# Plots of noisy signal and original signal

In Figure 1, the plot of the original signal versus time is plotted. The sampling frequency is 360 Hz, and the the time vector is established based on the sampling frequency, where the interval is equal to the sampling period and time span is 10s. Similarly, the plot of the noisy signal versus time is plotted in Figure 2.

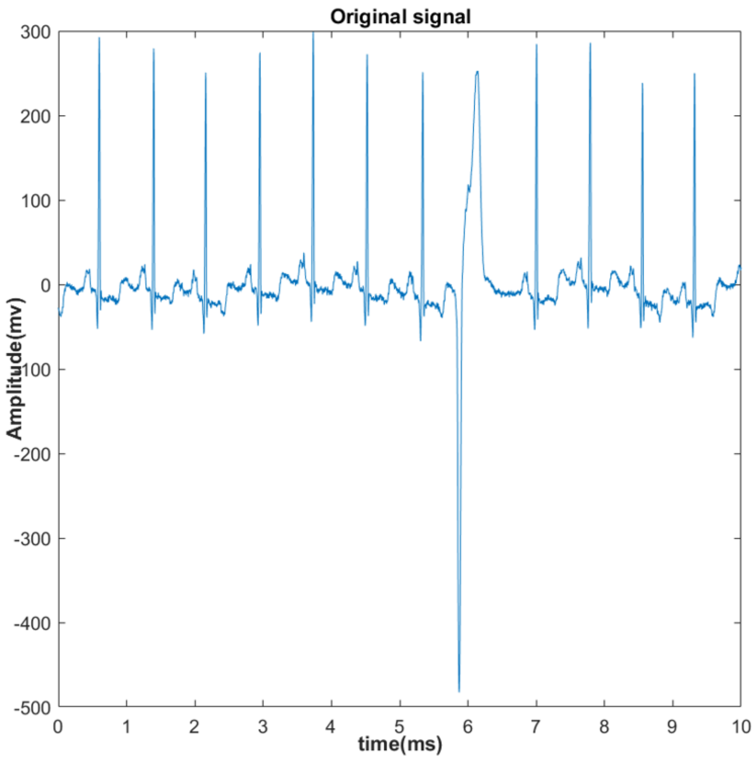


Figure 1: Original Signal versus time

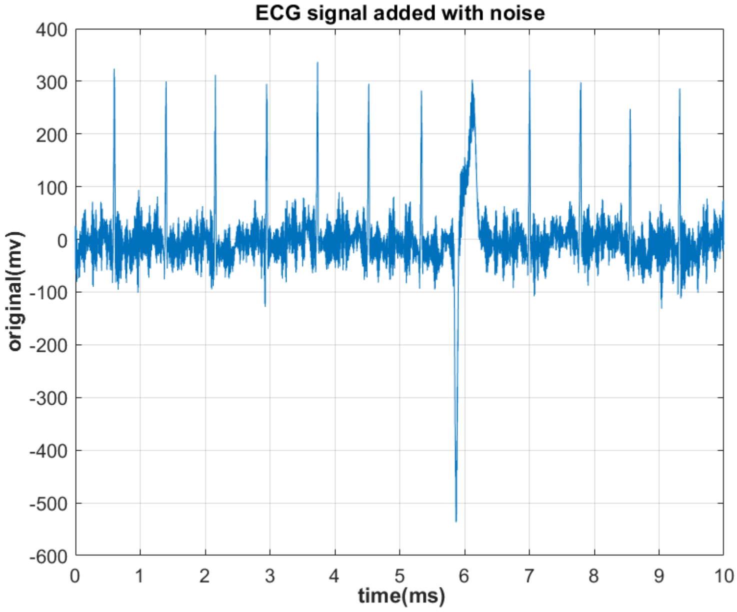


Figure 2: Noise versus time

# Heart Rate (Bpm)

The heart rate is calculated as beats per minute (Bpm). In the ECG signal, the average heart rate can be calculated by dividing the the number of extrema by duration in one minute:

$$Bpm = \frac{N}{D} \quad (1)$$

Where $N$ is the number of extrema and $D$ is the duration in one minute. The duration in one minute can be calculated by dividing the time span by 60 seconds. As mentioned before, the time span is 10 seconds then $D$ is $\frac{10}{60} = 0.1667s$. The extrema can be defined as the local maxima in ECG signal as shown in Figure 3. The final calculated Bpm is approximately 72.

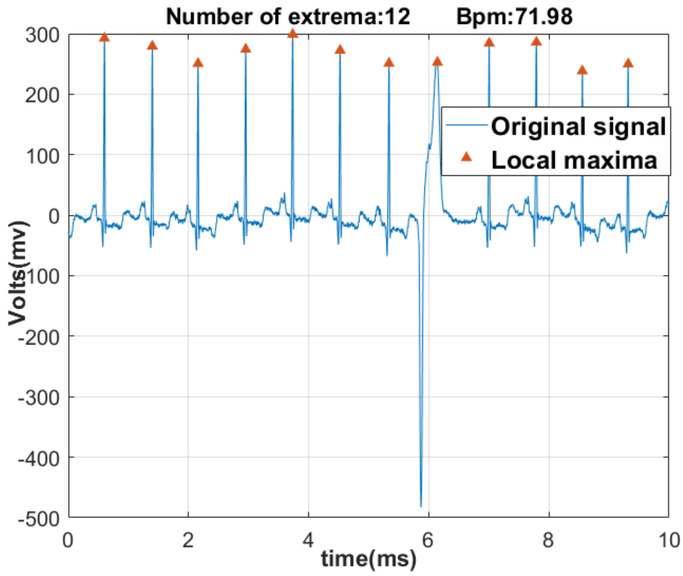


Figure 3: Number of extrema and calculated Bpm

## Frequency domain

Before designing the low pass filter, the original signal and Noise signal in frequency domain need to be shown so that we can find the frequency such that we can find the signal that we do not want. The plot of the ECG signal in frequency time domain is given by using fast fourier transform(fft) in Figure 4. As shown in this figure, the amplitude has been normalized, the frequency is established based on the sampling frequency, and it is also noted that the figure plotted by using fft is symmetrical about the origin

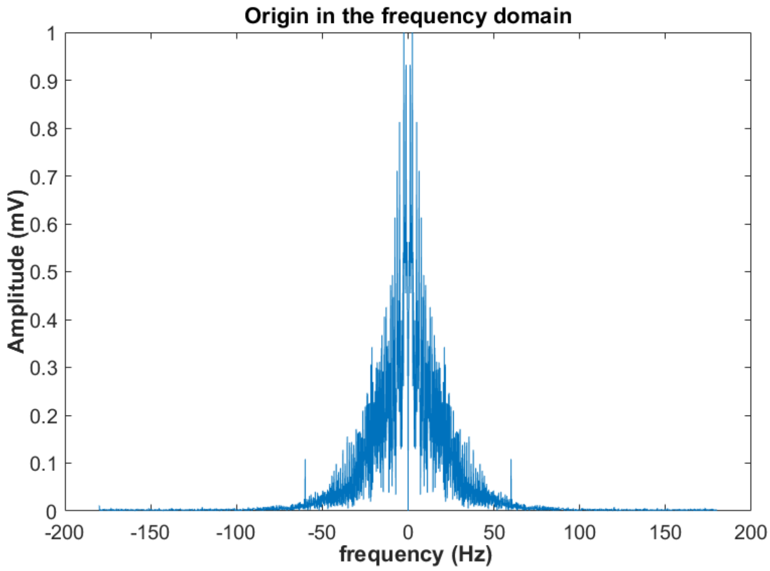


Figure 4: ECG signal in frequency domain

The reason why fft diagram is symmetrical is because of the nature of fourier transform and it can be proved in a mathematical way: Define a signal in time domain $h(t)$. Then the fourier transform of this signal can be expressed as:

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi ft}dt = \int_{-\infty}^{\infty} h(t)[\cos(2\pi ft) - jsin(2\pi ft)] \; (2)$$

And the fourier transform at the negative frequency can be expressed as:

$$H(-f) = \int_{-\infty}^{\infty} h(t)e^{2\pi ft}dt = \int_{-\infty}^{\infty} h(t)[\cos(2\pi ft) + jsin(2\pi ft)] \; (3)$$

Let the term $\int_{-\infty}^{\infty} h(t)\cos(2\pi ft)\,dt = F_R(f)$ and the term $\int_{-\infty}^{\infty} h(t)\sin(2\pi ft) = F_i(f)$.So, $H(f) = F_R(f) + iF_i(f)$, since the cosine term is even and sine term is odd, which can be expressed as:

$$F_R(f) = F_R(-f),\; F_i(f) = -F_i(-f) \; (4)$$

And the modulus of $F(f)$ can be expressed as:

$$H(f)^2 = |F_R(f)^2| + |F_i(f)|^2 \; (5)$$

Then clearly when the frequency is negative, the modulus is still the same:

$$H(-f)^2 = |F_R(-f)^2| + |F_i(-f)|^2 \; (6)$$

So (5) = (6), and this can also apply in fast fourier transform (fft). The plot of the Noise in frequency domain is given in Figure 5, where the mains hum is labeled at around 60 Hz, while the mains frequency in the US is also around 60 Hz. The wanted signal and unwanted signal are labeled in this figure as well.

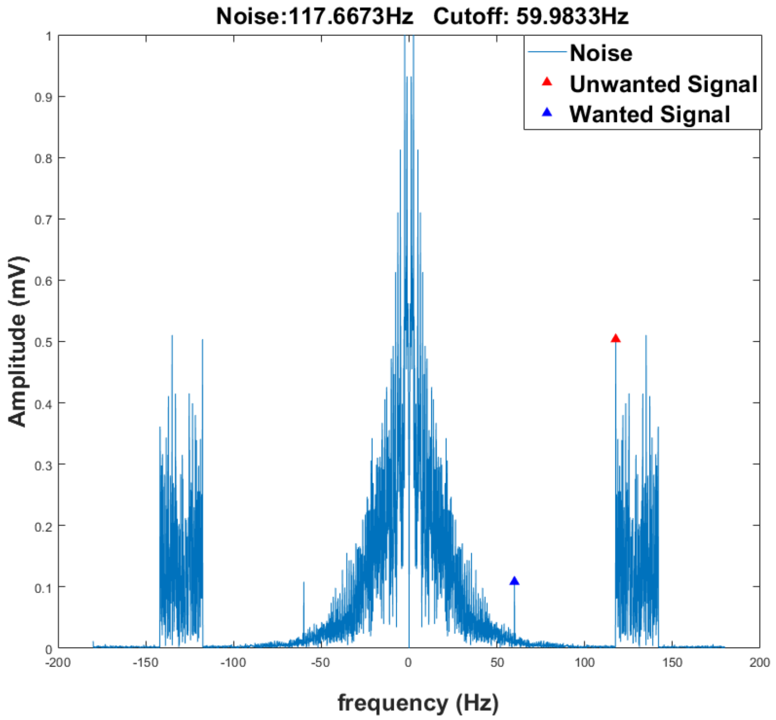


Figure 5: Noise in frequency domain

# Cut off frequency

To design a filter to filter the noise out, the cut off frequency needs to be determined. The cutoff frequency can be determined by comparing the Figure 5 with Figure 4 and it is noted that there is a line noise located at around 60 Hz, and after that, the signal can be ignored before reaching the noise located at around 117 Hz. So, the cutoff frequency $f_{cutoff} = 60\ Hz$ and it can be normalized as $f_{normalized} = \frac{f_{cutoff}}{f_s} = \frac{60}{360} = \frac{1}{6}\ cycles/sample$, where $f_s$ is the sampling frequency.

# Filter Design

## A. IIR filter

The infinite impulse response can be characterized as:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \quad (7)$$

Then the corresponding difference equation can be expressed as:

$$y(n) = \sum_{k=0}^{\mathrm{N}} a_k x(n-k) - \sum_{l=1}^{\mathrm{M}} b_l y(n-l) \quad (8)$$

Based on (8), the corresponding transfer function can be calculated as:

$$H(z) = \frac{a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_N z^{-N}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}} \quad (9)$$

Where $a_N$ and $b_M$ are the corresponding coefficients. To design a IIR low pass filter, we need to first determine the transfer function $H(s)$ of the low pass filter and the coefficients need to be determined. As shown in Figure 6, the graph of the frequency response in the analogue low pass filters using four different types of filters (Butter, cheby1, cheby2, ellip ) is given. The Butterworth filter has a flatter pass band and shorter transition band, which means that it will not remove the wanted signal while removing the noise. So, in this filter design, to achieve the transfer function and coefficients mentioned above, the Butterworth filter has been chosen.

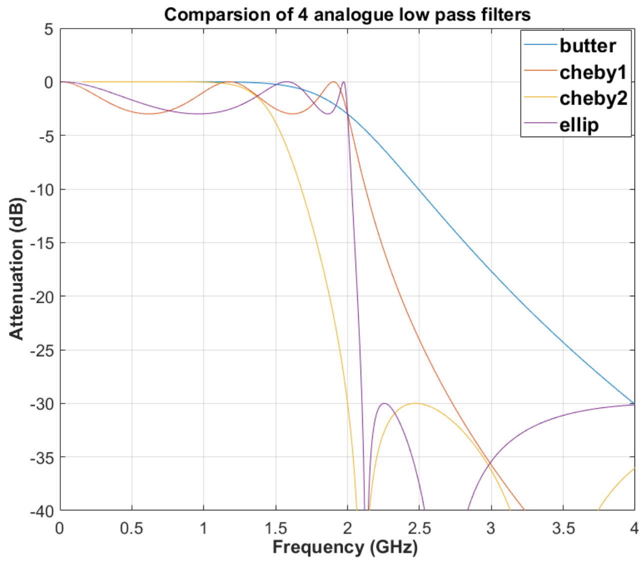


Figure 6: Comparison of four analogue low pass filters

The Butterworth polynomial can be expressed as:

$$B_n(s) = \prod_{l=1}^{\frac{n}{2}} \left[s^2 - 2s\cos\left(\frac{2l+n-1}{2l}\pi\right) + 1\right],\ n = even\ (10)$$

$$B_n(s) = (s+1)\prod_{l=1}^{\frac{n-1}{2}} \left[s^2 - 2s\cos\left(\frac{2l+n-1}{2l}\pi\right) + 1\right],\ n = odd\ (11)$$

According to bilinear transform, $s = \frac{2}{T}\frac{z-1}{z+1}$, where $z = e^{2\pi f_a T}$, and $f_a$ is the calculated cut-off frequency, which can be calculated as:

$$f_a = \frac{f_s}{\pi}\tan\left(\frac{\pi f_{cutoff}}{f_s}\right)\ (12)$$

And $T$ is the sampling period. $f_{cutoff}$ is the selected cut off frequency based on Figure 5. It is noted that when the sampling frequency $f_s \gg f_{cutoff}$, $f_{cutoff\_calculated} \approx f_{cutoff}$. According to the bilinear transform, the final transfer function can be expressed as $H(z) = H(s)$, where $s$ can be replaced by: $\frac{2}{T}\frac{z-1}{z+1}$.

## B. FIR filter

Compared with IIR filter, FIR filter is always stable, since its transfer function doesn't have any poles. The relationship between the input sequence $x[n]$ and output sequence $y[n]$ in a discrete-time FIR filter can be expressed as:

$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + \cdots b_Nx[n-N] \quad (13)$$

Where $N$ is the number of orders and $b_n$ is the coefficient, which can be expressed as:

$$b_n = \sum_{i=0}^{i=N} b_i\delta[n-i] = h[n] \; (14),$$

then the Z-transform of it is

$$H(z) = Z\{h[n]\} = \sum_{n=0}^{N} b_n z^{-n} \quad (15).$$

Compare Figure 5 with Figure 4, the wanted frequency is around 60 Hz. So, in this FIR filter design, to allow for some errors, the cut-off frequency is set as around 70 Hz and since the unwanted noise is located at around 118 Hz in the frequency domain as shown in Figure 5, the stop band frequency is set as around 115 Hz to allow for some errors as well.

In this report, the FIR filter is designed using window method. As shown in Figure 6, the frequency response of three different window methods are plotted. When frequency exceeds around 110 Hz, the attenuation of Hanning Window is bigger. Since in this simulation, the target is to remove the noise as much as possible, so, the attenuation factor should be as big as possible, so in this report, the window method used for FIR design is Hanning Winodw, whose attenuation factor at stop band is around 44dB.

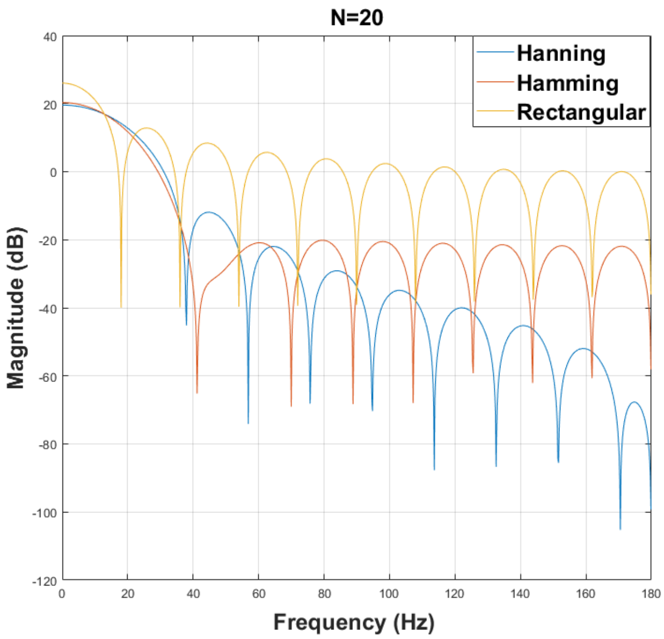


Figure 6: Frequency response of Three different Window Methods

In Hanning Window, the desired order can be calculated as:

$$N \geq \frac{8\pi}{w_{stop} - w_{pass}} (16)$$

Where $w_{stop}$ and $w_{pass}$ are the angular frequency of stop band and pass band respectively. As shown in Figure 7, the plot of the FIR using Hanning method with different order is given, when the cut off frequency and the range of the noise frequency are fixed. The attenuation at the same cutoff frequency starts to decrease, when the number of orders is increased. Since the noise is located at the frequency between around 117Hz and around 140Hz as shown in Figure 5, to analyse the performance of removing the noise in the FIR filter, the plot of attenuation between 117Hz and 140 Hz is given in Figure 8. When the order is 20, the attenuation range is bigger than 60dB, which will give the best performance of removing the target noise, and since the maximum number of order is 20, then in this FIR design, it will have 20 orders.

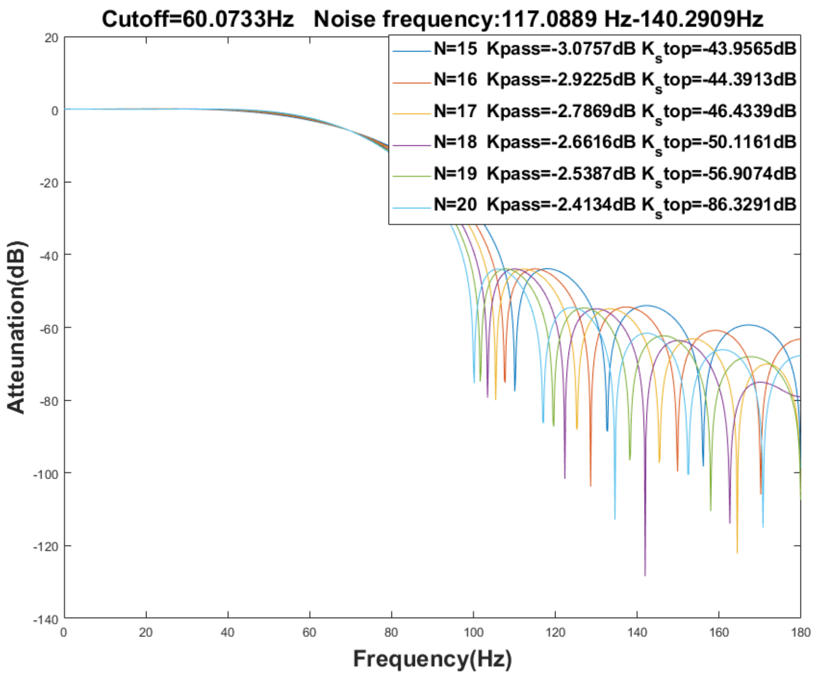


Figure 7: FIR with different number of orders

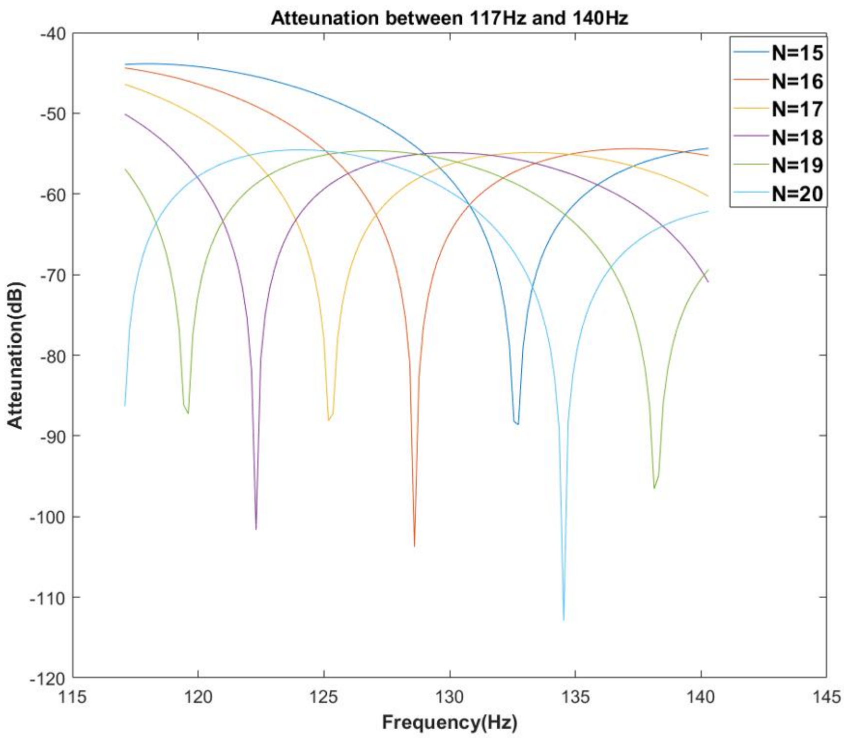


Figure 8: Attenuation within the noise range

## Frequency response

Figure 8 and Figure 9 plot the frequency response of IIR filter and FIR filter, respectively. As shown in Figure 8, the cut off frequency is at around 60 Hz, where the attenuation is around 3dB as expected, and when frequency continues to increase, the attenuation continuously decreases until it reaches the point at around $\frac{f_s}{2}$.

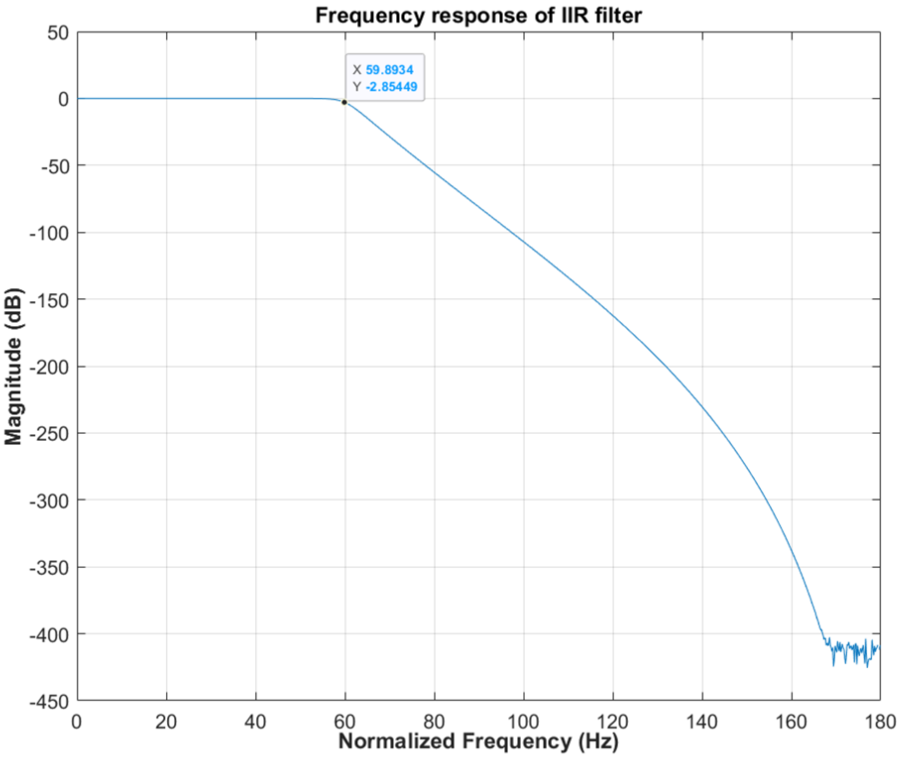


Figure 8: Frequency response of IIR filter

As shown in Figure 9, the cut off frequency is at around 62 Hz, where the attenuation is around 3 dB, and within the noise range(117Hz-140Hz), the attenuation is bigger than 55 dB, and as the frequency continuously increase, the attenuation continues to increase, which means noise will be cleared as much as possible, although there are some ripples.

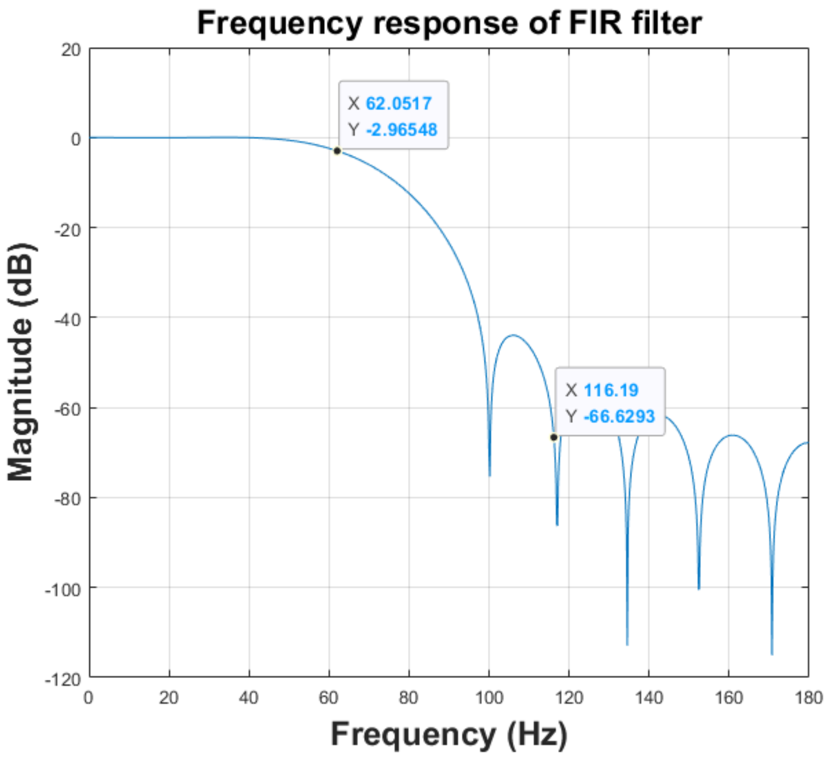


Figure 9: Frequency response of FIR filter

## Filtered Noise

As shown in Figure 10, compared with the original signal, most of the noise in the original signal is almost removed using IIR filter, and the filtered signal is quite close to the original signal. However, there are still spikes, since as shown in Figure 8, in this IIR filter, the attenuation within 80 Hz and 110 Hz already exceeds 50 dB, which almost removes the wanted signal within that range.

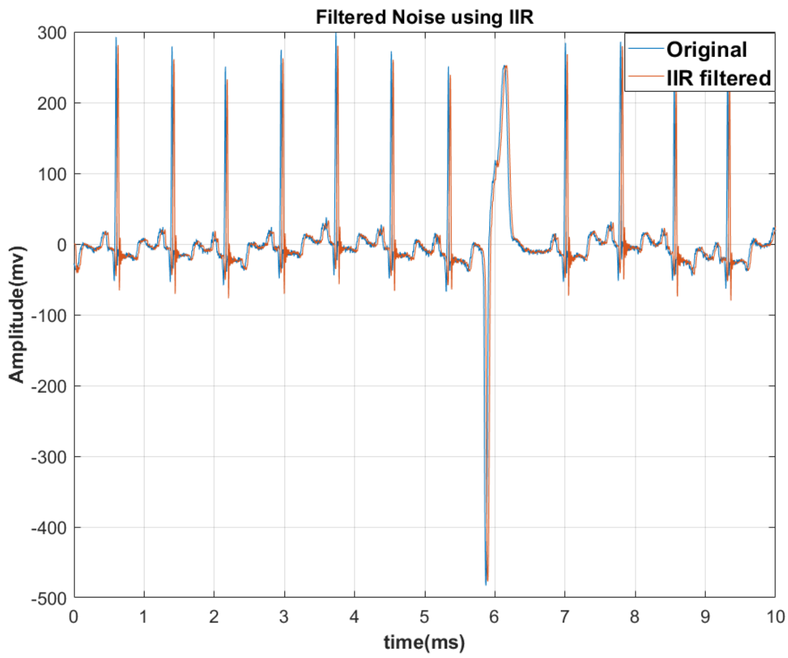


Figure 10: Filtered signal using IIR

In Figure 11 and 12, the plot of the filtered signal using FIR and the plot of that being zoomed in are given. Compared with the original signal, the filtered signal is quite close to the original signal, despite the little gaps between them.

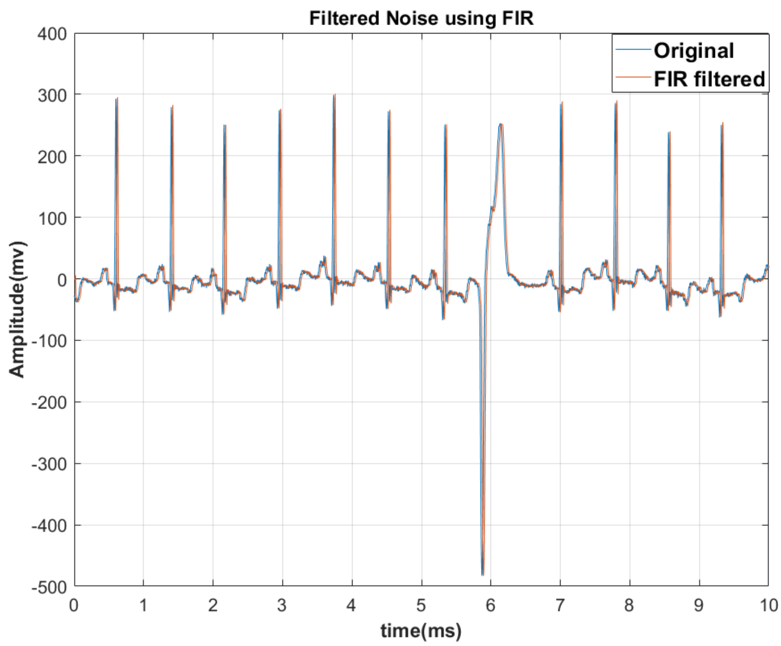


Figure 11: Filtered signal using FIR

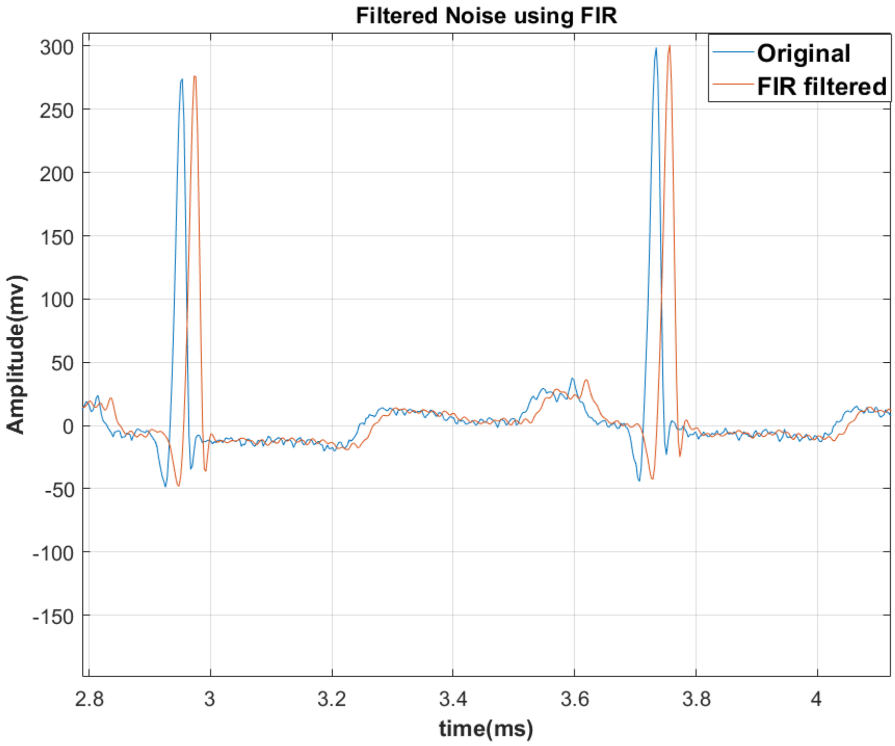


Figure 12: Filtered signal using FIR after zoomed in

## Comparison of FIR and IIR filter

According to the results from Figure 10 and 11, either the IIR filter or the FIR filter basically achieves the goal of removing the noise from the original signal, while FIR gives a better performance of removing the noise, since there are still some spikes of noise in filtered signal using IIR as shown in Figure 10. The reason for this case is because this removes all the signal after around 80 Hz, since its frequency response is smooth and decreasing monotonically and sharply as shown in Figure 8, and we can also see that in Figure 8, the frequency is not smooth anymore as the frequency increases, which also indicates that IIR is not quite stable for tapping of higher order.

## Improvement

As shown in Figure 1, the original signal is not a perfect signal, which still have some noises in it. In this section, some improvement will be given to filter the original signal to make the “PQRST” complex clearer to see.

When it comes to improvement of the IIR filter, IIR filter is suitable for tapping of lower order as mentioned before, since it can be unstable when increasing the order. A plot of the previously designed IIR which has different order is given in Figure 13, the transition starts to become shorter when the number of order decreases, which means that the attenuation is becoming smaller, and as mentioned in the original IIR design, the attenuation at around 80 Hz is a bit too large that some wanted signal is removed as well as the noise. So, reducing the number of the order can be a direction for improvement.

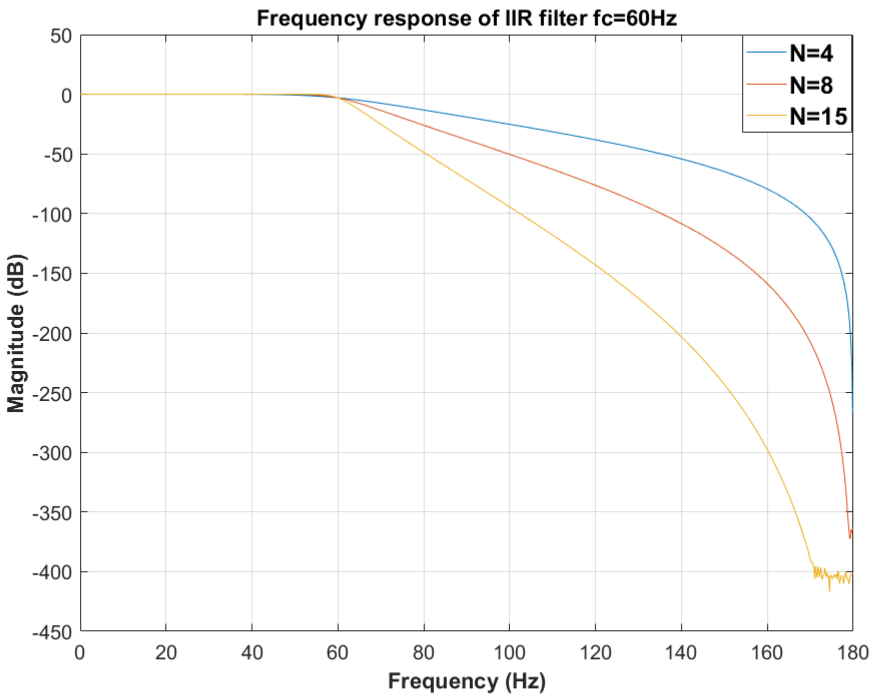


Figure 13: IIR with different number of order

As shown in Figure 14, the filtered signal using IIR with three picked orders is compared with the original signal, when the cut-off frequency is still set as 60 Hz. When order is 4, the filtered signal is closer to the original signal, while it is still not smooth. So, the order will be reduced to 4 for improvement and the next step is to change the frequency to make the filtered signal smoother.

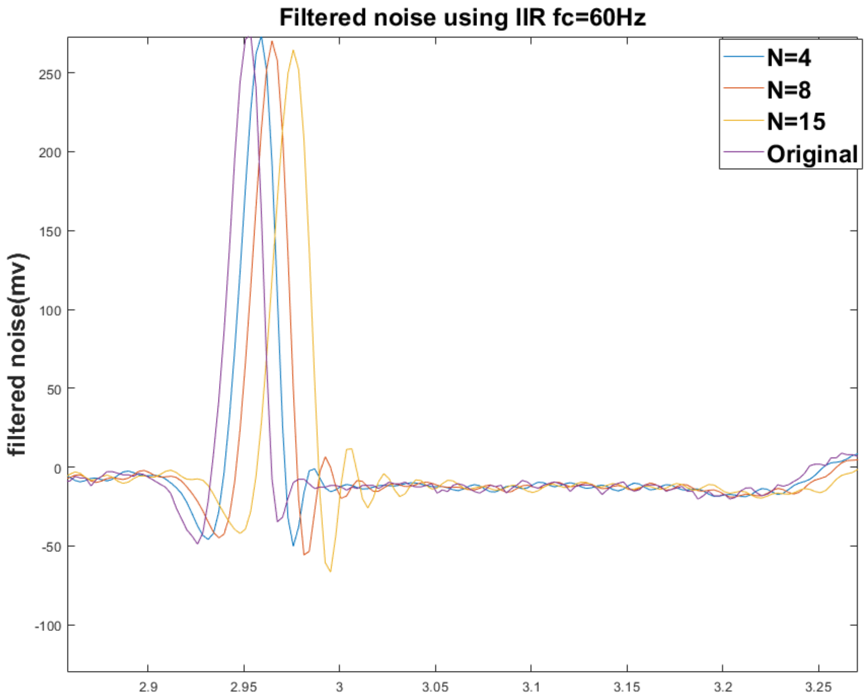


Figure 14: Filtered Signal using IIR with three orders

As shown in Figure 4, the obvious line noise is located at 60 Hz, while there are still some other spikes of noises located before this frequency. So, lowering the original cut-off frequency can also be a good direction for improvement. Plot of the filtered signal using IIR filter with three cut-off frequency is given in Figure 15. When the cut-off frequency is decreasing, the signal is becoming smoother, while there are also some signal which are sacrificed. So, a trade off needs to be found, which in this experiment is 30 Hz.

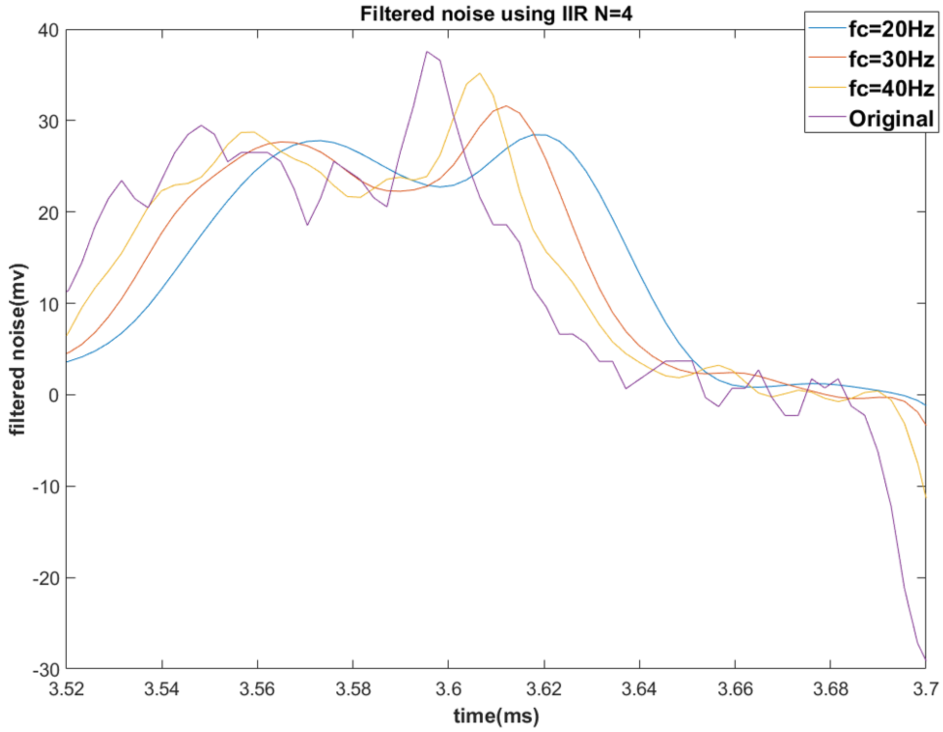


Figure 15: Filtered Signal using IIR with three cut-off frequency

The final parameter for improvement of the IIR design is given in Figure 16, the filtered signal using the improved IIR becomes smoother, compared with the original signal.

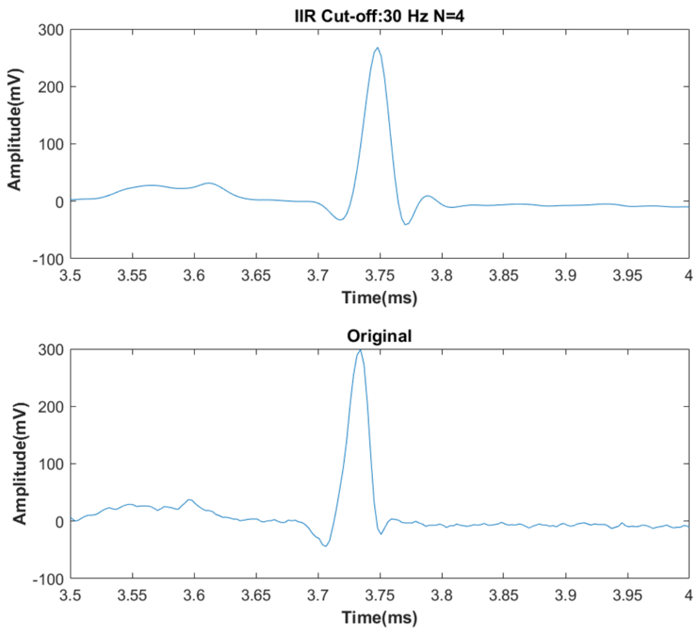


Figure 16: Final improvement of parameter in IIR filter