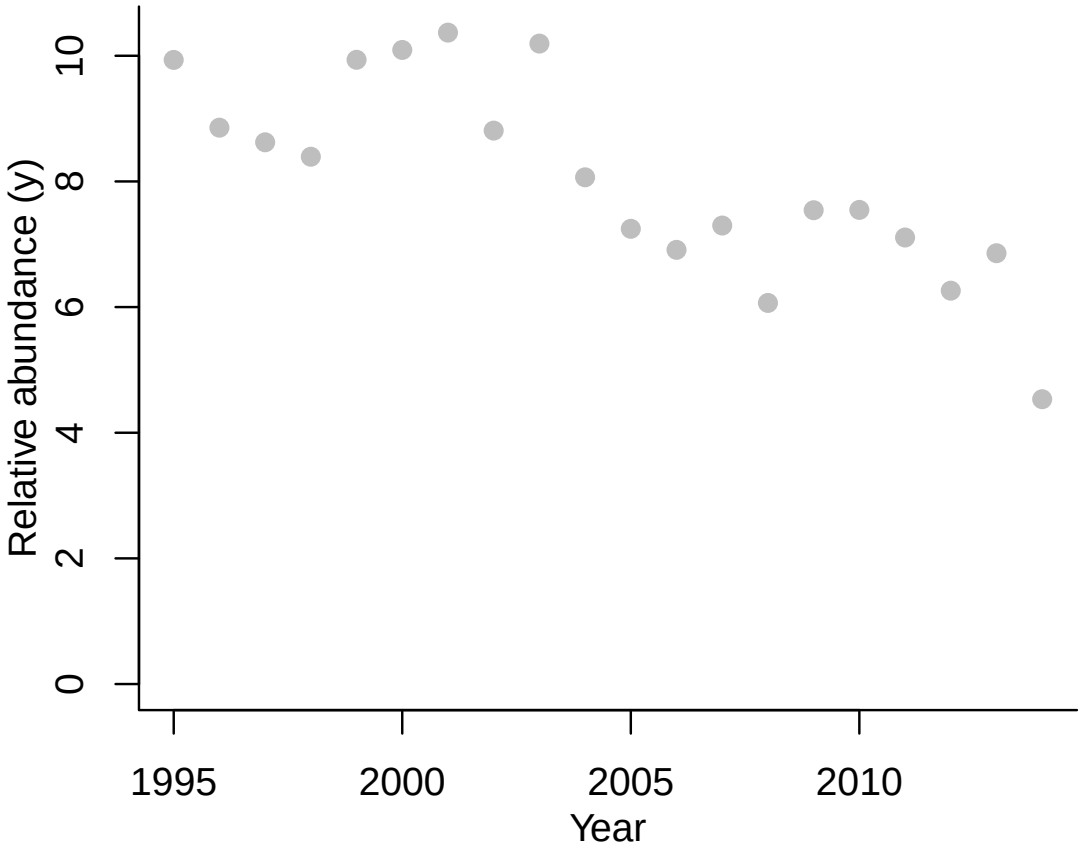
Likelihood surface of the two sources of variability



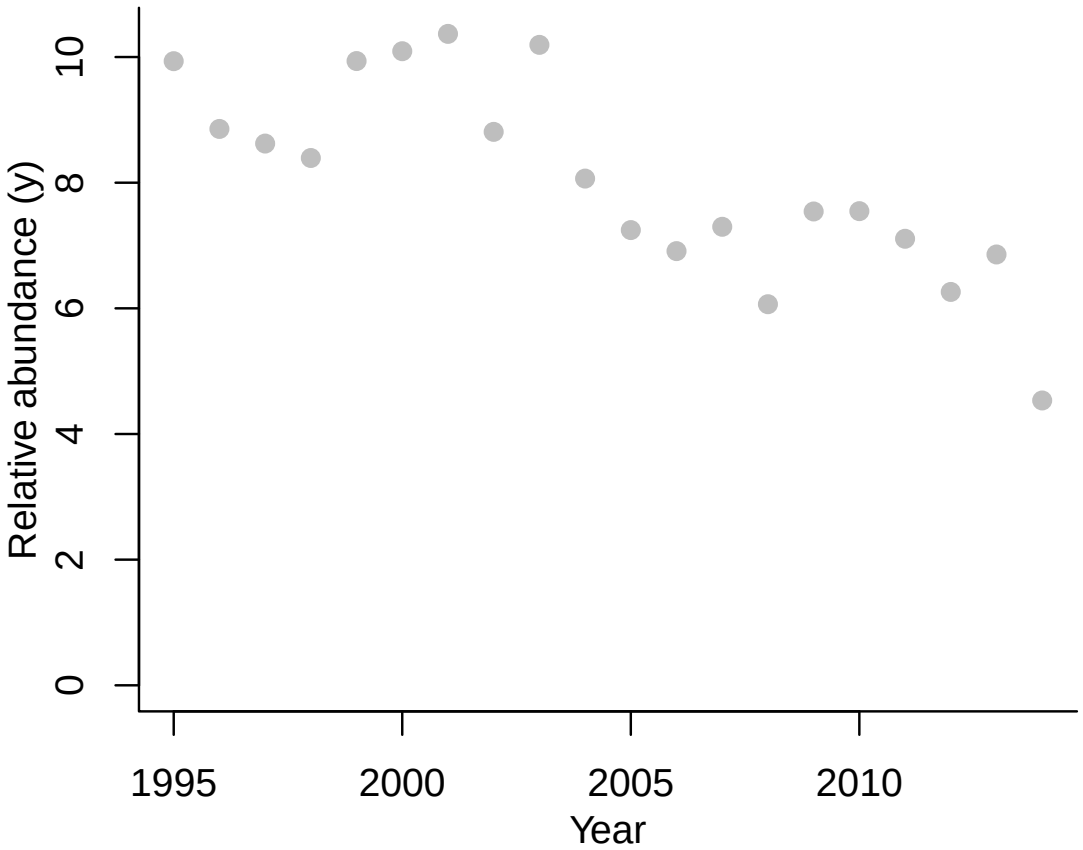
Separable here, not always the case though

Second state space model

Returning to the trend data



Second state space model



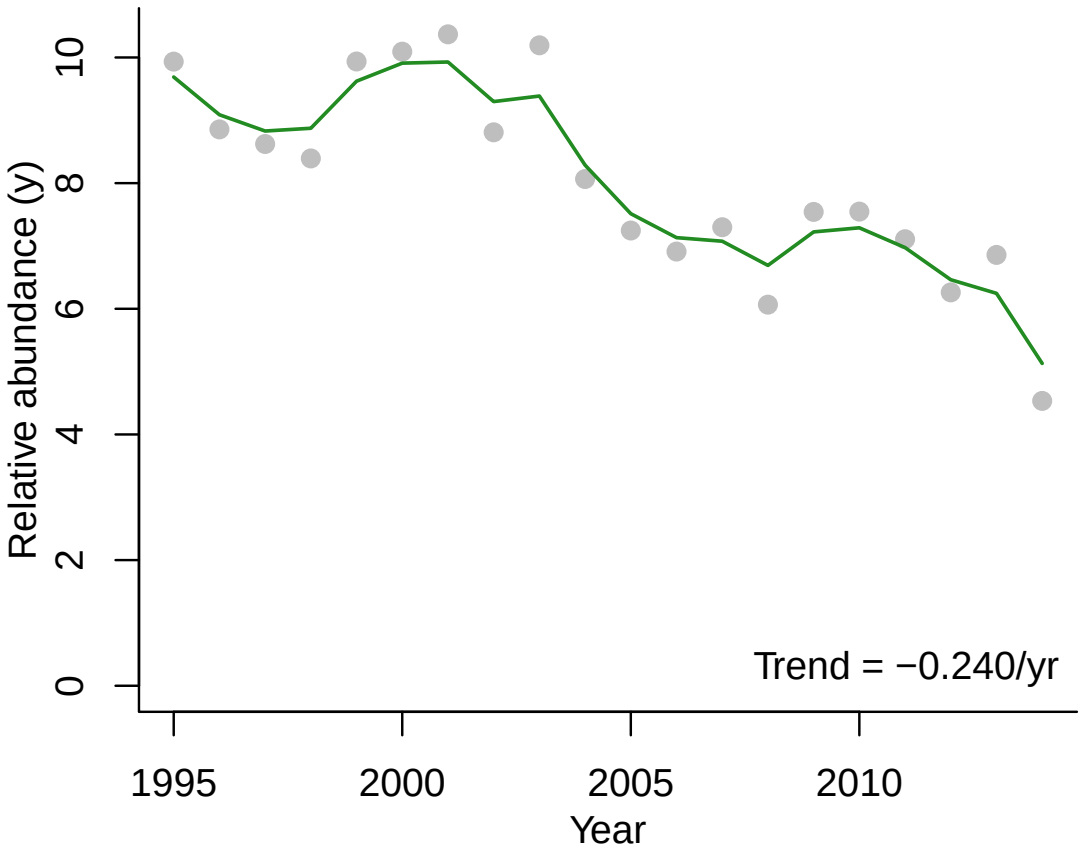
Set up a local linear trend model

$$\text{Process:} \quad x_t = x_{t-1} + \beta + \eta_t, \quad \eta_t \sim N(0, \sigma^2_{proc})$$

$$\text{Observation:} \quad y_t = x_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_{obs})$$

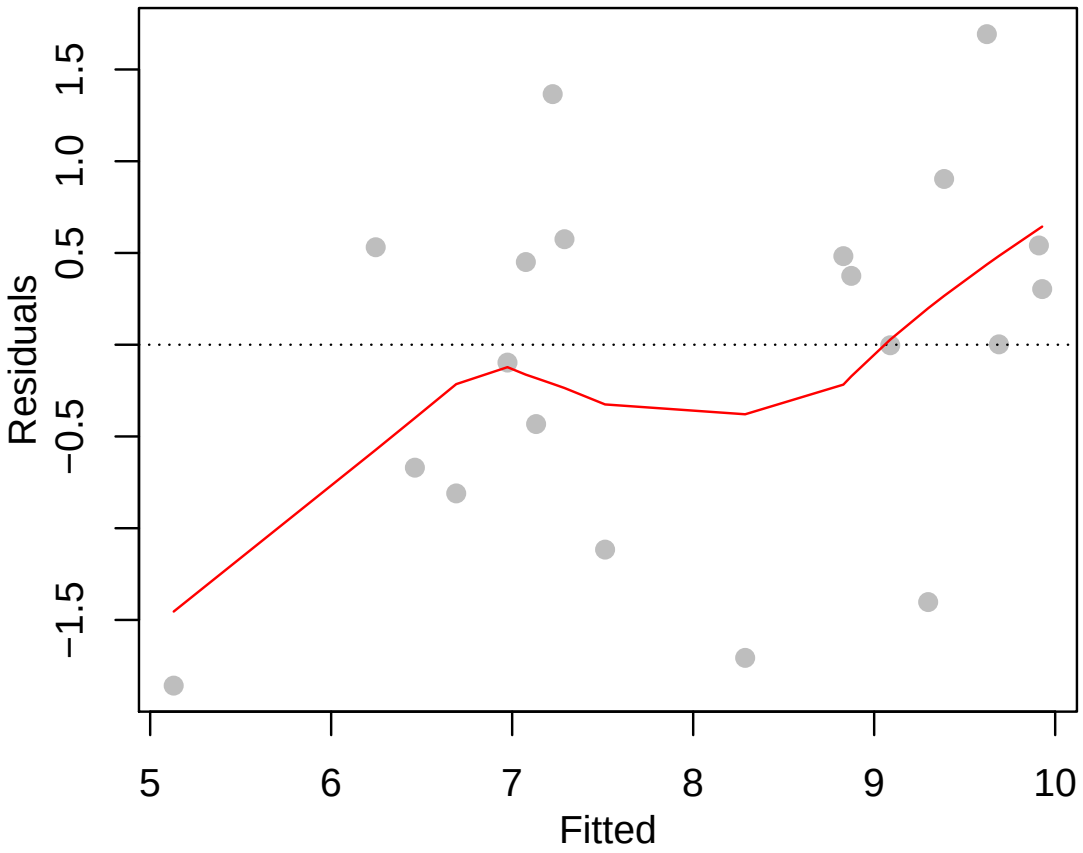
Second state space model

Fitted local linear trend model



Second state space model

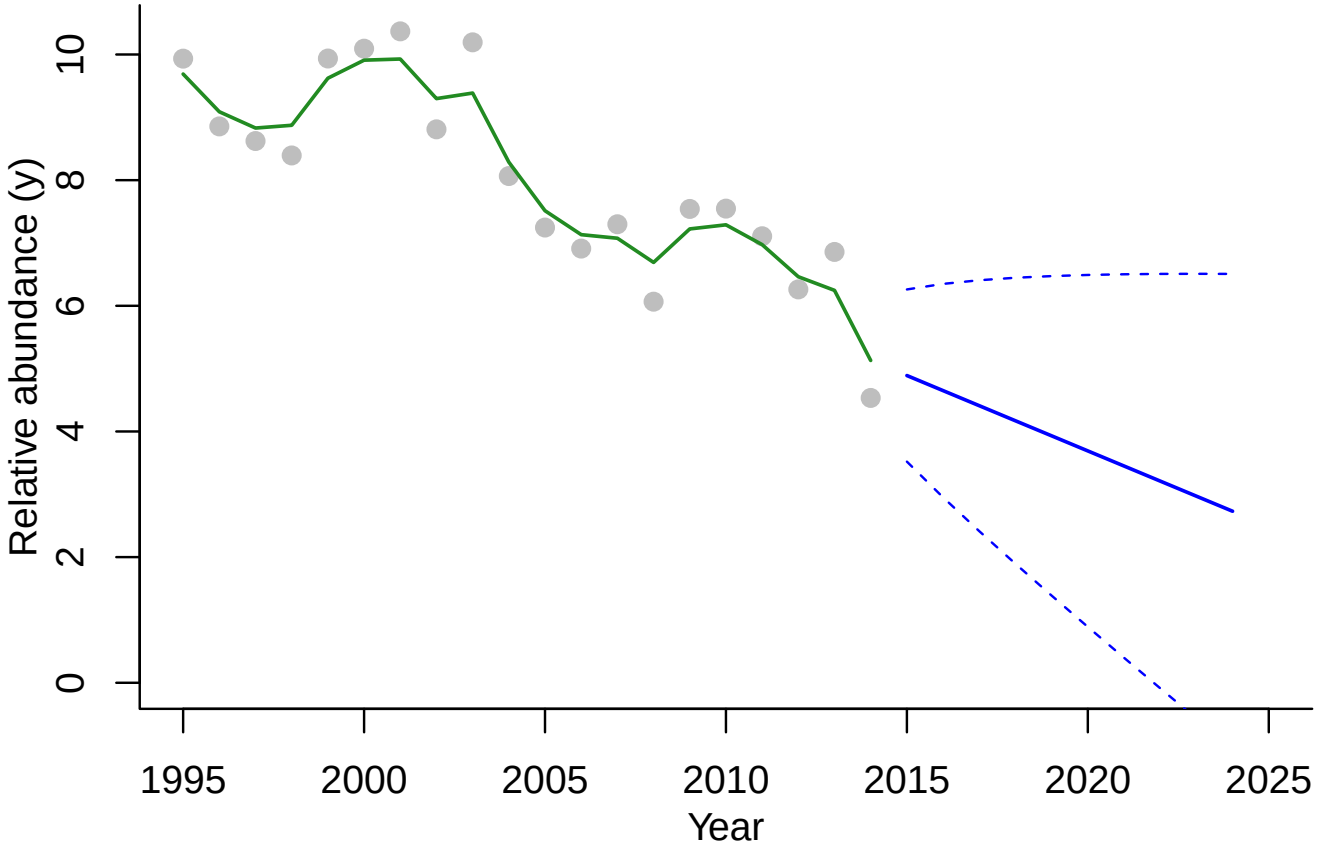
State space model residuals



Somewhat more evenly distributed, not stellar but typically better than measurement or process only fits

Second state space model

Forecasting is straight forward



Outline

Motivation

State space models

Spacing out

Extensions

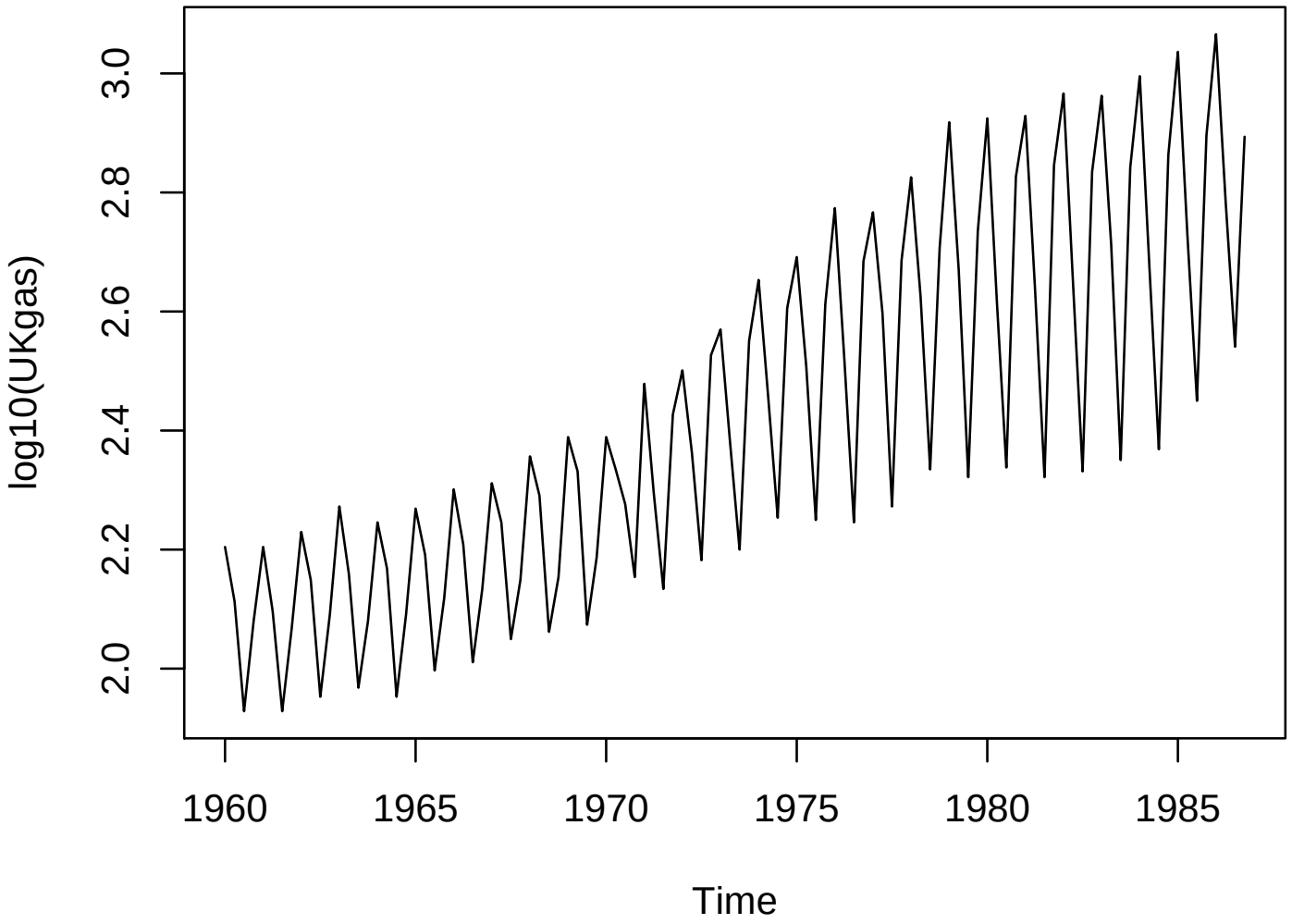
Time-varying trend

$$
\begin{aligned}
\text{Process:} \quad & x_t = x_{t-1} + \beta_t + \eta_t, && \eta_t \sim N(0, \sigma^2_{proc}) \\
& \beta_t = \beta_{t-1} + \zeta_t, && \zeta_t \sim N(0, \sigma^2_{\zeta}) \\
\text{Observation:} \quad & y_t = x_t + \varepsilon_t, && \varepsilon_t \sim N(0, \sigma^2_{obs})
\end{aligned}
$$

Note how the trend is also allowed to vary in time

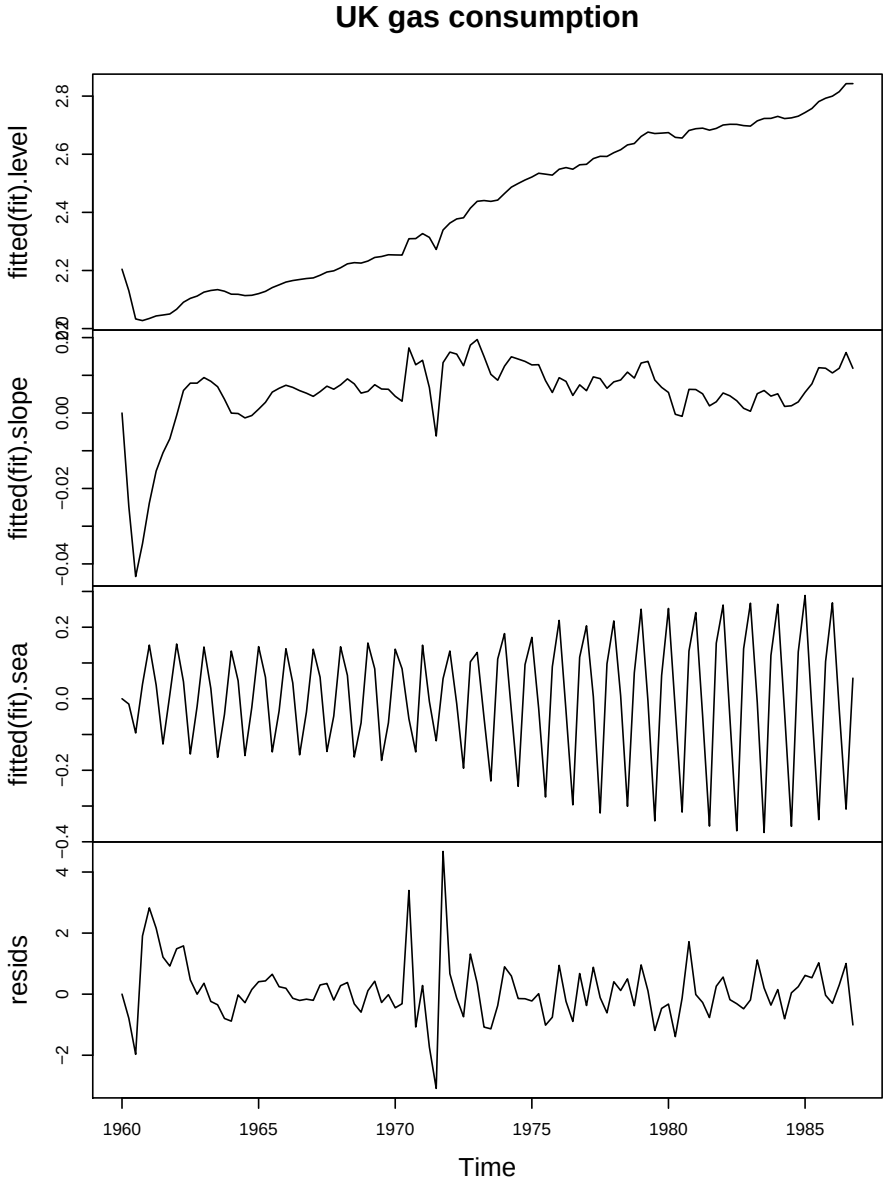
Seasonality

Can also accomodate time-varying seasonal components



Seasonality

Fitted *basic structural model*



Multivariate

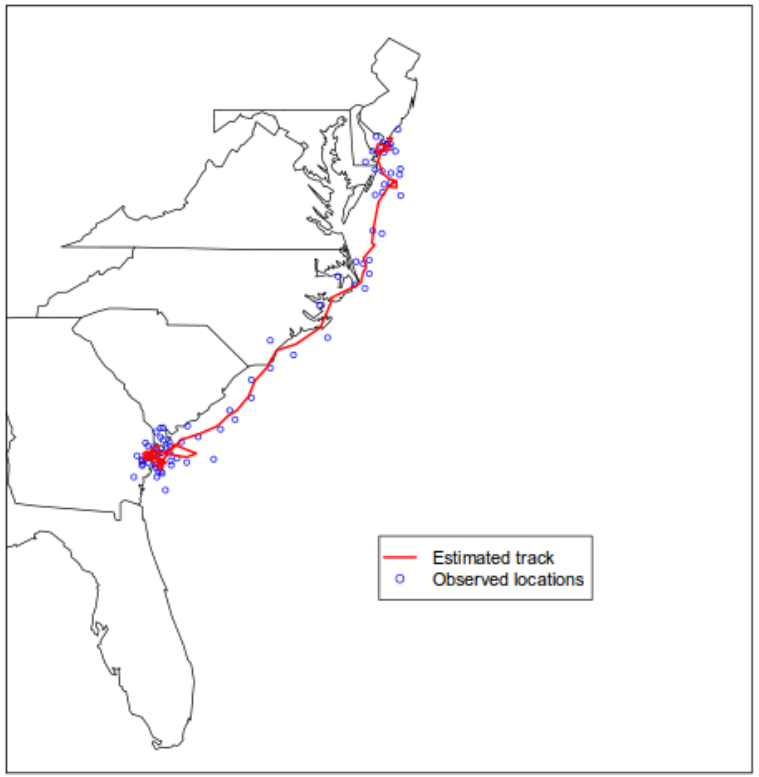
Can cast the model in multivariate form

$$\mathbf{Y_t} = \mathbf{F_t X_t} + \varepsilon_{\boldsymbol{t}}, \qquad \varepsilon_{\boldsymbol{t}} \sim \mathrm{N}(0, \mathbf{R})$$
$$\mathbf{X_t} = \mathbf{D X_{t-1}} + \eta_{\boldsymbol{t}}, \qquad \eta_{\boldsymbol{t}} \sim \mathrm{N}(0, \mathbf{Q}),$$

Where $\mathbf{Y_t}$ is a multivariate vector of observations
Where $\mathbf{Q}$ is the process <u>covariance</u> matrix
Interest may lie in how correlated the process errors are across stocks for example

Multivariate

Estimation and application of multivariate autoregressive state space models (MARSS) has been pioneered by Eli Holmes at NOAA



2D turtle track fit by a Kalman filter

Non-linear models

Straightforward Kalman filter assumes:

- Linearity
- Normally distributed observation and process errors

A state space model where:

$$\begin{aligned} x_{t+1} &= f(x_t, \eta_t) \\ y_t &= g(x_t, \varepsilon_t) \end{aligned}$$

and f and g are non-linear, non-normal functions requires alternative filtering methods

Non-linear models

Alternative estimation methods:

- Extended Kalman filters
- Particle filters
- Laplace/quadrature approximations (e.g., ADMB, TMB)
- Markov Chain Monte Carlo

These methods aim to estimate the unobserved states and parameters governing the observation and process equations for linear and non-linear state space models

Summary

- State space modelling aims to account for both process and observation
- Powerful and challenging class of models
- Increasingly used in many realms of science
- Kalman filter as an optimal method for linear, Gaussian systems
- Can be extended to multiple dimensions, opening a range of opportunities including empirical multispecies investigations
- Non-linear models feasible through extended/additional methods