MANAGEMENT SCIENCE
Vol. 13, No. 7, March, 1967
Printed in U.S.A.

ON NONLINEAR FRACTIONAL PROGRAMMING*†

WERNER DINKELBACH

University of Cologne, Germany‡

The main purpose of this paper is to delineate an algorithm for fractional programming with nonlinear as well as linear terms in the numerator and denominator. The algorithm presented is based on a theorem by Jagannathan [7] concerning the relationship between fractional and parametric programming. This theorem is restated and proved in a somewhat simpler way. Finally, it is shown how the given algorithm can be related to the method of Isbell and Marlow [6] for linear fractional programming and to the quadratic parametric approach by Ritter [10]. The Appendix contains a numerical example.

1. Introduction

From an early date, various parametrization techniques have been suggested for use in connection with extensions of linear programming. Thus, although the well-known Saaty-Gass methods were already publicly available by 1954 (see [11]) there are even earlier examples which could be cited as known to those who were active in such work even though they were not published.[1] Interest and research on this topic continues; as witness, for instance, the recent work by Jagannathan [7] which shows, *inter alia*, how such techniques may be employed in dealing with the kinds of problems that Charnes and Cooper [1] refer to as fractional programming.[2] Although the theory set forth in [1] now makes it possible to use any linear programming algorithm to solve such problems in linear fractional programming, there continues to be considerable interest, nevertheless, in methods that are specially designed for solving this particular class of problems as well as in further developments which can supply insight into the general nature and possibilities of parametrization methods *per se*. Cases in point are the papers by Martos[3] [8] and Jagannathan [7] which, respectively, supply

* Received July 1965 and revised July 1966.

† A first version of this paper was presented at the 4th Annual Meeting of the German Society for Operations Research (Deutsche Gesellschaft für Unternehmensforschung) in October 1965, at Mannheim, Germany. The author is indebted to Professor W. W. Cooper and the referees, Professors A. Charnes, R. Jagannathan, and B. Martos, whose constructive criticism has improved the content and presentation of this paper and has helped to eliminate errors and inadequacies that had appeared in an earlier version.

‡ Industrieseminar der Universität, 5000 Köln-Lindenthal, Germany.

[1] A case in point is the rotating hyperplane method developed by W. Jacobs for use in the work of Project SCOOP (circa 1950–51).

[2] The paper [1] by Charnes and Cooper provides a general theory which shows how to reduce all such problems to at most two ordinary linear programming problems by adjoining a specified constraint to the original set. Their paper [2] relates this to work that had been undertaken by others. See, e. g. [4], [5], [6], and [8].

[3] This appears to be the earliest example of such an approach although, in the published literature, Isbell and Marlow [6] had previously identified such fractional programming problems and, using a more restrictive set of assumptions than Martos, they suggested a solution procedure via a series of linear programming problems. See also Dinkelbach [4]

just such approaches and theoretical insight into the possibilities for developing and relating parametric programming to these and other types of problems.

In fact, in the subsequent Section 2 of this paper, we follow along the same lines as the theoretical development in Jagannathan's paper in restating his main theorem with a somewhat simpler proof. Then in Section 3, we make some specializing assumptions in order to delineate the resulting algorithmic possibilities for fractional functionals with nonlinear as well as linear terms in the numerator and denominator. Finally in Section 4, we show how the given algorithms can be related to the method of Isbell and Marlow [6] for linear fractional functionals and to the method of Ritter [10] for dealing with quadratic parametric functionals. A numerical example is outlined in the Appendix.

2. The Relationship between Nonlinear Fractional and Nonlinear Parametric Programming

Let E^n be the Euclidean space of dimension n and S be a compact and connected subset of E^n. Let $N(x)$ and $D(x)$ be continuous and real-valued functions of $x \varepsilon S$. Furthermore, the following assumption is also made:

$$D(x) > 0 \quad \text{for all} \quad x \varepsilon S.$$

We are interested in the following two problems:

$$\text{(I)} \qquad \max \{N(x)/D(x) \mid x \varepsilon S\} \qquad \text{and}$$

$$\text{(II)} \qquad \max \{N(x) - qD(x) \mid x \varepsilon S\} \quad \text{for} \quad q \varepsilon E^1.$$

The problems (I) and (II) have solutions, indeed, since $N(x)$ and $D(x)$ are continuous, S is compact, and the singular points defined by $D(x) = 0$ are excluded.

Lemma 1: $F(q) = \max \{N(x) - qD(x) \mid x \varepsilon S\}$ is convex over E^1.

Proof: Let x_t maximise $F(tq' + (1 - t)q'')$ with $q' \neq q''$ and $0 \leqq t \leqq 1$:

$$F(tq' + (1 - t)q'') = N(x_t) - (tq' + (1 - t)q'')D(x_t) = t[N(x_t) - q'D(x_t)] + (1 - t)[N(x_t) - q''D(x_t)] \leqq t \cdot \max \{N(x) - q'D(x) \mid x \varepsilon S\} + (1 - t) \cdot \max \{N(x) - q''D(x) \mid x \varepsilon S\} = tF(q') + (1 - t)F(q'').$$

Lemma 2: $F(q)$ is continuous for $q \varepsilon E^1$.

Proof: See Courant [3a], p. 290 or [3b], p. 326.

Lemma 3: $F(q) = \max \{N(x) - qD(x) \mid x \varepsilon S\}$ is strictly monotonic decreasing, i.e. $F(q'') < F(q')$, if $q' < q''$, $q', q'' \varepsilon E^1$.

Proof: Let x'' maximise $F(q'')$, then

---

and Dorn [5]. Of course, it is also possible that there may have been still earlier work by others. For instance, A. Charnes has called my attention to certain parts of his work in this area, *circa* 1957, which includes another parametric technique covered both in his lecture notes at Northwestern University as well as his use of it in a subsequent industrial application.

$$F(q'') = \max\{N(x) - q''D(x) \mid x \varepsilon S\} = N(x'') - q''D(x'') < N(x'') - q'D(x'') \leqq \max\{N(x) - q'D(x) \mid x \varepsilon S\} = F(q').$$

Lemma 4: $F(q) = 0$ has an unique solution, say q_0.

Proof: This assertion results from lemma 2, lemma 3, and the following fact: $\lim_{q \to -\infty} F(q) = +\infty$ and $\lim_{q \to +\infty} F(q) = -\infty$.

Lemma 5: Let $x^+ \varepsilon S$ and $q^+ = N(x^+)/D(x^+)$, then $F(q^+) \geqq 0$.

Proof: $F(q^+) = \max\{N(x) - q^+D(x) \mid x \varepsilon S\} \geqq N(x^+) - q^+D(x^+) = 0$. Hence $F(q^+) \geqq 0$.

For any $q = q^*$, the maximum of $\{N(x) - q^*D(x) \mid x \varepsilon S\}$ is taken on, for instance, at x^*; this may be indicated by writing $F(q^*, x^*)$. The following theorem can now be proved:

Theorem: $q_0 = N(x_0)/D(x_0) = \max\{N(x)/D(x) \mid x \varepsilon S\}$ if, and only if,

$$F(q_0) = F(q_0, x_0) = \max\{N(x) - q_0D(x) \mid x \varepsilon S\} = 0.$$

Proof:

a) Let x_0 be a solution of problem (I). We have

$$q_0 = N(x_0)/D(x_0) \geqq N(x)/D(x) \quad \text{for all} \quad x \varepsilon S.$$

Hence

$$(\alpha) \qquad N(x) - q_0D(x) \leqq 0 \quad \text{for all} \quad x \varepsilon S.$$

$$(\beta) \qquad N(x_0) - q_0D(x_0) = 0.$$

From (α) we have $F(q_0) = \max\{N(x) - q_0D(x) \mid x \varepsilon S\} = 0$. From (β) we see that the maximum is taken on, for instance, at x_0. Thus the first part of the proof is finished.

b) Let x_0 be a solution of problem (II) such that $N(x_0) - q_0D(x_0) = 0$. The definition of (II) implies

$$N(x) - q_0D(x) \leqq N(x_0) - q_0D(x_0) = 0 \quad \text{for all} \quad x \varepsilon S.$$

Hence

$$(\alpha) \qquad N(x) - q_0D(x) \leqq 0 \quad \text{for all} \quad x \varepsilon S$$

$$(\beta) \qquad N(x_0) - q_0D(x_0) = 0.$$

From (α) we have $q_0 \geqq N(x)/D(x)$ for all $x \varepsilon S$, that is q_0 is maximum of problem (I). From (β) we have $q_0 = N(x_0)/D(x_0)$, that is x_0 is a solution vector of (I).

Note that x_0 may not be unique. Furthermore, the theorem is still valid, if we replace "max" by "min".

3. A Method for Solving Nonlinear Fractional Programming Problems with Concave $N(x)$ and Convex $D(x)$

Additionally we assume, in this section, that $N(x)$ is concave and $D(x)$ is convex for $x \varepsilon S$, and consequently, S is assumed to be convex. Denoting an optimal solution of (I) by x_0, we formulate problem (I) as follows:

Find x_m, such that $q(x_0) - q(x_m) < \epsilon$ for any given $\epsilon > 0$.

Since $F(q)$ is continuous, we have a second formulation:

Find x_n and $q_n = N(x_n)/D(x_n)$, such that $F(q_n) - F(q_0)$

$$= F(q_n) < \delta \quad \text{for any given} \quad \delta > 0.$$

Furthermore, we assume $F(0) = \max\{N(x) \mid x \varepsilon S\} \geqq 0$. The algorithm[4] can be started by $q = 0$, (A_1), or by any feasible $x = x_1 \varepsilon S$ with $q(x) \geqq 0$, (A_2):

(A_1) Set $q_2 = 0$ and go to (B) with $k = 2$.

(A_2) Let $x_1 \varepsilon S$ and $q_2 = N(x_1)/D(x_1)$, proceed to (B) with $k = 2$.

(B) By means of any method of concave programming solve the following problem:

$$F(q_k) = \max\{N(x) - q_k D(x) \mid x \varepsilon S\}$$

and denote any solution point by x_k.

(B_1) If $F(q_k) < \delta$: Stop. If $F(q_k) > 0$, then $x_k = x_n$.
If $F(q_k) = 0$, then $x_k = x_0$.

(B_2) If $F(q_k) \geqq \delta$: Evaluate $q_{k+1} = N(x_k)/D(x_k)$ and go to (B) replacing q_k by q_{k+1}.

Proof of Convergence:

a) First we shall prove that $q_{k+1} > q_k$ for all k with $F(q_k) \geqq \delta$: Lemma 5 implies $F(q_k) > 0$. By definition we have

$$N(x_k) = q_{k+1}D(x_k), \quad \text{hence} \quad F(q_k) = N(x_k) - q_k D(x_k) = q_{k+1}D(x_k)$$
$$-q_k D(x_k) > 0. \quad \text{Since} \quad D(x_k) > 0, \quad \text{we have} \quad q_{k+1} > q_k.$$

b) Our second assertion is $\lim_{k\to\infty} q_k = q(x_0) = q_0$. If this is not true, we must have: $\lim_{k\to\infty} q_k = q^* < q_0$. By construction we have a sequence x_k^* with q_k^*, such that $\lim_{k\to\infty} F(q_k^*) = F(q^*) = 0$ (see (B_1) of the algorithm). Since $F(q)$ is strictly monotonic decreasing (lemma 3), we obtain

$$0 = F(q^*) > F(q_0) = 0,$$

which is a contradiction. Hence it follows that $\lim_{k\to\infty} F(q_k) = F(q_0)$, and then by lemma 2 we have $\lim_{k\to\infty} q_k = q_0$.

4. Concluding Remarks

Assuming $N(x)$ and $D(x)$ to be linear and S to be polyhedral, the algorithm presented reduced to that given by Isbell and Marlow [6]. Their algorithm generates a sequence of linear programs. The solutions of these programs converge to the solution of the fractional program. In the linear case, the algorithm always stops after a finite number of iterations, but this is not necessarily true for the nonlinear case.

[4] As one of the referees pointed out this formulation of the algorithm leads to an optimal solution starting with negative values of q as long as $N(x) - qD(x)$ remains concave. Moreover this formulation yields an optimal solution regardless of the sign of $F(0)$ if $D(x)$ is linear. It is obvious that starting with $q_2 = 0$ requires that q_0 is nonnegative.

The objective function for a quadratic fractional program has the following form

$$q(x) = (x'Ax + a'x + \alpha)/(x'Bx + b'x + \beta),$$

where A and B are negative (respectively positive) definite ($n \times n$)-matrices, a and b are column vectors of n components, α and β are scalars. Now, if the set S is polyhedral, the method for parametric quadratic programming by Ritter [10] is available to solve this quadratic fractional problem. Ritter investigates the following parametric problem:

$$\max \{x'(A - qB)x + (a' - qb')x \mid x \varepsilon S(q), S(q) \text{ polyhedral}\}.$$

By his method it is possible to determine optimal solutions for all values q of any given interval.

Finally, we point out that the function $N(x)/D(x)$ is quasi-concave if $N(x)$ concave, $D(x)$ convex, $D(x) > 0$ for $x \varepsilon S$ and $N(x) \geqq 0$ for $x \varepsilon S$ or $D(x)$ linear. Hence Martos' Theorem 2 ([9], p. 244) implies that any local maximum is a global maximum as well. Thus, it is also possible to solve (I) without parametrization by applying any gradient method of nonlinear programming.

Appendix

Let us consider the following simple numerical example:

$$\max\left\{q = \frac{N(x,y)}{D(x,y)} = \frac{-3x^2 - 2y^2 + 4x + 8y - 8}{x^2 + y^2 - 6y + 8} \,\middle|\, x + 3y \leqq 5, x \geqq 0, y \geqq 0\right\}.$$

Now, what about the assumptions of Sections 2 and 3? $D(x, y) = x^2 + y^2 - 6y + 8 = x^2 + (y - 3)^2 - 1$ is strictly convex. On the other hand, $N(x, y)$ is strictly concave. The intersection of $S = \{(x, y) \mid x + 3y \leqq 5, x \geqq 0, y \geqq 0\}$ and the area of the circle $D(x, y) \leqq 0$ is empty. Hence, $D(x, y) > 0$ for all $(x, y) \varepsilon S$.

Let us try to start the given algorithm with $q_2 = 0$ (starting with $(x_1, y_1) = (0, 0)$ is less efficient in this example). The subproblem

$$\mathrm{F}(0) = \max\{-3x^2 - 2y^2 + 4x + 8y - 8 \mid (x, y) \varepsilon S\}$$

has the solution

$$x_2 = 0.5517, \qquad y_2 = 1.4828; \qquad F(0) = 0.7586 > 0.001 \ (= \delta).$$

Now we determine $q_3 = N(x_2, y_2)/D(x_2, y_2) = 0.472$ and solve

$$F(0.472) = \max\{-3.472x^2 - 2.472y^2 + 4x + 10.832y - 11.776 \mid (x, y) \varepsilon S\}.$$

This subproblem leads to

$$x_3 = 0.4187, \qquad y_3 = 1.5271, \qquad F(0.472) = 0.0669 > 0.001.$$

With $q_4 = N(x_3, y_3)/D(x_3, y_3) = 0.522$ we have to solve

$$F(0.522) = \max\{-3.522x^2 - 2.522y^2 + 4x + 11.132y - 12.176 \mid (x, y) \varepsilon S\}.$$

The solution of this subproblem is

$$x_4 = 0.4066, \qquad y_4 = 1.5312, \qquad F(0.522) = 0.000451 < 0.001.$$

Thus let us take $(x_4, y_4) = (x_n, y_n)$ as an approximate solution of our problem.

For the concluding section we assume that the reader is familiar with Ritter's paper [10]. Applying his method, we consider $F(q)$ for a specific q, say $q = 0$, and after that we determine, in a first step of the algorithm, the resulting optimal solution as a function of q being an element of an interval containing $q = 0$. The result of this procedure is

$$x = x(q) = \frac{16 - 4q}{29 + 10q}, \qquad y = y(q) = \frac{43 + 18q}{29 + 10q},$$

$$F_1(q) = \frac{-6q^2 - 39q + 22}{29 + 10q}.$$

In a second step we determine that interval of q, for which $x(q)$ and $y(q)$ remains feasible and optimal. This interval is

$$-0.77526 \leqq q \leqq 4.$$

Since $F(-0.77526) > 0$ and $F(4) < 0$, $F_1(q) = 0$ has a solution in the mentioned interval. From $-6q^2 - 39q + 22 = 0$ we get

$$q_0 = -13/4 + (683/48)^{1/2} = 0.522157,$$

$$x_0 = x(q_0) = 0.40651, \qquad y_0 = y(q_0) = 1.53116.$$

The solution of this simple example satisfying the equation $x + 3y = 5$ can also be found by using classical Lagrange multipliers. By this method, it is possible to verify the above solution once more.

References

1. Charnes, A. and Cooper, W. W., "Programming with Linear Fractional Functionals," *Naval Research Logistics Quarterly*, Vol. 9, 1962, pp. 181–186.
2. —— and ——, "Programming with Linear Fractional Functionals, A Communication," *Naval Research Logistics Quarterly*, Vol. 10, 1963, pp. 273–274.
3. Courant, R., a: *Vorlesungen über Differential- und Integralrechnung*, 2. Bd., 3. Aufl., Berlin: Springer, 1955; b: *Differential and Integral Calculus*, Vol. 2, New York: Interscience, 1962.
4. Dinkelbach, W., "Die Maximierung eines Quotienten zweier linearer Funktionen unter linearen Nebenbedingungen," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, Band 1, 1962, S. 141–145.
5. Dorn, W. S., "Linear Fractional Programming," IBM Research Report RC-830, 1962.
6. Isbell, J. R. and Marlow, W. H., "Attrition Games," *Naval Research Logistics Quarterly*, Vol. 3, 1956, pp. 71–93.
7. Jagannathan, R., "On some Properties of Programming Problems in Parametric Form Pertaining to Fractional Programming," *Management Science*, Vol. 12, 1966, pp. 609–615.
8. Martos, B., "Hiperbolikus Programozas," *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, Vol. 5, Series B, 1960, pp. 383–406. A translation

by Andrew and Veronika Whinston appeared under the title "Hyperbolic Programming" in *Naval Research Logistics Quarterly*, Vol. 11, 1964, pp. 135–155.
9. ——, "The Direct Power of Adjacent Vertex Programming Methods," *Management Science*, Vol. 12, 1966, pp. 241–252.
10. RITTER, K., "Ein Verfahren zur Lösung parameterabhängiger, nichtlinearer Maximum-Probleme," *Unternehmensforschung*, Band 6, 1962, S. 149–166.
11. SAATY, T. AND GASS, S., "Parametric Objective Function (Part 1)," *Operations Research*, Vol. 2, 1954, pp. 316–319.
12. SWARUP, K., "Programming with Quadratic Fractional Functionals," *Opsearch*, Vol. 2, 1966, pp. 23–30.