Journal of Computational and Applied Mathematics 265 (2014) 173–186

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New a posteriori error estimate and quasi-optimal convergence of the adaptive nonconforming Wilson element ☆

Jun Hu [a,*], Longlong Jiang [a], Zhongci Shi [b]

[a] *LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, PR China*

[b] *LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China*

ARTICLE INFO

Article history:
Received 2 October 2012
Received in revised form 13 July 2013

Keywords:
Adaptive Wilson element
A posteriori error estimate
Optimal convergence

ABSTRACT

In this paper we establish the quasi-optimal convergence of the adaptive nonconforming Wilson element on the rectangular mesh. The main ingredients are a new a posteriori error estimator and a crucial observation that there is some special orthogonality between the conforming part and the nonconforming part in the energy inner product, which helps us to show the quasi-orthogonality and the discrete reliability. Finally we integrate these components in a usual way to achieve the quasi-optimal convergence.

1. Introduction

Adaptive finite element methods are a fundamental numerical instrument to approximate partial differential equations. The adaptive conforming finite element method for second order elliptic problems has been studied for many years following the pioneering work of Babuška [1], and its theory has in some sense become rather mature. For nonconforming methods, started with [2,3], where the nonconforming linear element method for the Poisson and Stokes equations is analyzed, the a posteriori error theory has been studied in the literature [4–7]. However, the convergence and optimality analysis are not established for most nonconforming methods in the literature.

The main difficulty for the convergence and optimality analysis of adaptive nonconforming finite element methods is the lack of the Galerkin-orthogonality, which is a key ingredient for the convergence and optimality analysis of adaptive conforming methods for second order elliptic problem [8–12]. For the nonconforming linear element of the Poisson equation, a quasi-orthogonality is obtained in [13,14] by using some special equivalency between the nonconforming linear element and the lowest order Raviart–Thomas element, which is extended to the nonconforming linear element for the Stokes-like problem in [15]. For the Morley element of the fourth order elliptic problem, a quasi-orthogonality is established in [16] based on a crucial local conservative property of the Morley element method, such an idea is generalized to the nonconforming linear element therein, see also [17,18]. However, these techniques cannot be extended to the nonconforming Wilson element under consideration, since the gradient of the functions in the Wilson element space is not a piecewise constant. Moreover, there is no local conservative property like the nonconforming linear element and the nonconforming Morley element.

The aim of this paper is to propose a new a posteriori error estimator and achieve the convergence and optimality of the adaptive Wilson element. The key observation is that the Wilson element space can be decomposed into a conforming part and a nonconforming part and that there is some special orthogonality between the two parts in the energy inner product. We use this property to prove the reliability and efficiency of the new estimator, and show a quasi-orthogonality,

☆ The research of the first author was partially supported by the NSFC projection 11271035, by the NSFC Key Project 11031006.

* Corresponding author. Tel.: +86 10 62757982.
E-mail addresses: hujun@math.pku.edu.cn (J. Hu), 1001210026@pku.edu.cn (L. Jiang), shi@lsec.cc.ac.cn (Z. Shi).

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.09.078

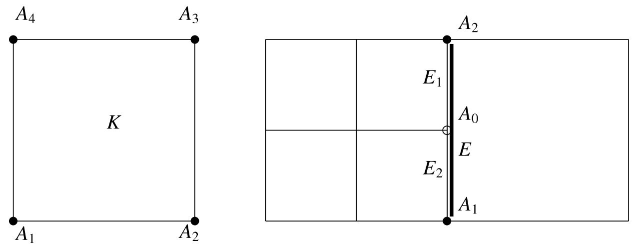

Fig. 1. (left) A rectangle K with its four nodes $A_1, \ldots, A_4$; (right) A hanging node A_0 and associated regular nodes A_1, A_2, a hanging edge E and its two children E_1 and E_2.

and establish the discrete reliability. We integrate these results to prove the quasi-optimal convergence of the adaptive nonconforming Wilson element method.

The rest of the paper is organized as follows. In Section 2, we present the second order elliptic equation and the 1-irregular mesh, and introduce the Wilson element as well as a new a posteriori error estimator. We give a priori analysis of the Wilson element on the 1-irregular mesh in Theorem 2.2, and then present a new a posteriori estimator with its reliability and efficiency proof in Theorem 2.6. In Section 3, we prove the quasi-orthogonality and show the convergence of the adaptive Wilson element method. To obtain the optimality of the adaptive algorithm, we establish the discrete reliability in Section 4. Consequently, we show the optimality of the adaptive Wilson element method in Section 5. Finally, we give some numerical examples in Section 6.

2. Notation and preliminaries

Let Ω be a polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$. We consider the following second order elliptic equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$.

Now we turn to the weak formulation of the problem (1). For a measurable set $G \subset \Omega$, let $(\cdot,\cdot)_{L^2(G)}$ and $\|\cdot\|_{L^2(G)}$ denote the inner product and the norm in $L^2(G)$, and if $G = \Omega$, we drop the index $L^2(\Omega)$ for simplicity. Then the weak formulation of the problem (1) reads

$$\begin{cases} \text{Find } u \in H_0^1(\Omega), & \text{such that} \\ a(u,v) = (f,v) & \text{for any } v \in H_0^1(\Omega) \end{cases} \tag{2}$$

with $a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$, where the symbol $\cdot$ is the inner product in the Euclidean space $\mathbb{R}^2$.

2.1. The 1-irregular mesh

Given an initial regular rectangular mesh $\mathcal{T}_0$ of Ω in the sense of Ciarlet [19], a rectangular mesh $\mathcal{T}$ is a set of rectangles obtained by a finite number L of refinements from $\mathcal{T}_0$, i.e., $\mathcal{T} = \mathcal{T}_L$, where for every $l = 1, \ldots, L$ there exists one $K \in \mathcal{T}_{l-1}$ and $\mathcal{T}_l$ is just the former partition except that K is refined into four elements $K_1, \ldots, K_4$ by connecting the midside points of the edges of K. Then, one says that $\mathcal{T}$ is some refinement of $\mathcal{T}_0$.

Given some element K of a rectangular mesh $\mathcal{T}_h$, $h_K = |K|^{1/2}$ denotes its size, $\mathcal{N}_h(K)$ its vertices, $\mathcal{E}_h(K)$ its edges. The set of nodes of $\mathcal{T}_h$ reads $\mathcal{N}_h := \cup_{K\in\mathcal{T}_h} \mathcal{N}_h(K)$, while the set of edges reads $\mathcal{E}_h := \cup_{K\in\mathcal{T}} \mathcal{E}(K)$. Besides, let $\mathcal{E}_h(\Omega)$ be the set of interior edges and $\mathcal{E}_h(\Gamma)$ be the set of boundary edges.

Let $\mathcal{T}_h$ be some refinement of $\mathcal{T}_0$, some node $z \in \mathcal{N}_h$ is called a hanging node if some element $K \in \mathcal{T}_h$ satisfies

$$z \in \partial K \setminus \mathcal{N}_h(K)$$

(i.e., z belongs to its boundary but not a vertex of it). Otherwise the node $z \in \mathcal{N}_h$ is called regular. In case any edge $E \in \mathcal{E}_h$ contains at most k hanging node in its inside, $\mathcal{T}_h$ is called k-irregular.

A 0-irregular mesh is a conforming mesh. In this paper, we restrict to conforming and 1-irregular meshes which allow for some local mesh-refinement.

An edge E of an element K is called a hanging edge if its midpoint A is a hanging node. The two edge E_1 and E_2 with vertex A which belong to the neighbor elements K_1 and K_2, are called children of E. Fig. 1 illustrates the definition of a hanging edge $E = \overline{A_1A_2}$ and its two children $E_1 = \overline{A_0A_2}$ and $E_2 = \overline{A_0A_1}$.

2.2. The Wilson element and its a priori error estimate

Let $\mathcal{T}_h$ be a rectangular mesh of Ω. We define $H^1(\mathcal{T}_h)$ as

$$H^1(\mathcal{T}_h) := \{v \in L^2(\Omega);\ \forall K \in \mathcal{T}_h, v|_K \in H^1(K)\},$$

and for $v \in H^1(\mathcal{T}_h)$, we denote by $\nabla_h v$ the gradient operator defined piecewise with respect to $\mathcal{T}_h$, i.e.,

$$\nabla_h v|_K := \nabla(v|_K).$$

Let K be an element of $\mathcal{T}_h$, and $\mathbf{x}_K^0 = (x_{0,K}, y_{0,K})$ be the center of K with the horizontal edge length $2h_{x,K}$ and vertical edge length $2h_{y,K}$. Define $\xi := \frac{x-x_{0,K}}{h_{x,K}}$ and $\eta := \frac{y-y_{0,K}}{h_{y,K}}$, then the rectangle K has another description

$$K = \{\mathbf{x} = (x, y)^T | x = x_{0,K} + \xi h_{x,K}, y = y_{0,K} + \eta h_{y,K}, -1 \leqslant \xi, \eta \leqslant 1\}. \tag{3}$$

For a measurable set $G \subset \Omega$, we use $P_k(G)$ to denote the space of all polynomials of degree no more than k and $Q_k(G)$ to denote the space of degree no more than k in each variable on the domain G. For a rectangular mesh, we first recall the conforming bilinear element space [1,20,21] before introducing the Wilson element space. Define the discontinuous finite element space on the 1-irregular mesh $\mathcal{T}_h$ as:

$$D_h := \{v \in L^2(\Omega); v|_K \in Q_1(K), \ \forall K \in \mathcal{T}_h\}, \tag{4}$$

then the conforming bilinear element space is $Q_h := D_h \cap H_0^1(\Omega)$. To keep the continuity of the functions in Q_h, we treat the unknowns corresponding to hanging nodes as spurious degrees of freedom, i.e., their values are fixed to be a suitable interpolation of the unknowns corresponding to neighboring regular nodes. Let v_1 be the nodal variable on the node A_1, and v_2 the nodal variable on the node A_2, then v_0, the nodal variable on the hanging node A_0, is determined by

$$v_0 = \frac{v_1 + v_2}{2}. \tag{5}$$

Define the nonconforming bubble function space

$$B_h := \{v \in L^2(\Omega); v|_K \in \text{span}\{1 - \xi^2, 1 - \eta^2\}, \ \forall K \in \mathcal{T}_h\}, \tag{6}$$

then the finite element space of the nonconforming Wilson element is defined as

$$V_h := Q_h + B_h.$$

The Wilson element approximation $u_h \in V_h$ of (2) then satisfies

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h dx = \int_\Omega f v_h dx, \quad \forall v_h \in V_h. \tag{7}$$

Remark 2.1. The Q_1 element space has another equivalent definition $Q_h = \{v \in H_0^1(\Omega); v|_K \in Q_1(K), \ \forall K \in \mathcal{T}_h\}$. However, the former is more convenient for implementation [21].

Let u_h be the solution to the discrete problem on the mesh $\mathcal{T}_h$, then u_h can be written as $u_h = u_h^c + u_h^b$, where $u_h^c \in Q_h$ and $u_h^b \in B_h$. The index c stands for the conforming part, and b, the bubble function which is the nonconforming part. For any $K \in \mathcal{T}_h$,

$$\begin{aligned} u_h^c|_K &= \frac{1}{4}(1-\xi)(1-\eta)u_h(A_1) + \frac{1}{4}(1+\xi)(1-\eta)u_h(A_2) + \frac{1}{4}(1-\xi)(1+\eta)u_h(A_3) + \frac{1}{4}(1+\xi)(1+\eta)u_h(A_4), \\ u_h^b|_K &= c_{x,K}(1-\xi^2) + c_{y,K}(1-\eta^2), \end{aligned} \tag{8}$$

where $c_{x,K} = -\frac{h_{x,K}^2}{2|K|}\int_K \frac{\partial^2 u_h}{\partial x^2}dxdy$, and $c_{y,K} = -\frac{h_{y,K}^2}{2|K|}\int_K \frac{\partial^2 u_h}{\partial y^2}dxdy$, and $u_h(A_i)$ are the values at the four vertices $A_i, \ i = 1, \ldots, 4$, of the element K as depicted in Fig. 1. Let $b_{x,K} = 1 - \xi^2$, $b_{y,K} = 1 - \eta^2$. In (7), we choose

$$v_h = \begin{cases} b_{x,K} & \mathbf{x} \in K, \\ 0 & \mathbf{x} \notin K. \end{cases}$$

This gives

$$(\nabla u_h, \nabla b_{x,K})_{L^2(K)} = (f, b_{x,K})_{L^2(K)}.$$

We recall that K is a rectangle. On the other hand, $u_h^c \in \text{span}\{1, \xi, \eta, \xi\eta\}$, $b_{x,K} \in \text{span}\{1 - \xi^2\}$, and $b_{y,K} \in \text{span}\{1 - \eta^2\}$. A direct calculation leads to the following important orthogonality:

$$(\nabla u_h^c, \nabla b_{x,K})_{L^2(K)} = 0, \qquad (\nabla b_{y,K}, \nabla b_{x,K})_{L^2(K)} = 0. \tag{9}$$

This leads to

$$c_{x,K}(\nabla b_{x,K}, \nabla b_{x,K})_{L^2(K)} = (f, b_{x,K})_{L^2(K)}.$$

Therefore,

$$c_{x,K} = \frac{3}{16}\frac{h_{x,K}}{h_{y,K}}\int_K f(1 - \xi^2)dxdy. \tag{10}$$

A similar argument shows that

$$c_{y,K} = \frac{3}{16}\frac{h_{y,K}}{h_{x,K}}\int_K f(1-\eta^2)dxdy. \tag{11}$$

Formulas (10) and (11) will play a crucial role in the analysis of this paper. We shall follow [22] to use the notation $\lesssim$ and $\approxeq$. When we write

$$A_1 \lesssim B_1, \quad \text{and} \quad A_2 \approxeq B_2,$$

then there exist possible constants C_1, c_2 and C_2 such that

$$A_1 \leqslant C_1 B_1, \quad \text{and} \quad c_2 B_2 \leqslant A_2 \leqslant C_2 B_2.$$

Define the canonical interpolation operator Π_K from $H^2(K)$ onto $P_K := Q_1(K) + \text{span}\{x^2, y^2\}$ as follows:

$$\begin{aligned}\Pi_K v := &\frac{1}{4}(1-\xi)(1-\eta)v(A_1) + \frac{1}{4}(1+\xi)(1-\eta)v(A_2)\\ &+ \frac{1}{4}(1-\xi)(1+\eta)v(A_3) + \frac{1}{4}(1+\xi)(1+\eta)v(A_4)\\ &+ (1-\xi^2)\left(-\frac{h_{x,K}^2}{2|K|}\int_K \frac{\partial^2 v}{\partial x^2}dxdy\right) + (1-\eta^2)\left(-\frac{h_{y,K}^2}{2|K|}\int_K \frac{\partial^2 v}{\partial y^2}dxdy\right),\end{aligned}$$

where $v(A_i)$ are the values at the four vertices A_i, $i = 1, \ldots, 4$, of the element K. We denote by Π_h the interpolation operator defined piecewise with respect to $\mathcal{T}_h$, i.e.,

$$\Pi_h v|_K := \Pi_K(v|_K), \quad \forall K \in \mathcal{T}_h.$$

The standard error estimate for the approximation of polynomials states

$$\|\nabla(v - \Pi_h v)\| \lesssim h_{\mathcal{T}_h}\|D^2 v\| \tag{12}$$

for any $v \in H^2(\Omega)$ where $D^2 v$ is the Hessian of v.

Theorem 2.2 (*A Priori Error Estimates*). *Let u and u_h be the solutions to problem (2) and problem (7), respectively. Suppose $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then it holds*

$$\|\nabla_h(u - u_h)\| \lesssim h_{\mathcal{T}_h}\|D^2 u\|, \tag{13}$$

where $h_{\mathcal{T}_h} := \max_{K\in\mathcal{T}_h} h_K$.

Proof. The Strang lemma gives the following estimate [19]

$$\|\nabla_h(u - u_h)\| \lesssim \inf_{v_h \in V_h}\|\nabla_h(u - v_h)\| + \sup_{0\neq w_h\in V_h}\frac{|(f, w_h) - a_h(u, w_h)|}{\|\nabla_h w_h\|}, \tag{14}$$

where $a_h(v, w) = \sum_{K\in\mathcal{T}_h} a(v|_K, w|_K)$, $\forall v, w \in H^1(\mathcal{T}_h)$. Since $-\Delta u = f$, integrating by parts yields

$$\begin{aligned}(f, w_h) - a_h(u, w_h) &= -(\Delta u, w_h) - a_h(u, w_h)\\ &= -\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial u}{\partial \nu}w_h ds,\end{aligned}$$

where $\nu = (\nu_x, \nu_y)$ is the unit normal vector of ∂K. By the definition of the space V_h, w_h has a decomposition $w_h = w_h^c + w_h^b$. Since $w_h^c \in H_0^1(\Omega)$, this implies

$$\begin{aligned}(f, w_h) - a_h(u, w_h) &= -\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial u}{\partial \nu}w_h^b ds\\ &= -\sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\frac{\partial u}{\partial x}\nu_x + \frac{\partial u}{\partial y}\nu_y\right)w_h^b ds.\end{aligned}$$

On the other hand, a direct calculation leads to

$$\begin{aligned}\int_{\partial K} w_h^b \nu_x ds &= \int_K \frac{\partial w_h^b}{\partial x}dxdy = 0,\\ \int_{\partial K} w_h^b \nu_y ds &= \int_K \frac{\partial w_h^b}{\partial y}dxdy = 0.\end{aligned}$$

Let P_K^0 be the orthogonal projection operator from $L^2(K)$ onto $P_0(K)$, this gives

$$\begin{aligned}|(f, w_h) - a_h(u, w_h)| &= \left|\sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\left(\frac{\partial u}{\partial x} - P_K^0\frac{\partial u}{\partial x}\right)(w_h - w_h^c)\nu_x + \left(\frac{\partial u}{\partial y} - P_K^0\frac{\partial u}{\partial y}\right)(w_h - w_h^c)\nu_y\right)ds\right| \\ &\leqslant \sum_{K\in\mathcal{T}_h}\left\|\frac{\partial u}{\partial x} - P_K^0\frac{\partial u}{\partial x}\right\|_{L^2(\partial K)}\|w_h - w_h^c\|_{L^2(\partial K)} + \sum_{K\in\mathcal{T}_h}\left\|\frac{\partial u}{\partial y} - P_K^0\frac{\partial u}{\partial y}\right\|_{L^2(\partial K)}\|w_h - w_h^c\|_{L^2(\partial K)} \\ &\lesssim \sum_{K\in\mathcal{T}_h} h_K^2\|D^2u\|_{L^2(K)}\|D^2w_h\|_{L^2(K)}.\end{aligned}$$

It follows from the inverse inequality and the Schwarz inequality that

$$|(f, w_h) - a_h(u, w_h)| \lesssim h_{\mathcal{T}_h}\|D^2u\|\cdot\|\nabla_h w_h\|. \tag{15}$$

A combination of (12), (14) and (15) completes the proof. □

Remark 2.3. The analysis herein is the extension of that in [23,24] to the mesh with hanging nodes.

2.3. A new a posteriori error estimator and its reliability and efficiency

Let ω_K denote the union of elements $K' \in \mathcal{T}_h$ that share a vertex, or an edge, or a child edge of an edge with K. Let ω_e denote the patch of elements having in common the edge e or one of the child edge of e, or share the hanging node as a vertex. Given any edge $e \in \mathcal{E}_h(\Omega)$ with the length h_e we assign one fixed unit normal $\nu_e := (\nu_x, \nu_y)$ and tangential vector $\tau_e := (-\nu_y, \nu_x)$. Once ν_e and τ_e have been fixed on e, in relation to ν_e one defines the element $K_- \in \mathcal{T}_h$ and $K_+ \in \mathcal{T}_h$, with $e = K_+ \cap K_-$. Given $e \in \mathcal{E}_h(\Omega)$ and some $\mathbb{R}^d$-valued function v defined in Ω with $d = 1, 2$, we denote by $[v] := (v|_{K_+})|_e - (v|_{K_-})|_e$ the jump of v across e.

Before introducing a new estimator, we first recall the residual-type a posteriori error estimator in the literature, see for instance [6]

$$\hat{\eta}^2(u_h, \mathcal{T}_h) := \sum_{K\in\mathcal{T}_h} h_K^2\|f + \Delta u_h\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h(\Omega)} h_e\|[\nabla u_h]\|_{L^2(e)}^2 + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e\left\|\frac{\partial u_h}{\partial\tau_e}\right\|_{L^2(e)}^2.$$

For this estimator, we have the following reliability and efficiency [6].

Lemma 2.4. *Let u be the solution of problem (2), and u_h be the solution of problem (7). Then*

$$\|\nabla_h(u - u_h)\|^2 \lesssim \hat{\eta}^2(u_h, \mathcal{T}_h) \lesssim \|\nabla_h(u - u_h)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_h). \tag{16}$$

For the estimator $\hat{\eta}$, we cannot show the convergence of the adaptive algorithm because the unknown u_h is involved in the interior residual $f + \Delta u_h$. The remedy is to propose a new estimator by dropping the term Δu_h. This results in the following estimator

$$\eta_K := h_K\|f\|_{L^2(K)} + \left(\sum_{\mathcal{E}_h(\Omega)\ni e\subset\partial K} h_e\|[\nabla_h u_h]\|_{L^2(e)}^2 + \sum_{\mathcal{E}_h(\Gamma)\ni e\subset\partial K} h_e\left\|\frac{\partial u_h}{\partial\tau_e}\right\|_{L^2(e)}^2\right)^{\frac{1}{2}}, \tag{17}$$

for any $K \in \mathcal{T}_h$. For any $S_h \subset \mathcal{T}_h$, we define the estimator over S_h by

$$\eta^2(u_h, S_h) := \sum_{K\in S_h}\eta_K^2.$$

In particular, for $S_h = \mathcal{T}_h$, we have

$$\eta^2(u_h, \mathcal{T}_h) := \sum_{K\in\mathcal{T}_h}\eta_K^2.$$

We further define the oscillation $\mathrm{osc}(f, \mathcal{T}_h)$ by

$$\mathrm{osc}^2(f, \mathcal{T}_h) := \sum_{K\in\mathcal{T}_h} h_K^2\|f - f_K\|_{L^2(K)}^2,$$

where $f_K \in P_0(K)$ is the constant projection of f over K.

Lemma 2.5. *Let u_h be the solution to the discrete problem (7) on the mesh $\mathcal{T}_h$. For any $K \in \mathcal{T}_h$, there exists a positive constant C such that*

$$\|f\|_{L^2(K)} - C\|f - f_K\|_{L^2(K)} \leqslant \|f + \Delta u_h\|_{L^2(K)} \leqslant \|f\|_{L^2(K)} + C\|f - f_K\|_{L^2(K)}. \tag{18}$$

Proof. Since u_h is the solution to the discrete problem (7) on the mesh $\mathcal{T}_h$, u_h has a decomposition $u_h = u_h^c + u_h^b$ with $u_h^c \in Q_h$ and $u_h^b \in B_h$. Since $\Delta u_h^c|_K = 0$,

$$\begin{aligned} f + \Delta u_h|_K &= f + \Delta u_h^b|_K \\ &= f + c_{x,K}\frac{\partial^2 b_{x,K}}{\partial x^2} + c_{y,K}\frac{\partial^2 b_{y,K}}{\partial y^2} \\ &= f - \frac{2}{h_{x,K}^2}c_{x,K} - \frac{2}{h_{y,K}^2}c_{y,K}. \end{aligned}$$

Note that $c_{x,K}$ and $c_{y,K}$ have already been computed in (10) and (11). This implies

$$\begin{aligned} f + \Delta u_h|_K &= f - \frac{3}{8}\frac{h_{x,K}}{h_{y,K}}\frac{1}{h_{x,K}^2}\int_K f(1-\xi^2)dxdy - \frac{3}{8}\frac{h_{y,K}}{h_{x,K}}\frac{1}{h_{y,K}^2}\int_K f(1-\eta^2)dxdy \\ &= 2(f - f_K) - f - \frac{3}{8}\frac{1}{h_{x,K}h_{y,K}}\int_K (f - f_K)(1-\xi^2)dxdy - \frac{3}{8}\frac{1}{h_{x,K}h_{y,K}}\int_K (f - f_K)(1-\eta^2)dxdy. \end{aligned}$$

Then

$$\|f + \Delta u_h\|_{L^2(K)} = \left\| 2(f - f_K) - f - \frac{3}{8}\frac{1}{h_{x,K}h_{y,K}}\int_K (f - f_K)(1-\xi^2)dxdy - \frac{3}{8}\frac{1}{h_{x,K}h_{y,K}}\int_K (f - f_K)(1-\eta^2)dxdy \right\|_{L^2(K)}.$$

An application of the triangle inequality completes the proof. □

By Lemmas 2.4, 2.5 and the fact $\operatorname{osc}(f, \mathcal{T}_h) \leqslant \eta(u_h, \mathcal{T}_h)$ we have the following reliability and efficiency of the estimator η.

Theorem 2.6. *Let u be the solution of problem (2), and u_h be the solution of problem (7). Then*

$$\|\nabla_h(u - u_h)\|^2 \leqslant C_{\mathrm{Rel}}\eta^2(u_h, \mathcal{T}_h) \leqslant C_{\mathrm{Eff}}\big(\|\nabla_h(u - u_h)\|^2 + \operatorname{osc}^2(f, \mathcal{T}_h)\big). \tag{19}$$

3. Convergence

In this section, we shall establish the convergence of our adaptive nonconforming finite element method. First we present our adaptive nonconforming finite element method. In doing this, we replace the dependence on the actual rectangular mesh $\mathcal{T}$ by the iteration counter k. Correspondingly, we use the notations Q_k, B_k, ∇_k instead of Q_h, B_h, ∇_h.

Algorithm 3.1. Given the initial mesh $\mathcal{T}_0$ and marking parameter $0 < \theta < 1$, set $k = 0$ and iterate

(1) Solve on $\mathcal{T}_k$, to get the solution u_k.
(2) Compute the error estimator $\eta = \eta(u_k, \mathcal{T}_k)$.
(3) Mark the minimal element set $\mathcal{M}_k$ such that

$$\eta^2(u_k, \mathcal{M}_k) \geqslant \theta\eta^2(u_k, \mathcal{T}_k). \tag{20}$$

(4) Refine each rectangle $K \in \mathcal{M}_k$, and refine any element for which any of the sides contains more than one irregular nodes; $k = k + 1$.

Next we show the quasi-orthogonality for the Wilson element. Quasi-orthogonality is essential for the convergence analysis, but for most nonconforming finite element methods, whether the quasi-orthogonality holds is not clear. In the known results, based on some special equivalency between the nonconforming linear element and the lowest order Raviart–Thomas element, the quasi-orthogonality for the nonconforming linear element of the Poisson equation was first established in [13]. For the Morley element of the fourth order elliptic problem, the conservative property is used to analyze the quasi-orthogonality. This is also extended to the nonconforming linear element therein, see also [17,18]. Neither could be used for the Wilson element under consideration, so we need to find a new way to handle with it. The key ingredient is to use the orthogonality of (9).

Lemma 3.2 (*Quasi-Orthogonality*). *Let $\mathcal{T}_k$ be some refinement of $\mathcal{T}_{k-\ell}$. Given any constant $\delta \in (0, 1)$, there exists a positive constant $C(\delta)$ such that*

$$(1-\delta)\|\nabla_k(u - u_k)\|^2 \leqslant \|\nabla_{k-\ell}(u - u_{k-\ell})\|^2 - \|\nabla_k(u_k - u_{k-\ell})\|^2 + C(\delta)\sum_{K\in\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k} h_K^2\|f\|_{L^2(K)}^2. \tag{21}$$

Proof. We use the decompositions $u_k = u_k^c + u_k^b$ and $u_{k-\ell} = u_{k-\ell}^c + u_{k-\ell}^b$ where $u_i^c \in Q_i$, $u_i^b \in B_i$, $i = k-\ell, k$. Since $(\nabla u - \nabla_k u_k, \nabla(u_k^c - u_{k-\ell}^c)) = 0$,

$$(\nabla u - \nabla_k u_k, \nabla_k u_k - \nabla_k u_{k-\ell}) = (\nabla u - \nabla_k u_k, \nabla_k u_k^b - \nabla_{k-\ell} u_{k-\ell}^b).$$

Notice that the restriction of the nonconforming part u_i^b on any element $K \in \mathcal{T}_i,\ i = k-\ell, k$ can be determined only by the source term f, so we have $\nabla_k u_k^b|_K = \nabla_{k-\ell} u_{k-\ell}^b|_K$ for $K \in \mathcal{T}_{k-\ell} \cap \mathcal{T}_k$. This gives

$$|(\nabla u - \nabla_k u_k, \nabla_k u_k - \nabla_k u_{k-\ell})| = \left| \sum_{K \in \mathcal{T}_{k-\ell} \setminus \mathcal{T}_k} (\nabla u - \nabla_k u_k, \nabla_k u_k^b - \nabla_{k-\ell} u_{k-\ell}^b) \right|. \tag{22}$$

For $K \in \mathcal{T}_{k-\ell} \setminus \mathcal{T}_k$, K is refined into elements $K_1, \dots, K_M,\ M \geqslant 4$, a combination of (8), (10) and (11) leads to

$$\begin{aligned}
\left\| \frac{\partial u_{k-\ell}^b}{\partial x} \right\|_{L^2(K)} &= \left\| c_{x,K} \frac{\partial b_{x,K}}{\partial x} \right\|_{L^2(K)} \\
&= \left\| -\frac{3\xi}{8h_{y,K}} \int_K f(1-\xi^2) dxdy \right\|_{L^2(K)} \\
&\lesssim \int_K f(1-\xi^2) dxdy \\
&= \|f\|_{L^2(K)} \|1-\xi^2\|_{L^2(K)} \\
&\lesssim h_K \|f\|_{L^2(K)},
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\left\| \frac{\partial u_k^b}{\partial x} \right\|_{L^2(K)}^2 &= \sum_{i=1}^M \left\| \frac{\partial u_k^b}{\partial x} \right\|_{L^2(K_i)}^2 \\
&\lesssim \sum_{i=1}^M h_{K_i}^2 \|f\|_{L^2(K_i)}^2 \\
&\lesssim h_K^2 \|f\|_{L^2(K)}^2.
\end{aligned} \tag{24}$$

From (22)–(24) we get

$$|(\nabla u - \nabla_k u_k, \nabla_k u_k - \nabla_k u_{k-\ell})| \leqslant C_{QO} \sum_{K \in \mathcal{T}_{k-\ell} \setminus \mathcal{T}_k} h_K \|\nabla u - \nabla_k u_k\|_{L^2(K)} \|f\|_{L^2(K)} \tag{25}$$

for some positive constant C_{QO}. The desired result follows from the Young inequality. □

Lemma 3.3 (*Corollary* 3.4, *[8], Lemma* 4.5, *[21]*). *Let $\mathcal{T}_k$ be some refinement of $\mathcal{T}_{k-1}$ with the bulk criterion (20), then there exist $\rho > 0$ and a positive constant $\beta \in (1-\rho\theta, 1)$ such that*

$$\eta^2(u_{k-1}, \mathcal{T}_k) \leqslant \beta \eta^2(u_{k-1}, \mathcal{T}_{k-1}) + (1-\rho\theta-\beta)\eta^2(u_{k-1}, \mathcal{T}_{k-1}). \tag{26}$$

Lemma 3.4. *Let $\mathcal{T}_k$ be some refinement of $\mathcal{T}_{k-\ell}$, then there exists $\rho > 0$ such that*

$$\sum_{K \in \mathcal{T}_k} h_K^2 \|f\|_{L^2(K)}^2 \leqslant \sum_{K \in \mathcal{T}_{k-\ell}} h_K^2 \|f\|_{L^2(K)}^2 - \rho \sum_{K \in \mathcal{T}_{k-\ell} \setminus \mathcal{T}_k} h_K^2 \|f\|_{L^2(K)}^2. \tag{27}$$

Proof. The proof immediately follows from the definition of the mesh size h_K. □

Lemma 3.5. *Let u_k and u_{k-1} be the solutions to the discrete problem (7) on the meshes $\mathcal{T}_k$ and $\mathcal{T}_{k-1}$, respectively. Given any positive constant ε, there exists a positive constant $\beta_2(\varepsilon)$ dependent on ε such that*

$$\eta^2(u_k, \mathcal{T}_k) \leqslant (1+\varepsilon)\eta^2(u_{k-1}, \mathcal{T}_k) + \frac{1}{\beta_2(\varepsilon)} \|\nabla_k (u_k - u_{k-1})\|^2. \tag{28}$$

Proof. Given any $K \in \mathcal{T}_k$, let

$$\eta_K(u_k) := h_K \|f\|_{L^2(K)} + \left(\sum_{\mathcal{E}_k(\Omega) \ni e \subset \partial K} h_e \|[\nabla_k u_k]\|_{L^2(e)}^2 + \sum_{\mathcal{E}_k(\Gamma) \ni e \subset \partial K} h_e \left\| \frac{\partial u_k}{\partial \tau_e} \right\|_{L^2(e)}^2 \right)^{\frac{1}{2}}. \tag{29}$$

It follows from the definitions of $\eta_K(u_k)$ and $\eta_K(u_{k-1})$ that

$$
\begin{aligned}
|\eta_K(u_k)-\eta_K(u_{k-1})| = &\Bigg|\left(\sum_{\mathcal{E}_k(\Omega)\ni e\subset\partial K} h_K\|[\nabla_k u_k]\|^2_{L^2(e)} + \sum_{\mathcal{E}_k(\Gamma)\ni e\subset\partial K} h_e\left\|\frac{\partial u_k}{\partial\tau_e}\right\|^2_{L^2(e)}\right)^{\frac12} \\
&-\left(\sum_{\mathcal{E}_k(\Omega)\ni e\subset\partial K} h_K\|[\nabla_{k-1} u_{k-1}]\|^2_{L^2(e)} + \sum_{\mathcal{E}_k(\Gamma)\ni e\subset\partial K} h_e\left\|\frac{\partial u_{k-1}}{\partial\tau_e}\right\|^2_{L^2(e)}\right)^{\frac12}\Bigg| \\
\leqslant &\left(\sum_{\mathcal{E}_k(\Omega)\ni e\subset\partial K} h_K\|[\nabla_k (u_k-u_{k-1})]\|^2_{L^2(e)} + \sum_{\mathcal{E}_k(\Gamma)\ni e\subset\partial K} h_e\left\|\frac{\partial (u_k-u_{k-1})}{\partial\tau_e}\right\|^2_{L^2(e)}\right)^{\frac12}.
\end{aligned}
$$

Let $e = K\cap K'$. Then we use the trace theorem and inverse inequality to get

$$\|\nabla_k(u_k-u_{k-1})|_K\|_{L^2(e)} \lesssim h_K^{-1/2}\|\nabla_k(u_k-u_{k-1})\|_{L^2(K)}.$$

The same argument holds for K'. This gives

$$\|[\nabla_k(u_k-u_{k-1})]\|_{L^2(e)} \lesssim h_K^{-1/2}\|\nabla_k(u_k-u_{k-1})\|_{L^2(\omega_e)}$$

which leads to

$$|\eta_K(u_k)-\eta_K(u_{k-1})| \lesssim \|\nabla_k(u_k-u_{k-1})\|_{L^2(\omega_K)}.$$

A summary over all elements in $\mathcal{T}_k$ plus the Young inequality complete the proof. □

With aforementioned lemmas, we shall show the convergence of our adaptive method. Generally, the energy error between two levels are not strictly monotone, so we borrow the concept of total error of [16] which contains the energy norm of the error, the scaled estimator and volume term. By showing reduction of the total error, we obtain the convergence.

Theorem 3.6. *Let u be the solution to the problem (2), and u_k and u_{k-1} be the solutions to the discrete problem (7) on the meshes $\mathcal{T}_k$ and $\mathcal{T}_{k-1}$, respectively. Then there exist positive constants γ, β_1 and $0<\alpha<1$ with*

$$\epsilon_k \leqslant \alpha\epsilon_{k-1} \tag{30}$$

where

$$\epsilon_k = \|\nabla_k(u-u_k)\|^2 + \gamma\eta^2(u_k,\mathcal{T}_k) + \beta_1\sum_{K\in\mathcal{T}_k} h_K^2\|f\|^2_{L^2(K)}. \tag{31}$$

Proof. By Lemmas 3.2–3.5, we have for any positive constant $0<\alpha<1$

$$
\begin{aligned}
\epsilon_k-\alpha\epsilon_{k-1} \leqslant &\left(\frac{1}{1-\delta}-\alpha\right)\|\nabla_{k-1}(u-u_{k-1})\|^2 + \gamma((1+\varepsilon)(1-\rho\theta-\beta)+\varepsilon\beta)\eta^2(u_{k-1},\mathcal{T}_{k-1}) \\
&+(\gamma\beta-\alpha\gamma)\eta^2(u_{k-1},\mathcal{T}_{k-1}) + \left(\frac{\gamma}{\beta_2(\varepsilon)}-\frac{1}{1-\delta}\right)\|\nabla_k(u_k-u_{k-1})\|^2 \\
&+(\beta_1-\alpha\beta_1)\sum_{K\in\mathcal{T}_{k-1}} h_K^2\|f\|^2_{L^2(K)} + \left(\frac{C(\delta)}{1-\delta}-\rho\beta_1\right)\sum_{K\in\mathcal{T}_{k-1}\backslash\mathcal{T}_k} h_K^2\|f\|^2_{L^2(K)}
\end{aligned}
$$

where δ, γ, β and β_1 are four positive constants to be chosen later. We first set

$$\gamma = \frac{\beta_2(\varepsilon)}{1-\delta},\qquad \beta_1 = \frac{C(\delta)}{\rho(1-\delta)},\quad \text{and}\quad \beta = (1-\rho\theta)(1+\varepsilon),$$

which leads to

$$\epsilon_k-\alpha\epsilon_{k-1} \leqslant \left(\frac{1}{1-\delta}-\alpha\right)\|\nabla_{k-1}(u-u_{k-1})\|^2 + (\gamma\beta-\alpha\gamma)\eta^2(u_{k-1},\mathcal{T}_{k-1}) \;+(\beta_1-\alpha\beta_1)\sum_{K\in\mathcal{T}_{k-1}} h_K^2\|f\|^2_{L^2(K)}.$$

We choose ε to be small enough such that $0<\beta<1$. So we obtain the reduction of the total error if the following inequality holds

$$\beta_1(1-\alpha)\sum_{K\in\mathcal{T}_{k-1}} h_K^2\|f\|^2_{L^2(K)} + \left(\frac{1}{1-\delta}-\alpha\right)\|\nabla_{k-1}(u-u_{k-1})\|^2 + \gamma(\beta-\alpha)\eta^2(u_{k-1},\mathcal{T}_{k-1}) \leqslant 0.$$

By applying the upper bound of Theorem 2.6

$$\|\nabla_{k-1}(u-u_{k-1})\|^2 \leqslant C_{\mathrm{Rel}}\eta^2(u_{k-1},\mathcal{T}_{k-1})$$

and the fact that

$$\sum_{K\in\mathcal{T}_{k-1}} h_K^2\|f\|_{L^2(K)}^2 \leqslant \eta^2(u_{k-1},\mathcal{T}_{k-1})$$

we get

$$\begin{aligned}&\beta_1(1-\alpha)\sum_{K\in\mathcal{T}_{k-1}} h_K^2\|f\|_{L^2(K)}^2+\left(\frac{1}{1-\delta}-\alpha\right)\|\nabla_{k-1}(u-u_{k-1})\|^2+\gamma(\beta-\alpha)\eta^2(u_{k-1},\mathcal{T}_{k-1})\\&\quad\leqslant\left(\left(\frac{1}{1-\delta}-\alpha\right)C_{\mathrm{Rel}}+\gamma(\beta-\alpha)+\beta_1(1-\alpha)\right)\eta^2(u_{k-1},\mathcal{T}_{k-1}),\end{aligned}$$

provided that $0<\delta<1$. This implies that we should choose the error reduction rate $\alpha=\frac{\beta_1+\gamma\beta+\frac{C_{\mathrm{Rel}}}{1-\delta}}{\gamma+\beta_1+C_{\mathrm{Rel}}}>\beta$. The choice of $0<\delta<\frac{\gamma-\gamma\beta}{\gamma-\gamma\beta+C_{\mathrm{Rel}}}$ assures that $\alpha<1$, which completes the proof. □

4. Discrete reliability

In this section we analyze the discrete reliability of the estimator η. Based on the decomposition of the Wilson element space, we can use the Scott–Zhang interpolation for estimating the conforming part and a similar method used in Lemma 3.2 for the nonconforming part.

Lemma 4.1. *Let $\mathcal{T}_k$ be some refinement of $\mathcal{T}_{k-\ell}$, and let u_k and $u_{k-\ell}$ be the solutions to the discrete problem (7) on the meshes $\mathcal{T}_k$ and $\mathcal{T}_{k-\ell}$, respectively. Then it holds that*

$$\|\nabla_k(u_k-u_{k-\ell})\|^2 \leqslant C_{\mathrm{Drel}}\eta^2(u_{k-\ell},\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k). \tag{32}$$

Proof. We use the decompositions $u_i=u_i^c+u_i^b$, where $u_i^c\in Q_i$, $u_i^b\in B_i$, $i=k-\ell,k$.

$$\begin{aligned}\|\nabla_k(u_k-u_{k-\ell})\|^2 &= (\nabla_k(u_k-u_{k-\ell}),\nabla_k(u_k^c-u_{k-\ell}^c))+(\nabla_k(u_k-u_{k-\ell}),\nabla_k(u_k^b-u_{k-\ell}^b))\\&=(f,v_k^c)-(\nabla_k u_{k-\ell},\nabla v_k^c)+(\nabla_k(u_k-u_{k-\ell}),\nabla_k(u_k^b-u_{k-\ell}^b))\end{aligned} \tag{33}$$

where $v_k^c=u_k^c-u_{k-\ell}^c$. To estimate the first part, we employ the Scott–Zhang interpolation operator $\mathcal{J}:Q_k\to Q_{k-\ell}$,which can be found in [20,25,21], which has the following properties:

$$\mathcal{J}v_k^c|_K=v_k^c|_K,\quad \text{for any } K\in\mathcal{T}_k\cap\mathcal{T}_{k-\ell}, \tag{34}$$

$$\|\nabla\mathcal{J}v_k^c\|_{L^2(K)}+\|h_K^{-1}(v_k^c-\mathcal{J}v_k^c)\|_{L^2(K)}\lesssim\|\nabla v_k^c\|_{L^2(\omega_K)}, \tag{35}$$

$$\|h_e^{-1/2}(v_k^c-\mathcal{J}v_k^c)\|_{L^2(e)}\lesssim\|\nabla v_k^c\|_{L^2(\omega_e)}. \tag{36}$$

Since $\mathcal{J}v_k^c\in Q_{k-\ell}\subset V_{k-\ell}$, from (34)–(36) we have

$$\begin{aligned}(f,v_k^c)-(\nabla_k u_{k-\ell},\nabla v_k^c) &= (f,v_k^c-\mathcal{J}v_k^c)-(\nabla_k u_{k-1},\nabla(v_k^c-\mathcal{J}v_k^c))\\&=(f+\Delta_k u_{k-\ell},v_k^c-\mathcal{J}v_k^c)-\sum_{K\in T_k}\int_{\partial K}\nabla_k u_{k-\ell}\cdot\nu(v_k^c-\mathcal{J}v_k^c)ds\\&\leqslant\sum_{K\in\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k}\|f+\Delta_k u_{k-\ell}\|_{L^2(K)}\|v_k^c-\mathcal{J}v_k^c\|_{L^2(K)}\\&\quad+\sum_{K\in\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k}\sum_e\|[\nabla_{k-\ell}u_{k-\ell}]\cdot\nu_e\|_{L^2(e)}\|v_k^c-\mathcal{J}v_k^c\|_{L^2(e)}\\&\lesssim\eta(u_{k-\ell},\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k)\|\nabla v_k^c\|\\&\lesssim\eta(u_{k-\ell},\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k)(\|\nabla_k v_k\|+\|\nabla_k v_k^b\|).\end{aligned} \tag{37}$$

In the second inequality, we use the fact $\|f+\Delta_k u_{k-\ell}\|_{L^2(K)}\lesssim\|f\|_{L^2(K)}$. From the proof of Lemma 3.2, we have

$$\|\nabla_k(u_k^b-u_{k-\ell}^b)\|\lesssim\eta(u_{k-\ell},\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k). \tag{38}$$

A summary of (33), (37) and (38) proves the desired result. □

By applying the discrete reliability we find some connection between the energy error and bulk criterion. We omit the proof here which can be found in [16].

Lemma 4.2. *Let $\mathcal{T}_k$ be some refinement of $\mathcal{T}_{k-\ell}$ such that the following reduction holds*

$$\|\nabla_k(u-u_k)\|^2 + \operatorname{osc}^2(f,\mathcal{T}_k) \leqslant \alpha'(\|\nabla_{k-\ell}(u-u_{k-\ell})\|^2 + \operatorname{osc}^2(f,\mathcal{T}_{k-\ell})), \tag{39}$$

for some $0<\alpha'<1$, *then there exists* $0<\theta_* = \frac{(1-\alpha')^2 C_{\text{Eff}}}{2(2\alpha'(C_{QO})^2+(1-\alpha')(C_{\text{Drel}}+1))} < 1$ *such that*

$$\theta_*\eta^2(u_{k-\ell},\mathcal{T}_{k-\ell}) \leqslant \eta^2(u_{k-\ell},\mathcal{T}_{k-\ell}\setminus\mathcal{T}_k).$$

5. Optimality

For the analysis of the optimality, we introduce some notation from nonlinear approximation theory. Let $\mathbb{T}_N$ be the set of all possible rectangular meshes satisfying one-irregular rule generated from $\mathcal{T}_0$ with at most N elements more than $\mathcal{T}_0$. For $s>0$ we define the nonlinear approximation class $\mathbb{A}_s$ as

$$\mathbb{A}_s := \{(u,f)\,|\,|u,f|_s := \sup_{N>0} N^s\sigma(N;u,f) < +\infty\}$$

with

$$\sigma(N;u,f) := \inf_{\mathcal{T}\in\mathbb{T}_N}\inf_{v\in V_{\mathcal{T}}}(\|\nabla_{\mathcal{T}}(u-v)\|^2 + R^2(u) + \operatorname{osc}^2(f,\mathcal{T}))$$

where

$$R(u) = \sup_{0\neq v\in V_{\mathcal{T}}}\frac{(\nabla u,\nabla_{\mathcal{T}}v)-(f,v)}{\|\nabla_{\mathcal{T}}v\|}.$$

Lemma 5.1. *Let $\mathcal{T}_k$ be some refinement of $\mathcal{T}_{k-\ell}$, u be the solution to the problem (2), and u_k and $u_{k-\ell}$ be the solutions to the discrete problem (7) on the meshes $\mathcal{T}_k$ and $\mathcal{T}_{k-\ell}$, respectively. Then there exists a constant $C_1>0$ such that*

$$\|\nabla(u-u_k)\|^2 + \operatorname{osc}^2(f,\mathcal{T}_k) \leqslant C_1\big(\|\nabla(u-u_{k-\ell})\|^2 + \operatorname{osc}^2(f,\mathcal{T}_{k-\ell})\big). \tag{40}$$

Proof. In view of inequalities (21) and (27), we get

$$\begin{aligned}(1-\delta)\|\nabla u-\nabla_k u_k\|^2 + \operatorname{osc}^2(f,\mathcal{T}_k) &\leqslant \|\nabla u-\nabla_{k-\ell}u_{k-\ell}\|^2 + (C(\delta)+1)\sum_{K\in\mathcal{T}_{k-\ell}} h_K^2\|f\|^2_{L^2(K)}\\ &\leqslant \|\nabla u-\nabla_{k-\ell}u_{k-\ell}\|^2 + (C(\delta)+1)\eta^2(u_{k-\ell},\mathcal{T}_{k-\ell}).\end{aligned}$$

This and inequality (19) complete the proof. □

Theorem 5.2. *Let $\mathcal{M}_k$ be a set of marked elements with minimal cardinality from Algorithm 3.1, u the solution of problem (7), and $(\mathcal{T}_k, V_k, u_k)$ the sequence of meshes, finite element spaces and discrete solutions produced by the adaptive finite element methods with* $0<\theta<\frac{C_{\text{Eff}}}{4(2(C_{QO})^2+C_{\text{Drel}}+1)}$. *It holds that*

$$\#\mathcal{M}_k \lesssim (\alpha')^{-\frac{1}{s}}|u,f|_s^{\frac{1}{s}}(C_1)^{\frac{1}{s}}(\|\nabla_k(u-u_k)\|^2 + \operatorname{osc}^2(f,\mathcal{T}_k))^{-\frac{1}{s}}, \tag{41}$$

for any $\alpha'\in(0,\frac{1}{2})$.

Proof. We set $\varepsilon = \alpha'(C_1)^{-1}(\|\nabla_k(u-u_k)\|^2+\operatorname{osc}^2(f,\mathcal{T}_k))$ with $0<\alpha'<\frac{1}{2}$. Since $(u,f)\in\mathbb{A}_s$, there exists a $\mathcal{T}_\varepsilon$ of the refinement of $\mathcal{T}_0$ and $u_\varepsilon\in V_{\mathcal{T}_\varepsilon}$ such that

$$\begin{aligned}&\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \leqslant |u,f|_s^{\frac{1}{s}}\varepsilon^{-\frac{1}{s}},\\ &\|\nabla_{\mathcal{T}_\varepsilon}(u-u_\varepsilon)\|^2 + \operatorname{osc}^2(f,\mathcal{T}_\varepsilon) < \varepsilon.\end{aligned} \tag{42}$$

Let $\mathcal{T}_*$ be the overlay of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_k$, and let u_* be the discrete solution of problem (7) on $\mathcal{T}_*$. Since $\mathcal{T}_*$ is a refinement of $\mathcal{T}_\varepsilon$, from (40) and (42) we have

$$\begin{aligned}\|\nabla_{\mathcal{T}_*}(u-u_*)\|^2 + \operatorname{osc}^2(f,\mathcal{T}_*) &\leqslant C_1(\|\nabla_{\mathcal{T}_\varepsilon}(u-u_\varepsilon)\|^2 + \operatorname{osc}^2(f,\mathcal{T}_\varepsilon))\\ &\leqslant C_1\varepsilon = \alpha'(\|\nabla_k(u-u_k)\|^2 + \operatorname{osc}^2(f,\mathcal{T}_k)).\end{aligned}$$

Hence, we deduce from Lemma 4.2 that

$$\theta_*\eta^2(u_k,\mathcal{T}_k) \leqslant \eta^2(u_k,\mathcal{T}_k\setminus\mathcal{T}_*),$$

where $\theta_*\in(0,1)$. We note that the step (3) in Algorithm 3.1 with $\theta\leqslant\theta_*$ chooses a subset of $\mathcal{M}_k\subset\mathcal{T}_k$ with minimal cardinality with the same property. Therefore, from [26, Lemma 4.3] and [27, Lemma 6.7],

$$\#\mathcal{M}_k \leqslant \#\mathcal{T}_* - \#\mathcal{T}_k \leqslant \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0. \tag{43}$$

Finally, by combining (43) and the definition of ε, we end up with

$$\#\mathcal{M}_k \lesssim (\alpha')^{-\frac{1}{s}} |u, f|_s^{\frac{1}{s}} (C_1)^{\frac{1}{s}} (\|\nabla_k(u-u_k)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_k))^{-\frac{1}{s}}. \quad \square$$

Theorem 5.3. *Let the marking step in Algorithm 3.1 select a set $\mathcal{M}_k$ of marked elements with minimal cardinality, u the solution to problem (2), and $(\mathcal{T}_k, V_k, u_k)$ the sequence of meshes, finite element spaces and discrete solutions produced by the adaptive finite element methods with $0 < \theta < \frac{C_{\mathrm{Eff}}}{4(2(C_{\mathrm{QO}})^2 + C_{\mathrm{Drel}}+1)}$. Then it holds that*

$$\|\nabla_{T_N}(u-u_N)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_N) \lesssim |u, f|_s (\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s} \quad \textit{for } (u, f) \in \mathbb{A}_s. \tag{44}$$

Proof. Let $\mu = (\alpha')^{-\frac{1}{s}} |u, f|_s^{\frac{1}{s}} (C_1)^{\frac{1}{s}}$. We use the result that $\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{k-1} \#\mathcal{M}_j$ from [27, Lemma 6.5] (see also [9] for meshes without hanging nodes) to obtain that

$$\#\mathcal{T}_N - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{N-1} \#\mathcal{M}_j \lesssim \mu \sum_{j=0}^{N-1} (\|\nabla_{T_j}(u-u_j)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_j))^{-\frac{1}{s}}. \tag{45}$$

The fact that

$$\|\nabla_j(u-u_j)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_j) \approxeq \epsilon_j$$

gives

$$\epsilon_j \lesssim \|\nabla_j(u-u_j)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_j). \tag{46}$$

For any $0 \leqslant j \leqslant N-1$, we use the convergence result from Theorem 3.6 to derive that

$$\epsilon_N \leqslant \alpha^{(N-j)} \epsilon_j. \tag{47}$$

A summary of (45)–(47) yields

$$\#\mathcal{T}_N - \#\mathcal{T}_0 \lesssim \mu (\|\nabla_{T_N}(u-u_N)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_N))^{-\frac{1}{s}} \sum_{j=1}^{N} \alpha^{\frac{j}{s}}.$$

Since $\alpha < 1$, the geometric series is bounded by the constant $C_\theta = \alpha^{1/s}(1-\alpha^{1/s})^{-1}$. This leads to

$$\|\nabla_N(u-u_N)\|^2 + \mathrm{osc}^2(f, \mathcal{T}_N) \lesssim |u, f|_s (\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s}$$

which completes the proof. $\square$

6. Numerical experiments

6.1. Example 1

Consider the domain $\Omega = [0, 1] \times [0, 1]$ and $f = 2\pi^2 \sin(\pi x)\sin(\pi y)$ with a vanishing Dirichlet boundary condition. The exact solution is $u = \sin(\pi x)\sin(\pi y)$. We scale the estimator η with the factor $\theta = 0.5$. Fig. 2 displays the grid when the degrees of freedom (DOFs) are more than 10^4. Fig. 3 displays experimental convergence rates for the true error and the estimator η for the adaptive refinement with the corresponding mesh depicted in Fig. 2. The convergence rate of the adaptive refinement is the optimal one, $O(n^{-1/2})$, with respect to the number of degrees of freedom.

6.2. Example 2

On the L-shaped domain $\Omega = [-0.5, 0.5] \times [-0.5, 0.5] \setminus [0, 0.5] \times [-0.5, 0]$, let $f = 0$ and u_D a smooth function such that in polar coordinates

$$u(r, \theta) = r^{2/3} \sin\left(\frac{2}{3}\theta\right)$$

is the exact solution of problem (1). Fig. 4 displays the grid when the degrees of freedom are more than 10^4. We find that there is a local higher refinement towards the reentrant corner. Fig. 5 shows the rate of convergence is optimal.

6.3. Example 3

Consider an interesting domain which is $\Omega = [0, 1] \times [0, 1] \setminus [0.25, 0.75] \times [0.5, 0.5] \setminus [0.5, 0.5] \times [0.25, 0.75]$. Let $u_D = 0$ and $f = 1$. The exact solution is unknown, but we can guess there must be a local higher refinement towards every reentrant corner. Fig. 6 confirms our guess.

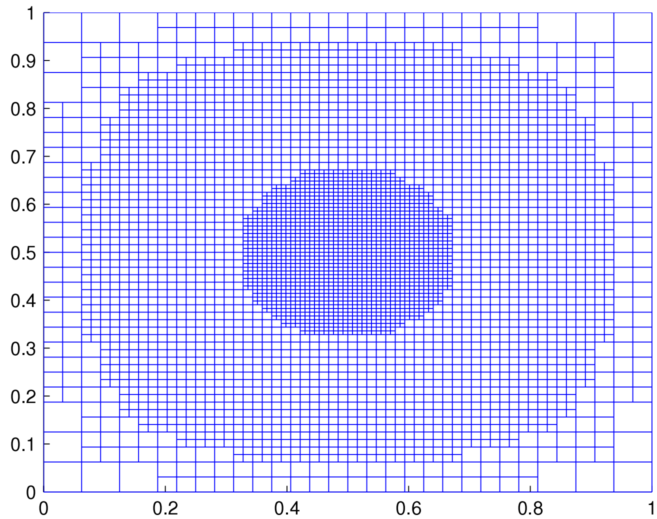

Fig. 2. Adaptive mesh refinement.

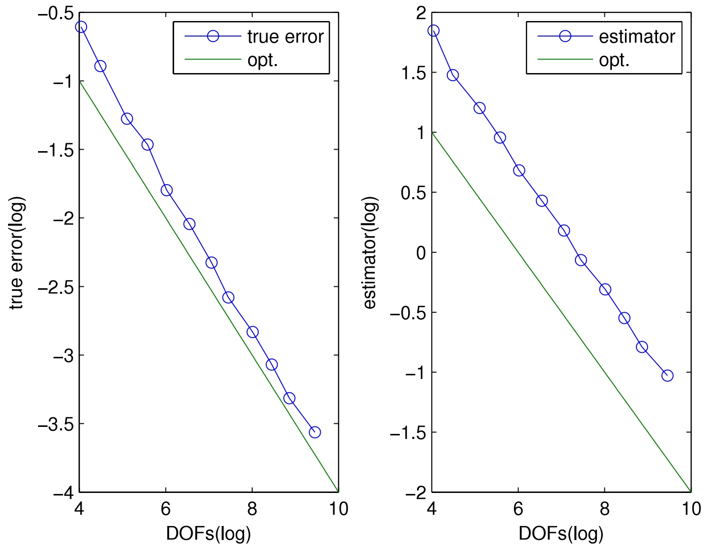

Fig. 3. True error and estimator: the optimal decay is indicated by the line with slope −1/2.

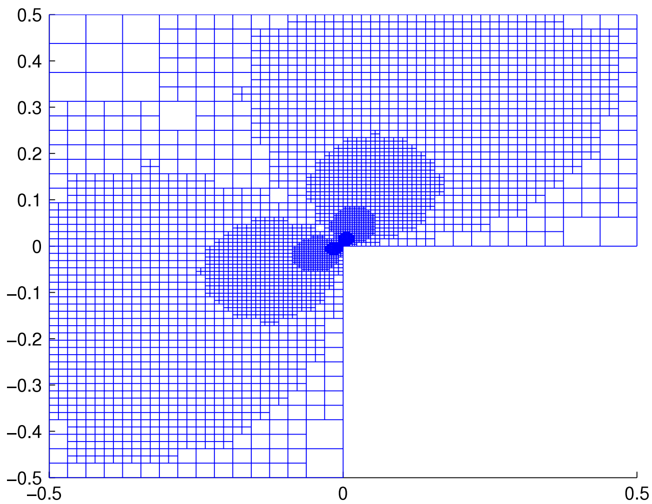

Fig. 4. Adaptive mesh refinement.

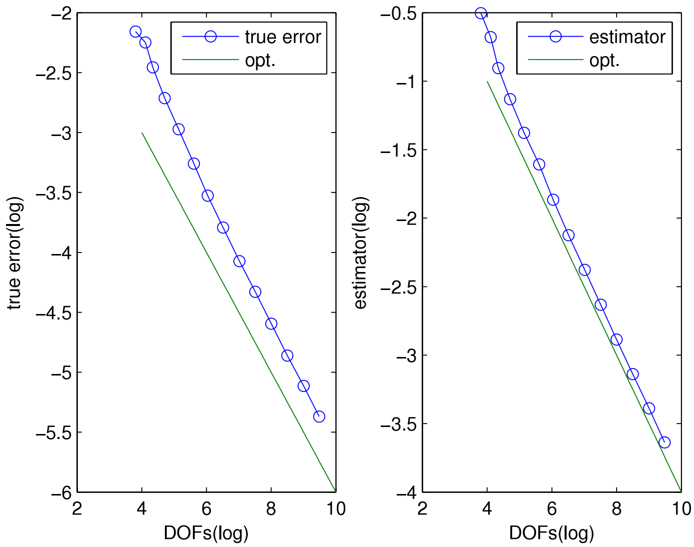

Fig. 5. True error and estimator: the optimal decay is indicated by the line with slope −1/2.

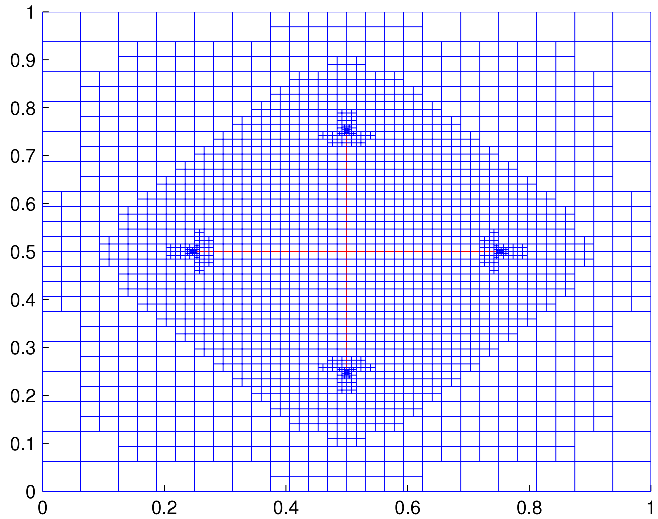

Fig. 6. Adaptive mesh refinement.

References

[1] I. Babuška, W.C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal. 15 (1978) 736–754.
[2] E. Dari, R. Duran, C. Padra, Error estimators for nonconforming finite-element approximations of the Strokes problem, Math. Comp. 64 (1995) 1017–1033.
[3] E. Dari, R. Duran, C. Padra, V. Vampa, A posteriori error estimators for nonconforming finite-element methods, M2AN Math. Model. Numer. Anal. 30 (1996) 385–400.
[4] C. Carstensen, A unifying theory of a posteriori finite element error control, Numer. Math. 100 (2005) 617–637.
[5] C. Carstensen, S. Bartels, S. Jansche, A posteriori error estimates for nonconforming finite-element methods, Numer. Math. 92 (2002) 233–256.
[6] C. Carstensen, J. Hu, A unifying theory of a posteriori error control for nonconforming finite-element methods, Numer. Math. 107 (2007) 473–502.
[7] C. Carstensen, J. Hu, A. Orlando, Framework for the a posteriori error analysis of nonconforming finite elements, SIAM J. Numer. Anal. 45 (2007) 68–82.
[8] J. Manuel Cascon, C. Kreizer, R.H. Nochetto, K.G. Siebert, Quasi-optimal convergenve rate for an adaptive finite-element method, SIAM J. Numer. Anal. 46 (2008) 2524–2550.
[9] R. Stevenson, Optimality of a standard adaptive finite element method, Found. Comput. Math. 7 (2007) 245–269.
[10] W. Dorfler, A convergent adaptive algorithm for Possion's equation, SIAM J. Numer. Anal. 33 (1996) 2169–2189.
[11] P. Morin, R. Nochetto, K. Siebert, Data oscillation and convergence of adaptive FEM, SIAM J. Numer. Anal. 38 (2000) 466–488.
[12] P. Morin, R.H. Nochetto, K.G. Siebert, Convergence of adaptive finite-element methods, SIAM Rev. 44 (2002) 631–658.
[13] C. Carstensen, R.H.W. Hoppe, Convergence analysis of an adaptive nonconforming finite-element method, Numer. Math. 103 (2006) 251–266.
[14] H. Rabus, A natural adaptive nonconforming FEM of quasi-optimal complexity, Comput. Methods Appl. Math. 10 (3) (2010) 315–325.
[15] J. Hu, J.C. Xu, Convergence of adaptive conforming and nonconforming finite element methods for the perturbed Stokes equation, Research Report 73, 2007, School of Mathematical Sciences and Institute of Mathematics, Peking University. Available at: www.math.pku.edu.cn:8000/var/preprint/7297.pdf.

[16] J. Hu, Z.C. Shi, J.C. Xu, Convergence and optimality of the adaptive Morley element method, Numer. Math. (2012) http://dx.doi.org/10.1007/s00211-012-0445-0; see also, J. Hu, Z.C. Shi, J.C. Xu, Convergence and optimality of adaptive nonconforming methods for high-order differential equations, Research Report 19, 2009, School of Mathematical Sciences and Institute of Mathematics, Peking University.
Available at: www.math.pku.edu.cn:8000/var/preprint/7280.pdf.
[17] R. Becker, S. Mao, Quasi-optimality of adaptive nonconforming finite element methods for the stokes equations, SIAM J. Numer. Anal. 49 (2011) 970–991.
[18] S.P. Mao, X.Y. Zhao, Z.C. Shi, Convergence of a standard adaptive nonconforming finite element method with optimal complexity, Appl. Numer. Math. 60 (2010) 673–688.
[19] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978, reprinted as Classis Appl. Math. 40, SIAM, Philadelphia, 2002.
[20] C. Carstensen, J. Hu, Hanging nodes in the unifying theory of a posteriori finite element error control, J. Comput. Math. 27 (2009) 215–236.
[21] X.Y. Zhao, S.P. Mao, Z.C. Shi, Adaptive quadrilateral and hexahedral finite element methods with hanging nodes and convergence analysis, J. Comput. Math. 28 (2009) 621–644.
[22] J.C. Xu, Iterative methods by space secomposition and subspace correction, SIAM Rev. 34 (1992) 581–613.
[23] Z.C. Shi, A convergence condition for the quadrilateral Wilson element, Numer. Math. 44 (1984) 349–361.
[24] R.L. Taylor, P.J. Bereford, E.L. Wilson, A nonconforming element for stress analysis, Internat. J. Numer. Methods Engrg. 10 (1976) 1211–1219.
[25] L.R. Scott, S.Y. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, Math. Comp. 54 (1990) 483–493.
[26] R.H. Nochetto, K.G. Siebert, A. Vesser, Theory of adaptive finite element methods: an introduction, in: Multiscale, Nonlinear and Adaptive Approximation, Springer, 2009.
[27] A. Bonito, R. Nochetto, Quasi-optimal convergence rate of an adaptive discontinuous Garlerkin method, SIAM J. Numer. Anal. 48 (2010) 734–771.