Inversion of the Laplace transform

Lasse Lybeck
013748498

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1 Introduction

Let $f : [0, \infty) \to \mathbb{R}$. The Laplace transform F of f is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt, \quad s \in \mathbb{C}, \tag{1}$$

provided that the integral converges.

The transform has many applications in physical sciences and is therefore widely used and studied. One example of the usage of the transform is in linear differential equations, which may in some cases be easily solved using the Laplace transform.

The direct problem is to determine F for a given function f according to (1). The inverse problem is: *given a Laplace transform F, find the corresponding function f*. In this study we will be looking at the inverse problem from the computational point of view.

2 Materials and Methods

2.1 The matrix model

Assume we know the values of F at these real-valued points:

$$0 < s_1 < s_2 < \ldots < s_n < \infty.$$

Then we may approximate the integral in (1) for example with the trapezoidal rule as

$$
\begin{aligned}
\int_0^\infty e^{-st} f(t)dt \approx \frac{t_k}{k} \bigg(&\frac{1}{2} e^{-st_1} f(t_1) + e^{-st_2} f(t_2) + e^{-st_3} f(t_3) + \dots \\
&+ e^{-st_{k-1}} f(t_{k-1}) + \frac{1}{2} e^{-st_k} f(t_k) \bigg),
\end{aligned} \tag{2}
$$

where vector $t = [t_1 \ t_2 \ \dots \ t_k]^T \in \mathbb{R}^k$, $0 \leq t_1 < t_2 < \dots < t_k$, contains the points at which the unknown function f will be evaluated. By denoting $f_\ell = f(t_\ell)$, $\ell = 1, \dots, k$, and $m_j = F(s_j)$, $j = 1, \dots, n$, and using (2), we get a linear model of the form $m = Af + \epsilon$ with

$$
A = \frac{t_k}{k} \begin{bmatrix}
\frac{1}{2} e^{-s_1 t_1} & e^{-s_1 t_2} & e^{-s_1 t_3} & \dots & e^{-s_1 t_{k-1}} & \frac{1}{2} e^{-s_1 t_k} \\
\frac{1}{2} e^{-s_2 t_1} & e^{-s_2 t_2} & e^{-s_2 t_3} & \dots & e^{-s_2 t_{k-1}} & \frac{1}{2} e^{-s_2 t_k} \\
\vdots & & & & & \vdots \\
\frac{1}{2} e^{-s_n t_1} & e^{-s_n t_2} & e^{-s_n t_3} & \dots & e^{-s_n t_{k-1}} & \frac{1}{2} e^{-s_n t_k}
\end{bmatrix}. \tag{3}
$$

2.2 The inversion method

In this study we will solve the inverse problem with the truncated singular value decomposition (henceforth referred to as SVD) method. This method is based on the fact that every matrix $A \in \mathbb{R}^{n \times m}$ can be decomposed into the product of three matrices

$$
A = UDV^T, \tag{4}
$$

where U and V are orthogonal matrices and D is a diagonal matrix. The diagonal elements $d_{i,i}$, $i = 1, \dots, min\{n, m\}$ of D are called the *singular values of D*.

The *pseudoinverse* of A (denoted as A^+) can be calculated via the SVD of A. Due to the fact that U and V are orthogonal, we know that $U^T = U^{-1}$ and $V^T = V^{-1}$. We define the pseudoinverse of the diagonal matrix $D \in \mathbb{R}^{n \times m}$ as the diagonal matrix $D^+ \in \mathbb{R}^{m \times n}$ where the diagonal elements have the values

$$
D^+_{i,i} = \begin{cases} 1/d_{i,i} & d_{i,i} \neq 0 \\ 0 & otherwise. \end{cases} \tag{5}
$$

3 Results

The Results section is for a detailed explanation of what happens when the methods of Section 2 are applied to the materials. There should be no interpretation of what the results might mean, just a dry and factual report with numbers, charts and plots.

(a) Compute numerically and plot the Laplace transform of

$$f(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Construct matrix A given by (3) for a suitable choice of points t_ℓ and s_j. Compute the singular values of A. Do you detect ill-posedness?

(c) Use truncated SVD to compute the inverse Laplace transform of f.

4 Discussion

This section is for interpretations of the results presented in Section 3. Sometimes this section is called Conclusions, or Discussion and Conclusions.