EEE309 (EEE339)
Topic 2: discrete Fourier transforms

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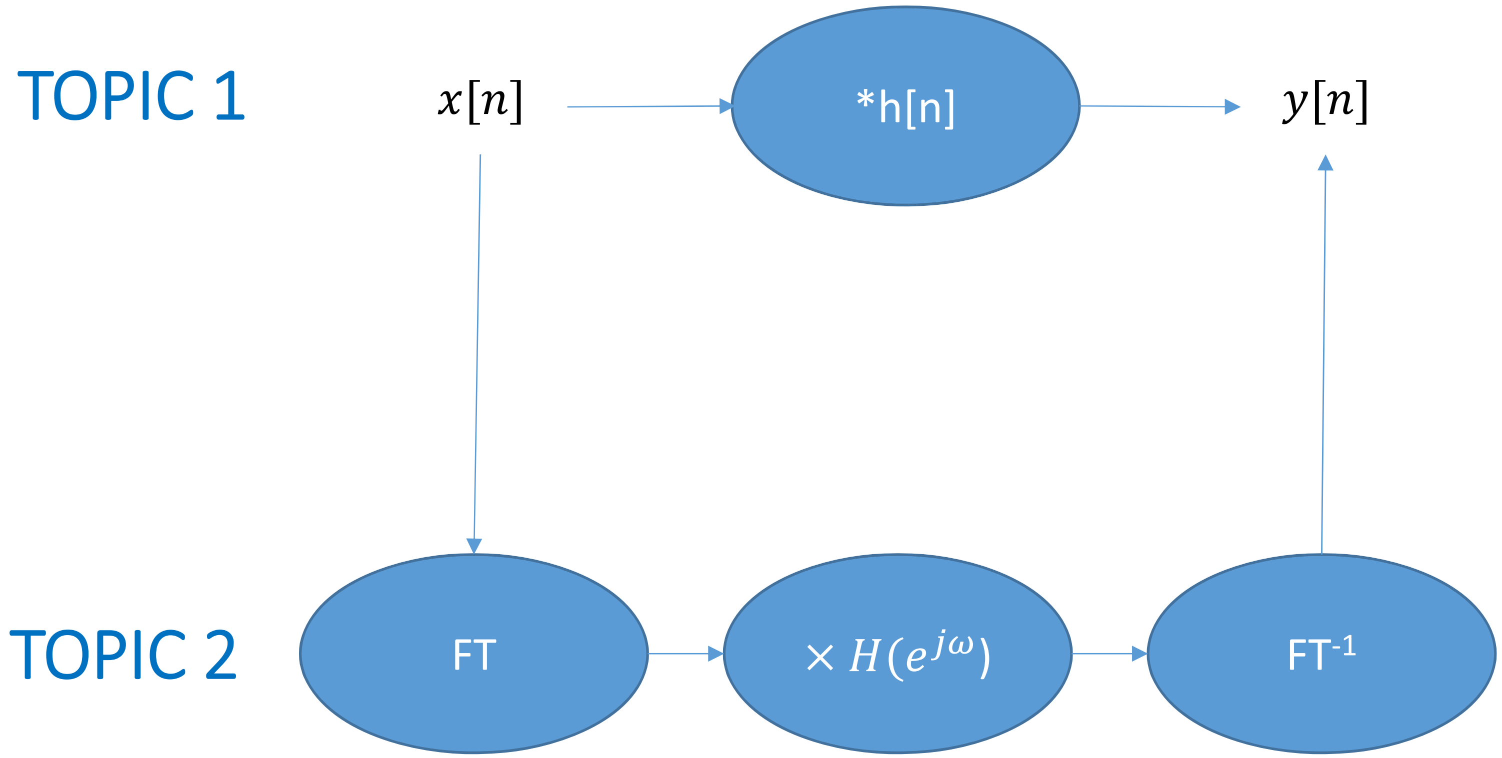
Two ways to represent digital systems

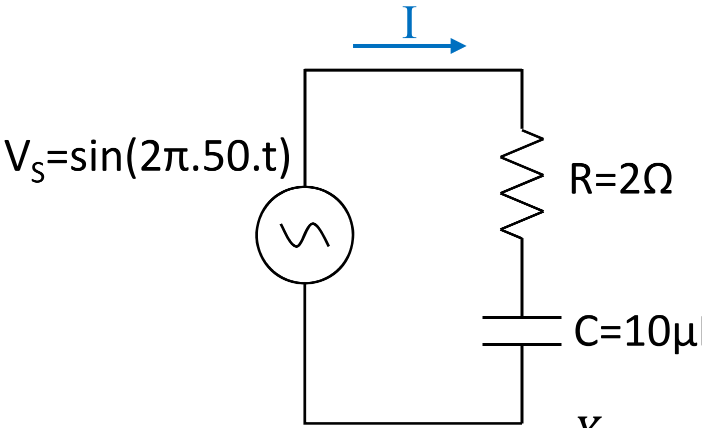


Overview

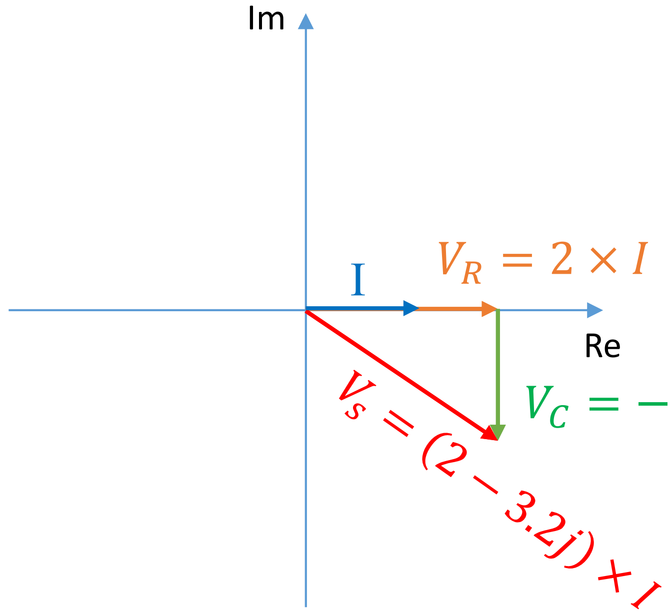
- Refresh of phasors and complex exponentials
- Understanding discrete time transforms
- Properties of discrete time transforms
- Relation to continuous transforms
- Transfer functions

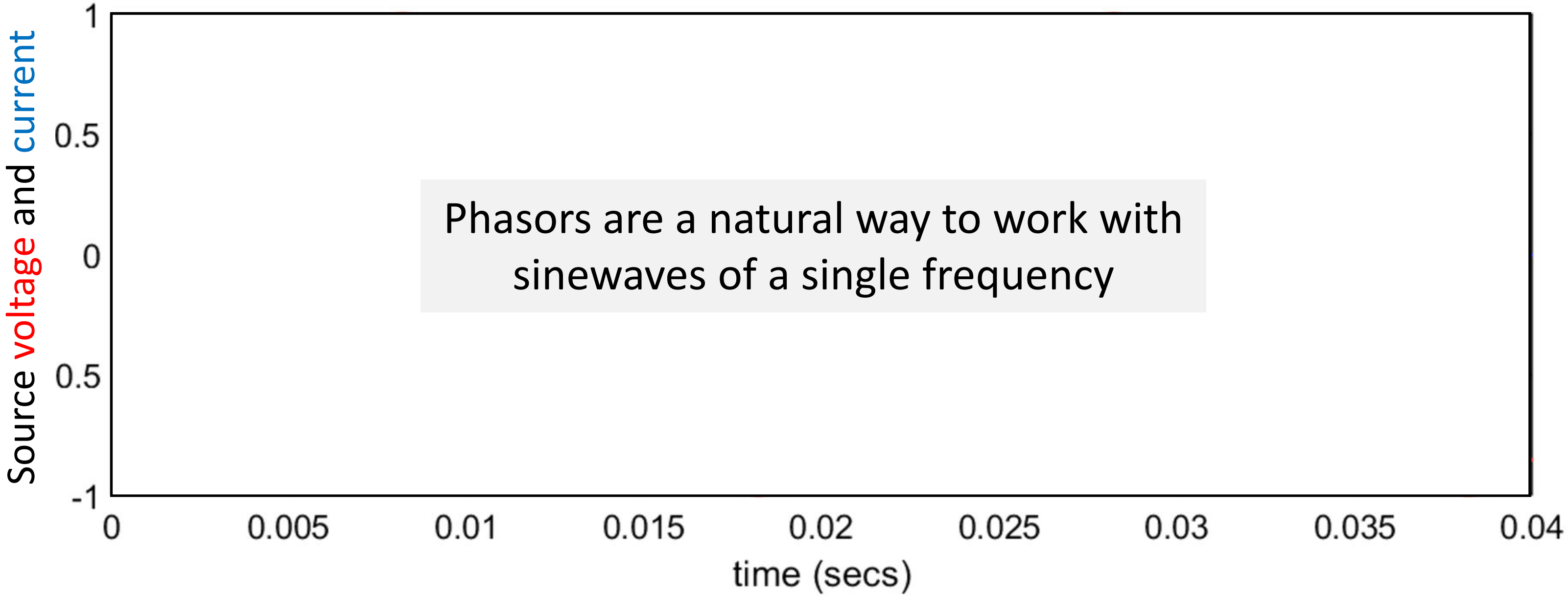
Refresh of complex exponentials

Phasors (recap)



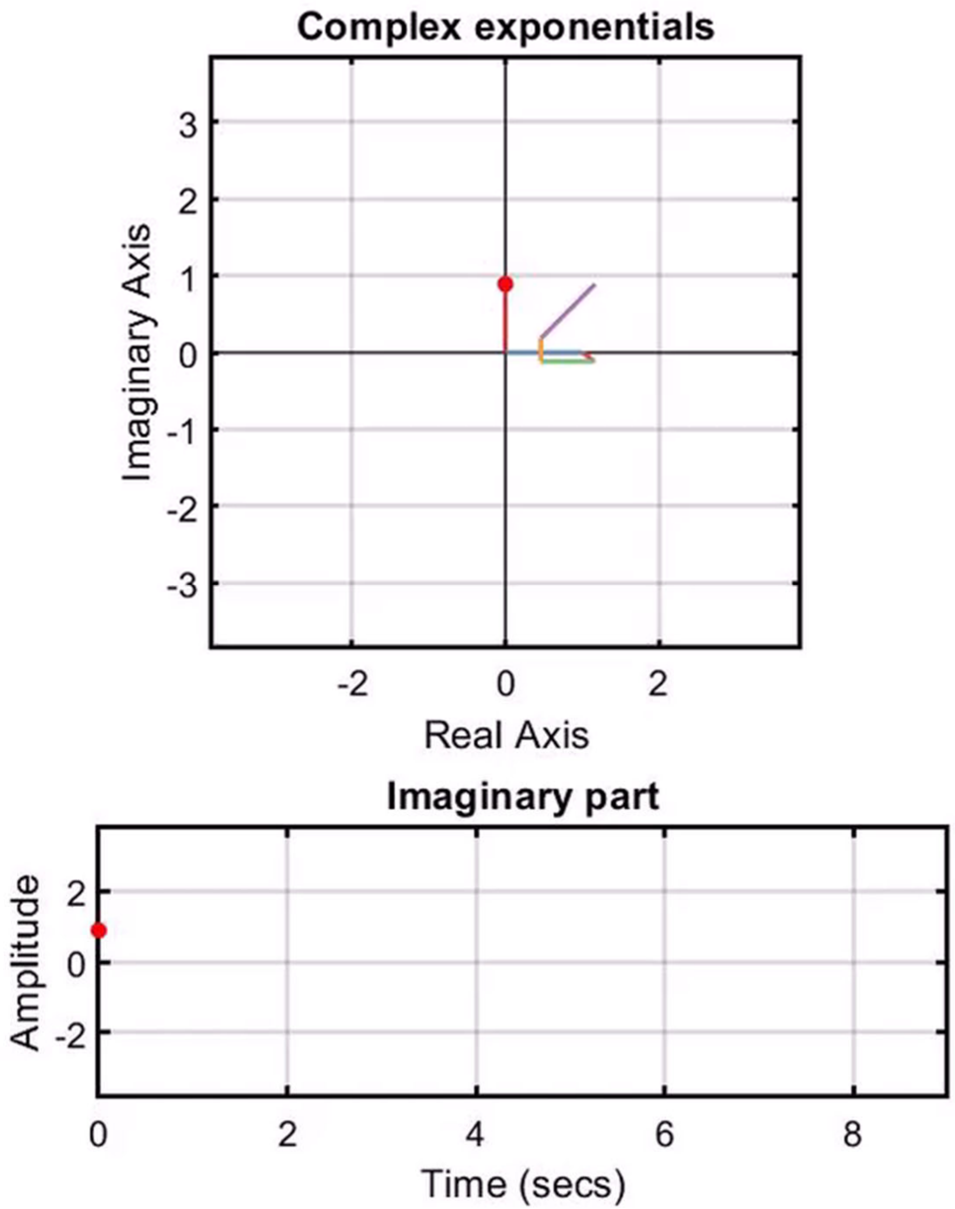
$$X_C = \frac{1}{j\omega C} = -j \times 3.2\Omega$$





Phasors are a natural way to work with sinewaves of a single frequency

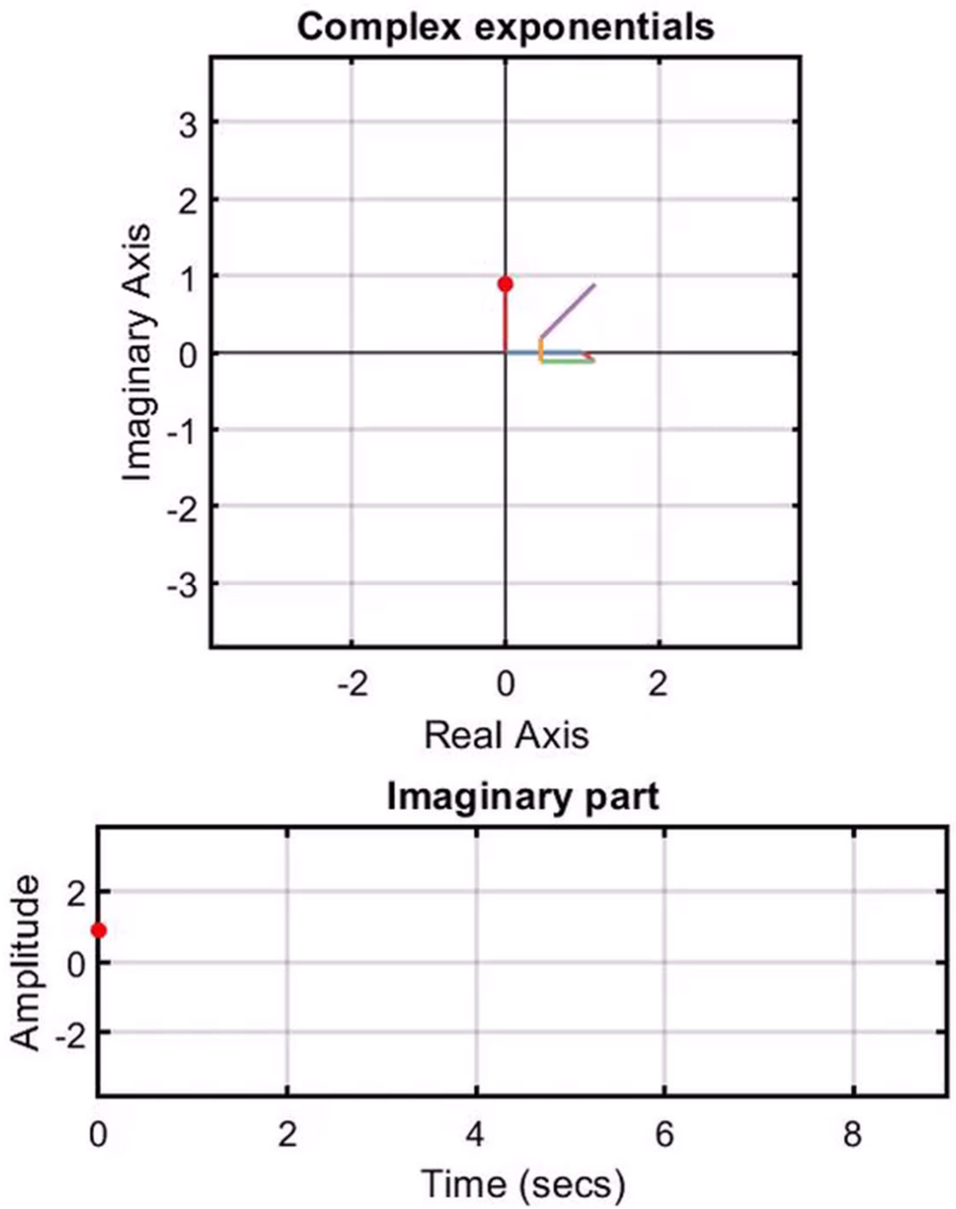
Phasors spinning at one frequency stay in the same alignment



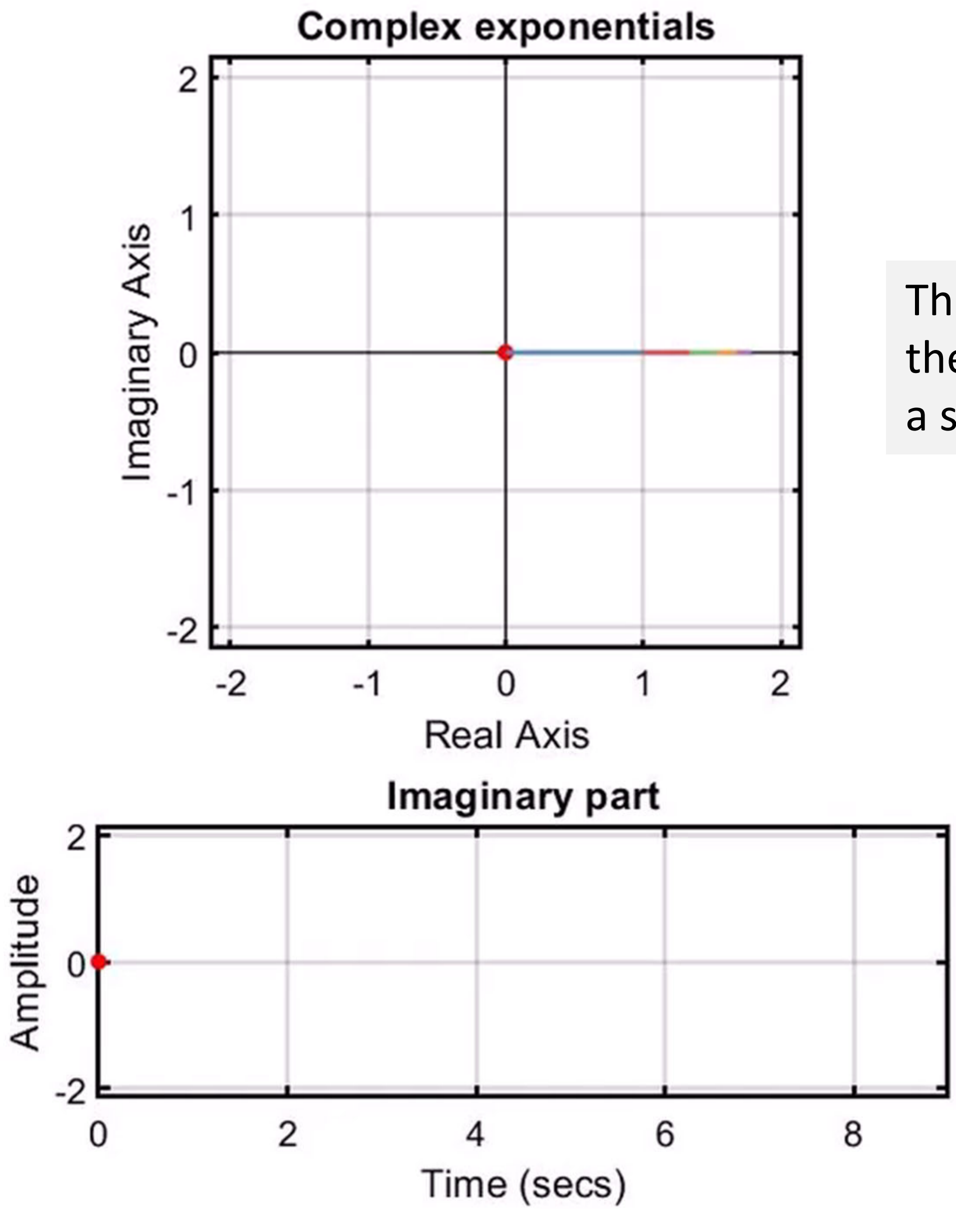
Whatever the phasors...

They ***Always*** add to make a sine wave

What if we have different frequencies?

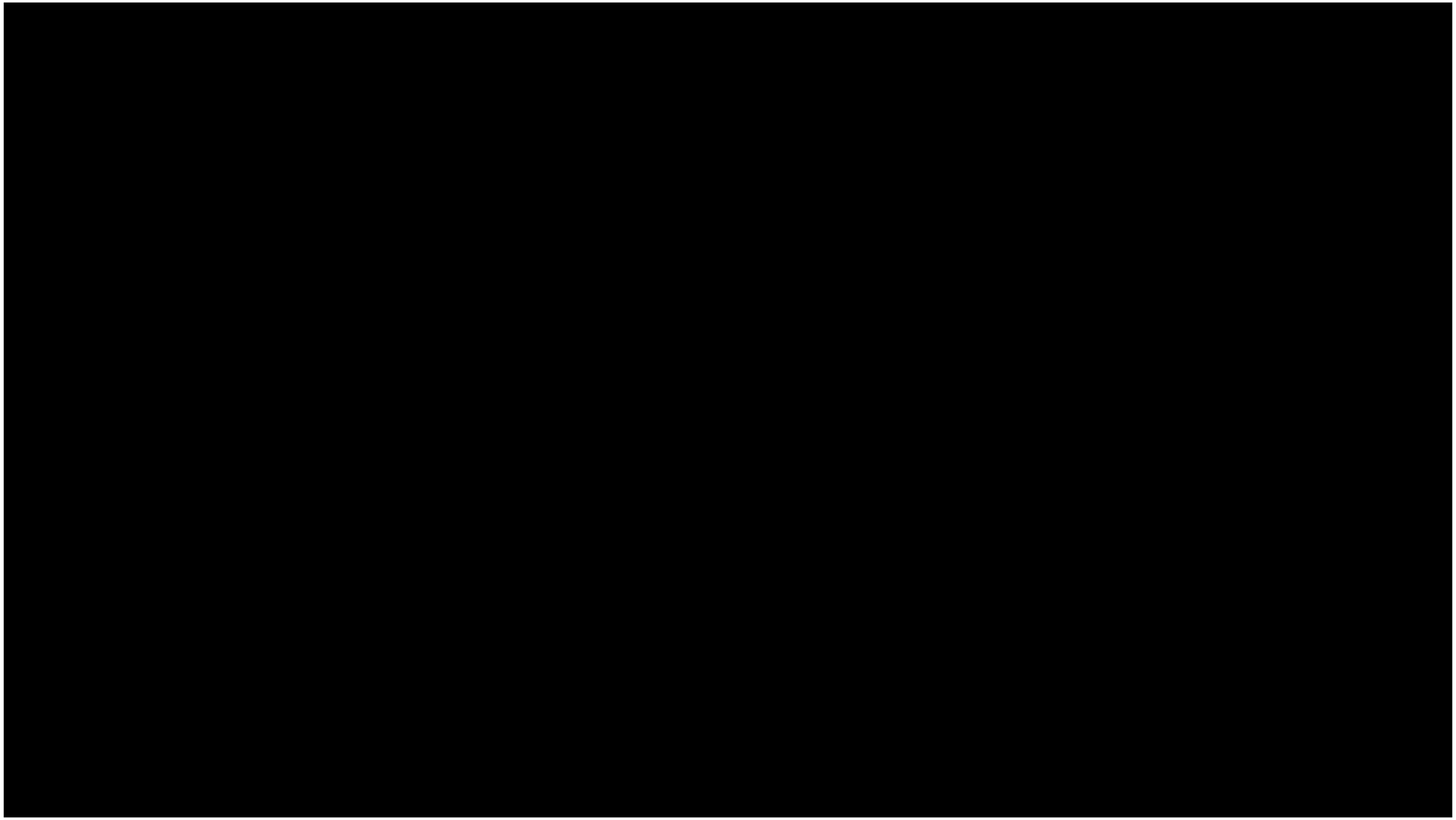


Which complex exponentials add up to a given signal?



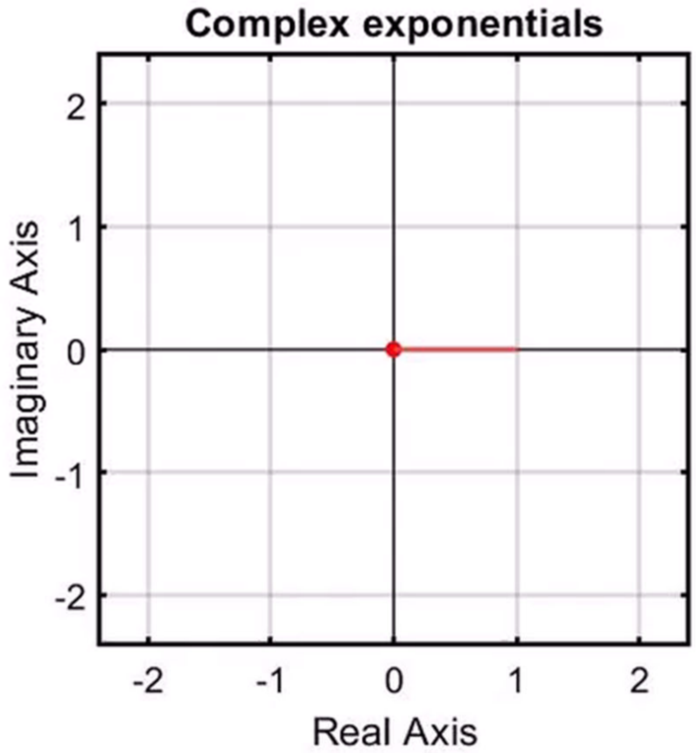
The first five terms in the Fourier series of a square wave

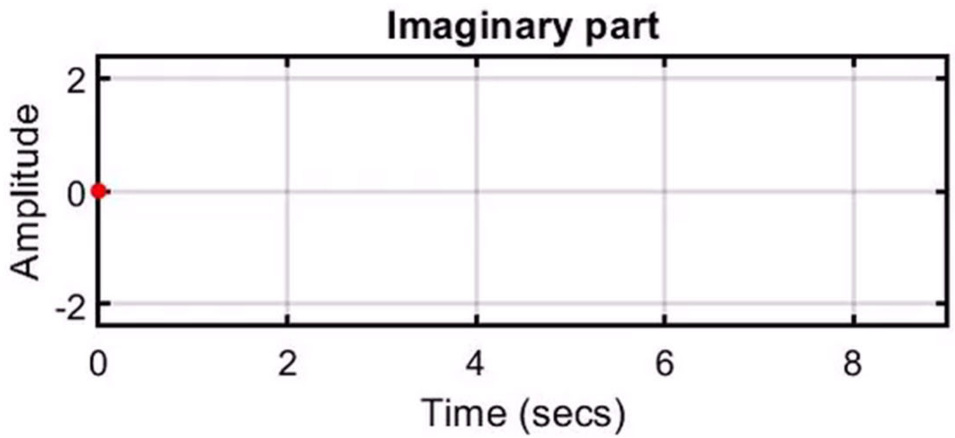
Any periodic signal is a sum of exponentials



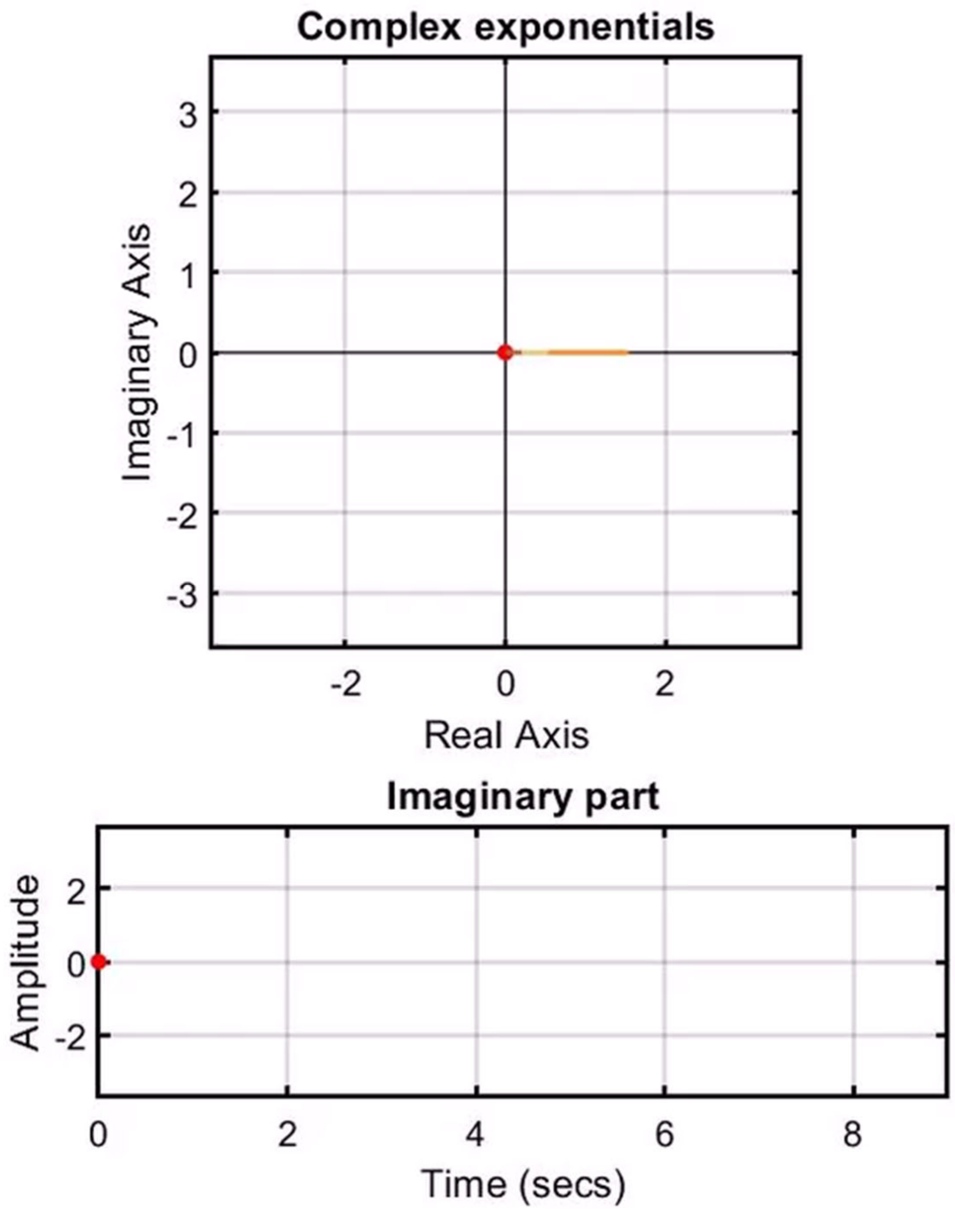
3Blue1Brown: https://youtu.be/-qgreAUpPwM

Real signals have equal clockwise and anticlockwise exponentials





Real signals have equal clockwise and anticlockwise exponentials



Discrete Fourier transforms

Eigenfunctions

An ***eigenfunction*** of an LTI system passes through the system unaffected apart from a change of scale. Recall that an LTI system is defined by convolution with the impulse response, h[n]:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Determine the scaling constant of the ***eigenfunction*** $x[n] = (0.1)^n$ for an LTI system with:

$$h[n] = [1\ 2\ 1]$$

Joseph Fourier

b. 1768 – d. 1830 (aged 62)

From "The analytical theory of heat"

we have dealt with a single case only of a more general problem, which consists in developing any function whatever in an infinite series of sines or cosines of multiple arcs. This problem is connected with the theory of partial differential equations, and has been attacked since the origin of that analysis. It was necessary to solve it, in order to integrate suitably the equations of the propagation of heat; we proceed to explain the solution. We shall examine, in the first place, the case in which it is required to reduce into a series of sines of multiple arcs, a function whose development contains only odd powers of the variable...

Engraving on Eiffel tower

Discrete time (co)sine transform

What Fourier showed was that signals can be decomposed into a sum of sines and cosines. For a sampled signal (a sequence), the amount of each sine or cosine 'in' the signal is computed as:

- The discrete time cosine transform is: $$X_c(\omega) = \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n)$$

- The discrete time sine transform is: $$X_s(\omega) = \sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)$$

VisFTScript.m

Moving to complex exponentials

Exponentials make life easy! Trig identities are impossible to remember but very naturally expressed as exponentials. To makes use of this, the cosine and sine transforms can be combined into the *Discrete Time Fourier Transform* (DTFT):

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)$$

Real part of the DTFT tells us about the *cosine* content of the sequence.

Imaginary part of the DTFT tells us about the ***sine*** content of the sequence.

The *inverse DTFT* is defined as: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}.\,d\omega$

Worked example 1

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Find the DTFT of the sequence: $x[n] = [1,2,3,4,5]$

Common DTFT pairs

Time domain	Frequency domain
$\delta[\mathrm{n}]$	1
$\delta[\mathrm{n} - n_0]$	$e^{-jn_0\omega}$
$e^{j\omega_0 n}$	$2\pi\delta(\omega - \omega_0)$
$\dfrac{\sin(\omega_c n)}{n}$	$\begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c < \lvert\omega\rvert < \pi \end{cases}$
$\sin(\omega_0 n)$	$j\pi\big(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\big)$
$\cos(\omega_0 n)$	$\pi\big(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\big)$
$\alpha^n u[n]$	$\dfrac{1}{1 - \alpha e^{-j\omega}}$

Properties of Fourier transforms

Theorem	sequence	Frequency domain
Linearity	$ax_1[\mathrm{n}]+bx_2[\mathrm{n}]$	$aX_1(e^{j\omega})+bX_2(e^{j\omega})$
Time delay	$x[\mathrm{n}-n_0]$	$e^{-jn_0\omega}X(e^{j\omega})$
Phase ramp	$e^{-j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Convolution	$x_1[\mathrm{n}]*x_2[\mathrm{n}]$ $x_1[\mathrm{n}]x_2[\mathrm{n}]$	$X_1(e^{j\omega})X_2(e^{j\omega})$ $\frac{1}{2\pi}X_1(e^{j\omega})*X_2(e^{j\omega})$
Symmetries	$x[\mathrm{n}]$ (real valued)	$X(e^{j\omega})=X^*(e^{-j\omega})$ $X_R(e^{j\omega})=X_R(e^{-j\omega})$ $X_I(e^{j\omega})=-X_I(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert$ $\angle X(e^{j\omega})=-\angle X(e^{-j\omega})$ R= real, I= imaginary part
Parseval	$\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{-j\omega})\rvert^2 .d\omega$

The geometric series

Two very useful identities for calculating Fourier transforms:

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \cdots = \frac{1}{1-a}, \qquad |a| < 1$$

Proof:

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}, \qquad \forall a$$

Proof:

Worked example 2

Find the DTFT of the sequence: $x[n] = u[n]\ (0.5)^n$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Passing a complex exponential through an LTI system

the input sequence: $$x[n] = e^{j\omega_0 n}$$

the impulse response: $$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

...is commutative: $$= \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} e^{j\omega_0(n-k)}h[k]$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}h[k] = x[n] \sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}h[k]$$

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

This is the DTFT of the impulse response of an LTI system: it is called its *transfer function* or *frequency response*.

SinewaveImpulseResponse.m

Worked example 3

HandoutExample2.m

An LTI system is given as: $y[n] = 0.8y[n-1] + x[n]$

Determine the output of the system when the input is: $x[n] = \cos(0.05\pi n)$

Worked example 3

HandoutExample2.m

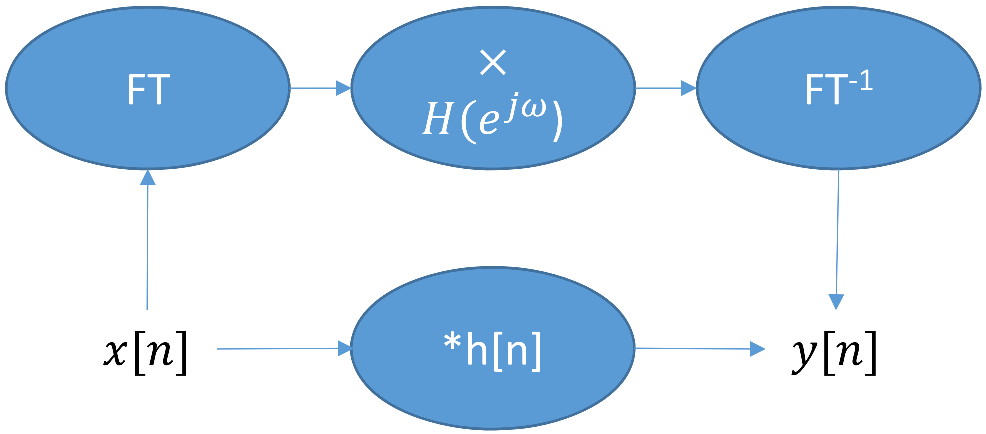
Generally, if $h[n]$ is real, and: $x[n] = \cos(\omega n)$

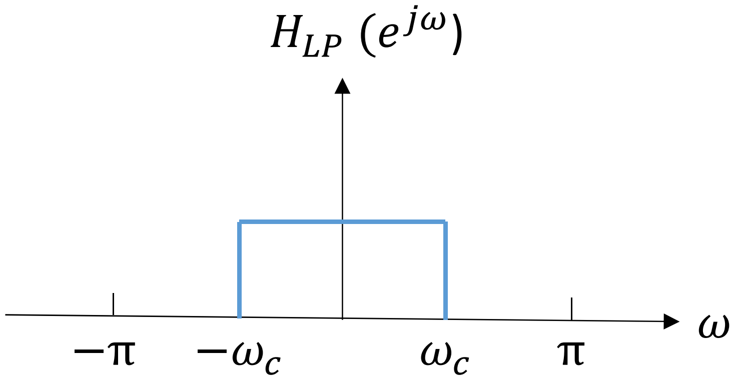
the output of the system it defines is: $y[n] = \left|H(e^{j\omega})\right|\cos(\omega n + \angle H(e^{j\omega}))$

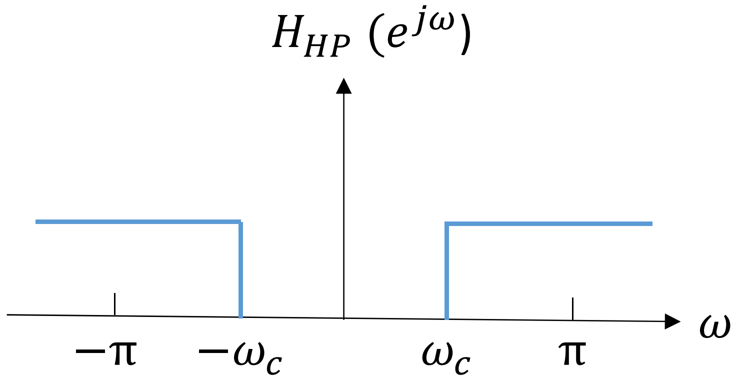
Passing a general sequence through an LTI system

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

is the DTFT of the impulse response of an LTI system: its ***transfer function*** or ***frequency response***.

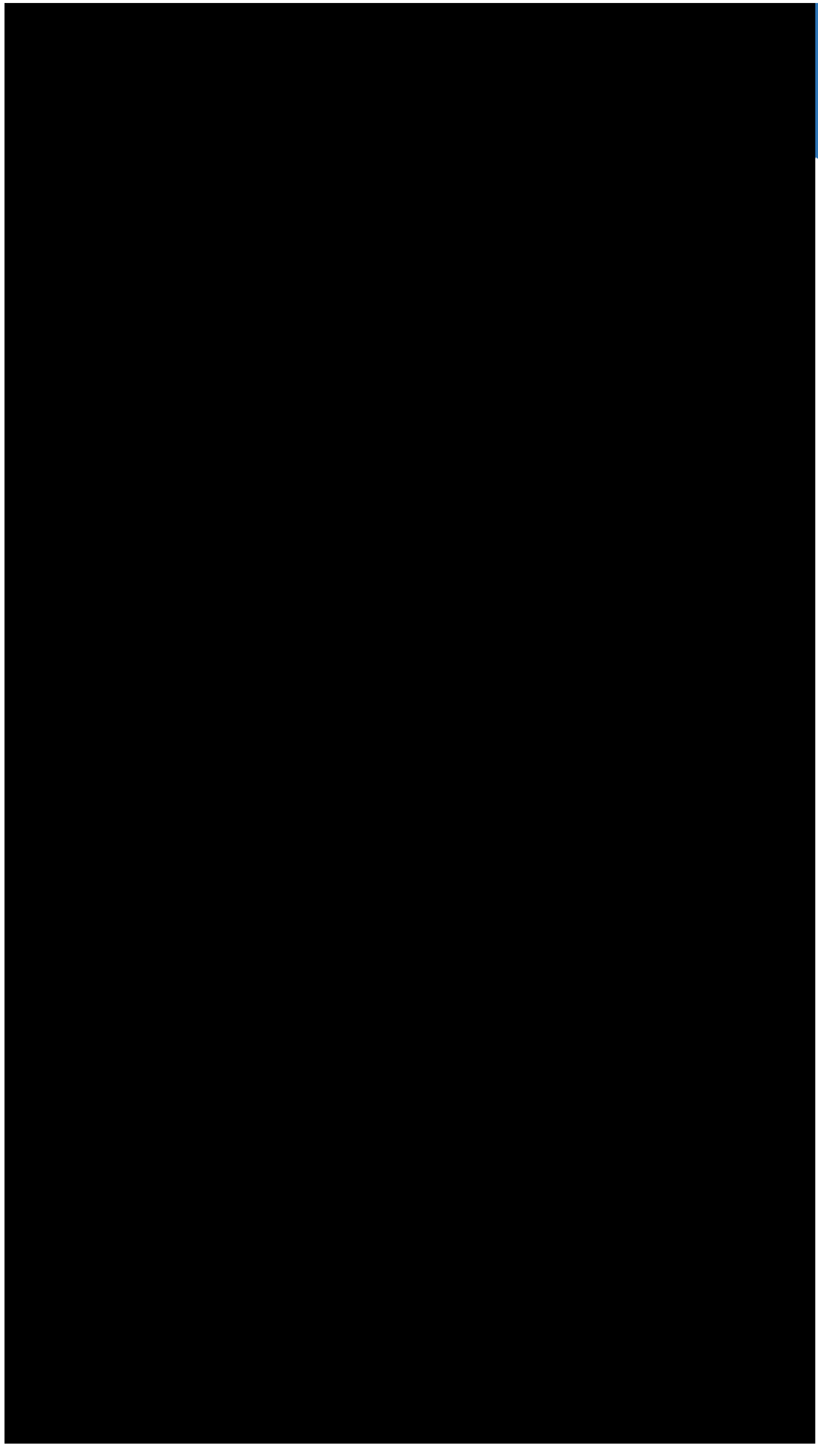






TransferFunctions.m

Aliasing and the Nyquist rate

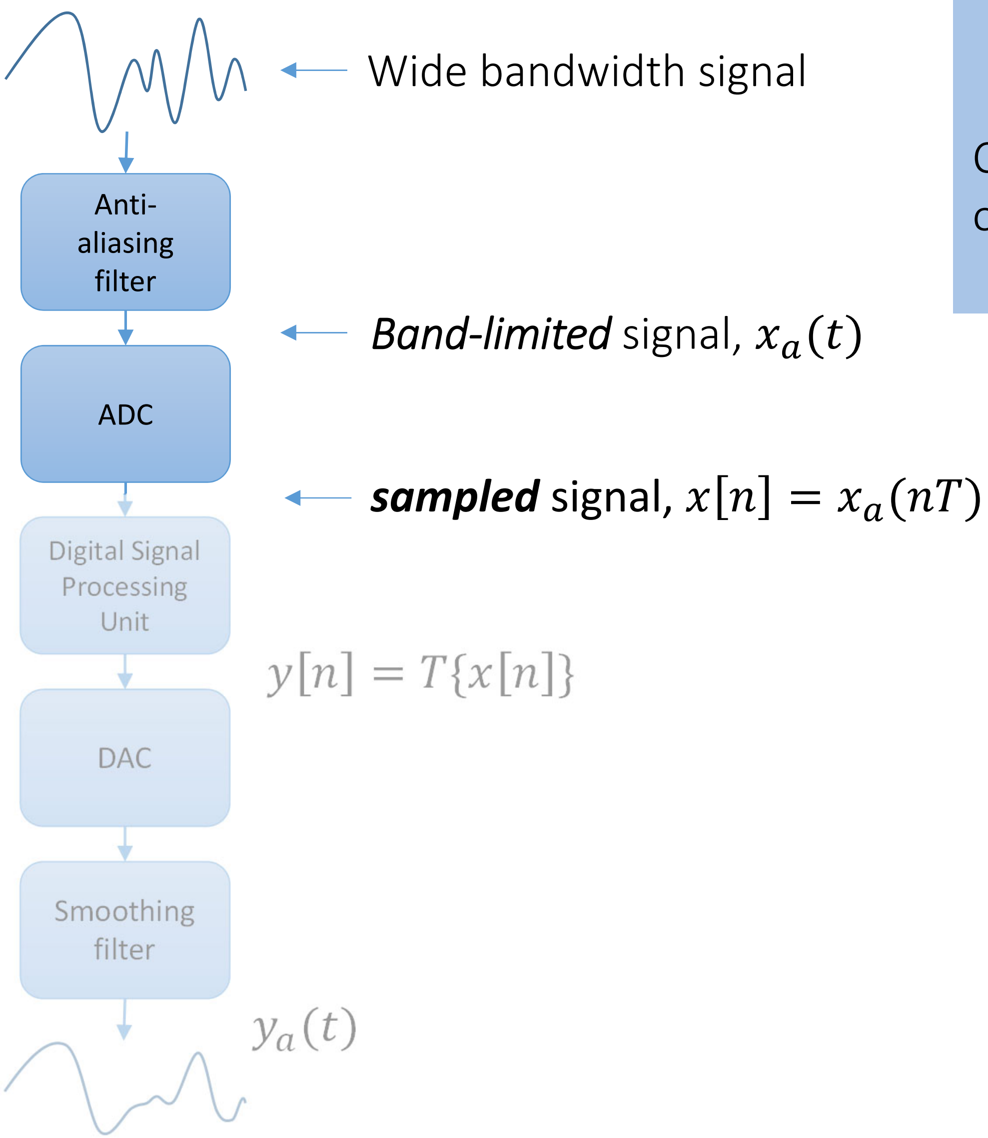


Aliasing

Search: “wagon wheel effect”

`Aliasing.m`

Aliasing



QUESTION:

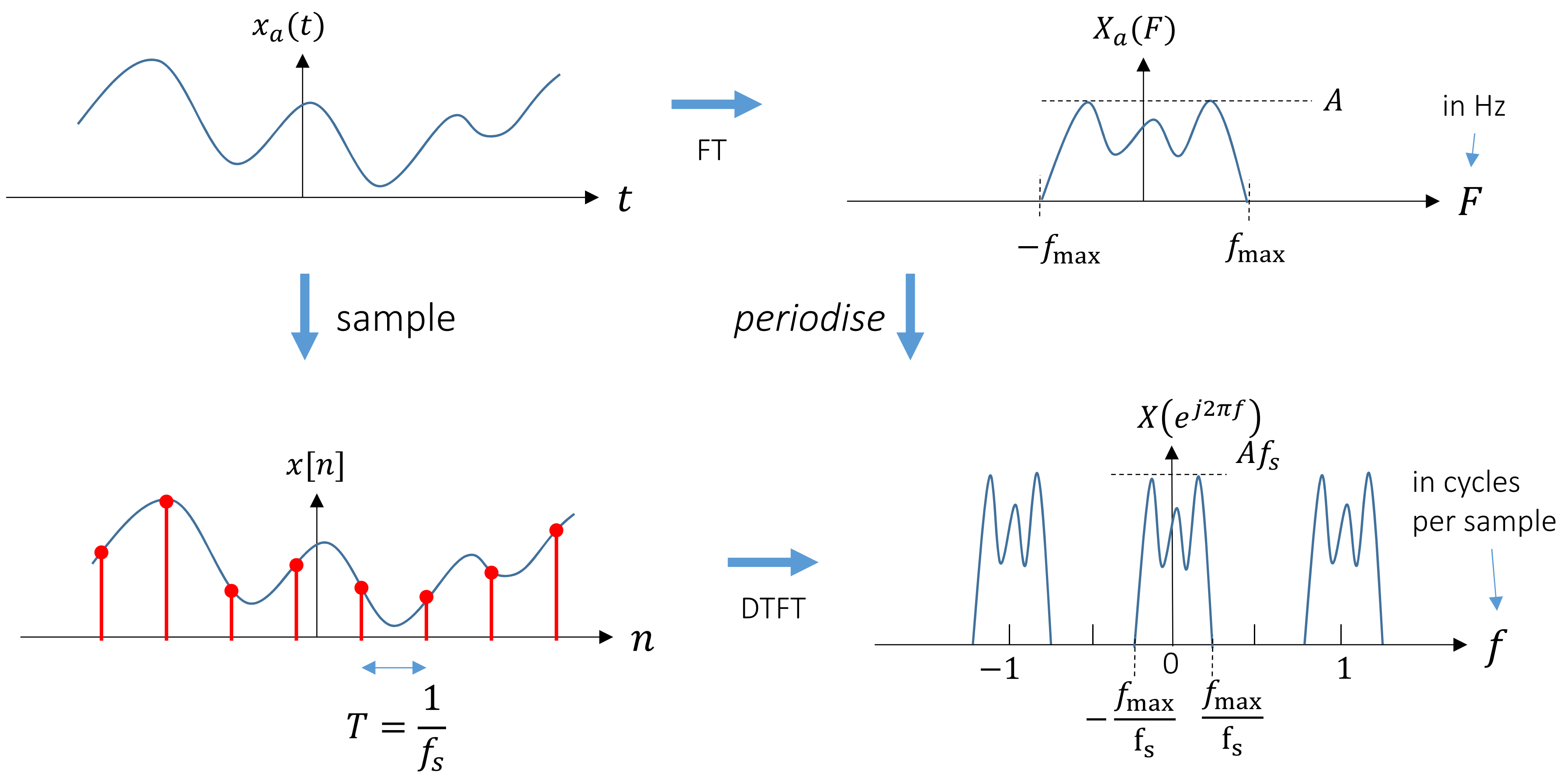
Can the sequence $x[n]$ capture ***every detail*** of the analogue signal, $x_a(t)$?

Aliasing formula

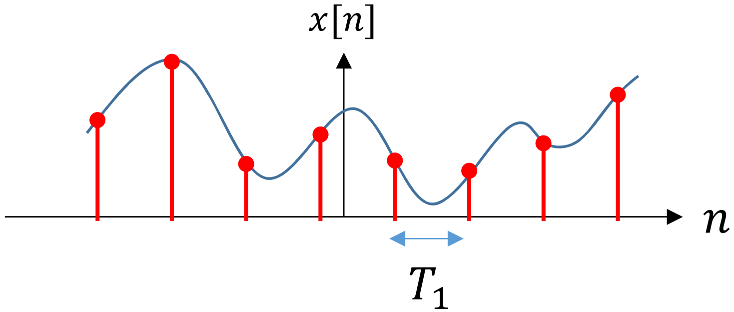
The aliasing formula relates the Fourier transform of an analogue signal to the Fourier transform of a sampled version of the signal.

$$X\left(e^{j2\pi f}\right) = f_s \sum_{k=-\infty}^{\infty} X_a(F - kf_s)$$

$(f = F/f_s)$

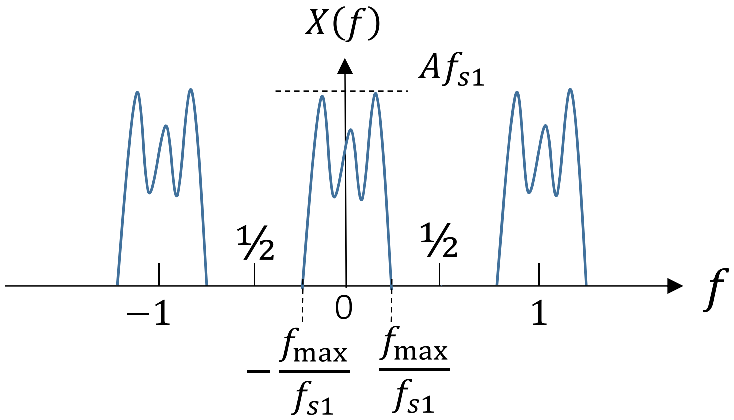


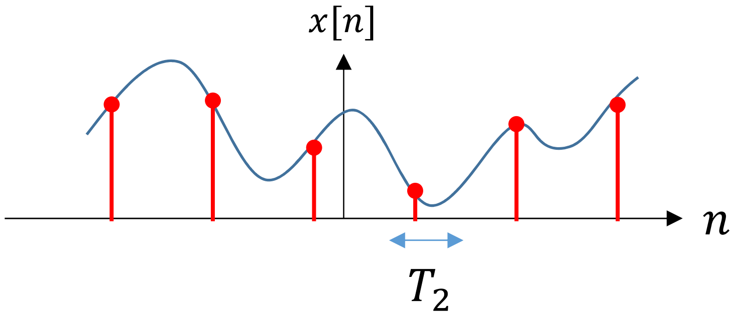
Aliasing conditions



$$1/T_1 = f_{s1} > 2f_{\max}$$
$$\therefore \frac{f_{max}}{f_{s1}} < \frac{1}{2}$$

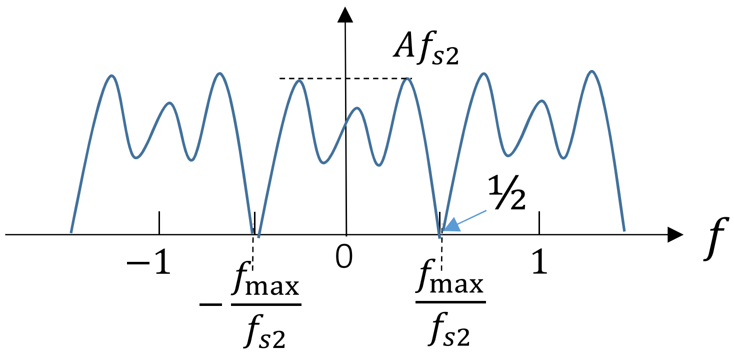
DTFT

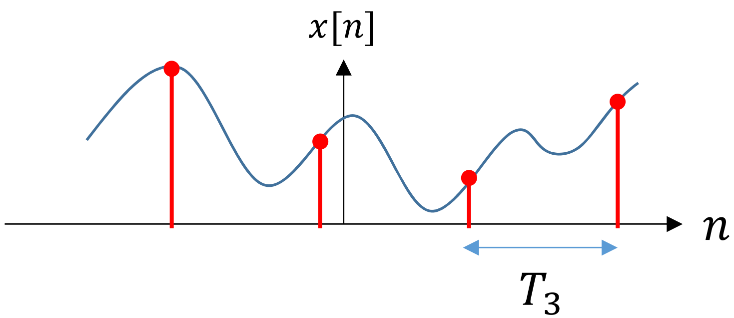




$$1/T_2 = f_{s2} = 2f_{\max},$$
$$\therefore \frac{f_{max}}{f_{s2}} = \frac{1}{2}$$

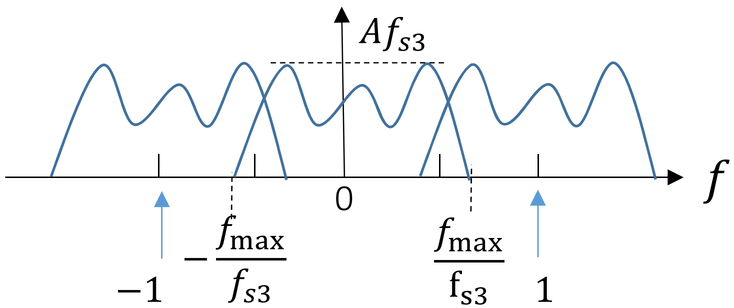
DTFT





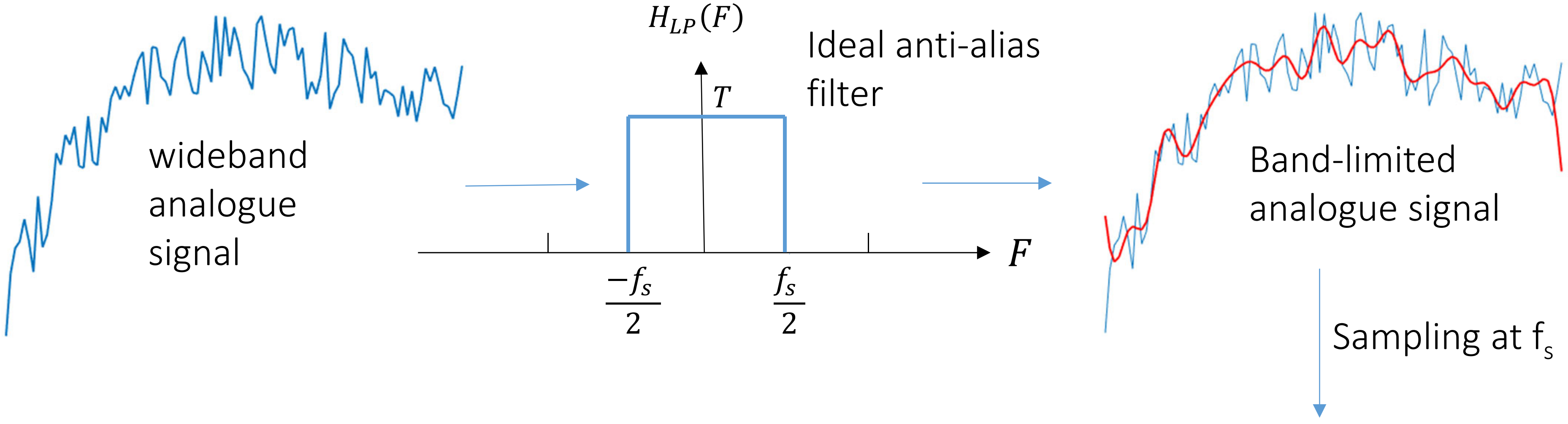
$$1/T_3 = f_{s3} < 2f_{\max},$$
$$\therefore \frac{f_{max}}{f_{s3}} > \frac{1}{2}$$

DTFT



Nyquist Sampling Theorem

AliasingSqWv.m



$$x[n] = [1\ 3\ 2\ 7\ 6\ \ldots]$$

Provided the input signal to a digital system is bandlimited, or pre-filtered, such that the maximum signal frequency, f_{max}, is smaller than half the sample frequency, i.e.:

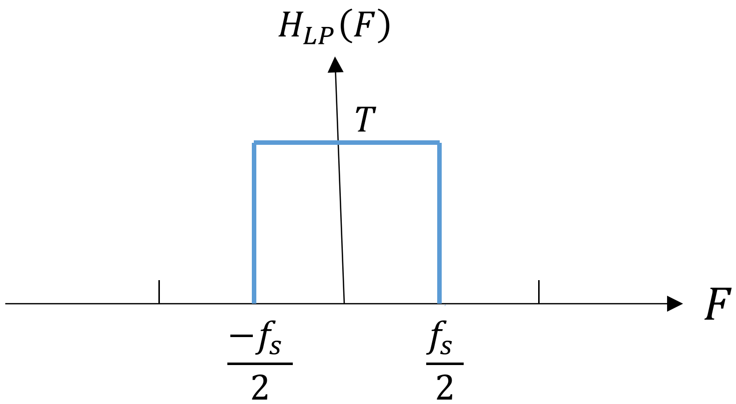
$$f_s > 2f_{max}$$

The sampled digital signal, x[n], contains every detail of the original analogue signal.
*The frequency $2f_{max}$ is known as the **Nyquist rate***

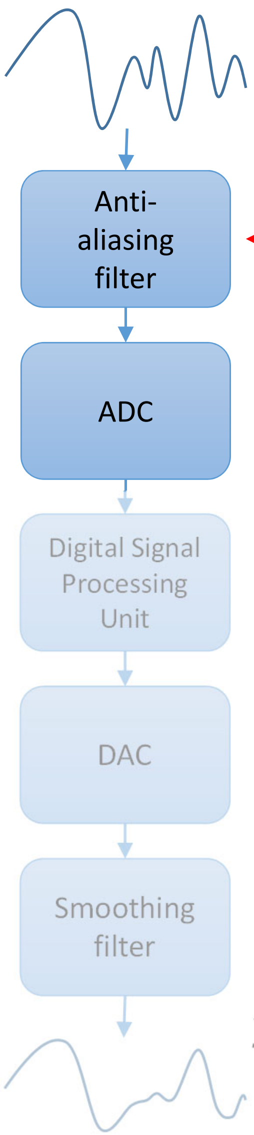
Fourier practice...

What is the time-domain representation of the perfect low pass filter shown?
Recall that the inverse Fourier transform is:

$$g(t) = \mathcal{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}.df$$



Aliasing



Wide bandwidth signal

$\mathrm{h(t)} = sinc\left(\frac{\mathrm{t}}{T}\right)$

Band-limited signal, $x_a(t)$

sampled signal, $x[n] = x_a(nT)$

$y[n] = T\{x[n]\}$

$y_a(t)$

QUESTION:

Can the sequence $x[n]$ capture ***every detail*** of the analogue signal, $x_a(t)$?

Yes!

But the ideal sinc filter is not physically realisable: it is non-causal

`SincRestoration.m`

`BoxVsSincFilters.m`

Worked example 4

Ingle & Proakis, Example 3.17:

The analogue signal $x_a\,(t) = 4 + 2\cos(150\pi t + \pi/3) + 4\sin(350\pi t)$ is sampled at $f_s = 200Hz$ to obtain the discrete-time signal $x[n]$.

Does this result in aliasing?

Determine $x[n]$ and its corresponding DTFT $X(e^{j\omega})$

$$e^{j\omega_0 n} \xrightarrow{\text{DTFT}} 2\pi\delta(\omega - \omega_0)$$

What you learned…

- The spectrum of a sequence is calculated using the DTFT
- The shape of a (co)sine or complex exponential passing through an LTI system remains unchanged, only its amplitude and phase change. (These are called ***eigenfunctions*** of LTI systems)
- An LTI system can be defined in terms of its ***frequency response***, or ***transfer function***:

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

- Provided a signal $x_a(t)$ is ***bandlimited*** and sampled above the ***Nyquist rate***, it can be recovered exactly from its sampled version $x[n]$

The DFT

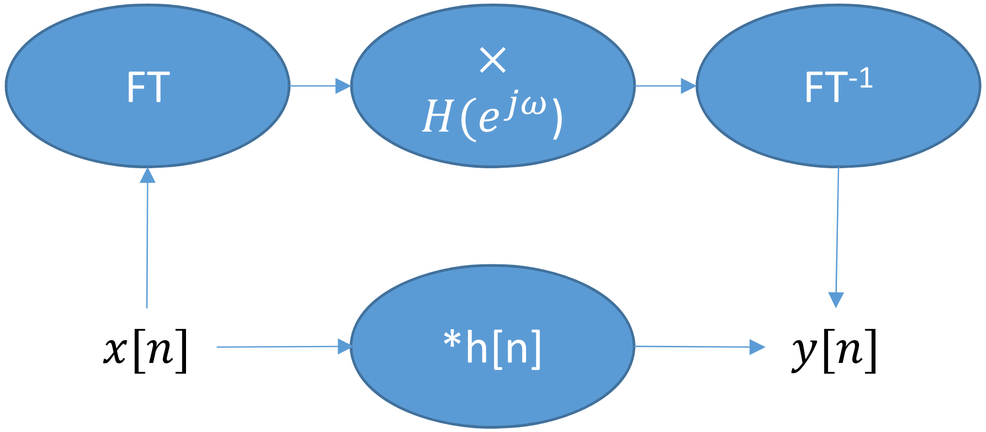
Overview

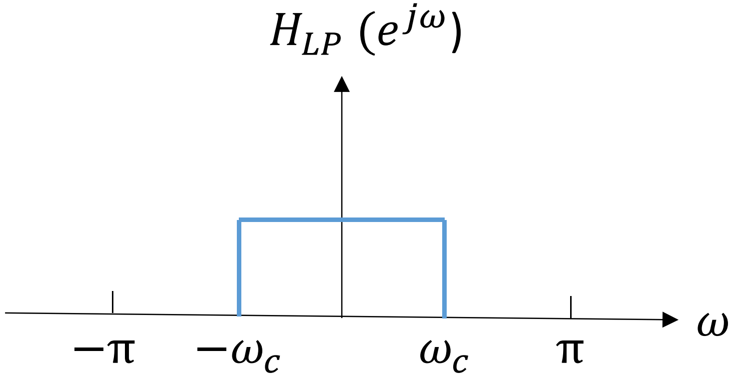
- (Correctly) Sampling the DTFT to realise the DFT
- Modulo and twiddle notation
- Important properties of the DFT
- Circular operations: shifts and convolution
- Applying the DFT to realise digital systems

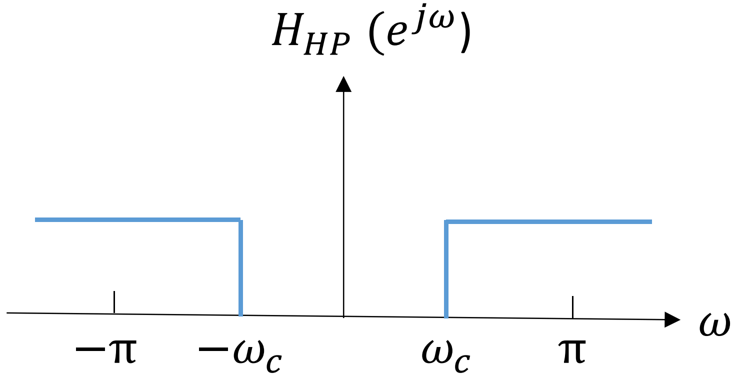
Recap: passing a general sequence through an LTI system

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

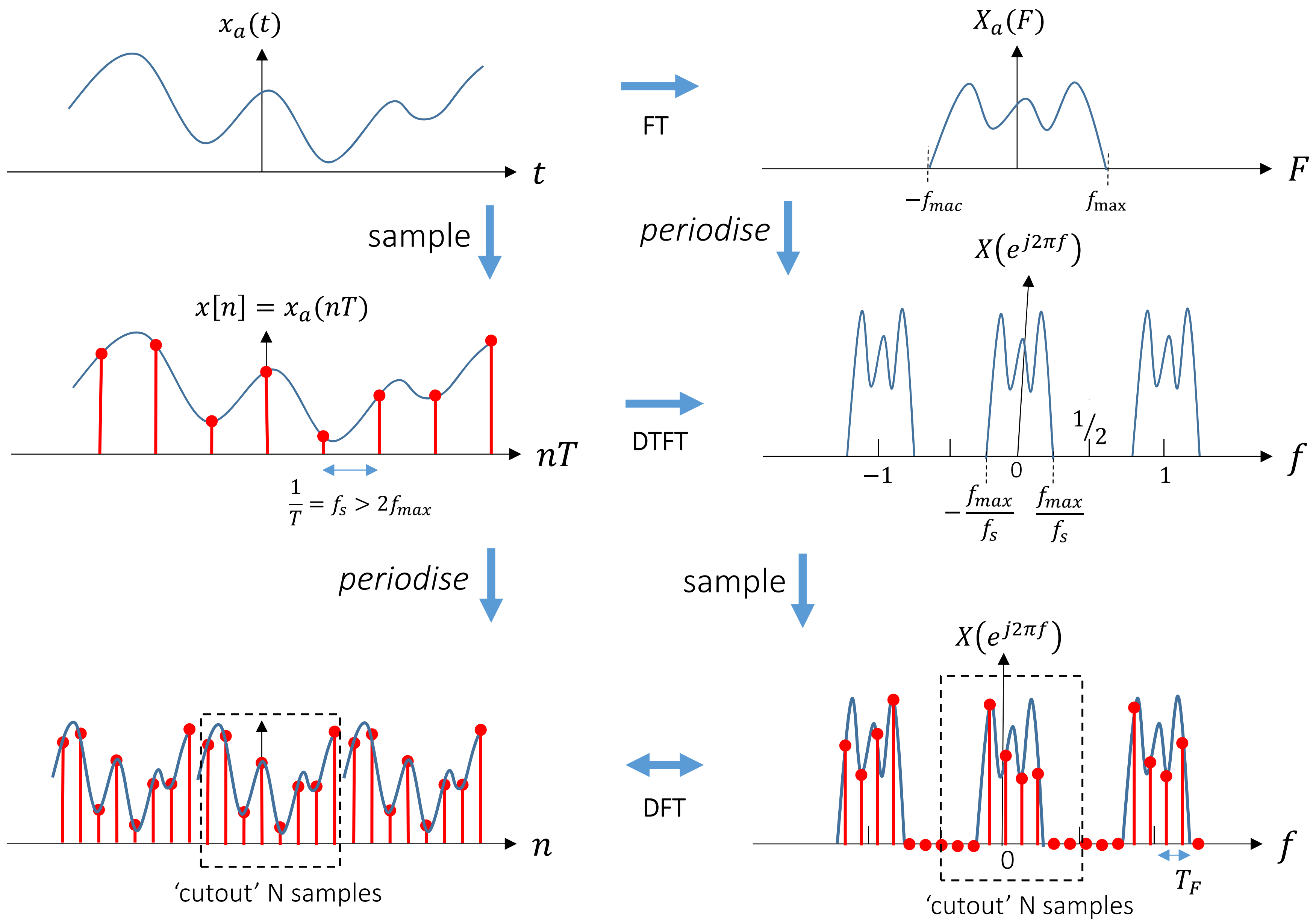
is the DTFT of the impulse response of an LTI system: its *transfer function* or *frequency response*.



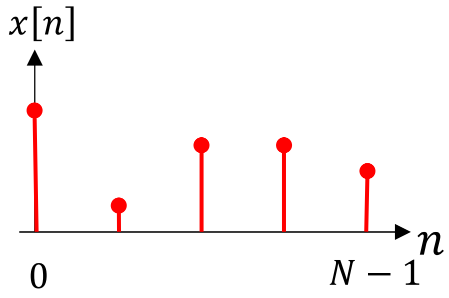




The set of Fourier transforms



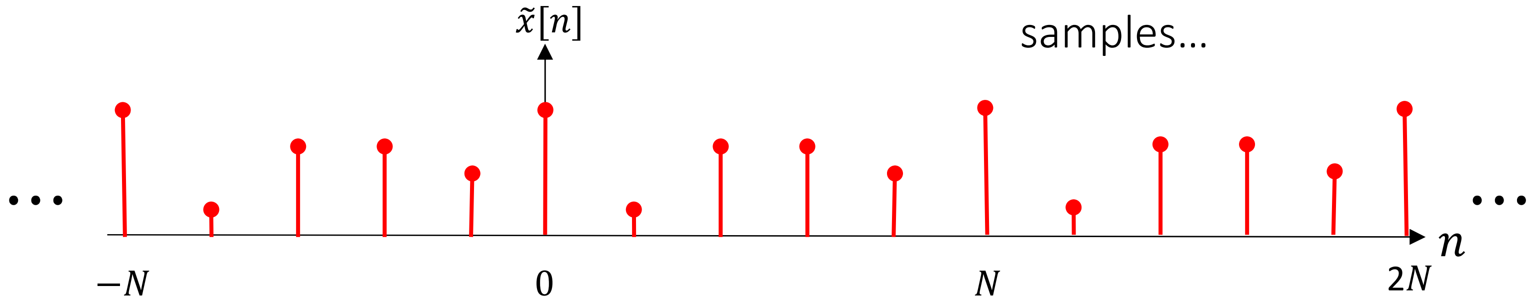
Limited length sequences



Finite length sequence of length N, e.g. N=5 here

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

Repeated every N samples…



What sampling period, T_F, in the DTFT will give this periodic sequence as its inverse FT?

The aliasing formula tells us: sampling a signal at a rate T *equates to a periodic Fourier transform, with period* $f_s = \frac{1}{T}$

It works the other way as well. Sampling the Fourier transform at a rate T_F *equates to a periodic inverse transform, with period* $T = \frac{1}{T_F}$.

∴ *a period of N samples in the periodic signal means a sampling in the Fourier transform of* $\boldsymbol{\frac{1}{N}}$

Discrete Fourier transform definition

- In the DTFT, the ***basis functions*** are (co)sines, represented by the complex exponentials: $e^{-j\omega} = e^{-j2\pi f}$
- In the discrete Fourier transform (DFT), the basis functions are sampled versions of the complex exponentials with sample rate $\mathrm{T_F} = 1/N$,
- So the sampled frequencies are: $f \rightarrow nT_F = n/N$, and the exponentials become: $e^{-j2\pi\frac{n}{N}}$, where N is the length of the sequence being transformed.

- The DFT is then: $$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-\frac{j2\pi}{N}nk}$$

- The *inverse DFT* is: $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{\frac{j2\pi}{N}nk}$$

- The DFT and the inverse DFT are finite length sequences of length N samples, where it is implicitly assumed that they are 'cutout' from their respective periodic sequences.

Modulo and twiddle notations

Instead of writing the periodic extension of the sequence x[n] using sigma, often ***modulo*** notation is used:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN] \longrightarrow \tilde{x}[n] = x[n \bmod N] = x[< n >_N]$$

Example: for the sequence $x[n] = [1\ 2\ 5\ 4\ 2]$, what is the 11th entry in the periodic version of the sequence?

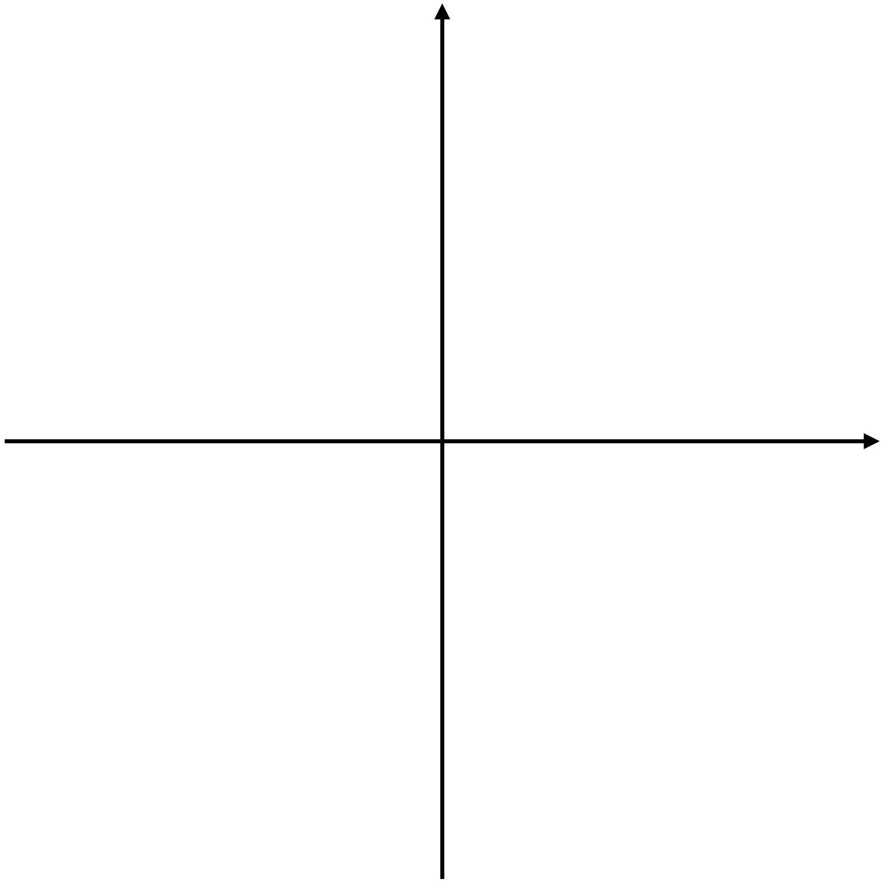
Often we will need to write complex exponentials with quite cumbersome exponents. To clean up the notation we can use so-called ***twiddle factors***, defined as:

$$W_N = e^{-j\frac{2\pi}{N}}$$

Example: What is the actual expression represented in twiddle notation as W_6^3?

More on twiddle factors

Twiddle factors are the Nth roots of unity – i.e. equally spaced points around the circumference of the unit circle:



We can rewrite the DFT and IDFT more compactly using twiddle notation:

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

IDFT

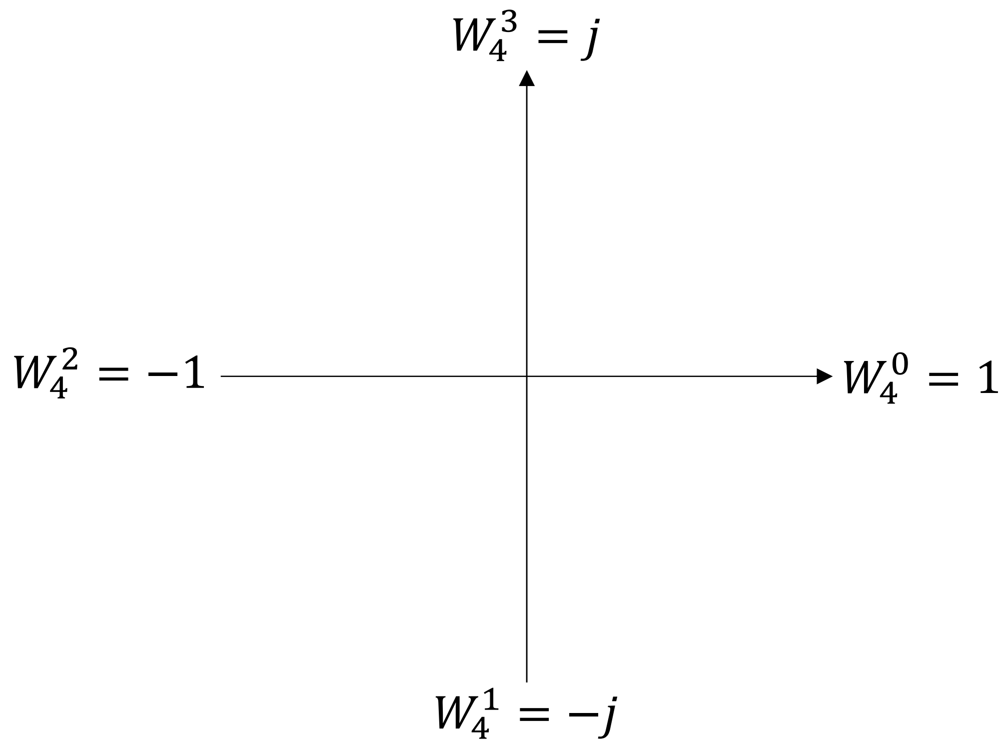
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}$$

Example

What is the DFT of the sequence: $x[n] = [1\ 2\ 6\ 4]$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$




```
fft([1 2 6 4])
```

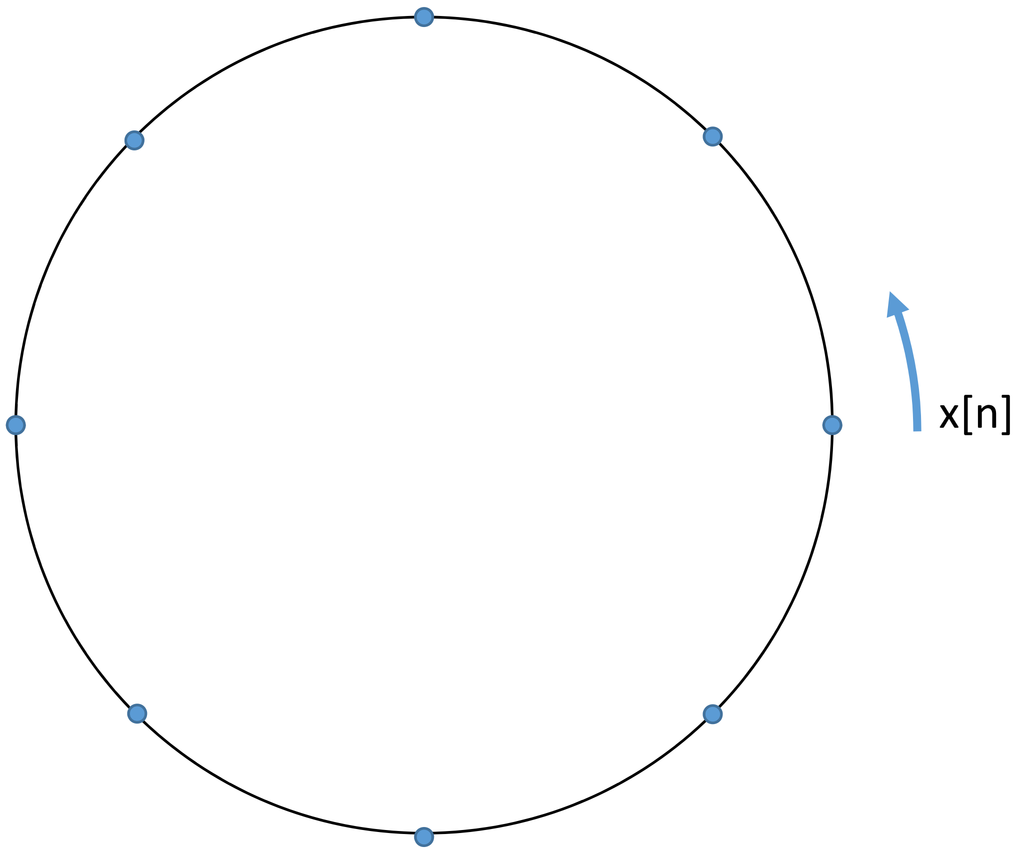
Some properties of the DFT

Theorem	sequence	Frequency domain
Linearity	$ax_1[\mathrm{n}]+bx_2[\mathrm{n}]$	$aX_1[k]+bX_2[k]$
Circular shift	$x[<\mathrm{n}-n_0>_N]$	$e^{-j\frac{2\pi}{N}kn_0}X[k]$
Convolution	$x_1[\mathrm{n}] \textcircled{N} x_2[\mathrm{n}]$	$X_1[k]X_2[k]$
Symmetries	$x[\mathrm{n}]$ (real valued)	$X[k]=X^*[<-k>_N]$ $X_R[k]=X_R[<-k>_N]$ $X_I[k]=-X_I[<-k>_N]$ $\lvert X[k]\rvert=\lvert X[<-k>_N]\rvert$ $\angle X[k]=-\angle X[<-k>_N]$ R = real, I = imaginary part

A wheel analogy

... A nice way to think about these properties is to represent the sequences around a wheel:

x[n] = [1 2 5 4 8 3 7 2]



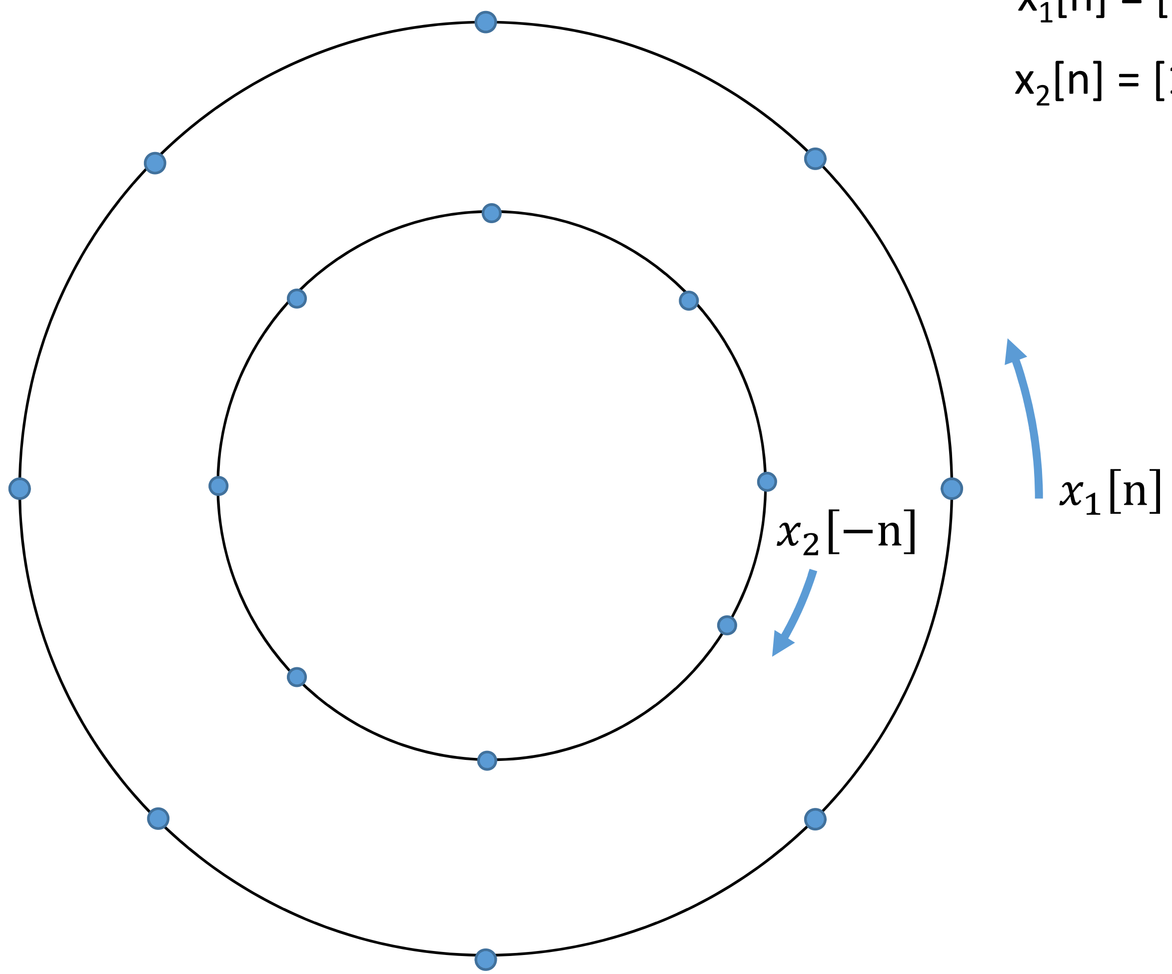
https://www.youtube.com/watch?v=vcn04gwFTyg

Circular convolution with DFTs

Circular convolution of two N-point sequences, $x_1[\mathrm{n}] \textcircled{N} x_2[\mathrm{n}]$, is simply standard convolution applied to the periodic sequences $\tilde{x}_1[\mathrm{n}]$, $\tilde{x}_2[\mathrm{n}]$. Or, it has a nice wheel representation too:

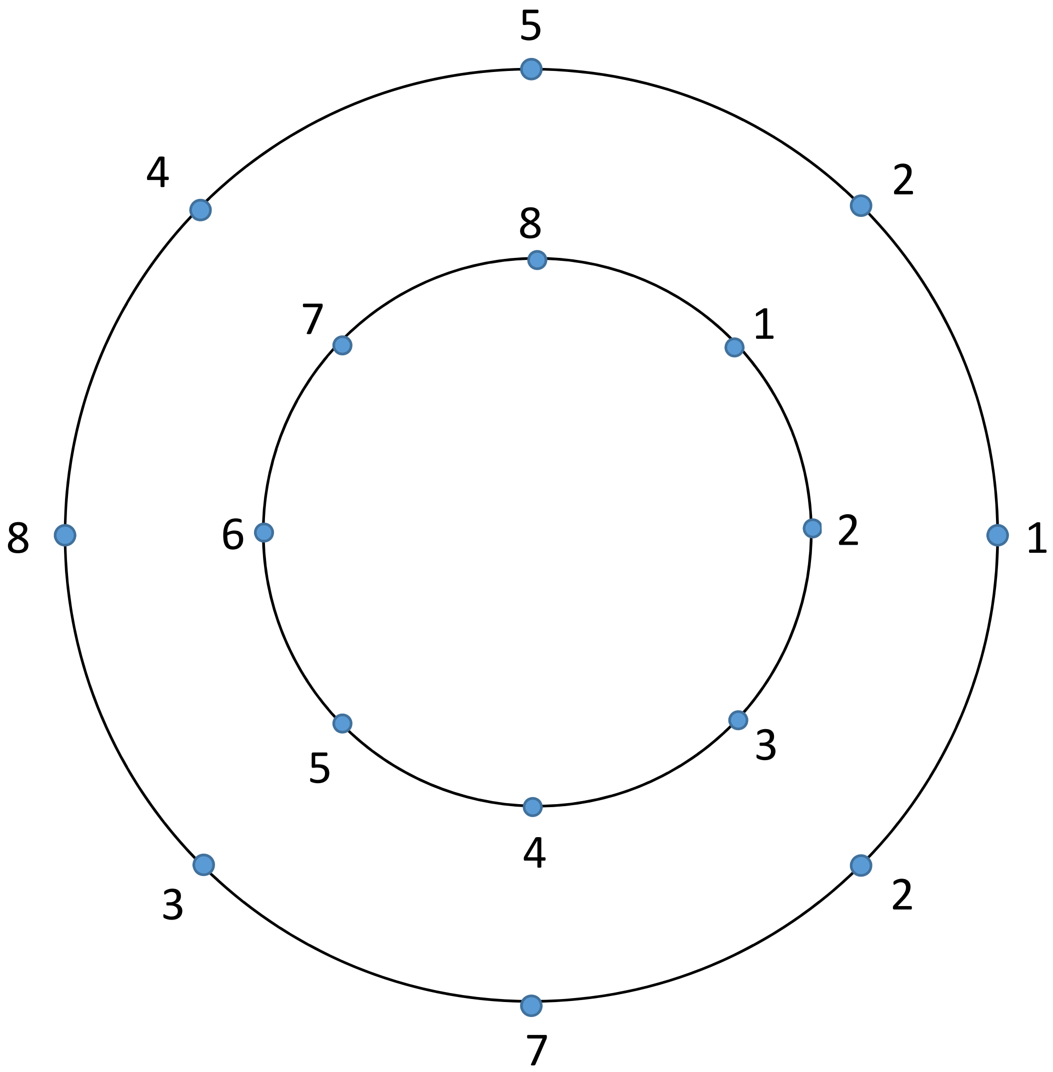
$x_1[n] = [1\ 2\ 5\ 4\ 8\ 3\ 7\ 2]$

$x_2[n] = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8]$



Circular convolution with DFTs

Circular convolution of two N-point sequences, $x_1[\mathrm{n}] \, \textcircled{N} \, x_2[\mathrm{n}]$, is simply standard convolution applied to the periodic sequences $\tilde{x}_1[\mathrm{n}]$, $\tilde{x}_2[\mathrm{n}]$. Or, it has a nice wheel representation too:



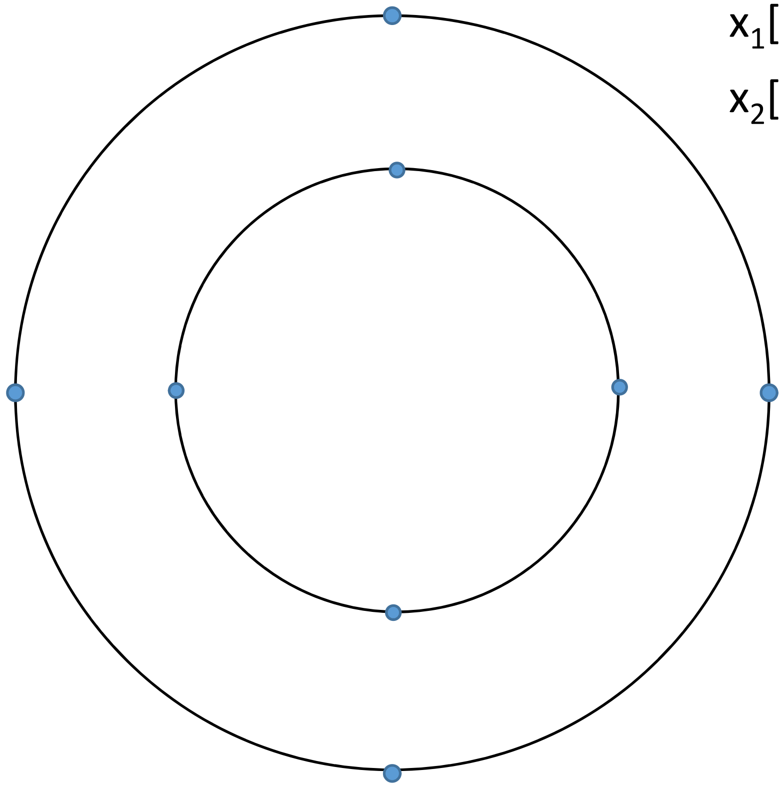
$x_1[n] = [1\ 2\ 5\ 4\ 8\ 3\ 7\ 2]$

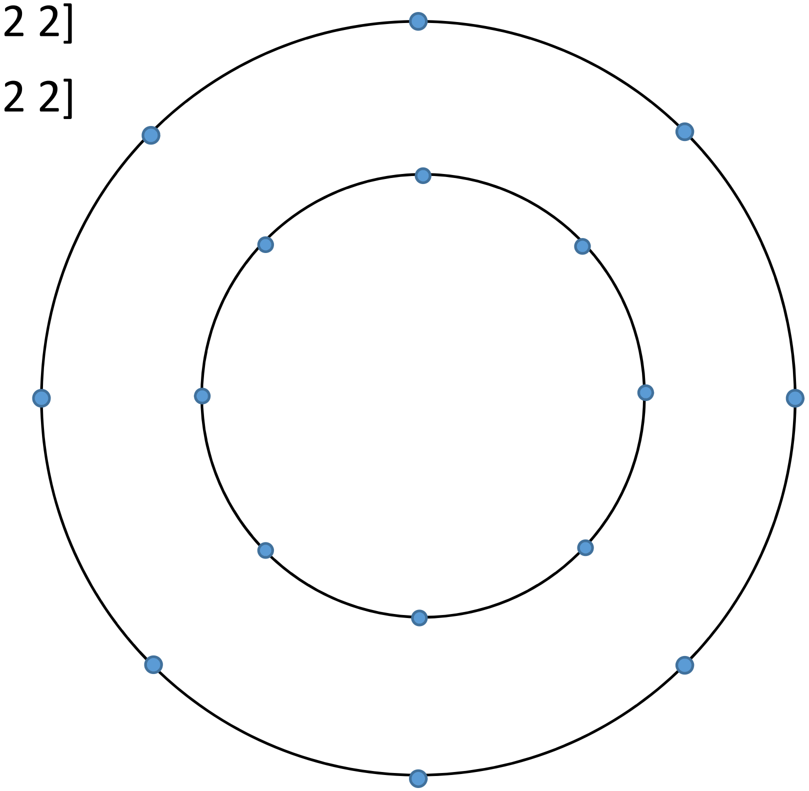
$x_2[n] = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8]$

Linear convolution with DFTs

$x_1[n] = [2\ 2\ 2\ 2]$

$x_2[n] = [2\ 2\ 2\ 2]$





$x_1[\text{n}] \; ④ \; x_2[\text{n}]$

$x_1[\text{n}] \; ⑧ \; x_2[\text{n}]$

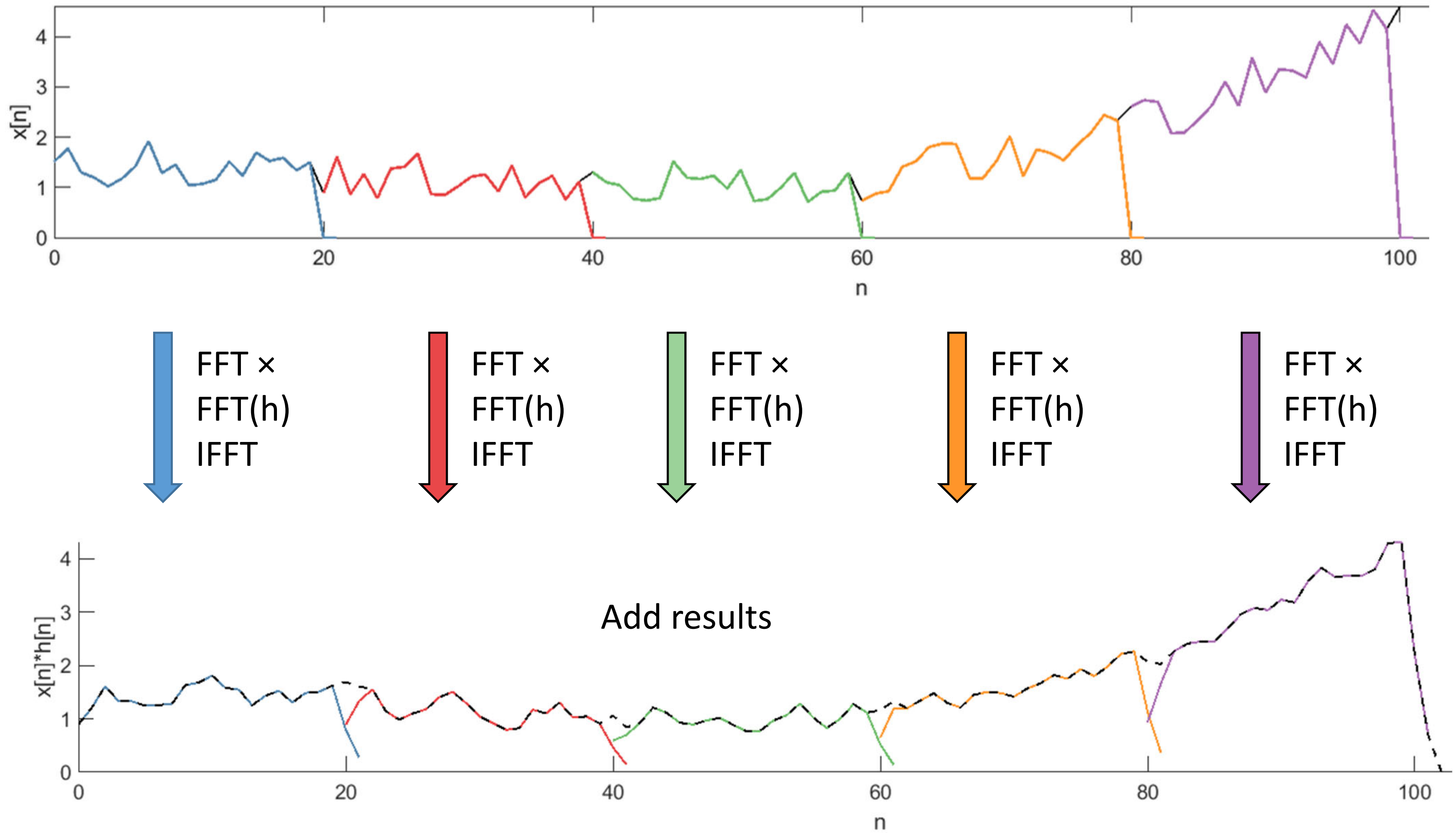
Linear convolution with DFTs

In general, to calculate the convolution of two sequences, $x_1[\mathrm{n}]$ of length L and $x_2[\mathrm{n}]$ of length P, we take the following steps:

- Calculate the length of the convolution that just eliminates the circular convolution overlap: N = L+P-1 (e.g. 4+4-1=7)
- Pad $x_1[\mathrm{n}]$ with N-L zeros and $x_2[\mathrm{n}]$ with N-P zeros
- Take the DFT (FFT) of both sequences
- Multiply the results
- Take the inverse DFT (IFFT) to find the result.

Implementing LTI systems using the DFT

Below is the ***multiply-add*** method, an alternative is the ***overlap-save*** method:



What you learned (DFT)…

- The DFT is the version of the Fourier transform that can be implemented digitally
- The DFT can be thought of as a 'cutout' from the DTFT of a periodic sequence
- The DFT is expressed in terms of twiddle factors: $W_N = e^{-j\frac{2\pi}{N}}$
- Convolution operations can be carried out digitally using DFTs
- This allows digital filters to be implemented by the "DFT -> multiply by transfer function -> IDFT" process