ODE SOLVERS: MULTI-STEP METHODS

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ABSTRACT.

CONTENTS

1. MULTI-STEPS METHODS

We can use more information on the previous steps to get a higher order methods. It will be useful to introduce the following symbols representing indices, function value and derivative at various locations.

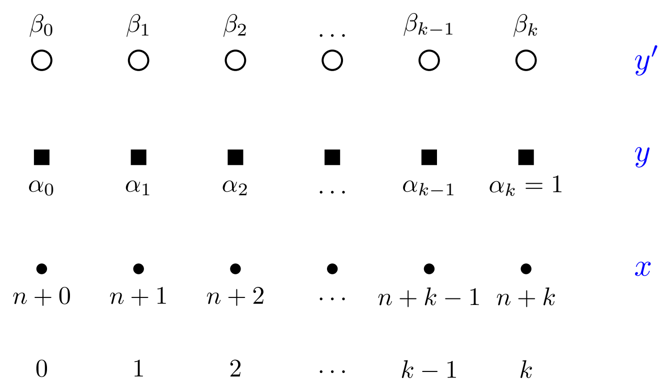

FIGURE 1. Notation for multi-step methods

We use subscript to indicates the function evaluated at the corresponding grid points. For example, for $i = 0, 1, \dots, k$

$$y_{n+i} = y(x_{n+i}), \quad f_{n+i} = f(x_{n+i}, y(x_{n+i})).$$

Depending on the context, sometimes f_{n+i} could represent $f(x_{n+i}, u_{n+i})$.

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1.1. **Adams-Bashforth method.** Recall that

$$y_{n+k} = y_{n+k-1} + \int_{x_{n+k-1}}^{x_{n+k}} y'(t)\,\mathrm{d}t = y_{n+k-1} + \int_{x_{n+k-1}}^{x_{n+k}} f(x, y(x))\,\mathrm{d}x. \tag{1}$$

Suppose we know function values y_{n+i} for $i = 0, \ldots, k-1$, we can evaluate to get f_{n+i} and fit the data (x_{n+i}, f_{n+i}) with a polynomial of degree $k-1$. For example, the Lagrange interpolant f_I to f can be written as

$$f_I(x) = \sum_{i=0}^{k-1} p_i(x) f_{n+i},$$

where $p_i(x) \in \mathbb{P}_{k-1}$ and $p_i(x_{n+j}) = \delta_{ij}$.

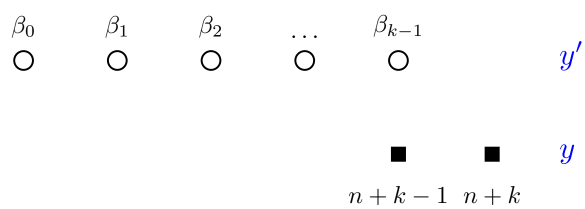

FIGURE 2. Adams-Bashforth method

Approximate f by f_I and let

$$\beta_i = \frac{1}{h}\int_{x_{n+k-1}}^{x_{n+k}} p_i(x)\,\mathrm{d}x,$$

we then obtain the Adams-Bashforth method

$$u_{n+k} = u_{n+k-1} + h\sum_{i=0}^{k-1} \beta_i f_{n+i}. \tag{2}$$

When studying the truncation error, we assume the function value y_{n+i} is known for $i = 0, 1, \ldots, k-1$. Then the truncation error

$$T_h^{\mathrm{AB}} := \frac{1}{h}(u_{n+k} - y_{n+k}) = \frac{1}{h}\int_{x_{n+k-1}}^{x_{n+k}} (f_I - f)\,\mathrm{d}x = \frac{1}{h}\int_{x_{n+k-1}}^{x_{n+k}} ((y')_I - y')\,\mathrm{d}x.$$

We switch the integrand to y' since now the remainder can be written as derivative of exact solution y. As the Lagrange interpolant preserves polynomial

1.2. **Adams-Moulton method.** The only difference is the point (x_{n+k}, f_{n+k}) is included to fit the polynomial.

Now the Lagrange interpolant f_I to f will be

$$f_I(x) = \sum_{i=0}^{k} p_i^*(x) f_{n+i},$$

where $p_i^*(x) \in \mathbb{P}_k$ and $p_i^*(x_{n+j}) = \delta_{ij}$ for $i, j = 0, 1, \ldots, k$. The superscript * is introduced to distinguish the same quantity used in A-B method.

Approximate f by f_I and let

$$\beta_i^* = \frac{1}{h}\int_{x_{n+k-1}}^{x_{n+k}} p_i^*(x)\,\mathrm{d}x,$$

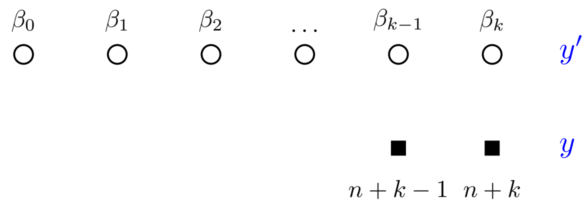

FIGURE 3. Adams-Moulton method

we then obtain the Adams-Moultion method

$$u_{n+k} = u_{n+k-1} + h\sum_{i=0}^{k-1} \beta_i^* f_{n+i} + h\beta_k^* f(x_{n+k}, u_{n+k}). \tag{3}$$

Here we single out the last term to emphasize A-M method is an implicit method and an iteration is needed to solve the nonlinear equation (3).