OLLI MARTIO
VLADIMIR RYAZANOV
URI SREBRO
EDUARD YAKUBOV

Moduli in Modern Mapping Theory

Springer Monographs in Mathematics

Springer

Springer Monographs in Mathematics

For other titles published in this series, go to
http://www.springer.com/series/3733

Olli Martio · Vladimir Ryazanov · Uri Srebro ·
Eduard Yakubov

Moduli in Modern Mapping Theory

With 12 Illustrations

Springer

Olli Martio
University of Helsinki
Helsinki
Finland
olli.martio@helsinki.fi

Vladimir Ryazanov
Institute of Applied Mathematics
and Mechanics of National Academy
of Sciences of Ukraine
Donetsk
Ukraine
vlryazanov1@rambler.ru

Uri Srebro
Technion - Israel Institute
of Technology
Haifa
Israel
srebro@math.technion.ac.il

Eduard Yakubov
H.I.T. - Holon Institute of Technology
Holon
Israel
yakubov@hit.ac.il

ISSN: 1439-7382
ISBN: 978-0-387-85586-8
e-ISBN: 978-0-387-85588-2
DOI 10.1007/978-0-387-85588-2

Library of Congress Control Number: 2008939873

Printed on acid-free paper

springer.com

Dedicated to 100 Years of Lars Ahlfors

Preface

The purpose of this book is to present modern developments and applications of the techniques of modulus or extremal length of path families in the study of mappings in $\mathbb{R}^n$, $n \geq 2$, and in metric spaces. The modulus method was initiated by Lars Ahlfors and Arne Beurling to study conformal mappings. Later this method was extended and enhanced by several other authors. The techniques are geometric and have turned out to be an indispensable tool in the study of quasiconformal and quasiregular mappings as well as their generalizations. The book is based on rather recent research papers and extends the modulus method beyond the classical applications of the modulus techniques presented in many monographs.

Helsinki *O. Martio*
Donetsk *V. Ryazanov*
Haifa *U. Srebro*
Holon *E. Yakubov*

2007

Contents

Chapter 1
Introduction and Notation

Mapping theory started in the 18th century. Beltrami, Caratheodory, Christoffel, Gauss, Hilbert, Liouville, Poincaré, Riemann, Schwarz, and so on all left their marks in this theory. Conformal mappings and their applications to potential theory, mathematical physics, Riemann surfaces, and technology played a key role in this development.

During the late 1920s and early 1930s, Grötzsch, Lavrentiev, and Morrey introduced a more general and less rigid class of mappings that were later named quasiconformal. Very soon quasiconformal mappings were applied to classical problems like the covering of Riemann surfaces (Ahlfors), the moduli problem of Riemann surfaces (Teichmüller), and the classification problem for simply connected Riemann surfaces (Volkovyski). Quasiconformal mappings were later defined in higher dimensions (Lavrentiev, Gehring, Väisälä) and were further extended to quasiregular mappings (Reshetnyak, Martio, Rickman, and Väisälä). The quasiregular mappings need not be injective and in many aspects are similar to analytic functions. The monographs [1,22,36,110,176,187,190,256,260,315,316,327–329] give a comprehensive account of the aforementioned theory and its more recent achievements.

Recently generalizations of quasiconformal mappings, mappings of finite distortion, have been studied intensively; see, e.g., the papers [19, 45, 46, 54, 79, 111, 115–117, 124, 132, 133, 145, 147–149, 153–156, 195, 196, 231–233, 237, 248–251] and the monograph [134]. Quasisymmetry has a natural interpretation in metric spaces and quasiconformality from a more analytic point of view has also been studied in these spaces; see, e.g., [21, 33, 107, 112, 201, 312]. These theories can be applied to mappings in the Carnot and Heisenberg groups; see, e.g., [108, 109, 166, 167, 197, 199, 221, 238, 314, 324–326].

The method of the modulus of a path family, or equivalently the method of extremal length, which was initiated by Ahlfors and Beurling in [5] for the study of conformal mapping, is one of the main tools in the theory of quasiconformal and quasiregular mappings. The conformal modulus can be used to define quasiconformal mappings in the plane and in space. It has also been employed in metric measure

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_1,

spaces, now called Loewner spaces; see [107] and [112]. However, it has not been used very much to study mappings of finite distortion and related mappings. The reason is that extremal metrics are more difficult to find and the estimates for the modulus of a path family become more complicated than in the quasiconformal case. In spite of these drawbacks, the modulus method has certain advantages since it is naturally connected to the metric and geometric behavior of the mapping.

In this monograph the modulus method is applied to the generalizations of quasiconformal mappings. The main goal is to study the classes of mappings with distortion of moduli dominated by a given measurable function Q. Functions Q like BMO (bounded mean oscillation), FMO (finite mean oscillation), L^1_{loc}, etc. are included and the principal tool is the modulus method. We concentrate on basic properties like differentiability, boundary behavior, removability of singularities, normal families, convergence, mapping problems, and distortion estimates.

We now recall the definition of the (conformal) modulus of a path family in $\mathbb{R}^n$, $n \geq 2$, and some of the basic inequalities. Let Γ be a path family in $\mathbb{R}^n$, $n \geq 2$. A Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called **admissible** for Γ, abbr. $\rho \in \operatorname{adm} \Gamma$, if

$$\int_\gamma \rho \, ds \geq 1 \tag{1.1}$$

for each $\gamma \in \Gamma$. Recall also that the **(conformal) modulus** of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int_{\mathbb{R}^n} \rho^n(x) \, dm(x), \tag{1.2}$$

where $dm(x)$ corresponds to the Lebesque measure in $\mathbb{R}^n$.

By the classical geometric definition of Väisälä (see, e.g., 13.1 in [316]), a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geq 2$, is K**-quasiconformal**, abbr. K**-qc mapping**, if

$$M(\Gamma)/K \leq M(f\Gamma) \leq K\, M(\Gamma) \tag{1.3}$$

for every path family Γ in D. A homeomorphism $f : D \to D'$ is called **quasiconformal**, abbr. **qc**, if f is K-quasiconformal for some $K \in [1,\infty)$, i.e., if the distortion of the moduli of path families under the mapping f is bounded.

By Theorem 34.3 in [316], a homeomorphism $f : D \to D'$ is quasiconformal if and only if

$$M(f\Gamma) \leq K\, M(\Gamma) \tag{1.4}$$

for some $K \in [1,\infty)$ and for every path family Γ in D. In other words, it is sufficient to verify that

$$\sup \frac{M(f\Gamma)}{M(\Gamma)} < \infty, \tag{1.5}$$

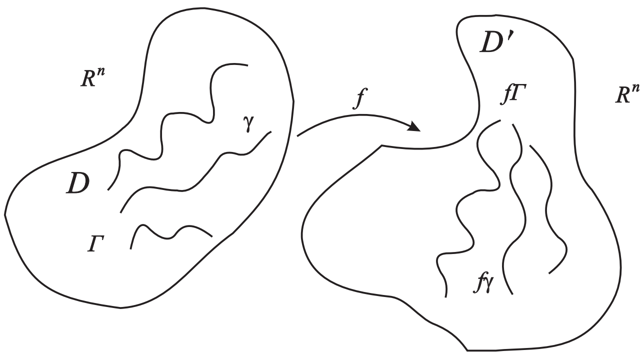

Figure 1

where the supremum is taken over all path families Γ in D for which $M(\Gamma)$ and $M(f\Gamma)$ are not simultaneously 0 or ∞. Then it is also

$$\sup \frac{M(\Gamma)}{M(f\Gamma)} < \infty. \tag{1.6}$$

Gehring was the first to note that the suprema in (1.5) and (1.6) remain the same if we restrict ourselves to families of paths connecting the boundary components of rings in D; see [73] or Theorem 36.1 in [316]. Thus, the geometric definition of a K-quasiconformal mapping by Väisälä is equivalent to Gehring's ring definition.

Moreover, condition (1.6) has been shown to be equivalent to the statement that f is ACL (absolutely continuous on lines), a.e. differentiable, and

$$\operatorname{ess\,sup} \frac{\|f'(x)\|^n}{J(x,f)} < \infty, \tag{1.7}$$

where $\|f'(x)\|$ denotes the matrix norm of the Jacobian matrix $f'(x)$ of the mapping f, i.e., $\max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$, and $J(x,f)$ its determinant at a point $x \in D$ [here the ratio is equal to 1 if $f'(x) = 0$]. Furthermore, it turns out that the suprema in (1.6) and (1.7) coincide; see Theorem 32.3 in [316]. The given three properties of f form the analytic definition for a quasiconformal mapping that is equivalent to the above geometric definition; see Theorem 34.6 in [316].

In the light of the interconnection between conditions (1.3) and (1.4), the following concept is a natural extension of the geometric definition of quasiconformality; see, e.g., [204–209]. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1,\infty]$ be a measurable function. We say that a homeomorphism $f : D \to \overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is a **Q-homeomorphism** if

$$M(f\Gamma) \leq \int_D Q(x) \cdot \rho^n(x)\, dm(x) \tag{1.8}$$

for every family Γ of paths in D and every admissible function ρ for Γ. This concept is related in a natural way to the theory of the so-called moduli with weights; see, e.g., [7, 8, 228, 229, 306].

Note that the estimate of type (1.8) was first established in the classical quasiconformal theory. Namely, in [190], p. 221, for quasiconformal mappings in the complex plane, the authors show that

$$M(f\Gamma) \leq \int\limits_{\mathbb{C}} K(z) \cdot \rho^2(z)\, dxdy, \tag{1.9}$$

where

$$K(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \tag{1.10}$$

is a (local) maximal dilatation of the mapping f at a point z. We later used inequality (1.9) in the study of the so-called BMO-quasiconformal mappings in the plane when

$$K(z) \leq Q(z) \in \text{BMO}; \tag{1.11}$$

see, e.g., [271–274]. Next, Lemma 2.1 in [26] shows that for quasiconformal mappings in space, $n \geq 2$,

$$M(f\Gamma) \leq \int\limits_{D} K_I(x, f)\, \rho^n(x)\, dm(x), \tag{1.12}$$

where K_I stands for the inner dilatation of f at x; see (1.16) ahead. Finally, we have come to the above general conception of a Q-homeomorphism.

An introduction to the main techniques in the geometric theory of quasiconformal mappings can be found in Chapters 2 and 3.

Chapter 4 is devoted to the basic theory of space Q-homeomorphisms f for $Q \in L^1_{\text{loc}}$. Differentiability a.e., absolute continuity on lines, estimates from below for distortion, removability of isolated singularities, extension to the boundary of the inverse mappings, and other properties are considered.

Chapter 5 includes estimates of distortion, removability of isolated singularities, theorems on continuous and homeomorphic extension to regular boundaries, and other results on Q-homeomorphisms for Q in the BMO class, where BMO refers to functions with bounded mean oscillation introduced by John–Nirenberg. Results on Q-homeomorphisms for Q in the FMO class (finite mean oscillation) and in more general classes are given in Chapter 6. Analogies of the Painleve theorem on removability of singularities of length zero and applications of the theory of Q-homeomorphisms to mappings in the Sobolev class $W^{1,n}_{\text{loc}}$ are presented.

Extensions of the quasiconformal theory to ring and lower Q-homeomorphisms and their applications to mappings with finite length and area distortion are found in Chapters 7–10. Existence theorems of ring Q-homeomorphisms in the plane case are given in Chapter 11. Some results on mappings quasiconformal in the mean related to the modulus techniques are contained in Chapter 12. Chapter 13 contains the theory of Q-homeomorphisms in general metric spaces with measures.

The Appendix at the end of the book includes the basic facts in the theory of moduli themselves.

Throughout this book, $\mathbb{R}^n$ denotes the **n-dimensional Euclidean space**, where we use the **Euclidean norm** $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ for points $x = (x_1, \ldots, x_n)$. $B^n(x,r)$ denotes the **open ball** in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r \in (0,\infty)$, i.e., $B^n(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $S^{n-1}(x,r)$ is its boundary sphere, i.e., $S^{n-1}(x,r) = \{y \in \mathbb{R}^n : |x-y| = r\}$. We also let $B^n(r) = B^n(0,r)$, $\mathbb{B}^n = B^n(1)$, and $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$.

In what follows, $\overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup\{\infty\}$ is the **one-point compactification** of $\mathbb{R}^n$, i.e., $\overline{\mathbb{R}^n}$ is a space obtained from $\mathbb{R}^n$ by joining only one "ideal" element ∞, which is called **infinity** and whose neighborhood base is formed by sets containing the complements of balls in $\mathbb{R}^n$ together with ∞. We use in $\overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup\{\infty\}$ the **spherical (chordal) metric** $h(x,y) = |\pi(x) - \pi(y)|$, where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:

$$\begin{aligned} h(x,y) &= \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \qquad x \neq \infty \neq y, \\ h(x,\infty) &= \frac{1}{\sqrt{1+|x|^2}}\,. \end{aligned} \tag{1.13}$$

Thus, by definition, $h(x,y) \leq 1$ for all x and $y \in \overline{\mathbb{R}^n}$. Note that $h(x,y) \leq |x-y|$ for all $x,y \in \mathbb{R}^n$ and $h(x,y) \geq |x-y|/2$ for all x and $y \in \mathbb{B}^n$. The **spherical (chordal) diameter** of a set $E \subset \overline{\mathbb{R}^n}$ is

$$h(E) = \sup_{x,y\in E} h(x,y). \tag{1.14}$$

Given a mapping $f : D \to \mathbb{R}^n$ with partial derivatives a.e., $f'(x)$ denotes the Jacobian matrix of f at $x \in D$ if it exists, $J(x) = J(x,f) = \det f'(x)$ is the Jacobian of f at x, and $|f'(x)|$ is the operator norm of $f'(x)$, i.e., $|f'(x)| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$. We also let $l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$. The **outer dilatation** of f at x is defined by

$$K_O(x) = K_O(x,f) = \begin{cases} \frac{|f'(x)|^n}{|J(x,f)|} & \text{if } J(x,f) \neq 0, \\ 1 & \text{if } f'(x) = 0, \\ \infty & \text{otherwise}, \end{cases} \tag{1.15}$$

the **inner dilatation** of f at x by

$$K_I(x) = K_I(x, f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n} \text{ if } J(x,f) \neq 0, \\ 1 \text{ if } f'(x) = 0, \\ \infty \text{ otherwise,} \end{cases} \tag{1.16}$$

and the **maximal dilatation**, or in short the **dilatation**, of f at x by

$$K(x) = K(x, f) = \max(K_O(x), K_I(x)). \tag{1.17}$$

Note that $K_I(x) \leq K_O(x)^{n-1}$ and $K_O(x) \leq K_I(x)^{n-1}$; see, e.g., Section 1.2.1 in [256], and, in particular, $K_O(x), K_I(x)$, and $K(x)$ are simultaneously finite or infinite. $K(x, f) < \infty$ a.e. is equivalent to the condition that a.e. either $\det f'(x) > 0$ or $f'(x) = 0$.

Recall that a (continuous) mapping $f : D \to \mathbb{R}^n$ is **absolutely continuous on lines**, abbr. $f \in$ **ACL**, if, for every closed parallelepiped P in D whose sides are perpendicular to the coordinate axes, each coordinate function of $f|P$ is absolutely continuous on almost every line segment in P that is parallel to the coordinate axes. Note that, if $f \in$ ACL, then f has the first partial derivatives a.e.

In particular, f is ACL if $f \in W^{1,1}_{\text{loc}}$. In general, mappings in the Sobolev classes $\mathbf{W}^{\mathbf{1,p}}_{\text{loc}}$, $p \in [1, \infty)$, with generalized first partial derivatives in L^p_{loc} can be characterized as mappings in $\mathbf{ACL}^{\mathbf{p}}_{\text{loc}}$, i.e. mappings in ACL whose usual first partial derivatives are locally integrable in the degree p; see, e.g., [215], p. 8.

Later on, for given sets A, B, and C in $\mathbb{R}^n$, $\Delta(A, B, C)$ denotes a collection of all paths $\gamma : [0,1] \to \mathbb{R}^n$ joining A and B in C, i.e., $\gamma(0) \in A$, $\gamma(1) \in B$, and $\gamma(t) \in C$ for all $t \in (0,1)$. Moreover, we use the abbreviation $\Delta(A, B)$ for the case $C = \mathbb{R}^n$.

Chapter 2
Moduli and Capacity

2.1 Introduction

In this chapter, we mainly follow the notes [201]; cf. also [107, 110, 112]. These notes are intended to be an introduction to the basic techniques in the geometric theory of quasiconformal maps. The main emphasis is on the concept of the p-modulus of a family of paths. The purpose is to relate this concept to other definitions of quasiconformality. An excellent account can be found in [316]. However, we have tried to develop the tools of quasiconformal theory beyond the usual Euclidean space $\mathbb{R}^n$. Such a development is rather recent. Quasisymmetric maps were considered by Tukia and Väisälä [311] in metric spaces and quasiconformality was characterized in local terms by Heinonen and Koskela [112]. The definitions of quasiconformality and the treatment of their equivalence in $\mathbb{R}^n$ very much follow the presentation in [316]. The concept of quasisymmetry is more thoroughly treated in [107]. The treatment of linear dilatation offers novel features. We hope that graduate students will find these tools applicable in new situations; see, e.g., Chapter 13.

The theory of quasiconformal maps essentially belongs to real analysis. This is very evident in Chapters 2 and 5–8, although no hard real analysis is needed.

The reference list is relatively short here. Further references can be found in the books [107] and [110] and in the paper [112].

2.2 Moduli in Metric Spaces

The length-area method was first used in the theory of conformal mappings. The name "extremal length" was used by Ahlfors and Beurling; see [5]. The name "modulus" or "p-modulus" is now widely used. The general theory for the p-modulus and the connections to function spaces was developed by Fuglede [64]. There is a similar theory of capacities of condensers.

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_2,

Paths and line integrals. Let (X,d) be a metric space. A **path** γ in X is a continuous map $\gamma\colon [a,b] \to X$. Sometimes we also consider "paths" γ that are defined on open intervals (a,b) of $\mathbb{R}$. The theory for these is similar.

The length of a path $\gamma\colon [a,b] \to X$ is

$$l(\gamma) = \sup \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is over all sequences $a = t_1 \le t_2 \le \cdots \le t_n \le t_{n+1} = b$. If the interval is not closed, then we define the length of γ to be the supremum of the lengths of all closed subcurves of γ. A curve γ is **rectifiable** if its length is finite, and a path γ is **locally rectifiable** if all of its closed subcurves are rectifiable. However, usually we assume that all paths are closed and nondegenerate, i.e., $\gamma([a,b])$ is not a point, unless otherwise stated.

Two important concepts are associated with a rectifiable path $\gamma\colon [a,b] \to X$: the **length function** $S_\gamma : [a,b] \to \mathbb{R}$ and **parameterization by arc length**. The length function is defined as

$$S_\gamma(t) = l(\gamma|[a,t]), \qquad a \le t \le b,$$

and the path $\tilde\gamma\colon [0, l(\gamma)] \to X$ is the unique 1-Lipschitz continuous map such that

$$\gamma = \tilde\gamma \circ S_\gamma.$$

In particular, $l(\tilde\gamma|[0,t]) = t,\ 0 \le t \le l(\gamma)$, and γ is obtained from $\tilde\gamma$ by an increasing change of parameter. The path $\tilde\gamma$ is called the parameterization of γ by arc length. For the construction of $\tilde\gamma$, see [316].

If $\gamma\colon [a,b] \to X$ is a path, then the set

$$|\gamma| = \{\gamma(t) : t \in [a,b]\}$$

is called a **locus** of the path. Often we shall not distinguish between a path and its locus, although this is dangerous in many occasions.

We recall that a set $J \subset X$ is a (closed) **arc** if it is homeomorphic to some interval $[a,b]$. For an arc J, the length (possibly infinite) is well defined: it is independent of the parameterization of J. In the theory of quasiconformal maps, mostly arcs or **Jordan curves** (homeomorphic images of the unit circle) are used, but paths are important in the theory of nonhomeomorphic quasiconformal maps (quasiregular maps) since the image of an arc need not be an arc; see, e.g., [210, 256, 260, 328].

Given a rectifiable curve γ in X, the line integral over γ of a Borel function $\rho : X \to [0,\infty]$ is

$$\int_\gamma \rho \, ds = \int_0^{l(\gamma)} \rho(\tilde\gamma(t)) \, dt\,.$$

Sometimes we write this as

$$\int_\gamma \rho \, |dx| \, .$$

If γ is only locally rectifiable, then we set

$$\int_\gamma \rho \, ds = \sup \int_{\gamma'} \rho \, ds,$$

where the supremum is taken over all rectifiable subcurves $\gamma' : [a', b'] \to X$ of γ. If $X = \mathbb{R}^n$ and a path $\gamma : [a,b] \to X$ has an absolutely continuous representation (this means that each coordinate function $\gamma_i : [a,b] \to \mathbb{R}$, $i = 1, \ldots, n$, of γ is absolutely continuous), then the line integral over γ is

$$\int_a^b \rho(\gamma(t)) \, |\gamma'(t)| \, dt,$$

where $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$ and $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$.

Let μ be a **Borel regular measure** in a metric space (X, d). Borel regularity means that open sets of X are μ-measurable and every μ-measurable set is contained in a Borel set of equal measure. For a given curve family Γ in X and a real number $p \geq 1$, we define the p-**modulus** of Γ by

$$M_p(\Gamma) = \inf \int_X \rho^p \, d\mu, \tag{2.1}$$

where the infimum is taken over all nonnegative Borel functions $\rho : X \to [0, \infty]$ satisfying

$$\int_\gamma \rho \, ds \geq 1$$

for all (locally) rectifiable curves $\gamma \in \Gamma$. Functions ρ that satisfy the latter condition are called **admissible functions**, or **metrics**, for the family Γ.

If X is the Euclidean n-space $\mathbb{R}^n$ equipped with the usual distance, then the measure μ will be the Lebesgue measure m in most cases.

By definition, the modulus of all curves in X that are not rectifiable is zero. If Γ contains a constant curve and the measure μ satisfies $\mu(\{x\}) = 0$ for all $x \in X$, then there are no admissible functions and the modulus is infinite. Further, the following properties are easily verified:

$$M_p(\emptyset) = 0, \tag{2.2}$$

$$M_p(\Gamma_1) \leq M_p(\Gamma_2) \tag{2.3}$$

if $\Gamma_1 \subset \Gamma_2$, and

$$M_p\left(\bigcup_{i=1}^{\infty}\Gamma_i\right) \le \sum_{i=1}^{\infty} M_p(\Gamma_i). \tag{2.4}$$

Moreover,

$$M_p(\Gamma) \le M_p(\Gamma_0) \tag{2.5}$$

if Γ is **minorized** by Γ_0, i.e., each path $\gamma \in \Gamma$ has a subpath $\gamma_0 \in \Gamma_0$.

Only (2.4) requires a proof. For (2.4), we may assume that every $M_p(\Gamma_i) < \infty$. For $\varepsilon > 0$, pick an admissible ρ_i for Γ_i such that

$$\int_X \rho_i^p \, d\mu < M_p(\Gamma_i) + \varepsilon/2^i.$$

Then the function $\rho = (\sum \rho_i^p)^{1/p}$ is admissible for $\Gamma = \cup\, \Gamma_i$ since $\rho \ge \rho_i$ for all $i = 1, 2, \dots$. Thus,

$$M_p(\Gamma) \le \int_X \rho^p \, d\mu = \sum_{i=1}^{\infty} \int_X \rho_i^p d\mu < \varepsilon + \sum_{i=1}^{\infty} M_p(\Gamma_i).$$

Letting $\varepsilon \to 0$ yields (2.4).

Conditions (2.2)–(2.4) mean that M_p is an outer measure on the set of curves in X.

Remark 2.1. Observe that ρ needs to be a Borel function [i.e. $\rho^{-1}((a,\infty])$ is a Borel set in X for each $a \in \mathbb{R}$] since otherwise the above line integrals can be undefined. In general, measurable admissible functions provide too restrictive a class. However, if $\rho \ge 0$ is μ-measurable, then there exists a Borel function ρ^* such that $\rho^* \ge \rho$ in X and $\rho^* = \rho$ a.e. with respect to the measure μ. This makes it possible to use μ-measurable functions as admissible functions on many occasions.

In general, it is difficult to compute $M_p(\Gamma)$ for a given curve family Γ. For example, let us compute the p-modulus of a curve family Γ that joins the bases of a cylinder in $\mathbb{R}^n$. In $\mathbb{R}^n$ we use the Lebesgue measure $\mu = m$. Let E be a Borel set in $\mathbb{R}^{n-1}$ and let $h > 0$. Set

$$G = \{x \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}) \in E \text{ and } 0 < x_n < h\}.$$

Then G is a cylinder with bases E and $F = E + he_n$ and height h. Let Γ be the family of all paths $\gamma\colon [a,b] \to \mathbb{R}^n$ such that $\gamma(t) \in G$, $t \in (a,b)$, $\gamma(a) \in E$, and $\gamma(b) \in F$.

We first make a simple observation:

Lemma 2.1. *Suppose that the curves γ of a family Γ lie in a Borel set $A \subset X$ and that $l(\gamma) \ge r > 0$ for each $\gamma \in \Gamma$. Then*

$$M_p(\Gamma) \le \frac{\mu(A)}{r^p}.$$

Proof. Set $\rho(x) = 1/r$ for $x \in A$ and $\rho(x) = 0,\ x \in X \setminus A$. Then ρ is admissible for Γ and the inequality follows.

Now, we show that in the cylinder

$$M_p(\Gamma) = \frac{m_{n-1}(E)}{h^{p-1}} = \frac{m(G)}{h^p}, \tag{2.6}$$

where m_{n-1} is the Lebesgue measure in $\mathbb{R}^{n-1}$.

Since $l(\gamma) \geq h$ for every $\gamma \in \Gamma$, Lemma 2.1 implies that $M_p(\Gamma) \leq m(G)/h^p$. Let ρ be an arbitrary admissible function for Γ. For each $y \in E$, let $\gamma_y : [0,h] \to \mathbb{R}^n$ be the vertical segment $\gamma_y(t) = y + te_n$. Then $\gamma_y \in \Gamma$. Assuming that $p > 1$, we obtain by Hölder's inequality

$$1 \leq \left(\int_{\gamma_y} \rho \, ds \right)^p \leq h^{p-1} \int_0^h \rho(y+te_n)^p \, dt\,.$$

Integration over $y \in E$ yields by Fubini's theorem

$$m_{n-1}(E) \leq h^{p-1} \int_E dm_{n-1} \int_0^h \rho(y+te_n)^p dt = h^{p-1} \int_G \rho^p dm \leq h^{p-1} \int \rho^p dm\,.$$

Since this holds for every admissible ρ, we obtain $M_p(\Gamma) \geq m_{n-1}(E)/h^{p-1}$. The proof for (2.6) in the case $p = 1$ is even simpler. □

In general, it is a relatively easy task to obtain upper bounds for $M_p(\Gamma)$; here one admissible ρ suffices. Obtaining nontrivial lower bounds is usually much more difficult.

2.3 Conformal Modulus

For quasiconformal maps, the most important modulus is the n-modulus $M_n(\Gamma)$ in $\mathbb{R}^n$, which is a conformal invariant and can also be used on Riemannian n-manifolds.

A diffeomorphism $f : \Omega \to \Omega'$ between two domains in $\mathbb{R}^n$ is **conformal** if at every point x its derivative $f'(x)$ is an orthogonal map, i.e., a **homothety**. This means that

$$\langle f'(x)h, f'(x)k \rangle = \lambda(x)\langle h,k \rangle \tag{2.7}$$

at each point of $x \in \Omega$ for every h and $k \in \mathbb{R}^n$, where $\lambda(x) > 0$ is a continuous function on Ω. Here $\langle h,k \rangle$ denotes the inner product of vectors h and k in $\mathbb{R}^n$. For maps $f : M^n \to N^n$ between two n-dimensional Riemannian manifolds M^n and N^n, (2.7) takes the form

$$\langle Df(x)X, Df(x)Y\rangle_{f(x)} = \lambda(x)\langle X, Y\rangle_x \tag{2.8}$$

at each point $x \in M$ for all tangent vectors X and Y in T_xM. Conditions (2.7) and (2.8) mean that the angles are preserved on the infinitesimal level.

The conformality of f can also be expressed in the form

$$\|f'(x)\|^n = |J(x,f)|, \qquad x \in \Omega. \tag{2.9}$$

Here

$$\|f'(x)\| = \sup_{|h|=1} |f'(x)h|$$

is the sup-norm of the linear map $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $J(x,f) = \det f'(x)$ is the Jacobian determinant of the $n \times n$ matrix of $f'(x)$. Indeed, the linear map $f'(x)$ maps the unit ball $B(0,1)$ of $\mathbb{R}^n$ onto an ellipsoid with semi-axis $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and $\|f'(x)\| = \lambda_n$ and $|J(x,f)| = \lambda_1 \cdot \ldots \cdot \lambda_n$. Condition (2.7) gives $\lambda_1 = \lambda_2 = \ldots = \lambda_n$ (an orthogonal linear map maps balls into balls). Since the correspondence $x \mapsto J(x,f)$ is continuous and does not vanish, it cannot change sign in Ω. In the theory of conformal maps, only **sense-preserving mappings**, i.e., $J(x,f) > 0$ a.e., are usually considered. Then (2.9) can be written without absolute signs. Note that another way to express (2.9) is $\|f'(x)\||h| = |f'(x)h|$ for all $h \in \mathbb{R}^n$.

For $n = 2$, (2.7) or (2.9) leads to the usual definition of a conformal map: A diffeomorphism $f : D \to D'$ between two plane domains D and D' is conformal if f has a "conformal" derivative at every point $x \in D$, i.e., $f'(x)$ is a sense-preserving homothety of the complex plane.

Theorem 2.1. *If $f : \Omega \to \Omega'$ is conformal, then*

$$M_n(\Gamma) = M_n(f\Gamma)$$

for each curve family $\Gamma \subset \Omega$ (for the measure μ, the Lebesgue measure is used).

Proof. If ρ is an admissible function for $f\Gamma$, then it is easily seen (this computation is done in the proof of Theorem 2.12 ahead) that

$$\int_\gamma \rho(f(x))\, \|f'(x)\|\, |dx| \geq \int_{f\circ\gamma} \rho\, ds \geq 1$$

for all $\gamma \in \Gamma$, so that $\rho(f(x))\, \|f'(x)\|$ is admissible for Γ. Thus,

$$M_n(\Gamma) \leq \int_\Omega \rho^n(f(x))\, \|f'(x)\|^n\, dx. \tag{2.10}$$

Using the change of variables in the right-hand side of (2.10) and the conformality condition (2.9), we transform the integral into

$$\int_{\Omega'} \rho(y)^n \, dy.$$

This shows that $M_n(\Gamma) \leq M_n(f\Gamma)$, and the rest follows by symmetry. $\square$

Remark 2.2. The n-modulus is often called the **conformal modulus**. In the literature the **extremal length** defined as $1/M_n(\Gamma)$ is also used.

Remark 2.3. In the plane the 2-modulus is a frequently used powerful tool in the study of conformal maps. In particular, it can be used to prove results like "a conformal map $f : B(0,1) \to \mathbb{R}^2$ has radial limits on $\partial B(0,1)$ except on a set of 2-capacity zero."

2.4 Geometric Definition for Quasiconformality

There are many equivalent ways to define quasiconformal maps. The one given in (1.3) is the strongest in the sense that many properties of quasiconformal maps can be derived rather directly from the definition and that it is impractical to check the quasiconformality of a given map by using (1.3). In particular, it follows from (1.3) that $f^{-1} : \Omega' \to \Omega$ is K-quasiconformal as well.

We shall discuss other definitions later, also those that generalize the notion of quasiconformality to spaces where modulus is not available. In particular, it would be useful to have a definition for quasiconformality that has a purely local character as in the case of conformal maps. Indeed, such definitions exist and are usually based on (2.9). There are also slightly different definitions based on metric concepts that will be discussed in Sections 2.8 and 2.10.

For a diffeomorphism $f : \Omega \to \Omega'$, when both f and f^{-1} belong to C^1, condition (2.9) can easily be relaxed, which also leads to a definition for quasiconformality. Set

$$K_0(f'(x)) = \|f'(x)\|^n / |J(x,f)|, \qquad K_I(f'(x)) = |J(x,f)| / l(f'(x))^n,$$

where $l(f'(x)) = \inf\{|f'(x)h| : |h| = 1\} = \lambda_1$ is the so-called minimal stretching of $f'(x)$. Note that

$$K_0(f'(x)) = \lambda_2/\lambda_1 = K_I(f'(x))$$

for $n = 2$, but these numbers are, in general, different for $n \geq 3$. Set

$$K(f) \;=\; \max\left(\sup_{x\in\Omega} K_0(f'(x)),\; \sup_{x\in\Omega} K_I(f'(x)) \right).$$

The number $K(f) \in [1,\infty]$ is called the **maximal dilatation** of the diffeomorphism f.

The same reasoning as in the proof of Theorem 2.1 yields the following result.

Theorem 2.2. *Suppose that $f : \Omega \to \Omega'$ is a diffeomorphism with $K(f) < \infty$. Then f satisfies (1.3) with $K = K(f)$, i.e., f is $K(f)$-quasiconformal.*

Example 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x) = (x_1, Kx_2),\ x = (x_1, x_2),\ K \geq 1$. Then f is a linear map and $K(f) = K$. In general, every nondegenerate linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal.

Remark 2.4. The converse of Theorem 2.2 is also true: If a diffeomorphism f satisfies (1.3), then the maximal dilatation $K(f)$ of f satisfies $K(f) \leq K$. The class of C^1-qc maps is not closed under locally uniform convergence. Hence, it is natural to study more general classes that also include nondiffeomorphic quasiconformal mappings.

2.5 Modulus Estimates

As we noted above, the p-modulus of a path family is difficult to compute exactly in most cases. However, in some cases the calculations are possible.

Lemma 2.2. *Let $B(x_0, r)$ be the open ball centered at $x_0 \in \mathbb{R}^n$ and radius $r > 0$. Let Γ be the family of all paths $\gamma : [a,b] \to \overline{A}$, where A is the open annulus*

$$A = B(x_0, R) \setminus \overline{B}(x_0, r), \qquad R > r,$$

with $\gamma(a) \in \partial B(x_0, r),\ \gamma(b) \in \partial B(x_0, R)$. Then

$$M_n(\Gamma) = \omega_{n-1} \left(\log \frac{R}{r} \right)^{1-n},$$

where ω_{n-1} is the area of the unit sphere $\partial B(0,1)$ in R^n.

Proof. The function

$$\rho(x) = \left(\log \frac{R}{r} \right)^{-1} |x_0 - x|^{-1}$$

restricted to A is admissible for Γ and, thus,

$$M_n(\Gamma) \leq \int_A \rho^n(x)\, dx = \left(\log \frac{R}{r} \right)^{-n} \int_{S^{n-1}} \int_r^R t^{-1} dt d\omega = \omega_{n-1} \left(\log \frac{R}{r} \right)^{1-n}.$$

On the other hand, if ρ is an arbitrary admissible function for Γ in A, then for each point ω on the unit sphere S^{n-1}, we have

$$1 \leq \int_r^R \rho(x_0 + t\omega)dt \leq \left(\int_r^R \rho(x_0 + t\omega)^n t^{n-1} dt \right)^{1/n} \left(\int_r^R t^{-1} dt \right)^{(n-1)/n}.$$

Hence,

$$\int_A \rho^n(x)dx \geq \omega_{n-1} \left(\log \frac{R}{r} \right)^{1-n}.$$

This completes the proof. □

Remark 2.5. The p-modulus of the path family joining the boundary components of an annulus can be computed exactly for any $p \geq 1$, i.e.,

$$M_p(\Gamma) = \omega_{n-1} \left(\frac{|n-p|}{p-1} \right)^{p-1} \left| R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}} \right|^{1-p}$$

if $p \neq n,\ p > 1$. In the case $p = 1$, we have

$$M_1(\Gamma) = \omega_{n-1} r^{n-1},$$

i.e., $M_1(\Gamma)$ is the area of the inner sphere. The computation is similar to the proof of Lemma 2.2; it is only necessary to guess the right admissible function; see [110].

Remark 2.6. The paths γ in Lemma 2.2 need not lie in $\overline{A}$. One can as well assume that $\gamma : [a,b] \to \mathbb{R}^n$ and the result is the same; see (2.3) and (2.5). Also, it is not necessary to consider all paths that join the boundary components of A: The radial rays emanating from x_0 and restricted to $\overline{A}$ are enough.

Corollary 2.1. *Let Γ be a family of (nonconstant) paths γ in $\mathbb{R}^n$ such that each γ meets a fixed point $x_0 \in \mathbb{R}^n$. Then $M_n(\Gamma) = 0$.*

Proof. Fix $r > 0$ and consider the annulus

$$A_i = B(x_0, r) \setminus \overline{B}(x_0, r/i), \quad i = 2, 3, \ldots.$$

Let $\Gamma_i(r)$ be the family paths γ in Γ that have a subpath whose endpoints lie in different boundary components of A_i. Then by (2.5) and Lemma 2.2 (see also Remark 2.6),

$$M_n(\Gamma_i(r)) \leq \omega_{n-1}(\log i)^{1-n} \to 0 \quad \text{as} \quad i \to \infty.$$

If $\Gamma(r)$ is the family of all paths $\gamma \in \Gamma$ that meet $\partial B(x_0, r)$, then

$$0 \leq M_n(\Gamma(r)) \leq M_n(\Gamma_i(r)), \quad i = 1, 2, \ldots,$$

and hence $M_n(\Gamma(r)) = 0$. The claim now follows from (2.4) because

$$M_n(\Gamma) \leq M_n \left(\bigcup_{j=1}^{\infty} \Gamma(1/j) \right) \leq \sum_{j=1}^{\infty} M_n(\Gamma(1/j)) = 0.$$

□

Remark 2.7. Corollary 2.1 remains true for $1 \le p \le n$ but is false for $p > n$.

Example. Consider the radial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = |x|^{\alpha-1}x$, where $0 < \alpha \le 1$. Then f is a homeomorphism of $\mathbb{R}^n$ and a diffeomorphism of $\mathbb{R}^n \setminus \{0\}$ onto itself. Although f is not C^1 for $\alpha < 1$ (f is α-Hölder only), f^{-1} is in $C^1(\mathbb{R}^n)$. Let $x \neq 0$. Then it is easy to see that $f'(x)$ maps $B(0,1)$ onto an ellipsoid with semi-axes $1/\alpha$ and α^{1-n} and we obtain from Theorem 2.2 that the mapping $f|\mathbb{R}^n \setminus \{0\}$ is α^{1-n}-quasiconformal. Now the mapping f is α^{1-n}-quasiconformal because by Corollary 2.1 the n-modulus of any path family passing through 0 is zero and, thus, f satisfies (1.3).

The most important modulus estimate in the theory of quasiconformal maps is the so-called Loewner estimate. We transfer it here in a general context of metric spaces.

Let (X,d) be a metric space with a Borel measure as before. For each real number $n > 1$, we define the **Loewner function** $\Phi_n : (0,\infty) \to [0,\infty)$ of X as

$$\Phi_n(t) = \Phi_{X,n}(t) = \inf\{M_n(\Gamma(E,F;X)) : \Delta(E,F) \le t\},$$

where E and F are disjoint nondegenerate continua in X with

$$\Delta(E,F) = \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}$$

and $\Gamma(E,F;X)$ is the family of all paths that join E to F in X. The number $\Delta(E,F)$ measures the relative position of E and F in X.

If one cannot find two disjoint continua in X, it is understood that $\Phi_{X,n}(t) \equiv 0$. Recall that a **continuum** is a compact connected set, and a continuum is **nondegenerate** if it is not a point and not empty; we shall assume that all continua are nondegenerate.

By definition, the function Φ_n is decreasing.

A pathwise connected metric measure space (X,μ) is said to be a **Loewner space of exponent** n , or an n**-Loewner space**, if the Loewner function $\Phi_{X,n}(t)$ is positive for all $t > 0$.

Note that the positivity of the Loewner function alone does not imply that the space X in question is pathwise connected; for instance, X can be a disjoint union of a Loewner space and a point.

In a Loewner space one finds a lot of rectifiable paths joining two disjoint continua, and the plenitude of paths is quantified by the function Φ_n. In particular, a space without rectifiable paths, such as $(\mathbb{R}^n, |x-y|^{1/2})$, cannot be a Loewner space. Also, notice the scale invariance of the condition.

The use of the exponent n in the definition is based on the result of Loewner [192].

Theorem 2.3. *$\mathbb{R}^n$ is an n-Loewner space.*

We shall come to the proof of this result in Chapter 10.

Remark 2.8. In $\mathbb{R}^n$ the function Φ_n has the following asymptotics:

$$\Phi_n(t) \approx (\log t)^{1-n}, \quad t \to \infty,$$

$$\Phi_n(t) \approx \log(1/t), \quad t \to 0.$$

The constants involved in these estimates depend only on n. For $n = 2$, the function Φ_2 has a representation in the form of an elliptic integral; see [190].

2.6 Upper Gradients and ACC$_p$ Functions

One of the most important properties of a C^1-function u defined in a domain Ω of $\mathbb{R}^n$ is that it can be recovered from its derivative. More precisely,

$$u(x) - u(y) = \int_\gamma \nabla u \cdot d\bar{s} = \int_0^{l(\gamma)} \langle \nabla u(\tilde{\gamma}(s)), \tilde{\gamma}'(s) \rangle \, ds, \tag{2.11}$$

where γ is any rectifiable path in Ω with endpoints x and y and $\tilde{\gamma}$ is the representation of γ by arc length. Now (2.11) leads to

$$|u(x) - u(y)| \leq \int_\gamma |\nabla u| \, ds. \tag{2.12}$$

As we will soon see, inequality (2.12) is almost as useful as equality (2.11).

We first extend (2.12) to a metric space. Let (X,d) be a metric space and $u : X \to \mathbb{R}$. A Borel function $\rho : X \to [0,\infty]$ is said to be an **upper gradient** of u if

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds \tag{2.13}$$

for each rectifiable path γ joining x and y in X.

Every function has an upper gradient, namely $\rho \equiv \infty$, and upper gradients are seldom unique. Note that $\rho = \infty$ could be the only upper gradient in the case! The constant function $\rho \equiv L$ is an upper gradient of every L-Lipschitz fucntion, but this is rarely the best choice. A constant function has an upper gradient $\rho \equiv 0$.

If X contains no nontrivial rectifiable paths, then $\rho \equiv 0$ is an upper gradient of any function. It follows that upper gradients are potentially useful objects only if the underlying space has plenty of rectifiable curves.

It is well known that, for a function $u : [a,b] \to \mathbb{R}$, a necessary and sufficient condition for

$$u(x) = u(a) + \int_a^x u'(t)dt, \quad x \in [a,b], \tag{2.14}$$

is that u is absolutely continuous. Unlike conformal maps, quasiconformal maps can be rather irregular. For this purpose an absolute continuity property in $\mathbb{R}^n$ is needed. This idea goes back to Tonelli [absolute continuity in the sense of Tonelli (ACT), nowadays absolute continuity on lines], and the idea from a different point of view was developed by Sobolev. In general, absolute continuity in $\mathbb{R}^n$ for $n \geq 2$ is much more problematic than in intervals $[a,b] \subset \mathbb{R}$. Here we develop the theory in a metric space X; for $X = \mathbb{R}^n$, this leads to the aforementioned theories.

Let γ be a path in a metric space X and let $l(\gamma)$ denote the length of γ. A function u is said to be **ACC**$_p$ or absolutely continuous on p-almost every curve if $u \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$ for p-almost every rectifiable arc-length parameterized path γ in X.

Next assume that μ is a Borel regular measure in X. The following definition is due to [38] and [291] and is a weakening of the concept of upper gradient.

Let u be an arbitrary real-valued function on X, and let ρ be a nonnegative Borel function on X. If there exists a family $\Gamma \subset \Gamma_{\text{rect}}$ such that $M_p(\Gamma) = 0$ and inequality (2.13) is true for all paths γ in $\Gamma_{\text{rect}} \setminus \Gamma$, then ρ is said to be a p-**weak upper gradient** of u. If inequality (2.13) holds for p-modulus almost all paths in a set $A \subset X$, then ρ is said to be a p-**weak upper gradient of** u **on** A. As the exponent p is usually fixed, in both cases ρ is simply called a weak upper gradient of u. Here Γ_{rect} denotes the family of all rectifiable paths $\gamma : [a,b] \to X$.

While the notion of upper gradients does not involve measures or the notion of p-modulus (and hence is independent of the index p), the notion of p-weak upper gradient is strongly dependent on the measure and concept of p-modulus.

Let $\tilde{N}^{1,p}(X) = \tilde{N}^{1,p}(X,d,\mu)$ be the set of all functions $u : X \to \mathbb{R}$ that belong to $L^p(X),\ p \geq 1$, and have a p-weak upper gradient $\rho \in L^p(X)$.

Note that $\tilde{N}^{1,p}$ is also a vector space since if α, β are real numbers and $u_1, u_2 \in \tilde{N}^{1,p}$ with respective weak upper gradients ρ_1, ρ_2, then $|\alpha|\rho_1 + |\beta|\rho_2$ is a weak upper gradient of $\alpha u_1 + \beta u_2$. Given a function u in $\tilde{N}^{1,p}$, let

$$\|u\|_{\tilde{N}^{1,p}} = \|u\|_{L^p} + \inf_\rho \|\rho\|_{L^p},$$

where the infimum is taken over all p-integrable weak upper gradients ρ of u.

It is easy to see that $\|\cdot\|_{\tilde{N}^{1,p}}$ satisfies the triangle inequality:

$$\|u+v\|_{\tilde{N}^{1,p}} \leq \|u\|_{\tilde{N}^{1,p}} + \|v\|_{\tilde{N}^{1,p}}.$$

Given functions u, v in $\tilde{N}^{1,p}$, let $u \approx v$ if $\|u - v\|_{\tilde{N}^{1,p}} = 0$. It can easily be seen that $\approx$ is an equivalence relation, partitioning $\tilde{N}^{1,p}$ into equivalence classes. This collection of equivalence classes under the norm of $\tilde{N}^{1,p}$ is a normed vector space.

The **Newtonian space** corresponding to the index p, $1 \le p < \infty$, denoted $N^{1,p}(X)$, is defined to be the normed space $\tilde{N}^{1,p}(X,d,\mu)/\approx$, with the norm $\|u\|_{N^{1,p}} := \|u\|_{\tilde{N}^{1,p}}$.

If u,v are functions in $\tilde{N}^{1,p}$, then the functions $\min\{u,v\}$, $\max\{u,v\}$, and $|u|$ are also in $\tilde{N}^{1,p}$. This follows from the corresponding properties of absolutely continuous functions. Thus, $N^{1,p}(X)$ also enjoys a lattice property. Also, if $\lambda \ge 0$, then $\min\{u,\lambda\}$ is in $\tilde{N}^{1,p}$, and if $\lambda \le 0$, then $\max\{u,\lambda\}$ is also in $\tilde{N}^{1,p}$.

The following lemma clarifies the connection between ACC$_p$-functions and functions in $\tilde{N}^{1,p}$.

Lemma 2.3. *If u is a function in $\tilde{N}^{1,p}$, then u is* ACC$_p$.

Proof. By the definition of $\tilde{N}^{1,p}$, u has a p-integrable weak upper gradient ρ. Let Γ be the collection of all paths in Γ_{rect} for which inequality (2.13) does not hold. Then, by the definition of weak upper gradients, $M_p(\Gamma) = 0$. Let Γ_1 be the collection of all paths in Γ_{rect} that have some subpath belonging to Γ. Then,

$$M_p(\Gamma_1) \le M_p(\Gamma) = 0.$$

Let Γ_2 be the collection of all paths γ in Γ_{rect} such that $\int_\gamma \rho\, ds = \infty$. As ρ is p-integrable, $M_p(\Gamma_2)$ is zero. Hence, $M_p(\Gamma_1 \cup \Gamma_2)$ is zero. If γ is a path in Γ_{rect} that is not in $\Gamma_1 \cup \Gamma_2$, then γ has no subpath in Γ_1, and hence for all x,y in $|\gamma|$,

$$|u(x) - u(y)| \le \int_{\gamma_{xy}} \rho\, ds < \infty.$$

Hence, if (a_i,b_i), $i = 1,2,\ldots,m$, are disjoint intervals in $[0,l(\gamma)]$, then

$$\sum_i |u(\gamma(b_i)) - u(\gamma(a_i))| \le \int_{\cup(a_i,b_i)} \rho(\gamma(s))\, ds,$$

and this clearly shows that $u \circ \gamma$ is absolutely continuous on $[0,l(\gamma)]$, as required. Thus, u is absolutely continuous on each path γ in $\Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2)$. $\square$

Note that the above lemma remains valid if the function u is required only to have a p-integrable upper gradient, without itself being p-integrable.

By Lemma 2.3, the space $\tilde{N}^{1,p}$ consists of ACC$_p$-functions u such that $u \in L^p(X)$ and u has a p-weak upper gradient $\rho \in L^p(X)$.

Next we prove a couple of lemmas that provide some further information on pointwise behavior of functions in $\tilde{N}^{1,p}$.

Lemma 2.4. *Suppose u is a function in $\tilde{N}^{1,p}$ such that $\|u\|_{L^p} = 0$. Then the family*

$$\Gamma = \{\gamma \in \Gamma_{\text{rect}} : u(x) \ne 0 \text{ for some } x \in |\gamma|\}$$

has zero p-modulus.

Proof. Since $\|u\|_{L^p} = 0$, the set $E = \{x \in X : u(x) \neq 0\}$ has measure zero. Given $C \subset X$, set

$$\Gamma_C = \{\gamma \in \Gamma_{\mathrm{rect}} : |\gamma| \cap C \neq \emptyset\}$$

and

$$\Gamma_C^+ = \{\gamma \in \Gamma_C : \gamma \text{ meets } C \text{ in a set of positive length}\}.$$

With this notation,

$$\Gamma = \Gamma_E^+ \cup (\Gamma_E \setminus \Gamma_E^+).$$

The subfamily Γ_E^+ can be disregarded since

$$M_p(\Gamma_E^+) \leq \|\infty \cdot \chi_E\|_{L^p} = 0.$$

Note that the set E need not be a Borel set, but it can be replaced by a Borel set E^* including E such that $\mu(E^* \setminus E) = 0$ and hence the function χ_E can be replaced by χ_{E^*}, which is a Borel function. Thus, $\infty \cdot \chi_{E^*}$ is an admissible function for Γ_E^+.

The paths γ in $\Gamma_E \setminus \Gamma_E^+$ intersect E only on a set of linear measure zero, and hence with respect to linear measure almost everywhere on γ the function u takes on the value of zero. By the fact that γ also intersects E, u is not absolutely continuous on γ since u is not even continuous on γ. By Lemma 2.3,

$$M_p(\Gamma_E \setminus \Gamma_E^+) = 0,$$

yielding $M_p(\Gamma) = 0$. □

This lemma indicates that functions in $\tilde{N}^{1,p}$ are well defined outside a small set. For example, not all sets E of zero measure in $\mathbb{R}^n$ have the property that the p-modulus of the family Γ_E is zero; hence, unlike L^p-functions, Newtonian functions on $\mathbb{R}^n$ cannot be arbitrarily changed on sets of measure zero. The above lemma yields the following:

Corollary 2.2. *If u_1, u_2 are two functions in $\tilde{N}^{1,p}(X)$ such that $\|u_1 - u_2\|_{L^p} = 0$, then u_1 and u_2 belong to the same equivalence class in $N^{1,p}(X)$.*

We shall not prove the following result of N. Shanmugalingam [291]; in fact, we do not need this result; see also [38] for a slightly different approach.

Theorem 2.4. *The space $N^{1,p}(X)$ with the norm $\|\ \|_{N^{1,p}}$ is a Banach space.*

The concepts of capacity and modulus are interlocked in some situations. Let $E, F \subset X$. We define the p-**capacity** of a **condenser** $(E,F) = (E,F;X)$ as follows:

$$\operatorname{cap}_p(E,F) = \inf \int_X \rho^p \, d\mu, \tag{2.15}$$

where the infimum is taken over all upper gradients of all real-valued functions u on X such that $u|E \leq 0$ and $u|F \geq 1$. Notice that no regularity assumption is made on u.

Next let (E,F) also stand for the family of paths γ that join E and F in X.

Theorem 2.5. $\mathrm{cap}_p(E,F) = M_p(E,F)$.

Proof. If u is a function on X with $u|E \le 0$ and $u|F \ge 1$, and if ρ is any upper gradient of u, then

$$1 \le |u(x) - u(y)| \le \int_\gamma \rho \, ds$$

for any rectifiable path γ joining a point $x \in E$ and a point $y \in F$. Therefore,

$$M_p(E,F) \le \mathrm{cap}_p(E,F).$$

On the other hand, if ρ is an admissible function for the family (E,F), then define

$$u(x) = \inf \int_{\gamma_x} \rho \, ds,$$

where the infimum is taken over all paths γ_x joining E to the point x in X. Then $u|E = 0$, $u|F \ge 1$, and ρ is an upper gradient of u. Indeed, let $x_0, y_0 \in X$ and let γ_0 be a path joining x_0 to y_0. Assuming $u(y_0) \ge u(x_0)$, we have

$$\begin{aligned} |u(y_0) - u(x_0)| = u(y_0) - u(x_0) &= \inf \int_{\gamma_{y_0}} \rho \, ds - \inf \int_{\gamma_{x_0}} \rho \, ds \\ &\le \inf \int_{\gamma_{x_0}} \rho \, ds + \int_{\gamma_0} \rho \, ds - \inf \int_{\gamma_{x_0}} \rho \, ds = \int_{\gamma_0} \rho ds \end{aligned}$$

because the path $\gamma_{x_0} + \gamma_0$ for each path γ_{x_0} joins E to y_0. The case $u(y_0) < u(x_0)$ follows by symmetry. This implies that $\mathrm{cap}_p(E,F) \le M_p(E,F)$ and the theorem follows. $\square$

Remark 2.9. If $X = \mathbb{R}^n$, then it is easy to see that the function u in the definition of the p-capacity can be assumed to be measurable provided that E and F are compact and disjoint (note that an upper gradient is a Borel function). Since u can be truncated so that $u(x) \in [0,1]$, we may assume that u is locally integrable in this case. With some extra work the function u can be made continuous, or even locally Lipschitz; see [107, 291].

2.7 ACC_p Functions in $\mathbb{R}^n$ and Capacity

It turns out that in $\mathbb{R}^n$ a much weaker condition implies the ACC_p condition. Such a condition is provided by the class of ACL or ACL^p (ACL = absolutely continuous on lines) functions.

Denote $\mathbb{R}^{n-1}_i = \{x \in \mathbb{R}^n | x_i = 0\}$. Furthermore, let P_i be the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$. Explicitly, $P_i x = x - x_i e_i$.

Let $Q = \{x \in \mathbb{R}^n | a_i \le x_i \le b_i\}$ be a closed n-interval. A mapping $f : Q \to \mathbb{R}$ is said to be ACL (**absolutely continuous on lines**) if f is absolutely continuous on almost every line segment in Q, parallel to the coordinate axes. More precisely, if E_i is the set of all $x \in P_i Q$ such that the mapping $t \to f(x + te_i)$ is not absolutely continuous on $[a_i, b_1]$, then $m_{n-1}(E_i) = 0$ for $1 \le i \le n$.

If U is an open set in $\mathbb{R}^n$, a mapping $f : U \to \mathbb{R}$ is called ACL if $f|Q$ is ACL for every closed n-interval $Q \subset U$.

If D and D' are domains in $\mathbb{R}^n$, a homeomorphism $f : D \to D'$ is called ACL if each coordinate function f_i of $f = (f_1, \dots, f_n)$ is ACL.

An ACL mapping $f : U \to \mathbb{R}$ (or $[-\infty, \infty]$) is said to be ACL^p, $p \ge 1$, if f is locally L^p-integrable in U and if the partial derivatives $\partial_i f$ (which exist a.e. and are measurable) of f are locally L^p-integrable as well.

A homeomorphism $f : D \to D'$ is ACL^p if each coordinate function of f is ACL^p.

Observe the following differences in the definitions of ACL^p and ACC$_p$ functions: In the space $N^{1,p}$ the functions u and their p-weak upper gradients belong to $L^p(X)$. For the ACL^p functions in an open set $U \subset \mathbb{R}^n$, this is required only locally.

Smoothing of functions. Here we have collected some (standard) approximation results; see [110, 316, 339].

Theorem 2.6. *Suppose that $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, is* ACL^p. *Then there is a sequence of functions $f_j \in C^1(U)$ such that for each compact subset $F \subset U$, $f_j \to f$ in $L^p(F)$ and $\partial_i f_j \to \partial_i f$ in $L^p(F)$ for each $i = 1, 2, \dots, n$.*

Remark 2.10. The proof of Theorem 2.6 is based on the standard convolution approximation of f. If $f, \partial_i f \in L^p(U)$ and the approximation is needed in $L^p(U)$, then the proof is much more difficult.

Remark 2.11. If $f \in C(U)$ [then $f \in L^p(F)$ for each compact set $F \subset U$ and each $p \ge 1$], then f_j can be chosen so that $f_j \to f$ uniformly on each compact subset of U.

Our aim now is to show that an ACL^p function is actually absolutely continuous on a p-a.e. path. This is a theorem of Fuglede [64]. We start with a lemma that we formulate in a general metric space X.

Lemma 2.5. *Suppose that E is a Borel set in X and that $f_k : E \to [-\infty, \infty]$ is a sequence of Borel functions that converge to a Borel function $f : E \to [-\infty, \infty]$ in $L^p(E)$. Then there is a subsequence $f_{k_1}, f_{k_2}, \dots$ such that*

$$\int_\gamma |f_{k_j} - f| ds \to 0 \tag{2.16}$$

for all rectifiable paths γ in E, except possibly for a family Γ such that $M_p(\Gamma) = 0$.

Proof. Choose a subsequence (f_{k_j}) such that

$$\int_E |f_{k_j} - f|^p dm < 2^{-j(p+1)}.$$

Set $g_j = |f_{k_j} - f|$, and let Γ be the family of all rectifiable paths γ such that $\gamma \subset E$ and $\int_\gamma g_j ds \nrightarrow 0$. We show that $M_p(\Gamma) = 0$.

Let Γ_j be the family of all rectifiable paths γ in E such that $\int_\gamma g_j ds > 2^{-j}$. Then $2^j g_j$ is admissible for Γ_j if we define $g_j(x) = 0$ for $x \notin E$. Thus,

$$M_p(\Gamma_j) \le 2^{pj} \int_E g_j^p dm < 2^{-j}.$$

On the other hand, $\Gamma \subset \bigcup_{j=i}^\infty \Gamma_j$ for every $i = 1, 2, \ldots$. Hence,

$$M_p(\Gamma) \le \sum_{j=i}^\infty M_p(\Gamma_j) < \sum_{j=i}^\infty 2^{-j} = 2^{-i+1}$$

for every $i = 1, 2, \ldots$. Consequently, $M_p(\Gamma) = 0$. □

Remark 2.12. Lemma 2.5 has an important consequence: If ρ_i is a Cauchy sequence of nonnegative Borel functions in L^p coverging to a Borel function ρ in L^p, then there is a subsequence ρ_{i_k} such that for p almost every path γ in Γ_{rect},

$$\lim_{k\to\infty} \int_\gamma \rho_{i_k} ds = \int_\gamma \rho ds < \infty.$$

We formulate the Fuglede theorem for continuous functions only (quasiconformal mappings are continuous). However, it holds for general ACL^p functions.

Theorem 2.7. (Fuglede's theorem). *Suppose that U is an open set in $\mathbb{R}^n$ and that $f : U \to \mathbb{R}$ is continuous and* ACL^p. *Let Γ be the family of all rectifiable paths in U on which f is not absolutely continuous. Then $M_p(\Gamma) = 0$.*

Proof. We express U as the union of an expanding sequence of open sets U_j such that each $\overline{U}_j$ is a compact subset of U. Let Γ_j be the family of all paths $\gamma \in \Gamma$ such that $\gamma \subset U_j$. Then $\Gamma \subset \cup \Gamma_j$, whence

$$M_p(\Gamma) \le \sum_{j=1}^\infty M_p(\Gamma_j).$$

It thus suffices to prove that $M_p(\Gamma_j) = 0$ for an arbitrary fixed j.

By Theorem 2.6, there is a sequence of C^1-functions $f_k : U \to \mathbb{R}$ such that $f_k \to f$ in $L^p(\overline{U}_j)$ and such that $\partial_i f_k \to \partial_i f$ in $L^p(\overline{U}_j)$, $1 \le i \le n$. Passing to a subsequence, we may assume, by Lemma 2.5 and by the fact that partial derivatives of a continuous function are Borel functions, that

$$\int_\gamma |\partial_i f_k - \partial_i f|\, ds \to 0$$

for all $1 \le i \le n$ and for all rectifiable paths γ in U_j except for a family Γ_0 with $M_p(\Gamma_0) = 0$. We show that $\Gamma_j \subset \Gamma_0$, which will prove that $M_p(\Gamma_j) = 0$.

Suppose that $\gamma \in \Gamma_j \setminus \Gamma_0$. Let $\beta : [0,c] \to U_j$ be the parameterization of γ by arc length. We write

$$\beta(t) = \sum_{i=1}^n \beta_i(t) e_i.$$

Since $f_k \circ \beta$ is absolutely continuous, we have for every $0 \le t \le c$,

$$\begin{aligned} f_k(\beta(t)) - f_k(\beta(0)) &= \int_0^t (f_k \circ \beta)'(u)\, du \qquad (2.17)\\ &= \int_0^t \sum_{i=1}^n \partial_i f_k(\beta(u))\, \beta_i'(u)\, du. \end{aligned}$$

Here $|\beta_i'(u)| \le |\beta'(u)| = 1$ for almost every $u \in [0,c]$. As $k \to \infty$, the left-hand side of (2.17) tends to $f(\beta(t)) - f(\beta(0))$; see Remark 2.11. On the other hand,

$$\left| \int_0^t \sum_{i=1}^n \partial_i f_k(\beta(u))\, \beta_i'(u)\, du - \int_0^t \sum_{i=1}^n \partial_i f(\beta(u))\, \beta_i'(u)\, du \right|$$

$$\begin{aligned} &\le \sum_{i=1}^n \int_0^t |\partial_i f_k(\beta(u)) - \partial_i f(\beta(u))|\, |\beta_i'(u)|\, du \\ &\le \sum_{i=1}^n \int_\gamma |\partial_i f_k - \partial_i f|\, ds \to 0. \end{aligned}$$

Hence, (2.17) yields

$$f(\beta(t)) - f(\beta(0)) = \int_0^t \sum_{i=1}^n \partial_i f(\beta(u))\, \beta_i'(u)\, du. \qquad (2.18)$$

As an integral, $f \circ \beta$ is absolutely continuous. In other words, f is absolutely continuous on γ. Since $\gamma \in \Gamma_j \subset \Gamma$, this is a contradiction. □

Remark 2.13. By Theorem 2.7, every continuous ACL^p function is ACC_p in U if $\nabla u \in L^p(U)$. Note that from (2.18) it follows that $\rho = |\nabla u|$ is a p-weak upper gradient of u. If u is not continuous (only ACL^p), the function ρ can be taken as a Borel function since for each measurable function $v : U \to [0,\infty]$, there is a Borel function $\rho : U \to [0,\infty]$ such that $\rho \geq v$ and $\rho = v$ a.e.

2.8 Linear Dilatation

Let (X,d) be a metric space. For $x \in X$ and $r > 0$, let $B(x,r)$ be the open ball $\{y \in X : d(y,x) < r\}$ centered at x and radius r. Let (Y,d') be another metric space and $f : X \to Y$ a map. For $x \in X$ and $r > 0$, we set

$$L(x,f,r) = \sup\,\{d'(f(y),f(x)) : y \in B(x,r)\}$$

and

$$l(x,f,r) = \inf\,\{d'(y,f(x)) : y \in Y \setminus B(x,r)\}$$

and $H(x,f,r) = L(x,f,r)/l(x,f,r)$. Note that when $L(x,f,r) = 0 = l(x,f,r)$, we put $H(x,f,r) = \infty$; we also interpret $\inf\,\emptyset = 0$. The **linear dilatation** of f at x is defined as

$$H(x,f) = \limsup_{r\to 0} H(x,f,r).$$

It has turned out that $H(x,f)$ is difficult with maps f between two metric spaces: The concept should be replaced by a more global concept called quasisymmetry. This will be studied in Chapter 10. However, the linear dilatation is one of the basic concepts for homeomorphisms between two domains in $\mathbb{R}^n$: A homeomorphism $f : D \to \mathbb{R}^n$ for a domain $D \subset \mathbb{R}^n$ is quasiconformal if and only if $H(x,f) \leq C < \infty$ at every point $x \in D$. This chapter is devoted to the study of the implications of various boundedness conditions on $H(x,f)$.

Let D be a domain in $\mathbb{R}^n$, $n \geq 1$, and $f : D \to \mathbb{R}^n$ a homeomorphism (embedding). For $x \in D$ and $0 < r < d(x,\partial D)$, we have

$$L(x,f,r) = \sup\,\{|f(y) - f(x)| : y \in \partial B(x,r)\}$$

and

$$l(x,f,r) = \inf\{|f(y) - f(x)| : u \in \partial B(x,r)\}.$$

Now at every point $x \in D$, $H(x,f) \in [1,\infty]$ and if f is differentiable at x, then $H(x,f) = \|f'(x)\|/l(f'(x)) = \lambda_n/\lambda_1$ provided that $l(f'(x)) > 0$. Here $\lambda_1 = l(f(x)) = \inf_{|h|=1} |f'(x)h|$ is the "minimal stretching" of the linear map $f'(x)$; see Section 2.2.

Mappings with $H(x,f) < \infty$ a.e. If a homeomorphism $f : D \to \mathbb{R}^n$ satisfies $H(x,f) < \infty$ a.e. in D or even $\operatorname{ess\,sup} H(x,f) < \infty$, then f need not be ACL. An example is constructed from the Cantor staircase function $g : [0,1] \to [0,1]$, i.e., g is a continually increasing function onto $[0,1]$ with the property $g'(x) = 0$ for a.e. $x \in [0,1]$. Let $g(x) = 0,\ x \leq 0$, and $g(x) = 1,\ x \geq 1$. Now $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x,y) = (g(x)+x,y)$ is a homeomorphism with $H(x,f) = 1$ a.e., but f is not ACL.

Note that the mapping f is not quasiconformal in the sense of the definition in Chapter 3. To see this, let $\Gamma = \{\{x\} \times [0,1] : x \in C\}$, where $C \subset [0,1]$ is the Cantor 1/3-set. Now $M_2(\Gamma) = 0$ since the function $\rho(x) = \infty$, $x \in C \times [0,1]$, $\rho(x) = 0$ otherwise, is admissible for Γ, but

$$\int_{\mathbb{R}^2} \rho^2 \, dx = 0$$

because $m(C \times [0,1]) = 0$. On the other hand, g maps the set C onto a set of positive linear measure; in fact, $m_1(gC) = 1$, and the same is true for the map $x \mapsto g(x) + x$. Thus,

$$f\Gamma = \{\{y\} \times [0,1] : y \in A\}$$

and A is a Borel set with $m_1(A) > 0$. By the example in Section 2.1, $M_2(f\Gamma) > 0$, which contradicts (1.3).

The case $n = 1$ is of special interest. An increasing homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is called K-**quasisymmetric** if it satisfies

$$\frac{1}{K} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq K \tag{2.19}$$

for all $x \in \mathbb{R}$ and $t > 0$. If f is K-quasisymmetric, then $H(x, f) \leq K$ for all $x \in \mathbb{R}$. Now Ahlfors and Beurling [5] constructed for each $K > 1$ a K-quasisymmetric mapping f that is not absolutely continuous. For more striking examples of such mappings; see [309]. Hence, no boundedness condition on $H(x, f)$, except $H(x, f) \equiv 1$, implies the absolute continuity for quasisymmetric maps.

Remark 2.14. Quasisymmetric maps on the line form an important class of mappings. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is quasiconformal and maps the real axis $\mathbb{R}$ onto itself (and is increasing there), then $f|\mathbb{R}$ is quasisymmetric. Conversely, every K-quasisymmetric map $f : \mathbb{R} \to \mathbb{R}$ can be extended to a K^2-quasiconformal map $f^* : \mathbb{R}^2 \to \mathbb{R}^2$; see [1].

Although a homeomorphism f with $H(x, f) < \infty$ a.e. can be irregular, it still has some nice properties.

Theorem 2.8. *Suppose that a homeomorphism $f : D \to \mathbb{R}^n$ satisfies $H(x, f) < \infty$ a.e. in D. Then f is a.e. differentiable.*

Proof. Fix an open set $G \subset\subset D$ and let $\Phi(E) = |f(E)|$ for each Borel set $E \subset G$. Here, and in the following, $|A|$ means the Lebesgue measure of a set $A \subset \mathbb{R}^n$. Then Φ is a finite Borel measure on G and hence has a finite derivative

$$\Phi'(x) = \lim_{r \to 0} \frac{\Phi(B(x,r))}{|B(x,r)|}$$

at a.e. $x \in G$.

Now at almost every point x of G, $\Phi'(x)$ exists and $H(x, f) < \infty$. Fix such a point x. Let $y \in G$ with $0 < |x - y| < d(x, \partial G)$. Now

$$\left(\frac{|f(y)-f(x)|}{|y-x|}\right)^n \leq \left(\frac{L(x,f,|y-x|)}{l(x,f,|y-x|)}\right)^n \left(\frac{l(x,f,|x-y|)}{|y-x|}\right)^n$$
$$\leq H(x,f,|y-x|)^n \Phi(B(x,|y-x|)/|B(x,|y-x|)|.$$

Letting $y \to x$, we see that

$$\limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|} \leq H(x,f)\ \Phi'(x)^{1/n} < \infty.$$

By the Rademacher–Stepanov theorem (see, e.g., [316]), the mapping f is a.e. differentiable in G and the theorem follows. □

Theorem 2.9. *Suppose that $H(x,f) \in L^s_{\mathrm{loc}}(D)$, $s \in [1,\infty]$, for a homeomorphism $f : D \to \mathbb{R}^n$. Then $f' \in L^p_{\mathrm{loc}}(D)$ with $p = sn/(n-1+s)$ and $p = n$ if $s = \infty$.*

Proof. We may assume that f is sense-preserving. Since $H(x,f) < \infty$ a.e. in D, Theorem 2.8 implies that $f'(x)$ exists a.e. If f is differentiable at x and $H(x,f) < \infty$, then an elementary argument shows that

$$\|f'(x)\|^n \leq H(x,f)^{n-1}J(x,f), \tag{2.20}$$

where $J(x,f)$ is the Jacobian of f, i.e., the determinant of $f'(x)$; see Section 2.2.

Fix an open set $G \subset\subset D$. For $s < \infty$, (2.20) and the Hölder inequality imply

$$\int_G \|f'(x)\|^p dx \leq \int_G H(x,f)^{p(n-1)/n} J(x,f)^{p/n} dx$$
$$\leq \left[\int_G H(x,f)^{p(n-1)/(n-p)} dx\right]^{(n-p)/n} \left[\int_G J(x,f) dx\right]^{p/n}$$
$$\leq \left[\int_G H(x,f)^s dx\right]^{(n-p)/n} |f(G)|^{p/n} < \infty,$$

as required. For $s = \infty$, the proof is similar. Note that the inequality

$$\int_G J(x,f) dx \leq |f(G)|$$

always holds for an a.e. differentiable homeomorphism; see, e.g., [246]. □

Linear dilatation and ACL. Here we prove a recent result in [146]; the result is an extension of an earlier result due to Gehring [65].

Theorem 2.10. *Suppose that a homeomorphism $f : D \to \mathbb{R}^n$ of a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and $s \in (1,\infty]$ satisfy the conditions:*

(a) $s > n/(n-1)$;
(b) $H(x,f) < \infty$ *for each* $x \in D$;
(c) $H(x,f) \in L^s_{\mathrm{loc}}(D)$.
Then f *is* ACL.

Remark 2.15. In (b) it suffices to assume that $H(x,f) < \infty$ for each $x \in D \setminus S$, where S has σ-finite $(n-1)$-dimensional Hausdorff measure. Note that condition (a) rules out the case $n = 1$.

Proof. Pick a closed cube $Q \subset\subset D$ whose sides are parallel to the coordinate axes and write $Q' = (1/2)Q$ for the cube with the same center as Q and side length half of that of Q. In order to show that f is ACL, it suffices to show that f is absolutely continuous on almost every line segment of Q' parallel to the coordinate axes. Renormalizing, we may assume that $Q = [-2,2]^n$ and by symmetry it is sufficient to consider segments parallel to the x_n-axis. Let $P : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection $P(x) = x - x_n \cdot e_n$ and for $y \in P(Q) \subset \mathbb{R}^{n-1}$ write $I = I(y) = Q' \cap P^{-1}(y)$ for the line segment parallel to the x_n-axis in Q'.

Next, for a Borel set $E \subset P(Q)$, one gets

$$\Phi(E) = |f(Q \cap P^{-1}(E))| \leq |f(Q)| < \infty.$$

Then Φ is a finite Borel measure in $P(Q)$ and hence has a finite derivative $\Phi'(y)$ for almost all $y \in P(Q')$. We choose $y \in P(Q')$ such that (1) $\Phi'(y)$ exists and (2) $H(x,f) \in L^s(I(y))$. The last assertion follows from the Fubini theorem. It suffices to show that f is absolutely continuous on $I(y)$.

To this end, let $F \subset I(y)$ be a compact set. For each $k = 0, 1, \dots$, set

$$F_k = \{x \in F : 2^k \leq H(x,f) < 2^{k+1}\}.$$

Then F_k is a Borel set and $F = \cup F_k$. Note also that $H(x,f) \geq 1$ for every x. We first derive the following estimate

$$\mathscr{H}^1(fF_k) \leq c2^k \mathscr{H}^1(F_k)^{(n-1)/n}, \tag{2.21}$$

where $c = (2^{2n+1}\Phi'(y))^{1/n}$. Here $\mathscr{H}^1$ stands for the 1-dimensional Hausdorff measure, i.e., $\mathscr{H}^1(S)$ is the length of the set S in $\mathbb{R}^n$.

For (2.21), fix k and, for each $j = 1, 2, \dots$, consider the set

$$F_{k,j} = \{x \in F_k : L(x,f,r)^n \leq 2^{n(k+1)}|fB(x,r)|/\Omega_n \ \text{ for } \ 0 < r < 1/j\},$$

where $\Omega_n = |B(0,1)|$. The sets $F_{k,j}$ are Borel sets and $F_{k,j} \subset F_{k,j+1}$ with

$$F_k = \bigcup_{j=1}^{\infty} F_{k,j}. \tag{2.22}$$

To see (2.22), let $x \in F_k$. Then $H(x,f) < 2^{k+1}$ and, hence, there is a j such that

$$L(x,f,r)/l(x,f,r) < 2^{k+1}$$

for all $0 < r < 1/j$ and we obtain

$$L(x,f,r)^n < 2^{n(k+1)} l(x,f,r)^n \leq 2^{n(k+1)} |fB(x,r)|/\Omega_n.$$

This shows that $x \in F_{k,j}$ and (2.22) follows.

By the monotonicity and (2.22), it suffices to prove (2.21) for $F_{k,j}$ instead of F_k. Fix j and let F' be an arbitrary compact subset of $F_{k,j}$. Let $\varepsilon > 0$ and $t > 0$. The continuity of the mapping $(x,r) \mapsto L(x,f,r)$ gives $\delta,\ 0 < \delta < 1/j$, such that $L(x,f,r) < t/2$ for $0 < r < \delta$ and for all $x \in F'$.

Next we use a Besicovitch-type covering lemma in $\mathbb{R}$: If $C \subset \mathbb{R}$ is a compact set and $\varepsilon, \delta > 0$, then there are $0 < r < \delta$ and points $x_i \in C,\ i = 1, \ldots, l$, such that

$$\cup\,(x_i - r, x_i + r) \supset C, \qquad lr \leq \mathscr{H}^1(C) + \varepsilon,$$

and each $x \in \mathbb{R}$ belongs to at most two different intervals $(x_i - r, x_i + r)$. This gives a covering F' by a finite number of balls $B_i = B(x_i, r),\ 0 < r < \delta,\ i = 1, \ldots, l$, where (i) $x_i \in F',\ i = 1, \ldots, l$, (ii) each point of $\mathbb{R}^n$ lies in at most two B_i, and (iii) $lr \leq \mathscr{H}^1(F') + \varepsilon$. Note that the renormalizing condition gives

$$B_i \subset Q \cap P^{-1}(B), \tag{2.23}$$

where $B = B^{n-1}(y,r)$.

The sets $f(B_i)$ cover $f(F')$ and

$$\operatorname{diam}(fB_i) \leq 2L(x_i,f,r) < t.$$

Hence,

$$\mathscr{H}^1_t(fF') \leq \sum_{i=1}^{l} \operatorname{diam}(fB_i),$$

where

$$\mathscr{H}^1_t(A) = \inf\left\{\sum \operatorname{diam}(A_i) : \cup A_i \supset A,\ \operatorname{diam}(A_i) < t\right\},$$

and the Hölder inequality together with the definition of $F_{k,j}$ yields

$$\begin{aligned} \mathscr{H}^1_t(fF')^n &\leq \left(\sum_{i=1}^{l} \operatorname{diam}(fB_i)\right)^n \leq l^{n-1} \sum_{i=1}^{l} \operatorname{diam}(fB_i)^n \\ &\leq l^{n-1} 2^n \sum_{i=1}^{l} L(x_i,f,r)^n \leq \frac{l^{n-1} 2^n 2^{n(k+1)}}{\Omega_n} \sum_{i=1}^{l} |fB_i|\,. \end{aligned} \tag{2.24}$$

Since f is a homeomorphism, we obtain from (ii) and (2.23) that

$$\sum_{i=1}^{l} |fB_i| \leq 2\left|\bigcup_{i=1}^{l} fB_i\right| \leq 2\Phi(B).$$

Thus, (2.24) and (iii) yield

$$\begin{aligned}\mathscr{H}_t^1(fF')^n &\le 2^{n(k+2)+1}(\mathscr{H}^1(F')+\varepsilon)^{n-1}\Phi(B)/m_{n-1}(B)\\ &\le 2^{n(k+2)+1}(\mathscr{H}^1(F_{k,j})+\varepsilon)^{n-1}\Phi(B)/m_{n-1}(B).\end{aligned}$$

Since $\mathscr{H}_t^1(fF') \to \mathscr{H}^1(fF')$ as $t \to 0$, letting first $r \to 0$, then $\varepsilon \to 0$, and finally $t \to 0$, we obtain

$$\mathscr{H}^1(fF')^n \le 2^{n(k+2)+1}\mathscr{H}^1(F_{k,j})^{n-1}\Phi'(y). \tag{2.25}$$

Now F' is an arbitrary compact subset of $F_{k,j}$. Hence, (2.25) holds for $F_{k,j}$ on the left-hand side of (2.25). This leads to estimate (2.21).

Since $fF = \cup\, fF_k$, (2.21) implies

$$\mathscr{H}^1(fF) \le \sum \mathscr{H}^1(fF_k) \le c\sum 2^k \mathscr{H}^1(F_k)^{(n-1)/n}. \tag{2.26}$$

The sets $F_k,\ k = 1,\dots,$ are disjoint and hence the integral estimate

$$\sum_{k=0}^{\infty} 2^{ks}\mathscr{H}^1(F_k) \le \int_F H(x,t)^s dx_n \tag{2.27}$$

is elementary. By (2.26), (2.27), and the Hölder inequality, we obtain

$$\begin{aligned}\mathscr{H}^1(fF) &\le c_1\left(\sum_{k=0}^{\infty} 2^{ks}\mathscr{H}^1(F_k)\right)^{(n-1)/n}\left(\sum_{k=0}^{\infty} 2^{k(n-s(n-1))}\right)^{1/n} \qquad (2.28)\\ &\le c_2\left(\int_F H(x,f)^s dx_n\right)^{(n-1)/n},\end{aligned}$$

where c_2 depends only on n, s, and $\Phi'(y)$. Note that the series

$$\sum_{k=0}^{\infty} 2^{k(n-s(n-1))}$$

converges because $s > n/(n-1)$ and hence $n - s(n-1) < 0$. Inequality (2.28) shows that f is absolutely continuous on $I(y)$, as required. □

Corollary 2.3. *Under the conditions of Theorem 2.10, f is a.e. differentiable and $f' \in L^p_{\mathrm{loc}}(D),\ p = sn/(n-1+s)$. In particular, f is* ACL^p.

Corollary 2.4. *Suppose that $f : D \to \mathbb{R}^n$ is a homeomorphism such that $H(x,f) \le c < \infty$ at every point $x \in D$. Then f is differentiable a.e. and* ACL^n.

Remark 2.16. Let $f : D \to \mathbb{R}^n$ be as in Corollary 2.4. Then f is not only ACL^n but *ACL*^{p} for some $p = p(n,c) > n$. This is the well-known result due to Bojarski [27] for $n = 2$ and Gehring [110] for $n \ge 3$. This follows from the fact that the condition

$H(x, f) \le c$ implies the quasiconformality of f (see Theorems 2.11 and 2.12), and the result is called the higher integrability of the derivative of a quasiconformal map. Many important properties (smoothness, change of Hausdorff measure under quasiconformal maps) can be derived from this result. The value $p(n, c)$ is known for $n = 2$ [16], but unknown for $n \ge 3$.

2.9 Analytic Definition for Quasiconformality

In this chapter we shall study still another definition, the so-called analytic definition for quasiconformality. According to this definition, a homeomorphism (embedding) $f : D \to \mathbb{R}^n$ for a domain D in $\mathbb{R}^n$ is quasiconformal if f is ACL^n and there is $K \in [1, \infty)$ such that

$$\|f'(x)\|^n \le KJ(x, f) \text{ a.e.} \tag{2.29}$$

It does not follow directly from this definition that f is a.e. differentiable. However, since f is ACL^n, the partial derivatives of the coordinate functions of f exist a.e. and hence the Jacobian matrix (the formal derivative of f at x)

$$f'(x) = \begin{pmatrix} \partial_1 f_1(x) & \dots & \partial_n f_1(x) \\ \vdots & & \vdots \\ \partial_1 f_n(x) & \dots & \partial_n f_n(x) \end{pmatrix}$$

exists a.e. Here $\|f'(x)\|$ stands for the supremum norm of the linear map $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $J(x, f) = \det f'(x)$. It will turn out that a quasiconformal mapping f is a.e. differentiable. Sometimes (2.29) is written as $\|f'(x)\|^n \le K|J(x, f)|$. This also includes sense-reversing maps. Definitions (2.7) and (1.3) include sense-reversing conformal and quasiconformal mappings, respectively.

Remark 2.17. If $f : D \to \mathbb{R}^n$ is continuous, ACL^n, and satisfies (2.29), then f is called quasiregular (or of bounded distortion). Note that then absolute values are not allowed on the right-hand side of (2.29). If $n = 2$ and $K = 1$, this definition leads to one of the most general definitions of analytic functions.

Next we show that the uniform boundedness of the linear dilatation leads to (2.29).

Theorem 2.11. *Suppose that $f : D \to \mathbb{R}^n$ is a homeomorphism such that*
(a) *$H(x, f) \le c$ for all $x \in D$,*
(b) *$H(x, f) \le c_0$ for a.e. $x \in D$,*
(c) *f is sense-preserving.*
Then f is ACL^n *and satisfies (2.29) with $K = c_0^{n-1}$, i.e., f is quasiconformal according to the analytic definition.*

Remark 2.18. The property that a mapping $f : D \to \mathbb{R}^n$ is sense-preserving can be defined for every continuous mapping f with the aid of the topological degree; cf.

[246]. However, since a map f satisfying (b) is differentiable a.e., property (c) can be defined as $\det f'(x) = J(x, f) \geq 0$ a.e.

Proof for Theorem 2.11. By Corollary 2.4, f is a.e. differentiable and ACL^n. Condition (c) implies $J(x, f) \geq 0$ a.e. It remains to show (2.29). If f is differentiable at x and $J(x, f) = 0$, then $\|f'(x)\| = 0$ because $H(x, f) \leq c$. Hence, (2.29) holds. If $J(x, f) > 0$, then at such points x, $H(x, f) = \lambda_n/\lambda_1$, where $\|f'(x)\| = \lambda_n$ and $l(f'(x)) = \lambda_1$; see Sections 2.2 and 2.7. This means that a.e. such a point x satisfies

$$\|f'(x)\|^n = \lambda_n^n \leq \lambda_n \lambda_1^{n-1} c_0^{n-1} \leq \lambda_1 \lambda_2 \cdots \lambda_n c_0^{n-1} = c_0^{n-1} J(x, f).$$

Hence, (2.29) holds with $K = c_0^{n-1}$, as required. □

Remark 2.19. Note that the values of c_0 and K do not quite fit, except for $n = 2$ when $c_0 = K$ in Theorem 2.11. The smallest K for which (2.29) holds is called the **outer dilatation** of f and denoted $K_0(f)$. The **inner dilatation** $K_I(f)$ of f is defined as the smallest K for which

$$J(x, f) \leq K l(f'(x))^n \tag{2.30}$$

holds a.e. in D. Note that if (2.29) holds for $K_0(f)$, then (2.30) holds for $K = K_0(f)^{n-1}$ because

$$\begin{aligned} J(x, f) &= \lambda_1 \lambda_2 \ldots \lambda_n \leq \lambda_1 \lambda_n^{n-1} \\ &= \lambda_1 \|f'(x)\|^{n-1} \leq \lambda_1 K_0(f)^{(n-1)/n} J(x, f)^{(n-1)/n}, \end{aligned}$$

and hence,

$$J(x, f) \leq K_0(f)^{n-1} \lambda_1^n = K_0(f)^{n-1} l(f'(x))^n.$$

This computation applies to the case $J(x, f) = 0$ as well.

It is also true that if (2.30) holds, then $K_0(f) \leq K_I(f)^{n-1}$. The number $K(f) = \max\,(K_0(f), K_I(f))$ is called the **maximal dilatation** of f.

These concepts also have an interpretation in the geometric definition of quasiconformality given in Chapter 1. Then (1.3) takes the form

$$M_n(\Gamma)/K_0(f) \leq M_n(f\Gamma) \leq K_I(f) M_n(\Gamma)$$

for each path family Γ in D.

Note that for $n = 2$, $K(f) = K_0(f) = K_I(f)$. For $n = 1$, these dilatations do not make much sense, since if $f : (a, b) \to \mathbb{R}$ is differentiable at x, then $\|f'(x)\| = |J(x, f)| = \lambda_1 = \lambda_n$ and $H(x, f) = 1$ provided that $f'(x) \neq 0$.

Next we show that the analytic definition implies the modulus definition, or at least another half of it.

Theorem 2.12. *Suppose that $f : D \to D'$ is a homeomorphism where D and D' are domains in $\mathbb{R}^n$. If f is* ACL^n *in D and satisfies (2.29), then*

$$M_n(\Gamma) \leq K M_n(f\Gamma) \tag{2.31}$$

for each family Γ of paths in D.

The proof requires a couple of results from real analysis whose proofs we omit; see [316] and [246]:

Lemma 2.6. *If $f : D \to \mathbb{R}^n$ is an ACL^p homeomorphism, $p > n-1$, then f is a.e. differentiable (in fact, it suffices that f is an open map).*

Lemma 2.7. *Suppose that $f : D \to D'$ is an a.e. differentiable homeomorphism and $u \geq 0$ is a measurable function in D'. Then*

$$\int_D u(f(x))\,|J(x,f)|\,dx \leq \int_{D'} u\,dy. \tag{2.32}$$

Remark 2.20. To obtain equality in (2.32), one has to assume that f is ACL^n. For a more detailed discussion; see [193].

Proof of Theorem 2.12. In order to prove (2.31), fix a family Γ of paths in D. Since f is ACL^n, the Fuglede theorem implies that f (the coordinate functions of f) is absolutely continuous on a path family Γ_0 of n-almost all paths in Γ. Then $M_n(\Gamma_0) = M_n(\Gamma)$. We need to show that

$$M_n(\Gamma_0) \leq K M_n(f\Gamma).$$

To this end, let ρ' be an admissible function for $f\Gamma$. Write

$$\rho(x) = \begin{cases} \rho'(f(x))L(x,f), & x \in D, \\ 0, & x \notin D, \end{cases}$$

where

$$L(x,f) = \limsup_{y \to x} \frac{|f(y)-f(x)|}{|y-x|}\,.$$

Now ρ is admissible for Γ_0 (note that ρ is a Borel function). To see this, let $\gamma \in \Gamma_0$ be parameterized by arc length $\gamma : [0, l(\gamma)] \to D$. Since f is absolutely continuous on γ, we have (see Section 2.2)

$$\int_{f\circ\gamma} \rho'\,ds = \int_0^{l(\gamma)} \rho'(f(\gamma(t)))\,|(f\circ\gamma)'(t)|\,dt. \tag{2.33}$$

If $(f\circ\gamma)'(t)$ and $\gamma'(t)$ exist (and a.e. $t \in [0, l(\gamma)]$ is such), then assuming $\gamma(t+\Delta t) \neq \gamma(t)$, we see that

$$\begin{aligned} |(f\circ\gamma)'(t)| &= \lim_{\Delta t \to 0} \left| \frac{f\circ\gamma(t+\Delta t) - f\circ\gamma(t)}{\Delta t} \right| \\ &\leq \limsup_{\Delta t \to 0} \frac{|f\circ\gamma(t+\Delta t) - f\circ\gamma(t)|}{|\gamma(t+\Delta t)-\gamma(t)|} \frac{|\gamma(t+\Delta t)-\gamma(t)|}{\Delta t} \\ &\leq L(\gamma(t),f)|\gamma'(t)| = L(\gamma(t),f) \end{aligned}$$

because $|\gamma'(t)| = 1$ a.e. If $\gamma(t+\Delta t) = \gamma(t)$, then the above inequality is clear. Hence, (2.33) yields

$$1 \leq \int_{f\circ\gamma} \rho'\, ds \leq \int_0^{l(\gamma)} \rho'(f(\gamma(t)))\, L(\gamma(t), f)\, dt = \int_\gamma \rho\, ds,$$

as required.

The rest of the proof now easily follows from Lemmas 2.6 and 2.7. Indeed, since ρ is admissible for Γ_0, we have

$$\begin{aligned} M_n(\Gamma_0) &\leq \int_D \rho^n dx = \int_D \rho'(f(x))^n L(x,f)^n dx = \int_D \rho'(f(x))^n \|f'(x)\|^n dt \\ &\leq K \int_D \rho'(f(x))^n |J(x,f)|\, dx \leq K \int_{D'} \rho'^n\, dy, \end{aligned}$$

where we have used (2.29) as well. Since ρ' was an arbitrary admissible function for $f\Gamma$, this shows that

$$M_n(\Gamma) = M_n(\Gamma_0) \leq K M_n(f\Gamma),$$

as required. □

Remark 2.21. In order to prove the upper bound

$$M_n(f\Gamma) \leq K M_n(\Gamma)$$

for each family Γ of paths in D, an obvious approach is to show that the inverse map $f^{-1} : D' \to D$ is quasiconformal in the sense of the analytic definition as well. This requires some work. The main steps are: If f is quasiconformal, then (a) f satisfies the Lusin condition (N) (maps sets of measure zero into sets of measure zero (see [193, 316])) and (b) $J(x, f) > 0$ a.e. (see [316]).

2.10 $\mathbb{R}^n$ as a Loewner Space

In this section we indicate how the Loewner lower bound for the n-modulus is obtained in $\mathbb{R}^n$, $n \geq 2$. Loewner was the first to observe that the 3-modulus of a family of paths joining two non-degenerate continua in $\mathbb{R}^3$ is positive; see [192]. We then derive an upper bound for the linear dilatation $H(x, f)$ of a quasiconformal map $f : D \to D'$ between domains D and D' in $\mathbb{R}^n$.

Real analysis: Maximal function and the Riesz potential. Let $f : \mathbb{R}^n \to [0, \infty]$ be a measurable function. If $A \subset \mathbb{R}^n$ is measurable, then

$$I_A(f)(x) = \int\limits_A \frac{f(y)}{|x-y|^{n-1}}\, dy, \qquad x \in \mathbb{R}^n,$$

is called the **Riesz potential** of f. For $R > 0$, the function

$$M_R(f)(x) = \sup_{0<r<R} \left(\frac{1}{r} \int\limits_{B(x,r)} f(y)^n dy \right)^{1/n}, \qquad x \in \mathbb{R}^n,$$

is called the (restricted) **maximal function** of f.

Remark 2.22. The classical **Hardy–Littlewood maximal function** $M(f)$ of f is defined as

$$M(f)(x) = \sup_{r>0} \left(\fint_{B(x,r)} f(y)\, dy \right).$$

Here

$$\fint_{B(x,r)} f(y)\, dy = \frac{1}{m(B(x,r)} \int\limits_{B(x,r)} f(y)\, dy = f_B$$

stands for the mean value f in B. Note that this is different from $M_\infty(f)$. The special form of $M_R(f)$ is needed to study the conformally invariant case.

In the following results and proofs, C stands for a constant that depends only on n. We start with a simple "mean value" estimate.

Lemma 2.8. *Suppose that u is a locally integrable function with an upper gradient ρ in the ball $B_0 = B(x_0, r_0)$. Then for all $x, y \in B(x_0, r_0/4)$,*

$$|u(x) - u(y)| \le C(I_{B_0}(\rho)(x) + I_{B_0}(\rho)(y)). \tag{2.34}$$

Proof. For the inequality (2.34), we first show that if $B = B(z,r) \subset B_0$ is any ball, then

$$|u(x) - u_B| \le C I_B(\rho)(x) \tag{2.35}$$

for all $x \in B$. Here u_B is the mean value of u in B.

Keep $x \in B$ fixed and let $y \in B$. Since ρ is an upper gradient of u,

$$|u(x) - u(y)| \le \int\limits_0^{|x-y|} \rho(x + r\omega)\, dr,$$

where $\omega = (y-x)/|y-x|$ is a unit vector in $\mathbb{R}^n$. Integrating over B with respect to y, we arrive at

$$m(B)\, |u(x) - u_B| \le \int\limits_B \int\limits_0^{|x-y|} \rho(x + r\omega)\, dr dy$$

[for the proof, assume that either $u(x) \geq u_B$ or $u(x) < u_B$]. Performing a change of variables, we obtain

$$|u(x) - u_B| \leq C \int_B \frac{\rho(y)}{|x-y|^{n-1}}\, dy$$

and (2.35) follows.

We can now finish the proof of the lemma. If $x, y \in B(x_0, r_0/4)$, then $B_x = B(x, 2|x-y|)$ lies in B_0. By (2.35) we obtain

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u_{B_x}| + |u(y) - u_{B_x}| \\ &\leq C(I_{B_x}(\rho)(x) + I_{B_x}(\rho)(y)) \leq C(I_{B_0}(\rho)(x) + I_{B_0}(\rho)(y)), \end{aligned}$$

as required. □

Remark 2.23. Inequality (2.35) is almost the same as the equality

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\langle \nabla u(y), (x-y) \rangle}{|x-y|^n}\, dy, \qquad x \in \mathbb{R}^n, \tag{2.36}$$

which holds for all functions $u \in C_0^1(\mathbb{R}^n)$ (compactly supported C^1-functions). In fact, (2.35) follows from (2.34) for these functions.

Remark 2.24. Inequalities like (2.34) are important in the theory of Newtonian spaces or more general function spaces on metric spaces. For example, if a function u belongs to the Newtonian space $N^{1,p}(\mathbb{R}^n),\ p > 1$, and if ρ is a p-weak upper gradient of u, then u satisfies

$$|u(x) - u(y)| \leq C|x-y|(M(\rho)(x) + M(\rho)(y)) \tag{2.37}$$

for a.e. $x, y \in \mathbb{R}^n$. Here $M(\rho)$ is the Hardy–Littlewood maximal function of ρ. Conversely, if $u \in L^p(\mathbb{R}^n)$ satisfies

$$|u(x) - u(y)| \leq |x-y|(g(x) + g(y)) \tag{2.38}$$

a.e. in $\mathbb{R}^n$ with some $g \in L^p(\mathbb{R}^n),\ g \geq 0$, then there is $\tilde{u} \in \tilde{N}^{1,p}(\mathbb{R}^n)$ such that $\tilde{u} = u$ a.e. and Cg can (essentially) be used as a p-weak upper gradient of $\tilde{u}$. See [101, 103, 107, 291] for more details.

Lemma 2.9. *For $0 < r \leq R$ and $x \in \mathbb{R}^n$,*

$$I_{B(x,r)}(f)(x) \leq C r^{1/n} M_R(f)(x).$$

Proof. Set

$$A_j = B(x, 2^{-j}r) \setminus B(x, 2^{-j-1}r), \quad j = 0, 1, \ldots.$$

Now

$$\begin{aligned}
I_{B(x,r)}(f)(x) &= \sum_j \int_{A_j} \frac{f(y)}{|x-y|^{n-1}}\, dy \leq C \sum_j (2^{-j}r)^{1-n} \int_{B(x,2^{-j}r)} f(y)\, dy \\
&\leq C \sum_j 2^{-j} r \left(\frac{1}{m(B(x,2^{-j}r))} \int_{B(x,2^{-j}r)} f(y)\, dy \right) \\
&\leq C \sum_j 2^{-j} r \left(\frac{1}{m(B(x,2^{-j}r))} \int_{B(x,2^{-j}r)} f(y)^n\, dy \right)^{1/n} \\
&\leq C r^{1/n} \sum_j 2^{-j/n} \left(\frac{1}{2^{-j}r} \int_{B(x,2^{-j}r)} f(y)^n\, dy \right)^{1/n} \leq C r^{1/n} M_R(f)(x),
\end{aligned}$$

as required. Here the Hölder inequality was also used. □

The Loewner property. Next we prove the Loewner property for $\mathbb{R}^n$, $n \geq 2$. In fact, we will show that each ball in $\mathbb{R}^n$ is a Loewner space. This, however, will follow from the corresponding property of $\mathbb{R}^n$. Let E and F be two nondegenerate continua in $\mathbb{R}^n$. Recall that $\Delta(E,F) = \operatorname{dist}(E,F)/\min(\operatorname{diam} E, \operatorname{diam} F)$ denotes the relative distance between E and F in $\mathbb{R}^n$.

Theorem 2.13. *If $\Delta(E,F) \leq t$, then*

$$M_n(\Gamma) \geq C/t, \tag{2.39}$$

where Γ is the family of paths joining E and F in $\mathbb{R}^n$.

Proof. Write $d = \operatorname{diam} E$. We may assume that

$$d \leq \operatorname{diam} F < \frac{1}{4} \operatorname{dist}(E,F)$$

(note that the n-modulus decreases if we make E and F smaller). Choose $x_0 \in E$ and $y_0 \in F$ such that

$$|x_0 - y_0| = \operatorname{dist}(E,F).$$

Then $E, F \subset B_0 = B(x_0, 8 \operatorname{dist}(E,F))$ and next we shall make use of Theorem 2.5 and Remark 2.9.

Let u be a locally integrable function in $\mathbb{R}^n$ with $u|E \leq 0$ and $u|F \geq 1$. Let ρ be an upper gradient of u. Observe that for all $x \in E$ and $y \in F$,

$$1 \leq |u(x) - u(y)| \leq C(I_{B_0}(\rho)(x) + I_{B_0}(\rho)(y))$$

by Lemma 2.8. Now either E or F belongs to the set

$$S = \{z \in \mathbb{R}^n : M_{r_0}(\rho)(z) > C\,\mathrm{dist}(E,F)^{-1/n}\},$$

where $r_0 = 8\ \mathrm{dist}(E,F)$. Here we have used Lemma 2.9. Suppose that, for example, $E \subset S$. We use a standard Besicovitch-type covering argument [107, 110]: If $\{B(x_i,r_i)\} = \mathscr{F}$ is any family of balls in $\mathbb{R}^n$ such that $r_i \le c < \infty$ for some c, then there is a countable (possibly finite) subfamily of disjoint balls $B(x_i,r_i),\ i = 1,2,\ldots,$ such that

$$\bigcup_{B\in\mathscr{F}} B \subset \bigcup_i B(x_i,5r_i).$$

By the definition of $M_{r_0}(\rho)$ and the covering theorem, the set E can be covered with balls $B(x_i,5r_i),\ i = 1,2,\ldots,$ such that the balls $B(x_i,r_i)$ are disjoint and

$$r_i \le Cd_0 \int\limits_{B(x_i,r_i)} \rho^n dz,$$

where $d_0 = \mathrm{dist}(E,F)$. Since E is covered with the balls $B(x_i,5r_i)$, we obtain

$$\begin{aligned}\mathrm{diam}\,E &\le 10\sum_i r_i \le Cd_0 \sum_i \int\limits_{B(x_i,r_i)} \rho^n dz \\ &= Cd_0 \int\limits_{\cup\, B(x_i,r_i)} \rho^n dz \le Cd_0 \int\limits_{B_0} \rho^n dz.\end{aligned}$$

This yields

$$\begin{aligned}1/t &\le \min(\mathrm{diam}\,E, \mathrm{diam}\,F/d_0) \\ &= \mathrm{diam}\,E/d_0 \le C\int\limits_{B_0} \rho^n dz \le C\int\limits_{\mathbb{R}^n} \rho^n dz.\end{aligned}$$

Taking the infimum over u and ρ, we obtain

$$C/t \le \mathrm{cap}_n(E,F) = M_n(\Gamma),$$

as required. □

Corollary 2.5. *Inequality (2.39) holds whenever the continua E and F lie in a ball $B(x_0,r_0) \subset \mathbb{R}^n$ and Γ is the family of paths that join E and F in $B(x_0,r_0)$.*

Proof. Let $T : \mathbb{R}^n \setminus \{x_0\} \to \mathbb{R}^n \setminus \{x_0\}$ be the reflection in the sphere $\partial B(x_0,r_0)$, i.e., $T(x) = x_0 + (x-x_0)/|x-x_0|^2$. Then T is conformal and $T = T^{-1}$. If ρ is admissible for Γ, then the function

$$\tilde\rho(x) = \begin{cases}\rho(x), & x \in B(x_0,r_0),\\ \rho(T(x))\|T'(x)\|, & x \in \mathbb{R}^n \setminus B(x_0,r_0)\end{cases}$$

is admissible for the family $\tilde{\Gamma}$ of paths that join E to F in $\mathbb{R}^n$ (note also that the n-modulus of the paths passing through x_0 is zero). Hence,

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{\rho}^n dy &= \int_{B(x_0,r_0)} \rho^n dy + \int_{\mathbb{R}^n \setminus B(x_0,r_0)} \rho(T(x))^n \|T'(x)\|^n dx \\ &= \int_{B(x_0,r_0)} \rho^n dy + \int_{\mathbb{R}^n \setminus B(x_0,r_0)} \rho(T(x))^n |J(x,T)|\, dx \\ &= \int_{B(x_0,r_0)} \rho^n dy + \int_{B(x_0,r_0)} \rho^n dy 2 \int_{B(x_0,r_0)} \rho^n dy,\end{aligned}$$

where we have used the analytic definition

$$\|T'(x)\|^n = |J(x,T)|$$

for the conformal map T. From (2.39) it thus follows that

$$C/t \leq M_n(\tilde{\Gamma}) \leq 2M_n(\Gamma),$$

as required. $\square$

Remark 2.25. Those metric spaces that satisfy the Loewner condition have been studied in [107]; see also [112].

Linear dilatation. Here we show that the linear dilatation of a quasiconformal map $f : D \to D'$ for domains $D, D' \subset \mathbb{R}^n$ is uniformly bounded. The global version of this result is studied in Section 2.11.

Theorem 2.14. *Suppose that $f : D \to D'$ is a K-quasiconformal map [see (1.3)]. Then for all $x \in D$,*

$$H(x,f) \leq C(n,K) < \infty. \tag{2.40}$$

Proof. Let $x \in D$ and choose $r_0 > 0$ such that $\overline{B}(x, 4r_0) \subset D$. Let $0 < r < r_0$. Choose $y, y' \in \partial B(x,r)$ such that

$$L = L(x,f,r) = |f(y') - f(x)|, \ \ l = l(x,f,r) = |f(y) - f(x)|$$

and let L'' be the line segment $[f(x), f(y)]$ and L' the half-open line segment in D' that is the continuation of the line segment $[f(x), f(y')]$ outside $B(f(x), L)$. Let Γ' be the family of all paths that join L'' to L' in D'. We may assume $L > l$; then

$$M_n(\Gamma') \leq \omega_{n-1}(\log(L/l))^{1-n};$$

see (2.5) and Lemma 2.2.

Next, let $E = f^{-1}(L'')$ and let F' be the connected part of $f^{-1}(L')$ that joins $\partial B(x,r)$ to $\partial B(x, 2r_0)$ in $\overline{B}(x_0, 4r_0)$. Since

$$\Delta(E,F') = \min(\text{diam}\, E,\ \text{diam}\, F')/\text{dist}\,(E,F') \geq r/r = 1,$$

we obtain from Corollary 2.5 that

$$M_n(f^{-1}\Gamma') \geq M_n(\Gamma) \geq C = C(n) > 0,$$

where Γ is the family of all paths joining E to F' in $B(x, 4r_0)$. By the quasiconformality of f,

$$C \leq M_n(f^{-1}\Gamma') \leq KM_n(\Gamma') \leq \omega_{n-1}(\log(L/l))^{1-n},$$

and hence

$$L(x, f, r)/l(x, f, r) = L/l \leq C = C(n, K).$$

Letting $r \to 0$, we see that $H(x, f) \leq C(n, K)$, as required. □

Remark 2.26. For $n = 2$, $C(2, K)$ is known; see [190]. The value $C(n, K)$, $n \geq 3$, was found very recently; see [285].

2.11 Quasisymmetry

In Section 2.9 we showed that the uniform bound for the linear dilatation of a homeomorphism $f : D \to D'$ between domains in the Euclidean n-space $\mathbb{R}^n$ implies quasiconformality [or at least the other half of the modulus definition (1.3)]. In Section 2.10 we proved that $\mathbb{R}^n$, and every ball $B(x, r) \subset \mathbb{R}^n$, is a Loewner space, which implies that every quasiconformal map $f : D \to D'$ in the sense of definition (1.3) satisfies $H(x, f) \leq C(n, K) < \infty$ at each point $x \in D$. Now it turns out that a more global version than $H(x, f) \leq C < \infty$ is true for quasiconformal maps f. This is called **quasisymmetry**. It can be expressed in the general context of metric spaces.

Let X and Y be metric spaces. We use a simplified notation $d(x, y) = |x - y|$, resp. $d'(x, y) = |x - y|$, for points $x, y \in X$, resp. $x, y \in Y$, although the difference $x - y$ has no meaning.

A mapping $f : X \to Y$ is called an **embedding** if f defines a homeomorphism of X onto $f(X)$. An embedding $f : X \to Y$ is called **quasisymmetric** if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$|x - a| \leq t|x - b| \quad \Rightarrow \quad |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)| \tag{2.41}$$

for all triples a, b, x of points in X, and for all $t > 0$. Thus, f is quasisymmetric if it distorts relative distances by a bounded amount. We also say that f is η-quasisymmetric if the function η needs to be mentioned. Note that a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ is nothing but a continuous strictly increasing function η on $[0, \infty)$ such that $\eta(0) = 0$ and

$$\lim_{t \to \infty} \eta(t) = \infty.$$

Observe that the inverse function η^{-1} of η is similar to η.

An embedding $f : X \to Y$ is said to be **bi-Lipschitz** if both f and f^{-1} are Lipschitz. The term L-bi-Lipschitz means that for all $x, y \in X$,

$$|x-y|/L \leq |f(x)-f(y)| \leq L|x-y|.$$

Notice the difference between quasisymmetric maps and bi-Lipschitz maps: The latter distort absolute distances by a bounded amount, which is a much stronger condition. It is easy to see that an L-bi-Lipschitz embedding is η-quasisymmetric with $\eta(t) = L^2 t$.

Examples. (a) The map $x \mapsto \lambda x$, $\lambda \neq 0$, in $\mathbb{R}^n$ is η-quasisymmetric with $\eta(t) = t$. The same is true for every conformal map $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$. Note that $x \mapsto \lambda x$ is not L-bi-Lipschitz with a constant L independent of λ.

(b) The map $f : [0,\infty) \to [0,\infty)$, $f(x) = x^2$, is η-quasisymmetric, $\eta(t) = t^2 + 2t$. Note that f is not bi-Lipschitz.

(c) The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, is quasisymmetric. In fact, every map $f(x) = |x|^{\alpha-1}x$, $f(0) = 0$, $\alpha > 0$, is quasisymmetric in $\mathbb{R}$.

There is a weaker condition than the η-quasisymmetry. We call an embedding $f : X \to Y$ **weakly** (H)**-quasisymmetric** if there is a constant $H \geq 1$ so that

$$|x-a| \leq |x-b| \text{ implies } |f(x)-f(a)| \leq H|f(x)-f(b)| \tag{2.42}$$

for all triples a, b, x of points in X.

Weakly quasisymmetric maps need not be quasisymmetric. This only takes place in badly disconnected spaces. Let $X = \mathbb{N} \times \{0, -1/4\} \subset \mathbb{R}^2$ and let $f : X \to \mathbb{R}^2$ be the embedding defined by $f(n,0) = (n,0)$, and $f(n,-1/4) = (n,-1/4n)$. Then f is weakly quasisymmetric but not quasisymmetric. Clearly, if f is η-quasisymmetric, then f is weakly $\eta(1)$-quasisymmetric.

As mentioned in the beginning, quasisymmetry provides a global version for linear dilatation. In particular, weakly quasisymmetric maps between Euclidean domains are quasiconformal.

Lemma 2.10. *Suppose that $D \subset \mathbb{R}^n$ is a domain and $f : D \to \mathbb{R}^n$ weakly H-quasisymmetric. Then the inequality*

$$H(x,f) \leq H$$

holds at each point $x \in D$.

Proof. Fix $x \in D$ and let $0 < r < d(x, \partial D)$. Pick $a \in \partial B(x,r)$ such that

$$|f(x)-f(a)| = \sup\{|f(x)-f(y)| : y \in \partial B(x,r)\} = L(x,f,r)$$

and $b \in \partial B(x,r)$ such that

$$|f(x)-f(b)| = \inf\{|f(x)-f(y)| : y \in \partial B(x,r)\} = l(x,f,r).$$

Now,

$$L(x,f,r) = |f(x) - f(a)| \le H|f(x) - f(b)| = Hl(x,f,r)$$

since $|x-a| = |x-b|$, showing that

$$\limsup_{r\to 0} L(x,f,r)/l(x,f,r) \le H,$$

as required.

$\square$

Properties of quasisymmetric maps. Here we list some basic properties of quasisymmetric maps. Most of these properties are easy to prove.
(a) If $f : X \to Y$ is η-quasisymmetric, then $f^{-1} : f(X) \to X$ is η'-quasisymmetric when $\eta(t) = 1/\eta^{-1}(t^{-1})$ for $t > 0$. Moreover, if $f : X \to Y$ and $g : Y \to Z$ are η_f- and η_g-quasisymmetric, respectively, then $g \circ f : X \to Z$ is $(\eta_g \circ \eta_f)$-quasisymmetric.
(b) The restriction to a subset of a quasisymmetric map is quasisymmetric with the same η.
(c) Quasisymmetric maps take Cauchy sequences to Cauchy sequences. In particular, every quasisymmetric image of a complete space is complete.
(d) Quasisymmetric embeddings map bounded spaces to bounded spaces. More quantitatively, if $f : X \to Y$ is η-quasisymmetric and if $A \subset B \subset X$ are such that $0 < \operatorname{diam} A \le \operatorname{diam} B < \infty$, then $\operatorname{diam} f(B)$ is finite and

$$\left(2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)\right)^{-1} \le \frac{\operatorname{diam} f(A)}{\operatorname{diam} f(B)} \le \eta\left(\frac{2\operatorname{diam} A}{\operatorname{diam} B}\right). \tag{2.43}$$

Doubling spaces. Quasisymmetry is intimately connected to a property of a metric space called a doubling property.

A metric space is called **doubling** if there is a constant $C_1 \ge 1$ so that every set of diameter d in the space can be covered by at most C_1 sets of diameter at most $d/2$. It is clear that subsets of doubling spaces are doubling.

Equivalent definitions for doubling spaces are often used. For instance, in the definition one may replace sets by balls. Moreover, doubling spaces have the following stronger covering property: There is a function $C_1 : (0,1/2] \to (0,\infty)$ such that every set of diameter d can be covered by at most $C_1(\varepsilon)$ sets of diameter at most εd. The function C_1, called a **covering function of** X, can be chosen to be in the form

$$C_1(\varepsilon) = C\varepsilon^{-\beta} \tag{2.44}$$

for some $C \ge 1$ and $\beta > 0$.

Given a doubling metric space X, the infimum of all numbers $\beta > 0$ such that a covering function of the form (2.44) can be found is called the **Assouad dimension** of X.

Doubling spaces are precisely the spaces of finite Assouad dimension.

It is easy to see that $\mathbb{R}^n$ is doubling with a constant depending only on n, and in fact the Assouad dimension of $\mathbb{R}^n$ is n. Thus, every subset of Euclidean space is doubling.

Lemma 2.11. *A quasisymmetric image of a doubling space is doubling.*

Proof. Let $f : X \to Y$ be an η-quasisymmetric homeomorphism. It suffices to show that every ball B of diameter d in Y can be covered by at most some fixed number C_2 of sets of diameter at most $d/4$. Let $B = B(y,R)$ and let

$$L = \sup_{z\in B} |f^{-1}(y) - f^{-1}(z)|.$$

Then we can cover $f^{-1}(B)$ by at most $C_1(\varepsilon)$ sets of diameter at most $\varepsilon 2L$ for any $\varepsilon \le 1/2$, where C_1 is a covering function of X. Let $A_1,\dots,A_p$ be such sets, so that $p = p(\varepsilon) \le C_1(\varepsilon)$. We may clearly assume that $A_i \subset f^{-1}(B)$ for all $i = 1,\dots,p$. Thus, $f(A_1),\dots,f(A_p)$ cover B and are contained in B, so that by (d) in (2.43), their diameters satisfy

$$\operatorname{diam} f(A_i) \le \operatorname{diam} B\eta\left(\frac{2\operatorname{diam} A_i}{\operatorname{diam} f^{-1}(B)}\right) \le d\,\eta\left(\frac{4\varepsilon L}{L}\right) \le d\,\eta(4\varepsilon)\,.$$

The lemma now follows upon choosing $\varepsilon = \varepsilon(\eta) > 0$ so small that $\eta(4\varepsilon) \le 1/4$.
□

The next theorem gives a sufficient condition for the equivalence of weak quasisymmetry and quasisymmetry. We omit the proof, which is somewhat tedious and uses a covering of a path from x to a together with a packing argument; see [311].

Theorem 2.15. *A weakly quasisymmetric embedding of a path-connected doubling space into a doubling space is quasisymmetric.*

Corollary 2.6. *A weakly quasisymmetric embedding of a path-connected subset of Euclidean space into another Euclidean space is quasisymmetric. In particular, a weakly quasisymmetric embedding of $\mathbb{R}^p$ into $\mathbb{R}^n$, $1 \le p \le n$, is quasisymmetric.*

Quasisymmetry in Euclidean domains. We have three equivalent definitions for the quasiconformality of a homeomorphism $f : D \to D'$ between domains D and D' in $\mathbb{R}^n$: is the modulus definition

$$M_n(\Gamma)/K \le M_n(f\Gamma) \le KM_n(\Gamma) \tag{2.45}$$

for each path family Γ in D; the boundedness of the linear dilatation

$$H(x,f) \le c \tag{2.46}$$

at every point $x \in D$; and the analytic definition: f is ACL^n and

$$\|f'(x)\|^n \le K|J(x,f)| \tag{2.47}$$

a.e. in D. Lemma 2.10 showed that if $f : D \to D'$ is quasisymmetric, then (2.46) holds and f is thus quasiconformal. There is also a converse statement, but, unfortunately, a K-quasiconformal map $f : D \to D'$ need not be η-quasisymmetric for any η. The reason for this is that quasisymmetry is a global condition (consider a Riemann mapping function of a disk onto a disk with a slit). However, there is a semiglobal version of this.

Theorem 2.16. *A homeomorphism $f : D \to D'$ between domains in $\mathbb{R}^n$, $n \ge 2$, is K-quasiconformal if and only if there is η such that f is η-quasisymmetric in each ball $B(x, 1/2\,\mathrm{dist}(x, \partial D))$ with $x \in D$. The function η depends only on n and K.*

Remark 2.27. The local quasisymmetry property in Theorem 2.16 for quasiconformal maps can be regarded as the quasiconformal version of the Koebe distortion theorem: If $f : B(0,1) \to \mathbb{R}^2$ is a conformal map normalized by the condition $f'(0) = 1$, then

$$(1-r)/(1+r)^3 \le |f'(z)| \le (1+r)/(1-r)^3$$

for $|z| = r < 1$; see [52], p. 32. The local quasisymmetry of a conformal map follows from this inequality by integration of f' along line segments.

Proof for Theorem 2.16. As noted earlier the boundedness of the linear dilatation, i.e., (2.46), already follows from the weak quasisymmetry, and so it remains to prove the converse.

For the converse, let $B = B(x,r)$ for some $x \in D$, where $r = \mathrm{dist}(x, \partial D)/2$. By Corollary 2.6, it suffices to show that f is weakly quasisymmetric in B.

Pick three distinct points a, b, and c in B with $|a-b| \le |a-c|$. We need to show that

$$|f(a) - f(b)| \le H|f(a) - f(c)| \tag{2.48}$$

for some $H = H(n,K) < \infty$.

Write $r = |a-b|$ and $R = |a-c|$. We first show that

$$L(a,f,R) \;\le\; H\, l(a,f,R), \tag{2.49}$$

where H depends only on n and K. Choose $y, y' \in \partial B(a,R)$ such that

$$L(a,f,R) \;=\; |f(a) - f(y)|\,, \qquad l(a,f,R) \;=\; |f(a) - f(y')|.$$

Let γ be the continuation of the ray from $f(a)$ to $f(y)$ in $D' \setminus f(B(a,R))$ and let γ' be the ray $[f(a), f(y')]$. Then $\gamma' \subset f(B(a,R)) \subset D'$. Set $\gamma_1 = f^{-1}(\gamma)$ and $\gamma_1' = f^{-1}(\gamma')$. Then γ_1 joins y to ∂D in $D \setminus B(a,R)$ and $\gamma_1' \subset B(a,R)$ joins a to y'. Let $\tilde{\gamma}_1$ be the component of γ_1 that contains y and a point in $\partial B(a, 3R/2)$ and lies in $\overline{B}(a, 3R/2)$. Note that $\overline{B}(a, 3R/2) \subset D$. Then $\Delta(\tilde{\gamma}_1, \gamma_1') \ge 1/2$ and since balls in $\mathbb{R}^n$ are Loewner spaces by Corollary 2.5, we obtain

$$M(\Gamma) \;\ge\; H' \;>\; 0,$$

where Γ is the family of paths that join $\tilde{\gamma}_1$ to γ_1' in $B(a,3R/2)$ and $H' < \infty$ depends only on n.

Since f is K-quasiconformal,

$$M_n(f\Gamma) \geq \frac{1}{K} M(\Gamma) \geq \frac{H'}{K} . \tag{2.50}$$

Now, each path in $f\Gamma$ has a subpath that joins $\overline{B}(f(a), l(a,f,R))$ and $\overline{B}(f(a), L(a,f,R))$. Hence,

$$M_n(f\Gamma) \leq \frac{\omega_{n-1}}{\ln\left(\frac{L(a,f,R)}{l(a,f,R)}\right)^{1-n}},$$

which together with (2.50), yields (2.49), as required.

To complete the proof, note that $L(a,f,r) \leq L(a,f,R)$; hence, (2.49) implies

$$|f(a) - f(b)| \leq L(a,f,r) \leq L(a,f,R) \leq H\,|f(a) - f(c)| \leq H\,|f(a) - f(c)|,$$

which is the required inequality (2.48). □

Although every Möbius map of the unit ball $B(0,1)$ onto itself is quasisymmetric, the family of all such maps is not η-quasisymmetric (or weakly H-quasisymmetric) for a fixed η (or for some $H < \infty$). As stated before, a conformal mapping $f: B(0,1) \to \mathbb{R}^2$ need not be quasisymmetric; in this case $f(B(0,1))$ is complicated. There is an interesting condition for a quasiconformal map $f: D \to D'$ which makes f quasisymmetric.

A domain D in $\mathbb{R}^n$ is C-**uniform** for some constant $C \geq 1$ if every pair of points $x, y \in D$ can be joined by a path $\gamma \subset D$ such that $l(\gamma) \leq C|x-y|$ and for each $z \in \gamma$

$$\operatorname{dist}(z, D) \geq \min\{|x-z|,\ |y-z|\}/C .$$

Uniform domains have turned out to be useful in many problems in analysis; see [81, 106, 142, 212].

The following theorem holds (for a more general version, see the next chapter):

Theorem 2.17. *A quasiconformal map $f: D \to D'$ between bounded uniform domains D and D' in $\mathbb{R}^n$ is quasisymmetric.*

Note that the slit domain $B(0,1) \setminus \{te_1 : 0 \leq t < 1\}$ in the plane is not uniform; it is uniform in $\mathbb{R}^n$, $n \geq 3$.

We omit the proof for Theorem 2.17. The proof is not difficult once it has been shown that a uniform domain is a Loewner space; this follows from a theorem of Jones [142] (see also [103]) stating that if $u \in N^{1,n}(D)$ in a uniform domain D, then there is $\tilde{u} \in \mathbb{N}^{1,n}(\mathbb{R}^n)$ such that

$$\|\tilde{u}\|_{N^{1,n}(\mathbb{R}^n)} \leq C\|u\|_{N^{1,n}(D)},$$

where $C < \infty$ is independent of u.

In the plane a simply connected domain $D \neq \mathbb{R}^2$ is uniform iff it is a quasidisk. This means that $D = f(B)$ for some quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and for some disk $B = B(x_0, r)$. See [212] for this result. This is not true in $\mathbb{R}^n$, $n \geq 3$, although a quasiball in $\mathbb{R}^n$ is a uniform domain.

A more detailed discussion of various types of domains and their interconnections can be found in the next chapter.

Chapter 3
Moduli and Domains

3.1 Introduction

Suppose that f is a quasiconformal mapping of a domain $D \subset \mathbb{R}^n$ onto D'. In this chapter we are interested in the conditions that guarantee an extension of f to ∂D or to $\mathbb{R}^n$. We consider quasiextremal distance (QED) domains and uniform domains. Our main source is [82].

A domain D in $\overline{\mathbb{R}^2}$ is said to be a K**-quasidisk** if it is the image of an open disk or half-plane under a K-quasiconformal self-mapping of $\overline{\mathbb{R}^2}$. The following two basic properties of quasidisks will be used to define two classes of domains in $\mathbb{R}^n$.

Extremal distance property. If D is a quasidisk and F_1 and F_2 are disjoint continua in D, then

$$\operatorname{mod} \Gamma \leq M \operatorname{mod} \Gamma_D,$$

where Γ and Γ_D are the families of paths that join F_1 and F_2 in $\overline{\mathbb{R}^2}$ and D, respectively, and where M is a constant that depends only on D.

Extension property. If D is a quasidisk and f is a quasiconformal mapping of D onto a domain D' in $\overline{\mathbb{R}^2}$, then f has a quasiconformal extension to $\overline{\mathbb{R}^2}$ if and only if D' is a quasidisk.

The first property is a consequence of a simple reflection principle for the moduli of path families; see Remark 3.4. The second property follows from the work of Ahlfors and Beurling [5].

For a domain in $\overline{\mathbb{R}^2}$, it turns out that these properties are related in the following sense. If D has the extremal distance property, then D and D' have the extension property if and only if D' has the extremal distance property. This is Corollary 3.4.

Section 3.2 is devoted to the study of quasiextremal distance (QED) exceptional sets and Section 3.3 to the study of QED domains. In Section 3.5 we derive several geometric properties of domains D in $\mathbb{R}^n$ that have the extremal distance property. It

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_3,

turns out that a simply connected plane domain of the hyperbolic type is QED if and only if it is a quasidisk. We then obtain in Section 3.5 a number of extension theorems for QED domains, including several generalizations of the above-mentioned result of Ahlfors and Beurling.

3.2 QED Exceptional Sets

A closed set E in $\overline{\mathbb{R}^n}$ is said to be an ***M*-quasiextremal distance** or ***M*-QED exceptional set**, $1 \le M < \infty$, if, for each pair of disjoint continua $F_1, F_2 \subset \overline{\mathbb{R}^n} \setminus E$,

$$\mathrm{mod}\ \Gamma \ \le\ M\ \ \mathrm{mod}\ \ \Gamma_E, \tag{3.1}$$

where Γ and Γ_E are families of paths joining F_1 and F_2 in $\overline{\mathbb{R}^n}$ and $\overline{\mathbb{R}^n} \setminus E$, respectively, and mod is the n-modulus. The class of QED exceptional sets contains the class of **NED** or **null-sets for extremal distances**; these are the sets E in $\overline{\mathbb{R}^n}$ for which (3.1) holds with $M = 1$ for all choices of F_1, F_2. See [5, 15, 317] and Remark 3.1. The class QED exceptional sets were introduced in [82], and we follow the presentation there.

The conformal or n-capacity can also be used to characterize QED exceptional sets. Let D be an open set in $\overline{\mathbb{R}^n}$ and C_1, C_2 compact disjoint sets in D. Set

$$\mathrm{cap}(C_1, C_2; D) = \inf_{u \in W} \int\limits_{D \cap \mathbb{R}^n} |\nabla u|^n dm, \tag{3.2}$$

where $W = W(C_1, C_2; D)$ is the family of all functions u that are continuous and ACL in D with $u(x) \le 0$ for $x \in C_1$ and $u(x) \ge 1$ for $x \in C_2$. Since a point has zero n-capacity, the point ∞ can be deleted in the definition for W and thus W in (3.2) can be replaced by the family $\tilde{W}$ of functions u that are continuous and ACL in $D \cap \mathbb{R}^n$ and satisfy $u(x) \le 0$ for $x \in C_1 \cap \mathbb{R}^n$ and $u(x) \ge 1$ for $x \in C_2 \cap \mathbb{R}^n$. The classes W and $\tilde{W}$ differ only if $\infty \in D$. It is well-known (see [122]) that cap $(C_1, C_2; D) = \mathrm{mod}\,\Gamma$, where Γ is the family of paths joining C_1 and C_2 in D. Hence (3.1) can be written as

$$\mathrm{cap}(F_1, F_2; \overline{\mathbb{R}^n}) \le M\, \mathrm{cap}(F_1, F_2; \overline{\mathbb{R}^n} \setminus E)\,. \tag{3.3}$$

Remark 3.1. If E is an M-QED exceptional set in $\overline{\mathbb{R}^n}$ with $m(E) = 0$, then E is NED. This follows from arguments in [15] although it is not explicitly mentioned there. To see this, let E be an M-QED exceptional set in $\overline{\mathbb{R}^n}$ with $m(E) = 0$ and let F_1, F_2 be two continua in $\overline{\mathbb{R}^n} \setminus E$. Then, for each $u \in W(F_1, F_2; \overline{\mathbb{R}^n} \setminus E)$, it follows from Lemmas 3 and 4 and the considerations on pp. 1220–1221 in [15] that there is a function $u^* \in W(F_1, F_2; \overline{\mathbb{R}^n})$ with

$$\int\limits_{\mathbb{R}^n} |\nabla u^*|^n dm \ = \int\limits_{\mathbb{R}^n \setminus E} |\nabla u^*|^n dm \ \le \int\limits_{\mathbb{R}^n \setminus E} |\nabla u|^n dm.$$

Hence, (3.3) holds with $M = 1$, and thus E is NED. This observation together with Corollary 3.1 below yields the following:

For M-QED exceptional sets E in $\overline{\mathbb{R}^n}$, the following conditions are equivalent:
(i) $m(E) = 0$.
(ii) $\operatorname{int} E = \emptyset$.
(iii) E is NED.

We shall derive some properties of QED exceptional sets. The first one is an immediate consequence of the quasi-invariance of the modulus under quasiconformal mappings; see [316].

Lemma 3.1. *Suppose that E is an M-*QED *exceptional set and that $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a quasiconformal mapping. Then $f(E)$ is an M'-*QED *exceptional set, where*

$$M' = K_I(f) K_O(f) M.$$

Here $K_I(f)$ and $K_O(f)$ denote the inner and outer dilatations of f, respectively.

We shall need the following estimate to establish several metric properties of QED sets.

Lemma 3.2. *Suppose that F_1 and F_2 are disjoint continua in $\overline{\mathbb{R}^n}$ and that*

$$\min_{j=1,2} \operatorname{diam} F_j \geq a \operatorname{dist}(F_1, F_2),$$

where a is a positive constant. If Γ is the family of paths that join F_1 and F_2 in $\overline{\mathbb{R}^n}$, then

$$\operatorname{mod} \Gamma \geq c > 0,$$

where c is a constant that depends only on n and a, respectively.

Proof. Choose $x_1 \in F_1$ and $x_2 \in F_2$ so that

$$|x_1 - x_2| \; = \; \operatorname{dist}(F_1, F_2).$$

By the hypothesis, we can choose points $y_j \in F_j$, $j = 1, 2$, such that

$$|y_j - x_j| \geq \frac{1}{2} \operatorname{diam} F_j \geq \frac{a}{2} \, |x_1 - x_2|.$$

By relabeling we may also assume that $|y_1 - x_1| \leq |y_2 - x_2|$ if necessary.

Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be a Möbius transformation with $f(y_2) = \infty$. Then

$$\begin{aligned}
\frac{|f(x_2) - f(x_1)|}{|f(y_1) - f(x_1)|} &= \frac{|x_2 - x_1|}{|y_1 - x_1|} \frac{|y_1 - y_2|}{|x_2 - y_2|} \\
&\leq \frac{2}{a} \, \frac{|y_1 - y_2|}{|x_2 - y_2|} \leq \frac{2}{a} \, \frac{|x_2 - y_2| + |x_1 - x_2| + |y_1 - x_1|}{|x_2 - y_2|} \\
&\leq \frac{2}{a} \left(1 + \frac{2}{a} + 1 \right) = \frac{4(a+1)}{a^2} = b > 0.
\end{aligned}$$

Hence, by Theorem 11.9 [317] (see also Theorem 4 in [66])

$$\operatorname{mod} \Gamma \;=\; \operatorname{mod} f(\Gamma) \;\geq\; \varphi_n(b) \;=\; c \;>\; 0,$$

where $\varphi_n : (0,\infty) \to (0,\infty)$ is a decreasing function depending only on n. $\square$

A set $A \subset \overline{\mathbb{R}^n}$ is said to be ***a*-quasiconvex**, $1 \leq a < \infty$, if each pair of points $x_1, x_2 \in A \setminus \{\infty\}$ can be joined in A by a rectifiable path γ whose length does not exceed $a|x_1 - x_2|$. If $A \subset \mathbb{R}^n$, then A is 1-quasiconvex if and only if A is convex in the usual sense.

Lemma 3.3. *Suppose that E is an M-*QED *exceptional set in $\overline{\mathbb{R}^n}$. Then $D = \overline{\mathbb{R}^n} \setminus E$ is a domain that is a-quasiconvex with*

$$a \leq \exp(bM^{1/(n-1)}),$$

where b depends only on n.

Proof. Since E is closed, D is open. Suppose that D is not connected. Let D_1, D_2 be two disjoint components of D. Choose non-degenerate continua $F_j \subset D_j$, $j = 1,2$, and let Γ and Γ_E denote the families of paths joining F_1 and F_2 in $\overline{\mathbb{R}^n}$ and D, respectively. Lemma 3.2 implies that $\operatorname{mod} \Gamma > 0$. On the other hand $\Gamma_E = \emptyset$ and hence $\operatorname{mod} \Gamma_E = 0$. These two conclusions contradict (3.1), and D must thus be connected.

We show next that D is a-quasiconvex. Fix $x_1, x_2 \in D \setminus \{\infty\}$ and let $r = |x_1 - x_2|$. Since $D \setminus \{\infty\}$ is a domain, there is a path α joining x_1 to x_2 in $D \setminus \{\infty\}$. Let F_j denote the component of $\alpha \cap \overline{\mathbb{B}^n}(x_j, r/4)$ that contains x_j, $j = 1,2$, and let Γ and Γ_E denote the families of paths joining F_1 and F_2 in $\overline{\mathbb{R}^n}$ and D, respectively. Then

$$\min_{j=1,2} \operatorname{diam} F_j \;\geq\; r/4 \;\geq\; \operatorname{dist}(F_1, F_2)/4,$$

and Lemma 3.2 yields

$$\operatorname{mod} \Gamma \geq c_0 > 0,$$

where c_0 depends only on n. Since E is an M-QED exceptional set,

$$\operatorname{mod} \Gamma_E \;\geq\; \frac{1}{M} \;\operatorname{mod} \Gamma \;\geq\; \frac{c_0}{M}. \tag{3.4}$$

Let Γ_1 consist of those paths in Γ_E that lie in $\mathbb{B}^n(x_2, s)$,

$$s = \frac{r}{4} \exp\left(\left(\frac{c_0}{2M\omega_{n-1}}\right)^{1/(1-n)}\right) = rc_1,$$

and let $\Gamma_2 = \Gamma_E \setminus \Gamma_1$. Suppose that each path γ in Γ_E has length $l(\gamma) \geq L > 0$. Then

$$\operatorname{mod} \Gamma_1 \leq \frac{\Omega_n s^n}{L^n} = \frac{\Omega_n r^n c_1^n}{L^n},$$

where Ω_n is the n-measure of $\mathbb{B}^n$. On the other hand, each $\gamma \in \Gamma_2$ meets $S^{n-1}(x_2, s)$ and, hence,

$$\text{mod } \Gamma_2 \leq \omega_{n-1} \left(\log \frac{4s}{r} \right)^{1-n} = \frac{c_0}{2M}.$$

These inequalities yield

$$\text{mod } \Gamma_E \leq \text{mod } \Gamma_1 + \text{mod } \Gamma_2 \leq \frac{\Omega_n r^n c_1^n}{L^n} + \frac{c_0}{2M},$$

and, thus, by (3.4),

$$L \leq r c_1 \left(\frac{2M\Omega_n}{c_0} \right)^{1/n} < r \exp(cM^{1/(n-1)}),$$

where

$$c = 2 \left(2\omega_{n-1}/c_0\right)^{1/(n-1)}$$

depends only on n. Set $c_2 = \exp(cM^{1/(n-1)})$. Then there is a rectifiable path $\gamma_0 \in \Gamma_E$ with

$$l(\gamma_0) \leq rc_2 = c_2|x_1 - x_2|$$

and with endpoints $y_1, y_2 \in \alpha$ such that

$$|x_j - y_j| \leq |x_1 - x_2|/4$$

for $j = 1, 2$.

Next, set $r_1 = |x_1 - y_1|$ and let F_1 and F_2 denote the two components of $\alpha \cap \overline{\mathbb{B}^n}(x_1, r_1/4)$ and $\gamma_0 \cap \overline{\mathbb{B}^n}(y_1, r_1/4)$ that contain x_1 and y_1, respectively. Then

$$\min_{j=1,2} \text{diam } F_j \geq \frac{r_1}{4} \geq \frac{\text{dist}(F_1, F_2)}{4};$$

arguing as above, we obtain a path γ_1 in D such that

$$l(\gamma_1) \leq c_2|x_1 - y_1| \leq c_2 \frac{|x_1 - x_2|}{4}$$

and such that γ_1 joins γ_0 to a point $z_1 \in \alpha$ with

$$|x_1 - z_1| \leq \frac{1}{4^2}|x_1 - x_2|.$$

Clearly, the paths γ_0 and γ_1 contain a rectifiable subpath joining z_1 to y_2. Now a continuation of this process and a similar construction starting from y_2 toward x_2 lead to two sequences of paths $\gamma_1, \gamma_2, \dots$ and $\overline{\gamma_1}, \overline{\gamma_2}, \dots$ whose union together with γ_0 contains a rectifiable path γ in D from x_1 to x_2 with

$$l(\gamma) \leq l(\gamma_0) + \sum_{i=1}^{\infty} l(\gamma_i) + \sum_{i=1}^{\infty} l(\overline{\gamma}_i)$$

$$\leq c_2 \left(|x_1 - x_2| + \sum_{i=1}^{\infty} \frac{|x_1 - x_2|}{4^i} + \sum_{i=1}^{\infty} \frac{|x_1 - x_2|}{4^i} \right)$$
$$= \frac{5}{3} c_2 |x_1 - x_2| .$$

Thus, D is a-quasiconvex with

$$a = \frac{5}{3} c_2 \leq \exp((c+1)M^{1/(n-1)}),$$

as desired. □

Remark 3.2. Lemma 3.3 is an extension of the following result due to Ahlfors and Beurling; see Theorem 10 in [5]. If E is an NED set in $\overline{\mathbb{R}^2}$, then $D = \overline{\mathbb{R}^2} \setminus E$ is a-quasiconvex for each $a > 1$.

3.3 QED Domains and Their Properties

If E is an M-QED exceptional set, then by Lemma 3.3, $D = \overline{\mathbb{R}^n} \setminus E$ is a domain; we call any such domain an ***M*-quasiextremal distance** or ***M*-QED domain**. A domain $D = \overline{\mathbb{R}^n} \setminus E$ is called a **quasiextremal distance (QED) domain** if it is M-QED for some $M \in [1, \infty)$. These domains were introduced in [82].

A set A in $\overline{\mathbb{R}^n}$ is c-**locally connected** (cf. [67]), if, for each $x_0 \in \mathbb{R}^n$ and $r > 0$,

(i) points in $A \cap \overline{\mathbb{B}^n}(x_0, r)$ can be joined in $A \cap \overline{\mathbb{B}^n}(x_0, cr)$,

(ii) points in $A \setminus \mathbb{B}^n(x_0, r)$ can be joined in $A \setminus \mathbb{B}^n(x_0, r/c)$.

The set A is **linearly locally connected** if it is c-locally connected for some c.

Remark 3.3. When A is open, it is easy to see that condition (i) holds for a given $x_0 \in \mathbb{R}^n$ and $r > 0$ if and only if $(i)'$ points in $A \cap \mathbb{B}^n(x_0, r)$ can be joined in $A \cap \mathbb{B}^n(x_0, cr)$, and similarly for condition (ii). Moreover, if condition (i) holds for A and its image under each Möbius transformation $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$, then condition (ii) holds. To see this, let $x_1, x_2 \in A \setminus \mathbb{B}^n(x_0, r)$ and let

$$f(x) = r^2 \frac{x - x_0}{|x - x_0|} + x_0 .$$

Then $f(x_1), f(x_2) \in f(A) \cap \overline{\mathbb{B}^n}(x_0, r)$ and, by hypothesis, these points can be joined by a path γ in $f(A) \cap \overline{\mathbb{B}^n}(x_0, cr)$. Hence, $f^{-1}(\gamma)$ joins x_1, x_2 in $A \setminus \mathbb{B}^n(x_0, r/c)$.

Finally, it is not difficult to show that the property of being linearly locally connected is invariant under quasiconformal self-mappings of $\overline{\mathbb{R}^n}$. In particular, if A is c-locally connected and $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is K-quasiconformal, then $f(A)$ is c'-locally connected, where c' depends only on n, c, and K; see Theorem 5.6 in [331].

Lemma 3.4. *Suppose that D is an M-*QED *domain. Then D is c-locally connected with*

$$c \le 1 + \exp(bM^{1/(n-1)}),$$

where b is the constant of Lemma 3.3.

Proof. Fix $x_0 \in \mathbb{R}^n$ and $r > 0$. By Lemma 3.3, D is a-quasiconvex with

$$a \le \exp(bM^{1/(n-1)}).$$

Hence, each pair of points $x_1, x_2 \in D \cap \overline{\mathbb{B}^n}(x_0, r)$ can be joined in $D \cap \overline{\mathbb{B}^n}(x_0, s)$, where

$$s \le r + a|x_1 - x_2|/2 \le r + ar = (1+a)r.$$

Since

$$1 + a \le 1 + \exp\left(bM^{1/(n-1)}\right),$$

the points x_1, x_2 can be joined in $D \cap \overline{\mathbb{B}^n}(x_0, cr)$ and c has the desired upper bound.

Next, if D' is the image of D under a Möbius transformation of $\overline{\mathbb{R}^n}$, then D' is M-QED by Lemma 3.1 and points in $D' \cap \overline{\mathbb{B}^n}(x_0, r)$ can be joined in $D' \cap \overline{\mathbb{B}^n}(x_0, cr)$ by what was proved above. Thus, D is c-locally connected by Remark 3.3. $\square$

Remark 3.4. Suppose that D is a ball or a half-space, that F_1, F_2 are disjoint continua in D, and that Γ and Γ_D are the families of paths joining F_1 and F_2 in $\overline{\mathbb{R}^n}$ and D, respectively. Let Γ^* denote the family of paths joining F_1^* and F_2^* in $\overline{\mathbb{R}^n}$, where $F_j^* = F_j \cup \varphi(F_j)$ and φ denotes reflection with respect to ∂D. Then a reflection of admissible functions for Γ_D shows that

$$\operatorname{mod} \Gamma^* = 2 \operatorname{mod} \Gamma_D$$

and since

$$\operatorname{mod} \Gamma \le \operatorname{mod} \Gamma^*,$$

we see that D is a 2-QED domain. It is easy to see that the constant 2 is the best possible choice.

Next, if

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

and D is the image of the exterior of a ball under the affine mapping

$$f(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n),$$

then Lemma 3.1 implies that D is M-QED where

$$M = 2(\lambda_n/\lambda_1)^n.$$

If, in particular, $\lambda_1 = 1$ and $\lambda_2 = \ldots = \lambda_n = t > 1$, then D is a-quasiconvex only if

$$a > t = (M/2)^{1/n}.$$

This observation yields lower bounds for the constants a and c in Lemmas 3.3 and 3.4.

Lemmas 3.3 and 3.4 give quantitative information about the connectivity of a QED domain. The following result yields a measure density condition for this class of domains.

Lemma 3.5. *Suppose that D is an M-*QED *domain in $\overline{\mathbb{R}^n}$. Then, for each $x_0 \in \overline{D} \cap \mathbb{R}^n$ and* $0 < r \leq \operatorname{diam} D$,

$$\frac{m(D \cap \mathbb{B}^n(x_0,r))}{m(\mathbb{B}^n(x_0,r))} \geq \frac{c}{M}, \tag{3.5}$$

where $c > 0$ depends only on n.

Proof. Fix $x_0 \in \overline{D} \cap \mathbb{R}^n$. Since $r \leq \operatorname{diam} D$, we can choose $x_3 \in \overline{D}$ so that $|x_3 - x_0| = r/2$. Set $s = r/10$, choose $x_1, x_2 \in D$ such that $|x_0 - x_1| < s, |x_2 - x_3| < s$, and let α be a path joining x_1 and x_2 in D. Let F_1 be the x_1-component of $\alpha \cap \overline{\mathbb{B}^n}(x_0, 2s)$ and F_2 the x_2-component of $\alpha \setminus \overline{\mathbb{B}^n}(x_0, 3s)$. Next, denote by Γ and Γ_D the families of paths that join F_1 and F_2 in $\overline{\mathbb{R}^n}$ and in D, respectively. Set

$$\rho(x) = \begin{cases} \frac{1}{s} & \text{in } D \cap \mathbb{B}^n(x_0,r), \\ 0 & \text{elsewhere.} \end{cases}$$

Since each $\gamma \in \Gamma_D$ contains a subpath β that joins $S^{n-1}(x_0, 2s)$ and $S^{n-1}(x_0, 3s)$ in D,

$$\int_\gamma \rho \, ds \geq \int_\beta \rho \, ds = \frac{1}{s} \, l(\beta) \geq 1,$$

ρ is admissible for Γ_D, and

$$\begin{aligned} \operatorname{mod} \Gamma_D &\leq \int_{\mathbb{R}^n} \rho^n dm = \frac{1}{s^n} \int_{D \cap \mathbb{B}^n(x_0,r)} dm \\ &= 10^n \Omega_n \, \frac{m(D \cap \mathbb{B}^n(x_0,r))}{m(\mathbb{B}^n(x_0,r))}, \end{aligned}$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. Next,

$$\min_{j=1,2} \operatorname{diam} F_j \geq s \geq \frac{1}{4} \operatorname{dist}(F_1, F_2),$$

and, thus, Lemma 3.2 implies that

$$\operatorname{mod} \Gamma \geq c_0 > 0,$$

where c_0 depends only on n. Since D is M-QED, we obtain

$$\frac{m(D \cap \mathbb{B}^n(x_0,r))}{m(\mathbb{B}^n(x_0,r))} \geq \frac{c}{M},$$

where $c = c_0/(10^n \Omega_n)$. □

Remark 3.5. Suppose that $t > 1$ and that D is the image of the unit ball $\mathbb{B}^n(0,1)$ under the map

$$f(x_1,\dots,x_{n-1},x_n) = (x_1,\dots,x_{n-1},tx_n).$$

Then as in Remark 3.4, D is M-QED, where $M = 2t^n$, while

$$\frac{m(D \cap \mathbb{B}^n(0,t))}{m(\mathbb{B}^n(0,t))} = \frac{1}{t^{n-1}} = \left(\frac{2}{M}\right)^{(n-1)/n}.$$

Hence, the exponent of M in (3.5) is asymptotically sharp for large n.

Corollary 3.1. *The boundary ∂D of a* QED *domain D in $\mathbb{R}^n$ has n-dimensional measure zero.*

Proof. If the measure of ∂D is positive, then $\partial D \setminus \{\infty\}$ contains a point x_0 of density. However, by Lemma 3.5, the point x_0 cannot be a point of density for $E = \overline{\mathbb{R}^n} \setminus D$ and hence not for ∂D. □

3.4 Uniform and Quasicircle Domains

Recall that a domain D in $\mathbb{R}^n$ is said to be uniform if there exist constants a, b such that each $x_1, x_2 \in D$ can be joined by a rectifiable path γ in D with

$$\begin{aligned} l(\gamma) &\leq a\,|x_1 - x_2| \\ \min(s,\ l(\gamma) - s) &\leq b\,\mathrm{dist}\,(\gamma(s), \partial D). \end{aligned} \tag{3.6}$$

Here γ is parameterized by arc length s. The uniform domains have been introduced in [212] and their various characterizations can be found in [70, 202, 226, 318–320, 328]. The next lemma is essentially due to P. Jones [142].

Lemma 3.6. *A uniform domain D is an M-*QED *domain where the constant M depends only on n and D.*

Proof. Let F_1 and F_2 be two disjoint continua in D. Let $\varepsilon > 0$ and choose $u \in W(F_1, F_2; D)$ such that

$$\int_D |\nabla u|^n dm \leq \mathrm{cap}\,(F_1, F_2; D) + \varepsilon/2.$$

Then, for small $t > 0$, the function $v = (1+t)(1-t)^{-1}(u-t)$ satisfies the inequality

$$\int_D |\nabla v|^n dm \le \operatorname{cap}(F_1, F_2; D) + \varepsilon$$

and $v(x) \le -t$ for $x \in F_1$, $v(x) \ge 1+t$ for $x \in F_2$. Uniform domains enjoy the Sobolev extension property by Theorem 2 in [142]; hence, there exists an ACL-function $v^* : \mathbb{R}^n \to \mathbb{R}^n$ such that $v^* = v$ in D and

$$M \int_D |\nabla v|^n dm \ge \int_{\mathbb{R}^n} |\nabla v^*|^n dm,$$

where M depends only on n and the constants for D. Choose a smooth convolution approximation φ of v^* with $\varphi \le 0$ on F_1, $\varphi \ge 1$ on F_2, and

$$\int_{\mathbb{R}^n} |\nabla v^*|^n dm \ge \int_{\mathbb{R}^n} |\nabla \varphi|^n dm - \varepsilon\,.$$

Then $\varphi \in \tilde{W}(F_1, F_2; \overline{\mathbb{R}^n})$ and the last three inequalities yield

$$\operatorname{cap}(F_1, F_2; \overline{\mathbb{R}^n})) \le \int_{\mathbb{R}^n} |\nabla \varphi|^n dm \le M \operatorname{cap}(F_1, F_2; D) + \varepsilon(M+1).$$

Letting $\varepsilon \to 0$, we obtain the desired result. □

Although the classes of QED, linearly locally connected, and uniform domains do not coincide, it is possible to obtain more precise relations between them when $n = 2$. In particular, we shall show that for finitely connected plane domains, these classes are the same.

We say that $D \subset \overline{\mathbb{R}^n}$ is a K**-quasiball** if D is the image of an open ball or half-space under a K-quasiconformal self-mapping of $\overline{\mathbb{R}^n}$ and that $S \subset \overline{\mathbb{R}^n}$ is a K**-quasisphere** if it is a boundary of a K-quasiball. Next, a domain $D \subset \overline{\mathbb{R}^n}$ is said to be a K**-quasisphere domain** if each component of ∂D is either a point or a K-quasisphere. We use the more standard terms "quasidisk" and "quasicircle" when $n = 2$.

We shall show that every quasisphere domain is linearly locally connected and that this property characterizes this class of domains when $n = 2$. We require first the following result.

Lemma 3.7. *If $G_1, \dots G_k$ are pairwise disjoint K-quasiballs that all meet $S^{n-1}(x_0, r_1)$ and $S^{n-1}(x_0, r_2)$, then*

$$k \le a \left(\frac{r_2 + r_1}{|r_2 - r_1|} \right)^{n-1},$$

where a depends only on n and K.

Proof. The proof employs a standard packing argument. We may assume $r_2 > r_1$. Set $t = |r_2 - r_1|/2$. For each $i = 1, \dots, k$, choose $x_i \in G_i$ such that

$$|x_i - x_0| = \frac{r_1 + r_2}{2}.$$

By Lemma 3.1 (see also Remark 3.4), each G_i is an M-QED domain, where M depends only on K. For $i = 1, \ldots, k$, Lemma 3.5 yields

$$m(G_i \cap \mathbb{B}^n(x_i, t)) \geq \frac{c}{M} m(\mathbb{B}^n(x_i, t)) = \frac{c\Omega_n t^n}{M},$$

where $c > 0$ depends only on n. Since the quasiballs G_i are disjoint,

$$\begin{aligned} \Omega_n(r_2^n - r_1^n) &= m(\mathbb{B}^n(x_0, r_2) \setminus \overline{\mathbb{B}^n}(x_0, r_1)) \\ &\geq \sum_{i=1}^{k} m(G_i \cap \mathbb{B}^n(x_i, t)) \geq \frac{c\Omega_n k t^n}{M} = \frac{c\Omega_n k (r_2 - r_1)^n}{M 2^n}. \end{aligned}$$

Thus,

$$k \leq a\, \frac{r_2^n - r_1^n}{(r_2 - r_1)^n} = a\, \frac{1 - s^n}{(1 - s)^n},$$

where $s = r_1/r_2 < 1$ and $a = M2^n/c$ depends only on n and K. The elementary inequality

$$1 - s^n \leq (1 - s)(1 + s)^{n-1}$$

follows easily by induction and, hence,

$$k \leq a \left(\frac{r_2 + r_1}{|r_2 - r_1|} \right)^{n-1},$$

as desired. □

Lemma 3.8. *If D is a K-quasisphere domain , then D is c-locally connected, where c depends only on n and K.*

Proof. Let C_0 be a nondegenerate component of ∂D and let D_0 denote the component of $\overline{\mathbb{R}^n} \setminus C_0$ that contains D. Then D_0 is a K-quasiball and hence $c = c(n, K)$-locally connected by, for example, Remark 3.4 and Lemmas 3.1 and 3.4.

Fix $x_0 \in \mathbb{R}^n$, $r > 0$, and $d > c$. We shall show that D is d-locally connected. Since each image of D under a Möbius transformation is again a K-quasisphere domain, it suffices by the remarks in Section 3.3 to show that each pair of points $x_1, x_2 \in D \cap \mathbb{B}^n(x_0, r)$ can be joined in $D \cap \mathbb{B}^n(x_0, r)$. Suppose that this is not true for a given pair x_1, x_2. Then these points are separated by

$$F = \partial D \cup S^{n-1}(x_0, dr).$$

By Theorem Y.14.3 in [227], there is a component E of F that does this.

Now observe that E meets $S^{n-1}(x_0,dr)$ since otherwise $E \subset \partial D$ and hence could not separate x_1 and x_2. Let

$$E_0 = S^{n-1}(x_0,dr) \cup \left(\bigcup_\alpha C_\alpha \right),$$

where $\{C_\alpha\}$ is the collection of all components of ∂D that meet $S^{n-1}(x_0,dr)$. Then E_0 is a connected subset of F,

$$E \cap E_0 \supset E \cap S^{n-1}(x_0,dr) \neq \emptyset,$$

and hence $E_0 \subset E$. Suppose that there exists a point $y \in \partial D \setminus E_0$. Then y lies in a component C of ∂D with

$$C \cap S^{n-1}(x_0,dr) = \emptyset .$$

Choose $\varepsilon > 0$ so that

$$\varepsilon < q(C, S^{n-1}(x_0,dr)),$$

where q is the chordal metric in $\overline{\mathbb{R}^n}$. Then, by Corollary 1 in [227], p. 83, there is a set $H \subset \partial D$ such that H is both open and closed in ∂D with

$$C \subset H \subset \{x : q(x,C) < \varepsilon\} .$$

Thus,

$$H \cap S^{n-1}(x_0,dr) = \emptyset$$

and H is closed in F. On the other hand,

$$F \setminus H = S^{n-1}(x_0,dr) \cup (\partial D \setminus H)$$

is also closed in F. Hence, y does not belong to the same component of F as $S^{n-1}(x_0,dr)$, i.e., $y \notin E$. It follows that $E = E_0$ or

$$E \;=\; S^{n-1}(x_0,dr) \;\cup\; \left(\bigcup_\alpha C_\alpha \right) .$$

For each non-degenerate component C_α, let D_α and G_α denote the components of $\overline{\mathbb{R}^n} \setminus C_\alpha$ labeled so that $D \subset D_\alpha$. Then the sets G_α are pairwise disjoint K-quasiballs and hence, by Lemma 3.7, at most k of the C_α meet $S^{n-1}(x_0,cr)$, where

$$k \leq a \left(\frac{d+c}{d-c} \right)^{n-1}, \qquad a = a(n,K) .$$

By relabeling we may assume that these are the components $C_1, \ldots, C_k$. Then, for $i = 1, \ldots, k$, x_1 and x_2 lie in $D_i \cap \mathbb{B}^n(x_0,r)$ and hence x_1 and x_2 can be joined in $D_i \cap \mathbb{B}^n(x_0,cr)$. This says that x_1 and x_2 are not separated by

$$F_i = S^{n-1}(x_0, cr) \cup C_i.$$

For $j = 1, \dots, k$, let

$$E_j = \sum_{i=1}^{j} F_j$$

and suppose that x_1, x_2 are not separated by E_j for some $j < k$. Then, since

$$E_j \cap F_{j+1} = S^{n-1}(x_0, cr),$$

we can apply Theorem II.5.18 in [335] to conclude that x_1, x_2 are not separated by E_{j+1} and hence not by

$$E_k = S^{n-1}(x_0, cr) \cup \left(\bigcup_{i=1}^{k} C_i \right).$$

In particular, there is an arc γ that joins x_1 and x_2 in $\mathbb{B}^n(x_0, cr)$ and does not meet any C_i, $i = 1, \dots, k$. Choose C_α with $\alpha \notin \{1, ..., k\}$. Then C_α meets $S^{n-1}(x_0, dr)$ and not $S^{n-1}(x_0, cr)$. Hence, $C_\alpha \cap \gamma = \emptyset$ and we conclude that

$$E \cap \gamma = (S^{n-1}(x_0, dr) \cap \gamma) \cup \left(\bigcup_{\alpha} C_\alpha \cap \gamma \right) = \emptyset.$$

This means that E does not separate x_1 and x_2 and the proof is complete. $\square$

Lemma 3.9. *Suppose that D is b-locally connected and that ∂D is connected and contains at least two points. Then ∂D is a K-quasiconformal circle, where K depends only on b.*

Proof. Suppose that p is a point in $\overline{D}$. With each neighborhood U of p we associate a second neighborhood V as follows. If $p = z_0 \in \mathbb{C}$, choose $r \in (0, \infty)$ so that $\overline{B}(z_0, br) \subset U$ and let $V = B(z_0, r)$; if $p = \infty$, choose $r \in (0, \infty)$ so that $C(B(0, r/b)) \subset U$ and let $V = C(\overline{B}(0, r))$. In each case, the fact that D is b-locally connected implies that points are in $D \cap U$. Thus, D is uniformly locally connected and ∂D is a Jordan path γ by Theorem VI.16.2 in [227].

We show next that for any pair of finite points $z_1, z_2 \in \gamma$,

$$\min(\operatorname{diam}(\gamma_1), \operatorname{diam}(\gamma_2)) \leq b^2 |z_1 - z_2|, \tag{3.7}$$

where γ_1, γ_2 denote the components of $\gamma \setminus \{z_1, z_2\}$. By a theorem of Ahlfors, inequality (3.7) will then imply that γ is a K-quasiconformal circle, where K depends only on b, thus completing the proof; see, for example, Theorem II.8.6 in [190].

To this end, fix $z_1, z_2 \in \gamma$, set

$$z_0 = \frac{1}{2}(z_1 + z_2), \qquad r = \frac{1}{2}|z_1 - z_2|,$$

and suppose that (3.7) does not hold. Then there exist $t \in (r,\infty)$ and finite points w_1, w_2 such that

$$w_i \in \gamma_i \setminus B(z_0, b^2 t) \tag{3.8}$$

for $i = 1,2$. Choose $s \in (r,t)$. Since $z_1, z_2 \in \gamma \cap B(z_0,s)$, we can find for $i = 1,2$ an endcut α_i of D joining z_i to $z_i' \in D$ in $\overline{B}(z_0,s)$. Next, since D is b-locally connected, we can find an arc α_3 joining z_1' to z_2' in $D \cap \overline{B}(z_0,bs)$. Then $\alpha_1 \cup \alpha_2 \cup \alpha_3$ contains a crosscut α of D from z_1 to z_2 with

$$\alpha \subset \overline{B}(z_0,bs). \tag{3.9}$$

By virtue of (3.8), the same argument can be applied to obtain a crosscut β of D from w_1 to w_2 with

$$\beta \subset C(B(z_0,bt)). \tag{3.10}$$

But (3.9) and (3.10) imply that $\alpha \cap \beta = \emptyset$, contradicting the fact that z_1 and z_2 separate w_1 and w_2 in γ. Thus, (3.7) holds and the proof of Lemma 3.9 is complete. □

Lemma 3.10. *Suppose that D is b-locally connected. Then each component of ∂D is either a point or a K-quasiconformal circle where K depends only on b.*

Proof. Let B_0 be a component of ∂D, let C_0 denote the component of $C(D)$ that contains B_0, and let $D_0 = C(C_0)$. Then D_0 is a domain with $\partial D_0 = B_0$; see, e.g., the proof of Theorem VI.16.3 in [227]. To complete the proof, we need only show that D_0 is b-locally connected, for then, by Lemma 3.9, ∂D_0 will be a point or a K-quasiconformal circle where $K = K(b)$.

Fix $z_0 \in \mathbb{C}$ and $r \in (0,\infty)$. Given $z_1, z_2 \in D_0 \cap \overline{B}(z_0,r)$, we must find an arc γ joining these points in $D_0 \cap \overline{B}(z_0,br)$. For this let α be any arc joining z_1 and z_2 in $\overline{B}(z_0,r)$. If $\alpha \subset D_0$, we may take $\gamma = \alpha$. Suppose that $\alpha \not\subset D_0$ and for $i = 1,2$, let α_i denote the component of $\alpha \cap D_0$ that contains z_i. Then for each i there exists a point w_i such that

$$w_i \in \alpha_i \cap D. \tag{3.11}$$

If $z_i \in D$, we may take $w_i = z_i$; otherwise, $z_i \in C_i$, a component of $C(D)$ different from C_0, and the fact that

$$\overline{\alpha}_i \cap C_0 \neq \emptyset, \qquad \alpha_i \cap C_i \neq \emptyset$$

imply that α_i must meet D and hence contain a point w_i satisfying (3.11). Since D is b-locally connected and since

$$w_1, w_2 \in \alpha \cap D \subset D \cap \overline{B}(z_0,r),$$

we can join w_1 and w_2 by an arc β in $D \cap \overline{B}(z_0,br)$. Then $\alpha_1 \cup \beta \cup \alpha_2$ will contain an arc γ joining z_1 and z_2 in $D_0 \cap \overline{B}(z_0,br)$.

Next, the same argument shows that each pair of points in $D_0 \setminus B(z_0, r)$ can be joined in $D_0 \setminus B(z_0, r/b)$. Hence, D_0 is b-locally connected and the proof is complete. □

Theorem 3.1. *A domain D in $\overline{\mathbb{R}^2}$ is a quasicircle domain if and only if it is linearly locally connected.*

Proof. Suppose that D is a domain in $\overline{\mathbb{R}^2}$. If D is linearly locally connected, then, by Lemma 3.10, D is a quasicircle domain. The converse follows from Lemma 3.8. □

Theorem 3.2. *If D is a finitely connected domain in $\overline{\mathbb{R}^2}$, then the following conditions are equivalent.*

(i) D is a QED *domain.*

(ii) D is linearly locally connected.

(iii) D is a quasicircle domain.

(iv) D is uniform.

Proof. That (i) implies (ii) follows from Lemma 3.4; that (ii) implies (iii) is a consequence of Theorem 3.1. By Theorem 5 in [234] and Theorem 5 in [83], a finitely connected quasicircle domain is uniform. Finally, (iv) implies (i) by Lemma 3.6. □

Remark 3.6. Suppose that $D \neq \mathbb{R}^2$ is a simply connected domain in $\mathbb{R}^2$. Then Theorem 3.2 implies the well-known equivalence of the following conditions.

(i) D is a QED domain.

(ii) D is linearly locally connected.

(iii) D is a quasidisk.

(iv) D is uniform.

The equivalence of (i) and (iii) was first proved by V. Gol'dstein and S. Vodop'janov in [93]. For the equivalence of (iii) and (iv), see Corollary 2.33 in [212], while the equivalence of (ii) and (iii) follows from Lemmas 4 and 5 in [68]; cf. also [70].

Remark 3.7. Finally, for a domain $D \subset \overline{\mathbb{R}^n}$, $n \geq 2$, we have the following relations between the classes of domains considered above.

(i) If D is uniform, then D is QED.

(ii) If D is QED, then D is linearly locally connected.

(iii) If D is a quasisphere domain, then D is linearly locally connected.

(iv) There exists a QED domain D that is not uniform.

(v) There exists a quasisphere domain D that is not QED, and hence not uniform.

(vi) For $n > 2$, there exists a domain D that is uniform, and hence QED and linearly locally connected, but not a quasisphere domain.

The first three conclusions follow from Lemmas 3.6, 3.4, and 3.8, respectively. For (iv), let E be the set in $\mathbb{R}^n$ whose points have integer coordinates. Then E is NED set because E is countable and hence the family of all paths meeting E has zero modulus. Thus, $D = \mathbb{R}^n \setminus E$ is a QED domain, but D cannot be uniform because the second condition in *(3.6)* fails. For (v), choose a closed, totally disconnected set in $\mathbb{R}^n$ with $m(E) > 0$. Then $D = \mathbb{R}^n \setminus E$ is a 1-quasisphere domain with $\partial D = E \cup \{\infty\}$, and hence D is not QED by Corollary 3.1. Finally, when $n > 2$, then $D = \mathbb{R}^n \setminus \mathbb{R}^1$ is a uniform domain while $\mathbb{R}^1 \cup \{\infty\}$ is neither a point nor a quasisphere.

3.5 Extension of Quasiconformal and Quasi-Isometric Maps

We shall show in this chapter that a quasiconformal mapping between QED domains in $\overline{\mathbb{R}^n}$ has a homeomorphic extension to the closures of the domains when $n \geq 2$ and a quasiconformal extension to $\overline{\mathbb{R}^n}$ when $n = 2$. These results first appeared in [82] as a result of extensive studies of the corresponding extension problems in uniform domains. Chapter 2 then yields several extension theorems for quasiconformal mappings on various subclasses of QED domains. We also prove corresponding results for injective local quasi-isometries.

We begin with the following result.

Theorem 3.3. *Suppose that D and D' are domains in $\overline{\mathbb{R}^n}$, that D is M-*QED *and that D' is c'-locally connected. If f is a K-quasiconformal mapping of D onto D', then f has a homeomorphic extension to $\bar{D}$. Moreover, if x_1, x_2, x_3, x_4 are distinct points in $\bar{D}$ with*

$$\frac{|x_1 - x_2|}{|x_3 - x_2|} \frac{|x_3 - x_4|}{|x_1 - x_4|} \leq a,$$

then

$$\frac{|f(x_1) - f(x_2)|}{|f(x_3) - f(x_2)|} \frac{|f(x_3) - f(x_4)|}{|f(x_1) - f(x_4)|} \leq b, \tag{3.12}$$

where b is a constant that depends only on n, K, M, c', and a.

Proof. We begin by deriving *(3.12)* whenever $x_1, x_2, x_3, x_4 \in D$. By composing f with a pair of Möbius transformations and appealing to Lemma 3.1 and Remark 3.3, we see that it is sufficient to consider the case where $x_4 = \infty$ and $f(x_4) = \infty$; then we must show that

$$\frac{|x_1 - x_2|}{|x_3 - x_2|} \leq a \quad \Rightarrow \quad \frac{|y_1 - y_2|}{|y_3 - y_2|} \leq b, \tag{3.13}$$

where $y_j = f(x_j), j = 1, 2, 3, 4$.

First we choose t so that

$$|y_1 - y_2| = c'^2 t |y_3 - y_2| = c'^2 tr$$

and we suppose that $t > 1$. Because D' is c'-locally connected, there exist continua F_1' and F_2', which join y_2 to y_3 in $D' \cap \overline{\mathbb{B}^n}(y_2, c'r)$ and y_1 to $y_4 = \infty$ in $D' \setminus \mathbb{B}^n(y_2, c'tr)$, respectively. Set $F_j = f^{-1}(F_j')$ and let Γ and Γ_D denote the families of paths joining F_1 and F_2 in $\overline{\mathbb{R}^n}$ and D, respectively. If $\gamma \in \Gamma_D$, then $f(\gamma)$ joins $S^{n-1}(y_2, c'r)$ to $S^{n-1}(y_2, c'tr)$ and, thus,

$$\operatorname{mod} \Gamma_D \le K \operatorname{mod} f(\Gamma_D) \le K\, \omega_{n-1}\, (\log t)^{1-n}.$$

Next,

$$\min_{j=1,2} \operatorname{diam} F_j \ge |x_3 - x_2| \ge \frac{1}{a}\, |x_1 - x_2| \ge \frac{1}{a} \operatorname{dist}(F_1, F_2),$$

and by Lemma 3.2,

$$\operatorname{mod} \Gamma \ge c,$$

where $c > 0$ depends only on n and a. Since D is an M-QED domain, these inequalities yield

$$c \le \operatorname{mod} \Gamma \le M \operatorname{mod} \Gamma_D \le MK\omega_{n-1}\, (\log t)^{1-n}$$

or

$$t \le \exp\left(\left(\frac{MK\omega_{n-1}}{c} \right)^{1/(n-1)} \right).$$

Now this inequality holds trivially whenever $t \le 1$. Hence, we obtain *(3.13)* with

$$b = c'^2 \exp\left(\left(\frac{MK\omega_{n-1}}{c} \right)^{1/(n-1)} \right).$$

Next, we show that f has a homeomorphic extension to $\overline{D}$. Again, it suffices to consider the case where $\infty \in D$ and $f(\infty) = \infty$. Fix $x_0 \in \partial D$ and choose points $x_j \in D$ so that $x_j \to x_0$ and $f(x_j) \to y_0$ as $j \to \infty$. Then $y_0 \in \partial D' \subset \mathbb{R}^n$. Given $\varepsilon > 0$, fix k such that

$$|f(x_k) - y_0| \le \varepsilon.$$

Suppose that $x \in D$ and

$$|x - x_0| \le \frac{1}{3} |x_k - x_0| = \delta.$$

For large j, $|x_j - x_0| \le \delta$ and

$$\begin{aligned} |x - x_j| &\le |x - x_0| + |x_j - x_0| \le 3\delta - |x_j - x_0| \\ &= |x_k - x_0| - |x_j - x_0| \le |x_k - x_j| \end{aligned} \tag{3.14}$$

and, applying *(3.12)* with $x_1 = x, x_2 = x_j$, $x_3 = x_k$, and $x_4 = \infty$, we conclude that

$$|f(x) - f(x_j)| \le b |f(x_k) - f(x_j)|,$$

where $b = b(n,K,M,c')$. Letting $j \to \infty$, we obtain

$$|f(x) - y_0| \leq b|f(x_k) - y_0| \leq b\varepsilon,$$

which shows that $f(x) \to y_0$ as $x \to x_0$ in D. Thus, f has a continuous extension to $\overline{D}$, which we again denote by f. By continuity, *(3.12)* holds whenever $x_1, x_2, x_3, x_4 \in \overline{D}$, where b is the original constant corresponding to $a+1$, and this, in turn, implies that f is injective in $\overline{D}$ and hence a homeomorphism. □

Theorem 3.3, Lemma 3.4, and Lemma 3.6 imply the following results.

Corollary 3.2. *If D and D' are* QED *domains in $\overline{\mathbb{R}^n}$, then each quasiconformal mapping of D onto D' has a homeomorphic extension to $\overline{D}$.*

Corollary 3.3. *If D and D' are uniform domains in $\overline{\mathbb{R}^n}$, then each quasiconformal mapping of D onto D' has a homeomorphic extension to $\overline{D}$.*

Remark 3.8. In the case of bounded uniform domains, Corollary 3.3 also follows from Corollary 3.30 in [70] since then both f and f^{-1} belong to some Lipschitz class Lip_α, $\alpha > 0$.

Remark 3.9. In the plane, Theorem 3.3 can be considerably sharpened. We first require the following results on quasidisks.

Lemma 3.11. *Suppose that G is a K-quasidisk in $\mathbb{R}^2$, that $z_0 \in \mathbb{R}^2 \setminus G$, and that α is a component of $G \cap S^1(z_0, r)$. Then*

$$\operatorname{diam} \alpha \leq c\, |z_1 - z_2|,$$

where z_1, z_2 are the endpoints of α and c depends only on K.

Proof. Let θ be the angle subtended by α at z_0. If $0 < \theta \leq \pi$, then

$$\operatorname{diam} \alpha = |z_1 - z_2| .$$

If $\theta > \pi$, then consider the ray from z_0 through the point $2z_0 - z_1$ on the opposite side of z_1 in $S^1(z_0, r)$. Since G lies in $\mathbb{R}^2$, this ray meets each of the components γ_1 and γ_2 of $\partial G \setminus \{z_1, z_2\}$; thus,

$$\operatorname{diam} \gamma_j \geq |z_1 - z_0| = r$$

for $j = 1, 2$. On the other hand, since ∂G is a K-quasicircle,

$$\min_{j=1,2} \operatorname{diam} \gamma_j \leq a\, |z_1 - z_2|,$$

where $a = a(K)$, and hence

$$\operatorname{diam} \alpha \leq 2r \leq 2a|z_1 - z_2|.$$

□

Lemma 3.12. *If G_j is an infinite sequence of pairwise disjoint K-quasidisks, then*

$$\lim_{j\to\infty} q(G_j) = 0,$$

where $q(G_j)$ is the chordal diameter of G_j.

Proof. The proof follows from the fact that a quasidisk cannot be very thin. Indeed, if the lemma does not hold, then after passing to a subsequence if necessary, choose $z_j, w_j \in G_j$ such that $z_j \to z_0 \neq \infty$ and $w_j \to w_0 \neq z_0$. Fix $0 < r_1 < r_2 < |z_0 - w_0|$. Then there exists j_0 such that $|z_j - z_0| < r_1$ and $|w_j - z_0| > r_2$ for $j \geq j_0$. This says that infinitely many G_j meet both $S^1(z_0, r_1)$ and $S^1(z_0, r_2)$, contradicting the conclusion of Lemma 3.7. $\square$

Theorem 3.4. *Suppose that D and D' are domains in $\overline{\mathbb{R}^2}$, that D is M-*QED*, and that D' is c'-locally connected. If f is a K-quasiconformal mapping of D onto D', then f has a K^*-quasiconformal extension to $\overline{\mathbb{R}^2}$, where K^* depends only on the constants K, M, and c'.*

Proof. By Theorem 3.3, f has a homeomorphic extension, denoted again by f, which maps $\overline{D}$ onto $\overline{D'}$. Next, by Lemma 3.4, D is c-locally connected, where $c = c(M)$, and it follows from Theorem 3.1 that D and D' are K_1-quasicircle domains, where K_1 depends only on M and c'.

Let C be a quasicircle component of ∂D. Then $C' = f(C)$ is also a quasicircle and there exist K_1-quasiconformal mappings g and g' of $\overline{\mathbb{R}^2}$ onto itself such that $g(C) = \overline{\mathbb{R}^1}, g'(C') = \overline{\mathbb{R}^1}$, and $g' \circ f \circ g^{-1}(\infty) = \infty$. Moreover, we may assume that g maps the component G of $\overline{\mathbb{R}^2} \setminus \overline{D}$ bounded by C onto the lower half-plane H and that g' does the same for the corresponding component G' of $\overline{\mathbb{R}^2} \setminus \overline{D'}$. Then $h = g' \circ f \circ g^{-1}$ is a homeomorphism that maps $\overline{g(D)}$ onto $\overline{g'(D')}$ and $\mathbb{R}^1$ onto $\mathbb{R}^1$ and is K_2-quasiconformal in $g(D), K_2 = KK_1^2$. Now $g(D)$ is M_1-QED with $M_1 = K_1^2 M$ and by Remark 3.3, $g'(D')$ is c_1'-locally connected, where c_1' depends only on c' and K_1. Choose $x \in \mathbb{R}^1$, $t > 0$, and let

$$x_1 = x + t,\ x_2 = x,\ x_3 = x - t,\ x_4 = \infty.$$

Then, by Theorem 3.3 applied to h,

$$\frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq b,$$

where b depends only on K_1, M_1, and c_1'. By interchanging the roles of x_1 and x_3 above, we conclude that $h \mid \mathbb{R}^1$ is b-quasisymmetric and hence, by a theorem of Ahlfors and Beurling in [6], there exists a homeomorphism $h^* : \overline{H} \to \overline{H}$ that agrees with h on ∂H and is K_3-quasiconformal in $H, K_3 = K_3(b)$.

Mapping back, we obtain a homeomorphism f_G of $\overline{D} \cup G$ onto $\overline{D'} \cup G'$ that extends f and that is K^*-quasiconformal in D and in G, where K^* depends only on

K, M, and c'. Define $f^* : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ as $f^*(z) = f(z)$ when $z \in \overline{D}$ and $f^*(z) = f_G(z)$ when z belongs to a quasidisk component of G of $\overline{\mathbb{R}^2} \setminus \overline{D}$. Next we show that f^* is a homeomorphism. Since f^* is injective, it suffices to show that f^* is continuous, and this clearly follows if we establish the continuity of f^* at $z_0 \in \partial D$.

Let $z_j \to z_0$ and suppose that $f^*(z_j) \to w_0$. We want to show that $w_0 = f^*(z_0)$. If infinitely many z_j belong either to $\overline{D}$ or to a single component G of $\overline{\mathbb{R}^2} \setminus \overline{D}$, then this follows from the fact that f is continuous in $\overline{D}$ and f_G in $\overline{G}$, respectively. Suppose that the points z_j lie in infinitely many distinct components G_j of $\overline{\mathbb{R}^2} \setminus \overline{D}$. Passing to a subsequence, if necessary, we may assume that $z_j \in G_j$, where the G_j are pairwise disjoint. For each j choose $w_j \in \partial G_j \subset \partial D$. Since the K_1-quasidisks G_j are pairwise disjoint, Lemma 3.12 implies that $q(G_j) \to 0$ as $j \to \infty$. Thus, $w_j \to z_0$, and, hence, $f^*(w_j) \to f^*(z_0)$ by the continuity of f in $\overline{D}$. Next, because the K_1-quasidisks $f^*(G_j)$ are pairwise disjoint, the same reasoning shows that $f^*(z_j)$ approaches the same limit as $f(w_j)$. Thus, $w_0 = f(z_0)$.

It remains to show that f^* is K^*-quasiconformal in $\overline{\mathbb{R}^2}$. Suppose first that $\infty \in D$ and $f^*(\infty) = \infty$. By Corollary 3.1, ∂D has zero planar measure. Hence, by a well-known characterization for quasiconformality, it suffices to show that there is a constant c such that

$$L(z_0, r) \le c\, l(z_0, r) \tag{3.15}$$

for all $z_0 \in \partial D \setminus \{\infty\}$ and $0 < r < \infty$, where

$$L(z_0, r) = \max_{|z - z_0| = r} |f^*(z) - f^*(z_0)|,$$

$$l(z_0, r) = \min_{|z - z_0| = r} |f^*(z) - f^*(z_0)|.$$

By making a pair of change of variables in the image and preimage of f, we may assume that $z_0 = 0$ and $f^*(z_0) = 0$.

Suppose first that $z_1, z_2 \in \overline{D}$ with $|z_1| = |z_2| = r$. Then, by *(3.12)*,

$$|w_2| \le b_1 |w_1|, \tag{3.16}$$

where $w_j = f^*(z_j)$ for $j = 1, 2$ and $b_1 = b_1(K, M, c')$.

Suppose next that $z_3 \in \mathbb{R}^2 \setminus \overline{D}$ with $|z_3| = r$. Then $z_3 \in G$, where G is a K_1-quasidisk in $\mathbb{R}^2$ with $0 \notin G$; let z_1, z_2 denote the endpoints of the component α of $G \cap S^1(0, r)$ that contains z_3, labeled so that $|w_1| \le |w_2|$. Here again, $w_j = f^*(z_j)$ for $j = 1, 2, 3$. We shall show that

$$\frac{1}{b_2}\,|w_1| \le |w_3| \le b_2\,|w_2|, \tag{3.17}$$

where b_2 depends only on K, M, and c'.

Choose $z_4 \in \partial G \subset \overline{D}$ so that $|w_4| = |w_3|$, and suppose first that $|z_3 - z_4| \le \frac{1}{3}|z_1 - z_4|$. Then

$$|z_4| \le \frac{1}{3}\,|z_1 - z_4| + |z_3| \le \frac{4}{3}\,|z_3| + \frac{1}{3}\,|z_4|,$$

$$|z_4| \ge |z_3| - \frac{1}{3}\,|z_1 - z_4| \ge \frac{2}{3}\,|z_1| - \frac{1}{3}\,|z_4|.$$

Hence,

$$\frac{1}{2}\,|z_1| \le |z_4| \le 2|z_2|,$$

and Theorem 3.3 applied to $f^* \mid \overline{D}$ yields

$$\frac{1}{b_3}\,|w_1| \le |w_3| = |w_4| \le b_3\,|w_2|,$$

where $b_3 = b_3(K, M, c')$. Suppose next that $|z_3 - z_4| > \frac{1}{3}|z_1 - z_4|$. Then, by Lemma 3.11,

$$\frac{|z_1 - z_4|}{|z_3 - z_4|}\frac{|z_3 - z_2|}{|z_1 - z_2|} \le 3\,\frac{\text{diam}\,\alpha}{|z_1 - z_2|} \le 3c,$$

where $c = c(K_1)$. Since G and $G' = f^*G$ are K_1-quasidisks and hence $2K_1^2$-QED domains, we can apply Theorem 3.3 to $f^* \mid \overline{G}$ with $a = 3c$ to obtain

$$\frac{|w_1 - w_4|}{|w_3 - w_4|}\frac{|w_3 - w_2|}{|w_1 - w_2|} \le b_4, \tag{3.18}$$

where $b_4 = b_4(K, M, c')$. If $|w_4| \ge 2|w_1|$, then

$$|w_3 - w_4| \le 2|w_4| \le 4|w_1 - w_4|,$$

and from *(3.18)* we obtain

$$\begin{aligned} |w_3| &\le 4b_4|w_1 - w_2| + |w_2| \le 4b_4(|w_1| + |w_2|) + |w_2| \\ &\le (8b_4 + 1)|w_2|, \end{aligned}$$

where the inequality $|w_1| \le |w_2|$ has also been used. Similarly, if $|w_3| \le |w_1|/2$ and hence $|w_3| \le |w_2|/2$, then

$$|w_1 - w_2| \le 2|w_2| \le 4|w_3 - w_2|$$

and

$$|w_1| \le 4b_4|w_3 - w_4| + |w_4| \le (8b_4 + 1)|w_3|,$$

where *(3.18)* and the equality $|w_3| = |w_4|$ have been used. Thus, we obtain *(3.17)* with

$$b_2 = \max(b_3, 2,\ 8b_4 + 1).$$

Finally, *(3.16)* and *(3.17)* imply *(3.15)* with $c = b_1 b_2^2$ and $z_0 = 0$, completing the proof for the case where $\infty \in D$ and $f(\infty) = \infty$. The general case can then be reduced to the special case by composing f with two auxiliary Möbius transformations. □

The first corollary below follows from Theorem 3.4 and Lemmas 3.1 and 3.4. The second corollary is due to the first and to Remark 3.6. The second corollary was proved in [212] (see also [83]), and the first corollary appeared in [82].

Corollary 3.4. *If D is a* QED *domain in $\overline{\mathbb{R}^2}$ and f is a quasiconformal mapping of D onto D', then f has a quasiconformal extension to $\overline{\mathbb{R}^2}$ if and only if D' is* QED.

Corollary 3.5. *If D is a uniform domain in $\mathbb{R}^2$ and f is a quasiconformal mapping of D onto a domain D' in $\mathbb{R}^2$, then f has a quasiconformal extension to $\overline{\mathbb{R}^2}$ if and only if D' is uniform.*

For finitely connected domains D in $\overline{\mathbb{R}^2}$, we obtain the following statement.

Corollary 3.6. *Suppose that D is a linearly locally and finitely connected domain in $\overline{\mathbb{R}^2}$. If f is a quasiconformal mapping of D onto a domain D', then f has a quasiconformal extension to $\overline{\mathbb{R}^2}$ if and only if D' is linearly locally connected.*

Proof. If $D = \overline{\mathbb{R}^2}$, then there is nothing to prove and in the case $D \neq \overline{\mathbb{R}^2}$ we can compose f with two auxiliary Möbius transformations and hence assume $D, D' \subset \mathbb{R}^2$. Now Corollary 3.4 or Corollary 3.5 together with Theorem 3.2 yields the result. □

Remark 3.10. (a) Since a quasidisk $D \subset \mathbb{R}^2$ is uniform, linearly locally connected, and QED, all three corollaries are generalizations of the known Beurling–Ahlfors extension theorem; see [5].

(b) Corollaries 3.4 and 3.5 do not hold for $n \geq 3$. A counterexample is provided by a quasiconformal mapping of a smooth knotted torus onto one that is not knotted.

(c) If D is a QED domain in $\overline{\mathbb{R}^n}$, $n \geq 2$, with $\overline{D} = \overline{\mathbb{R}^n}$, then $E = \overline{\mathbb{R}^n} \setminus D$ is NED; see Remark 3.1. In this case it follows from results of Ahlfors and Beurling [5] when $n = 2$ and Aseev and Sychev [15] when $n \geq 3$ that every K-quasiconformal mapping of D into $\overline{\mathbb{R}^n}$ has a K-quasiconformal extension to $\overline{\mathbb{R}^n}$.

(d) We give an example in Chapter 7 to show that Corollary 3.6 does not hold when D is infinitely connected.

Remark 3.11. Theorem 3.4 can be used to interpret the geometric structure of QED and uniform domains in $\overline{\mathbb{R}^2}$.

Suppose that D and D' are domains in $\overline{\mathbb{R}^2}$. If there exists a quasiconformal mapping of $\overline{\mathbb{R}^2}$ that carries D onto D', then D is QED if and only if D' is so. This statement is false if we know only that there exists a quasiconformal mapping of D that carries D onto D'; for an example, let D be the upper half-plane and $f(z) = z^2$. On the other hand, if we know that D and D' are linearly locally connected and that there exists a quasiconformal mapping of D that carries D onto D', then Theorem 3.4 implies that D is QED if and only if D' is. Thus, the collection of QED domains is invariant under quasiconformal mappings in the class of plane domains that are linearly locally connected, i.e., in the class of quasicircle domains.

Alternatively, we may think of a domain $D \subset \overline{\mathbb{R}^2}$ as being determined by the shape of its boundary components and by their relative position and size as measured by its conformal moduli. Then Theorem 3.4 implies that D is QED if and only if D is a quasicircle domain whose conformal geometry is quasiconformally equivalent to that of another QED domain. In particular, it is natural to ask for geometric conditions on the boundary components of a quasicircle domain D that are necessary and sufficient to guarantee that D is QED.

Obviously, the same remarks and questions hold for uniform domains in $\overline{\mathbb{R}^2}$.

3.6 Extension of Local Quasi-Isometries

Suppose that f is a mapping of $E \subset \overline{\mathbb{R}^n}$ into $\overline{\mathbb{R}^n}$. We say that f is an **L-quasi-isometry** in E if

$$\frac{1}{L}\,|x_1 - x_2| \leq |f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|$$

for each pair of points $x_1, x_2 \in E \setminus \{\infty\}$ and if $f(\infty) = \infty$ whenever $\infty \in E$. We say that f is a **local L-quasi-isometry** in E if, for each $L' > L$, every $x \in E$ has a neighborhood U such that f is an L'-quasi-isometry in $E \cap U$.

The next theorem is a counterpart of Theorem 3.3 for injective local quasi-isometries.

Theorem 3.5. *If f is an injective local L-quasi-isometry of a quasiconvex domain $D \subset \overline{\mathbb{R}^n}$ into a domain $D' \subset \overline{\mathbb{R}^n}$, then f extends to a quasi-isometry f^* of $\overline{D}$ onto $\overline{D'}$ if and only if D' is a quasiconvex. In this case f^* is an L^*-quasi-isometry with $L^* = \max(a, a')$, where a and a' are the constants for D and D'.*

Proof. Suppose first that f extends to an L^*-quasi-isometry f^* of $\overline{D}$ onto $\overline{D'}$. Let $y_1, y_2 \in D' \setminus \{\infty\}$. Since D is an a-quasiconvex domain, there is a path γ in D joining $f^{-1}(y_1)$ to $f^{-1}(y_2)$ with

$$l(\gamma) \;\leq\; a\,|f^{-1}(y_1) - f^{-1}(y_2)|.$$

Now, $f(\gamma)$ joins y_1 to y_2 in D' and

$$l(f(\gamma)) \;\leq\; L^*\,l(\gamma) \;\leq\; L^* a\,|f^{-1}(y_1) - f^{-1}(y_2)| \;\leq\; L^{*2} a\,|y_1 - y_2|.$$

Thus, D' is a'-quasiconvex with $a' = L^{*2}a$.

Next, suppose that D' is a'-quasiconvex and that f is an injective local L-quasi-isometry of an a-quasiconvex domain D onto D'. Fix $x_1, x_2 \in D' \setminus \{\infty\}$. There is a rectifiable path γ joining x_1 and x_2 in D with

$$l(\gamma) \le a\,|x_1 - x_2|.$$

Thus,

$$|f(x_1) - f(x_2)| \le l(f(\gamma)) \le L\,l(\gamma) \le La\,|x_1 - x_2|.$$

Since f is injective, f^{-1} is a local L-quasi-isometry in D' and arguing as above yields

$$|x_1 - x_2| \le La'\,|f(x_1) - f(x_2)|.$$

Hence, f is an L^*-quasi-isometry in D, where $L^* = L\max(a, a')$, and we can extend f to $\overline{D}$ by continuity. □

Remark 3.12. Theorem 3.5 together with Section 3.2 yields several extension results for injective local quasi-isometries. For example, *if f is an injective local quasi–isometry of a uniform domain $D \subset \mathbb{R}^n$ onto a domain $D' \subset \mathbb{R}^n$, then f extends to a quasi-isometry of $\overline{D}$ onto $\overline{D'}$ if and only if D' is uniform.* If D and D' are uniform, then the extension follows from Theorem 3.5 and from the fact that uniform domains are quasiconvex; cf. (3.6). On the other hand, it is easy to see that the image of a uniform domain D under a quasi-isometry $f : D \to \mathbb{R}^n$ is again a uniform domain.

We conclude this chapter with the following analogue of Theorem 3.4 for injective local quasi-isometries.

Theorem 3.6. *Suppose that D and D' are domains in $\overline{\mathbb{R}^2}$, that D is M-*QED*, and that D' is c'-locally connected. If f is an injective local L-quasi-isometry of D onto D' and, in the case $\infty \notin D$, the unbounded complementary components of D and D' correspond under f, then f has an L^*-quasi-isometric extension to $\overline{\mathbb{R}^2}$, where L^* depends only on the constants L, M, and c'.*

The formulation of this result requires a word of explanation. If f is an injective local quasi-isometry, then f defines a homeomorphism of D onto D'. In this case, for each component E of $\overline{\mathbb{R}^2} \setminus D$ there exists a unique component E' of $\overline{\mathbb{R}^2} \setminus D'$ such that $f(x) \to E'$ if and only if $x \to E$ in D. The second hypothesis on f in Theorem 3.6 requires that $\infty \in E'$ whenever $\infty \in E$. This condition is clearly necessary for f to have a quasi-isometric extension to $\overline{\mathbb{R}^2}$.

Proof for Theorem 3.6. The hypotheses imply that f is a K-quasiconformal mapping of D onto D', where $K = L^2$. Hence, by Theorem 3.4, f has a K^*-quasiconformal extension to $\overline{\mathbb{R}^2}$, where K^* depends only on L, M, and c'; hence, D' is M$'$-QED, where $M' = K^{*2}M$. By Lemma 3.3, D and D' are a-quasiconvex and a depends only on M and M'. Theorem 3.5 implies that f has an extension, denoted again by f, as an L'-quasiisometry of D onto D', where L' depends only on L and a and, thus, only on L, M, and c'.

Next, let C be a nondegenerate component of ∂D. Then, cf. the proof of Theorem 3.4, the boundary component C is a K-quasicircle, where K depends only on M. Let

C' be the boundary component of D' that corresponds to C under f. Again C' is a K'-quasicircle and K' depends only on c'. Let G and G' denote the components of $\overline{\mathbb{R}^2} \setminus \overline{D}$ and $\overline{\mathbb{R}^2} \setminus \overline{D'}$ bounded by C and C', respectively. Then $G \subset \mathbb{R}^2$ if and only if $G' \subset \mathbb{R}^2$, and we can apply Theorem 7 in [69] to get L^*-quasi-isometry of $\overline{G}$ onto $\overline{G'}$, which agrees with f on C. Moreover, L^* depends only on L', K, and K' and thus only on L, M, and c'.

Proceeding in this way, we obtain an injective mapping $f : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ that extends f, maps ∞ onto ∞, and satisfies the inequality

$$|z_1 - z_2|/L^* \leq |f^*(z_1) - f^*(z_2)| \leq L^* |z_1 - z_2| \tag{3.19}$$

whenever z_1 and z_2 are finite points in the closure of the same component of $\overline{\mathbb{R}^2} \setminus \partial D$. A trivial argument then yields *(3.19)* for all $z_1, z_2 \in \mathbb{R}^2$ and, thus, completes the proof. $\square$

Finally, the following consequences of Theorem 3.6 extend Corollary 1 in [69] in precisely the same way that Corollaries 3.4 and 3.5 extend the aforementioned theorem of Ahlfors and Beurling.

Corollary 3.7. *If D is a* QED *domain in $\overline{\mathbb{R}^2}$ and f is an injective local quasi-isometry of D onto D', then f has a quasi-isometric extension to $\overline{\mathbb{R}^2}$ if and only if D' is* QED *and the unbounded complementary components of D and D' correspond under f.*

Corollary 3.8. *If D is a uniform domain in $\mathbb{R}^2$ and f is an injective local quasi-isometry of D onto D', then f has a quasi-isometric extension to $\overline{\mathbb{R}^2}$ if and only if D' is uniform and the unbounded complementary components of D and D' correspond under f.*

3.7 Quasicircle Domains and Conformal Mappings

Here we give two infinitely connected domains D, D' in $\mathbb{R}^2$ and a conformal mapping f of D onto D' that has no quasiconformal extension to $\mathbb{R}^2$. This example (see [82]) shows that the hypothesis that D be finitely connected is essential in Corollary 3.6. The example is closely connected to the fact that zero-dimensional sets are not invariant under conformal mapping, i.e., if $D = \mathbb{R}^2 \setminus E \to \mathbb{R}^2$ is conformal and E is a totally disconnected closed set in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus f(\mathbb{R}^2 \setminus E)$ need not be totally disconnected; see [5].

Theorem 3.7. *There exist a compact, totally disconnected set E in $\mathbb{R}^2$ and a conformal mapping f of $D = \overline{\mathbb{R}^2} \setminus E$ onto $D' = \mathbb{B}^2 \setminus F$, where F is a closed, totally disconnected subset of $\mathbb{B}^2$.*

Since D and D' are 1-quasicircle domains, this theorem yields the desired example. The proof of Theorem 3.7 is based on the following results due to Grötzsch (see also [144]) and to Ahlfors and Beurling Theorem 16 in [5], respectively.

Lemma 3.13. *Suppose G is a domain in $\overline{\mathbb{R}^2}$ and that $z_0 \in \partial G \setminus \{\infty\}$. Then the following conditions are equivalent.*

(i) $\lim_{z\to z_0} f(z)$ *exists for all conformal mappings f of G into $\overline{\mathbb{R}^2}$.*

(ii) *For each $r > 0, \operatorname{mod}\Gamma = \infty$, where Γ is the family of all closed paths γ in $G \cap \mathbb{B}(z_0, r)$ that have nonzero winding number about z_0.*

Lemma 3.14. *There exists a compact, totally disconnected set F in $\mathbb{R}^2$ such that $m(F) > 0$ and such that $\lim_{z\to z_0} f(z)$ exists for each $z_0 \in F$ and each conformal mapping f of $\overline{\mathbb{R}^2} \setminus F$ into $\overline{\mathbb{R}^2}$.*

We require the following easy consequence of the above two results.

Corollary 3.9. *Suppose that G is a domain in $\mathbb{R}^2$ with $m(G) < \infty$ and $0 < \varepsilon < 1$. Then there exists a compact set E in G such that $m(G \setminus E) < \varepsilon m(G)$ and such that $\lim_{z\to z_0} f(z)$ exists for each $z_0 \in E$ and each conformal mapping f of $G \setminus E$ into $\mathbb{R}^2$.*

Proof. Let F be the set described in Lemma 3.14. Since $m(F) > 0, F$ has a point of density and we can pick an open disk B_0 and a compact set $E_0 \subset F \cap B_0$ such that

$$m(B_0 \setminus E_0) \; < \; \frac{\varepsilon}{2}\, m(B_0). \tag{3.20}$$

Then, from Lemmas 3.13 and 3.14, we see that $\lim\_z \to z_0 f(z)$ exists for each $z_0 \in E_0$ and each conformal mapping f of $B_0 \setminus E_0$ into $\overline{\mathbb{R}^2}$.

Because $m(G) < \infty$, we can choose disjoint open disks B_j in G, $j = 1, 2, \dots, n$, such that

$$m\left(G \setminus \bigcup_{j=1}^{n} B_j\right) \; < \; \frac{\varepsilon}{2}\, m(G). \tag{3.21}$$

Let E_j denote the image of E_0 under the similarity mapping that carries B_0 onto B_j. Then

$$E \; = \; \bigcup_{j=1}^{n} E_j$$

is a compact subset of G,

$$m(G \setminus E) \; = \; m\left(G \setminus \bigcup_{j=1}^{n} B_j\right) \; + \; \sum_{j=1}^{n} m(B_j \setminus E_j) \; < \; \varepsilon\, m(G)$$

by *(3.20)* and *(3.21)*, and $\lim_{z\to z_0} f(z)$ exists for each $z_0 \in E$ and each conformal mapping f of $G \setminus E$ into $\overline{\mathbb{R}^2}$. $\square$

Proof of Theorem 3.7. For $j = 1,2,\dots$, let $G_j = \{z : 2^{-(j+1)} < |z| < 2^{-j}\}$ and let E_j denote the compact subset of G_j given in Corollary 3.9 corresponding to $\varepsilon = 2^{-3j}$. Next, let $D = \overline{\mathbb{R}^2} \setminus E$, where

$$E = \bigcup_{j=1}^{\infty} E_j \cup \{0\},$$

let Γ denote the family of closed paths in $D \cap B^2$ that have nonzero winding number about 0, and set

$$\rho(z) = \begin{cases} \frac{1}{2\pi|z|} & \text{if } z \in D \cap B^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{\gamma} \rho \, ds \ \geq \ \frac{1}{2\pi} \left| \int_{\gamma} \frac{dz}{z} \right| \ = \ |n(\gamma,0)| \ \geq \ 1$$

for each rectifiable path γ in Γ and

$$\begin{aligned} \operatorname{mod} \Gamma \ &\leq \int_{\mathbb{R}^2} \rho^2 \, dm \ = \ (2\pi)^{-2} \sum_{j=1}^{\infty} \int_{G_j \setminus E_j} \frac{dm}{|z|^2} \\ &\leq \ (2\pi)^{-2} \sum_{j=1}^{\infty} 2^{2(j+1)} m(G_j \setminus E_j) \ < \ \infty. \end{aligned}$$

Hence, by Lemma 3.13, there exists a conformal mapping g of D into $\overline{\mathbb{R}}^2$ such that $\lim_{z\to 0} g(z)$ does not exist; since D is locally connected at 0, this implies that the cluster set $C(g,0)$ of g at 0 is a nongenerate continuum. Next, by Corollary 3.9, $\lim_{z\to z_0} g(z)$ does exist for each $z_0 \in E \setminus \{0\}$, and hence g has a homeomorphic extension to $\overline{\mathbb{R}}^2 \setminus \{0\}$. Thus, $G = g(\overline{\mathbb{R}}^2 \setminus \{0\})$ is a simply connected subdomain of $\overline{\mathbb{R}}^2 \setminus C(g,0)$ and the Riemann mapping theorem yields a conformal mapping h of G onto B^2. The conclusion of Theorem 3.7 then follows with $f = h \circ g$ and $F = f(E \setminus \{0\})$. $\square$

3.8 On Weakly Flat and Strongly Accessible Boundaries

We complete this chapter with a new class of domains in $\mathbb{R}^n$, $n \geq 2$, which are wider than the class of QED domains described earlier. The significance of such a type of domain is that conformal and quasiconformal mappings as well as many of the generalizations between them admit homeomorphic extensions to their boundary.

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{R}^n$ defined below are localizations and generalizations of the corresponding notions introduced in [208,209]; cf. with the properties P_1 and P_2 by Väisälä in [316] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [224]. Lemma 3.15 establishes the relation of weak flatness formulated in terms of moduli of path families with the general topological notion of local connectedness on the boundary; see [163].

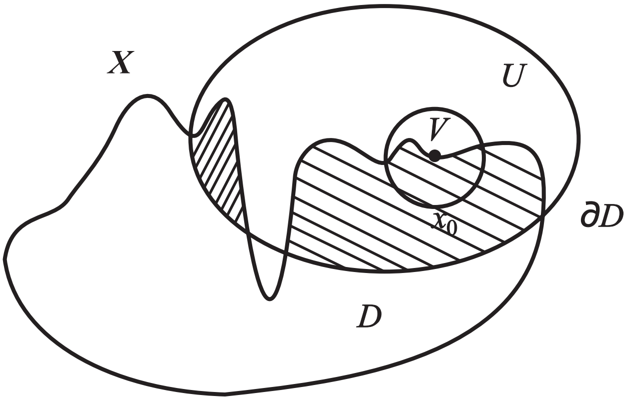

Figure 2

Recall that a domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be **locally connected at a point** $x_0 \in \partial D$ if, for every neighborhood U of the point x_0, there is a neighborhood $V \subseteq U$ of x_0 such that $V \cap D$ is connected [in other words, for every ball $B_0 = B(x_0, r_0)$, there is a component of connectivity of $B_0 \cap D$ that includes $B \cap D$, where $B = B(x_0, r)$ for some $r \in (0, r_0)$]. Note that every Jordan domain D in $\mathbb{R}^n$ is locally connected at each point of ∂D; see, e.g., [335], p. 66.

We say that ∂D is **weakly flat at a point** $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of x_0 such that

$$M(\Delta(E,F;D)) \geq P \tag{3.22}$$

for all continua E and F in D intersecting ∂U and ∂V. Here and later on, $\Delta(E,F;D)$ denotes the family of all paths $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ connecting E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma(t) \in D$ for all $t \in (a,b)$. We say that the boundary ∂D is **weakly flat** if it is weakly flat at every point in ∂D.

We also say that a point $x_0 \in \partial D$ is **strongly accessible** if, for every neighborhood U of the point x_0, there exist a compactum E, a neighborhood $V \subset U$ of x_0, and a number $\delta > 0$ such that

$$M(\Delta(E,F;D)) \geq \delta \tag{3.23}$$

for all continua F in D intersecting ∂U and ∂V. We say that the boundary ∂D is **strongly accessible** if every point $x_0 \in \partial D$ is strongly accessible.

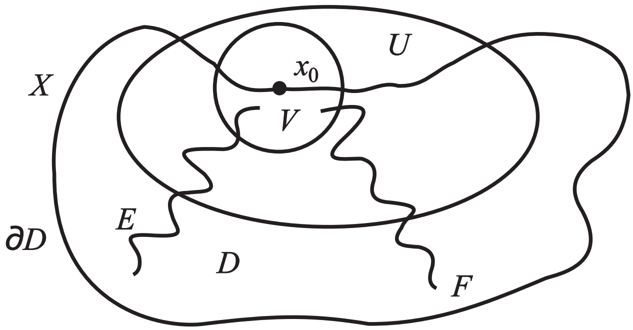

Figure 3

Remark 3.13. Here, in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods U and V of a point x_0 only balls (closed or open) centered at x_0 or only neighborhoods of x_0 in another fundamental system of its neighborhoods. These conceptions can also be extended in a natural way to the case of $\overline{\mathbb{R}^n}$, $n \geq 2$, and $x_0 = \infty$. Then we must use the corresponding neighborhoods of ∞.

Proposition 3.1. *If a domain D in $\mathbb{R}^n$, $n \geq 2$, is weakly flat at a point $x_0 \in \partial D$, then the point x_0 is strongly accessible from D.*

Proof. Indeed, let $U = B(x_0, r_0)$ where $0 < r_0 < d_0 = \sup_{x \in D} |x - x_0|$ and $P_0 \in (0, \infty)$. Then, by the condition of weak flatness, there is $r \in (0, r_0)$ such that

$$M(\Delta(E,F;D)) \geq P_0 \tag{3.24}$$

for all continua E and F in D intersecting $\partial B(x_0, r_0)$ and $\partial B(x_0, r)$. Choose an arbitrary path connecting $\partial B(x_0, r_0)$ and $\partial B(x_0, r)$ in D as a compactum E. Then, for every continuum F in D intersecting $\partial B(x_0, r_0)$ and $\partial B(x_0, r)$, inequality (3.24) holds. □

Corollary 3.10. *Weakly flat boundaries of domains in $\mathbb{R}^n$, $n \geq 2$, are strongly accessible.*

Lemma 3.15. *If a domain D in $\mathbb{R}^n$, $n \geq 2$, is weakly flat at a point $x_0 \in \partial D$, then D is locally connected at x_0.*

Proof. Indeed, let us assume that the domain D is not locally connected at the point x_0. Then there is a positive number $r_0 < d_0 = \sup_{x \in D} |x - x_0|$ such that, for every neighborhood $V \subseteq U := B(x_0, r_0)$ of x_0, one of the following two conditions holds:

(1) $V \cap D$ has at least two connected components K_1 and K_2 with $x_0 \in \overline{K_1} \cap \overline{K_2}$;

(2) $V\cap D$ has a sequence of connected components K_1, K_2, ..., K_m, ... such that $x_m\to x_0$ as $m\to\infty$ for some $x_m\in K_m$. Note that $\overline{K_m}\cap\partial V\neq\varnothing$ for all $m=1,2,\dots$ in view of the connectivity of D.

In particular, this is true for the neighborhood $V=U=B(x_0,r_0)$. Let r_* be an arbitrary number in the interval $(0,r_0)$. Then, for all $i\neq j$,

$$M(\Delta(K_i^*,K_j^*;D))\ \le\ M_0\ :=\ \frac{|D\cap B(x_0,r_0)|}{[2(r_0-r_*)]^n}\ <\ \infty, \tag{3.25}$$

where $K_i^*=K_i\cap\overline{B(x_0,r_*)}$ and $K_j^*=K_j\cap\overline{B(x_0,r_*)}$. Note that the following function is admissible for the path family $\Gamma_{ij}=\Delta(K_i^*,K_j^*;D)$:

$$\rho(x)\ =\ \begin{cases}\frac{1}{2(r_0-r_*)} & \text{for } x\in B_0\setminus\overline{B_*}\,,\\ 0 & \text{for } x\in\mathbb{R}^n\setminus(B_0\setminus\overline{B_*})\,,\end{cases}$$

where $B_0=B(x_0,r_0)$ and $B_*=B(x_0,r_*)$ because K_i and K_j as components of connectivity for $D\cap B_0$ cannot be connected by a path in B_0 and hence every path connecting K_i^* and K_j^* must go through the ring $B_0\setminus\overline{B_*}$ at least twice.

However, in view of (1) and (2), we obtain a contradiction between (3.25) and the weak flatness of ∂D at x_0. Indeed, by the condition, there is $r\in(0,r_*)$ such that

$$M(\Delta(E,F;D))\ \ge\ 2M_0 \tag{3.26}$$

for all continua E and F in D intersecting the spheres $|x-x_0|=r_*$ and $|x-x_0|=r$. By (1) and (2) there is a pair of components K_{i_0} and K_{j_0} of $D\cap B_0$ that intersect both spheres. Let us choose points $x_0\in K_{i_0}\cap B$ and $y_0\in K_{j_0}\cap B$, where $B=B(x_0,r)$, and connect them by a path C in D. Let C_1 and C_2 be the components of $C\cap K_{i_0}^*$ and $C\cap K_{j_0}^*$ including the points x_0 and y_0, respectively. Then, by (3.25),

$$M(\Delta(C_1,C_2;D))\ \le\ M_0$$

and, by (3.26),

$$M(\Delta(C_1,C_2;D))\ \ge\ 2M_0.$$

The contradiction disproves the assumption that D is not locally connected at x_0.

□

Corollary 3.11. *A domain D in $\mathbb{R}^n$, $n\ge 2$, with a weakly flat boundary is locally connected at every boundary point.*

Remark 3.14. As is well known (see, e.g., 10.12 in [316]),

$$M(\Delta(E,F;\mathbb{R}^n))\ \ge\ c_n\log\frac{R}{r} \tag{3.27}$$

for all sets E and F in $\mathbb{R}^n$, $n\ge 2$, intersecting all the spheres $S(x_0,\rho)$, $\rho\in(r,R)$. Hence, it follows directly from the definitions that a QED domain has a weakly flat boundary.

Corollary 3.12. *Every* QED *domain D in $\mathbb{R}^n$, $n \geq 2$, is locally connected at each boundary point and ∂D is strongly accessible.*

By Theorem 3.2, the QED domains coincide with the uniform domains in the class of finitely connected plane domains. The following example shows that, even among simply connected plane domains, the class of domains with weakly flat boundaries is wider than the class of QED domains.The example is based on Lemma 3.5, which says that a QED domain has the measure density property at every boundary point. Furthermore, the example shows that the weaker property on doubling measure is, generally speaking, not valid for domains with weakly flat boundaries.

Example. Consider a simply connected plane domain D of the form

$$D = \bigcup_{k=1}^{\infty} R_k,$$

where

$$R_k = \{ (x,y) \in \mathbb{R}^2 : 0 < x < w_k,\ 0 < y < h_k\}$$

is a sequence of rectangles with quickly decreasing widths $w_k = 2^{-\alpha 2^k} \to 0$ as $k \to \infty$, where $\alpha > 1/(\log 2) > 1$, and monotonically increasing heights $h_k = 2^{-1} + \ldots + 2^{-k} \to 1$ as $k \to \infty$.

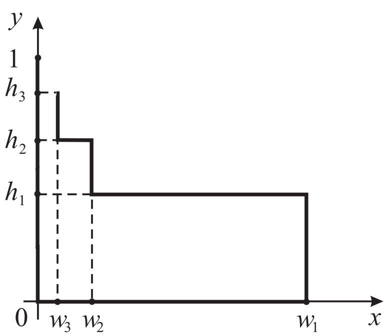

Figure 4

It is easy to see that D has a weakly flat boundary. This fact is not obvious only for its boundary point $z_0 = (0,1)$. According to Remark 3.13, take as a fundamental system of neighborhoods of the point z_0 the rectangles centered at z_0:

$$P_k = \{ (x,y) \in \mathbb{R}^2 : |x| < w_k,\ |y-1| \leq 1 - h_{k-1} = 2^{-(k-1)} \},$$

$k = 1,2,\ldots$. Note that

$$P_k \cap D = \bigcup_{l=k}^{\infty} S_l$$

for all $k > 1$, where

$$S_l \;=\{(x,y)\in\mathbb{R}^2:\ 0<x<w_l,\ h_{l-1}\le y<h_l\}.$$

Let E and F be an arbitrary pair of continua in D intersecting ∂S_l, i.e., intersecting the horizontal lines $y=h_{l-1}$ and $y=h_l$. Denote by S_l^0 the interiority of S_l. Then $\Delta(E,F,S_l^0)\subset\Delta(E,F,D)$ and $\Delta(E,F,S_l^0)$ minorizes the family Γ_l of all paths joining the vertical sides of S_l^0 in S_l^0. Hence (see, e.g., Proposition A.1),

$$M(\Delta(E,F,D))\;\ge\;2^{-l}/w_l\;\ge\;2^{(\alpha-1)l}\;\to\infty$$

as $l\to\infty$. Thus, the domain D really has a weakly flat boundary.

Now, set $r_k=1-h_{k-1}=2^{-k}(1+2^{-1}+\ldots)=2^{-(k-1)}$ and $B_k=B(z_0,r_k)$. Then

$$\lim_{k\to\infty}\frac{|D\cap P_k|}{|D\cap B_k|}\;=\;1$$

because $w_k/r_k\le 2^{-(\alpha-1)k}\to 0$. However,

$$|D\cap P_k|\;=\;\sum_{l=k}^{\infty}|S_l|\;=\;\sum_{l=k}^{\infty}w_l\cdot(h_l-h_{l-1})\;=\;\sum_{l=k}^{\infty}w_l\,2^{-l},$$

and hence

$$\begin{aligned}\frac{|D\cap P_k|}{|D\cap P_{k+1}|}\;&=\;\frac{\sum\limits_{l=k}^{\infty}w_l2^{-l}}{\sum\limits_{l=k+1}^{\infty}w_l2^{-l}}\;=\;\frac{w_k2^{-k}+\sum\limits_{l=k+1}^{\infty}w_l2^{-l}}{\sum\limits_{l=k+1}^{\infty}w_l2^{-l}}\\&=\;1\;+\;\frac{1}{\sum\limits_{m=1}^{\infty}\frac{w_{k+m}}{w_k}\,2^{-m}}\;\ge\;1\;+\;\frac{1}{\frac{w_{k+1}}{w_k}}\\&=\;1\;+\;\frac{w_k}{w_{k+1}}\;=\;1\;+\;2^{\alpha2^k}\;\to\infty.\end{aligned}$$

Consequently,

$$\lim_{k\to\infty}\frac{|D\cap B_k|}{|D\cap B_{k+1}|}\;=\;\infty.$$

Thus, the domain D does not have the doubling measure property at the point $z_0\in\partial D$, and then, by Lemma 3.5, D is not a QED domain.

Finally, we would also like to compare our notions of weak flatness and strong accessibility with other close notions.

First, we recall the corresponding notions in [209], p. 60. There the weak flatness of ∂D means that the condition

$$M(\Delta(E,F;D))\;=\;\infty \tag{3.28}$$

holds for all nondegenerate continua E and F in $\overline{D}$ with $E \cap F \neq \varnothing$. Note that (3.28) always holds if $E \cap F \cap D \neq \varnothing$ because of (3.27). It is clear that (3.28) implies (3.22).

The strong accessibility of ∂D in [209] means that

$$M(\Delta(E,F;D)) > 0 \tag{3.29}$$

for all nondegenerate continua E and F in $\overline{D}$. Note that (3.29) always holds for continua E and F in D, see Theorem 5.2 in [225].

The property P_1 in [316], p. 54, and the quasiconformal flatness of D at a point $x_0 \in \partial D$ by Näkki in [224], p. 12, mean that

$$M(\Delta(E,F;D)) = \infty \tag{3.30}$$

for all connected sets E and F in D with $x_0 \in \overline{E} \cap \overline{F}$. It easy to see that (3.30) implies (3.22) for connected sets in D but not only for continua.

The property P_2 by Väisälä (or quasiconformal accessibility by Näkki) of D at $x_0 \in \partial D$ means that, for every neighborhood U of x_0, there are a compactum (continuum) $E \subset D$ and a number $\delta > 0$ such that

$$M(\Delta(E,F;D)) \geq \delta \tag{3.31}$$

for all connected sets F in D with $x_0 \in \overline{F}$ and $F \cap \partial U \neq \varnothing$. It is easy also to see that (3.31) implies (3.23) not only for continua but also for connected sets.

We show later that all theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries (3.22). The condition of strong accessibility (3.23) plays a similar role for a continuous extension of the mappings to the boundary.

Chapter 4
Q-Homeomorphisms with $Q \in L^1_{\text{loc}}$

Various modulus inequalities play a great role in the theory of quasiconformal mappings and their generalizations. Along these lines, we introduced and studied the concept of Q-homeomorphisms. In this class we study differentiability, absolute continuity, distortion theorems, boundary behavior, removability, and mapping problems. Our proofs are based on extremal length methods. In this chapter we give some results for Q-homeomorphisms with locally integrable Q; see, e.g., [204, 205, 207–209, 282].

4.1 Introduction

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1,\infty]$ be a measurable function. Recall that a homeomorphism $f : D \to \overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is said to be a **Q-homeomorphism** if

$$M(f\Gamma) \leq \int\limits_D Q(x) \cdot \rho^n(x)\, dm(x) \tag{4.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ.

The subject of Q-homeomorphisms is interesting on its own right and has applications to many classes of mappings that we also investigate ahead. In particular, the theory of Q-homeomorphisms can be applied to mappings in local Sobolev classes (see, e.g., Sections 6.3 and 6.10) to the mappings with finite length distortion (see Sections 8.6 and 8.7) and to the finitely bi-Lipschitz mappings; see Section 10.6.

The main goal of the theory of Q-homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In this chapter we first study various properties of Q-homeomorphisms for $Q \in L^1_{\text{loc}}$. Examples of $Q(x)$-homeomorphisms are provided by a class of homeomorphisms $f \in \mathrm{W}^{1,n}_{\text{loc}}$ having either the locally integrable inner dilatation $K_I(x,f)$ or the outer dilatation $K_O(x,f) \in L^{n-1}_{\text{loc}}(D)$; see Theorem 4.1. The base for it is the following statement.

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_4,

Proposition 4.1. *Let $f : D \to \mathbb{R}^n$ be a sense-preserving homeomorphism in the class $W^{1,n}_{\mathrm{loc}}$. Then*

(i) f is differentiable a.e.,
(ii) f satisfies Lusin's property (N),
(iii) $J_f(x) \geq 0$ a.e.

If, in addition, $f^{-1} \in W^{1,n}_{\mathrm{loc}}$, in particular, if either $K_I(x,f) \in L^1_{\mathrm{loc}}$ or $K_O(x,f) \in L^{n-1}_{\mathrm{loc}}$, then

(iv) f^{-1} is differentiable a.e.,
(v) f^{-1} has the property (N),
(vi) $J_f(x) > 0$ a.e.

Proof. (i) and (ii) follow from the corresponding results for $W^{1,n}_{\mathrm{loc}}$ homeomorphisms by Reshetnyak; see [257] and [258]. In view of (i) and the fact that f is sense-preserving, (iii) follows by Rado–Reichelderfer [246], p. 333. Finally, if either $K_I(x,f) \in L^1_{\mathrm{loc}}$ or $K_O(x,f) \in L^{n-1}_{\mathrm{loc}}(D)$, then $f^{-1} \in \mathrm{W}^{1,n}_{\mathrm{loc}}(f(D))$ by Corollary2.3 in [154] and Theorem 6.1 in [111], correspondingly, and thus (iv)–(vi) follow. □

Remark 4.1. Note that by [246] every homeomorphism in $\mathbb{R}^n$ is either sense-preserving or sense-reversing. Moreover, the latter can be obtained from the former by reflections with respect to hyperplanes and conversely. Thus, Proposition 4.1 is also applicable to the latter but with the opposite sign of the Jacobian.

4.2 Examples of Q-homeomorphisms

The next theorem provides examples of Q-homeomorphisms; cf. Theorem 6.1, Corollaries 6.4 and 6.5.

Theorem 4.1. *Let $f : D \to \mathbb{R}^n$ be a homeomorphism in the class $W^{1,n}_{\mathrm{loc}}$ with $f^{-1} \in W^{1,n}_{\mathrm{loc}}$, in particular, with either $K_I(x,f) \in L^1_{\mathrm{loc}}$ or $K_O(x,f) \in L^{n-1}_{\mathrm{loc}}$. Then, for every family Γ of paths in D and every $\rho \in \mathrm{adm}\,\Gamma$,*

$$M(f\Gamma) \;\leq\; \int\limits_D K_I(x,f)\,\rho^n(x)\,dm(x), \tag{4.2}$$

i.e., f is a Q-homeomorphism with $Q(x) = K_I(x,f)$.

Proof. Since either $K_I(x,f) \in L^1_{\mathrm{loc}}$ or $K_O(x,f) \in L^{n-1}_{\mathrm{loc}}$, we may apply Proposition 4.1. Thus, $f^{-1} \in \mathrm{W}^{1,n}_{\mathrm{loc}}(f(D))$, and hence $f^{-1} \in \mathrm{ACL}^n_{\mathrm{loc}}(f(D))$; see, e.g., [215], p. 8. Therefore, by Fuglede's theorem (see [64] and [316], p. 95), if $\tilde{\Gamma}$ is the family of all paths $\gamma \in f\Gamma$ for which f^{-1} is absolutely continuous on all closed subpaths of γ, then $M(f\Gamma) = M(\tilde{\Gamma})$. Also, by Proposition 4.1, f^{-1} is differentiable a.e. Hence, as in the qc case (see [316], p.110), given a function $\rho \in \mathrm{adm}\,\Gamma$, we let $\tilde{\rho}(y) =$

$\rho(f^{-1}(y))|(f^{-1})'(y)|$ for $y \in f(D)$ and $\tilde{\rho}(y) = 0$ otherwise. Then we obtain, that for $\tilde{\gamma} \in \tilde{\Gamma}$,

$$\int_{\tilde{\gamma}} \tilde{\rho}\, ds \ \ge \int_{f^{-1}\circ\tilde{\gamma}} \rho\, ds \ \ge\ 1,$$

and consequently $\tilde{\rho} \in \operatorname{adm} \tilde{\Gamma}$.

By Proposition 4.1 and Remark 4.1, both f and f^{-1} are differentiable a.e. and have the (N)-property and $J(x,f) > 0$ a.e., and we may apply the integral transformation formula to obtain

$$\begin{aligned} M(f\Gamma) = M(\tilde{\Gamma}) &\le \int_{f(D)} \tilde{\rho}^n dm(y) \\ &= \int_{f(D)} \rho(f^{-1}(y))^n |(f^{-1})'(y))|^n dm(y) = \int_{f(D)} \frac{\rho(f^{-1}(y))^n}{l(f'(f^{-1}(y))^n} dm(y) \\ &= \int_{f(D)} \rho(f^{-1}(y))^n K_I(f^{-1}(y), f) J(y, f^{-1}) dm(y) \le \int_D K_I(x,f)\rho(x)^n dm(x). \end{aligned}$$

The proof follows. □

4.3 Differentiability and $K_O(x,f) \le C_n Q^{n-1}(x)$ a.e.

Theorem 4.2. *Let D and D' be domains in $\mathbb{R}^n$, $n \ge 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then f is differentiable a.e. in D.*

Let us consider the set function $\Phi(\mathscr{B}) = m(f(\mathscr{B}))$ defined over the algebra of all the Borel sets $\mathscr{B}$ in D. Recall that by the Lebesgue theorem on the differentiability of nonnegative, subadditive locally finite set functions (see, e.g., III.2.4 in [246] or 23.5 in [316]), there exists a finite limit for a.e. $x \in D$

$$\varphi(x) = \lim_{\varepsilon \to 0} \frac{\Phi(B(x,\varepsilon))}{\Omega_n \varepsilon^n}, \tag{4.3}$$

where $B(x,\varepsilon)$ is a ball in $\mathbb{R}^n$ centered at $x \in D$ with radius $\varepsilon > 0$. The quantity $\varphi(x)$ is called the **volume derivative** of f at x.

Given x and $y \in D$, we set

$$L(x,f) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}\ . \tag{4.4}$$

By the Rademacher–Stepanov theorem (see, e.g., [281], p. 311), the proof of Theorem 4.2 is reduced to the following lemma.

Lemma 4.1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\rm loc}$. Then a.e.*

$$L(x,f) \ \leq\ \gamma_n\, \varphi^{\frac{1}{n}}(x)\, Q^{\frac{n-1}{n}}(x), \tag{4.5}$$

where the constant γ_n depends only on n.

Proof. Consider the spherical ring $R_\varepsilon(x) = \{y :\ \varepsilon < |x-y| < 2\varepsilon\}$, $x \in G$, with $\varepsilon > 0$ such that $R_\varepsilon(x) \subset D$. Since $\left(fB(y,2\varepsilon), \overline{fB(y,\varepsilon)}\right)$ are ringlike condensers in D', according to [71] (cf. also [122] and [293]; see Section A.1),

$$\operatorname{cap}(fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) = M(\triangle(\partial fB(x,2\varepsilon), \partial fB(x,\varepsilon); fR_\varepsilon(x)))$$

and, in view of the homeomorphism of f,

$$\triangle\left(\partial fB\left(x,2\varepsilon\right), \partial fB\left(x,\varepsilon\right); fR_\varepsilon(x)\right) = f\left(\triangle\left(\partial B(x,2\varepsilon), \partial B(x,\varepsilon); R_\varepsilon(x)\right)\right).$$

Thus, since f is a Q-homeomorphism, we obtain

$$\operatorname{cap}(fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) \leq \int\limits_G Q(x)\cdot\rho^n(x)\, d\overline{m}(x)$$

for every admissible function ρ of $\triangle(\partial B(x,2\varepsilon), \partial B(x,\varepsilon); K_\varepsilon(x))$. The function

$$\rho(x) = \begin{cases} \frac{1}{\varepsilon} & \text{if } x \in R_\varepsilon(x), \\ 0 & \text{if } x \in G \setminus R_\varepsilon(x), \end{cases}$$

is admissible and, thus,

$$\operatorname{cap}(fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) \leq \frac{2^n\Omega_n}{m(B(x,2\varepsilon))} \int\limits_{B(x,2\varepsilon)} Q(y)\, d\overline{m}(y). \tag{4.6}$$

On the other hand, by Lemma 5.9 in [210] (see Section A.2), we have

$$\operatorname{cap}\ (fB(x,2\varepsilon), \overline{fB(x,\varepsilon)}) \geq \left(C_n \frac{d^n(fB(x,\varepsilon))}{m(fB(x,2\varepsilon))}\right)^{\frac{1}{n-1}}, \tag{4.7}$$

where C_n is a constant depending only on n, and $d(A)$ and $m(A)$ denote the diameter and Lebesgue measure of a set A in $\mathbb{R}^n$, respectively.

Combining (4.6) and (4.7), we obtain

$$\frac{d(fB(x,\varepsilon))}{\varepsilon} \leq \gamma_n \left(\frac{m(fB(x,2\varepsilon))}{m(B(x,2\varepsilon))}\right)^{1/n} \left(\frac{1}{m(B(x,2\varepsilon))} \int\limits_{B(x,2\varepsilon)} Q(y)\, dm(y)\right)^{(n-1)/n},$$

and hence a.e.

$$L(x,f) \le \limsup_{\varepsilon \to 0} \frac{d(fB(x,\varepsilon))}{\varepsilon} \le \gamma_n \varphi^{1/n}(x) Q^{(n-1)/n}(x).$$

□

Corollary 4.1. *Let D and D' be domains in $\mathbb{R}^n$, $n \ge 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then f has locally integrable partial derivatives.*

Proof. For $L(x,f)$ given by (4.4) and a compact set $V \subset G$, we have by (4.5)

$$\int_V L(x,f)\, dm(x) \le \gamma_n \int_V \varphi^{1/n}(x) Q^{(n-1)/n}(x)\, dm(x)$$

and, applying the Hölder inequality (see e.g. (17.3) in [20]) with $p = n$ and $q = n/(n-1)$, we obtain

$$\int_V \varphi^{1/n}(x) Q^{(n-1)/n}(x)\, dm(x) \le \left(\int_V \varphi(x)\, dx \right)^{1/n} \left(\int_V Q(x)\, dm(x) \right)^{(n-1)/n}.$$

Finally, in view of $Q \in L^1_{\mathrm{loc}}$, by the Lebesgue theorem, we see that

$$\int_V L(x,f)\, dx \le \gamma_n\, (mfV)^{1/n} \left(\int_V Q(x)\, dm(x) \right)^{(n-1)/n} < \infty.$$

□

Corollary 4.2. *Let D and D' be domains in $\mathbb{R}^n$, $n \ge 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then a.e.*

$$K_O(x,f) \ \le \ C_n\, Q^{n-1}(x), \tag{4.8}$$

where the constant C_n depends only on n.

Remark 4.2. Note also that f^{-1} has Lusin's (N)-property and $J(x,f) \neq 0$ a.e. for every Q-homeomorphism f with $Q \in L^1_{\mathrm{loc}}$. Indeed, by Corollary 4.2,

$$\int_C K_0^{n'-1}(x,f)\, dm(x) \ \le \ \gamma_n \int_C Q^{(n'-1)(n-1)}(x,f)\, dm(x) \ = \gamma_n \int_C Q(x,f)\, dm(x) \ < \ \infty \tag{4.9}$$

for each compact set C in D where $1/n + 1/n' = 1$ and γ_n depends only on n. Thus, $|E| = 0$ whenever $|fE| = 0$ by Corollary 4.3 and [152]. By [244] the latter is equivalent to the condition $J(x,f) \neq 0$ a.e.

4.4 Absolute Continuity on Lines and $W^{1,1}_{\mathrm{loc}}$

Theorem 4.3. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then $f \in$ ACL.*

Proof. Let $I = \{x \in \mathbb{R}^n : a_i < x_i < b_i,\ i = 1, \ldots, n\}$ be an n-dimensional interval in $\mathbb{R}^n$ such that $\bar{I} \subset D$. Then $I = I_0 \times J$, where I_0 is an $(n-1)$-dimensional interval in $\mathbb{R}^{n-1}$ and J is an open segment of the axis x_n, $J = (a_n, b_n)$. Next we identify $\mathbb{R}^{n-1} \times \mathbb{R}$ with $\mathbb{R}^n$. We prove that for almost every segment $J_z = \{z\} \times J$, $z \in I_0$, the mapping $f|_{J_z}$ is absolutely continuous.

Consider the set function $\Phi(\mathscr{B}) = m(f(\mathscr{B} \times J))$ defined over the algebra of all the Borel sets $\mathscr{B}$ in I_0. Note that by the Lebesgue theorem on differentiability for nonnegative, subadditive locally finite set functions (see, e.g., III.2.4 in [246]), there is a finite limit for a.e. $z \in I_0$

$$\varphi(z) = \lim_{r \to 0} \frac{\Phi\,(B(z,r))}{\Omega_{n-1} r^{n-1}}, \tag{4.10}$$

where $B(z,r)$ is a ball in $I_0 \subset \mathbb{R}^{n-1}$ centered at $z \in I_0$ of the radius $r > 0$.

Let Δ_i, $i = 1, 2, \ldots$, be some enumeration S of all intervals in J such that $\overline{\Delta_i} \subset J$ and the ends of Δ_i are rational numbers. Set

$$\varphi_i(z) := \int\limits_{\Delta_i} Q(z, x_n)\, dx_n.$$

Then by the Fubini theorem (see, e.g., III. 8.1 in [281]), the functions $\varphi_i(z)$ are a.e. finite and integrable in $z \in I_0$. In addition, by the Lebesgue theorem on the differentiability of the indefinite integral, there is a.e. a finite limit

$$\lim_{r \to 0} \frac{\Phi_i(B(z,r))}{\Omega_{n-1} r^{n-1}} = \varphi_i(z), \tag{4.11}$$

where Φ_i for a fixed $i = 1, 2, \ldots$ is the set function

$$\Phi_i(\mathscr{B}) = \int\limits_{\mathscr{B}} \varphi_i(\zeta)\, dm(\zeta)$$

given over the algebra of all the Borel sets $\mathscr{B}$ in I_0.

Let us show that the mapping f is absolutely continuous on each segment $J_z, z \in I_0$, where the finite limits *(4.10)* and *(4.11)* exist. Fix one such point z. We have to prove that the sum of diameters of the images of an arbitrary finite collection of mutually disjoint segments in $J_z = \{z\} \times J$ tends to zero together with the total length of the segments. In view of the continuity of the mapping f, it suffices to verify this fact only for mutually disjoint segments with rational ends in J_z. So, let $\Delta_i^* = \{z\} \times \overline{\Delta_i} \subset J_z$, where $\Delta_i \in S$, $i = 1, \ldots, k$ under the corresponding renumbering of S, are mutually disjoint intervals. Without loss of generality, we may assume that $\overline{\Delta_i}\,, i=1,\dots,k$, are also mutually disjoint.

Let $\delta>0$ be an arbitrary rational number that is less than half of the minimum of the distances between $\Delta_i^*\,, i=1,\dots,k$, and also less than their distances to the endpoints of the interval J_z. Let $\Delta_i^*=\{z\}\times[\alpha_i,\beta_i]$ and $A_i=A_i(r)=B(z,r)\times(\alpha_i-\delta,\beta_i+\delta)$, $i=1,\dots,k$, where $B(z,r)$ is an open ball in $I_0\subset\mathbb{R}^{n-1}$ centered at point z of the radius $r>0$. For small $r>0$, $(A_i,\Delta_i^*), i=1,\dots,k$, are ringlike condensers in I , hence $(fA_i,f\Delta_i^*), i=1,\dots,k$, are also ringlike condensers in D'.

According to [71] (cf. also [122] and [293]; see Section A.1),

$$\operatorname{cap}(fA_i,f\Delta_i^*)\ =\ M\left(\triangle\left(\partial fA_i,f\Delta_i^*;fA_i\right)\right)$$

and, in view of the homeomorphism of f,

$$\triangle\left(\partial fA_i,f\Delta_i^*;fA_i\right)=f\left(\triangle\left(\partial A_i,\Delta_i^*;A_i\right)\right).$$

Thus, since f is a Q-homeomorphism, we obtain

$$\operatorname{cap}(fA_i,f\Delta_i^*)\le\int\limits_D Q(x)\cdot\rho^n(x)\,dm(x)$$

for every function $\rho\in\operatorname{adm}\triangle(\partial A_i,\Delta_i^*;A_i)$. In particular, the function

$$\rho(x)\ =\ \begin{cases}\frac{1}{r}, & x\in A_i,\\ 0, & x\in\mathbb{R}^n\setminus A_i\end{cases}$$

is admissible under $r<\delta$ and, thus,

$$\operatorname{cap}(fA_i,f\Delta_i^*)\le\frac{1}{r^n}\int\limits_{A_i} Q(x)\,dm(x)\,. \tag{4.12}$$

On the other hand, by Lemma 5.9 in [210] (see Section A.2),

$$\operatorname{cap}(fA_i,\,f\Delta_i^*)\ge\left(C_n\,\frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}}, \tag{4.13}$$

where d_i is a diameter of the set $f\Delta_i^*$, m_i is a volume of the set fA_i, and C_n is a constant depending only on n.

Combining *(4.12)* and *(4.13),* we have the inequalities

$$\left(\frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}}\le\frac{c_n}{r^n}\int\limits_{A_i} Q(x)\,dm(x)\,,\qquad i=1,\dots,k \tag{4.14}$$

where the constant c_n depends only on n.

By the discrete Hölder inequality (see, e.g., (17.3) in [20] with $p = n/(n-1)$ and $q = n$, $x_k = d_k/m_k^{1/n}$ and $y_k = m_k^{1/n}$), we obtain

$$\sum_{i=1}^{k} d_i \leq \left(\sum_{i=1}^{k} \left(\frac{d_i^n}{m_i} \right)^{\frac{1}{n-1}} \right)^{\frac{n-1}{n}} \left(\sum_{i=1}^{k} m_i \right)^{\frac{1}{n}} , \tag{4.15}$$

i.e.,

$$\left(\sum_{i=1}^{k} d_i \right)^n \leq \left(\sum_{i=1}^{k} \left(\frac{d_i^n}{m_i} \right)^{\frac{1}{n-1}} \right)^{n-1} \Phi(B(z,r)) , \tag{4.16}$$

and in view of (4.14),

$$\left(\sum_{i=1}^{k} d_i \right)^n \leq \gamma_n \frac{\Phi(B(z,r))}{\Omega_{n-1} r^{n-1}} \left(\sum_{i=1}^{k} \frac{\int_{A_i} Q(x)\, dm(x)}{\Omega_{n-1} r^{n-1}} \right)^{n-1} , \tag{4.17}$$

where γ_n depends only on n. First letting $r \to 0$ and then $\delta \to 0$, we get by Lebesgue's theorem that

$$\left(\sum_{i=1}^{k} d_i \right)^n \leq \gamma_n \varphi(z) \left(\sum_{i=1}^{k} \varphi_i(z) \right)^{n-1} . \tag{4.18}$$

Finally, in view of (4.18), the absolute continuity of the indefinite integral of Q over the segment J_z implies the absolute continuity of the mapping f over the same segment. Hence, $f \in$ ACL. $\square$

Corollary 4.3. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$. Then f belongs to $W^{1,1}_{\mathrm{loc}}$.*

The conclusion follows by Theorem 4.3 and Corollary 4.1; see also [215].

Remark 4.3. By the way, from the proof of Theorem 4.3, the estimate of the variation of the mapping f on the segment I_z and the length of the path fI_z follow:

$$l(fI_z) \leq \gamma_n^* \, \varphi^{\frac{1}{n}}(z) \left(\int_{a_n}^{b_n} Q(z,x_n)\, dx_n \right)^{\frac{n-1}{n}} , \tag{4.19}$$

where the constant γ_n^* depends only on n and the function $\varphi(z)$ is defined by (4.10).

4.5 Lower Estimate of Distortion

Theorem 4.4. *Let $f:\mathbb{B}^n\to\overline{\mathbb{R}^n}$ be a Q-homeomorphism with $Q\in L^1(\mathbb{B}^n)$, $f(0)=0$, $h(\overline{\mathbb{R}^n}\setminus f(\mathbb{B}^n))\ge\delta>0$, and $h(f(x_0),f(0))\ge\delta$ for some $x_0\in\mathbb{B}^n$. Then, for all $|x|<r=\min(|x_0|/2,1-|x_0|)$,*

$$|f(x)|\ge\psi(|x|),\tag{4.20}$$

where $\psi(t)$ is a strictly increasing function with $\psi(0)=0$ that depends only on the L^1-norm of Q in $\mathbb{B}^n$, n, and δ.

Proof. Given y_0 with $|y_0|<r$, choose a continuum E_1 that contains the points 0 and x_0 and a continuum E_2 that contains the points y_0 and $\partial\mathbb{B}^n$, so that $\mathrm{dist}(E_1,E_2\cup\partial\mathbb{B}^n)=|y_0|$. More precisely, denote by L the straight line generated by the pair of points 0 and x_0 and by P the plane defined by the triple of the points 0, x_0, and y_0 (if $y_0\in L$, then P is an arbitrary plane passing through L). Let C be the circle under the intersection of P and the sphere $S^{n-1}(y_0,|y_0|)\subset B^n(|x_0|)$. Let t_0 be the tangency point to C of the ray starting from x_0 that is opposite to y_0 with respect to L (an arbitrary one of the two possible if $y_0\in L$). Then $E_1=[x_0,t_0]\cup\gamma(0,t_0)$, where $\gamma(0,t_0)$ is the shortest arc of C joining 0 and t_0, and $E_2=[y_0,i_0]\cup S^{n-1}$, where $S^{n-1}=\partial\mathbb{B}^n$ is the unit sphere and i_0 is the point (opposite to t_0 with respect to L) of the intersection of S^{n-1} with the straight line in P, that is perpendicular to L and passes through y_0.

Let Γ denote the family of paths that join E_1 and E_2. Then

$$\rho(x)\;=\;|y_0|^{-1}\,\chi_{\mathbb{B}^n}(x)\;\in\;\mathrm{adm}\;\Gamma,$$

and hence,

$$\begin{aligned} M(f\Gamma)\;&\le\;\int\rho^n(x)Q(x)\,dm(x)\\ &\le\;|y_0|^{-n}\int\limits_{\mathbb{B}^n}Q(x)\,dm(x)\;=\;\frac{||Q||_1}{|y_0|^n}.\end{aligned}\tag{4.21}$$

The ring domain $A'=f(\mathbb{B}^n\backslash(E_1\cup E_2))$ separates the continua $E_1'=f(E_1)$ and $E_2'=\overline{\mathbb{R}^n}\backslash f(\mathbb{B}^n\backslash E_2)$, and since

$$h(E_1')\ge h(f(x_0),\quad f(0))\ge\delta,\qquad h(E_2')\ge h(\overline{\mathbb{R}^n}\backslash f(\mathbb{B}^n))\ge\delta$$

and

$$h(E_1',E_2')\le h(f(y_0),f(0),$$

it follows that

$$M(f(\Gamma))\ge\lambda(h(f(y_0),f(0))),\tag{4.22}$$

where $\lambda(t)=\lambda_n(\delta,t)$ is a strictly decreasing positive function with $\lambda(t)\to\infty$ as $t\to0$ (see 12.7 in [316]). Hence, by (4.21) and (4.22),

$$|f(y_0)| > h(f(y_0), \quad f(0)) \geq \psi(|y_0|),$$

where $\psi(t) = \lambda^{-1}\left(\frac{||Q||_1}{t^n}\right)$ has the required properties. □

4.6 Removal of Singularities

Theorem 4.5. *Let $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be a Q-homeomorphism. If*

$$\limsup_{r\to 0} \frac{1}{|B^n(r)|} \int\limits_{B^n(r)} Q(x)\, dm(x) < \infty, \tag{4.23}$$

then f has an extension to $\mathbb{B}^n$ that is a Q-homeomorphism.

Proof. As the modulus of a family of paths that pass through the origin vanishes, it suffices to show that f has a continuous extension to $\mathbb{B}^n$. Suppose that this is not the case. Since f is a homeomorphism, $\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n \setminus \{0\})$ consists of two connected compact sets F_1 and F_2 in $\overline{\mathbb{R}^n}$, where F_1 contains the cluster set $E = C(0, f)$ of f at 0. Here F_1 is a nondegenerate continuum. Using an arbitrary Möbius transformation, we may assume that $F_1 \subset \mathbb{R}^n$. $U = F_1 \cup f(\mathbb{B}^n \setminus \{0\})$ is a neighborhood of E. Thus, there exists $\delta > 0$ such that all balls $B_z = B^n(z, \delta), z \in F_1$, are contained in U. Let $V = \cup B_z$.

Now, choose a point $y \in F_1$ such that $\operatorname{dist}(y, \partial V) = \delta$, and a point $z \in B_y \setminus F_1$. Next, choose a path $\beta : [0,1] \to B_y$ with $\beta(0) = y, \beta(1) = z$ and $\beta(t) \in B_y \setminus F_1$ for $t \in (0,1]$. Let $\alpha = f^{-1} \circ \beta$. For $r \in (0, |f^{-1}(z)|)$, let α_r denote the connected component of the path $\alpha(I) \setminus B^n(r)$, $I = [0,1]$, that contains the point $f^{-1}(z) = \alpha(1)$, and let Γ_r denote the family of all paths joining α_r and the point 0 in $\mathbb{B}^n \setminus \{0\}$. Then the function $\rho(x) = 1/r$ if $x \in B^n(r) \setminus \{0\}$ and $\rho = 0$ otherwise is in adm Γ_r, and by (4.23),

$$\limsup_{r\to 0} \int\limits_{B^n(r)\setminus\{0\}} Q(x)\, \rho^n(x)\, dm(x) \tag{4.24}$$

$$= \Omega_n \limsup_{r\to 0} \frac{1}{|B^n(r)|} \int\limits_{B^n(r)\setminus\{0\}} Q(x)\, dm(x) < \infty.$$

On the other hand, if Γ_r' denotes the family of all paths joining two continua $f(\alpha_r)$ and E in $B_y \setminus E$, then $\Gamma_r' \subset f(\Gamma_r)$, and thus

$$M(\Gamma_r') \leq M(f\Gamma_r). \tag{4.25}$$

Evidently, $\operatorname{dist}(f(\alpha_r), E) \to 0$, and the diameter of $f(\alpha_r)$ increases as $r \to 0$. As both $f(\alpha_r)$ and E are subsets of a ball, $M(f\Gamma_r) \to \infty$ as $r \to 0$. This, together with (4.24) and (4.25), contradicts the modulus inequality (4.1). □

4.7 Boundary Behavior

Theorem 4.6. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let f be a Q-homeomorphism of D onto D' with $Q \in L^1(D)$. If D is locally connected at ∂D and $\partial D'$ is weakly flat, then f^{-1} has a continuous extension to $\overline{D'}$.*

It is necessary to stress here that the extension problem for the direct mappings f has a more complicated nature; see Chapter 5 and 6 and, especially, the example in Proposition 6.3. The proof of this theorem is reduced to the following lemma.

Lemma 4.2. *Let D and D' be domains in $\mathbb{R}^n$ and let f a Q-homeomorphism of D onto D' with $Q \in L^1(D)$. If D is locally connected at ∂D and $\partial D'$ is weakly flat, then $C(x_1, f) \cap C(x_2, f) = \emptyset$ for every two distinct points x_1 and x_2 in ∂D.*

Proof. Without loss of generality, we may assume that the domain D is bounded. For $i = 1, 2$, let E_i denote the cluster sets $C(x_i, f)$ and suppose that $E_1 \cap E_2 \neq \emptyset$.

Write $d = |x_1 - x_2|$. Since D is locally connected in ∂D, there are neighborhoods U_i of x_i such that $W_i = D \cap U_i$ are connected and $U_i \subset B^n(x_i, d/3)$, $i = 1, 2$. Then the function $\rho(x) = 3/d$ if $x \in D \cap B^n((x_1 + x_2)/2, d)$ and $\rho(x) = 0$ elsewhere is admissible for the family $\Gamma = \Gamma(W_1, W_2; D)$. Thus,

$$M(f\Gamma) \leq \int_D Q(x)\rho^n(x)\, dm(x) \leq \frac{3^n}{d^n} \int_D Q(x)\, dm(x) < \infty. \tag{4.26}$$

The last estimate contradicts, however, the weak flatness condition of $\partial D'$ if there is a point $y_0 \in E_1 \cap E_2$. Indeed, then $y_0 \in \overline{fW_1} \cap \overline{fW_2}$ and, in the domains $W_1^* = fW_1$ and $W_2^* = fW_2$, there exist paths intersecting arbitrary small prescribed spheres $|y - y_0| = r_0$ and $|y - y_0| = r_*$. Thus, the assumption that $E_1 \cap E_2 \neq \emptyset$ was not true. $\square$

In particular, by Theorem 4.6, we obtain the following important conclusion.

Theorem 4.7. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$. If D' is locally connected at $\partial D'$ and ∂D is weakly flat, then any quasiconformal mapping $f : D \to D'$ admits a continuous extension to the boundary $\overline{f} : \overline{D} \to \overline{D'}$.*

Combining Theorem 4.7 with Lemma 3.15, we come to the following statement.

Corollary 4.4. *If D and D' are domains with weakly flat boundaries, then any quasiconformal mapping $f : D \to D'$ admits a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Note that these results on the extension to weakly flat boundaries are new even for the class of conformal mappings in the plane. Here we do not assume that domains are simply connected.

4.8 Mapping Problems

We may consider the following two questions.

(a) Are there any proper subsets of $\mathbb{R}^n$ that can be mapped under a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$ onto $\mathbb{R}^n$?

(b) Are there any nondegenerate continua E in $\mathbb{B}^n$ such that $\mathbb{B}^n \setminus E$ can be mapped under a Q-homeomorphism with $Q \in L^1_{\mathrm{loc}}$ onto $\mathbb{B}^n \setminus \{0\}$?

Here we give partial answers to these questions; see also the next chapter.

Theorem 4.8. *Let D be a domain in $\mathbb{R}^n, D \neq \mathbb{R}^n$, $n \geq 2$, and $f : D \to \mathbb{R}^n$ a Q-homeomorphism. If there exist a point $b \in \partial D$ and a neighborhood U of b such that $Q|_{D\cap U} \in L^1$, then $f(D) \neq \mathbb{R}^n$.*

Proof. The statement is trivial if D is not a topological ball. Suppose that D is a topological ball. By the Möbius invariance, we may assume that $b = 0$ and $\infty \in \partial D$. Let $r > 0$ be such that $B^n(r) \subset U$. Then Q is integrable in $B^n(r) \cap D$. Choose two arcs E and F in $B^n(r/2) \cap D$ each having exactly one endpoint in ∂D such that $0 < \mathrm{dist}(E,F) < r/2$. Such arcs exist. Indeed, since ∂D is connected and 0 and ∞ belong to ∂D, the sphere $\partial B^n(r/2)$ meets ∂D and contains a point x_0 that belongs to D. Then one can take E as a maximal line segment in $(0,x_0] \cap D$ with one endpoint at x_0 and the other one in ∂D, and F as a circular arc in the maximal spherical cap in $\partial B^n(r/2) \cap D$ that is centered at x_0, so that F has one end-point in ∂D and the other one in D.

Now, let Γ denote the family of all paths that join E and F in D. Then $\rho(x) = \mathrm{dist}(E,F)^{-1}$ if $x \in B^n(r) \cap D$ and $\rho(x) = 0$ otherwise is admissible for Γ. Then, by (4.1),

$$M(f\Gamma) \leq \int_D Q\rho^n \, dm \leq \frac{1}{\mathrm{dist}(E,F)^n} \int_{B^n(r)\cap D} Q \, dm < \infty. \tag{4.27}$$

On the other hand, if $f(D) = \mathbb{R}^n$, then $f(E)$ and $f(F)$ meet at ∞ and $f\Gamma$ is the family of paths joining $f(E)$ and $f(F)$ in $\mathbb{R}^n$. Thus, $M(f\Gamma) = \infty$. The contradiction shows that $f(D) \neq \mathbb{R}^n$. □

By the techniques used in the proof of Theorem 4.8, one can establish the following.

Theorem 4.9. *Let E be a nondegenerate continuum in $\mathbb{B}^n$, $D = \mathbb{B}^n \setminus E$, and $f : D \to \mathbb{R}^n$ a Q-homeomorphism. If there exist a point $x_0 \in \partial D \cap \mathbb{B}^n$ and a neighborhood U of x_0 such that $Q|_{D\cap U} \in L^1$, then $f(D)$ is not a punctured topological ball.*

Corollary 4.5. *Let E be a nondegenerate continuum in $\mathbb{B}^n$ and $Q \in L^1(\mathbb{B}^n \setminus E)$. Then there exists no Q-homeomorphism of $\mathbb{B}^n \setminus E$ onto $\mathbb{B}^n \setminus \{0\}$.*

Chapter 5
Q-homeomorphisms with Q in BMO

Spatial BMO-quasiconformal mappings satisfy a special modulus inequality that was used in the previous chapter to define the class of Q-homeomorphisms. In this chapter we study distortion theorems, boundary behavior, removability, and mapping problems for Q-homeomorphisms with Q $\in$ BMO; see [204–209].

5.1 Introduction

Given a function $Q : D \to [1,\infty]$, we say that a sense-preserving homeomorphism $f : D \to \mathbb{R}^n$ is $Q(x)$-**quasiconformal**, abbr. $Q(x)$-**qc**, if $f \in \mathrm{W}^{1,n}_{\mathrm{loc}}(D)$ and

$$K(x,f) \leq Q(x) \quad \text{a.e.} \tag{5.1}$$

We say that $f : D \to \mathbb{R}^n$ is **BMO-quasiconformal**, abbr. **BMO-qc**, if f is $Q(x)$-qc for some BMO function $Q : D \to [1,\infty]$. Here BMO stands for the function space by John and Nirenberg [140]; see also [255] and Section B.

By Corollary B.1 and Theorem 4.1, we have the following conclusion.

Corollary 5.1. *Every* BMO-*qc mapping is a Q-homeomorphism with some $Q \in$* BMO.

Since $\mathrm{L}^\infty(D) \subset$ BMO, the class of BMO-qc mapings obviously contains all qc mappings. We show that many properties of qc mappings hold for BMO-qc mappins. Note that Q-homeomorphisms, $Q(x)$-qc and BMO-qc mappings are Möbius invariants, and hence the concepts are extended to mappings $f : D \to \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ as in the standard qc theory.

The study of related maps for $n = 2$ was started by David [48] and Tukia [310]. Recently, Astala, Iwaniec, Koskela, and Martin considered mappings with dilatation controlled by BMO functions for $n \geq 3$; see, e.g., [19]. It is necessary to note the activity of the related investigations of mappings of finite distortion; see, e.g., [131,

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_5,

132, 134, 147, 148, 153, 156, 195, 196]. This chapter is a continuation of our study of BMO–qc mappings in the plane [271–274] (see also [9, 284, 298]) and a similar geometric approach is used throughout.

The following lemma provides the standard lower bound for the modulus of all paths joining two continua in $\mathbb{R}^n$; see [71] or Corollary 7.37 in [328] or Section A.1. The lemma involves the constant λ_n, which depends only on n and appears in the asymptotic estimates of the modulus of the Teichmüller ring $R_n(t) = \mathbb{R}^n \setminus ([-1,0] \cup [t,\infty]),\, t > 0$.

Lemma 5.1. *Let E and F be two continua in $\overline{\mathbb{R}^n}$, $n \geq 2$, with $h(E) \geq \delta_1 > 0$ and $h(F) \geq \delta_2 > 0$, and let Γ be the family of paths joining E and F. Then*

$$M(\Gamma) \geq \frac{\omega_{n-1}}{(\log \frac{2\lambda_n}{\delta_1\delta_2})^{n-1}}, \tag{5.2}$$

where ω_{n-1} is the $(n-1)$-measure of the unit sphere S^{n-1} in $\mathbb{R}^n$.

5.2 Main Lemma on BMO

Lemma 5.2. *Let Q be a positive* BMO *function in $\mathbb{B}^n, n \geq 3$, and let $A(t) = \{x \in \mathbb{R}^n : t < |x| < e^{-1}\}$. Then, for all $t \in (0, e^{-2})$,*

$$\int\limits_{A(t)} \frac{Q(x)dm(x)}{(|x| \ \log\ 1/|x|)^n} \leq c, \tag{5.3}$$

where $c = c_1||Q||_* + c_2Q_1$, and c_1 and c_2 are positive constants that depend only on n. Here $||Q||_*$ is the BMO *norm of Q and Q_1 is the average of Q over $B^n(1/e)$.*

Proof. Fix $t \in (0, e^{-2})$ and set

$$\eta(t) = \int\limits_{A(t)} \frac{Q(x)dm(x)}{(|x| \ \log\ 1/|x|)^n} \,. \tag{5.4}$$

For $k = 1, 2, \ldots$, write $t_k = e^{-k}$, $A_k = \{x \in \mathbb{R}^n : t_{k+1} < |x| < t_k\}, B_k = B^n(t_k)$, and let Q_k be the mean value of $Q(x)$ in B_k. Choose an integer N such that $t_{N+1} \leq t < t_N$. Then $A(t) \subset A(t_{N+1}) = \cup_{k=1}^{N+1} A_k$ and

$$\eta(t) \leq \int\limits_{A(t_{N+1})} \frac{Q(x)}{|x|^n \ \log^n\ 1/|x|} dm(x) = S_1 + S_2, \tag{5.5}$$

where

$$S_1 = \sum_{k=1}^{N+1} \int\limits_{A_k} \frac{Q(x) - Q_k}{|x|^n \ \log^n\ 1/|x|} dm(x) \tag{5.6}$$

and

$$S_2 = \sum_{k=1}^{N+1} Q_k \int\limits_{A_k} \frac{dm(x)}{|x|^n \ \log^n \ 1/|x|} \ . \tag{5.7}$$

Since $A_k \subset B_k$, and for $x \in A_k$, $|x|^{-n} \le \Omega_n e^n/|B_k|$, where $\Omega_n = |\mathbb{B}^n|$, and since $\log 1/|x| > k$, it follows that

$$|S_1| \le \Omega_n \sum_{k=1}^{N+1} \frac{1}{k^n} \frac{e^n}{|B_k|} \int\limits_{B_k} |Q(x) - Q_k| dx \le \Omega_n e^n ||Q||_* \sum_{k=1}^{N+1} \frac{1}{k^n}$$

and, thus,

$$|S_1| \le 2\Omega_n e^n ||Q||_* \tag{5.8}$$

because, for $p \ge 2$,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} < 2. \tag{5.9}$$

To estimate S_2, we first obtain from the triangle inequality

$$Q_k = |Q_k| \le \sum_{l=2}^{k} |Q_l - Q_{l-1}| + Q_1. \tag{5.10}$$

Next we show that, for $l \ge 2$,

$$|Q_l - Q_{l-1}| \le e^n ||Q||_*. \tag{5.11}$$

Indeed,

$$\begin{aligned} |Q_l - Q_{l-1}| &= \frac{1}{|B_l|} \left| \int\limits_{B_l} (Q(x) - Q_{l-1})\, dm(x) \right| \\ &\le \frac{e^n}{|B_{l-1}|} \int\limits_{B_{l-1}} |Q(x) - Q_{l-1}| dm(x) \le e^n ||Q||_*. \end{aligned}$$

Thus, by (5.10) and (5.11),

$$Q_k \le Q_1 + \sum_{l=2}^{k} e^n ||Q||_* \le Q_1 + k e^n ||Q||_*, \tag{5.12}$$

and, since

$$\int\limits_{A_k} \frac{dm(x)}{|x|^n \ \log^n 1/|x|} \le \frac{1}{k^n} \int\limits_{A_k} \frac{dm(x)}{|x|^n} = \omega_{n-1} \frac{1}{k^n}, \tag{5.13}$$

where ω_{n-1} is the $(n-1)$-measure of S^{n-1}, it follows that

$$S_2 \leq \omega_{n-1} \sum_{k=1}^{N+1} \frac{Q_k}{k^n} \leq \omega_{n-1} Q_1 \sum_{k=1}^{N+1} \frac{1}{k^n} + \omega_{n-1} e^n ||Q||_* \sum_{k=1}^{N+1} \frac{1}{k^{(n-1)}}.$$

Thus, for $n \geq 3$, we have by (5.9) that

$$S_2 \leq 2\omega_{n-1} Q_1 + 2\omega_{n-1} e^n ||Q||_*. \tag{5.14}$$

Finally, from (5.8) and (5.14), we obtain (5.3), where $c = c_1 Q_1 + c_2 ||Q||_*$, and c_1 and c_2 are constants that depend only on n. □

Remark 5.1. It is easy to follow by the above proof that in the case $n = 2$,

$$\int\limits_{A(t)} \frac{\varphi(x)\, dm(x)}{\left(|x| \ \log \frac{1}{|x|}\right)^2} = O\left(\log \ \log \frac{1}{t}\right) \tag{5.15}$$

as $t \to \infty$. For $n \geq 2,\ 0 < t < e^{-2}$, and $A(t)$ as in Lemma 5.2, let Γ_t denote the family of all paths joining the spheres $|x| = t$ and $|x| = e^{-1}$ in $A(t)$. Then the function ρ given by

$$\rho(x) = \frac{1}{(\log \ \log 1/t)|x| \ \log 1/|x|} \tag{5.16}$$

for $x \in A(t)$ and $\rho(x) = 0$ otherwise belongs to adm Γ_t.

5.3 Upper Estimate of Distortion

Theorem 5.1. *Let $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ be a Q-homeomorphism with $Q \in \mathrm{BMO}(\mathbb{B}^n)$. If $h(\overline{\mathbb{R}^n} \setminus f(B^n(1/e))) \geq \delta > 0$, then for all $|x| < e^{-2}$,*

$$h(f(x), f(0)) \leq \frac{C}{(\log 1/|x|)^\alpha}, \tag{5.17}$$

where C and α are positive constants that depend only on n, δ, the BMO *norm $||Q||_*$ of Q, and the average Q_1 of Q over the ball $|x| < 1/e$.*

Proof. Fix $t \in (0, e^{-2})$. Let $A(t), \Gamma_t$, and ρ be as in Remark 5.1 and let $\delta_t = h(f(B^n(t)))$. Then, by Remark 5.1, $\rho \in$ adm Γ_t, and

$$M(f\Gamma_t) \leq \int\limits_{\mathbb{R}^n} Q\rho^n dm. \tag{5.18}$$

In view of (5.3) (see also Remark 5.1),

$$\int\limits_{\mathbb{R}^n} Q\rho^n dm = \int\limits_{A(t)} Q\rho^n dm \leq \frac{c}{(\log \ \log 1/t)^{n-1}}, \tag{5.19}$$

where c is the constant from Lemma 5.2. On the other hand, Lemma 5.1 applied to $M(f\Gamma_t)$ with $E = f(\overline{B^n(t)})$ and $F = \overline{\mathbb{R}^n} \setminus f(B^n(1/e))$ yields

$$M(f\Gamma_t) \geq \frac{\omega_{n-1}}{(\log \frac{2\lambda_n}{\delta\delta_t})^{n-1}}, \tag{5.20}$$

and the result follows by (5.18)–(5.20) and from the fact that $h(f(x), f(0)) \leq \delta_t$ for $|x| = t$. □

Corollary 5.2. *Let $\mathscr{F}$ be a family of Q-homeomorphisms $f : D \to \mathbb{R}^n$, with $Q \in$ BMO(D), and let $\delta > 0$. If every $f \in \mathscr{F}$ omits two points a_f and b_f in $\mathbb{R}^n$ with $h(a_f, b_f) \geq \delta$, then $\mathscr{F}$ is equicontinuous.*

5.4 Removal of Isolated Singularities

Theorem 5.2. *Let $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be a Q-homeomorphism with $Q \in$ BMO$(\mathbb{B}^n \setminus \{0\})$. Then f has a $Q(x)$-homeomorphic extension to $\mathbb{B}^n$.*

Proof. Fix $t \in (0, e^{-2})$ and let $A(t), \Gamma_t$, and ρ be as in Remark 5.1. Then, by Lemma 5.1,

$$\frac{\omega_{n-1}}{(\log \frac{2\lambda_n}{\delta\delta_t})^{n-1}} \leq M(f\Gamma_t) \leq \int\limits_{A(t)} Q\rho^n dm, \tag{5.21}$$

where $\delta = h(f(\partial B^n(e^{-1})))$ and $\delta_t = h(f(\partial B^n(t)))$. Since isolated singularities are removable for BMO functions (see [255]), we may assume that Q is defined in $\mathbb{B}^n$ and that $Q \in$ BMO$(\mathbb{B}^n)$. Thus, by Lemma 5.2 and Remark 5.1,

$$\int\limits_{A(t)} Q(x)\rho^n dm \leq \frac{c}{(\log\ \log 1/t)^{n-1}}. \tag{5.22}$$

Since here c depends only on $n, ||Q||_*$, and $Q_1 = Q_{B^n(1/e)}$, we obtain by (5.21)–(5.22) that $\delta_t \to 0$ as $t \to 0$, and hence that $\lim\limits_{x\to 0} f(x)$ exists. □

Corollary 5.3. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a* BMO*-qc mapping, then f has a homeomorphic extension to $\overline{\mathbb{R}^n}$ and, in particular, $f(\mathbb{R}^n) = \mathbb{R}^n$.*

5.5 On Boundary Correspondence

Lemma 5.3. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in$ BMO$(\overline{D})$. If D is locally connected at ∂D and $\partial D'$ is strongly accessible, then f has a continuous extension $\tilde{f} : \overline{D} \to \overline{D}'$.*

Proof. Let $x_0 \in \partial D$. As BMO functions and Q-homeomorphisms are Möbius invariant, we may assume that $x_0 = 0$. We will show that the cluster set $E = C(0, f)$ of f at 0 is a point that will prove that $f(x)$ has a limit at 0. We may also assume that $0 \in E$. Note that $E \neq \varnothing$ in view of the compactness of $\overline{\mathbb{R}^n}$.

Now, let us assume that there is one more point $y^* \in E$. Set $U = B(r_0) = B(0, r_0)$, where $0 < r_0 < |y^*|$.

By the local connectivity of D at ∂D, there is a sequence of neighborhoods V_m of 0 with connected $D_m = D \cap V_m$ and $\delta(V_m) \to 0$ as $m \to \infty$. Choose in the domains $D'_m = fD_m$ points y_m and y^*_m with $|y_m| < r_0$ and $|y^*_m| > r_0$, $y_m \to 0$ and $y^*_m \to y^*$ as $m \to \infty$. Let C_m be paths connecting y_m and y^*_m in D'_m. Note that by the construction, $\partial U \cap C_m \neq \varnothing$.

By the condition of the strong accessibility of $\partial D'$, there are compactum C in D' and a number $\delta > 0$ such that

$$M(\Delta(C, C_m; D')) \geq \delta$$

for large m. Note that $K = f^{-1}C$ is a compactum in D and hence $\varepsilon_0 = \operatorname{dist}(0, K) > 0$. Set $\delta_0 = \min\{\varepsilon_0, 1/e\}$.

Let Γ_t be the family of all paths joining K with the ball $B(t)$ in D. As in Lemma 5.2, we let $A(t)$ denote the spherical ring $t < |x| < \delta_0$. Then the function $\rho(x)$ defined in (5.16) is admissible for Γ_t, and hence

$$M(f\Gamma_t) \leq \int_D Q(x)\rho^n(x)\,dm(x). \tag{5.23}$$

For $Q \in \mathrm{BMO}(\overline{D})$, by Lemma 5.2 and Remark 5.1,

$$\int_D Q(x)\rho^n(x)\,dm(x) = \int_{A(t)} Q(x)\rho^n(x)\,dm(x) \to 0 \tag{5.24}$$

as $t \to 0$. On the other hand, for every fixed $t \in (0, \delta_0)$, $D_m \subset B(t)$, hence $C_m \subset fB(t)$ for large m, and thus

$$M(f\Gamma_t) \;\geq\; M(\Delta(C, C_m; D')) \;\geq\; \delta. \tag{5.25}$$

The obtained contradiction disproves the assumption that E contains more than one point. $\square$

Combining Lemma 5.3 and Theorem 4.6, we obtain the following.

Corollary 5.4. *Let $f : D \to D' \subset \mathbb{R}^n$ be a Q-homeomorphism onto D' with $Q \in \mathrm{BMO}(\overline{D})$. If D locally connected at ∂D and $\partial D'$ is weakly flat, then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

By Lemma 3.15, we also obtain the following corollary.

Corollary 5.5. *Let $f : D \to D' \subset \mathbb{R}^n$ be a Q-homeomorphism onto D' with $Q \in \mathrm{BMO}(\overline{D})$. If ∂D and $\partial D'$ are weakly flat, then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

These and the next theorems extend the known Gehring–Martio results (see [81], p. 196, and [214], p. 36) from qc mappings to Q-homeomorphisms with $Q \in \mathrm{BMO}(\overline{D})$ and to BMO-qc mappings, respectively.

Theorem 5.3. *Let $f : D \to D'$ be a Q-homeomorphism between* QED *domains D and D' with $Q \in \mathrm{BMO}(\overline{D})$. Then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

Theorem 5.4. *Let $f : D \to D'$ be a* BMO*-qc mapping between uniform domains D and D'. Then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

Corollary 5.6. *Let $f : D \to D'$ be a* BMO*-qc mapping between bounded convex domains D and D'. Then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

Corollary 5.7. *If D is a domain in $\mathbb{R}^n$ that is locally connected at ∂D and D is not a Jordan domain, then D cannot be mapped onto $\mathbb{B}^n$ by a Q-homeomorphism with $Q \in \mathrm{BMO}(\overline{D})$.*

Corollary 5.8. *If a domain D in $\mathbb{R}^n$ is uniform but not Jordan, then there is no* BMO*-qc mapping of D onto $\mathbb{B}^n$.*

In Section 5.7, for every $n \geq 3$, we give an example of a uniform domain that is not Jordan although it is a topological ball inside $\mathbb{B}^n$.

5.6 Mapping Problems

In Section 5.4 we showed that there are no BMO-qc mappings of $\mathbb{R}^n$ onto a proper subset of $\mathbb{R}^n$, nor BMO-qc mappings of a punctured ball onto a domain that has two nondegenerate boundary components. We may consider the following two questions.

(a) Are there any proper subsets of $\mathbb{R}^n$ that can be mapped BMO-quasiconformally onto $\mathbb{R}^n$?

(b) Are there any nondegenerate continua E in $\mathbb{B}^n$ such that $\mathbb{B}^n \setminus E$ can be mapped BMO-quasiconformally onto $\mathbb{B}^n \setminus \{0\}$?

In [273] we showed that the answer to these questions is negative if $n = 2$. The proofs were based on the Riemann mapping theorem and on the existence of a homeomorphic solution to the Beltrami equation

$$w_{\bar{z}} = \mu(z) w_z$$

for measurable functions μ with $||\mu||_\infty \le 1$ that satisfy $(1+|\mu(z)|)/(1-|\mu(z)|) \le Q(z)$ a.e. for some BMO function Q. One may modify questions (a) and (b), substituting the word "BMO-quasiconformally" for "by a Q-homeomorphism." Ahead, we provide a negative answer to questions (a) and (b) in some special cases when $n > 2$.

We say that a proper subdomain D of $\mathbb{R}^n$ is an L^1**-BMO domain** if $u \in L^1(D)$ whenever $u \in$ BMO(D). Evidently, D is an L^1-BMO domain if D is a bounded uniform domain. By [299], pp. 106–107, cf. [94], p. 69, D is an L^1-BMO domain if and only if $k_D(\cdot, x_0) \in L^1(D)$ where k_D is the **quasihyperbolic metric** on D,

$$k_D(x,x_0) = \inf_\gamma \int\limits_\gamma \frac{ds}{d(y,\partial D)}, \tag{5.26}$$

where ds denotes the Euclidean length element, $d(y,\partial D)$ denotes the Euclidean distance from $y \in D$ to ∂D, and the infimum is taken over all rectifiable paths $\gamma \in D$ joining x to x_0. L^1-BMO domains are not invariant under quasiconformal mappings of $\mathbb{R}^n$, but they are invariant under quasi-isometries; see [299], pp. 119 and 112.

In particular, every John domain is an L^1-BMO domain; see Theorem 3.14 in [299], p. 115. A domain $D \subset \mathbb{R}^n$ is called a **John domain** if there exist $0 < \alpha \le \beta < \infty$ and a point $x_0 \in D$ such that, for every $x \in D$, there is a rectifiable path $\gamma\colon [0,l] \to D$ parameterized by arc length such that $\gamma(0) = x$, $\gamma(l) = x_0$, $l \le \beta$, and

$$d(\gamma(t),\partial D) \ge \frac{\alpha}{l} \cdot t \tag{5.27}$$

for all $t \in [0,l]$. A John domain need not be uniform, but a bounded uniform domain is a John domain; see [212], p. 387. Note also that John domains are invariant under qc mappings of $\mathbb{R}^n$; see [212], p. 388. A convex domain D is a John domain if and only if D is bounded. For various characterizations of John domains, see [106, 212, 226].

More generally, the Hölder domains are also L^1-BMO domains. A domain $D \subset \mathbb{R}^n$ is said to be a **Hölder domain** if there exist $x_0 \in D$, $\delta \ge 1$, and $C > 0$ such that

$$k_D(x,x_0) \le C + \delta \cdot \log \frac{d(x_0,\partial D)}{d(x,\partial D)} \tag{5.28}$$

for all $x \in D$. It is known that D is a Hölder domain if and only if the quasihyperbolic metric $k_D(x,x_0)$ is exponentially integrable in D; see [295]. Thus, a Hölder domain is also an L^1-BMO domain.

As a consequence of Theorem 4.8, we have the following corollaries, which say that a proper subdomain D of $\mathbb{R}^n$ having a nice boundary at least at one point of ∂D cannot be mapped BMO-quasiconformally onto $\mathbb{R}^n$.

Corollary 5.9. *Let D be a domain in $\mathbb{R}^n, D \neq \mathbb{R}^n$, $n \geq 2$, and let $f : D \to \mathbb{R}^n$ be a Q-homeomorphism with $Q \in \mathrm{BMO}(D)$. If there exist a point $b \in \partial D$ and a neighborhood U of b such that $D \cap U$ is an L^1-*BMO *domain or, in particular, $\partial(D \cap U)$ is a quasisphere, then $f(D) \neq \mathbb{R}^n$.*

Remark 5.2. In particular, Theorem 4.8 implies that if a BMO-qc mapping f of D is onto $\mathbb{R}^n$, then either $D = \mathbb{R}^n$ or the domain D cannot be (even locally at a boundary point) convex, uniform, John, or Hölder.

By Theorem 4.9, we are able to give partial answers to (b).

Corollary 5.10. *Let E be a nondegenerate continuum in $\mathbb{B}^n$ and $D = \mathbb{B}^n \setminus E$. If there exist a point $x_0 \in \partial D \cap \mathbb{B}^n$ and a neighborhood U of x_0 such that $U \setminus E$ is an L^1-*BMO *domain or, in particular, $\partial(U \setminus E)$ is a quasisphere, then D cannot be mapped* BMO-*quasiconformally onto $\mathbb{B}^n \setminus \{0\}$.*

Remark 5.3. As we mentioned above, the condition $Q|_{D\cap U} \in L^1$ for $Q \in \mathrm{BMO}(D)$ in Theorems 4.8 and 4.9 has the explicit characterization in terms of integrability of the quasihyperbolic metric $k_{D\cap U}$. In particular, there exist examples in which $k_{D\cap U} \in L^1$ under $|\partial D \cap U| > 0$ (see [299]), although the latter is impossible for convex, uniform, QED, as well as John domains; see [202], p. 204, [81], p. 189, and [214], p. 33.

5.7 Some Examples

We say that a domain D in $\mathbb{R}^n$, $n \geq 2$, is a **quasiball**, respectively, **BMO-quasiball**, if there exists a homeomorphism of D onto $\mathbb{B}^n$ that is qc, respectively, BMO-qc . We say that a set S in $\overline{\mathbb{R}^n}$ is a **quasisphere**, respectively, **BMO-quasisphere**, if there exists a qc mapping, respectively, BMO-qc mapping, f of $\overline{\mathbb{R}^n}$ onto itself such that $f(S) = \partial\mathbb{B}^n$.

The following example shows that there is a BMO-quasicircle γ that is not a quasicircle.

Example 1. Consider the path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where $\gamma_1 = [0, \infty]$, $\gamma_2 = [-\infty, -1/e]$, and

$$\gamma_3 = \{te^{i\pi/\log 1/t} : 0 < t < 1/e\}.$$

Clearly, γ does not satisfy Ahlfors's three-points condition, and hence it is not a quasicircle. However, γ is a BMO-quasicircle. Indeed, the map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which is identity in $\overline{\mathbb{C}} \setminus \mathbb{B}^2$ and is given for $|z| < 1$ by

$$f(re^{i\theta}) = \begin{cases} r \exp i(\theta \log 1/r) & \text{if } 0 \leq \theta \leq \frac{\pi}{\log 1/r}, \\ r \ \exp\, i\pi(1 + \frac{1-\theta/\pi \log 1/r}{1-2\log 1/r}) & \text{if} \frac{\pi}{\log 1/r} \leq \theta < 2\pi, \end{cases}$$

is $Q(z)$-qc with $Q(re^{i\theta}) = \max(1, \log 1/r)$, which is BMO-qc in $\overline{\mathbb{C}}$ and maps γ onto $\overline{\mathbb{R}}$.

Note that $\mathbb{R}^n$ is a topological ball that cannot be mapped by a BMO-qc mapping onto $\mathbb{B}^n$; see Corollary 5.3. In view of Corollary 5.8, the following example shows that, for every $n \geq 3$, there exists a proper subdomain of $\mathbb{B}^n$ that is a topological ball but not a BMO-quasiball.

Example 2. Let $B = \mathbb{B}^n \setminus C^n(\varepsilon)$, where $C^n(\varepsilon)$ is a cone with its vertex v at the point of $S^{n-1} = \partial\mathbb{B}^n$ in the hyperplane $x_n = 1$ and with the disk $B^{n-1}(\varepsilon)$, $0 < \varepsilon < 1$, in the hyperplane $x_n = 0$ as its base. For $n \geq 3$, the domain B is uniform. Evidently, B is a topological ball. However, the boundary of B is not homeomorphic to the sphere S^{n-1}, because the point v splits ∂B into two components.

Chapter 6
More General Q-Homeomorphisms

In this chapter we continue the development of the theory of Q-homeomorphisms. More advanced results on Q-homeomorphisms for the case of $Q \in$ FMO and more general situations are proved here. For this goal, we develop a general method of singular functional parameters; see [127, 128].

6.1 Introduction

Our study concerns isolated boundary points, thin parts of the boundary in terms of Hausdorff measures, and domains with regular boundaries such as the quasiextremal distance domains of Gehring–Martio, uniform, convex, smooth, etc. Our results on continuous and homeomorphic extensions of Q-homeomorphisms to boundary points are formulated in terms of various conditions on the majorant $Q(x)$, e.g., if $Q(x)$ has finite mean oscillation at the corresponding points.

In particular, we show that an isolated singularity is removable for Q-homeomorphisms provided that $Q(x)$ has finite mean oscillation at this point. An analogue of the well-known Painleve theorem for analytic functions also follows if $Q(x)$ has finite mean oscillation at each point of a singular set of the length zero. The well-known Gehring–Martio theorem on the homeomorphic extension to the boundary of quasiconformal mappings is also generalized to Q-homeomorphisms with $Q \in$FMO. The results are applied to certain classes of Sobolev homeomorphisms.

Let D be a domain in $\mathbb{R}^n$, $n \geq 1$. Following [127], we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation at a point** $x_0 \in \overline{D}$ if

$$\overline{\lim_{\varepsilon \to 0}} \;\; -\!\!\!\!\!\!\int_{D(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, dm(x) \; < \; \infty, \tag{6.1}$$

where

$$\overline{\varphi}_\varepsilon \; = \; -\!\!\!\!\!\!\int_{D(x_0,\varepsilon)} \varphi(x) \, dm(x) \; = \; \frac{1}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} \varphi(x) \, dm(x) \tag{6.2}$$

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_6,

is the mean value of the function $\varphi(x)$ over $D(x_0,\varepsilon) = D\bigcap B(x_0,\varepsilon)$, $\varepsilon > 0$. Here

$$B(x_0,\varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}, \tag{6.3}$$

and condition (6.1) includes the assumption that φ is integrable in $D(x_0,\varepsilon)$ for small ε. In particular, if $x_0 \in \partial D$, then it is assumed nothing on the boundary in the definition.

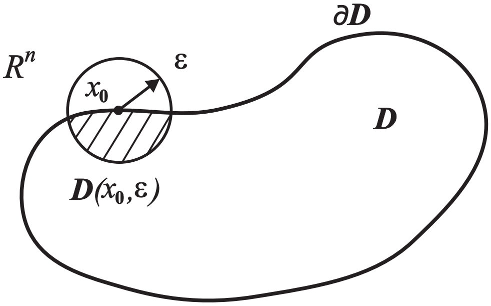

Figure 5

Note that under (6.1) it is possible that $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$. In Section 6.2 we construct a nonnegative function $\varphi : \mathbb{B}^n \to \mathbb{R}$, $n \geq 3$, that has finite mean oscillation at 0 but is not of BMO in each neighborhood of 0.

6.2 Lemma on Finite Mean Oscillation

Proposition 6.1. *If, for some collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \;\fint_{D(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon|\, dm(x) \;<\; \infty, \tag{6.4}$$

then φ has finite mean oscillation at x_0.

Indeed, by the triangle inequality,

$$\fint_{D(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\, dm(x)$$

$$\leq \fint_{D(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon|\, dm(x) \;+\; |\varphi_\varepsilon - \overline{\varphi}_\varepsilon| \;\leq\; 2\cdot \fint_{D(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon|\, dm(x).$$

Corollary 6.1. *If, for a point $x_0 \in \overline{D}$,*

$$\overline{\lim_{\varepsilon\to 0}} \quad \fint_{D(x_0,\varepsilon)} |\varphi(x)| \, dm(x) \; < \; \infty, \tag{6.5}$$

then φ has finite mean oscillation at x_0.

A point $x_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of x_0 and

$$\lim_{\varepsilon\to 0} \; \fint_{B(x_0,\varepsilon)} |\varphi(x) - \varphi(x_0)| \, dm(x) \; = 0. \tag{6.6}$$

It is well known that, for every function $\varphi \in L^1(D)$, almost all points D are its Lebesgue points; see, e.g., [281].

Corollary 6.2. *Every function $\varphi : D \to \mathbb{R}$ that is locally integrable has finite mean oscillation at almost every point in D.*

We say that a domain $D \subset \mathbb{R}^n$, $n \geq 2$, satisfies the **condition of doubling (Lebesgue) measure** at a point $x_0 \in \partial D$ if

$$|B(x_0, 2\varepsilon) \cap D| \leq c \cdot |B(x_0, \varepsilon) \cap D| \tag{6.7}$$

for some $c > 0$ and for all small enough $\varepsilon > 0$; cf. [107] and [110]. Note that the condition of doubling measure holds, in particular, at all boundary points of bounded convex domains and bounded domains with smooth boundaries in $\mathbb{R}^n$.

For inner points, a version of the next lemma was first proved for the BMO class in the planar case in [273, 274] (cf. Corollary 6.3 ahead) and then in the spatial case in [208, 209].

Lemma 6.1. *Let a domain $D \subset \mathbb{R}^n$, $n \geq 3$, satisfy the condition of doubling measure at $0 \in \partial D$. If a nonnegative function $\varphi : D \to \mathbb{R}$ has finite mean oscillation at 0, then*

$$\int\limits_{|x|<\varepsilon_0} \frac{\varphi(x) \, dm(x)}{|x \, \log|x||^n} \; < \; \infty, \tag{6.8}$$

i.e., the singular integral is convergent for some $\varepsilon_0 > 0$.

Proof. Take $\varepsilon_0 \in (0, 2^{-1})$ such that φ is integrable over $D_1 = D \bigcap B$, where $B = B(0, \varepsilon_0)$, and

$$\delta = \sup_{r\in(0,\varepsilon_0)} \; \fint_{D(r)} |\varphi(x) - \overline{\varphi_r}| \, dm(x) \; < \infty,$$

where $D(r) = D \bigcap B(r)$, $B(r) = B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$. Further, let $\varepsilon < 2^{-1}\varepsilon_0$, $\varepsilon_k = 2^{-k}2^{-1}\varepsilon_0$, $A_k = \{x \in \mathbb{R}^n : \varepsilon_{k+1} \leq |x| < \varepsilon_k\}$, $B_k = B(\varepsilon_k)$, and let φ_k be the mean value of $\varphi(x)$ over $D_k = D \bigcap B_k$, $k = 1, 2 \ldots$. Take a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$ and denote $\alpha(t) = (t \log_2 1/t)^{-n}$, $0 < t < 1$. Then $D \bigcap A(\varepsilon, \varepsilon_0) \subset \Delta(\varepsilon) = \bigcup_{k=1}^{N} \Delta_k$ where $\Delta_k = D \bigcap A_k$, and

$$\eta(\varepsilon) = \int\limits_{\Delta(\varepsilon)} \varphi(x)\, \alpha(|x|)\, dm(x) \le |S_1| + S_2,$$

where

$$S_1(\varepsilon) = \sum_{k=1}^{N} \int\limits_{\Delta_k} (\varphi(x) - \varphi_k)\, \alpha(|x|)\, dm(x),$$

$$S_2(\varepsilon) = \sum_{k=1}^{N} \varphi_k \int\limits_{\Delta_k} \alpha(|x|)\, dm(x).$$

Since $\Delta_k \subset D_k \subset B_k$, $|x|^{-n} \le \Omega_n 2^n/|D_k|$ for $x \in \Delta_k$, where Ω_n is the volume of the unit ball in $\mathbb{R}^n$, and $\log_2(1/|x|) > k$ in Δ_k, then

$$|S_1| \le \delta \Omega_n e^n \sum_{k=1}^{N} \frac{1}{k^n} < \delta \Omega_n 2^{n+1}$$

because

$$\sum_{k=2}^{\infty} \frac{1}{k^n} < \int\limits_1^{\infty} \frac{dt}{t^n} = \frac{1}{n-1} \le 1.$$

Now,

$$\int\limits_{\Delta_k} \alpha(|x|)\, dm(x) \le \frac{1}{k^n} \int\limits_{A_k} \frac{dm(x)}{|x|^n} \le \frac{\omega_{n-1}}{k^n},$$

where ω_{n-1} is the $(n-1)$-dimensional area of the unit sphere in $\mathbb{R}^n$. Moreover,

$$\begin{aligned} |\varphi_k - \varphi_{k-1}| &= \frac{1}{|D_k|} \left| \int\limits_{D_k} (\varphi(x) - \varphi_{k-1})\, dm(x) \right| \\ &\le \frac{c}{|D_{k-1}|} \int\limits_{D_{k-1}} |(\varphi(x) - \varphi_{k-1})|\, dm(x) \le \delta c, \end{aligned}$$

where c is the constant from the condition of doubling measure, and by the triangle inequality,

$$\varphi_k = |\varphi_k| \le \varphi_1 + \sum_{l=1}^{k} |\varphi_l - \varphi_{l-1}| \le \varphi_1 + k\delta c.$$

Hence,

$$S_2 = |S_2| \le \omega_{n-1} \sum_{k=1}^{N} \frac{\varphi_k}{k^n} \le 2\varphi_1 \omega_{n-1} + \delta \omega_{n-1} c \sum_{k=1}^{N} \frac{1}{k^{(n-1)}}$$

and with the estimate

$$\sum_{k=2}^{\infty} \frac{1}{k^{(n-1)}} < \int_1^{\infty} \frac{dt}{t^{n-1}} = \frac{1}{n-2} \leq 1$$

for $n \geq 3$, the proof of Lemma 6.1 is complete. □

Since

$$\sum_{k=2}^{N} \frac{1}{k} < \int_1^{N} \frac{dt}{t} = \log N < \log_2 N$$

and, for $\varepsilon_0 \in (0, 2^{-1})$ and $\varepsilon < \varepsilon_N$,

$$N < N + \log_2 \left(\frac{1}{2\varepsilon_0}\right) = \log_2 \left(\frac{1}{\varepsilon_N}\right) < \log_2 \left(\frac{1}{\varepsilon}\right),$$

then, for $n \geq 2$,

$$\sum_{k=1}^{N} \frac{1}{k^{(n-1)}} \leq \sum_{k=1}^{N} \frac{1}{k} < 1 + \log_2 \log_2 \left(\frac{1}{\varepsilon}\right),$$

and we have the following consequence of the proof.

Corollary 6.3. *Under the conditions of Lemma 6.1, for $n \geq 2$,*

$$\int\limits_{D \bigcap A(\varepsilon,\varepsilon_0)} \frac{\varphi(x)\, dm(x)}{\left(|x| \log \frac{1}{|x|}\right)^n} = O\left(\log \log \frac{1}{\varepsilon}\right) \tag{6.9}$$

for some $\varepsilon_0 > 0$ as $\varepsilon \to 0$, where

$$A(\varepsilon, \varepsilon_0) = \{x \in \mathbb{R}^n : \varepsilon < |x| < \varepsilon_0 < 1\}. \tag{6.10}$$

Examples. By the John–Nirenberg lemma, the function $\varphi(x) = \log(1/|x|)$ belongs to BMO in the unit ball $\mathbb{B}^n$ (see, e.g., [255], p. 5), but $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$. Thus, condition (6.5) is only sufficient but not necessary for a function φ to have finite mean oscillation at x_0. The example also shows that condition (6.8) cannot be extended to $n = 2$.

Note that any power of $\log(1/|x|)$ is integrable in the unit ball. However, for $n \geq 3$, the function

$$\varphi(x) = \left[\log \frac{1}{|x|}\right]^{n-1} \tag{6.11}$$

does not have finite mean oscillation at $x = 0$ because it does not satisfy the necessary condition (6.8). Simultaneously, function (6.11) satisfies (6.9) for every $n \geq 2$. Hence, condition (6.9) is necessary but not sufficient for φ to have finite mean oscillation at 0.

Now, take an arbitrary sequence of disjoint balls $B_k = B(x_k, r_k) \subset \mathbb{B}^n$, $k = 1, 2, \ldots$, such that $x_k \to 0$ and $r_k \to 0$ as $k \to \infty$, and set

$$\varphi^*(x) = c_k \varphi\left(\frac{x - x_k}{r_k}\right), \qquad x \in B_k, \tag{6.12}$$

and $\varphi^*(x) = 0$ outside $\cup B_k$, where the c_k are chosen in such a way that

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(0,\varepsilon)} \varphi^*(x)\, dm(x) \;<\; \infty. \tag{6.13}$$

Then φ^* has finite mean oscillation at 0 by Corollary 6.1. By the construction, φ^* does not have finite mean oscillation at every point x_k. Hence, φ^* is not of BMO in any neighborhood of 0.

6.3 On Super Q-Homeomorphisms

In this section we start to study **super Q-homeomorphisms** , i.e., such Q-homeomorphisms $f : D \to \mathbb{R}^n$, $n \geq 2$, that inequality (1.8) holds not only for all families Γ of continuous paths $\gamma : (0,1) \to D$ but also for **dashed lines** $\gamma : \Delta \to D$, i.e., continuous mappings γ of open subsets Δ of the real axis $\mathbb{R}$ into D. Recall that every open set Δ in $\mathbb{R}$ consists of a countable collection of mutually disjoint intervals $\Delta_i \subset \mathbb{R}$, $i = 1, 2, \ldots$. This fact gives reasons for the term "dashed line."

We say that a family Γ of dashed lines is **minorized** by another such family Γ^*, abbr. $\Gamma \geq \Gamma^*$, if, for every line $\gamma \in \Gamma, \gamma : \Delta \to \mathbb{R}^n$, there is a line $\gamma^* \in \Gamma^*, \gamma^* : \Delta^* \to \mathbb{R}^n$, that is a restriction of γ, i.e., $\Delta^* \subset \Delta$ and $\gamma^* = \gamma|_{\Delta^*}$. Later on, the following property is useful; see Theorem 1(c) in [64], p. 178.

Proposition 6.2. *Let Γ and Γ^* be families of dashed lines. If $\Gamma \geq \Gamma^*$, then $M(\Gamma) \leq M(\Gamma^*)$.*

We say that a property P holds for **almost every (a.e.)** dashed line γ in a family Γ if the subfamily of all lines in Γ for which P fails has modulus zero. In particular, almost every dashed line in $\mathbb{R}^n$ is rectifiable; see, e.g., Theorem 2 in [64]. All definitions of the modulus, rectifiability, and so on for dashed lines are perfectly similar to the corresponding notions for paths and hence are omitted. Many results for dashed lines are also similar, and it is not necessary for our goals to formulate all of them explicitly here. For the advanced theory of more general systems of measures in metric space, see [64].

Theorem 6.1. *Let $f : D \to \mathbb{R}^n$ be a homeomorphism in the class $W^{1,n}_{\mathrm{loc}}$ with $f^{-1} \in W^{1,n}_{\mathrm{loc}}$. Then f is a super Q-homeomorphism with $Q(x) = K_I(x, f)$.*

Proof. First, $f^{-1} \in \mathrm{ACL}^n_{\mathrm{loc}}$; see, e.g., [215], p. 8. Hence, by the Fuglede theorem, the modulus of all locally rectifiable paths in $f(D)$ with at least one closed subpath where f^{-1} is not absolutely continuous has modulus zero; see [64] and [316]. This family of paths minorizes the corresponding family of dashed lines; thus, by Proposition 6.2, the latter family also has modulus zero.

Let Γ be an arbitrary family of dashed lines in D. Let us denote by Γ^* the family of all dashed lines $\gamma^* \in f\Gamma$ for which f^{-1} is absolutely continuous on every closed subpath of γ^*. Then $M(f\Gamma) = M(\Gamma^*)$.

For $\rho \in \operatorname{adm}\Gamma$, set $\rho^*(y) = \rho(f^{-1}(y)) \cdot |(f^{-1})'(y)|$ if $f^{-1}(y)$ is differentiable and $\rho^*(y) = \infty$ otherwise at $y \in f(D)$ and $\rho^*(y) = 0$ outside $f(D)$. Then

$$\int\limits_{\gamma^*} \rho^* ds^* \geq \int\limits_{f^{-1}\circ\gamma^*} \rho\, ds \geq 1 \tag{6.14}$$

for all $\gamma^* \in \Gamma^*$, i.e., $\rho^* \in \operatorname{adm}\Gamma^*$.

By Proposition 4.1 and Remark 4.1, f^{-1} has the (N)-property and is differentiable with $J(y, f^{-1}) \neq 0$ a.e. Hence, using a change of variables (see, e.g., Theorem 6.4 in [222], cf. also Corollary 8.1 and Proposition 8.3 ahead), we have

$$\begin{aligned} M(f\Gamma) = M(\Gamma^*) \leq & \int\limits_{f(D)} \rho^*(y)^n dm(y) \\ = & \int\limits_{f(D)} \rho(f^{-1}(y))^n K_O(y, f^{-1}) J(y, f^{-1})\, dm(y) \\ = & \int\limits_{D} \rho(x)^n K_I(x, f)\, dm(x), \end{aligned}$$

i.e., f is a super Q-homeomorphism with $Q(x) = K_I(x, f)$. □

It is known that homeomorphisms of the class $W^{1,n}_{\text{loc}}$ with $K_I \in L^1_{\text{loc}}$ have the inverse f^{-1} in the same class; see Corollary 2.3 in [154]. Thus, we have the next assertion.

Corollary 6.4. *Let $f : D \to \mathbb{R}^n$ be a homeomorphism in the class $W^{1,n}_{\text{loc}}$ with $K_I \in L^1_{\text{loc}}$. Then f is a super Q-homeomorphism with $Q(x) = K_I(x, f)$.*

Since $K_I(x, f) \leq K_O^{n-1}(x, f)$, we also have the following statement.

Corollary 6.5. *Under the conditions of Theorem 6.1, f is a super Q-homeomorphism with $Q(x) = K_O^{n-1}(x, f)$.*

Theorem 6.1 shows that super Q-homeomorphisms form a wide subclass of Q-homeomorphisms including many mappings with finite distortion.

6.4 Removal of Isolated Singularities

It is well known that isolated singularities are removable for conformal as well as quasiconformal mappings. The following statement shows that any power of integrability of $Q(x)$ cannot guarantee the removability of isolated singularities of Q-homeomorphisms. This is a new phenomenon.

Proposition 6.3. *For any $p \in [1,\infty)$, there is a super Q-homeomorphism $f : \mathbb{B}^n\backslash\{0\} \to \mathbb{R}^n$, $n \geq 2$, with $Q \in L^p(\mathbb{B}^n)$ that has no continuous extension to $\mathbb{B}^n$. Moreover, a $Q(x)$-qc mapping can be chosen as f.*

Here $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ denotes the unit ball in $\mathbb{R}^n$.

Proof. The desired homeomorphism f can be given in the explicit form

$$y = f(x) = \frac{x}{|x|}\,(1+|x|^{\alpha}),$$

where $\alpha \in (0, n/p(n-1))$. Note that f maps the punctured unit ball $\mathbb{B}^n\backslash\{0\}$ onto the spherical ring $1 < |y| < 2$ in $\mathbb{R}^n$ and f has no continuous extension onto $\mathbb{B}^n$.

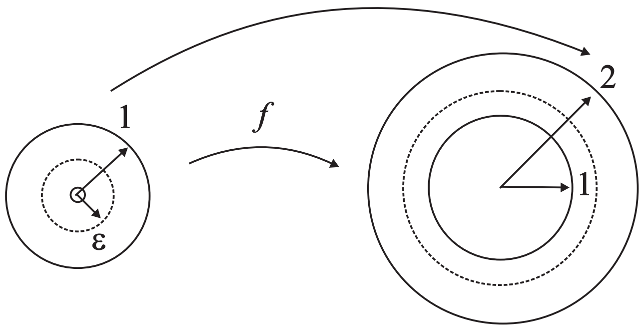

Figure 6

On the sphere $|x| = r$, the tangent and radial distortions are

$$\delta_\tau = \frac{|y|}{|x|} = \frac{1+r^{\alpha}}{r}, \qquad \delta_r = \frac{\partial|y|}{\partial|x|} = \alpha r^{\alpha-1},$$

respectively. Without loss of generality, we may assume that p is great enough so that $\alpha < 1$. Thus, $\delta_\tau \geq \delta_r$ and, consequently,

$$K_O = \frac{\delta_\tau^n}{\delta_\tau^{n-1}\delta_r} = \frac{\delta_\tau}{\delta_r}, \qquad K_I = \frac{\delta_\tau^{n-1}\delta_r}{\delta_r^n} = \left(\frac{\delta_\tau}{\delta_r}\right)^{n-1}$$

(see, e.g., Section 1.2.1 in [256]), and hence the maximal dilatation is

$$Q(x) = K_I(x,f) = \left(\frac{1+r^{\alpha}}{\alpha r^{\alpha}}\right)^{n-1} \leq \frac{C}{r^{\alpha(n-1)}}, \qquad |x| = r,$$

where $C = (2/\alpha)^{n-1}$. Note that $Q \in L^p(\mathbb{B}^n\backslash\{0\})$ because $\alpha(n-1)p < n$ by the choice of α. It remains to note that $f \in C^1 \subset W^{1,n}_{\mathrm{loc}}$ in $\mathbb{B}^n\backslash\{0\}$, and hence f is a super Q-homeomorphism; see Theorem 6.1. $\Box$

However, as the next lemma shows, it is sufficient for the removability of isolated singularities of Q-homeomorphisms to require that $Q(x)$ be integrable with suitable weights.

Lemma 6.2. *Let $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$, $n \geq 2$, be a Q-homeomorphism. If*

$$\int\limits_{\varepsilon<|x|<1} Q(x) \cdot \psi^n(|x|) \ dm(x) = o(I(\varepsilon)^n) \tag{6.15}$$

as $\varepsilon \to 0$, where $\psi(t)$ is a nonnegative measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon) := \int\limits_{\varepsilon}^{1} \psi(t) \ dt < \infty, \qquad \varepsilon \in (0,1)\,, \tag{6.16}$$

then f has a continuous extension to $\mathbb{B}^n$ that is a Q-homeomorphism.

Note the conditions (6.15) and (6.16) imply that $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This follows immediately from arguments by contradiction.

Remark 6.1. Note also that (6.15) holds, in particular, if

$$\int\limits_{\mathbb{B}^n} Q(x) \cdot \psi^n(|x|) \ dm(x) < \infty \tag{6.17}$$

and $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the removability of a singularity at $x = 0$, it is sufficient that integral (6.17) converges for some nonnegative function $\psi(t)$ that is locally integrable over $(0,1)$ but has a nonintegrable singularity at 0. The functions $Q(x) = \log^{\lambda}(e/|x|)$, $\lambda \in (0,1)$, $x \in \mathbb{B}^n$, $n \geq 2$, and $\psi = 1/(t\ \log(e/t))$, $t \in (0,1)$, show that condition (6.17) is compatible with the condition $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. By Lemma 6.1, condition (6.17) holds with the given ψ for every function $Q(x) \geq 1$ in $L^1(\mathbb{B}^n)$ having finite mean oscillation at 0 if $n \geq 3$.

Proof. Since the modulus of a family of paths passing through a fixed point equals 0, it is sufficient to show that $f(x)$ has a limit as $x \to 0$.

Let Γ_ε be a family of all paths joining the spheres $S_\varepsilon = \{x \in \mathbb{R}^n : |x| = \varepsilon\}$ and $S_0 = \{x \in \mathbb{R}^n : |x| = 1\}$ in the ring $A_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}$. Also let ψ^* be a Borel function such that $\psi^*(t) = \psi(t)$ for a.e. $t \in (0,\infty)$. Such a function ψ^* exists by Lusin's theorem; see, e.g., Section 2.3.5 in [55] and [284], p. 69. Then the function

$$\rho_\varepsilon(x) = \begin{cases} \psi^*(|x|)/I(\varepsilon) & \text{if } x \in A_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus A_\varepsilon \end{cases}$$

is admissible for Γ_ε and, hence,

$$M(f\Gamma_\varepsilon) \leq \int\limits_{0<|x|<1} Q(x) \cdot {\rho_\varepsilon}^n(|x|) \, dm(x)\,,$$

i.e., $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (6.15).

By the Jordan–Brouwer theorem, the images of the above spheres fS_t, $t \in (0,1)$, split the space $\mathbb{R}^n$ into two components and, thus, $\mathbb{R}^n \backslash fA_\varepsilon$ consists of exactly two components; see, e.g., [50], p. 358, [123], p. 363, and [335], p. 63. Denote by Γ_ε^* the family of all paths in $\mathbb{R}^n$ joining the images of the spheres fS_ε and fS_0. Then

$$M(\Gamma_\varepsilon^*) = M(f\Gamma_\varepsilon)$$

because $f\Gamma_\varepsilon \subset \Gamma_\varepsilon^*$. Hence, $M(f\Gamma_\varepsilon) \leq M(\Gamma_\varepsilon^*)$, and, on the other hand, $f\Gamma_\varepsilon < \Gamma_\varepsilon^*$ (i.e., every path in Γ_ε^* contains a subpath in $f\Gamma_\varepsilon$ as fA_ε separates the two components of $\mathbb{R}^n \setminus fA_\varepsilon$), and consequently, $M(f\Gamma_\varepsilon) \geq M(\Gamma_\varepsilon^*)$; see, e.g., either Theorem 1(c) in [64] or 6.4 in [316].

By Gehring's lemma in [71] (see also (7.19), Lemma 7.22 and Corollary 7.37 in [328], and Section, A.1), we have

$$M(\Gamma_\varepsilon^*) \geq a_n \Bigg/ \left(\log \frac{b_n}{\delta_0 \delta \varepsilon}\right)^{n-1} .$$

Here, the constants a_n and b_n depend only on n. The numbers δ_0 and δ_ε denote the spherical (chordal) diameter of fS_0 and fS_ε, respectively. Thus, $\delta_\varepsilon \to 0$ and fS_ε are contracted to a point as $\varepsilon \to 0$. □

In particular, choosing in Lemma 6.2 $\psi(t) = 1/(t \log 1/t)$, we obtain by Corollary 6.3 the following theorem.

Theorem 6.2. *Let $f : D\backslash\{x_0\} \to \mathbb{R}^n$, $n \geq 2$, be a Q-homeomorphism where $Q(x)$ has finite mean oscillation at a point $x_0 \in D$. Then f has a Q-homeomorphic extension to D.*

In other words, an isolated singularity of a Q-homeomorphism is removable if $Q(x)$ has finite mean oscillation at the point. In particular, this is the case if $Q(x)$ is continuous at x_0. As consequences of Theorem 6.2, Proposition 6.1, and Corollary 6.1, we also obtain the following statements.

Corollary 6.6. *A Lebesgue point of Q is a removable isolated singularity for Q-homeomorphisms.*

Corollary 6.7. *If $f : \mathbb{B}^n\backslash\{0\} \to \mathbb{R}^n$, $n \geq 2$, is a Q-homeomorphism with*

$$\overline{\lim_{\varepsilon \to 0}} \;\; -\!\!\!\!\!\!\int_{B(\varepsilon)} Q(x)\, dm(x) \; < \; \infty, \tag{6.18}$$

then f has a Q-homeomorphic extension to $\mathbb{B}^n$.

Similarly, choosing in Lemma 6.2 the function $\psi(t) = 1/t$ as a weight, we come to the following more general statement.

Theorem 6.3. *Let* $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n$*,* $n \geq 2$*, be a Q-homeomorphism. If*

$$\int\limits_{\varepsilon<|x|<1} Q(x) \ \frac{dm(x)}{|x|^n} = o\left(\left[\log\frac{1}{\varepsilon}\right]^n\right) \tag{6.19}$$

as $\varepsilon \to 0$*, then* f *has a Q-homeomorphic extension to* $\mathbb{B}^n$*.*

Corollary 6.8. *Condition (6.19) and the assertion of Theorem 6.3 hold if*

$$Q(x) = o\left(\left[\log\frac{1}{|x|}\right]^{n-1}\right) \tag{6.20}$$

as $x \to 0$*. The same holds if*

$$q(r) = o\left(\left[\log\frac{1}{r}\right]^{n-1}\right) \tag{6.21}$$

as $r \to 0$*, where* $q(r)$ *is the mean value of the function* $Q(x)$ *over the sphere* $|x| = r$*.*

Remark 6.2. Choosing in Lemma 6.2 the function $\psi(t) = 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (6.19) by

$$\int\limits_{\varepsilon<|x|<1} \frac{Q(x)\, dm(x)}{\left(|x|\log\frac{1}{|x|}\right)^n} = o\left(\left[\log\,\log\frac{1}{\varepsilon}\right]^n\right), \tag{6.22}$$

and (6.21) by

$$q(r) = o\left(\left[\log\frac{1}{r}\log\,\log\frac{1}{r}\right]^{n-1}\right). \tag{6.23}$$

Thus, it is sufficient to require that

$$q(r) = O\left(\left[\log\frac{1}{r}\right]^{n-1}\right). \tag{6.24}$$

In general, we could give here the whole scale of the corresponding conditions in logarithms using functions $\psi(t)$ of the form $1/(t\log\cdots\log 1/t)$. However, we prefer to give conditions of other types that are often met in the mapping theory (see, e.g., [189] and [220]) and that can be obtained directly from Lemma 6.2.

Theorem 6.4. *Let* $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n$*,* $n \geq 2$*, be a Q-homeomorphism and, for some* $\beta \geq 1/(n-1)$*, let*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{rq^{\beta}(r)} = \infty, \tag{6.25}$$

where $q(r)$ *is the mean integral value of the function* $Q(x)$ *over the sphere* $|x| = r$*. Then* f *has a Q-homeomorphic extension to* $\mathbb{B}^n$*.*

Proof. Indeed, for the function

$$\psi(t) = \begin{cases} 1/[tq^{\beta}(t)], & t \in (0, \varepsilon_0), \\ 0, & t \in (\varepsilon_0, 1), \end{cases} \tag{6.26}$$

we have

$$\int\limits_{\varepsilon<|x|<1} Q(x) \cdot \psi^n(|x|)\, dm(x) = \omega_{n-1} \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{rq^{\beta n-1}(r)} \leq \omega_{n-1} \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{rq^{\beta}(r)}, \tag{6.27}$$

where ω_{n-1} is the $(n-1)$-dimensional area of the unit sphere $|x| = 1$ in $\mathbb{R}^n$. Thus, the assertion follows immediately from Lemma 6.2 by condition (6.25). □

Corollary 6.9. *Every Q-homeomorphism $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$, $n \geq 2$, with*

$$\int\limits_{0}^{\varepsilon_0} \frac{dr}{rq(r)} = \infty \tag{6.28}$$

can be extended to a Q-homeomorphism of $\mathbb{B}^n$ into $\overline{\mathbb{R}^n}$.

Corollary 6.10. *If, for some $\beta \geq 1/(n-1)$ and $\alpha \geq 1$,*

$$\int\limits_{0}^{\varepsilon_0} \frac{dr}{rq_{\alpha}^{\beta}(r)} = \infty, \tag{6.29}$$

where

$$q_{\alpha}(r) \;=\; \left(\fint_{|x|=r} Q^{\alpha}(x) \right)^{\frac{1}{\alpha}}, \tag{6.30}$$

then every Q-homeomorphism $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ allows extension to $\mathbb{B}^n$.

Indeed, by the Jensen inequality, $q_{\alpha}(r) \geq q_1(r) = q(r)$ (see, e.g., [339], p. 20), and thus condition (6.29) implies condition (6.25).

In summary, Lemma 6.2 is a rich source of various conditions for the removability of isolated singularities of Q-homeomorphisms.

6.5 Topological Lemmas

In this section we prove lemmas that will replace the Jordan–Brouwer theorem in the cases to be considered. Instead of an isolated singular point, a singular set that is infinite and even uncountable (of the continuum cardinality) will be examined.

Recall that by the well known Alexandroff–Borsuk theorem (see, e.g., [50], p. 357, [126], p. 100, [11], and [29]), a compact set $K \subset \mathbb{R}^n$, $n \geq 2$, disconnects $\mathbb{R}^n$ if and only if there is a continuous mapping $f : K \to S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ that is not homotopic to a constant mapping. Conversely, the statement that a compactum K does not disconnect $\mathbb{R}^n$ is equivalent to the statement that each continuous mapping $f : K \to S^{n-1}$ is homotopic to a constant mapping. Thus, we obtain the following simple corollary of the Alexandroff–Borsuk theorem.

Proposition 6.4. *Let K_1 and K_2 be disjoint compact sets in $\overline{\mathbb{R}^n}$, $n \geq 2$, each of which does not separate $\overline{\mathbb{R}^n}$. Then the compactum $K = K_1 \bigcup K_2$ does not separate $\overline{\mathbb{R}^n}$.*

On this basis we prove the following statement.

Lemma 6.3. *Let D be a domain in $\overline{\mathbb{R}^n}$, $n \geq 2$. Then the boundary of every component of its complement $\overline{\mathbb{R}^n} \backslash D$ is a component of ∂D.*

Proof. Let us assume that the conclusion of Lemma 6.3 is not true, i.e., there is a domain D in $\overline{\mathbb{R}^n}$, $n \geq 2$, such that the boundary of a component C_0 of $\overline{\mathbb{R}^n} \backslash D$ does not coincide with a component of ∂D.

As known, $\partial C_0 \subset \partial D$; see, e.g., [50], p. 356. Moreover, if a component of K of the boundary ∂D has a nonempty intersection with ∂C_0, then $K \subset \partial C_0$. Thus, the negation of the statement of Lemma 6.3 is equivalent to the property of ∂C_0 to consist of more than one component of ∂D, i.e., to the statement that ∂C_0 is not a connected set.

Every component of $\overline{\mathbb{R}^n} \backslash D$ is a compact set. Hence, joining to the domain D, if it is necessary, all components of its complement $\overline{\mathbb{R}^n} \backslash D$ except C_0, we may by the Alexandroff–Borsuk theorem consider without loss of generality that $\overline{\mathbb{R}^n} \backslash D$ has only one component C_0 and that $\overline{\mathbb{R}^n} \backslash C_0 = D$ is a domain.

By the above assumption, ∂C_0 can be split into two disjoint compact sets S_1 and S_2. Let Ω_1 be a component of the complement $\overline{\mathbb{R}^n} \backslash S_1$ including the domain D. Then the compactum $K_1 = \overline{\mathbb{R}^n} \backslash \Omega_1$ includes S_1 and does not separate $\overline{\mathbb{R}^n}$. Moreover, by the construction, $K_1 \subset C_0$ and $S_2 \subset \Omega_1$ [the latter because every neighborhood of each point in $S_2 \subset \partial C_0$ must involve points of $D \subset \Omega_1$, but the spherical distance $h(S_2, \partial\Omega_1) \geq h(S_2, S_1) > 0$ since $\partial\Omega_1 \subset S_1$; see [50], p. 356]. The sets Ω_2 and K_2 are defined in a similar way through S_2.

Note that $K_1 \bigcap K_2 = \emptyset$. Indeed,

$$\partial K_1 = \partial\Omega_1 \subset S_1 \subset \Omega_2 = \overline{\mathbb{R}^n} \backslash K_2$$

and

$$\partial K_2 = \partial\Omega_2 \subset S_2 \subset \Omega_1 = \overline{\mathbb{R}^n} \backslash K_1 .$$

Hence, $\partial K_1 \bigcap K_2 = \emptyset$ and $\partial K_2 \bigcap K_1 = \emptyset$. Thus, if

$$K_1 \bigcap K_2 \neq \emptyset, \tag{6.31}$$

then

$$\mathrm{Int}\, K_1 \bigcap \mathrm{Int}\, K_2 \neq \emptyset,$$

i.e., there exist components $\Omega^{(1)} \neq \Omega_1$, $\Omega^{(1)} \subset \mathrm{Int}\, K_1$ and $\Omega^{(2)} \neq \Omega_2$, $\Omega^{(2)} \subset \mathrm{Int}\, K_2$ of the complements $\overline{\mathbb{R}^n} \backslash S_1$ and $\overline{\mathbb{R}^n} \backslash S_2$, respectively, such that $\Omega^{(1)} \bigcap \Omega^{(2)}$

$\neq \emptyset$. If $\Omega^{(1)} = \Omega^{(2)}$, then $\partial\Omega^{(1)} = \partial\Omega^{(2)} \subset S_1 \bigcap S_2 = \emptyset$, i.e., $\partial\Omega^{(1)} = \partial\Omega^{(2)} = \emptyset$ and $\Omega^{(1)} = \Omega^{(2)} = \overline{\mathbb{R}^n}$ (because $\overline{\mathbb{R}^n}$ is a connected space), which is impossible by the construction. For the definiteness, let $\Omega^{(1)} \setminus \Omega^{(2)} \neq \emptyset$. Then $\partial\Omega^{(2)} \bigcap \Omega^{(1)} \neq \emptyset$ because $\emptyset \neq \Omega^{(1)} \bigcap \Omega^{(2)} \subset \Omega^{(1)}$ and $\Omega^{(1)}$ is connected. However, $\partial\Omega^{(2)} \subset S_2$ and $\Omega^{(1)} \subset K_1$. Consequently, assumption (6.31) contradicts the inclusion $S_2 \subset \Omega_1 = \overline{\mathbb{R}^n} \setminus K_1$.

Then, by Proposition 6.4, the compactum $K = K_1 \bigcup K_2 \subset C_0$ does not separate $\overline{\mathbb{R}^n}$, i.e., $\Omega = \overline{\mathbb{R}^n} \setminus K = \Omega_1 \bigcap \Omega_2$ is a domain in $\overline{\mathbb{R}^n}$.

Further, since $\partial C_0 = S_1 \bigcup S_2 \subset K_1 \bigcup K_2 = K$ and $K_1 \bigcap K_2 = \emptyset$ and C_0 is connected, then $\Omega_0 = \Omega \bigcap \operatorname{Int} C_0 \neq \emptyset$ (if $\operatorname{Int} C_0 \subset K$, then $C_0 = K = K_1 \bigcup K_2$). However, by the construction, $D \subset \Omega$, $D \bigcup C_0 = \overline{\mathbb{R}^n}$, $D \bigcap C_0 = \emptyset$, and $\Omega = D \bigcup \Omega_0$. The latter contradicts the connectivity of Ω. Thus, the assumption that ∂C_0 is not connected is really not true and we come to the assertion of the lemma. □

Corollary 6.11. *The interior* $\operatorname{Int} C^*$ *of every component* C^* *of the complement* $\overline{\mathbb{R}^n} \setminus D$ *is separated from* $\overline{\mathbb{R}^n} \setminus C^*$ *by a single component* K^* *of* ∂D*, i.e., any path joining* $x \in \operatorname{Int} C^*$ *and* $y \in \overline{\mathbb{R}^n} \setminus C^*$ *in* $\overline{\mathbb{R}^n}$ *intersects* K^**.*

Indeed, by Lemma 6.3, the boundary of C^* consists of one component K^* of ∂D. If some path $\gamma : (0,1) \to \overline{\mathbb{R}^n}$ joining $x \in \operatorname{Int} C^*$ and $y \in \overline{\mathbb{R}^n} \setminus C^*$ does not intersect ∂D, then $(0,1)$ is split into the two disjoint open sets $\gamma^{-1}(\operatorname{Int} C^*)$ and $\gamma^{-1}(\overline{\mathbb{R}^n} \setminus C^*)$, which contradicts the connectivity of the interval $(0,1)$.

Lemma 6.4. *Let* D *be a domain in* $\overline{\mathbb{R}^n}$*,* $n \geq 2$*, and let* C^* *be a component of its complement* $\overline{\mathbb{R}^n} \setminus D$*. Then, for every* $\varepsilon > 0$*, there is a neighborhood* N_ε *of* C^* *such that* $D_\varepsilon = D \bigcap N_\varepsilon \subset C_\varepsilon$ *is a domain where*

$$C_\varepsilon = \{x \in \overline{\mathbb{R}^n} : h(x, C^*) < \varepsilon\} \tag{6.32}$$

is the ε*-neighborhood of* C^* *with respect to the spherical (chordal) metric* h *in* $\overline{\mathbb{R}^n}$*.*

Proof. Denote by S_ε the union of all components of the compact $C = \overline{\mathbb{R}^n} \setminus D$ that intersects the compact $\overline{\mathbb{R}^n} \setminus C_\varepsilon$. Note that the set S_ε is closed and hence compact in $\overline{\mathbb{R}^n}$.

Indeed, let us assume that S_ε is not closed, i.e., there exist a point $x_0 \in \overline{\mathbb{R}^n} \setminus S_\varepsilon$ and a sequence $x_l \in S_l$ such that $x_l \to x_0$ as $l \to \infty$. Let $C_l \subset S_\varepsilon$ be the corresponding sequence of the components of C containing x_l. Then

$$C_0 = \overline{\lim_{l \to \infty}} C_l = \{y \in \overline{\mathbb{R}^n} : y = \lim_{l \to \infty} y_l,\, y_l \in C_l,\, l = 1, 2, \ldots\} \tag{6.33}$$

is a connected (closed) subset of C (see, e.g., (9.12) in [334], p. 15), which contains x_0 and intersects $\overline{\mathbb{R}^n} \setminus C_\varepsilon$. The contradiction disproves the above assumption.

Now, let N_ε be a component of $\overline{\mathbb{R}^n} \setminus S_\varepsilon$ containing C^*. Then, by the construction, every component of C is completely contained either in the domain N_ε or in its complement. Note also that the open set $D_\varepsilon = D \bigcap N_\varepsilon$ is not empty. It remains to show that D_ε is connected.

Let us assume that D_ε is not connected. Then there is a component $D_0 \subset D_\varepsilon$ such that $D_\varepsilon \setminus D_0 \neq \emptyset$ and, by Corollary 6.11, there is one component K_0 of ∂D_0 that separates D_0 from another component D^* of $D_\varepsilon \setminus D_0$. By the construction, K_0 is contained in one component K^* of the boundary of either D or N_ε.

If $K^* \subset \partial D$, then points $x \in D_0$ and $y \in D^*$ can be joined by a path γ in D. On the other hand, by Corollary 6.11, γ must intersect K_0, which is impossible because $K_0 \subset C = \overline{\mathbb{R}^n} \setminus D$. Similarly, if $K^* \subset \partial N_\varepsilon$, then points $x \in D_0$ and $y \in D^*$ can be joined by a path γ in N_ε. Again by Corollary 6.11, γ must intersect K_0, which contradicts the inclusion $K_0 \subset \overline{\mathbb{R}^n} \setminus N_\varepsilon$. □

Lemma 6.5. *Let D be domain in $\overline{\mathbb{R}^n}$, $n \geq 2$, and let $f : D \to \overline{\mathbb{R}^n}$ be a homeomorphism. Then $D' = f(D)$ is a domain and there is a natural one-to-one correspondence between components K and K' of the boundaries ∂D and $\partial D'$ such that $C(f,K) = K'$ and $C(f^{-1},K') = K$.*

Here we use the notation of the **cluster set** of the mapping f for $E \subset \partial D$:

$$C(f,E) = \{y \in \overline{\mathbb{R}^n} : y = \lim_{l\to\infty} f(x_l), \ \ x_l \to x \in E\}. \tag{6.34}$$

Proof. $D' = f(D)$ is a domain by the well known Brouwer theorem; see, e.g., [50], p. 358.

Further, for every set $E \subset \partial D$, $C(f,E) \subset \partial D'$, and, similarly, for every set $E' \subset \partial D'$, $C(f^{-1},E') \subset \partial D$.

Indeed, by the definition, $C(f,E) \subset \overline{D'}$. Let us assume that there is a point $y_0 \in C(f,E) \bigcap D'$. Set $x_0 = f^{-1}(y_0)$. Then $x_0 \in D$ and hence $\delta_0 = \mathrm{dist}\,(x_0, \partial D)/2 > 0$. Let $x_k \in D$ be such that $f(x_k) \to y_0$ and $\mathrm{dist}\,(x_k, E) \to 0$ as $k \to \infty$. Then $x_k \overline{\in} B(x_0, \delta_0)$ for great enough k and, simultaneously, $x_k = f^{-1}(f(x_k)) \to f^{-1}(y_0) = x_0$ as $k \to \infty$ by the continuity of f^{-1}. The contradiction disproves the assumption.

Let K be a component of the boundary of D. It is clear that K is a closed subset of ∂D that is a compact set in $\overline{\mathbb{R}^n}$ and, hence, that K is a continuum. Further, let

$$\delta_\varepsilon = \{x \in D : h(x,K) < \varepsilon\},$$

where h is a spherical (chordal) distance in $\overline{\mathbb{R}^n}$. Then

$$C(f,K) = \bigcap_{\varepsilon>0} \overline{f(\delta_\varepsilon)}.$$

Denote by D'_ε a component of $\overline{f(\delta_\varepsilon)}$ including $C(f,K)$. The existence of such a component follows from Lemma 6.4. The sets D'_ε form a decreasing family of continua and

$$C(f,K) = \bigcap_{\varepsilon>0} D'_\varepsilon.$$

Thus, $C(f,K)$ is a continuum; see, e.g., (9.4) in [334], p. 15. □

Denote by K' the component of $\partial D'$ including $C(f,K)$. Then, arguing as above, we obtain that $C(f^{-1},K')$ is a continuum, which, by the construction, includes K

and hence $K = C(f^{-1}, K')$. In view of the symmetry of the conditions of the lemma with respect to f and f^{-1}, it is also $K' = C(f, K)$.

Corollary 6.12. *Let $f : D \setminus X \to D'$ be a homeomorphism where X is a closed, totally disconnected subset of D. If f has a continuous extension $\overline{f}$ to D, then $\overline{f}$ is a homeomorphism of D onto $\overline{f}(D)$.*

Here the set $X \subset \mathbb{R}^n$ is called **totally disconnected** if every (connected) component of X consists of one point. Closed subsets X of $\mathbb{R}^n$ are locally compact spaces and hence, for such X, total disconnectness is equivalent to the condition $\dim X = 0$; see [126], p. 22.

Remark 6.3. In view of the well-known Menger–Urysohn theorem, if D is a domain in $\mathbb{R}^n$, $n \geq 2$, and a point $x_0 \in \partial D$ has a neighborhood U where the dimension of ∂D is less than $n-1$, then D is locally connected at x_0, i.e., there exists arbitrarily small neighborhoods V_ε, $\operatorname{diam} V_\varepsilon < \varepsilon$, of x_0 such that the set $D_\varepsilon = V_\varepsilon \bigcap D = V_\varepsilon \setminus \partial D$ is connected; see, e.g., [126], p. 48. In this case, the cluster set

$$C(f, x_0) = \bigcap_{\varepsilon > 0} C_\varepsilon\,, \quad C_\varepsilon = \overline{f(D_\varepsilon)}\,,$$

is a continuum for every continuous mapping $f : D \to \mathbb{R}^n$ by (9.4) in [334], p. 15.

6.6 On Singular Sets of Length Zero

In this section we consider the problem of removability of singularities for super Q-homeomorphisms. A set X in $\mathbb{R}^n$ is called a **set of length zero** if X can be covered by a sequence of balls in $\mathbb{R}^n$ with an arbitrary small sum of diameters. As known, such sets have the (Lebesgue) measure zero,

$$\dim X = 0, \tag{6.35}$$

and hence they are totally disconnected; see, e.g., [126], pp. 22 and 104. A classical example of such sets is the set C of the Cantor type obtained by deleting a sequence of open sets, known as middle halfs, from a closed unit interval. Note that C is **perfect**, i.e., it is closed and without isolated points. Hence, by the well-known theorem of W. H. Young, each neighborhood of a point in C contains a subset of C of the continuum cardinality; see [337].

By the theorem of Menger and Urysohn, condition (6.35) guarantees that X does not disconnect a domain D in $\mathbb{R}^n$, $n \geq 2$, and, thus, if X is closed in D, then $D^* = D \setminus X$ is also a domain.

Lemma 6.6. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let X be a closed subset of D of length zero, and let $f : D \setminus X \to \mathbb{R}^n$ be a super Q-homeomorphism. If, for $x_0 \in X$,*

$$\int\limits_{\varepsilon<|x|<\delta(x_0)} Q(x_0+x)\cdot\psi^n_{x_0,\varepsilon}(|x|)\ \ dm(x) = o(I(x_0,\varepsilon)^n) \tag{6.36}$$

as $\varepsilon\to 0$*, where* $0<\delta(x_0)<\operatorname{dist}(x_0,\partial D)$ *and* $\psi_{x_0,\varepsilon}(t),\ \ \varepsilon\in(0,\delta(x_0))$*, is a family of nonnegative measurable (by Lebesgue) functions on* $(0,\infty)$ *such that*

$$0<I(x_0,\varepsilon)=\int\limits_{\varepsilon}^{\delta(x_0)}\psi_{x_0,\varepsilon}(t)\ \ dt<\infty, \tag{6.37}$$

then f *has a continuous extension to* x_0*.*

Proof. Let Γ_ε be the family of all open arcs (injective paths) joining $B_0=\overline{\mathbb{R}^n}\backslash B(x_0,\varepsilon_0)$ and $B_\varepsilon=\overline{B(x_0,\varepsilon)}$ in the ring $A_\varepsilon=\{x\in\mathbb{R}^n:\varepsilon<|x-x_0|<\varepsilon_0\}$, where $\varepsilon_0=\delta(x_0)$, and let Γ°_ε be the family of the corresponding dashed lines in $A_\varepsilon\backslash X$ obtained from the arcs of Γ_ε by the rejection of all the points in X. Let $\psi^*_{x_0,\varepsilon}$ be Borel functions such that $\psi^*_{x_0,\varepsilon}(t)=\psi_{x_0,\varepsilon}(t)$ for a.e. $t\in(0,\infty)$; see Section 2.3.5 in [55] and [284], p. 69. Then the function

$$\rho_\varepsilon(x)=\begin{cases}\psi^*_{x_0,\varepsilon}(|x-x_0|)/I(x_0,\varepsilon) & \text{if } x\in A_\varepsilon\setminus X,\\ 0 & \text{if } x\in\mathbb{R}^n\backslash(A_\varepsilon\setminus X)\end{cases} \tag{6.38}$$

is admissible for Γ°_ε because X is of length zero (see, e.g., Remark 30.11 in [316]), and, hence,

$$M(f\Gamma^\circ_\varepsilon)\le\int\limits_{D\backslash X}Q(x)\cdot\rho^n_\varepsilon(|x|)\ \ dm(x). \tag{6.39}$$

Consequently,

$$\lim_{\varepsilon\to 0}M(f\Gamma^\circ_\varepsilon)=0 \tag{6.40}$$

by condition (6.36).

Denote by Γ^*_ε the family of all open arcs in $\overline{\mathbb{R}^n}$ joining the continua $\overline{f(B_0)}$ and $\overline{f(B_\varepsilon)}$. Then, as is clear from Corollary 6.11,

$$f\Gamma^\circ_\varepsilon\le\Gamma^*_\varepsilon \tag{6.41}$$

and, consequently,

$$M(f\Gamma^\circ_\varepsilon)\ge M(\Gamma^*_\varepsilon); \tag{6.42}$$

see Proposition 6.2.

Recall also that

$$M(\Gamma^*_\varepsilon)=M(\Gamma'_\varepsilon), \tag{6.43}$$

where Γ'_ε is the family of all paths joining $\overline{f(B_\varepsilon)}$ and $\overline{f(B_0)}$ in $\overline{\mathbb{R}^n}$; see, e.g., Remark 7.11 in [316]. On the other hand, by the Gehring lemma (see [71]; see also (7.19), Lemma 7.22, and Corollary 7.37 in [328] and Section A.1),

$$M(\Gamma'_\varepsilon) \geq a_n \Big/ \left(\log \frac{b_n}{\delta_0 \delta_\varepsilon}\right)^{n-1}, \tag{6.44}$$

where a_n and b_n are constants depending only on n, and δ_ε and δ_0 are the diameters of $\overline{f(B_\varepsilon)}$ and $\overline{f(B_0)}$ in the spherical (chordal) metric in $\overline{\mathbb{R}^n}$.

Finally, relations (6.40)-(6.44) imply that $\delta_\varepsilon \to 0$, i.e., $\overline{f(B_\varepsilon)}$ is contracted to a point and, thus, the assertion of the lemma follows. $\square$

Choosing in Lemma 6.6 $\psi(t) = 1/(t \log 1/t)$, we obtain by Corollaries 6.3 and 6.12 the following result.

Theorem 6.5. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let X be a closed subset of D of length zero, and let $f : D\backslash X \to \mathbb{R}^n$ be a super Q-homeomorphism. If the function $Q(x)$ has finite mean oscillation at every point $x_0 \in X$, then f has a homeomorphic extension to D.*

Corollary 6.13. *In particular, if the function $Q(x)$ is integrable in a neighborhood of every point $x_0 \in X$ and has a finite limit in the mean as $x \to x_0$, then there is a limit $f(x)$ as $x \to x_0$ for every $x_0 \in X$.*

By Corollary 6.1, we also come to the following two consequences of Theorem 6.5.

Corollary 6.14. *Let X be a closed subset of length zero in D and let*

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0,\varepsilon)} Q(x)\ dm(x)\ <\ \infty \tag{6.45}$$

for every $x_0 \in X$. Then every super Q-homeomorphism $f : D\backslash X \to \mathbb{R}^n$ has a homeomorphic extension to D.

Corollary 6.15. *Let X be a closed subset of length zero in D and let $Q(x)$ be integrable in a neighborhood of X, where every point of X is a Lebesgue point of $Q(x)$. Then every super Q-homeomorphism $f : D\backslash X \to \mathbb{R}^n$ has a homeomorphic extension to D.*

Choosing in Lemma 6.6 $\psi(t) = 1/t$, we obtain the next theorem.

Theorem 6.6. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let X be a closed subset of D of length zero, and for every $x_0 \in X$, let*

$$\int\limits_{\varepsilon<|x|<\delta(x_0)} Q(x_0+x)\ \frac{dm(x)}{|x|^n} = o\left(\left[\log \frac{1}{\varepsilon}\right]^n\right) \tag{6.46}$$

as $\varepsilon \to 0$, where $0 < \delta(x_0) < \operatorname{dist}(x_0, \partial D)$. Then every super Q-homeomorphism $f : D\backslash X \to \mathbb{R}^n$ has a homeomorphic extension to D.

Corollary 6.16. *In particular, if the singular integral*

$$\int\limits_{U} \frac{Q(y)-Q(x)}{|y-x|^n}\, dm(y) \tag{6.47}$$

is convergent for every $x \in X$ over a neighborhood U of the set X of length zero, then f has a homeomorphic extension to D.

Remark 6.4. Conditions of the type (6.20)–(6.29) can also be used with singularities of length zero. As is clear from the well-known example of the conformal mapping f of the complement of a segment onto the complement of the unit disk in $\mathbb{C}$, the condition of length zero for singular sets is essential and the results cannot be extended (without additional geometric conditions) to singular sets of a finite positive length even under the best possible maximal dilatation $K(x, f) \equiv 1$.

In this context, note the interesting work [219], which proved the removability, for bounded quasiconformal mappings in domains of $\mathbb{R}^n$, of closed sets whose projections into all coordinate hyperplanes have $(n-1)$-dimensional measure zero and which can be of positive length. However, this is possible only because of the additional geometric condition on zero projections.

The above results on the homeomorphic continuability of super Q-homeomorphisms can be extended to Q-homeomorphisms and to singular sets X of positive length only under additional conditions on the size of the cluster sets $f(X)$; see Sections 6.9.

6.7 Main Lemma on Extension to Boundary

Lemma 6.7. *Let $f : D \to \mathbb{R}^n$, $n \geq 2$, be a Q-homeomorphism and let the domain D be locally connected at $x_0 \in \partial D$ and the domain $D' = f(D)$ have a strongly accessible boundary. If*

$$\int\limits_{D_{x_0,\varepsilon}} Q(x) \cdot \psi^n_{x_0,\varepsilon}(|x-x_0|)\, dm(x) = o(I^n_{x_0}(\varepsilon)) \tag{6.48}$$

as $\varepsilon \to 0$, where $D_{x_0,\varepsilon} = \{x \in D : \varepsilon < |x-x_0| < \varepsilon_0\}$, $\varepsilon_0 < \delta(x_0) = \sup_{x\in D}|x-x_0|$, and $\psi_{x_0,\varepsilon}(t)$ are nonnegative measurable (by Lebesgue) functions on $(0,\infty)$ such that

$$0 < I_{x_0}(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{x_0,\varepsilon}(t)\, dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0)\,, \tag{6.49}$$

then f can be extended to x_0 by continuity in $\overline{\mathbb{R}^n}$.

Proof. We must show that the cluster set

$$E = C(x_0, f) = \{y \in \overline{\mathbb{R}^n} : y = \lim_{k\to\infty} f(x_k), x_k \to x_0, x_k \in D\}$$

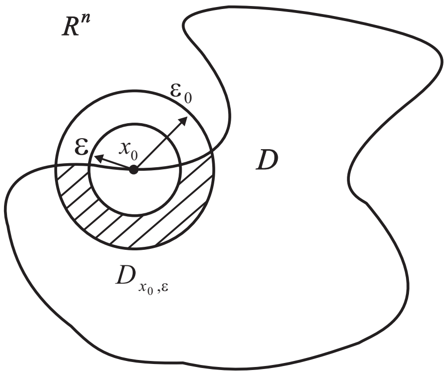

Figure 7

is a singleton. Note that E is not empty, because of the compactness of $\overline{\mathbb{R}^n}$. Let us assume that E is not degenerate, i.e., there are at least two points y_0 and $y^* \in E$. Set $U = B(x_0, r_0)$, where $0 < r_0 < |y_0 - y^*|$.

In view of the connectedness of the D at x_0, there is a sequence of neighborhoods V_m of x_0 such that $D_m = D \cap V_m$ are domains with $\delta(V_m) \to 0$ as $m \to \infty$. Then there exist points y_m and $y_m^* \in D_m' = fD_m$ close enough to y_0 and y^*, respectively, for which $|y_0 - y_m| < r_0$ and $|y_0 - y_m^*| > r_0$, $y_m \to y_0$ and $y_m^* \to y^*$ as $m \to \infty$. Let C_m be paths joining y_m and y_m^* in D_m'. Note that by the construction, $C_m \cap \partial B(x_0, r_0) \neq \varnothing$.

By the condition of strong accessibility of $\partial D'$, there are a compactum C in D' and a number $\delta > 0$ such that

$$M(\Delta(C, C_m; D')) \geq \delta$$

for large m because dist$\,(y_0, C_m) \to 0$ as $m \to \infty$. Note that $K = f^{-1}C$ is a compactum in D as a continuous image of the compactum C. Thus, $\varepsilon_0 = \mathrm{dist}(0, K) > 0$.

Let Γ_ε be the family of all paths joining K with the ball $B(\varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$ in D. Let $\psi^*_{x_0,\varepsilon}$ be a Borel function such that $\psi^*_{x_0,\varepsilon}(t) = \psi_{x_0,\varepsilon}(t)$ for a.e. $t \in (\varepsilon, \infty)$. Such functions $\psi^*_{x_0,\varepsilon}$ exist by the Lusin theorem; see, e.g., Section 2.3.5 in [55] and [284], p. 69. Then the function

$$\rho_\varepsilon(x) = \begin{cases} \psi^*_{x_0,\varepsilon}(|x - x_0|)/I_{x_0}(\varepsilon) & \text{if } x \in D_{x_0,\varepsilon}, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus D_{x_0,\varepsilon} \end{cases}$$

is admissible for Γ_ε and, hence,

$$M(f\Gamma_\varepsilon) \leq \int\limits_D Q(x) \cdot {\rho_\varepsilon}^n(x) \; dm(x),$$

i.e., $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (6.48).

On the other hand, for every fixed $\varepsilon \in (0, \varepsilon_0)$, $D_m \subset B_\varepsilon$ for large m. Hence, $C_m \subset fB_\varepsilon$ for such m and, thus, $M(f\Gamma_\varepsilon) \geq M(\Delta(C, C_m; D') \geq \delta$.

The contradiction disproves the above assumption that E is not degenerate. □

Corollary 6.17. *If, in addition, D is locally connected at ∂D and condition (6.48) holds at every point $x_0 \in \partial D$, $Q \in L^1(D)$, and $\partial D'$ is weakly flat, then f is extended to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$.*

The latter is a direct consequence of Lemmas 4.2 and 6.7.

Remark 6.5. Furthermore, by the same arguments, the assertion of Corollary 6.17 is valid if, instead of the condition $Q \in L^1(D)$, the condition $Q \in L^1(D \cap U)$ holds for some neighborhood U of ∂D.

6.8 Consequences for Quasiextremal Distance Domains

By Section 3.8, Lemmas 4.2 and 6.7, Corollary 6.17, and Remark 6.5, we obtain the following theorems.

Lemma 6.8. *Let f be a Q-homeomorphism between* QED *domains D and D' in $\mathbb{R}^n$, $n \geq 2$. If condition (6.48) holds at a point $x_0 \in \partial D$, then there is a limit of $f(x)$ as $x \to x_0$ in $\overline{\mathbb{R}^n}$.*

Corollary 6.18. *If, under the conditions of Lemma 6.8, in addition $Q \in L^1(D \cap U)$, where U is a neighborhood of ∂D and (6.48) holds at every point $x_0 \in \partial D$, then f admits a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

In particular, taking in (6.48) $\psi(t) = 1/t$, we have as a consequence of Lemma 6.8 the following theorem; cf. Remark 6.2.

Theorem 6.7. *Let f be a Q-homeomorphism between* QED *domains D and D' in $\mathbb{R}^n$, $n \geq 2$. If, at every point $x \in \partial D$,*

$$q(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right) \tag{6.50}$$

as $r \to 0$, where $q(r)$ is the mean value of $Q(y)$ over the intersection of the sphere $|y - x| = r$ with the domain D, then f extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$.

Corollary 6.19. *In particular, the assertion holds if, for every $x \in \partial D$,*

$$Q(y) = O\left(\left[\log \frac{1}{|y - x|}\right]^{n-1}\right) \tag{6.51}$$

as $y \to x$.

Similarly, choosing in (6.48) $\psi(t) = 1/tq^{\beta}(t)$, we have the next consequence of Lemma 6.8; cf. the calculations under the proof of Theorem 6.4.

Corollary 6.20. *The assertion of Theorem 6.7 remains valid if condition (6.50) is replaced by*

$$\int_0^{\varepsilon_0} \frac{dr}{rq^{\beta}(r)} = \infty \tag{6.52}$$

for some $\beta \geq 1/(n-1)$, in particular, by

$$\int_0^{\varepsilon_0} \frac{dr}{rq(r)} = \infty. \tag{6.53}$$

Corollary 6.21. *The assertion is valid if, for some $\beta \geq 1/(n-1)$ and $\alpha \geq 1$, at every point $x \in \partial D$, we have*

$$\int_0^{\varepsilon_0} \frac{dr}{rq_{\alpha}^{\beta}(r)} = \infty, \tag{6.54}$$

where

$$q_{\alpha}(r) = q_{\alpha}(x,r) = \left(\fint_{S(r)} Q^{\alpha}(y) \right)^{\frac{1}{\alpha}}, \tag{6.55}$$

$$S(r) = S_D(x,r) = \{y \in D : |y-x| = r\}. \tag{6.56}$$

Indeed, by the Jensen inequality, $q_{\alpha}(r) \geq q_1(r) = q(r)$ (see, e.g., [339], p. 20) and, thus, (6.54) implies (6.52).

Remark 6.6. Theorem 6.7 and its corollaries are valid if we take as $q(r)$ and $q_{\alpha}(r)$ the means of $Q(y)$ over the whole spheres $|y-x| = r$ formally extending $Q(y)$ by zero outside the domain D.

Bounded domains with smooth boundaries and bounded convex domains in $\mathbb{R}^n$, $n \geq 2$, satisfy the condition of doubling measure (6.7) at every boundary point and hence, choosing in (6.48) $\psi(t) = 1/(\log 1/t)$, we obtain by Lemma 6.8 and Corollary 6.3 the following theorem.

Theorem 6.8. *Let f be a Q-homeomorphism between bounded domains D and D' in $\mathbb{R}^n$, $n \geq 2$, with smooth boundaries. If $Q(x) \in L^1(D)$ has finite mean oscillation at every point $x_0 \in \partial D$, then f has a homeomorphic extension to the closure of D in $\overline{\mathbb{R}^n}$.*

Theorem 6.9. *Let f be a Q-homeomorphism between bounded convex domains D and D' in $\mathbb{R}^n$, $n \geq 2$. If $Q(x) \in L^1(D)$ has finite mean oscillation at every point $x_0 \in \partial D$, then f has a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Corollary 6.22. *If f is a Q-homeomorphism of the unit ball $\mathbb{B}^n$, $n \geq 2$, onto itself, where $Q \in L^1(\mathbb{B}^n)$ has finite mean oscillation at every point $x_0 \in \partial\mathbb{B}^n$, then f admits a homeomorphic extension $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$.*

6.9 On Singular Null Sets for Extremal Distances

Recall that a closed set $X \subset \mathbb{R}^n$, $n \geq 2$, is called a **null set for extremal distances**, abbr. an **NED set**, if

$$M(\Delta(E,F;\mathbb{R}^n)) = M(\Delta(E,F;\mathbb{R}^n \backslash X)) \tag{6.57}$$

for every pair of disjoint continua E and $F \subset \mathbb{R}^n \backslash X$.

Remark 6.7. It is known that if $X \subset \mathbb{R}^n$ is an NED set, then

$$|X| = 0 \tag{6.58}$$

and X does not locally disconnect $\mathbb{R}^n$, i.e., see [126],

$$\dim X \leq n-2. \tag{6.59}$$

Conversely, if $X \subset \mathbb{R}^n$ is closed and

$$\Lambda_{n-1}(X) = 0, \tag{6.60}$$

then X is an NED set; see [317].

Here $\Lambda_{n-1}(X)$ denotes the $(n-1)$-dimensional Hausdorff measure of a subset X in $\mathbb{R}^n$. We also denote by $f(X)$ the cluster set of a mapping $f : D \to \overline{\mathbb{R}^n}$ for a set $X \subset \overline{D}$,

$$C(X,f) := \{ y \in \overline{\mathbb{R}^n} : y = \lim_{k \to \infty} f(x_k),\, x_k \to x_0 \in X \}. \tag{6.61}$$

Note that the complements of NED sets in $\mathbb{R}^n$ are a very particular case of QED domains considered in the previous section. Thus, arguing locally, we obtain as in Section 6.8 by Lemmas 4.2 and 6.7, Corollary 6.17, and Remark 6.5, the following statement.

Lemma 6.9. *Let f be a Q-homeomorphism of $D \backslash X$ into $\overline{\mathbb{R}^n}$, $n \geq 2$, and let $X \subset D$. Suppose that X and $C(X,f)$ are* NED *sets and Q is integrable in a neighborhood of the set X. If condition (6.48) holds at every point $x_0 \in X$, then f has a homeomorphic extension to D in $\overline{\mathbb{R}^n}$.*

By Corollaries 6.1 and 6.3, choosing $\psi(t) = 1/(t \log 1/t)$ in (6.48), we have as a consequence of Lemma 6.9 the following theorem.

Theorem 6.10. *If $Q \in L^1_{\mathrm{loc}}(D)$ has finite mean oscillation at each point of an* NED *set $X \subset D$, then every Q-homeomorphism f of $D \backslash X$ into $\overline{\mathbb{R}^n}$ with an* NED *set $C(X,f)$ has a homeomorphic extension to D in $\overline{\mathbb{R}^n}$.*

In view of Remark 6.7, we obtain the following consequence of Theorem 6.10.

Corollary 6.23. *In particular, the assertion of Theorem 6.10 holds if X is a closed subset of D with*

$$\Lambda_{n-1}(X) = \Lambda_{n-1}(C(X,f)) = 0 \tag{6.62}$$

and $Q \in L^1_{\mathrm{loc}}(D)$ has finite mean oscillation at every point $x \in X$.

In particular, by Corollary 6.1, we come to the next consequence.

Corollary 6.24. *If all points of a closed set $X \subset D$ with condition (6.62) are Lebesgue points for the function $Q \in L^1_{\mathrm{loc}}(D)$, then the Q-homeomorphism f of $D \backslash X$ into $\overline{\mathbb{R}^n}$ admits a homeomorphic extension to D in $\overline{\mathbb{R}^n}$.*

By Lemma 6.9 under $\psi(t) = 1/t$, we also have the next statement.

Corollary 6.25. *If the singular integral*

$$\int\limits_U \frac{Q(y) - Q(x)}{|y-x|^n}\, dm(y) \tag{6.63}$$

is convergent for every x of a closed set $X \subset D$ over a neighborhood U of the set X, then under condition (6.62), every Q-homeomorphism f of $D \backslash X$ into $\overline{\mathbb{R}^n}$ has homeomorphic extension to D in $\overline{\mathbb{R}^n}$.

Remark 6.8. In the same way, by Lemma 6.9, analogies of all other theorems in Sections 6.4 and 6.8 can be obtained, too. In particular, if at least one of the conditions (6.50), (6.51), (6.52), (6.53), and (6.54) holds at every point x of a closed set $X \subset D$, then under condition (6.62), every Q-homeomorphism f of $D \backslash X$ into $\overline{\mathbb{R}^n}$ admits a homeomorphic extension to D in $\overline{\mathbb{R}^n}$.

Using the known term from the theory of analytic functions, we say that the given types of singularities at X of a Q-homeomorphism f of $D \backslash X$ into $\overline{\mathbb{R}^n}$ are **unessential**, i.e., f is extended to X by continuity to a homeomorphism of D into $\overline{\mathbb{R}^n}$.

6.10 Applications to Mappings in Sobolev Classes

The results of this chapter are applicable, in particular, to BMO-quasiconformal mappings and homeomorphisms of finite length distortion and finite area distortion; see Chapters 4, 8, and 10. They have also a number of consequences for other classes of mappings with finite distortion. Let us give some of these applications explicitly. They are based on Corollary 6.4 to Theorem 6.1 and the corresponding results in Sections 6.4 and 6.6–6.9 of this chapter. Recall that singular sets of $(n-1)$-dimensional Hausdorff measure zero are removable for the Sobolev class $W^{1,n}$; see, e.g., [216], p. 16.

Let us begin with isolated singularities of homeomorphisms in the local Sobolev class $W^{1,n}_{\mathrm{loc}}$. Directly by Corollary 6.4 and Theorem 6.2, we have the following theorem.

Theorem 6.11. *Let f be a homeomorphism of $D\setminus\{x_0\}$ into $\overline{\mathbb{R}^n}$, $n \geq 2$, of the class $W^{1,n}_{\mathrm{loc}}$ and let its inner dilatation $K_I(x,f)$ be majorized by a function $Q(x)$ with finite mean oscillation at $x_0 \in D$. Then f is extended to a Q-homeomorphism of D into $\overline{\mathbb{R}^n}$.*

By Proposition 6.1 and Corollary 6.1, respectively, we obtain the following two consequences of Theorem 6.11.

Corollary 6.26. *In particular, the assertion holds if x_0 is a Lebesgue point either of K_I or of a majorant of K_I in a neighborhood of x_0.*

Corollary 6.27. *If f is a homeomorphism of $\mathbb{B}^n\setminus\{0\}$ into $\overline{\mathbb{R}^n}$, $n \geq 2$, of the class $W^{1,n}_{\mathrm{loc}}$ with*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B(\varepsilon)} K_I(x,f)\, dm(x) \;<\; \infty, \tag{6.64}$$

then f is extended to a Q-homeomorphism of $\mathbb{B}^n$ into $\overline{\mathbb{R}^n}$.

Analogies of the known Painleve theorem also take place for such classes; cf. [24]. The following theorem is a direct consequence of Corollary 6.4 and Theorem 6.5.

Theorem 6.12. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let X be a closed subset of D of length zero, and let f be a homeomorphism of $D\setminus X$ into $\overline{\mathbb{R}^n}$ of the class $W^{1,n}_{\mathrm{loc}}$. If $K_I(x,f) \leq Q(x)$ and the majorant $Q(x)$ has finite mean oscillation at every point $x_0 \in X$, then f is extended to a homeomorphism of D into $\overline{\mathbb{R}^n}$.*

The following two corollaries of Theorem 6.12 follow by Proposition 6.1 and Corollary 6.1, respectively.

Corollary 6.28. *Let X be a closed subset of length zero in D and let f be a homeomorphism of $D\setminus X$ into $\overline{\mathbb{R}^n}$, $n \geq 2$, of the class $W^{1,n}_{\mathrm{loc}}$ such that every point of X is a Lebesgue point for $K_I(x,f)$. Then f is extended to a homeomorphism of D into $\overline{\mathbb{R}^n}$.*

Corollary 6.29. *Let X be a closed subset of length zero in D and let*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B(x_0,\varepsilon)} Q(x)\;\; dm(x) \;<\; \infty \tag{6.65}$$

for every $x_0 \in X$. Then every homeomorphism f of $D\setminus X$ into $\overline{\mathbb{R}^n}$ of the class $W^{1,n}_{\mathrm{loc}}$ with $K_I(x,f) \leq Q(x)$ a.e. is extended to a homeomorphism of D into $\overline{\mathbb{R}^n}$.

For a singular set X with positive length, it is necessary to request additional conditions on its cluster set $C(X,f)$ under the mapping f; see (6.61). The following theorem is obtained directly by Corollaries 6.4 and 6.23.

Theorem 6.13. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let f be a homeomorphism of $D\setminus X$ into $\overline{\mathbb{R}^n}$, $n \geq 2$, in the class $W^{1,n}_{\mathrm{loc}}$, and let X be a closed subset of D such that*

$$\Lambda_{n-1}(X) = \Lambda_{n-1}(C(X,f)) = 0. \tag{6.66}$$

If $K_I(x,f) \le Q(x)$ and the majorant $Q \in L^1_{\mathrm{loc}}$ has finite mean oscillation at every point $x_0 \in X$, then f can be extended to a homeomorphism of D into $\overline{\mathbb{R}^n}$.

By Proposition 6.1 and Corollary 6.1, we also have the following two corollaries of Theorem 6.13.

Corollary 6.30. *If all points of a closed set $X \subset D$ with condition (6.66) are Lebesgue points for $K_I(x,f) \in L^1_{\mathrm{loc}}(D)$, then the homeomorphism f of $D\backslash X$ into $\overline{\mathbb{R}^n}$ of the class $W^{1,n}_{\mathrm{loc}}(D\backslash X)$ admits a homeomorphic extension to D in $\overline{\mathbb{R}^n}$.*

Corollary 6.31. *If a closed set $X \subset D$ with condition (6.66) also satisfies the condition*

$$\overline{\lim_{\varepsilon\to 0}} \ -\!\!\!\!\!\!\int_{B(x_0,\varepsilon)} K_I(x,f)\ dm(x) \ < \ \infty \tag{6.67}$$

for every $x_0 \in X$, then the homeomorphism $f \in W^{1,n}_{\mathrm{loc}}(D\backslash X)$ has a homeomorphic extension to D in $\overline{\mathbb{R}^n}$.

Finally, the homeomorphic extension of homeomorphisms $f \in W^{1,n}_{\mathrm{loc}}$ to hard boundaries is also possible under the corresponding conditions on $K_I(x,f)$ at the boundary points but with suitable geometric conditions on the boundaries. We restrict ourselves to the simplest cases; cf. Section 6.8. Namely, bounded domains with smooth boundaries and bounded convex domains satisfy the condition of doubling measure (6.7) at all boundary points. Hence, combining Lemma 6.1 and Corollary 6.17 with Theorems 6.8 and 6.9, respectively, we obtain the following two theorems.

Theorem 6.14. *Let $f \in W^{1,n}_{\mathrm{loc}}(D)$ be a homeomorphism between bounded domains D and D' in $\mathbb{R}^n$, $n \ge 2$, with smooth boundaries. If $K_I(x,f) \le Q(x)$, where $Q(x) \in L^1(D)$ has finite mean oscillation at every point $x_0 \in \partial D$, then f has a homeomorphic extension to the closure of D onto the closure of D'.*

Theorem 6.15. *Let $f \in W^{1,n}_{\mathrm{loc}}(D)$ be a homeomorphism between bounded convex domains D and D' in $\mathbb{R}^n$, $n \ge 2$. If $K_I(x,f) \le Q(x)$, where $Q(x) \in L^1(D)$ has finite mean oscillation at every point $x_0 \in \partial D$, then f has a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Corollary 6.32. *If $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n)$ is a homeomorphism of the unit ball $\mathbb{B}^n$, $n \ge 2$, onto itself such that $K_I(x,f) \le Q(x)$, where $Q \in L^1(\mathbb{B}^n)$ has finite mean oscillation at every point $x_0 \in \partial\mathbb{B}^n$, then f admits a homeomorphic extension $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$.*

In particular, by Corollary 6.1, we come to the following statement.

Corollary 6.33. *If a homeomorphism $f \in W^{1,n}(\mathbb{B}^n)$ of the unit ball $\mathbb{B}^n$, $n \ge 2$, onto itself, $f(0) = 0$, satisfies the condition*

$$\overline{\lim_{\varepsilon\to 0}} \ -\!\!\!\!\!\!\int_{B^*(x_0,\varepsilon)} K_I(x,f)\ dm(x) \ < \ \infty\,, \qquad \forall x_0 \in \partial\mathbb{B}^n, \tag{6.68}$$

where $B^*(x_0,\varepsilon) = B(x_0,\varepsilon) \cap \mathbb{B}^n$, *then its extension by reflection through* $\partial\mathbb{B}^n$ *is a homeomorphism of* $\mathbb{R}^n$ *of the class* $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$.

Remark 6.9. Of course, the list of consequences could be continued. By Theorem 6.1 and Corollary 6.4, all of the above results for Q-homeomorphisms in Sections 6.4 and 6.6-6.9 hold for homeomorphisms f in the Sobolev class $W^{1,n}_{\mathrm{loc}}$ such that $f^{-1} \in W^{1,n}_{\mathrm{loc}}$ and, in particular, for $f \in W^{1,n}_{\mathrm{loc}}$ with $K_I \in L^1_{\mathrm{loc}}$ if

$$K_I(x,f) \le Q(x). \tag{6.69}$$

Chapter 7
Ring Q-Homeomorphisms

In this chapter we develop the theory of normal families of ring Q-homeomorphisms including Q-homeomorphisms that was first started in the plane (see [275,277,280]) and then in space (see [267–270]). As is well known, normal families take an important role in the research of the local and boundary behavior of mappings as well as the problem of existence of solutions for various differential equation; see, e.g., Chapter 11. Their investigation is closely related with equicontinuous families and, thus, with estimations of the distortion in the corresponding classes of mappings.

7.1 Introduction

Given a domain D and two sets E and F in $\overline{\mathbb{R}^n}$, $n \geq 2$, $\Delta(E,F,D)$ denotes the family of all paths $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ that join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma(t) \in D$ for $a < t < b$. We set $\Delta(E,F) = \Delta(E,F,\overline{\mathbb{R}^n})$ if $D = \overline{\mathbb{R}^n}$. A **ring domain**, or shortly a **ring** in $\overline{\mathbb{R}^n}$, is a doubly connected domain R in $\overline{\mathbb{R}^n}$. Let R be a ring in $\overline{\mathbb{R}^n}$. If C_1 and C_2 are the connected components of $\overline{\mathbb{R}^n} \setminus R$, we write $R = R(C_1,C_2)$. The **(conformal) capacity** of R can be defined by the equality

$$\operatorname{cap} R(C_1,C_2) \;=\; M(\Delta(C_1,C_2,R)); \tag{7.1}$$

see, e.g., [122] and Section A.3. Note that

$$M(\Delta(C_1,C_2,R)) \;=\; M(\Delta(C_1,C_2)); \tag{7.2}$$

see, e.g., Theorem 11.3 in [316].

Motivated by the ring definition of quasiconformality in [66], we introduce the following notion that in a natural way localizes and generalizes the notion of a Q-homeomorphism. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [0,\infty]$ a measurable function. Set

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_7,

$$A(r_1,r_2,x_0) = \{x \in \mathbb{R}^n : r_1 < |x-x_0| < r_2\}, \tag{7.3}$$
$$S(x_0,r_i) = \{x \in \mathbb{R}^n : |x-x_0| = r_i\}, \quad i=1,2. \tag{7.4}$$

Given domains D in $\mathbb{R}^n$ and D' in $\overline{\mathbb{R}^n}$, $n \geq 2$, we say that a homeomorphism $f : D \to D'$ is a **ring Q-homeomorphism** at a point $x_0 \in D$ if

$$M(\Delta(fS_1, fS_2, fD)) \leq \int\limits_A Q(x) \cdot \eta^n(|x-x_0|)\, dm(x) \tag{7.5}$$

for every ring $A = A(r_1,r_2,x_0)$, $0 < r_1 < r_2 < d_0 = \mathrm{dist}(x_0, \partial D)$, and for every measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1. \tag{7.6}$$

Note that every Q-homeomorphism $f : D \to D'$ is a ring Q-homeomorphism at every point $x_0 \in D$, but the inverse conclusion, generally speaking, is not true.

7.2 On Normal Families of Maps in Metric Spaces

First give some general facts on normal families of mappings in metric spaces. Let (X,d) and $(X^{'},d^{'})$ be metric spaces with distances d and $d^{'}$, respectively. A family $\mathfrak{F}$ of continuous mappings $f : X \to X^{'}$ is said to be **normal** if every sequence of mappings $f_m \in \mathfrak{F}$ has subsequence f_{m_k} converging uniformly on each compact set $C \subset X$ to a continuous mapping. Normality is closely related to the following. A family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ is said to be **equicontinuous at a point** $x_0 \in X$ if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $d^{'}(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x,x_0) < \delta$. The family $\mathfrak{F}$ is **equicontinuous** if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

Proposition 7.1. *Let (X,d) and $(X^{'},d^{'})$ be arbitrary metric spaces and let $\mathfrak{F}$ be a normal family of mappings $f : X \to X^{'}$. Then $\mathfrak{F}$ is equicontinuous.*

Proof. Indeed let us assume that there exist $x_0 \in X$, $\varepsilon_0 > 0$ and sequences of mappings $f_m \in \mathfrak{F}$ and points $x_m \in X$ such that $x_m \to x_0$ and $d^{'}(f_m(x_m), f_m(x_0)) \geq \varepsilon_0$. Without loss of generality, we may consider that $f_m \to f$ uniformly on each compact set $C \subset X$, where f is a continuous mapping. However, $\bigcup\{x_m\}$ is a compact set. Hence, $d^{'}(f_m(x_0), f(x_0)) < \varepsilon_0/3$ and $d^{'}(f_m(x_m), f(x_m)) < \varepsilon_0/3$ for all great enough m. Moreover, $d^{'}(f(x_m), f(x_0)) < \varepsilon_0/3$ by the continuity of the mapping f and, consequently, $d^{'}(f_m(x_m), f_m(x_0)) \leq \varepsilon_0$ by the triangle inequality. The latter contradicts the above assumption. $\square$

A family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ is said to be **uniformly equicontinuous on a set** $E \subset X$ if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $d^{'}(f(x), f(x^{'})) < \varepsilon$ for all $f \in \mathfrak{F}$ and for all x and $x^{'} \in E$ with $d(x,x^{'}) < \delta$.

Lemma 7.1. *Let (X,d) and $(X^{'},d^{'})$ be metric spaces and let $\mathfrak{F}$ be a family of equicontinuous mappings $f : X \to X^{'}$. Then $\mathfrak{F}$ is uniformly equicontinuous on every compact set $C \subset X$.*

Corollary 7.1. *Normal families of mappings between metric spaces are uniformly equicontinuous on compacts.*

Proof of Lemma 7.1. Let us assume that there exist a compact set $C \in X$, a number $\varepsilon_0 > 0$, and sequences of mappings $f_m \in \mathfrak{F}$ and of points $x_m, x_m^{'} \in C$ such that $d(x_m, x_m^{'}) \to 0$ as $m \to \infty$ and $d^{'}(f_m(x_m), f_m(x_m^{'})) \geq \varepsilon_0$. Without loss of generality, we may assume that $x_m \to x_0$ and $x_m^{'} \to x_0 \in C$ because C is compact. Then $d^{'}(f_m(x_m), f_m(x_0)) < \varepsilon_0/2$ and $d^{'}(f_m(x_0), f_m(x_m^{'})) < \varepsilon_0/2$ for great enough m, which contradicts the above assumption. □

The function

$$\omega_E(t) = \omega_E^{\mathfrak{F}}(t) = \sup d^{'}(f(x), f(z)), \tag{7.7}$$

where the supremum is taken over all $x, z \in E$ such that $d(x,z) \leq t$ and $f \in \mathfrak{F}$, is called the **modulus of continuity of the family $\mathfrak{F}$ on the set** E.

Similarly, the function

$$\omega_{x_0}(t) = \omega_{x_0}^{\mathfrak{F}}(t) = \sup d^{'}(f(x_0), f(x)), \tag{7.8}$$

where the supremum is taken over all $x \in X$ and $f \in \mathfrak{F}$ such that $d(x,x_0) \leq t$, is called the **modulus of continuity of $\mathfrak{F}$ at the point** $x_0 \in X$.

Note that, by definition, ω_E and ω_{x_0} are nonnegative, nondecreasing, and continuous from the right. Note also that $\omega_{x_0}(t) \to 0$ as $t \to 0$ for every $x_0 \in X$ if the family $\mathfrak{F}$ is equicontinuous. Moreover, the following statement follows from Lemma 7.1.

Corollary 7.2. *If a family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ is equicontinuous, then $\omega_C(t) \to 0$ as $t \to 0$ for every compact set $C \subset X$.*

The next statement is also obvious.

Proposition 7.2. *Let (X,d) and $(X^{'},d^{'})$ be metric spaces and let $\overline{\mathfrak{F}}$ be a closure of a family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ with respect to the pointwise convergence in X. Then the moduli of continuity (7.7) and (7.8) of $\overline{\mathfrak{F}}$ and $\mathfrak{F}$ coincide.*

Corollary 7.3. *If a sequence of mappings $f_m : X \to X^{'}$, $m = 1,2\ldots$ is equicontinuous and $f_m(x) \to f(x)$ as $m \to \infty$ for every $x \in X$, then the limit function $f : X \to X^{'}$ is continuous.*

A sequence of mappings $f_m : X \to X^{'}$, $m = 1,2\ldots$, is called **continuously convergent** to $f : X \to X^{'}$, if $f_m(x_m) \to f(x_0)$ as $m \to \infty$ for every convergent sequence of points $x_m \to x_0$ in X.

Remark 7.1. The uniform convergence of continuous mappings on compact sets always implies the continuous convergence because $\cup_{m=0}^{\infty}\{x_m\}$ is a compact set as $x_m \to x_0$ and because, by the triangle inequality,

$$d^{'}(f_m(x_m), f(x_0)) \leq d^{'}(f_m(x_m), f(x_m)) + d^{'}(f(x_m), f(x_0)). \tag{7.9}$$

If the second space $X^{'}$ has a countable basis at each point, say if $X^{'}$ is separable, then the convergences are equivalent; see, e.g., [50], p. 268. It is also obvious that the continuous convergence implies pointwise convergence. The converse conclusion is, generally speaking, not true, as shown by the example $f_m(x) = x^m$, $x \in [0,1]$: $f_m(x) \to 0$ for $x < 1$ and $f_k(1) \to 1$, but $f_m(x_m) \equiv 1/2$ for $x_m = 2^{-1/m} \to 1$ as $m \to \infty$.

The following theorem shows that all three convergences are equivalent for equicontinuous families of mappings in arbitrary metric spaces.

Theorem 7.1. *Let (X,d) and $(X^{'},d^{'})$ be metric spaces and let $\mathfrak{F}$ be an equicontinuous family of mappings $f: X \to X^{'}$. Then the following statements are equivalent for all sequences $f_m \in \mathfrak{F}$:*

(1) f_m converges uniformly on every compact set;

(2) f_m converges continuously;

(3) f_m converges at every point $x \in X$.

Corollary 7.4. *The closures $\overline{\mathfrak{F}}$ of equicontinuous families $\mathfrak{F}$ with respect to the pointwise convergence and the uniform convergence on compact sets coincide in arbitrary metric spaces.*

Proof of Theorem 7.1. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious; see Remark 7.1. Thus, it remains to prove the implication $(3) \Rightarrow (1)$. Indeed, let us assume there is a sequence $f_m \in \mathfrak{F}$ such that $f_m(x) \to f(x)$ as $m \to \infty$ for every $x \in X$ and, simultaneously, for a compact set $C \subset X$, there is a number $\varepsilon_0 > 0$ such that $d^{'}(f_m(x_m), f(x_m)) \geq \varepsilon_0$ for some sequence of points $x_m \in C$. Without loss of generality, we may consider that $x_m \to x_0 \in C$ as $m \to \infty$. However, by the triangle inequality, $d^{'}(f_m(x_m), f(x_m)) \leq d^{'}(f_m(x_m), f_m(x_0)) + d^{'}(f_m(x_0), f(x_0)) + d^{'}(f(x_0), f(x_m))$ and by Corollaries 7.2 and 7.3, we come to the contradiction with the above assumption. □

Lemma 7.2. *Let (X,d) be a metric space, let a set $E \subset X$ be dense everywhere in X, and let $(X^{'},d^{'})$ be a complete metric space. If an equicontinuous sequence of mappings $f_m : X \to X^{'}$ is pointwise convergent on the set E, then f_m converges uniformly on every compact set $C \subset X$.*

Proof. In view of Theorem 7.1, it is sufficient to prove that the pointwise convergence of f_m on E implies the pointwise convergence of f_m on X. Indeed, for every $x_0 \in X \backslash E$, there is a sequence $x_k \in E$ such that $x_k \to x_0$ as $k \to \infty$ because E is dense in X. By the equicontinuity of f_m, for every $\varepsilon > 0$, there is $K = K(\varepsilon)$ such that $d^{'}(f_m(x_k), f_m(x_0)) < \varepsilon/3$ for all $k \geq K$ and all $m = 1, 2, \ldots$. Let us fix $k_0 \geq K$. By the Cauchy criterion, we have $d^{'}(f_n(x_{k_0}), f_m(x_{k_0})) < \varepsilon/3$ for all n and $m \geq N = N(\varepsilon, k_0)$. Finally, by the triangle inequality,

$$\begin{aligned} d^{'}(f_n(x_0), f_m(x_0)) \; &\leq d^{'}(f_n(x_0), f_n(x_{k_0})) \, + \, d^{'}(f_n(x_{k_0}), f_m(x_{k_0})) \\ &+ d^{'}(f_m(x_{k_0}), f_m(x_0)) \, < \, \varepsilon \end{aligned}$$

for all n and $m \geq N$, i.e., the sequence $f_m(x_0)$ is fundamental and hence is convergent by the completeness of the space $X^{'}$. □

As is well known, every compact metric space is complete; see, e.g., Theorem 3 in Section 33, II, [185]. Thus, using the diagonal process, we obtain the following consequence of Proposition 7.1 and Lemma 7.2.

Corollary 7.5. *If (X, d) is a separable metric space and $(X^{'}, d^{'})$ is a compact metric space, then a family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

7.3 Characterization of Ring Q-Homeomorphisms

Here we use the standard conventions $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0 \cdot \infty = 0$; see, e.g., [281], p. 6.

Lemma 7.3. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to [0, \infty]$ a measurable function, and $q_{x_0}(r)$ the mean of $Q(x)$ over the sphere $|x - x_0| = r$. Set*

$$I \, = \, I(r_1, r_2) \, = \int\limits_{r_1}^{r_2} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \tag{7.10}$$

and $S_j = \{x \in \mathbb{R}^n : |x - x_0| = r_j\}$, $j = 1, 2$, where $x_0 \in D$ and $0 < r_1 < r_2 < d_0 =$ $\operatorname{dist}(x_0, \partial D)$. Then

$$M(\Delta(fS_1, fS_2, fD)) \, \leq \, \frac{\omega_{n-1}}{I^{n-1}} \tag{7.11}$$

whenever $f : D \to \mathbb{R}^n$ is a ring Q-homeomorphism, where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. With no loss of generality, we may assume that $I \neq 0$ because otherwise (7.11) is trivial, and that $I \neq \infty$ because otherwise we can replace $Q(z)$ by $Q(z) + \delta$ with arbitrarily small $\delta > 0$ and then take the limit as $\delta \to 0$ in (7.11). The condition $I \neq \infty$ implies, in particular, that $q(r) \neq 0$ a.e. in (r_1, r_2). Set

$$\psi(t) = \begin{cases} 1/[t q_{x_0}^{\frac{1}{n-1}}(t)], & t \in (r_1, r_2), \\ 0, & \text{otherwise}. \end{cases} \tag{7.12}$$

Then

$$\int_A Q(x) \cdot \psi^n(|x-x_0|)\, dm(x) \;=\; \omega_{n-1} I, \tag{7.13}$$

where

$$A \;=\; A(r_1,r_2,x_0) \;=\; \{x \in \mathbb{R}^n : r_1 < |x-x_0| < r_2\}. \tag{7.14}$$

Let Γ be a family of all paths joining the spheres S_1 and S_2 in A. Also let ψ^* be a Borel function such that $\psi^*(t) = \psi(t)$ for a.e. $t \in [0,\infty]$. Such a function ψ^* exists by the Lusin theorem; see, e.g., Section 2.3.5 in [64] and [281], p. 69. Then the function

$$\rho(x) \;= \psi^*(|x-x_0|)/I$$

is admissible for $\Gamma$and, since f is a ring Q-homeomorphism, we get by (7.13) that

$$M(f\Gamma) \;\le\; \int_A Q(x) \cdot \rho^n(x)\, dm(x) \;=\frac{\omega_{n-1}}{I^{n-1}}\,.$$

□

The following lemma shows that the estimate (7.11) cannot be improved for ring Q-homeomorphisms.

Lemma 7.4. *Fix* $x_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < r_0$, $A = \{x \in \mathbb{R}^n : r_1 < |x-x_0| < r_2\}$, $B = B(x_0,r_0) = \{x \in \mathbb{R}^n : |x-x_0| < r_0\}$, *and suppose that* $Q : D \to [0,\infty]$ *is a measurable function. Set*

$$\eta_0(r) = \frac{1}{Irq_{x_0}^{\frac{1}{n-1}}(r)}, \tag{7.15}$$

where $q(r)$ *is the mean of* $Q(x)$ *over the sphere* $|x-x_0| = r$ *and* I *is as in Lemma 7.3. Then*

$$\frac{\omega_{n-1}}{I^{n-1}} \;=\; \int_A Q(x) \cdot \eta_0^n(|x-x_0|)\, dm(x) \;\le\; \int_A Q(x) \cdot \eta^n(|x-x_0|)\, dm(x) \tag{7.16}$$

whenever $\eta : (r_1,r_2) \to [0,\infty]$ *is such that*

$$\int_{r_1}^{r_2} \eta(r)\, dr \;= 1. \tag{7.17}$$

Proof. If $I = \infty$, then the left-hand side in (7.16) is equal to zero and the inequality is obvious. If $I = 0$, then $q_{x_0}(r) = \infty$ for a.e. $r \in (r_1,r_2)$ and both sides in (7.16) are equal to ∞. Hence, we may assume below that $0 < I < \infty$.

Now, by (7.10) and (7.17), $q_{x_0}(r) \neq 0$ and $\eta(r) \neq \infty$ a.e. in (r_1,r_2). Set $\alpha(r) = rq_{x_0}^{\frac{1}{n-1}}(r)\eta(r)$ and $w(r) = 1/rq_{x_0}^{\frac{1}{n-1}}(r)$. Then, by the standard conventions, $\eta(r) = \alpha(r)w(r)$ a.e. in (r_1,r_2) and

$$C := \int_A Q(x) \cdot \eta^n(|x-x_0|)\, dm(x) = \omega_{n-1} \int_{r_1}^{r_2} \alpha^n(r) \cdot w(r)\, dr. \tag{7.18}$$

By Jensen's inequality with weights (see, e.g., Theorem 2.6.2 in [252]), applied to the convex function $\varphi(t) = t^n$ in the interval $\Omega = (r_1, r_2)$ with the probability measure

$$\nu(E) \;=\; \frac{1}{I} \int_E w(r)\, dr, \tag{7.19}$$

we obtain

$$\left(\fint \alpha^n(r) w(r)\, dr \right)^{1/n} \;\geq\; \fint \alpha(r) w(r)\, dr \;=\; \frac{1}{I}, \tag{7.20}$$

where we also used the fact that $\eta(r) = \alpha(r)w(r)$ satisfies (7.17). Thus,

$$C \;\geq\; \frac{\omega_{n-1}}{I^{n-1}}, \tag{7.21}$$

and the proof is complete. □

Theorem 7.2. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \mathbb{R}^n$ is a ring Q-homeomorphism at a point x_0 if and only if, for every* $0 < r_1 < r_2 < d_0 = \operatorname{dist}(x_0, \partial D)$,

$$M(\Delta(fC_1, fC_2, fD)) \;\leq\; \frac{\omega_{n-1}}{I^{n-1}}, \tag{7.22}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $q_{x_0}(r)$ is the mean value of $Q(x)$ over the sphere $|x - x_0| = r$, $S_j = \{x \in \mathbb{R}^n : |x - x_0| = r_j\}$, $j = 1, 2$, and

$$I \;=\; I(r_1, r_2) \;=\; \int_{r_1}^{r_2} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)}$$

Moreover, the infimum from the right-hand side in (7.5) holds for the function

$$\eta_0(r) = \frac{1}{I r q_{x_0}^{\frac{1}{n-1}}(r)}\;. \tag{7.23}$$

7.4 Estimates of Distortion

Lemma 7.5. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let x_0 be a point in D, let $r_0 <$* $\operatorname{dist}(x_0, \partial D)$*, and let D' be a domain in $\overline{\mathbb{R}^n}$ with $h(\overline{\mathbb{R}^n} \setminus D') \geq \Delta > 0$. Then, for every homeomorphism $f : D \to D'$ and $z \in B(x_0, r_0)$,*

$$h(f(z),f(x_0)) \leq \frac{\alpha_n}{\Delta} \cdot \exp\left(-\left\{\frac{\omega_{n-1}}{M(\Delta(fS,fS_0,fD))}\right\}^{\frac{1}{n-1}}\right), \tag{7.24}$$

where $S_0 = \{x \in \mathbb{R}^n : |x-x_0| = r_0\}$ *and* $S = \{x \in \mathbb{R}^n : |x-x_0| = |z-x_0|\}$, ω_{n-1} *is the area of the unit sphere in* $\mathbb{R}^n$, *and* α_n *depends only on* n.

Proof. Let E denote the component of $\overline{\mathbb{R}^n} \setminus fA$ containing $f(x_0)$ and F the component containing ∞, where $A = \{x \in \mathbb{R}^n : |z-x_0| < |x-x_0| < r_0\}$. By the known Gehring lemma,

$$\operatorname{cap} R(E,F) \;\geq\; \operatorname{cap} R_T\left(\frac{1}{h(E)h(F)}\right), \tag{7.25}$$

where $h(E)$ and $h(F)$ denote the spherical diameters of the continua E and F, respectively, and $R_T(t)$ is the Teichmüller ring

$$R_T(t) \;=\; \mathbb{R}^n \setminus ([-1,0] \cup [t,\infty])\,, \qquad t > 0; \tag{7.26}$$

see, e.g., Corollary 7.37 in [328] or [71], Section A.1. It is also known that

$$\operatorname{cap} R_T(t) = \frac{\omega_{n-1}}{(\log \Phi(t))^{n-1}}, \tag{7.27}$$

where the function Φ admits the good estimates:

$$t+1 \leq \Phi(t) \leq \lambda_n^2 \cdot (t+1) < 2\lambda_n^2 \cdot t, \quad t > 1\,; \tag{7.28}$$

see, e.g., [71], pp. 225–226, (7.19) and Lemma (7.22) in [328], and Section A.1. Hence, inequality (7.25) implies that

$$\operatorname{cap} R(E,F) \;\geq\; \frac{\omega_{n-1}}{\left(\log \frac{2\lambda_n^2}{h(E)h(F)}\right)^{n-1}}\,. \tag{7.29}$$

Thus,

$$h(E) \;\leq \frac{2\lambda_n^2}{h(F)}\,\exp\left(-\left\{\frac{\omega_{n-1}}{\operatorname{cap} R(E,F)}\right\}^{\frac{1}{n-1}}\right), \tag{7.30}$$

which implies the desired statement. □

Now, combining Lemmas 7.3, 7.4, and 7.5, we have the following lemma.

Lemma 7.6. *Let D be a domain in* $\mathbb{R}^n$, $n \geq 2$, *let D' be a domain in* $\overline{\mathbb{R}^n}$ *with* $h(\overline{\mathbb{R}^n} \setminus D') \geq \Delta > 0$, *and let* $f : D \to D'$ *be a ring Q-homeomorphism at a point* $x_0 \in D$. *If, for* $0 < \varepsilon_0 < \operatorname{dist}(x_0, \partial D)$,

$$\int\limits_{\varepsilon<|x-x_0|\,<\,\varepsilon_0} Q(x) \cdot \psi_\varepsilon^n(|x-x_0|)\, dm(x) \;\leq\; c \cdot I^p(\varepsilon)\,, \qquad \varepsilon \in (0,\varepsilon_0), \tag{7.31}$$

where $p \le n$ and $\psi_\varepsilon(t)$ is nonnegative on $(0,\infty)$ such that

$$0 \; < \; I(\varepsilon) \; = \; \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t)\, dt < \infty, \qquad \varepsilon \in (0,\varepsilon_0), \tag{7.32}$$

then

$$h(f(x),f(x_0)) \; \le \; \frac{\alpha_n}{\Delta} \; \exp\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|)\} \tag{7.33}$$

for all $x \in B(x_0,\varepsilon_0)$, where α_n depends only on n,

$$\beta_n = \left(\frac{\omega_{n-1}}{c}\right)^{\frac{1}{n-1}}, \quad \gamma_{n,p} = 1 - \frac{p-1}{n-1}\,. \tag{7.34}$$

Corollary 7.6. *Under the conditions of Lemma 7.6 and for $p = 1$,*

$$h(f(x),f(x_0)) \; \le \; \frac{\alpha_n}{\Delta} \; \exp\{-\beta_n I(|x-x_0|)\}. \tag{7.35}$$

By Lemmas 7.3 and 7.5, we obtain the following estimate.

Theorem 7.3. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, let D' be a domain in $\overline{\mathbb{R}^n}$ with $h(\overline{\mathbb{R}^n} \setminus D') \ge \Delta > 0$, and let $f : D \to D'$ be a ring Q-homeomorphism at a point $x_0 \in D$. Then*

$$h(f(x),f(x_0)) \; \le \; \frac{\alpha_n}{\Delta} \; \exp\left\{ - \int\limits_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \right\} \tag{7.36}$$

for $x \in B(x_0,\varepsilon(x_0))$, where $\varepsilon(x_0) < \operatorname{dist}(x_0,\partial D)$, α_n depends only on n, and $q_{x_0}(r)$ is the mean integral value of $Q(x)$ over the sphere $|x-x_0| = r$.

Remark 7.2. Of course, the mean value $q_{x_0}(r)$ of $Q(x)$ over some spheres $|x-x_0| = r$ can be infinite. However, $q_{x_0}(r)$ is measurable in the parameter r because $Q(x)$ is measurable in x, say by the Fubini theorem. Moreover, at every point $x \ne x_0$,

$$\int\limits_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \; < \; \infty \tag{7.37}$$

for any ring Q-homeomorphism because in the contrary case we would have from (7.36) that $f(x) = f(x_0)$. The integral in (7.37) can be 0 if $q_{x_0}(r) = \infty$ a.e., but then inequality (7.36) is obvious because $\alpha_n \ge 32$ and Δ as well as $h(f(x),f(x_0))$ are less than or equal to 1.

Note also that if $Q(x) \ge 1$ or at least $q_{x_0}(r) \ge 1$ a.e., then one may use any degree $\beta \ge 1/(n-1)$ and, in particular, $\beta = 1$ instead of $1/(n-1)$, in inequalities (7.36) and (7.37). Indeed, for the function

$$\psi(t) = \begin{cases} \frac{1}{tq_{x_0}^{\beta}(t)}, & t \in (0, \varepsilon_0), \\ 0, & t \in [\varepsilon_0, \infty), \end{cases} \tag{7.38}$$

we have

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \cdot \psi^n(|x-x_0|)\, dm(x) \;=\; \omega_{n-1} \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{rq_{x_0}^{\beta n-1}(r)} \;\leq\; \omega_{n-1} \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{rq^{\beta}(r)}, \tag{7.39}$$

and, thus, the conclusion follows immediately from Corollary 7.6.

Corollary 7.7. *If*

$$q_{x_0}(r) \leq \left[\log \frac{1}{r}\right]^{n-1} \tag{7.40}$$

for $r < \varepsilon(x_0) < \operatorname{dist}(x_0, \partial D)$, *then*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}} \tag{7.41}$$

for all $x \in B(x_0, \varepsilon(x_0))$.

Corollary 7.8. *If*

$$Q(x) \leq \left[\log \frac{1}{|x-x_0|}\right]^{n-1}, \quad x \in B(x_0, \varepsilon(x_0)), \tag{7.42}$$

then (7.41) holds in the ball $B(x_0, \varepsilon(x_0))$.

Remark 7.3. If, instead of (7.40) and (7.42), we have the conditions

$$q_{x_0}(r) \leq c \cdot \left[\log \frac{1}{r}\right]^{n-1} \tag{7.43}$$

and, correspondingly,

$$Q(x) \leq c \cdot \left[\log \frac{1}{|x-x_0|}\right]^{n-1}, \tag{7.44}$$

then

$$h(f(x), f(x_0)) \;\leq\; \frac{\alpha_n}{\Delta} \left[\frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}}\right]^{1/c^{1/(n-1)}}. \tag{7.45}$$

Choosing in Lemma 7.6 $\psi(t) = 1/t$ and $p = 1$, we also have the following conclusion.

Corollary 7.9. *Let* $f : \mathbb{B}^n \to \mathbb{B}^n$ *be a ring* Q*-homeomorphism such that* $f(0) = 0$ *and*

$$\int\limits_{\varepsilon<|x|<1} Q(x)\ \frac{dm(x)}{|x|^n}\ \leq\ c\ \log\frac{1}{\varepsilon}, \qquad \varepsilon\in(0,1). \tag{7.46}$$

Then

$$|f(x)|\ \leq\ 2\alpha_n\cdot|x|^{\beta_n}, \tag{7.47}$$

where α_n depends only on n and β_n is defined by (7.34).

Finally, by Lemma 7.6 and Corollary 6.3, we obtain the following estimate.

Theorem 7.4. *Let D be a domain in $\mathbb{R}^n$, $n\geq 2$, let D' be a domain in $\overline{\mathbb{R}^n}$ with $h(\overline{\mathbb{R}^n}\setminus D')\geq\Delta>0$, and let $f:D\to D'$ be a ring Q-homeomorphism at a point $x_0\in D$. If $Q(x)$ has finite mean oscillation at the point $x_0\in D$, then*

$$h(f(x),f(x_0))\leq\frac{\alpha_n}{\Delta}\left\{\frac{\log\frac{1}{\varepsilon_0}}{\log\frac{1}{|x-x_0|}}\right\}^{\beta_0} \tag{7.48}$$

for some $\varepsilon_0<\operatorname{dist}(x_0,\partial D)$ and every $x\in B(x_0,\varepsilon_0)$, where α_n depends only on n and $\beta_0>0$ depends only on the function Q.

7.5 On Normal Families of Ring Q-Homeomorphisms

Given a domain D in $\mathbb{R}^n$, $n\geq 2$, and a measurable function $Q:D\to[0,\infty]$, let $\mathfrak{R}_{Q,\Delta}(D)$ be the class of all ring Q-homeomorphisms f in D with $h(\overline{\mathbb{R}^n}\setminus fD)\geq\Delta>0$. The above results now yield the following:

Theorem 7.5. *If $Q\in$ FMO, then $\mathfrak{R}_{Q,\Delta}(D)$ is a normal family.*

Corollary 7.10. *The class $\mathfrak{R}_{Q,\Delta}(D)$ is normal if*

$$\overline{\lim_{\varepsilon\to 0}}\ \fint_{B(x_0,\varepsilon)} Q(x)\ dm(x)\ <\ \infty \qquad \forall\, x_0\in D. \tag{7.49}$$

Corollary 7.11. *The class $\mathfrak{R}_{Q,\Delta}(D)$ is normal if every $x_0\in D$ is a Lebesgue point of $Q(x)$.*

Theorem 7.6. *Let $\Delta>0$ and let $Q:D\to[0,\infty]$ be a measurable function such that*

$$\int\limits_0^{\varepsilon(x_0)}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}=\infty \tag{7.50}$$

holds at every point $x_0\in D$, where $\varepsilon(x_0)=\operatorname{dist}(x_0,\partial D)$ and $q_{x_0}(r)$ denotes the mean integral value of $Q(x)$ over the sphere $|x-x_0|=r$. Then $\mathfrak{R}_{Q,\Delta}$ forms a normal family.

Corollary 7.12. *The class $\mathfrak{R}_{Q,\Delta}(D)$ is normal if $Q(x)$ has singularities of the logarithmic type of order not greater than $n-1$ at every point $x\in D$.*

Remark 7.4. In view of Remark 7.2, if $Q(x) \geq 1$ a.e. in D, then one may use any degree $\beta \geq 1/(n-1)$, say $\beta = 1$, instead of $1/(n-1)$ in condition (7.50).

Note that all the above results hold for homeomorphisms f of the Sobolev class $W^{1,n}_{\mathrm{loc}}$ with $f^{-1} \in W^{1,n}_{\mathrm{loc}}$ under the condition that

$$K_I(x,f) \leq Q(x) \quad \text{a.e.,} \tag{7.51}$$

where K_I is the inner dilatation of the mapping f; see, e.g., Theorem 6.1. In particular, this is valid for homeomorphisms $f \in W^{1,n}_{\mathrm{loc}}$ with $K_I \in L^1_{\mathrm{loc}}$; see Corollary 6.4.

Postscript. As in the case of Q-quasiconformal mappings (cf. Theorem II.5.1 in [190]), a family $\mathfrak{R}$ of ring Q-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ in a domain $D \subset \mathbb{R}^n$, $n \geq 2$, is normal under every condition given above for Q if there is a number $\Delta > 0$ such that one of the following conditions holds:

(1) Every mapping $f \in \mathfrak{R}$ omits two values whose spherical distance is greater than Δ.

(2) Every mapping $f \in \mathfrak{R}$ omits one value w_0 and $h(w(x_i), w_0) > \Delta$, $i = 1, 2$, at two fixed points x_1 and $x_2 \in D$.

(3) At three fixed points x_1, x_2, and $x_3 \in D$, $h(w(x_i), w(x_j)) > \Delta$, $i \neq j$, $i, j = 1, 2, 3$.

In particular, $\mathfrak{R}$ is normal if all mappings $f \in \mathfrak{R}$ omit two fixed values in $\overline{\mathbb{R}^n}$.

7.6 On Strong Ring Q-Homeomorphisms

Given domains D and D' in $\overline{\mathbb{R}^n}$, $n \geq 2$, a measurable function $Q : D \to [0, \infty]$, we say that a homeomorphism $f : D \to D'$ is a **strong ring Q-homeomorphism** if

$$M(\Delta(fC_1, fC_2, fD)) \leq \int\limits_D Q(x)\, \rho^n(x)\, dm(x) \tag{7.52}$$

for two arbitrary continua C_1, C_2 in D and $\rho \in \operatorname{adm} \Delta(C_1, C_2, D)$.

Lemma 7.7. *Let f be a strong ring Q-homeomorphism of $\mathbb{B}^n$ into $\overline{\mathbb{R}^n}$ with $Q \in L^1(\mathbb{B}^n)$, $f(0) = 0$, $h\left(\overline{\mathbb{R}^n} \setminus f\mathbb{B}^n\right) \geq \delta > 0$, and $h(f(z_0), 0) \geq \delta$ for some $z_0 \in \mathbb{B}^n$. Then*

$$h(f(x), f(0)) \geq \psi(|x|) \tag{7.53}$$

for all $|x| < r = \min(|z_0|/2, 1 - |z_0|)$, where $\psi(t) : [0, \infty) \to [0, \infty)$ is a continuously increasing function with $\psi(0) = 0$ depending only on n, δ, and $\|Q\|_1$.

The proof of Lemma 7.7 is completely similar to the proof of Theorem 4.4 and hence is omitted.

Corollary 7.13. *In particular, (7.53) implies that*

$$|f(x)| \geq \psi(|x|). \tag{7.54}$$

The following statement is a generalization of the well-known theorem by Weierstrass on the locally uniform convergence of analytic functions.

Theorem 7.7. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and f_m a sequence of strong ring Q-homeomorphisms of D into $\overline{\mathbb{R}^n}$ with $Q \in L^1_{\mathrm{loc}}$ converging locally uniformly to a mapping $f : D \to \overline{\mathbb{R}^n}$. Then f is either a strong ring Q-homeomorphism or $f \equiv \mathrm{const}$ in D.*

Proof. As a locally uniform limit of continuous mappings, f is continuous. Let $f \not\equiv \mathrm{const}$.

Let us first show that f is a discrete mapping. Indeed, if f is not discrete, then there are a point $x_0 \in D$ and a sequence $x_k \in D$, $x_k \neq x_0$, with $f(x_k) = f(x_0)$, $k = 1,2,\dots$, such that $x_k \to x_0$ as $k \to \infty$. Note that the set $E_0 = \{x \in D : f(x) = f(x_0)\}$ is closed in D because f is continuous. Note also that E_0 does not coincide with D because $f \not\equiv \mathrm{const}$. Thus, we can replace x_0 by a boundary nonisolated point of the set E_0.

Without loss of generality, we may assume that $x_0 = 0$, $f_m(0) = f(0) = 0$, $\overline{\mathbb{B}^n} \subset D$ and there is, at least, one point $z_0 \in \mathbb{B}^n$ where $f(z_0) \neq 0$. By the continuity of the chordal metric,

$$h(f_m(z_0),0) \geq \delta_0/2 \qquad \forall\, m \geq M_0,$$

where $\delta_0 = h(f(z_0),0) > 0$. Since $\overline{\mathbb{B}^n}$ is a compactum in D and $f_m \to f$ uniformly in $\overline{\mathbb{B}^n}$,

$$h\left(\overline{\mathbb{R}^n} \setminus f_m\left(\overline{\mathbb{B}^n}\right)\right) \geq \delta_*/2 \qquad \forall\, m \geq M_*,$$

where $\delta_* = h\left(\overline{\mathbb{R}^n} \setminus f\left(\overline{\mathbb{B}^n}\right)\right)$. Setting $\delta = \min\{\delta_0/2,\, \delta_*/2\}$ and $M = \max\{M_0, M_*\}$, we have by Lemma 7.7 that

$$|f_m(x)| \geq \psi(|x|) \qquad \forall\, m \geq M$$

for all $x \in B(0,r)$ and $r = \min\{|z_0|/2, 1 - |z_0|\}$, where ψ is an increasing function with $\psi(0) = 0$ depending only on $\|Q\|_1$, n, and δ. Thus,

$$|f(x)| \geq \psi(|x|) \qquad \forall\, x \in B(0,r). \tag{7.55}$$

Then, in particular,

$$0 = |f(x_k)| \geq \psi(|x_k|) \qquad \forall\, k \geq k_0,$$

and, consequently, $\psi(r_k) = 0$ for $r_k = |x_k| \neq 0$, $k \geq k_0$. This contradiction shows that f is discrete.

Now, let us show that f is injective in D. Indeed, assume that there exist $x_1, x_2 \in D$, $x_1 \neq x_2$, with $f(x_1) = f(x_2)$. Let $x_2 \notin \overline{B(x_1,t)} \subset D$ for all $t \in (0,t_0]$. Then every $f_m(\partial B(x_1,t))$, $t \in (0,t_0]$, separates $f_m(x_1)$ from $f_m(x_2)$ and, consequently,

$$h(f_m(x_1), f_m(\partial B(x_1,t)) < h\left(f_m(x_1), f_m(x_2)\right).$$

Hence,

$$h(f(x_1), f(\partial B(x_1,t)) \;\leq\; h\left(f(x_1), f(x_2)\right). \tag{7.56}$$

Since $f(x_1) = f(x_2)$, it follows from *(7.56)* that, for every $t \in (0,t_0]$, there is a point $x_t \in \partial B(x_1,t)$ such that $f(x_t) = f(x_1)$. The latter contradicts the discreteness of the mapping f. The continuity of the inverse mapping f^{-1} also follows from *(7.55)*. Thus, f is a homeomorphism.

Finally, condition *(7.52)* follows by Theorem A.12; see [71]. The proof is complete. $\square$

Chapter 8
Mappings with Finite Length Distortion (FLD)

In this chapter we investigate mappings with finite length distortion, which are a natural extension of quasiregular mappings and mappings with bounded length distortion; see [207, 287, 288]; cf. also [210, 213, 256].

8.1 Introduction

For $x \in E \subset \mathbb{R}^n$ and a mapping $\varphi : E \to \mathbb{R}^n$, we set

$$L(x,\varphi) = \limsup_{y \to x, y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y-x|} \tag{8.1}$$

and

$$l(x,\varphi) = \liminf_{y \to x, y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y-x|}. \tag{8.2}$$

We assume here that D is a domain in $\mathbb{R}^n$, $n \geq 2$, and that all mappings $f : D \to \mathbb{R}^n$ are continuous.

We say that a mapping $f : D \to \mathbb{R}^n$ is of **finite metric distortion**, abbr. $f \in$ **FMD**, if f has the Lusin (N)-property and

$$0 < l(x,f) \leq L(x,f) < \infty \quad \text{a.e.} \tag{8.3}$$

Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have the (N)**-property** if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f has the (N^{-1})**-property** if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$, where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. It is said that a property P holds for **almost every (a.e.) path** γ in a family Γ if the subfamily of all paths in Γ for which P fails has modulus zero.

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_8,

If $\gamma : \Delta \to \mathbb{R}^n$ is a locally rectifiable path, then there is the unique increasing length function l_γ of Δ onto a length interval $\Delta_\gamma \subset \mathbb{R}$ with a prescribed normalization $l_\gamma(t_0) = 0 \in \Delta_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subpath $\gamma|_{[t_0,t]}$ of γ if $t > t_0$, $t \in \Delta$, and $l_\gamma(t)$ is equal to $-l(\gamma|_{[t,t_0]})$ if $t < t_0$, $t \in \Delta$. Let $g : |\gamma| \to \mathbb{R}^n$ be a continuous mapping, and suppose that the path $\tilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there is a unique increasing function $L_{\gamma,g} : \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ such that

$$L_{\gamma,g}(l_\gamma(t)) = l_{\tilde{\gamma}}(t) \quad \text{for all } t \in \Delta. \tag{8.4}$$

We say that a mapping $f : D \to \mathbb{R}^n$ has the (L)**-property** if the following two conditions hold:

(L_1) for a.e. path γ in D, $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable, and the function $L_{\gamma,f}$ has the (N)-property;

(L_2) for a.e. path $\tilde{\gamma}$ in $f(D)$, each lifting γ of $\tilde{\gamma}$ is locally rectifiable, and the function $L_{\gamma,f}$ has the (N^{-1})-property.

A path γ in D is a **lifting** of a path $\tilde{\gamma}$ in $\mathbb{R}^n$ under $f : D \to \mathbb{R}^n$ if $\tilde{\gamma} = f \circ \gamma$. Note that condition (L_2) applies only to paths $\tilde{\gamma}$ that have the lifting.

We say that a mapping $f : D \to \mathbb{R}^n$ is of **finite length distortion**, abbr. $f \in$ **FLD**, if f is of FMD and has the (L)-property.

The class of all FLD mappings includes all nonconstant quasiregular mappings and, in turn, every FLD mapping f belongs to the following class for some Q that is determined by f; see Theorem 8.2 and Corollary 8.7.

Let $Q(x,y) = (Q_1(x), Q_2(y))$ be a pair of measurable functions $Q_1 : D \to [1,\infty]$ and $Q_2 : D' \to [1,\infty]$. We say that $f : D \to \mathbb{R}^n$, $n \geq 2$, $f(D) = D'$, is a Q**-mapping** if

$$M(f\Gamma) \leq \int\limits_D Q_1(x)\rho^n(x)\,dm(x) \tag{8.5}$$

and

$$M(\Gamma) \leq \int\limits_{D'} Q_2(y)\rho_*^n(y)\,dm(y) \tag{8.6}$$

for every family Γ of paths in D and all $\rho \in \operatorname{adm}\Gamma$ and $\rho_* \in \operatorname{adm}\ f\Gamma$.

If, in addition, f is discrete and open, we say that f is a Q**-covering**. The more restrictive notion of a Q-homeomorphism, when f is a homeomorphism and $Q_2 \equiv \infty$, has been introduced in [204, 205]; see Chapters 4–6.

Recall that a mapping $f : D \to \mathbb{R}^n$ is **open** if the image of every open subset of D under f is an open set in $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^n$ is **discrete** if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points. B_f denotes the **branch set** of a mapping f, i.e., a set of all points $x \in D$ at which f fails to be a local homeomorphism.

8.2 Moduli of Cuttings and Extensive Moduli

We adopt the following conventions. Given a set $E \in \mathbb{R}^n$ and a path $\gamma : \Delta \to \mathbb{R}^n$, we identify $\gamma \cap E$ with $\gamma(\Delta) \cap E$. If γ is locally rectifiable, we set

$$l(\gamma \cap E) = |E_\gamma|, \tag{8.7}$$

where

$$E_\gamma = l_\gamma(\gamma^{-1}(E)). \tag{8.8}$$

Here $|A|$ means the length (Lebesgue) measure of a set $A \subset \mathbb{R}$ and $l_\gamma : \Delta \to \Delta_\gamma$ is as in Section 8.1. In general, for sets A in $\mathbb{R}^n$, $|A|$ will denote the n-Lebesgue measure of A. Note that

$$E_\gamma = \gamma_0^{-1}(E), \tag{8.9}$$

where $\gamma_0 : \Delta_\gamma \to \mathbb{R}^n$ is the natural parameterization of γ, and that

$$l(\gamma \cap E) = \int\limits_{\Delta} \chi_E(\gamma(t))|dx| = \int\limits_{\Delta_\gamma} \chi_{E_g}(s)ds. \tag{8.10}$$

We say that $\gamma \cap E$ is measurable on γ if E_γ is measurable in Δ_γ.

Remark 8.1. The definition of the modulus immediately implies that

(1) a.e. path in $\mathbb{R}^n$ is rectifiable,

(2) given a Borel set B in $\mathbb{R}^n$ of measure zero,

$$l(\gamma \cap B) = 0 \tag{8.11}$$

for a.e. rectifiable path γ in $\mathbb{R}^n$,

(3) for every Lebesgue measurable set E in $\mathbb{R}^n$, there exist Borel sets B_* and B^* in $\mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $|B^* \setminus B_*| = 0$. Thus, by (2), E_γ and χ_{E_γ} are a measurable set and function, respectively, in the length interval Δ_γ for a.e. γ in $\mathbb{R}^n$.

The following lemma extends Theorem 33.1 in [316] from Borel sets to arbitrary sets (cf. also Theorem 3 in [64]) and is based on (3).

Lemma 8.1. *Let E be a set in a domain $D \subset \mathbb{R}^n$, $n \geq 2$. Then E is measurable if and only if $\gamma \cap E$ is measurable for a.e. path γ in D. Moreover, $|E| = 0$ if and only if*

$$l(\gamma \cap E) = 0 \tag{8.12}$$

for a.e. path γ in D.

Proof. Suppose first that E is measurable. Then by (3) in Remark 8.1, $\gamma \cap E$ is measurable for a.e. path γ in D.

For the other direction, let C be a closed cube in D with edges parallel to the coordinate axes. By that assumption, $\gamma \cap E$ is measurable for a.e. line segments γ

joining opposite faces of C and parallel to the edges. Thus, by the Fubini theorem, E is measurable.

Next, suppose that $|E| = 0$. Then there is a Borel set B such that $|B| = 0$ and $E \subset B$. By Remark 8.1, (8.11) and hence (8.12) hold for a.e. path γ in D.

The sufficiency of (8.12) follows from the corresponding result for Borel sets in Theorem 33.1 in [316] by virtue of (3) in Remark 8.1. This completes the proof. $\square$

Proposition 8.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a (Lebesgue) measurable function. Then φ is a measurable function on a.e. rectifiable path in $\mathbb{R}^n$ with respect to the length measure.*

Proof. Indeed, for every measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$, in view of the Lusin theorem and the regularity of the Lebesgue measure (see, e.g., Section 2.3.5 in [55] and [281], p. 69), there is a Borel function $\varphi_0 : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi \equiv \varphi_0$ outside a Borel set B with $|B| = 0$ and $\varphi_0 \equiv 0$ on B. Thus, the statement follows by (2) in Remark 8.1. $\square$

Given a Lebesgue measurable function $\rho : \mathbb{R}^n \to [0,\infty]$, there is a Borel function $\rho^* : \mathbb{R}^n \to [0,\infty]$ such that $\rho^* = \rho$ a.e. in $\mathbb{R}^n$; see, e.g, Section 2.3.5 in [55] and [281], p. 69. This suggests an alternative definition of the modulus. A Lebesgue measurable function $\rho : \mathbb{R}^n \to [0,\infty]$ is **extensively admissible** for a path family Γ in $\mathbb{R}^n$, abbr. $\rho \in \operatorname{ext\,adm}\,\Gamma$, if

$$\int\limits_{\gamma} \rho(x)|dx| \geq 1 \tag{8.13}$$

for a.e. $\gamma \in \Gamma$. Note that (8.13) includes the assumption that the function $s \mapsto \rho(\gamma(s))$ is measurable in the interval $[0, l(\gamma)]$; the path is parameterized by arc length. The **extensive modulus** $\overline{M}(\Gamma)$ of Γ is defined as

$$\overline{M}(\Gamma) = \inf \int\limits_{\mathbb{R}^n} \rho^n(x)\, dm(x), \tag{8.14}$$

where the infimum is taken over all $\rho \in \operatorname{ext\,adm}\Gamma$.

Proposition 8.2. *For every family Γ of paths in $\mathbb{R}^n$,*

$$\overline{M}(\Gamma) = M(\Gamma). \tag{8.15}$$

Proof. Indeed, $\overline{M}(\Gamma) \leq M(\Gamma)$ because $\operatorname{adm}\,\Gamma \subset \operatorname{ext\,adm}\,\Gamma$. Let $\overline{M}(\Gamma)$ be realized by a sequence $\rho_m \in \operatorname{ext\,adm}\,\Gamma$, $m = 1, 2, \ldots$, i.e.,

$$\overline{M}(\Gamma) = \lim_{m\to\infty} \int\limits_{\mathbb{R}^n} \rho_m^n(x)\, dm(x).$$

Then there is a sequence of Borel functions $\varphi_m : \mathbb{R}^n \to [0,\infty]$ such that $\varphi_m = \rho_m$ a.e. in $\mathbb{R}^n$. By Lemma 8.1, $\varphi_m \in \operatorname{adm}\,\Gamma \setminus \Gamma_m$, where $M(\Gamma_m) = 0$. Since $\{\varphi_m\}_{m=1}^{\infty} \subset \operatorname{adm}\,\Gamma_0$, where $\Gamma_0 = \Gamma \setminus \Gamma_*$, $\Gamma_* = \cup \Gamma_m$, we obtain $\overline{M}(\Gamma) \geq M(\Gamma_0)$. Note also that

$M(\Gamma_*) = 0$ by the subadditivity of the modulus. Consequently, by monotonicity and subadditivity,

$$M(\Gamma_0) \leq M(\Gamma) \leq M(\Gamma_0) + M(\Gamma_*) = M(\Gamma_0),$$

i.e., $M(\Gamma_0) = M(\Gamma)$ and, thus, $\overline{M}(\Gamma) \geq M(\Gamma)$. The proof is complete. $\square$

8.3 FMD Mappings

A map $\varphi : X \to Y$ between metric spaces X and Y is said to be **Lipschitz** provided

$$\mathrm{dist}(\varphi(x_1), \varphi(x_2)) \leq M \cdot \mathrm{dist}(x_1, x_2) \tag{8.16}$$

for some $M < \infty$ and for all x_1 and $x_2 \in X$. The map φ is called **bi-Lipschitz** if, in addition,

$$M^* \mathrm{dist}(x_1, x_2) \leq \mathrm{dist}(\varphi(x_1), \varphi(x_2)) \tag{8.17}$$

for some $M^* > 0$ and for all x_1 and $x_2 \in X$. Later on X and Y will be subsets of $\mathbb{R}^n$ with the Euclidean distance.

Lemma 8.2. *Let $E \subset \mathbb{R}^n$ be a measurable set and let a mapping $\varphi : E \to \mathbb{R}^n$ satisfy the condition $L(x, \varphi) < \infty$ a.e. Then there is a countable collection of compact sets $C_k \subset E$ such that*

$$\left| E \setminus \bigcup_{k=1}^{\infty} C_k \right| = 0 \tag{8.18}$$

and $\varphi|_{C_k}$ is Lipschitzian for every $k = 1, 2, \ldots$.

Proof. Since $L(x, \varphi) < \infty$ a.e., we have by Section 3.1.8 in [55] that there is a countable collection of measurable sets $C_i \subset E$ such that $|E \setminus B| = 0$, $B = \cup_{i=1}^{\infty} C_i$, and $\varphi|_{C_i}$ is Lipschitzian for every $i = 1, 2, \ldots$. By regularity of the Lebesgue measure, we have that, for every $i = 1, 2, \ldots$, there is a countable collection of closed sets $C_{ij} \subset C_i$ such that $|C_i \setminus \cup_{j=1}^{\infty} C_{ij}| = 0$. Moreover, for every fixed $i, j, l = 1, 2, \ldots$, the set $C_{ijl} = C_{ij} \cap \overline{B^n(l)}$ is compact and $C_{ij} = \cup_{l=1}^{\infty} C_{ijl}$. Thus, the countable collection of the sets C_{ijl}, $i, j, l = 1, 2, \ldots$, is a desired collection. $\square$

For $f : D \to \mathbb{R}^n$ and $E \subset D$, we use the **multiplicity functions**

$$N(y, f, E) = \mathrm{card}\{x \in E : f(x) = y\}, \tag{8.19}$$

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E). \tag{8.20}$$

Proposition 8.3. *Let $f : D \to \mathbb{R}^n$ be an* FMD *mapping. Then* *(i) f is differentiable a.e. and*

$$J(x,f) \neq 0 \quad \text{a.e.,} \tag{8.21}$$

(ii) f has the (N^{-1})-property,

(iii) the change-of-variables formula

$$\int_E g(f(x))|J(x,f)|\,dm(x) = \int_{\mathbb{R}^n} g(y)N(y,f,E)\,dm(y) \tag{8.22}$$

holds for every measurable function $g : \mathbb{R}^n \to [0,\infty)$ and every measurable set $E \subset D$.

Proof. (i) f is differentiable a.e. by Rademacher–Stepanoff's theorem because $L(x,f) < \infty$ a.e.; see, e.g., Section 3.1.9 in [55]. Since $l(x,f) > 0$ a.e., we have

$$l(f'(x)) > 0 \quad \text{a.e.,} \tag{8.23}$$

where, for a given linear mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, $l(T)$ denotes the minimum of the modulus $|Tz|$ over all unit vectors $z \in \mathbb{R}^n$. Hence, (8.21) follows from (8.23).

(ii) By the Ponomarev result for a.e. differentiable mappings f, the condition (8.21) is equivalent to the (N^{-1})-property; see Theorem 1 in [244], p. 412.

(iii) By Lemma 8.2, $f|_{C_k}$, $k = 1,2,\ldots$, is Lipschitzian, where we may assume that the C_k are mutually disjoint bounded Borel sets. Every $f|_{C_k}$ can be extended to a Lipschitz map of $\mathbb{R}^n$ by applying to each component of the mapping f Theorem 1 in [218]. Thus, (8.22) follows from Section 3.2.5 in [55] by the countable additivity of the integral, (8.18), the (N)-property, and (ii). □

Note that here we show a preference the application of McShane's theorem instead of Kirszbraun's theorem in [150] (see also Section 2.10.43 in [55]), because the proof of the latter is based on the axiom of choice.

Remark 8.2. As is clear from the proof, the differentiability of f a.e. is equivalent to the condition

$$L(x,f) < \infty \quad \text{a.e.} \tag{8.24}$$

If f is differentiable a.e., then the conditions

$$l(x,f) > 0 \quad \text{a.e.,} \tag{8.25}$$

(8.21), (8.23), and the (N^{-1})-property are all equivalent.

Corollary 8.1. *A mapping $f : D \to \mathbb{R}^n$ is of finite metric distortion if and only if f is differentiable a.e. and has the (N)- and (N^{-1})-properties.*

Corollary 8.2. *Compositions of* FMD *mappings are* FMD *mappings. In particular, the* FMD *mappings remain* FMD *mappings under compositions with nonconstant quasiregular mappings. The inverse mapping f^{-1} of an* FMD *homeomorphism f is an* FMD *homeomorphism.*

Denote by $J(D)$ the collection of all subdomains G of D such that $\overline{G} \subset D$ is compact. Given a mapping $f : D \to \mathbb{R}^n$, $G \in J(D)$ and $y \in \mathbb{R}^n \setminus f(\partial G)$, let $\mu(y, f, G)$ be the **topological index** of the triple (y, f, G); see, e.g., [246], [256] and [62]. A mapping $f : D \to \mathbb{R}^n$ is said to be sense-preserving (sense-reversing) if $\mu(y, f, G) > 0$ ($\mu(y, f, G) < 0$) for all such triples (y, f, G) with $y \in f(G) \setminus f(\partial G)$. It is well known that if f is one-to-one, then f is either sense-preserving or sense-reversing and, moreover, $\mu(y, f, G) \equiv 1$ or $\mu(y, f, G) \equiv -1$, respectively, for all $y \in fG$; see, e.g., [246], pp. 133–134.

Note that the topological dimension of the branch set B_f for a discrete and open mapping f is not more than $n-2$ by the Chernavskii theorem; see [39] and [40]. Hence, B_f does not separate D by the Menger–Urysohn theorem; see, e.g., [126], p. 48. Thus, every discrete and open mapping is either sense-preserving or sense-reversing; cf. also Theorem 9 in [308], p. 336. By Lemma 2.14 in [210], every discrete, open, and sense-preserving mapping f satisfies $J(x, f) = 0$ whenever $x \in B_f$ is a point of differentiability of f.Hence, we have the following consequence of (i) in Proposition 8.3.

Proposition 8.4. *For every discrete open* FMD *mapping $f : D \to \mathbb{R}^n$,*

$$|B_f| = |f(B_f)| = |f^{-1}(f(B_f))| = 0. \tag{8.26}$$

Recall that a mapping f is **light** if $f^{-1}(y)$ is totally disconnected for every $y \in \mathbb{R}^n$, i.e., if every component of the preimage $f^{-1}(y)$ consists of a single point. It is well known that every light sense-preserving mapping is open and discrete; see [308], p. 333. Consequently, we have the following corollary; see [246], p. 333.

Corollary 8.3. *If an* FMD *mapping $f : D \to \mathbb{R}^n$ is light and sense-preserving, then*

$$J(x, f) > 0 \quad a.e. \tag{8.27}$$

Lemma 8.3. *Let $f : D \to \mathbb{R}^n$ be an* FMD *mapping. Then there is a countable collection of compact sets $C_k^* \subset D$ such that $|B| = 0$, where $B = D \setminus \cup_{k=1}^{\infty} C_k^*$, and $f|_{C_k^*}$ is one-to-one and bi-Lipschitz for every $k = 1, 2, \ldots$; moreover, f is differentiable at all points C_k^* with $J(x, f) \neq 0$.*

Proof. First, let C_k be as in Lemma 8.2. By Proposition 8.3(i) and the regularity of Lebesgue's measure (see, e.g., III(6.6)(i) in [281], p. 69), we can always replace C_k by compact sets where f is differentiable, $J(x, f) \neq 0$, and $l(x, f) > 0$; see Remark 8.2. Then every C_k is the union of a countable collection of Borel sets where f is one-to-one; see [53], p. 94. Hence, there is a countable collection of Borel sets C_k' such that $f|_{C_k'}$ is one-to-one and Lipschitzian for every $k = 1, 2, \ldots$ and $|D \setminus \cup_{k=1}^{\infty} C_k'| = 0$. By the regularity of Lebesgue's measure, we may assume that the C_k' are compact sets. Finally, since $l(x,f) > 0$ on C_k' and hence $L(y, f_k^{-1}) < \infty$ on $f(C_k')$, we are able to apply Lemma 8.2 to f_k^{-1}, where $f_k = f|_{C_k'}$, to derive the bi-Lipschitz property of $f|_{C_k^*}$, see also Proposition 8.3(ii). □

8.4 FLD Mappings

Recall that $f : D \to \mathbb{R}^n$ is of FLD if it is of FMD and has the (L)-property. We begin with the latter property. In the final section, we will give examples showing that the (L)-property does not imply openness and discreteness. Later on, we will often apply the following simple remark. We recall that if a family of paths contains a degenerate path, then its modulus is infinite.

Remark 8.3. (a) If $f : D \to \mathbb{R}^n$ satisfies the (L_2)-condition, then $f^{-1}(y)$ cannot contain a nondegenerate path. We call a mapping with the latter property **weakly light**. (b) If f is weakly light, then the correspondence $L_{\gamma,f} : \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ between the natural parameters of locally rectifiable paths γ and $\tilde{\gamma} = f \circ \gamma$ is a homeomorphism and $L_{\gamma,f}^{-1}$ is well defined.

In view of Remark 8.3, the (L_1)-property implies **absolute continuity on paths,** abbr. **ACP**, i.e., $L_{\gamma,f}$ is absolutely continuous on closed subintervals of Δ_γ for a.e. path γ in D; see, e.g., Section 2.10.13 in [55]. In particular, the (L)-property implies absolute continuity on lines because the following obvious inclusion holds:

$$\mathrm{ACP} \subset \mathrm{ACL}\,. \tag{8.28}$$

Now, let a mapping $f : D \to \mathbb{R}^n$ be weakly light. Then the following notion is well defined in view of Remark 8.3. We say that f is **absolute continuous on paths in the inverse direction,** abbr. **ACP^{-1}**, if $L_{\gamma,f}^{-1}$ is absolutely continuous on closed subintervals of $\Delta_{\tilde{\gamma}}$ for a.e. path $\tilde{\gamma}$ in $f(D)$ and for each lifting γ of $\tilde{\gamma}$. By Section 2.10.13 in [55], the (L_2)-condition implies ACP^{-1}.

Proposition 8.5. *A mapping $f : D \to \mathbb{R}^n$ has the (L)-property if and only if f is weakly light and*

$$f \in \mathrm{ACP} \cap \mathrm{ACP}^{-1}. \tag{8.29}$$

Here $f \in \mathrm{ACL}^p$ means that the mapping $f : D \to \mathbb{R}^n$ is ACL and its partial derivatives belong to $L^p(D)$, $p \geq 1$.

Remark 8.4. It is known that

(1) if $f \in \mathrm{ACL}^n$, then $f \in \mathrm{ACP}$; see Theorem 3 in [64] and Theorem 28.2 in [316];

(2) if $f \in \mathrm{ACL}^p$, $p > n-1$, is an open mapping, then f is differentiable a.e. in D; see Lemma 3 in [322];

(3) if $f \in W^{1,n}$ is open, then f has the (N)-property; see [193];

(4) for every $n \geq 2$, there is a function $f \in \mathrm{ACL}^n$ that is nowhere differentiable; see [286], p. 371;

(5) there is a homeomorphism $f \in W^{1,p}$ for all $p < n$ that does not possess the (N)-property; see [243], p. 140.

Combining (1)–(3) with Proposition 8.5 and Corollary 8.1, we have the following statement.

Theorem 8.1. *Let $f : D \to \mathbb{R}^n$ be a homeomorphism such that $f \in \mathrm{ACL}^n_{\mathrm{loc}}$ and $f^{-1} \in \mathrm{ACL}^n_{\mathrm{loc}}$. Then f is of finite length distortion.*

Similarly, by Lemma 6 in [242] on the ACP^{-1} property of quasiregular mappings, we come to the following corollary.

Theorem 8.2. *Every nonconstant quasiregular mapping is a mapping of finite length distortion.*

A mapping $f : D \to \mathbb{R}^n$ is said to be of **finite distortion** if $f \in W^{1,n}_{\mathrm{loc}}$ and

$$|f'(x)|^n \leq K(x) \cdot J(x,f) \quad \text{a.e.} \tag{8.30}$$

for some finite-valued function $K(x) : D \to [1,\infty)$; see, e.g., [79, 111, 137]; cf. also [134]. As is well known by [195] and [196], a mapping f of finite distortion is discrete and open if $K \in L^p_{\mathrm{loc}}$, $p > n-1$. By [111], if f is a homeomorphism of finite distortion with $K \in L^{n-1}$, then $f^{-1} \in W^{1,n}$.

Corollary 8.4. *Every homeomorphism of finite distortion with $K \in L^{n-1}_{\mathrm{loc}}$ is a mapping of finite length distortion.*

Remark 8.5. Applying the arguments in the proof of Lemma 6 in [242], one can show that a discrete open mapping $f : D \to \mathbb{R}^n$ of finite distortion with $K \in L^{n-1}_{\mathrm{loc}}$ is of FLD provided that $|fB_f| = 0$. In the literature the condition $f \in W^{1,n}_{\mathrm{loc}}$ in the definition of the mappings of finite distortion is sometimes replaced by the weaker condition $f \in W^{1,1}_{\mathrm{loc}}$; see, e.g., [147, 148, 154].

It is known that homeomorphisms of the class $W^{1,n}_{\mathrm{loc}}$ with $K_I \in L^1_{\mathrm{loc}}$ have the inverse f^{-1} in the same class; see Corollary 2.3 in [154]. Thus, we have the following consequence from Theorem 8.1.

Corollary 8.5. *Every homeomorphism $f : D \to \mathbb{R}^n$ of finite distortion with $K_I \in L^1_{\mathrm{loc}}$ is of finite length distortion.*

Lemma 8.4. *Let a mapping $f : D \to \mathbb{R}^n$ be of finite length distortion. If $y \mapsto N(y,f,C)$ is integrable in $\mathbb{R}^n$ for every compact set C in D and if*

$$K_O(x,f) \in L^q_{\mathrm{loc}}, \qquad q \geq 1/(n-1), \tag{8.31}$$

then $f \in W^{1,s}_{\mathrm{loc}}$ where $s = nq/(1+q)$.

Note that $s \geq 1$ under $q \geq 1/(n-1)$.

Proof. Since f satisfies the (L_1)-property, f is ACL and it suffices to establish the local L^s-inequality of the partial derivatives $\partial_i f$, $i = 1, 2, \ldots, n$. For this let C be a compact set in D. From (8.22) with $g \equiv 1$ and the Hölder inequality, we obtain

$$\begin{aligned} \|\partial_i f\|_s &\leq \|K_O^{1/n}(x,f)\|_p \cdot \|J^{1/n}(x,f)\|_n \\ &\leq \|K_O(x,f)\|_q^{1/n} \cdot \left(\int_{\mathbb{R}^n} N(y,f,C)\,dm(y) \right)^{1/n}, \end{aligned} \tag{8.32}$$

where $1/s = 1/n + 1/p$ and $p = qn$. □

In view of (iii) in Proposition 8.3, the Jacobian $J(x,f)$ of an FMD mapping f is locally integrable in D if and only if the multiplicity function $N(y,f,C)$ is integrable for every compact set C in D. Thus, by Theorem 1.3 in [147], p. 137 (see also Lemma 3.1 in [148], p. 174), and Corollary 8.1 we obtain the following.

Corollary 8.6. *If f is a sense-preserving* FLD *mapping with $N(y,f,C) \in L^1(\mathbb{R}^n)$ for every compact set $C \subset D$ and $K_O(x,f) \in \mathrm{BMO}_{\mathrm{loc}}$, then f is discrete and open.*

A discrete open mapping $f : D \to \mathbb{R}^n$ has the bounded multiplicity $N(y,f,C)$ for every compact set $C \subset D$ and $N(y,f,C)$ is integrable over $\mathbb{R}^n$ because $f(C)$ is also compact.

Theorem 8.3. *Let a mapping $f : D \to \mathbb{R}^n$ be of finite length distortion. If f is discrete and open and*

$$K_O(x,f) \leq K < \infty \quad a.e., \tag{8.33}$$

then f is a quasiregular mapping. For $n \geq 3$ and $K = 1$, the mapping f is a Möbius transformation.

8.5 Uniqueness Theorem

We use Theorem 8.3 to establish uniqueness results for FLD mappings with prescribed characteristics. We employ the normalized Jacobian matrix

$$M_f(x) = f'(x)/|J(x,f)|^{1/n} \tag{8.34}$$

and the symmetrized normalized Jacobian matrix

$$G_f(x) = M_f^*(x) M_f(x), \tag{8.35}$$

where $M_f^*(x)$ denotes the transpose of $M_f(x)$. Note that $M_f(x)$ of f is defined by (8.34) a.e. for a mapping f of FMD. We set $M_f(x) = I =$ identity at the rest of the points in D. We call $M_f(x)$ and $G_f(x)$ the **matrix dilatation** and the **dilatation tensor** of the mapping f at x, respectively; see [2,3]. It is clear that $|\det M_f(x)| = 1 = \det G_f(x)$ and that $K_O(x, f) = \|M_f(x)\|^n$ a.e.

If an FMD mapping f is sense-preserving, then $M_f(x)$ is a **unimodular matrix**, i.e., $\det M_f(x) = 1$ for all $x \in D$. Moreover, if f and g are FMD mappings and $f \circ g$ is well defined, then, in view of Corollary 8.1, the composition rule

$$M_{f\circ g}(x) = M_f(g(x))M_g(x) \tag{8.36}$$

holds a.e. The dilatation tensor $G_f(x)$ of the mapping f at x is symmetric, positive definite, and unimodular.

The **special linear group** $SL(n)$ is the multiplicative group of all $n \times n$ matrices M over $\mathbb{R}$ with $\det M = 1$. The collection of all the symmetric positive definite matrices of $SL(n)$ is denoted by $S(n)$. Note that $S(n), n \geq 2$, is not a group, because the product of symmetric matrices need not be symmetric; see, e.g., [23], p. 24. For a matrix $M \in SL(n)$, set

$$G = M^*M \in S(n), \tag{8.37}$$

where M^* is the transpose of M. The matrix G is called the **symmetrization** of M. The following statement follows immediately from the definition of the symmetrization.

Proposition 8.6. *Let M_1 and $M_2 \in SL(n)$, $n \geq 2$, and let G_1 and G_2 be the symmetrization of M_1 and M_2, respectively. Then $G_1 = G_2$ if and only if $M_2 = UM_1$, where $U \in SL(n)$ with the unit norm $|U| = 1$.*

In other words, the symmetrization G determines the corresponding M from (8.37) up to left rotations. Below $\mathscr{O}^+(n)$ denotes just the group of $n \times n$ orthogonal matrices U, defined by $U^*U = I = UU^*$, with determinant 1.

Lemma 8.5. *Let $f, g : D \to \mathbb{R}^n$, $n \geq 3$, be sense-preserving homeomorphisms of finite length distortion. If*

$$G_f(x) = G_g(x) \quad \text{a.e.}, \tag{8.38}$$

then

$$f = h \circ g, \tag{8.39}$$

where h is a Möbius transformation of $\overline{\mathbb{R}^n}$.

Proof. Indeed, $h = f \circ g^{-1}$ is an FLD homeomorphism. By Proposition 8.6, $M_f(x) = U(x)M_g(x)$ with $U(x) \in \mathscr{O}^+(n)$ a.e. Hence, by the chain rule (8.36) and the (N)- and (N^{-1})-properties, we have

$$M_h(x) = U(g^{-1}(y)) \in \mathscr{O}^+(n) \quad \text{a.e.}, \tag{8.40}$$

i.e., $K_h(y) = 1$ a.e. By Theorem 8.3, h is a Möbius transformation. □

Note that, for $n = 2$, Lemma 8.5 is valid with a conformal mapping h.

Theorem 8.4. *Suppose that discrete and open mappings $f,g : D \to \mathbb{R}^n$, $n \geq 3$, are of finite length distortion. If f and g have a.e. equal dilatation tensors, then $f = h \circ g$, where h is a Möbius transformation of $\overline{\mathbb{R}^n}$.*

Proof. By the Chernavskii theorem, $\dim(B_f \cup B_g) \leq n-2$ (see [39, 40]), and then $\Omega = D \setminus (B_f \cup B_g)$ is a domain by the Menger–Urysohn theorem; see, e.g., [126], p. 48. Arguing locally in Ω, we obtain from Lemma 8.5 that $f|_\Omega = h \circ g|_\Omega$, where h is a Möbius transformation of $\overline{\mathbb{R}^n}$; by continuity, this holds in D because Ω is everywhere dense in D. $\square$

8.6 FLD and Q-Mappings

The following theorem extends the so-called K_0-inequality in the theory of quasiregular mappings to FLD mappings; cf. [210], p. 16, [260], p. 31, [328], p. 130, and [154].

Theorem 8.5. *Let a mapping $f : D \to \mathbb{R}^n$ be of finite length distortion and let $E \subset D$ be a measurable set. Then*

$$M(\Gamma) \leq \int\limits_{f(E)} K_I(y, f^{-1}, E) \cdot \rho_*^n(y)\ dm(y) \tag{8.41}$$

for every path family Γ in E and $\rho_* \in \operatorname{adm} f\Gamma$, where

$$K_I(y, f^{-1}, E) = \sum_{x \in E \cap f^{-1}(y)} K_O(x, f). \tag{8.42}$$

In particular, for $E = D$, we have

$$K_I(y, f^{-1}, D) = K_I(y, f^{-1}) = \sum_{x \in f^{-1}(y)} K_O(x, f). \tag{8.43}$$

Proof. In view of the regularity of the Lebesgue measure, we may assume that $f(E)$ is Borel and that $\rho_* \equiv 0$ outside $f(E)$; see, e.g., III(6.6)(i) in [281], p. 69 and Lemma 8.1. Let B and C_k^*, $k = 1, 2, \ldots$, be as in Lemma 8.3. Setting by induction $B_0 = B$, $B_1 = C_1^*$, $B_2 = C_2^* \setminus B_1$, and

$$B_k = C_k^* \setminus \bigcup_{l=1}^{k-1} B_l, \tag{8.44}$$

we obtain the countable covering of D consisting of mutually disjoint Borel sets $B_k, k = 0, 1, 2, \ldots$, with $|B_0| = 0$. Then, by Remark 8.1, $l(\gamma \cap B_0) = 0$ for a.e. $\gamma \in \Gamma$ and hence by the (L_1)-property $l(f \circ \gamma \cap f(B_0)) = 0$ also for a.e. $\gamma \in \Gamma$.

Given $\rho_* \in \operatorname{adm} f\Gamma$, set

$$\rho(x) = \begin{cases} \rho_*(f(x))|f'(x)|, & \text{for } x \in D \setminus B_0, \\ 0, & \text{otherwise.} \end{cases} \tag{8.45}$$

Arguing piecewise on B_k, we have by Section 3.2.5 for $m = 1$ in [55] and the additivity of integrals, that

$$\int_\gamma \rho ds \geq \int_{f\circ\gamma} \rho_* ds \geq 1 \tag{8.46}$$

for a.e. $\gamma \in \Gamma$, i.e., $\rho \in \text{adm } \Gamma \setminus \Gamma_0$, where $M(\Gamma_0) = 0$. Therefore, by the subadditivity of the modulus,

$$M(\Gamma) \leq \int_D \rho^n(x) dm(x). \tag{8.47}$$

Note that $\rho = \Sigma_{k=1}^{\infty} \rho_k$, where $\rho_k = \rho \cdot \chi_{B_k}$ have mutually disjoint supports. By Section 3.2.5 for $m = n$ in [55], we obtain

$$\int_{f(B_k \cap E)} K_O(f_k^{-1}(y), f) \cdot \rho_*^n(y)\ dm(y) = \int_D \rho_k^n(x)\ dm(x), \tag{8.48}$$

where every $f_k = f|_{B_k}$, $k = 1, 2, \ldots$, is injective by the construction.

Finally, by the Lebesgue positive convergence theorem (see, e.g., [281], p. 27), we conclude from (8.47) and (8.48) that

$$\int_{f(E)} K_I(y, f^{-1}, E) \cdot \rho_*^n(y)\ dm(y) = \int_D \sum_{k=1}^{\infty} \rho_k^n(x)\ dm(x) \geq M(\Gamma).$$

□

The next inequality is a generalized form of the K_I-inequality and is also known as Poletskii's inequality; cf. [242], [260], pp. 49–51, and [328], p. 131.

Theorem 8.6. *Let $f : D \to \mathbb{R}^n$ be an* FLD *mapping. Then*

$$M(f\Gamma) \leq \int_D K_I(x, f) \cdot \rho^n(x)\ dm(x) \tag{8.49}$$

for every path family Γ in D and $\rho \in$ adm *Γ.*

Proof. Let B_k, $k = 0, 1, 2, \ldots$, be Borel sets as given as by (8.44). By the construction and (N)-property, $|f(B_0)| = 0$. Thus, by Lemma 8.1, $l\,(\tilde{\gamma} \cap f(B_0)) = 0$ for a.e. $\tilde{\gamma} \in f\Gamma$ and hence by the (L_2)-property, $l\,(\gamma \cap B_0) = 0$ also for a.e. $\tilde{\gamma} \in f\Gamma$, $\tilde{\gamma} = f \circ \gamma$.

Let $\rho \in \text{adm } \Gamma$. Set

$$\tilde{\rho}(y) = \sup_{x \in f^{-1}(y) \cap D \setminus B_0} \rho^*(x), \tag{8.50}$$

where
$$\rho^*(x) = \begin{cases} \rho(x)/l(f'(x)) & \text{for } x \in D \setminus B_0, \\ 0 & \text{otherwise.} \end{cases} \tag{8.51}$$

Note that $\tilde{\rho} = \sup \rho_k$, where

$$\rho_k(y) = \begin{cases} \rho^*(f_k^{-1}(y)) & \text{for } y \in f(B_k), \\ 0 & \text{otherwise,} \end{cases} \tag{8.52}$$

and every $f(B_k)$ is Borel and $f_k = f|_{B_k}$, $k = 1, 2, \ldots$, is injective. Thus, the function $\tilde{\rho}$ is Borel; see, e.g., Theorem I(8.5) in [281], p. 15.

Arguing piecewise on B_k, we obtain by Section 3.2.5 for $m = 1$ in [55] and the additivity of integrals, that
$$\int_{\tilde{\gamma}} \tilde{\rho}\, ds \geq \int_{\gamma} \rho\, ds \geq 1 \tag{8.53}$$

for a.e. $\tilde{\gamma} = f \circ \gamma \in f\Gamma$ and, thus, $\tilde{\rho} \in \operatorname{adm} f\Gamma \setminus \Gamma_0$, where $M(\Gamma_0) = 0$. Hence,

$$M(f\Gamma) \leq \int_{f(D)} \tilde{\rho}^n(y)\, dm(y). \tag{8.54}$$

Moreover, by Section 3.2.5 for $m = n$ in [55], we have

$$\int_{B_k} K_I(x, f) \cdot \rho^n(x)\ dm(x) = \int_{f(D)} \rho_k^n(y)\ dm(y). \tag{8.55}$$

Finally, by Lebesgue's theorem, we obtain from (8.52) and (8.55) the desired inequality:

$$\begin{aligned} \int_D K_I(x, f) \cdot \rho^n(x)\ dm(x) &= \sum_{k=1}^{\infty} \int_{f(D)} \rho_k^n(y)\ dm(y) \\ &= \int_{f(D)} \sum_{k=1}^{\infty} \rho_k^n(y)\ dm(y) \geq M(f\Gamma). \end{aligned}$$

□

Remark 8.6. If $K_I(f) = \operatorname{ess\ sup} K_I(x, f) < \infty$, then (8.49) yields the standard Poletskii inequality:
$$M(f\Gamma) \leq K_I(f) \cdot M(\Gamma) \tag{8.56}$$

for every path family in D. If $K_O(f) = \operatorname{ess\ sup} K_O(x, f) < \infty$ and E is a Borel set with $N(f, E) < \infty$, then we have from (8.41) the usual form of the K_O-inequality:

$$M(\Gamma) \leq N(f, E) \cdot K_O(f) \cdot M(f\Gamma) \tag{8.57}$$

for every path family in E.

Finally, combining Theorems 8.5 and 8.6, we obtain the following:

Corollary 8.7. *Every* FLD *mapping f is a Q-mapping with*

$$Q(x,y) = (K_I(x,f), K_I(y,f^{-1})), \tag{8.58}$$

where $K_I(y,f^{-1})$ is defined in (8.43).

8.7 On FLD Homeomorphisms

By Corollary 8.7, every FLD homeomorphism f is a $Q(x)$-homeomorphism with $Q(x) = K_I(x,f)$. Furthermore, it is easy to show similarly to the proofs of Theorems 6.1 and 8.6 the following statement. A strict proof of a more general statement, Lemma 10.2, can be found in Chapter 10 on mappings with finite area distortion because surfaces are not assumed to be connected there. Note that the family of all dashed lines is "minorized" by the family of all paths and, thus, the (L)-property also holds for the dashed lines and not only for the family of paths in the class of FLD mappings.

Theorem 8.7. *Every* FLD *homeomorphism $f: D \to \mathbb{R}^n$ is a super Q-homeomorphism with*

$$Q(x) = K_I(x,f). \tag{8.59}$$

Thus, the whole theory developed in Chapters 4–7 can be applied to FLD homeomorphisms. Let us give formulations of some corollaries in the explicit form. We begin with the removability of isolated singularities.

Theorem 8.8. *Let $f: \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be an FLD homeomorphism with*

$$K_I(x,f) \ \le \ Q(x), \tag{8.60}$$

where the majorant Q has finite mean oscillation at 0. *Then f has an* FLD *homeomorphic extension to $\mathbb{B}^n$.*

Theorem 8.9. *Let $f: \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be an* FLD *homeomorphism. If*

$$\limsup_{r \to 0} \fint_{B^n(r)} K_I(x,f)\, dm(x) \ < \ \infty, \tag{8.61}$$

then f has an extension to $\mathbb{B}^n$ that is an FLD *homeomorphism.*

Corollary 8.8. *Isolated points are removable for* FLD *homeomorphisms with $K_I(x,f) \le Q(x) \in$* BMO *and, in particular, if $f: \mathbb{R}^n \to \mathbb{R}^n$, then f has a homeomorphic extension to $\overline{\mathbb{R}^n}$ and $f(\mathbb{R}^n) = \mathbb{R}^n$.*

Similarly, we have the following statement.

Theorem 8.10. *Let $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n$, $n \geq 2$, be an* FLD *homeomorphism. If*

$$\int_{\varepsilon<|x|<1} K_I(x,f) \ \frac{dm(x)}{|x|^n} = o\left(\left[\log \frac{1}{\varepsilon}\right]^n\right) \tag{8.62}$$

as $\varepsilon \to 0$, then f has an FLD *homeomorphic extension to $\mathbb{B}^n$.*

Corollary 8.9. *Condition (8.62) and the assertion of Theorem 8.10 hold if*

$$K_I(x,f) \ = \ o\left(\left[\log \frac{1}{|x|}\right]^{n-1}\right) \tag{8.63}$$

as $x \to 0$. The same holds if

$$k(r) = o\left(\left[\log \frac{1}{r}\right]^{n-1}\right) \tag{8.64}$$

as $r \to 0$, where $k(r)$ is the mean value of the function $K_I(x,f)$ over the sphere $|x| = r$.

Remark 8.7. We may replace (8.62) by

$$\int_{\varepsilon<|x|<1} \frac{K_I(x,f)\, dm(x)}{\left(|x| \log \frac{1}{|x|}\right)^n} = o\left(\left[\log \ \log \frac{1}{\varepsilon}\right]^n\right) \tag{8.65}$$

and (8.64) by

$$k(r) = o\left(\left[\log \frac{1}{r} \log \ \log \frac{1}{r}\right]^{n-1}\right). \tag{8.66}$$

Thus, it is sufficient to require that

$$k(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right). \tag{8.67}$$

In general, we are able to formulate the whole scale of the corresponding conditions using functions of the form $\log \cdots \log 1/t$. However, we prefer to give other interesting conditions here that are met in the mapping theory.

Theorem 8.11. *Let $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n$, $n \geq 2$, be an* FLD *homeomorphism and for some $\beta \geq 1/(n-1)$, let*

$$\int_0^{\varepsilon_0} \frac{dr}{rk^{\beta}(r)} = \infty, \tag{8.68}$$

where $k(r)$ is the mean integral value of the function $K_I(x,f)$ over the sphere $|x| = r$. Then f has an FLD *homeomorphic extension to $\mathbb{B}^n$.*

Corollary 8.10. *Every* FLD *homeomorphism* $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$, $n \geq 2$, *with*

$$\int_0^{\varepsilon_0} \frac{dr}{rk(r)} = \infty \tag{8.69}$$

can be extended to an FLD *homeomorphism of* $\mathbb{B}^n$ *into* $\overline{\mathbb{R}^n}$.

The removability theorems for inverse mappings of FLD homeomorphisms can be formulated under much weaker conditions.

Proposition 8.7. *Let E be a nondegenerate continuum in $\mathbb{B}^n$. Then there exists no* FLD *homeomorphism of* $\mathbb{B}^n \setminus E$ *onto* $\mathbb{B}^n \setminus \{0\}$ *with*

$$K_I(x, f) \in L^1(\mathbb{B}^n \setminus E). \tag{8.70}$$

Analogies of the known Painleve theorem also take place for FLD homeomorphisms.

Theorem 8.12. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let X be a closed subset of D of length zero, and let $f : D \setminus X \to \mathbb{R}^n$ be an* FLD *homeomorphism. If $K_I(x, f) \leq Q(x)$ and the majorant $Q(x)$ has finite mean oscillation at every point $x_0 \in X$, then f has an* FLD *homeomorphic extension to D.*

Corollary 8.11. *Let X be a closed subset of length zero in D and let $f : D \setminus X \to \mathbb{R}^n$ be an* FLD *homeomorphism such that every point of X is a Lebesgue point for $K_I(x, f)$. Then f has an* FLD *homeomorphic extension to D.*

Corollary 8.12. *Let X be a closed subset of length zero in D and let*

$$\overline{\lim_{\varepsilon \to 0}} \; -\!\!\!\!\!\!\int_{B(x_0,\varepsilon)} Q(x) \;\; dm(x) \; < \; \infty \tag{8.71}$$

for every $x_0 \in X$. Then every FLD *homeomorphism $f : D \setminus X \to \mathbb{R}^n$ with $K_I(x, f) \leq Q(x)$ a.e. extends to an* FLD *homeomorphism of D into $\overline{\mathbb{R}^n}$.*

For a singular set X with positive length, it is necessary to request additional conditions on its cluster set $f(X) = C(X, f)$ under the mapping f.

Theorem 8.13. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let $f : D \setminus X \to \mathbb{R}^n$ be an* FLD *homeomorphism, and let X be a closed subset of D such that*

$$\Lambda_{n-1}(X) = \Lambda_{n-1}(f(X)) = 0\,. \tag{8.72}$$

If $K_I(x, f) \leq Q(x)$ and the majorant $Q(x) \in L^1_{\mathrm{loc}}(D)$ has finite mean oscillation at every point $x_0 \in X$, then f has an FLD *homeomorphic extension to D.*

Corollary 8.13. *If all points of a closed set $X \subset D$ with condition (8.72) are Lebesgue points for $K_I(x, f) \in L^1_{\mathrm{loc}}(D)$, then the* FLD *homeomorphism $f : D \setminus X \to \mathbb{R}^n$ admits an* FLD *homeomorphic extension to D.*

Corollary 8.14. *If a closed set $X \subset D$ with condition (8.72) satisfies the condition*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B(x_0,\varepsilon)} K_I(x,f)\ dm(x) \ < \ \infty \tag{8.73}$$

for every $x_0 \in X$, then the FLD *homeomorphism $f : D\backslash X \to \mathbb{R}^n$ has an* FLD *homeomorphic extension to D.*

Moreover, the homeomorphic extension of FLD homeomorphisms to hard boundaries is also possible under the corresponding conditions on $K_I(x,f)$ at the boundary points but with suitable geometric conditions on the boundaries. The next theorem extends the Gehring–Martio results in [81], p. 196, on the boundary correspondence from quasiconformal mappings to FLD homeomorphisms; cf. Corollaries 3.2 and 3.3.

Theorem 8.14. *Let $f : D \to D'$ be an* FLD *homeomorphism between* QED*, in particular, uniform domains D and D' with*

$$K_I(x,f) \ \leq \ Q(x), \tag{8.74}$$

where Q has finite mean oscillation at every boundary point. Then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.

Corollary 8.15. *If a domain D in $\mathbb{R}^n$ is uniform but not a Jordan domain, then there is no* FLD *homeomorphism of D onto $\mathbb{B}^n$ with $K_I(x,f) \leq Q(x) \in \mathrm{BMO}(D)$.*

We restrict ourselves ahead to the simplest cases, namely, to bounded domains with smooth boundaries and bounded convex domains.

Theorem 8.15. *Let f be an* FLD *homeomorphism between bounded domains D and D' in $\mathbb{R}^n$, $n \geq 2$, with smooth boundaries. If $K_I(x,f) \leq Q(x)$, where $Q(x) \in L^1(D)$ has finite mean oscillation at every point $x_0 \in \partial D$, then f has a homeomorphic extension to the closure of D.*

Theorem 8.16. *Let f be an* FLD *homeomorphism between bounded convex domains D and D' in $\mathbb{R}^n$, $n \geq 2$. If $K_I(x,f) \leq Q(x)$, where $Q(x) \in L^1(D)$ has finite mean oscillation at every point $x_0 \in \partial D$, then f has a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Corollary 8.16. *If f is an* FLD *homeomorphism of the unit ball $\mathbb{B}^n, n \geq 2$, onto itself such that $K_I(x,f) \leq Q(x)$, where $Q \in L^1(\mathbb{B}^n)$ has finite mean oscillation at every point $x_0 \in \partial\mathbb{B}^n$, then f admits an* FLD *homeomorphic extension $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$.*

In particular, we have the following statement.

Corollary 8.17. *If an* FLD *homeomorphism f of the unit ball $\mathbb{B}^n$, $n \geq 2$, onto itself, $f(0) = 0$, satisfies the condition*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B^*(x_0,\varepsilon)} K_I(x,f)\ dm(x) \ < \ \infty \qquad \forall x_0 \in \partial\mathbb{B}^n, \tag{8.75}$$

where $B^*(x_0,\varepsilon) = B(x_0,\varepsilon) \cap \mathbb{B}^n$, *then its extension by reflection in* $\partial\mathbb{B}^n$ *is an* FLD *homeomorphism of* $\mathbb{R}^n$.

In the following two theorems we state some mapping properties of FLD homeomorphisms.

Theorem 8.17. *Let* D *be a domain in* $\mathbb{R}^n, D \neq \mathbb{R}^n$, *and let* $f : D \to \mathbb{R}^n$ *be an* FLD *homeomorphism. If there exist a point* $b \in \partial D$ *and a neighborhood* U *of* b *such that* $K_I(x,f)|_{D\cap U} \in L^1$, *then* $f(D) \neq \mathbb{R}^n$.

Theorem 8.18. *Let* E *be a nondegenerate continuum in* $\mathbb{B}^n$, $D = \mathbb{B}^n \setminus E$, *and let* $f : D \to \mathbb{R}^n$ *be an* FLD *homeomorphism. If there exist a point* $x_0 \in \partial D \cap \mathbb{B}^n$ *and a neighborhood* U *of* x_0 *such that* $K_I(x,f)|_{D\cap U} \in L^1$, *then* $f(D)$ *is not a punctured topological ball.*

Moreover, we have the following estimations of distortions under FLD homeomorphisms.

Theorem 8.19. *Let* $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ *be an* FLD *homeomorphism with* $K_I(x,f) \in L^1(\mathbb{B}^n)$, $f(0) = 0$, $h(\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n)) \geq \delta > 0$, *and, for some* $x_0 \in \mathbb{B}^n$, $h(f(x_0), f(0)) \geq \delta$. *Then*

$$|f(x)| \geq \psi(|x|) \tag{8.76}$$

for all $|x| < r = \min(|x_0|/2, 1-|x_0|)$, *where* $\psi(t)$ *is a strictly increasing function with* $\psi(0) = 0$ *that depends only on the* L^1*-norm of* K_I *in* $\mathbb{B}^n$, n, *and* δ.

Similar upper estimates are possible in terms of $K_I(y, f^{-1})$. However, the estimates in terms of $K_I(x,f)$ are more interesting.

Theorem 8.20. *Let* $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, *be an* FLD *homeomorphism such that* $D' = f(D)$ *omits at least two points* v *and* $w \in \overline{\mathbb{R}^n}$ *with* $h(v,w) \geq \delta > 0$. *Then, for every* $x_0 \in D$ *and* $x \in B(x_0, \varepsilon(x_0))$, $\varepsilon(x_0) \leq \operatorname{dist}(x_0, \partial D)$,

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\delta} \exp\left\{ -\int\limits_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r k_{x_0}^{\frac{1}{n-1}}(r)} \right\}, \tag{8.77}$$

where α_n *depends only on* n *and* $k_{x_0}(r)$ *is the mean integral value of* $K_I(x,f)$ *over the sphere* $|x - x_0| = r$.

Theorem 8.21. *Let* $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, *be an* FLD *homeomorphism such that* $D' = f(D)$ *omits at least two points* v *and* $w \in \overline{\mathbb{R}^n}$ *with* $h(v,w) \geq \delta > 0$. *If* $K_I(x,f) \leq Q(x)$, *where* Q *has finite mean oscillation at a point* $x_0 \in D$, *then*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\delta} \left\{ \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}} \right\}^{\beta_0} \tag{8.78}$$

for $x \in B(x_0, \varepsilon_0)$ *where* α_n *depends only on* n, $\varepsilon_0 < \operatorname{dist}(x_0, \partial D)$, *and* $\beta_0 > 0$ *depends only on the function* Q.

Let D be a domain in $\mathbb{R}^n$ and let $Q : D \to [1,\infty]$ be a measurable function. Let $\mathfrak{F}_{Q,\Delta}(D)$ be the class of all FLD homeomorphisms $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, such that $K_I(x,f) \leq Q(x)$ and $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta > 0$.

Theorem 8.22. *If* $Q \in \mathrm{FMO}$*, then* $\mathfrak{F}_{Q,\Delta}(D)$ *is a normal family.*

Corollary 8.18. *The class* $\mathfrak{F}_{Q,\Delta}(D)$ *is normal if, for every* $x_0 \in D$*,*

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0,\varepsilon)} Q(x) \ dm(x) < \infty. \tag{8.79}$$

Corollary 8.19. *The class* $\mathfrak{F}_{Q,\Delta}(D)$ *is normal if every* $x_0 \in D$ *is a Lebesgue point of* $Q(x)$*.*

Theorem 8.23. *Let* $\Delta > 0$ *and let* $Q : D \to [1,\infty]$ *be a measurable function such that*

$$\int\limits_0^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \infty \tag{8.80}$$

holds at every point $x_0 \in D$*, where* $\varepsilon(x_0) = \mathrm{dist}(x_0, \partial D)$ *and* $q_{x_0}(r)$ *denotes the mean integral value of* $Q(x)$ *over the sphere* $|x - x_0| = r$*. Then* $\mathfrak{F}_{Q,\Delta}$ *forms a normal family.*

Corollary 8.20. *The class* $\mathfrak{F}_{Q,\Delta}(D)$ *is normal if* $Q(x)$ *has singularities only of the logarithmic type of order not greater than* $n-1$ *at every point* $x \in D$*.*

8.8 On Semicontinuity of Outer Dilatations

Recall that by Corollary 8.7, every FLD homeomorphism f is a $Q(x)$-homeomorphism with $Q(x) = K_I(x,f)$ and hence is a strong ring $Q(x)$-homeomorphism with the same $Q(x)$. Thus, the following statement follows immediately from Theorem 7.5.

Theorem 8.24. *Let* D *be a domain in* $\mathbb{R}^n$*,* $n \geq 2$*, and* $f_m : D \to \mathbb{R}^n$ *a sequence of homeomorphisms with finite length distortion converging locally uniformly to a mapping* f*. If*

$$K_I(x, f_m) \ \leq \ Q(x) \in L^1_{\mathrm{loc}} \,, \tag{8.81}$$

then f *is either a homeomorphism or* $f \equiv \mathrm{const}$ *in* D*.*

Since $K_I(x,g) \leq K_O^{n-1}(x,g)$, for the conclusion of Theorem 8.24, it is sufficient to suppose that $K_O(x,f_m) \leq K(x) \in L^{n-1}_{\mathrm{loc}}$ instead of (8.81). Set

$$P_O(x,f) = (K_O(x,f))^{\frac{1}{n-1}}. \tag{8.82}$$

Lemma 8.6. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f_j : D \to \mathbb{R}^n$, $j = 1,2,\ldots$, a sequence of* FLD *homeomorphisms in D converging locally uniformly to a mapping $f : D \to \mathbb{R}^n$. Then at each point x_0 of differentiability of the mapping f,*

$$P_O(x_0, f) \leq \liminf_{h \to 0} \liminf_{j \to \infty} \frac{1}{h^n} \int\limits_{C(x_0,h)} P_O(y, f_j)\, dy, \tag{8.83}$$

where $C(x_0,h)$ denotes the cube in $\mathbb{R}^n$ centered at x_0 whose edges are oriented along the principal axes of quadratic form $(f'(x_0)z, f'(x_0)z)$ and have length h.

The proof of Lemma 8.6 is similar to the proof of Lemma 4.7 for mappings with bounded distortion in [97].

Proof for Lemma 8.6. We may assume that $x = 0$, $f(0) = 0$, and $f_j(0) = 0$, $j = 1,2,\ldots$. Let $e_1,\ldots,e_n$ be an orthonormal basis in $\mathbb{R}^n$ formed by the eigenvectors of $f'(0)^*f'(0)$. Now $f'(0)\mathbb{B}^n$ is an ellipsoid whose semiaxes $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the positive square roots of the corresponding eigenvalues of $f'(0)^*f'(0)$. We also abbreviate $C(h) = C(0;h)$. For $\varepsilon > 0$, we can choose $\delta = \delta(\varepsilon) > 0$ such that for $h \in (0,\delta)$ and all $y \in C(h)$,

$$|f(y) - f'(0)y| < h\varepsilon$$

since f is differentiable at 0. Since $f_j \to f$ locally uniformly, we have, for all $y \in C(h)$,

$$|f_j(y) - f'(0)y| < h\varepsilon \tag{8.84}$$

for $j > j_0$. The set $f'(0)C(h)$ is the rectangular parallelopiped

$$(-\lambda_1 h/2, \lambda_1 h/2) \times \cdots \times (-\lambda_n h/2, \lambda_n h/2),$$

where the edges are directed along the basis vectors $\tilde{e}_1,\ldots,\tilde{e}_n$, of $\mathbb{R}^n$:

$$\tilde{e}_i = \frac{f'(0)e_i}{|f'(0)e_i|}, \quad i = 1,2,\ldots,n,$$

that are orthogonal in view of the choice of the $e_1,\ldots,e_n$. Inequality (8.84) yields that the points $f_j(y)$, $y \in C(h)$, all lie in the parallelopiped

$$\left(-\left(\frac{\lambda_1}{2}+\varepsilon\right)h,\ \left(\frac{\lambda_1}{2}+\varepsilon\right)h\right) \times \cdots \times \left(-\left(\frac{\lambda_n}{2}+\varepsilon\right)h,\ \left(\frac{\lambda_n}{2}+\varepsilon\right)h\right).$$

Here $\mathbb{R}^n$ is again equipped with the basis $\tilde{e}_1,\ldots,\tilde{e}_n$. Thus,

$$\operatorname{mes}(f_j(C(h))) \leq h^n(\lambda_1 + 2\varepsilon)(\lambda_2 + 2\varepsilon)\cdots(\lambda_n + 2\varepsilon). \tag{8.85}$$

Further, by the Lebesgue theorem for locally finite measures (see [281], pp. 115 and 119), we obtain from (8.85) the inequality

$$\int_{C(h)} |J(y,f_j)|\, dy \;\leq\; \operatorname{mes}(f_j(C(h))) \leq h^n[|J(0,f)| + \triangle(\varepsilon)], \tag{8.86}$$

where $\triangle(\varepsilon) \to 0$ as $\varepsilon \to 0$ because $J(0,f) = \lambda_1\lambda_2 \cdot \ldots \cdot \lambda_n$.

Next consider the $(n-1)$-dimensional cube $C^*(h)$ with center at $x = 0$ and edges (of length h) oriented along $e_1,\ldots,e_{n-1}$. Consider a segment $l(z)$, $z \in C^*(h)$, perpendicular to $C^*(h)$ inside $C(h)$ and write $l_j(z)$ for the length of the path $f_j(l(z))$. Since f is ACP, we have

$$l_j(z) \;=\; \int_{-h/2}^{h/2} |f_j'(z,y_n)e_n|\, dy_n \tag{8.87}$$

for almost every $z \in C^*(h)$ with respect to the $(n-1)$-dimensional Lebesgue measure. On the other hand, (8.84) implies

$$l_j(z) \geq (|f'(0)e_n| - 2\varepsilon)h = (\lambda_n - 2\varepsilon)h.$$

Hence, (8.87) yields

$$\int_{-h/2}^{h/2} |f_j'(z,y_n)e_n|\, dy_n \;\geq\; h(\lambda_n - 2\varepsilon)$$

for a.e. $z \in C^*(h)$. Integrating over $C^*(h)$ and using the Fubini theorem, we obtain

$$\int_{C(h)} |f_j'(y)e_n|\, dy \;\geq\; h^n(\lambda_n - 2\varepsilon). \tag{8.88}$$

Next, the Hölder inequality gives

$$\begin{aligned} \int_{C(h)} |f_j'(y)e_n|\, dy &\leq \int_{C(h)} \|f_j'(y)\|\, dy \\ &\leq \int_{C(h)} K^{1/n}(y,f_j)|J(y,f_j)|^{1/n}\, dy \\ &\leq \left(\int_{C(h)} K^{\frac{1}{n-1}}(y,f_j)\, dy\right)^{\frac{n-1}{n}} \left(\int_{C(h)} |J(y,f_j)|dy\right)^{\frac{1}{n}}. \end{aligned} \tag{8.89}$$

Here the equality $\|f_j'(y)\|^n = K(y,f_j)|J(y,f_j)|$ a.e. has also been used. Now (8.89) together with (8.86) and (8.88) yields

$$\left(\frac{(\lambda_n-2\varepsilon)^n}{|J(0,f)|+\triangle(\varepsilon)}\right)^{\frac{1}{n-1}} \leq \frac{1}{h^n}\int\limits_{C(h)} K^{\frac{1}{n-1}}(y,f_j)dy.$$

Letting first $j\to\infty$ and then $h\to 0$ and finally $\varepsilon\to 0$, we complete the proof. $\square$

Applying the Jensen inequality to (8.83), we obtain the following conclusion.

Corollary 8.21. *Under the hypothesis and terms of Lemma 8.6,*

$$\Phi\left(P_O\left(x_0,f\right)\right) \leq \liminf_{h\to 0}\ \liminf_{j\to\infty}\frac{1}{h^n}\int\limits_{C(x_0,h)}\Phi\left(P_O\left(y,f_j\right)\right)\,dy \tag{8.90}$$

for every increasing convex function $\Phi(t):[1,+\infty]\to[0,+\infty]$.

In particular, for $\Phi(t)=t^{n-1}$, we have the next conclusion.

Corollary 8.22. *Under the assumptions of Lemma 8.6,*

$$K_O(x_0,f) \leq \liminf_{h\to 0}\ \liminf_{j\to\infty}\frac{1}{h^n}\int\limits_{C(x_0,h)}K_O(y,f_j)\,dy\,. \tag{8.91}$$

Theorem 8.25. *Let D be a domain in $\mathbb{R}^n$, $n\geq 2$, and $f_m:D\to\mathbb{R}^n$ a sequence of* FLD *homeomorphisms converging locally uniformly to an* FLD *homeomorphism f. If*

$$K_O\left(x,f_m\right) \leq K\left(x\right)\in L^1_{\mathrm{loc}} \qquad m=1,2,3,\ldots, \tag{8.92}$$

then

$$K_O\left(x,f\right)\leq\limsup_{j\to\infty}K_O\left(x,f_j\right)\quad a.e. \tag{8.93}$$

Proof. Applying Corollary 8.22 and the theorem on term-by-term integration, we have

$$K_O\left(x,f\right)\leq\liminf_{h\to 0}\frac{1}{h^n}\int\limits_{C(x,h)}\limsup_{j\to\infty}K_O(y,f_j)\,dy. \tag{8.94}$$

Now, by the theorem on the differentiability of the indefinite Lebesgue integral and (8.92), we obtain a.e. the equality

$$\lim_{h\to 0}\frac{1}{h^n}\int\limits_{C(x,h)}\limsup_{j\to\infty}K_O\left(y,f_j\right)\,dy=\limsup_{j\to\infty}K_O\left(x,f_j\right). \tag{8.95}$$

Combining *(8.94)* and *(8.95),* we come to *(8.93).* $\square$

Theorem 8.26. *Let D be a domain in $\mathbb{R}^n$, $n\geq 2$, and $f_j:D\to\mathbb{R}^n$, $j=1,2,\ldots$, a sequence of* FLD *homeomorphisms converging locally uniformly to an* FLD *homeomorphism f. Suppose that $\Phi:[1,+\infty]\to[0,\infty]$ is a convex increasing function and a.e.*

$$P_O(x, f_j) \leq K(x), \tag{8.96}$$

where

$$\Phi\,(K(x)) \in L^1_{\mathrm{loc}}. \tag{8.97}$$

Then

$$\int\limits_E \Phi\,(P_O\,(x,f))\,dx \leq \liminf_{j\to\infty} \int\limits_E \Phi\,(P_O\,(x,f_j))\,dx \tag{8.98}$$

for every measurable set $E \subset D$ *with* mes $E < \infty$.

Proof. By Corollary 8.21 and (8.96), we get $\Phi\,(P_O\,(x,f)) \leq \Phi\,(K\,(x))$ a.e. and hence $\Phi\,(P_O\,(x,f)) \in L^1_{\mathrm{loc}}\,(D)$ by (8.97). Consequently, by the theorem of the differentiability of an indefinite integral a.e.,

$$\lim_{h\to 0} \frac{1}{h^n} \int\limits_{C(x,h)} \Phi\,(P_O(y,f))\,dy = \Phi\,(P_O(x,f))\,. \tag{8.99}$$

Let E_0 be the set of all $x \in D$, where either f is not differentiable or *(8.99)* does not hold. Note that mes $E_0 = 0$. By Corollary 8.21,

$$\Phi\,(P_O(x,f)) \leq \liminf_{h\to 0}\ \liminf_{j\to\infty} \frac{1}{h^n} \int\limits_{C(x,h)} \Phi\,(P_O(y,f_j))\,dy \quad \forall\, x \in D \backslash E_0\,.$$

Hence, we have $\forall\, \varepsilon > 0 : \exists\, \delta = \delta\,(x,\varepsilon) : \forall\, h < \delta$:

$$h^n \Phi\,(P_O(x,f)) < \liminf_{j\to\infty} \int\limits_{C(x,h)} \Phi\,(P_O(y,f_j))\,dy + \varepsilon h^n\,, \tag{8.100}$$

and by (8.99),

$$\int\limits_{C(x,h)} \Phi\,(P_O(y,f))\,dy < \liminf_{j\to\infty} \int\limits_{C(x,h)} \Phi\,(P_O(y,f_j))\,dy + \varepsilon h^n \tag{8.101}$$

for $h < \delta = \delta(x,\varepsilon)$.

Let $\Omega \subset D$ be an open set. The system of cubes $C(x,h)$, $x \in \Omega \backslash E_0$, forms the Vitali covering of the set $\Omega \backslash E_0$. Thus, by the Vitali theorem (see IV(3.1) in [281]), there is a sequence of mutually disjoint cubes $C_m = C(x_m, h_m) \subseteq \Omega$ such that

$$\text{mes}\,\left(\Omega \backslash \bigcup C_m\right) = 0\,.$$

Thus, by (8.101), we obtain

$$\int\limits_\Omega \Phi\,(P_O(y,f))\,dy \leq \liminf_{j\to\infty} \int\limits_\Omega \Phi\,(P_O(y,f_j))\,dy + \varepsilon\,\text{mes}\ \Omega\,, \tag{8.102}$$

and, since $\varepsilon > 0$ is arbitrary, (8.98) has been proved for an arbitrary open set $\Omega \subset D$ with mes $\Omega < \infty$.

Now, let E be a measurable set in D with mes $E < \infty$. Then, for every $\varepsilon > 0$, there is an open set $\Omega = \Omega_\varepsilon \supseteq E$ with mes $(\Omega_\varepsilon \setminus E) < \varepsilon$; see III(6.6) in [281]. From inequality (8.98) for open Ω, we have

$$\int_E \Phi\left(P_O(y,f)\right) dy \le \liminf_{j\to\infty} \int_E \Phi\left(P_O(y,f_j)\right) dy + \int_{\Omega_\varepsilon \setminus E} \Phi\left(K(y)\right) dy .$$

Finally, in view of (8.97), we obtain (8.98) by the absolute continuity of integrals. □

8.9 On Convergence of Matrix Dilatations

Theorem 8.27. *Let $f, f_j : D \to \mathbb{R}^n$ be* FLD *homeomorphisms such that $f_j \to f$ as $j \to \infty$ locally uniformly in D, M, M_j their matrix dilatations and let a.e. $K(x, f_j) \le K(x) \in L^1_{\mathrm{loc}}$. Suppose that*

$$U_j(x) M_j(x) \to M_0(x) \qquad a.e. \tag{8.103}$$

as $j \to \infty$ for some sequence of orthogonal matrices $U_j(x)$. Then

$$M(x) = U(x) M_0(x) \qquad a.e. \tag{8.104}$$

for some orthogonal matrix $U(x)$.

Proof. Let $A_m : \mathbb{R}^n \to \mathbb{R}^n$ be some enumeration of all matrices with rational elements that satisfy the condition det $A_m = 1$. Note that each matrix A_m defines a quasiconformal linear mapping $x \to A_m x$, where $x \in \mathbb{R}^n$ is interpreted as a vector-column. Let N and N_j be matrix dilatations of the mappings $f \circ A_m^{-1}$ and $f_j \circ A_m^{-1}$, respectively. Then, by the composition rule [see (8.36)],

$$N(y) = M\left(A_m^{-1} y\right) A_m^{-1}, \quad N_j(y) = M_j\left(A_m^{-1} y\right) A_m^{-1} . \tag{8.105}$$

Note that the mappings $f \circ A_m^{-1}$ and $f_j \circ A_m^{-1}$ are also FLD homeomorphisms such that $f_j \circ A_m^{-1} \to f \circ A_m^{-1}$ locally uniformly as $j \to \infty$ and

$$K_O\left(y, f_j \circ A_m^{-1}\right) \le K_O\left(A_m^{-1} y, f_j\right) K_O\left(A_m^{-1}\right) \le K\left(A_m^{-1} y\right) \cdot c_m^n \in L^1_{\mathrm{loc}},$$

where $c_m = \|A_m^{-1}\|$ is the matrix norm of A_m^{-1}; see (2.11), Chapter I in [256]. Thus, for each fixed $m \in \mathbf{N}$, the mappings $f \circ A_m^{-1}$ and $f_j \circ A_m^{-1}$ satisfy the conditions of Theorem 8.25. By Theorem *8.25,* for each fixed $m = 1, 2, \ldots,$

$$\|M\left(A_m^{-1}y\right)A_m^{-1}\| \le \limsup_{j\to\infty} \|M_j\left(A_m^{-1}y\right)A_m^{-1}\| \qquad \text{a.e.} \tag{8.106}$$

because $K_O(x, f) = \|M_f(x)\|^n$. Consequently, we also have for each fixed $m = 1,2,\ldots,$

$$\|M(x)A_m^{-1}\| \le \limsup_{j\to\infty} \|M_j(x)A_m^{-1}\| \qquad \text{a.e.} \tag{8.107}$$

Since the collection of the matrices A_m is countable, (8.107) holds for a.e. $x \in D$ just for all $m = 1,2,\ldots$. Note that the set of the matrices $\{A_m\}_{m=1}^{\infty}$ is dense in the space of all matrices with $\det A = 1$ and it follows from (8.103) that $\det M_0(x) = 1$ a.e. Thus, we have from (8.107) that

$$\|M(x)M_0^{-1}(x)\| \le \limsup_{j\to\infty} \|M_j(x)M_0^{-1}(x)\| \quad \text{a.e.} \tag{8.108}$$

Since the U_j are orthogonal matrices and $U_j(x)M_j(x) \to M_0(x)$ a.e., we have from (8.108) that

$$\|M(x)M_0^{-1}(x)\| \le 1 \qquad \text{a.e.} \tag{8.109}$$

On the other hand (see, e.g., (2.6), Chapter I, in [256]),

$$\|M(x)M_0^{-1}(x)\| \ge 1. \tag{8.110}$$

Thus,

$$\|M(x)M_0^{-1}(x)\| = 1 \qquad \text{a.e.} \tag{8.111}$$

Finally, from (8.111), taking into account that $\det M(x)M_0^{-1}(x) = 1$ a.e., we have that $M(x)M_0^{-1}(x)$ is an orthogonal matrix $U(x)$, i.e., $M(x) = U(x)M_0(x)$ a.e. The proof is complete. □

In what follows, for an arbitrary matrix M, denote

$$G = M^*M, \tag{8.112}$$

where M^* is the transpose of M. First, though, we observe the following algebraic fact.

Proposition 8.8. *Let M_1 and M_2 be real nonsingular square matrices. Then*

$$G_2 = G_1 \tag{8.113}$$

if and only if

$$M_2 = UM_1, \tag{8.114}$$

where $U \in \mathscr{O}(n)$.

Here we write $U \in \mathscr{O}(n)$ if a matrix U is orthogonal, i.e.,

$$U^*U = I = UU^*.$$

Geometrically, (8.114) means that G determines M from (8.112) up to left rotations. Indeed, (8.114) implies

$$G_2 = M_2^* M_2 = M_1^* U^* U M_1 = M_1^* M_1 = G_1 .$$

Conversely, let (8.113) take place. Then, for $M = M_1 M_2^{-1}$, we have

$$G = M^* M = (M_2^*)^{-1} M_1^* M_1 M_2^{-1} = (M_2^*)^{-1} G_1 M_2^{-1}$$
$$= (M_2^*)^{-1} G_2 M_2^{-1} = (M_2^*)^{-1} M_2^* M_2 M_2^{-1} = I,$$

i.e., M is an orthogonal matrix.

Further, applying the well-known diagonalization theorem for the symmetric matrices and the continuous dependence of the roots for the characteristic polynomial, we have the following conclusion, which was first proved as Lemma 3.18 in [97].

Lemma 8.7. *Let $M_j, j = 0, 1, 2, \ldots,$ be real nonsingular square matrices. Then*

$$\lim_{j\to\infty} G_j = G_0 \tag{8.115}$$

is equivalent to

$$\lim_{j\to\infty} U_j M_j = M_0 \tag{8.116}$$

for some orthogonal matrices U_j.

Here the convergence of matrices is understood in the element-wise sense.

Proof. Indeed, (8.116) obviously implies (8.115) because

$$G_j = M_j^* M_j = M_j^* U_j^* U_j M_j = (U_j M_j)^* (U_j M_j).$$

Conversely, let (8.115) take place. Then for $N_j = M_j M_0^{-1}$, we have

$$D_j = N_j^* N_j = (M_0^*)^{-1} M_j^* M_j M_0^{-1} = (M_0^*)^{-1} G_j M_0^{-1} \to (M_0^*)^{-1} G_0 M_0^{-1} = I$$

as $j \to \infty$ element-wise.

It is well known from algebra (see, e. g., [23], p. 54) that

$$D_j = V_j^* \Lambda_j^2 V_j ,$$

where $V_j \in \mathscr{O}(n), j = 0, 1, 2, \ldots,$ and the Λ_j^2 are diagonal matrices with eigenvalues of D_j on their diagonals. By Proposition 8.8,

$$N_j = W_j \Lambda_j V_j ,$$

where $W_j \in \mathscr{O}(n), j = 1, 2, \ldots.$

Further, the eigenvalues of D_j are roots of the so-called characteristic equations for D_j where the coefficients are continuous functions of the elements of D_j; see, e.g., [23], p. 34. Next, in view of the continuous dependence of roots of polynomials from their coefficients (see, e.g., [125], p. 634, and [235]), we obtain $\Lambda_j \to I$ as $j \to \infty$.

Now, set

$$U_j = V_j{}^* W_j{}^* \in \mathscr{O}(n). \tag{8.117}$$

Then the maximal element of the matrix

$$\Delta_j = U_j N_j - I = V_j^* (\Lambda_j - I) V_j$$

does not exceed the maximal element of the matrix $(\Lambda_j - I)$, because $V_j \in \mathscr{O}(n)$. Thus, we obtain (8.116) with U_j given by (8.117). □

From Lemma 8.7 we arrive at the following consequence of Theorem 8.27, which was first established for quasiregular mappings in [129]; cf. [97].

Corollary 8.23. *Let $f, f_j : D \to \mathbb{R}^n$ be* FLD *homeomorphisms such that $f_j \to f$ locally uniformly as $j \to \infty$. Let G and G_j be the dilatation tensors of f and f_j, respectively, and let $K_O(x, f_j) \le K(x) \in L^1_{\mathrm{loc}}$ a.e. If*

$$G_j(x) \to G_0(x) \qquad a.e. \tag{8.118}$$

as $j \to \infty$, then

$$G(x) = G_0(x) \qquad a.e. \tag{8.119}$$

8.10 Examples and Subclasses

It is easy to give examples of FLD mappings that are not discrete, open, or sense-preserving; see, e.g., [296] and [217]. Consider one more interesting mapping below.

Example. Let C be a Cantor set in $[0,1]$. Then $(0,1) \setminus C = \cup I_j$, where the intervals I_j are open and disjoint. Define $f : (0,1) \to \mathbb{R}$ as $f(x) = 0$ for $x \in C$ and

$$f(x) = \frac{|b_j - a_j|}{2} - \left| x - \frac{b_j + a_j}{2} \right| \tag{8.120}$$

for $x \in I_j = (a_j, b_j)$, $j = 1, 2, \ldots$, $f(x) = x$ for $x \le 0$, and $f(x) = x - 1$ for $x \ge 1$. Then f is an FLD mapping provided that $|C| = 0$. In this case the mapping f is length-preserving:

$$l(f \circ \gamma) = l(\gamma) \quad \text{for every } \gamma. \tag{8.121}$$

Moreover,

$$l(x,f) = L(x,f) = 1 \quad \text{a.e.} \tag{8.122}$$

For $|C| > 0$, f is not an FLD mapping because $l(x,f) = 0$ for all $x \in C$.

The mapping $f : \mathbb{R} \to \mathbb{R}$ has the natural extension f^* to $\mathbb{R}^n$, $n \geq 2$,

$$f^*(z,x) = (z, f(x)), \quad z \in \mathbb{R}^{n-1},\ x \in \mathbb{R}, \tag{8.123}$$

with similar properties. Note that f as well as f^* are neither discrete nor open as well as sense-preserving.

General considerations for length-preserving mappings in metric spaces can be found in [35]. However, the main results there deal with locally injective mappings.

Proposition 8.9. *Let $f : D \to \mathbb{R}^n$ be a mapping satisfying the condition*

$$L^{-1} \cdot l(\gamma) \leq l(f \circ \gamma) \leq L \cdot l(\gamma) \tag{8.124}$$

for some $L \geq 1$ and for every path γ in D and let

$$l(x,f) > 0 \quad \textit{a.e.} \tag{8.125}$$

Then f is of finite length distortion.

Proof. Indeed, (8.121) implies the (L)-property and the inequality

$$L(x,f) \leq L < \infty \quad \text{for all } x \in D. \tag{8.126}$$

Since f is L-Lipschitz, f clearly satisfies the (N)-property. □

Remark 8.8. Note that condition (8.121) implies that f is weakly light. If f is light,

$$L(x,f) < \infty \quad \text{for all } x \in D \tag{8.127}$$

and $J(x,f) \geq 0$ a.e. in D or $J(x,f) \leq 0$ a.e. in D, then, by Theorem 8 in [44], f is open. However, the above example shows that, in general, the mappings in Proposition 8.9 need not be discrete, open, as well as sense-preserving; hence, they are not of bounded length distortion in the sense of Martio–Väisälä [213].

As is clear from Remark 8.4, points (1) and (4), and the Rademacher–Stepanoff theorem, the (L)-property does not imply that $L(x,f) < \infty$ a.e. in $\mathbb{R}^n, n \geq 2$. On the other hand, it is easy to give examples of FMD mappings $\mathbb{R}^n, n \geq 2$, without the (L)-property; see, e.g., [65] and [190].

Chapter 9
Lower Q-Homeomorphisms

So far the upper estimates of moduli have played a major role in the theory of quasiconformal mappings and their generalizations. In the present chapter, we elucidate possibilities of lower estimates of moduli for families of $(n-1)$-dimensional surfaces under mappings with finite distortion; see [161–164]. In particular, this makes it possible for us to investigate the boundary behavior of homeomorphisms with finite area distortion in the next chapter.

9.1 Introduction

The following concept is motivated by Gehring's ring definition of quasiconformality in [73].

Given domains D and D' in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geq 2$, $x_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, we say that a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at the point** x_0 if

$$M(f\Sigma_\varepsilon) \geq \inf_{\rho \in \operatorname{ext} \operatorname{adm} \Sigma_\varepsilon} \int\limits_{D \cap R_\varepsilon} \frac{\rho^n(x)}{Q(x)}\, dm(x) \tag{9.1}$$

for every ring

$$R_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \qquad \varepsilon \in (0, \varepsilon_0),\ \varepsilon_0 \in (0, d_0),$$

where

$$d_0 = \sup_{x \in D} |x - x_0|, \tag{9.2}$$

and Σ_ε denotes the family of all intersections of the spheres

$$S(r) = S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \qquad r \in (\varepsilon, \varepsilon_0),$$

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_9,

with D. As usual, the notion can be extended to the case $x_0 = \infty \in \overline{D}$ by applying the inversion T with respect to the unit sphere in $\overline{\mathbb{R}^n}$, $T(x) = x/|x|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at** $\infty \in \overline{D}$ if $F = f \circ T$ is a lower Q_*-homeomorphism with $Q_* = Q \circ T$ at 0.

We also say that a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a **lower Q-homeomorphism in** D if f is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$.

We show here that condition (9.1) is equivalent to the inequality

$$M(f\Sigma_\varepsilon) \ \geq \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)}\,, \tag{9.3}$$

where

$$||Q||_{n-1}(r) \ = \left(\int\limits_{D(x_0,r)} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}}\,, \tag{9.4}$$

$d\mathscr{A}$ corresponds to the area on the surface $D(x_0,r) = D \bigcap S(x_0,r)$. Note that the infimum from the right-hand side in (9.1) is attained for the function

$$\rho_0(x) \ = \ \frac{Q(x)}{||Q||_{n-1}(|x|)}\,. \tag{9.5}$$

Later we often assume that $Q \equiv 0$ outside D and take the integrals in (9.4) over the whole spheres $S(x_0,r)$.

This allows us to find the corresponding estimates of distortion for distance under lower Q-homeomorphisms and to investigate the removability of isolated singularities and the boundary behavior of such mappings.

Let Σ_ε^* be the family of all $(n-1)$-dimensional surfaces in D that separate the spheres $|x-x_0| = \varepsilon$ and $|x-x_0| = \varepsilon_0$ in D. Note that (9.3) implies the corresponding lower estimate for Σ_ε^* because $\Sigma_\varepsilon \subset \Sigma_\varepsilon^*$ and, hence, $\mathrm{adm}\, f\Sigma_\varepsilon^* \subset \mathrm{adm}\, f\Sigma_\varepsilon$. However, inequality (9.3) for Σ_ε^* is not precise. The same is true for Σ_ε^{**} consisting of all closed sets C in D that separate the given spheres in D. Indeed, $\Sigma_\varepsilon \subseteq \Sigma_\varepsilon^{**}$ and hence $\mathrm{adm}\, f\Sigma_\varepsilon^{**} \subset \mathrm{adm}\, f\Sigma_\varepsilon$; cf. [340]. Thus, $M(f\Sigma_\varepsilon)$ is majorized by $M(f\Sigma_\varepsilon^*)$ as well as by $M(f\Sigma_\varepsilon^{**})$. However, estimate (9.3) is not precise for such families of surfaces. Simultaneously, condition (9.1) gives the widest class of mappings satisfying (9.3) and hence to which the whole upcoming theory is applicable.

9.2 On Moduli of Families of Surfaces

In this section H^k, $k = 1, \dots, n-1$ denotes the **k-dimensional Hausdorff measure** in $\mathbb{R}^n$, $n \geq 2$. More precisely, if A is a set in $\mathbb{R}^n$, then

$$H^k(A) = \sup_{\varepsilon>0} H^k_\varepsilon(A), \tag{9.6}$$

$$H^k_\varepsilon(A) = \Omega_k \inf \sum_{i=1}^{\infty} \left(\frac{\delta_i}{2}\right)^k, \tag{9.7}$$

where the infimum is taken over all countable collections of numbers $\delta_i \in (0,\varepsilon)$ such that some sets A_i in $\mathbb{R}^n$ with diameters δ_i cover A. Here Ω_k denotes the volume of the unit ball in $\mathbb{R}^k$. H^k is an **outer measure in the sense of Caratheodory**, i.e.,

(1) $H^k(X) \le H^k(Y)$ whenever $X \subseteq Y$,

(2) $H^k(\Sigma X_i) \le \Sigma H^k(X_i)$ for each sequence X_i of sets,

(3) $H^k(X \cup Y) = H^k(X) + H^k(Y)$ whenever $\operatorname{dist}(X,Y) > 0$.

A set $E \subset \mathbb{R}^n$ is called **measurable** with respect to H^k if $H^k(X) = H^k(X \cap E) + H^k(X \setminus E)$ for every set $X \subset \mathbb{R}^n$. It is well known that every Borel set is measurable with respect to any outer measure in the sense of Caratheodory; see, e.g., [281], p. 52. Moreover, H^k is Borel regular, i.e., for every set $X \subset \mathbb{R}^n$, there is a Borel set $B \subset \mathbb{R}^n$ such that $X \subset B$ and $H^k(X) = H^k(B)$; see, e.g., [281], p. 53, and Section 2.10.1 in [55]. The latter implies that for every measurable set $E \subset \mathbb{R}^n$, there exist Borel sets B_* and $B^* \subset \mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $H^k(B^* \setminus B_*) = 0$; see, e.g., Section 2.2.3 in [55]. In particular, $H^k(B^*) = H^k(E) = H^k(B_*)$.

Let ω be an open set in $\overline{\mathbb{R}^k}$, $k = 1, \dots, n-1$. A (continuous) mapping $S : \omega \to \mathbb{R}^n$ is called a k-dimensional surface S in $\mathbb{R}^n$. Sometimes we call the image $S(\omega) \subseteq \mathbb{R}^n$ the surface S, too. The number of preimages

$$N(S,y) = \operatorname{card} S^{-1}(y) = \operatorname{card}\{x \in \omega : S(x) = y\} \tag{9.8}$$

is said to be a **multiplicity function** of the surface S at a point $y \in \mathbb{R}^n$. In other words, $N(S,y)$ denotes the multiplicity of covering of the point y by the surface S. It is known that the multiplicity function is lower semi continuous, i.e.,

$$N(S,y) \ge \liminf_{m\to\infty} N(S,y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1,2,\dots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$; see, e.g., [246], p. 160. Thus, the function $N(S,y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure H^k; see, e.g., [281], p. 52.

k-dimensional Hausdorff area in $\mathbb{R}^n$ (or simply **area**) associated with a surface $S : \omega \to \mathbb{R}^n$ is given by

$$\mathscr{A}_S(B) = \mathscr{A}^k_S(B) := \int_B N(S,y)\, dH^k y \tag{9.9}$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set that is measurable with respect to H^k in $\mathbb{R}^n$. The surface S is **rectifiable** if $\mathscr{A}_S(\mathbb{R}^n) < \infty$.

If $\rho : \mathbb{R}^n \to [0,\infty]$ is a Borel function, then its **integral over** S is defined by the equality

$$\int\limits_S \rho \, d\mathscr{A} \; := \int\limits_{\mathbb{R}^n} \rho(y)\, N(S,y)\, dH^k y. \tag{9.10}$$

Given a family Γ of k-dimensional surfaces S, a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called **admissible** for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_S \rho^k \, d\mathscr{A} \; \geq \; 1 \tag{9.11}$$

for every $S \in \Gamma$. Given $p \in (0,\infty)$, the **p-modulus** of Γ is the quantity

$$M_p(\Gamma) \; = \; \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x). \tag{9.12}$$

We also set

$$M(\Gamma) \; = \; M_n(\Gamma) \tag{9.13}$$

and call the quantity $M(\Gamma)$ the **modulus of the family** Γ. The modulus is itself an outer measure on the collection of all families Γ of k-dimensional surfaces.

We say that Γ_2 is **minorized** by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every $S \subset \Gamma_2$ has a subsurface that belongs to Γ_1. It is known that $M_p(\Gamma_1) \geq M_p(\Gamma_2)$; see [64], pp. 176–178. We also say that a property P holds for **p-a.e.** (almost every) k-dimensional surface S in a family Γ if a subfamily of all surfaces of Γ, for which P fails, has the p-modulus zero. If $0 < q < p$, then P also holds for q-a.e. S; see Theorem 3 in [64]. In the case $p = n$, we write simply a.e.

Remark 9.1. The definition of the modulus immediately implies that, for every $p \in (0,\infty)$ and $k = 1,\ldots,n-1$

(1) p-a.e. k-dimensional surface in $\mathbb{R}^n$ is rectifiable,

(2) given a Borel set B in $\mathbb{R}^n$ of (Lebesgue) measure zero,

$$\mathscr{A}_S(B) \; = \; 0 \tag{9.14}$$

for p-a.e. k-dimensional surface S in $\mathbb{R}^n$.

Lemma 9.1. *Let $k = 1,\ldots,n-1$, $p \in [k,\infty)$, and let C be an open cube in $\mathbb{R}^n$, $n \geq 2$, whose edges are parallel to coordinate axes. If a property P holds for p-a.e. k-dimensional surface S in C, then P also holds for a.e. k-dimensional plane in C that is parallel to a k-dimensional coordinate plane H.*

The latter a.e. is related to the Lebesgue measure in the corresponding $(n-k)$-dimensional coordinate plane $H^{\perp}$ that is perpendicular to H.

Proof. Let us assume that the conclusion is not true. Then, by the regularity of the Lebesgue measure m_{n-k} in $H^{\perp}$, there is a Borel set B such that $m_{n-k}(B) > 0$ and P fails for a.e. k-dimensional plane S in C that is parallel to H and intersects B. If a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is admissible for the given family Γ of surfaces S such that $\rho \equiv 0$ outside $C_0 \times B$, where C_0 is the projection of C on H, then, by the Hölder inequality,

$$\int\limits_{C_0\times B} \rho^k(x)\,dm(x) \;\le\; \left(\int\limits_{C_0\times B} \rho^p(x)\,dm(x)\right)^{\frac{k}{p}} \left(\int\limits_{C_0\times B} dm(x)\right)^{\frac{p-k}{p}}$$

and, hence, by the Fubini theorem,

$$\int\limits_{\mathbb{R}^n} \rho^p(x)\,dm(x) \;\ge\; \frac{\left(\int\limits_{C_0\times B} \rho^k(x)\,dm(x)\right)^{\frac{p}{k}}}{\left(\int\limits_{C_0\times B} dm(x)\right)^{\frac{p-k}{k}}} \;\ge\; \frac{(m_{n-k}(B))^{\frac{p}{k}}}{(h^k\cdot m_{n-k}(B))^{\frac{p-k}{k}}},$$

i.e.,

$$M_p(\Gamma) \;\ge\; \frac{m_{n-k}(B)}{h^{p-k}},$$

where h is the length of the edge of cube C. Thus, $M_p(\Gamma) > 0$, which contradicts the lemma's hypothesis. $\Box$

The following statement is an analogue of the Fubini theorem; cf., e.g., [281], p. 77. It extends Theorem 33.1 in [316]; cf. also Theorem 3 in [64], Lemma 2.13 in [207], and Lemma 8.1 here.

Theorem 9.1. *Let $k = 1,\dots,n-1$, $p \in [k,\infty)$, and let E be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \ge 2$. Then E is measurable by Lebesgue in $\mathbb{R}^n$ if and only if E is measurable with respect to area on p-a.e. k-dimensional surface S in Ω. Moreover, $|E| = 0$ if and only if*

$$\mathscr{A}_S(E) \;=\; 0 \tag{9.15}$$

on p-a.e. k-dimensional surface S in Ω.

Proof. By the Lindelöf property in $\mathbb{R}^n$ and the minorant property of M_p, we may assume without loss of generality that Ω is an open cube C in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes.

Suppose first that E is Lebesgue measurable in $\mathbb{R}^n$. Then, by the regularity of the Lebesgue measure, there exist Borel sets B_* and B^* in $\mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $|B^*\setminus B_*| = 0$. Thus, by (2) in Remark 9.1, $\mathscr{A}_S(B^*\setminus B_*) = 0$ and hence E is measurable by area on p-a.e. k-dimensional surface S in C. Conversely, if the latter is true, then E is measurable by area on a.e. k-dimensional plane H in C that is parallel to a k-dimensional coordinate plane; see Lemma 9.1. Thus, E is measurable by the Fubini theorem.

Now, suppose that $|E| = 0$. Then there is a Borel set B such that $|B| = 0$ and $E \subset B$. Then, by (2) in Remark 9.1, the relation (9.15) holds for p-a.e. k-dimensional surface S in C. Conversely, if the latter is true, then, in particular, $\mathscr{A}_S(E) = 0$ on a.e. k-dimensional plane H in C, which is parallel to a k-dimensional coordinate plane; see Lemma 9.1. Thus, $|E| = 0$, again by the Fubini theorem. $\square$

Remark 9.2. Say by the Lusin theorem (see, e.g., Section 2.3.5 in [55]), for every measurable function $\rho : \mathbb{R}^n \to [0,\infty]$, there is a Borel function $\rho^* : \mathbb{R}^n \to [0,\infty]$ such that $\rho^* = \rho$ a.e. in $\mathbb{R}^n$. Thus, by Theorem 9.1, ρ is measurable on p-a.e. k-dimensional surface S in $\mathbb{R}^n$ for every $p \in (0,\infty)$ and $k = 1,\ldots,n-1$.

We say that a Lebesgue measurable function $\rho : \mathbb{R}^n \to [0,\infty]$ is **p-extensively admissible** for a family Γ of k-dimensional surfaces S in $\mathbb{R}^n$, abbr. $\rho \in \mathrm{ext}_p \,\mathrm{adm}\,\Gamma$, if

$$\int_S \rho^k \, d\mathscr{A} \;\geq\; 1 \tag{9.16}$$

for p-a.e. $S \in \Gamma$. The p**-extensive modulus** $\overline{M}_p(\Gamma)$ of Γ is the quantity

$$\overline{M}_p(\Gamma) \;=\; \inf \int_{\mathbb{R}^n} \rho^p(x)\, dm(x), \tag{9.17}$$

where the infimum is taken over all $\rho \in \mathrm{ext}_p \,\mathrm{adm}\,\Gamma$. In the case $p = n$, we use the notations $\overline{M}(\Gamma)$ and $\rho \in \mathrm{ext}\,\mathrm{adm}\,\Gamma$, respectively. For every $p \in (0,\infty)$, $k = 1,\ldots,n-1$, and every family Γ of k-dimensional surfaces in $\mathbb{R}^n$,

$$\overline{M}_p(\Gamma) \;=\; M_p(\Gamma)\,. \tag{9.18}$$

9.3 Characterization of Lower Q-Homeomorphisms

We start first from the following general statement.

Lemma 9.2. *Let (X,μ) be a measure space with finite measure μ, $p \in (1,\infty)$, and let $\varphi : X \to (0,\infty)$ be a measurable function. Set*

$$I(\varphi,p) \;=\; \inf_{\alpha} \int_X \varphi\, \alpha^p \, d\mu, \tag{9.19}$$

where the infimum is taken over all measurable functions $\alpha : X \to [0,\infty]$ such that

$$\int_X \alpha \, d\mu \;=\; 1\,. \tag{9.20}$$

Then

$$I(\varphi,p) \;=\; \left[\int\limits_X \varphi^{-\lambda}\, d\mu\right]^{-\frac{1}{\lambda}}, \tag{9.21}$$

where

$$\lambda = \frac{q}{p}, \qquad \frac{1}{p}+\frac{1}{q}=1\,, \tag{9.22}$$

i.e., $\lambda = 1/(p-1) \in (0,\infty)$. *Moreover, the infimum in (9.19) is attained only under the function*

$$\alpha_0 \;=\; C\cdot\varphi^{-\lambda}, \tag{9.23}$$

where

$$C=\left(\int\limits_X \varphi^{-\lambda}\, d\mu\right)^{-1}. \tag{9.24}$$

Proof. First let $\int \varphi^{-\lambda}\, d\mu<\infty$. Then, by the Hölder inequality,

$$1 \;=\; \int\limits_X \alpha\, d\mu \;=\; \int\limits_X (\varphi^{-\frac{q}{p}})^{\frac{1}{q}}\,[\varphi\,\alpha^p]^{\frac{1}{p}}\, d\mu \;\le\; \left[\int\limits_X \varphi^{-\frac{q}{p}}\, d\mu\right]^{\frac{1}{q}}\cdot\left[\int\limits_X \varphi\,\alpha^p\, d\mu\right]^{\frac{1}{p}}$$

and the equality holds if and only if

$$c\cdot\varphi^{-\frac{q}{p}} \;=\; \varphi\cdot\alpha^p \quad \text{a.e.};$$

see, e.g., [105] or [261]. $C=c^{1/p}$ in (9.24), i.e.,

$$C \;=\; \left(\int\limits_X \varphi^{-\frac{1}{p-1}}\, d\mu\right)^{-1}$$

and

$$\alpha_0(x) \;=\; \left(\int\limits_X \varphi^{-\frac{1}{p-1}}\, d\mu\right)^{-1}\cdot\varphi^{-\frac{1}{p-1}}(x)\,.$$

In the case $\int \varphi^{-\lambda}\, d\mu=\infty$, the above arguments are applicable to the function

$$\varphi_\varepsilon(x) \;=\; \begin{cases} \varphi(x) & \text{if } \varphi(x)>\varepsilon\,,\\ 1 & \text{if } \varphi(x)\le\varepsilon \end{cases}$$

with arbitrary small $\varepsilon>0$. Note that $I(\varphi,p)\le I(\varphi_\varepsilon,p)\to 0$ as $\varepsilon\to 0$ in the given case. □

Theorem 9.2. *Let D and D' be domains in $\overline{\mathbb{R}^n}$, $n\ge 2$, let $x_0\in\overline{D}\setminus\{\infty\}$, and let $Q:D\to(0,\infty)$ be a measurable function. A homeomorphism $f:D\to D'$ is a lower*

Q-homeomorphism at x_0 if and only if

$$M(f\Sigma_\varepsilon) \ \geq \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)} \quad \forall\, \varepsilon \in (0,\varepsilon_0)\,, \quad \varepsilon_0 \in (0,d_0), \tag{9.25}$$

where

$$d_0 \ = \ \sup_{x\in D} |x-x_0|, \tag{9.26}$$

Σ_ε denotes the family of all the intersections of the spheres $S(r) = \{x \in \mathbb{R}^n : |x-x_0| = r\}$, $r \in (\varepsilon,\varepsilon_0)$, with D, and

$$||Q||_{n-1}(r) = \left(\int\limits_{D(x_0,r)} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}} \tag{9.27}$$

is the L_{n-1}-norm of Q over $D(x_0,r) = \{x \in D : |x-x_0| = r\} = D\cap S(x_0,r)$. The infimum of the expression from the right-hand side in (9.1) is attained only for the function

$$\rho_0(x) \ = \ \frac{Q(x)}{||Q||_{n-1}(|x|)}\,.$$

Proof. For every $\rho \in \operatorname{ext}\operatorname{adm}\Sigma_\varepsilon$,

$$A_\rho(r) \ = \int\limits_{D(x_0,r)} \rho^{n-1}(x)\, d\mathscr{A} \ \neq \ 0 \quad \text{a.e.}$$

is a measurable function in the parameter r, say by the Fubini theorem. Thus, we may required the equality $A_\rho(r) \equiv 1$ a.e. instead of (9.16) and

$$\inf_{\rho\in\operatorname{ext}\operatorname{adm}\Sigma_\varepsilon} \int\limits_{D\cap R_\varepsilon} \frac{\rho^n(x)}{Q(x)}\, dm(x) \ = \int\limits_{\varepsilon}^{\varepsilon_0} \left(\inf_{\alpha\in I(r)} \int\limits_{D(x_0,r)} \frac{\alpha^p(x)}{Q(x)}\, d\mathscr{A} \right) dr,$$

where $p = n/(n-1) > 1$ and $I(r)$ denotes the set of all measurable functions α on the surface $D(x_0,r) = S(x_0,r)\cap D$ such that

$$\int\limits_{D(x_0,r)} \alpha(x)\, d\mathscr{A} \ = \ 1.$$

Hence, Theorem 9.2 follows by Lemma 9.2 with $X = D(x_0,r)$, the $(n-1)$-dimensional area as a measure μ on $D(x_0,r)$, $\varphi = \frac{1}{Q}|_{D(x_0,r)}$, and $p = n/(n-1) > 1$. $\square$

Corollary 9.1. *Let D and D' be domains in $\overline{\mathbb{R}^n}$, $n \geq 2$, $x_0 \in \overline{D}\setminus\{\infty\}$, $Q : D \to (0,\infty)$ a measurable function, and $f : D \to D'$ a lower Q-homeomorphism at x_0. Then*

$$M(f\Sigma_\varepsilon) \geq \omega_{n-1}^{\frac{1}{1-n}} \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r \cdot q_{n-1}(r)} \quad \forall\, \varepsilon \in (0, \varepsilon_0),\ \varepsilon_0 \in (0, d_0), \tag{9.28}$$

where

$$d_0 \;=\; \sup_{x \in D} |x - x_0|, \tag{9.29}$$

$$q_{n-1}(r) \;=\; \left(\fint_{S(x_0,r)} q^{n-1}(x)\, d\mathscr{A} \right)^{1/(n-1)}, \tag{9.30}$$

$$q(x) \;=\; \begin{cases} Q(x) & x \in D, \\ 0 & x \in \mathbb{R}^n \setminus D. \end{cases} \tag{9.31}$$

9.4 Estimates of Distortion

Lemma 9.3. *Let D and D' be domains in $\overline{\mathbb{R}^n}$, $n \geq 2$, $f : D \to D'$ a lower Q-homeomorphism at $x_0 \in \overline{D} \setminus \{\infty\}$, and $0 < \varepsilon < \varepsilon_0 < \operatorname{dist}(x_0, \partial D)$. Then*

$$h(fS_\varepsilon) \;\leq\; \frac{\alpha_n}{h(fS_{\varepsilon_0})} \cdot \exp\left(-\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r\, q_{n-1}(r)} \right), \tag{9.32}$$

where $\alpha_n = 2\lambda_n^2$ with $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, and $\lambda_n^{1/n} \to e$ as $n \to \infty$,

$$q_{n-1}(r) \;=\; \left(\fint_{|x-x_0|=r} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}}, \tag{9.33}$$

and S_ε and S_{ε_0} denote the spheres in $\mathbb{R}^n$ centered at x_0 with radii ε and ε_0, respectively.

Proof. Set $E = fS_\varepsilon$ and $F = fS_{\varepsilon_0}$. By the Gehring lemma,

$$\operatorname{cap} R(E,F) \;\geq\; \operatorname{cap} R_T\left(\frac{1}{h(E)\, h(F)} \right), \tag{9.34}$$

where $h(E)$ and $h(F)$ denote the spherical diameters of E and F, respectively, and $R_T(s)$ is the Teichmüller ring

$$R_T(s) \;=\; \mathbb{R}^n \setminus ([-1,0] \cup [s, \infty]), \quad s > 1; \tag{9.35}$$

see, e.g., [71] or Corollary 7.37 in [328], Section A.1. We know that

$$\operatorname{cap} R_T(s) \;=\; \frac{\omega_{n-1}}{(\log \Psi(s))^{n-1}}, \tag{9.36}$$

where the function Ψ admits the estimates

$$s+1 \;\le\; \Psi(s) \;\le\; \lambda_n^2 \cdot (s+1) \;<\; 2\lambda_n^2 \cdot s, \quad s>1; \tag{9.37}$$

see, e.g., [71], pp. 225–226, and (7.19) and Lemma 7.22 in [328], Section A.1. Hence, inequality (9.34) implies that

$$\operatorname{cap} R(E,F) \;\ge\; \frac{\omega_{n-1}}{\left(\log \frac{2\lambda_n^2}{h(E)h(F)}\right)^{n-1}}. \tag{9.38}$$

Denote by Σ_ε the family of all surfaces $D(r) = \{x \in D : |x - x_0| = r\}$, $r \in (\varepsilon, \varepsilon_0)$. By Theorem 3.13 in [340] and (9.28), we have

$$\operatorname{cap} R(E,F) \;\le\; \frac{1}{M^{n-1}(f\Sigma_\varepsilon)} \;\le\; \frac{\omega_{n-1}}{\left(\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r \cdot q_{n-1}(r)}\right)^{n-1}} \tag{9.39}$$

because $f\Sigma_\varepsilon \subset \Sigma(fS_\varepsilon, fS_{\varepsilon_0})$, where $\Sigma(fS_\varepsilon, fS_{\varepsilon_0})$ consists of all $(n-1)$-dimensional surfaces in fD that separate fS_ε and fS_{ε_0}.

Finally, combining (9.38) and (9.39), we obtain (9.32). □

9.5 Removal of Isolated Singularities

By Theorem 9.2, similarly to the proof of Lemma 9.3, we obtain the following statement.

Theorem 9.3. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, $x_0 \in D$, $Q : D \to (0,\infty)$ a measurable function, and f a lower Q-homeomorphism at x_0 of $D \setminus \{x_0\}$ into $\overline{\mathbb{R}^n}$. Suppose that*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r \cdot q_{n-1}(r)} \;=\; \infty, \tag{9.40}$$

where $\varepsilon_0 < \operatorname{dist}(x_0, \partial D)$, and

$$q_{n-1}(r) \;=\; \left(⨍_{|x-x_0|=r} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}}. \tag{9.41}$$

Then f has a continuous extension to D in $\overline{\mathbb{R}^n}$.

Corollary 9.2. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, $x_0 \in D$, and f a lower Q-homeomorphism at x_0 of $D \setminus \{x_0\}$ into $\overline{\mathbb{R}^n}$. If*

$$⨍_{|x-x_0|=r} Q^{n-1}(x)\, d\mathscr{A} \;=\; O\left(\log^{n-1} \frac{1}{r}\right) \tag{9.42}$$

as $r \to 0$*, then* f *has a continuous extension to* D *in* $\overline{\mathbb{R}^n}$.

Corollary 9.3. *Let* D *be a domain in* $\mathbb{R}^n$*,* $n \geq 2$*,* $x_0 \in D$*, and* f *a lower* Q*-homeomorphism at* x_0 *of* $D \setminus \{x_0\}$ *into* $\overline{\mathbb{R}^n}$*. If*

$$\fint_{|x-x_0|=r} Q^{n-1}(x)\, d\mathscr{A} = O\left(\left[\log\frac{1}{r}\cdot\log\log\frac{1}{r}\cdot\ldots\cdot\log\ldots\log\frac{1}{r}\right]^{n-1}\right) \tag{9.43}$$

as $r \to 0$*, then* f *has a continuous extension to* D *in* $\overline{\mathbb{R}^n}$.

9.6 On Continuous Extension to Boundary Points

Lemma 9.4. *Let* D *and* D' *be domains in* $\mathbb{R}^n$*,* $n \geq 2$*,* $x_0 \in \partial D$*, let* $Q : D \to (0,\infty)$ *a measurable function, and* $f : D \to D'$ *a lower* Q*-homeomorphism at* x_0*. Suppose that the domain* D *is locally connected at* x_0 *and* $\partial D'$ *is strongly accessible at least at one point of the cluster set*

$$L = C(x_0, f) = \{y \in \overline{\mathbb{R}^n} : y = \lim_{k\to\infty} f(x_k), x_k \to x_0\}. \tag{9.44}$$

If

$$\int_0^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)} = \infty, \tag{9.45}$$

where

$$0 < \varepsilon_0 < d_0 = \sup_{x\in D} |x - x_0| \tag{9.46}$$

and

$$||Q||_{n-1}(r) = \left(\int_{D\cap S(x_0,r)} Q^{n-1}(x)\, d\mathscr{A}\right)^{\frac{1}{n-1}}, \tag{9.47}$$

then f *extends by continuity to* x_0 *in* $\overline{\mathbb{R}^n}$.

Proof. Note that $L \neq \varnothing$, in view of the compactness of the extended space $\overline{\mathbb{R}^n}$. By the condition, $\partial D'$ is strongly accessible at a point $y_0 \in L$. Let us assume that there is one more point $z_0 \in L$ and set $U = B(x_0, r_0)$, where $0 < r_0 < |y_0 - z_0|$.

In view of the local connectedness of D at x_0, there is a sequence of neighborhoods V_k of x_0 with connected $D_k = D \cap V_k$ and $\delta(V_k) \to 0$ as $k \to \infty$. Choose in the domains $D'_k = fD_k$ points y_k and z_k with $|y_0 - y_k| < r_0$ and $|y_0 - z_k| > r_0$, $y_k \to y_0$ and $z_k \to z_0$ as $k \to \infty$. Let C_k be paths connecting y_k and z_k in D'_k. Note that by the construction, $\partial U \cap C_k \neq \varnothing$.

By the condition of strong accessibility, the point y_0 from D', there are a compactum $E \subseteq D'$ and a number $\delta > 0$ such that

$$M(\Delta(E,C_k;D')) \geq \delta$$

for large k. Without loss of generality, we may assume that the last condition holds for all $k = 1,2,\ldots$. Note that $C = f^{-1}E$ is a compactum in D' and hence $\varepsilon_0 = \mathrm{dist}(x_0,C) > 0$.

Let Γ_ε be a family of all paths connecting the spheres $S_\varepsilon = \{x \in \mathbb{R}^n : |x-x_0| = \varepsilon\}$ and $S_0 = \{x \in \mathbb{R}^n : |x-x_0| = \varepsilon_0\}$ in D. Note that $C_k \subset fB_\varepsilon$ for every fixed $\varepsilon \in (0,\varepsilon_0)$ for large k, where $B_\varepsilon = B(x_0,\varepsilon)$. Thus, $M(f\Gamma_\varepsilon) \geq \delta$ for all $\varepsilon \in (0,\varepsilon_0)$.

By [122], [293], and [340] (see Section A.3, A.4, and A.6),

$$M(f\Gamma_\varepsilon) \leq \frac{1}{M^{n-1}(f\Sigma_\varepsilon)},$$

where Σ_ε is the family of all surfaces $D(r) = \{x \in D : |x-x_0| = r\}$, $r \in (\varepsilon,\varepsilon_0)$. Thus, $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ by Theorem 9.2 in view of (9.45). The contradiction disproves the above assumption. □

9.7 On One Corollary for QED Domains

By Section 3.8, we obtain the following consequence of Lemma 9.4.

Theorem 9.4. *Let D and D' be* QED *domains in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \partial D$, $Q : D \to (0,\infty)$ a measurable function, and $f : D \to D'$ a lower Q-homeomorphism at x_0. If*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)} = \infty, \tag{9.48}$$

where

$$0 < \varepsilon_0 < d_0 = \sup_{x\in D} |x-x_0| \tag{9.49}$$

and

$$||Q||_{n-1}(r) = \left(\int\limits_{D\cap S(x_0,r)} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}}, \tag{9.50}$$

then f extends by continuity to x_0 in $\overline{\mathbb{R}^n}$.

9.8 On Singular Null Sets for Extremal Distances

In this section $C(X,f)$ denotes the **cluster set** of the mapping $f : D \to \overline{\mathbb{R}^n}$ for a set $X \subset \overline{D}$, i.e.,

$$C(X,f) \;=\; \{y\in\overline{\mathbb{R}^n}:\; y=\lim_{k\to\infty} f(x_k),\; x_k\to x\in X\}.$$

Note that the complements of NED sets in $\mathbb{R}^n$ give very particular cases of QED domains considered in the previous section. Thus, arguing locally, by Theorem 9.4, we obtain the following statement.

Theorem 9.5. *Let D be a domain in $\mathbb{R}^n$, $n\geq 2$, $X\subset D$, and f a lower Q-homeomorphism at $x_0\in X$ of $D\backslash X$ into $\overline{\mathbb{R}^n}$. Suppose that X and $C(X,f)$ are* NED *sets. If*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)} \;=\; \infty, \tag{9.51}$$

where

$$0\;<\;\varepsilon_0\;<\;d_0\;=\;\sup_{x\in D}|x-x_0| \tag{9.52}$$

and

$$||Q||_{n-1}(r)=\left(\int\limits_{D\cap S(x_0,r)} Q^{n-1}(x)\,d\mathscr{A}\right)^{\frac{1}{n-1}}, \tag{9.53}$$

then f can be extended by continuity to x_0 in $\overline{\mathbb{R}^n}$.

9.9 Lemma on Cluster Sets

Lemma 9.5. *Let D and D' be domains in $\mathbb{R}^n$, $n\geq 2$, z_1 and z_2 distinct points in ∂D, $z_1\neq\infty$, and f a lower Q-homeomorphism at z_1 of D onto D', and let the function Q be integrable with degree $n-1$ on the surfaces*

$$D(r)\;=\;\{x\in D:|x-z_1|=r\}\;=\;D\cap S(z_1,r)$$

for some set E of numbers $r<|z_1-z_2|$ of a positive linear measure. If D is locally connected at z_1 and z_2 and $\partial D'$ is weakly flat, then

$$C(z_1,f)\cap C(z_2,f)\;=\;\varnothing. \tag{9.54}$$

Proof. Without loss of generality, we may assume that the domain D is bounded. Let $d=|z_1-z_2|$. Choose $\varepsilon_0\in(0,d)$ such that

$$E_0\;=\;\{r\in E: r\in(\varepsilon,\varepsilon_0)\}$$

has a positive measure. The choice is possible because of the countable subadditivity of the linear measure and because of the exhaustion

$$E = \bigcup_{m=1}^{\infty} E_m,$$

where

$$E_m = \{r \in E : r \in (1/m, d-1/m)\}.$$

Note that each of the spheres $S(z_1,r)$, $r \in E_0$, separates the points z_1 and z_2 in $\mathbb{R}^n$ and $D(r)$, $r \in E_0$, in D. Thus, by Theorem 9.2, we have

$$M(f\Sigma_\varepsilon) > 0, \tag{9.55}$$

where Σ_ε denotes the family of all intersections of the spheres

$$S(r) = S(z_1,r) = \{x \in \mathbb{R}^n : |x-z_1| = r\},\ r \in (\varepsilon,\varepsilon_0),$$

with D.

For $i = 1,2$, let C_i be the cluster set $C(z_i,f)$ and suppose that $C_1 \cap C_2 \neq \varnothing$. Since D is locally connected at z_1 and z_2, there exist neighborhoods U_i of z_i such that $W_i = D \cap U_i$, $i = 1,2$ are connected and $U_1 \subset B^n(z_1,\varepsilon)$ and $U_2 \subset \mathbb{R}^n \setminus B^n(z_1,\varepsilon_0)$.

Set $\Gamma = \Delta(\overline{W_1},\overline{W_2};D)$. By [122], [293], [340] (see Sections A.3, A.4, and A.6), and (9.55),

$$M(f\Gamma) \leq \frac{1}{M^{n-1}(f\Sigma_\varepsilon)} < \infty. \tag{9.56}$$

Let $y_0 \in C_1 \cap C_2$. Without loss of generality, we may assume that $y_0 \neq \infty$ because in the contrary case one can use an additional Möbius transformation. Choose $r_0 > 0$ such that $S(y_0,r_0) \cap fW_1 \neq \varnothing$ and $S(y_0,r_0) \cap fW_2 \neq \varnothing$.

By the condition, $\partial D'$ is weakly flat and hence, given a finite number $M_0 > M(f\Gamma)$, there is $r_* \in (0,r_0)$ such that

$$M(\Delta(E,F;D')) \geq M_0$$

for all continua E and F in D' intersecting the spheres $S(y_0,r_0)$ and $S(y_0,r_*)$. However, these spheres can be connected by paths P_1 and P_2 in domains fW_1 and fW_2, respectively, and for these paths

$$M_0 \leq M(\Delta(P_1,P_2;D')) \leq M(f\Gamma).$$

The contradiction disproves the earlier assumption that $C_1 \cap C_2 \neq \varnothing$. The proof is complete. □

As an immediate consequence of Lemma 9.5, we have the following statement.

Theorem 9.6. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, D locally connected on ∂D and $\partial D'$ weakly flat. If f is a lower Q-homeomorphism of D onto D' with $Q \in L^{n-1}(D)$, then f^{-1} has a continuous extension to $\overline{D'}$ in $\overline{\mathbb{R}^n}$.*

Proof. By the Fubini theorem, the set

$$E \;=\; \{r \in (0,d) : Q|_{D(r)} \in L^{n-1}(D(r))\}$$

has a positive linear measure because $Q \in L^{n-1}(D)$. $\square$

Remark 9.3. It is sufficient to request in Theorem 9.6 that Q be integrable with degree $n-1$ in a neighborhood of ∂D only.

Lemma 9.6. *Let D and D' be domains in $\overline{\mathbb{R}^n}$, $n \geq 2$, $Q : D \to (0,\infty)$ a measurable function, and $f : D \to D'$ a lower Q-homeomorphism at $x_0 \in \overline{D} \setminus \{\infty\}$. Then*

$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)} \;<\; \infty \;\; \forall\, \varepsilon \in (0,\varepsilon_0), \quad \varepsilon_0 \in (0,d_0), \tag{9.57}$$

where

$$d_0 \;=\; \sup_{x\in D} |x - x_0| \tag{9.58}$$

and

$$||Q||_{n-1}(r) = \left(\int\limits_{D(x_0,r)} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}} \tag{9.59}$$

is the L_{n-1}-norm of Q over $D(r) = \{x \in D : |x - x_0| = r\} = D \cap S(x_0, r)$.

Proof. Let $x_1 \in D(\varepsilon)$ and $x_2 \in D(\varepsilon_0)$. Denote by C_1 and C_2 the continua $f(D(\varepsilon) \cap B(x_1,r_1))$ and $f(D(\varepsilon_0) \cap B(x_2,r_2))$, respectively, where $r_1 < \mathrm{dist}(x_1,\partial D)$ and $r_2 < \mathrm{dist}(x_2,\partial D)$ and, moreover, r_1 and $r_2 < |x_1 - x_2|/2$. Then, by [122] and [340] (see Sections A.3 and A.6),

$$M(\Delta(C_1,C_2;fD)) \;\leq\; \frac{1}{M^{n-1}(f\Sigma_\varepsilon)},$$

where $\Sigma_\varepsilon = \{D(r)\}_{r\in(\varepsilon,\varepsilon_0)}$ and by Theorem 5.2 in [225], p. 234, $M(\Delta(C_1,C_2;fD)) > 0$ because C_1 and C_2 are nondegenerate mutually disjoint continua in the domain $D' = fD$. Consequently, $M(f\Sigma_\varepsilon) < \infty$ and, thus, the conclusion of the lemma follows by Theorem 9.2. $\square$

Corollary 9.4. *If $f : D \to \mathbb{R}^n$ is a lower Q-homeomorphism at $x_0 \in \overline{D}$ with*

$$\int\limits_{0}^{\delta_0} \frac{dr}{||Q||_{n-1}(r)} \;=\; \infty \tag{9.60}$$

for some $\delta_0 \in (0,d_0)$, then

$$\int\limits_{0}^{\delta} \frac{dr}{||Q||_{n-1}(r)} \;=\; \infty \tag{9.61}$$

for all $\delta \in (0, d_0)$.

Combining Lemmas 9.5 and 9.6, we immediately have the following statement.

Theorem 9.7. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, D locally connected on ∂D and $\partial D'$ weakly flat, $Q : D \to (0, \infty)$ a measurable function such that the condition*

$$\int_0^{\delta(x_0)} \frac{dr}{||Q||_{n-1}(x_0, r)} = \infty \tag{9.62}$$

hold for all $x_0 \in \partial D$ with some $\delta(x_0) \in (0, d(x_0))$, where

$$d(x_0) = \sup_{x \in D} |x - x_0|, \tag{9.63}$$

and

$$||Q||_{n-1}(x_0, r) = \left(\int_{D(x_0, r)} Q^{n-1}(x) \, d\mathscr{A} \right)^{\frac{1}{n-1}} \tag{9.64}$$

is the L_{n-1}-norm of Q over $D(x_0, r) = \{x \in D : |x - x_0| = r\} = D \cap S(x_0, r)$. Then there is a continuous extension of f^{-1} to $\overline{D'}$ in $\overline{\mathbb{R}^n}$ for every lower Q-homeomorphism f of D onto D'.

9.10 On Homeomorphic Extensions to Boundaries

Combining the above results, we obtain the following statements.

Theorem 9.8. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0, \infty)$ a measurable function, and $f : D \to D'$ a lower Q-homeomorphism in D. Suppose that the domain D is locally connected on ∂D and that the domain D' has a weakly flat boundary. If, at every point $x_0 \in \partial D$,*

$$\int_0^{\delta(x_0)} \frac{dr}{||Q||_{n-1}(x_0, r)} = \infty \tag{9.65}$$

for some $\delta(x_0) \in (0, d(x_0))$ where

$$d(x_0) = \sup_{x \in D} |x - x_0| \tag{9.66}$$

and

$$||Q||_{n-1}(x_0,r) = \left(\int_{D\cap S(x_0,r)} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}}, \tag{9.67}$$

then f has a homeomorphic extension to $\overline{D}$.

The following theorem is a far-reaching generalization of the known Gehring–Martio result (see [81], p. 196) from quasiconformal mappings to lower Q-homeomorphisms; cf. Corollaries 3.2 and 3.3.

Theorem 9.9. *Let D and D' be bounded* QED *domains in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0,\infty)$ a measurable function, and $f : D \to D'$ a lower Q-homeomorphism in D. If condition (9.65) holds at every point $x_0 \in \partial D$, then f has a homeomorphic extension to $\overline{D}$.*

Theorem 9.10. *Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0,\infty)$ a measurable function, $X \subset D$, and f a lower Q-homeomorphism of $D \setminus \{X\}$ into $\overline{\mathbb{R}^n}$. Suppose that X and $C(X,f)$ are* NED *sets. If condition (9.65) holds at every point $x_0 \in X$ for $\delta(x_0) < \mathrm{dist}(x_0,\partial D)$, where*

$$||Q||_{n-1}(x_0,r) = \left(\int_{|x-x_0|=r} Q^{n-1}(x)\, d\mathscr{A} \right)^{\frac{1}{n-1}}, \tag{9.68}$$

then f has a homeomorphic extension to D.

Remark 9.4. In particular, the conclusion of Theorem 9.10 is valid if X is a closed set with

$$H^{n-1}(X) \; = \; 0 \; = \; H^{n-1}(C(X,f)). \tag{9.69}$$

Chapter 10
Mappings with Finite Area Distortion

We show that mappings in $\mathbb{R}^n$ with finite area distortion (FAD) in all dimensions $k = 1,\ldots,n-1$ satisfy certain modulus inequalities in terms of their inner and outer dilatations and, in particular, we prove generalizations of the well-known Poletskii inequality for quasiregular mappings; see [159], [160]. Moreover, we show that homeomorphisms f with finite area distortion of dimension $n-1$ are lower Q-homeomorphisms with $Q(x) = K_O(x, f)$, and on this basis we study their boundary behavior. The developed theory is applicable, for example, to the class of finitely bi-Lipschitz mappings, which is a natural generalization of the well-known classes of bi-Lipschitz mappings as well as isometries and quasi-isometries in $\mathbb{R}^n$; see [157, 158]. The mappings with finite area distortion extend the mappings with finite length distortion; see [207] and Chapter 8 in this volume.

10.1 Introduction

Here we assume that Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, and that all mappings $f : \Omega \to \mathbb{R}^n$ are continuous.

Given a mapping $\varphi : E \to \mathbb{R}^n$ and a point $x \in E \subseteq \mathbb{R}^n$, recall that

$$L(x,\varphi) \;=\; \limsup_{y\to x\, y\in E} \frac{|\varphi(y)-\varphi(x)|}{|y-x|} \tag{10.1}$$

and

$$l(x,\varphi) \;=\; \liminf_{y\to x\, y\in E} \frac{|\varphi(y)-\varphi(x)|}{|y-x|}, \tag{10.2}$$

and a mapping $f : \Omega \to \mathbb{R}^n$ is said to be of **finite metric distortion**, abbr. $f \in$FMD, if f has the (N)-property and

$$0 \;<\; l(x,f) \;\leq\; L(x,f) \;<\; \infty \quad \text{a.e.}; \tag{10.3}$$

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_10,

see [207]. By Corollary 8.1, a mapping $f : \Omega \to \mathbb{R}^n$ is of FMD if and only if f is differentiable a.e. and has the (N)- and (N^{-1})-properties.

We say that a mapping $f : \Omega \to \mathbb{R}^n$ has the $(\mathbf{A_k})$**-property** if the following two conditions hold:
$(A_k^{(1)})$: for a.e. k-dimensional surface S in Ω, the restriction $f|_S$ has the (N)-property with respect to area;
$(A_k^{(2)})$: for a.e. k-dimensional surface S_* in $\Omega_* = f(\Omega)$, the restriction $f|_S$ has the (N^{-1})-property for each lifting S of S_* with respect to area.

Here a surface S in Ω is a **lifting** of a surface S_* in $\mathbb{R}^n$ under a mapping $f : \Omega \to \mathbb{R}^n$ if $S_* = f \circ S$. We also say that a mapping $f : \Omega \to \mathbb{R}^n$ is of **finite area distortion in dimension** $k = 1, \ldots, n-1$, abbr. $f \in \mathrm{FAD}_k$, if $f \in$FMD and has the (A_k)-property. Note that analogues of (A_k)-properties and the classes FAD_k were first formulated in the [207] for $k = 1$; see Chapter 8, which discusses the connectivity and local rectifiability of S_* and S in the $(A_k^{(1)})$- and $(A_k^{(2)})$-properties, respectively. Finally, we say that a mapping $f : \Omega \to \mathbb{R}^n$ is of **finite area distortion**, abbr. $f \in$FAD, if $f \in \mathrm{FAD}_k$ for every $k = 1, \ldots, n-1$.

As in Chapter 8, given a pair $Q(x,y) = (Q_1(x), Q_2(y))$ of measurable functions $Q_1 : \Omega \to [1,\infty]$ and $Q_2 : \Omega_* \to [1,\infty]$ and $k = 1, \ldots, n-1$, we say that a mapping $f : \Omega \to \mathbb{R}^n$, $f(\Omega) = \Omega_*$, is a **hyper Q-mapping in dimension** $k = 1, \ldots, n-1$ if

$$M(f\Gamma) \le \int\limits_{\Omega} Q_1(x) \cdot \rho^n(x)\, dm(x) \tag{10.4}$$

and

$$M(\Gamma) \le \int\limits_{\Omega_*} Q_2(y) \cdot \rho_*^n(y)\, dm(y) \tag{10.5}$$

for every family Γ of k-dimensional surfaces S in Ω and all $\rho \in \operatorname{adm}\Gamma$ and $\rho_* \in \operatorname{adm} f\Gamma$. We also say that a mapping $f : \Omega \to \mathbb{R}^n$ is a **hyper Q-mapping** if f is a hyper Q-mapping in all dimensions $k = 1, \ldots, n-1$.

We show that every mapping f with finite area distortion is a hyper Q-mapping with

$$Q(x,y) = \left(K_I(x,f), \sum_{z \in f^{-1}(y)} K_O(z,f) \right). \tag{10.6}$$

10.2 Upper Estimates of Moduli

The following lemma makes it possible to extend the so-called K_0-inequality from the theory of quasiregular mappings to FAD mappings; see, e.g., [210], p. 16, [260], p. 31, [328], p. 130; cf. also [207] and Section 8.6.

Lemma 10.1. *Let a mapping $f:\Omega\to\mathbb{R}^n$ be of finite metric distortion with the $(A_k^{(1)})$-property for some $k=1,\dots,n-1$ and let a set $E\subset\Omega$ be Borel. Then*

$$M(\Gamma)\ \leq\ \int\limits_{f(E)} K_I(y,f^{-1},E)\cdot\rho_*^n(y)\,dm(y) \tag{10.7}$$

for every family Γ of k-dimensional surfaces S in E and $\rho_*\in\operatorname{adm} f\Gamma$, where

$$K_I(y,f^{-1},E)\ =\ \sum_{x\in E\cap f^{-1}(y)} K_O(x,f). \tag{10.8}$$

In particular, here we have in the case $E=\Omega$

$$K_I(y,f^{-1},\Omega)\ =\ K_I(y,f^{-1})\ :=\sum_{x\in f^{-1}(y)} K_O(x,f). \tag{10.9}$$

Proof. Let B be a (Borel) set of all points x in Ω where f has a differential $f'(x)$ and $J(x,f)=\det f'(x)\neq 0$. Then $B_0=\Omega\setminus B$ has the Lebesgue measure zero in $\mathbb{R}^n$ because $f\in$FMD. It is known that B is the union of a countable collection of Borel sets B_l, $l=1,2,\ldots$, such that $f_l=f|_{B_l}$ is a bi-Lipschitz homeomorphism; see, e.g., point 3.2.2 in [55]. For example, setting $B_1^*=B_1$, $B_2^*=B_2\setminus B_1$, and

$$B_l^*\ =\ B_l\setminus\bigcup_{m=1}^{l-1}B_m,$$

we may assume that the B_l are no-empty and mutually disjoint. Note that by (2) in Remark 9.1, $\mathscr{A}_S(B_0)=0$ for a.e. k-dimensional surface S in Ω and by the $(A_k^{(1)})$-property $\mathscr{A}_{S_*}(f(B_0))=0$, where $S_*=f\circ S$ also for a.e. k-dimensional surface S.

Given $\rho_*\in\operatorname{adm} f\Gamma$, set

$$\rho(x)\ =\ \begin{cases}\rho_*(f(x))\|f'(x)\|, & \text{for } x\in\Omega\setminus B_0,\\ 0, & \text{otherwise.}\end{cases} \tag{10.10}$$

We may assume without loss of generality that $\rho_*\equiv 0$ outside $f(E)$. Arguing piecewise on B_l, we have by point 3.2.20 and 1.7.6 in [55] and Theorem 9.1 (see also Remark 9.2)

$$\int\limits_S \rho^k\,d\mathscr{A}\ \geq\ \int\limits_{S_*}\rho_*^k\,d\mathscr{A}\ \geq\ 1 \tag{10.11}$$

for a.e. $S\in\Gamma$, i.e., $\rho\in\operatorname{ext}\operatorname{adm}\Gamma$. Hence, by (9.18),

$$M(\Gamma)\leq\int\limits_{\Omega}\rho^n(x)\,dm(x). \tag{10.12}$$

Now, by a change of variables [see, e.g., Proposition 8.3(iii)], we obtain

$$\int\limits_{f(B_l \cap E)} K_O(f_l^{-1}(y), f) \cdot \rho_*^n(y)\, dm(y) \;=\; \int\limits_{\Omega} \rho_l^n(x)\, dm(x), \tag{10.13}$$

where $\rho_l = \rho \cdot \chi_{B_l}$ and every $f_l = f|_{B_l}$, $l = 1, 2, \ldots$, is injective by the construction.

Thus, by the Lebesgue monotone convergence theorem (see, e.g., [281], p. 27),

$$\int\limits_{f(E)} K_I(y, f^{-1}, E) \cdot \rho_*^n(y)\, dm(y) \;=\; \int\limits_{\Omega} \sum_{l=1}^{\infty} \rho_l^n(x)\, dm(x) \;\geq\; M(\Gamma).$$

$\square$

The next inequality is a generalized form of the K_I-inequality, also known as Poletskii's inequality; see [242], [260], pp. 49–51, and [328], p. 131; cf. also Section 8.6.

Lemma 10.2. *Let $f : \Omega \to \mathbb{R}^n$ be an* FMD *mapping with the $(A_k^{(2)})$-property for some $k = 1, \ldots, n-1$. Then*

$$M(f\Gamma) \;\leq\; \int\limits_{\Omega} K_I(x, f) \cdot \rho^n(x)\, dm(x) \tag{10.14}$$

for every family Γ of k-dimensional surface S in Ω and $\rho \in \operatorname{adm}\Gamma$.

Proof. Let B_l, $l = 0, 1, 2, \ldots$, be given as in the proof of Lemma 10.1. By the construction and the (N)-property, $|f(B_0)| = 0$. Thus, by Theorem 9.1, $\mathscr{A}_{S_*}(f(B_0)) = 0$ for a.e. $S_* \in f\Gamma$ and, hence, by the $(A_k^{(2)})$-property, $\mathscr{A}_S(B_0) = 0$ for a.e. $S_* \in f\Gamma$, where S is an arbitrary lifting of S_* under the mapping f, i.e., $S_* = f \circ S$.

Let $\rho \in \operatorname{adm}\Gamma$ and let

$$\tilde{\rho}(y) \;=\; \sup_{x \in f^{-1}(y) \cap \Omega \setminus B_0} \rho_*(x), \tag{10.15}$$

where

$$\rho_*(x) \;=\; \begin{cases} \rho(x)/l(f'(x)), & \text{for } x \in \Omega \setminus B_0, \\ 0, & \text{otherwise.} \end{cases} \tag{10.16}$$

Note that $\tilde{\rho} = \sup \rho_l$, where

$$\rho_l(y) \;=\; \begin{cases} \rho_*(f_l^{-1}(y)), & \text{for } y \in f(B_l), \\ 0, & \text{otherwise,} \end{cases} \tag{10.17}$$

and every $f_l = f|_{B_l}$, $l = 1, 2, \ldots$, is injective. Thus, the function $\tilde{\rho}$ is Borel; see, e.g., [281], p. 15.

Arguing as in the proof of (10.11), we obtain

$$\int\limits_{S_*} \tilde{\rho}^{\,k}\, d\mathscr{A} \;\geq \int\limits_{S} \rho^k\, d\mathscr{A} \;\geq\; 1 \tag{10.18}$$

for a.e. $S_* = f \circ S \in f\Gamma$ and, thus, $\tilde{\rho} \in \operatorname{ext}\operatorname{adm} f\Gamma$. Hence, (9.18) yields

$$M(f\Gamma) \;\leq \int\limits_{f(\Omega)} \tilde{\rho}^{\,n}(y)\, dm(y). \tag{10.19}$$

Further, by a change of variables, we have

$$\int\limits_{B_l} K_I(x,f)\cdot \rho^n(x)\, dm(x) \;=\; \int\limits_{f(\Omega)} \rho_l(y)\, dm(y). \tag{10.20}$$

Finally, by Lebesgue's theorem, we obtain the desired inequality:

$$\begin{aligned}\int\limits_{\Omega} K_I(x,f)\cdot \rho^n(x)\, dm(x) \;&=\; \sum_{l=1}^{\infty} \int\limits_{f(\Omega)} \rho_l(y)\, dm(y)\\ &= \int\limits_{f(\Omega)} \sum_{l=1}^{\infty} \rho_l(y)\, dm(y) \;\geq\; M(f\Gamma).\end{aligned}$$

□

Combining Lemmas 10.1 and 10.2, we come to the main result of this section.

Theorem 10.1. *Let a mapping $f : \Omega \to \mathbb{R}^n$ belong to class* FAD_k *for some* $k = 1,\ldots,n-1$. *Then f is a hyper Q-mapping in dimension k with*

$$Q(x,y) \;=\; (K_I(x,f), K_I(y,f^{-1})). \tag{10.21}$$

Corollary 10.1. *Every* FAD *mapping f is a hyper Q-mapping with Q given by* (10.21).

Remark 10.1. If $K_I(f) = \operatorname{ess\,sup} K_I(x,f) < \infty$, then (10.14) for $k=1$ yields the Poletskii inequality:

$$M(f\Gamma) \;\leq\; K_I(f)\, M(\Gamma) \tag{10.22}$$

for every path family in Ω. If $K_O(f) = \operatorname{ess\,sup} K_O(x,f) < \infty$ and E is a Borel set with $N(f,E) < \infty$, then we have from (10.7) the usual form of the K_O-inequality:

$$M(\Gamma) \;\leq\; N(f,E)\, K_O(f)\, M(f\Gamma) \tag{10.23}$$

for every path family in E.

10.3 On Lower Estimates of Moduli

Lemma 10.3. *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, and $f : \Omega \to \mathbb{R}^n$ an* FMD *homeomorphism with the $(A_k^{(1)})$-property for some $k = 1, \dots, n-1$. Then*

$$M(f\Gamma) \;\geq\; \inf_{\rho \in \operatorname{ext}\operatorname{adm}\Gamma} \int\limits_{\Omega} \frac{\rho^n(x)}{K_O(x,f)}\, dm(x) \tag{10.24}$$

for every family Γ of k-dimensional surfaces S in Ω.

Proof. Let B be a (Borel) set of all points x in Ω where f has a differential $f'(x)$ and $J(x,f) = \det f'(x) \neq 0$. As we know, B is the union of a countable collection of Borel sets B_l, $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ is bi-Lipschitz; see, e.g., point 3.2.2 in [55]. Without loss of generality, we may assume that the B_l are mutually disjoint. Note that $B_0 = \Omega \setminus B$ and $f(B_0)$ have Lebesgue measure zero in $\mathbb{R}^n$ for $f \in$FMD; see Corollary 8.1. Thus, by Theorem 9.1, $\mathscr{A}_S(B_0) = 0$ for a.e. $S \in \Gamma$ and hence by $(A_k^{(1)})$-property, $\mathscr{A}_{S_*}(f(B_0)) = 0$ for a.e. $S \in \Gamma$, where $S_* = f \circ S$.

Let $\rho_* \in \operatorname{adm} f\Gamma$, $\rho_* \equiv 0$ outside $f(\Omega)$, and set $\rho \equiv 0$ outside Ω and

$$\rho(x) \;=\; \rho_*(f(x))\,||f'(x)||, \quad \text{a.e.} \quad x \in \Omega$$

Arguing piecewise on B_l, we have by points 3.2.20 and 1.7.6 in [55] that

$$\int\limits_{S} \rho^k \, d\mathscr{A} \;\geq\; \int\limits_{S_*} \rho_*^k \, d\mathscr{A} \;\geq\; 1$$

for a.e. $S \in \Gamma$ and, thus, $\rho \in \operatorname{ext}\operatorname{adm}\Gamma$.

By a change of variables for the class FMD (see Proposition 8.3),

$$\int\limits_{\Omega} \frac{\rho^n(x)}{K_O(x,f)}\, dm(x) \;=\; \int\limits_{f(\Omega)} \rho_*^n(y)\, dm(y),$$

and (10.24) follows. □

Combining Lemmas 10.2 and 10.3, we have the following statement.

Theorem 10.2. *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, and let a homeomorphism $f : \Omega \to \mathbb{R}^n$ belong to* FAD_k *for some $k = 1, \dots, n-1$. Then, for every family Γ of k-dimensional surfaces S in Ω, f satisfies the double inequality*

$$\inf \int\limits_{\Omega} \frac{\rho^n(x)}{K_O(x,f)}\, dm(x) \;\leq\; M(f\Gamma) \;\leq\; \inf \int\limits_{\Omega} K_I(x,f)\rho^n(x)\, dm(x),$$

where the infimums are taken over all $\rho \in \operatorname{ext}\operatorname{adm}\Gamma$ and $\rho \in \operatorname{adm}\Gamma$, respectively.

Corollary 10.2. *Every homeomorphism $f : \Omega \to \mathbb{R}^n$ of class* FAD_{n-1} *is a lower Q-homeomorphism with*

$$Q(x) \;=\; K_O(x,f). \tag{10.25}$$

10.4 Removal of isolated singularities

By Corollary 10.2 and Section 9.5, we have the following conclusions.

Theorem 10.3. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, and $f : D \setminus \{x_0\} \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *Suppose that*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r \cdot k_{n-1}(r)} \;=\; \infty, \tag{10.26}$$

where $\varepsilon_0 < \mathrm{dist}(x_0, \partial D)$ and

$$k_{n-1}(r) \;=\; \left(⨍_{|x-x_0|=r} K_O^{n-1}(x,f))\, d\mathscr{A} \right)^{\frac{1}{n-1}}. \tag{10.27}$$

Then f has a homeomorphic extension to D of class FAD_{n-1}.

Corollary 10.3. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, and $f : D\setminus\{x_0\} \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *If*

$$⨍_{|x-x_0|=r} K_O^{n-1}(x,f)\, d\mathscr{A} \;=\; O\left(\log^{n-1}\frac{1}{r}\right) \tag{10.28}$$

as $r \to 0$, then f has a homeomorphic extension to D of class FAD_{n-1}.

Corollary 10.4. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, and $f : D\setminus\{x_0\} \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *If*

$$⨍_{|x-x_0|=r} K_O^{n-1}(x,f)\, d\mathscr{A} = O\left(\left[\log\frac{1}{r}\cdot\log\log\frac{1}{r}\cdot\ldots\cdot\log\cdot\ldots\cdot\log\frac{1}{r}\right]^{n-1}\right) \tag{10.29}$$

as $r \to 0$ then f has a homeomorphic extension to D of class FAD_{n-1}.

Remark 10.2. In particular, (10.28) holds if

$$K_O(x,f) \;=\; O\left(\log\frac{1}{|x-x_0|}\right) \tag{10.30}$$

as $x \to x_0$. Note that the continuous extension of f in Theorem 10.3 and Corollaries 10.3 and 10.4 is a homeomorphism by Corollary 6.12.

10.5 Extension to Boundaries

On the basis of Corollary 10.2 and Sections 9.6–9.10, here we review of results on the boundary behavior of mappings with finite area distortion.

Lemma 10.4. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \partial D$, and $f : D \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *Suppose that the domain D is locally connected at x_0 and the domain $D' = f(D)$ has a strongly accessible boundary. If*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{||K_O||_{n-1}(r)} \;=\; \infty, \tag{10.31}$$

where

$$0 \;<\; \varepsilon_0 \;<\; d_0 \;=\; \sup_{x\in D} |x-x_0| \tag{10.32}$$

and

$$||K_O||_{n-1}(r) = \left(\int\limits_{D\cap S(x_0,r)} K_O^{n-1}(x,f)(x)\,d\mathscr{A} \right)^{\frac{1}{n-1}}, \tag{10.33}$$

then f extends by continuity to x_0.

Theorem 10.4. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \partial D$, and $f : D \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *Suppose that D and $D' = f(D)$ are* QED *domains. If condition (10.31) holds, then f extends by continuity to x_0.*

The complements of NED sets in $\mathbb{R}^n$ give a very particular case of QED domains. Hence, arguing locally, we obtain by Theorem 10.4 the following statement.

Theorem 10.5. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $X \subset D$, and $f : D\backslash X \to \overline{\mathbb{R}^n}$ a homeomorphism of class* FAD_{n-1}. *Suppose that X and $C(X,f)$ are* NED *sets. If condition (10.31) holds, then f extends by continuity to x_0.*

Lemma 10.5. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, z_1 and z_2 distinct points in ∂D, $z_1 \neq \infty$, f a homeomorphism of class* FAD_{n-1} *of D onto D', and $K_O(x,f)$ integrable with degree $n-1$ on the surfaces*

$$D(r) \;=\; \{x \in D : |x-z_1| = r\} \;=\; D\cap S(z_1,r)$$

for some set E of numbers $r < |z_1 - z_2|$ of a positive linear measure. If D is locally connected at z_1 and z_2 and $\partial D'$ is weakly flat, then

$$C(z_1,f)\cap C(z_2,f) \;=\; \varnothing\,. \tag{10.34}$$

As usual, here $C(z_i,f)$ denotes the cluster sets at the points $z_i, i = 1,2$.

As an immediate consequence of Lemma 10.5, we have the following statement.

Theorem 10.6. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, D locally connected on ∂D and $\partial D'$ weakly flat. If f is a homeomorphism of class* FAD_{n-1} *of D onto D' with $K_O \in L^{n-1}(D)$, then f^{-1} has a continuous extension to $\overline{D'}$.*

Remark 10.3. It is sufficient to request in Theorem 10.6 that $K_O(x, f)$ be integrable with degree $n-1$ in a neighborhood of ∂D only.

Theorem 10.7. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, D locally connected on ∂D and $\partial D'$ weakly flat, and f a homeomorphism of D onto D' of class* FAD_{n-1}. *If condition (10.31) holds, then there is a continuous extension f^{-1} to $\overline{D'}$.*

Combining the above results, we obtain the following statements.

Theorem 10.8. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *Suppose that the domain D is locally connected on ∂D and that the domain $D' = f(D)$ has a weakly flat boundary. If condition (10.31) holds at every point $x_0 \in \partial D$, then f has a homeomorphic extension to $\overline{f} : \overline{D} \to \overline{D'}$.*

The next theorem extends the Gehring–Martio results in [81], p. 196, on the boundary correspondence from quasiconformal mappings to homeomorphisms with finite area distortion; cf. Corollaries 3.2 and 3.3.

Theorem 10.9. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *Suppose that D and $D' = f(D)$ are* QED *domains. If condition (10.31) holds at every point $x_0 \in \partial D$, then f has a homeomorphic extension to $\overline{D}$.*

Theorem 10.10. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $X \subset D$, and $f : D \backslash X \to \overline{\mathbb{R}^n}$, a homeomorphism of class* FAD_{n-1}. *Suppose that X and $C(X, f)$ are* NED *sets. If condition (10.31) holds at every point $x_0 \in X$, then f has a homeomorphic extension to D in class* FAD_{n-1}.

Corollary 10.5. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *Suppose that the domain D is locally connected on ∂D and that the domain $D' = f(D)$ has a weakly flat boundary. If, at every point $x_0 \in \partial D$,*

$$K_O(x, f) \; = \; O\left(\log \frac{1}{|x - x_0|}\right) \tag{10.35}$$

as $x \to x_0$, then f has a homeomorphic extension to $\overline{D}$.

Corollary 10.6. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to \mathbb{R}^n$ a homeomorphism of class* FAD_{n-1}. *Suppose that D and $D' = f(D)$ are* QED *domains. If condition (10.35) holds at every point $x_0 \in \partial D$, then f has a homeomorphic extension to $\overline{D}$.*

Corollary 10.7. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f : D \backslash X \to \overline{\mathbb{R}^n}$ a homeomorphism of class* FAD_{n-1}. *Suppose that X and $C(X, f)$ are* NED *sets. If condition (10.35) holds at every point $x_0 \in X$, then f has a homeomorphic extension to D that belongs to class* FAD_{n-1}.

Remark 10.4. In particular, the conclusion of Theorem 10.10 and Corollary 10.7 is valid if X is a closed set with

$$H^{n-1}(X) \;=\; 0 \;=\; H^{n-1}(C(X,f)). \tag{10.36}$$

10.6 Finitely Bi-Lipschitz Mappings

Recall that, given a set $A \subseteq \mathbb{R}^n$, $n \geq 1$, a mapping $f: A \to \mathbb{R}^n$ is called **Lipschitz** if there is number $L > 0$ such that the inequality

$$|f(x) - f(y)| \leq L\,|x-y| \tag{10.37}$$

holds for all x and y in A.

Given an open set $\Omega \subseteq \mathbb{R}^n$, we say that a mapping $f: \Omega \to \mathbb{R}^n$ is **finitely Lipschitz** if $L(x,f) < \infty$ holds for all $x \in \Omega$ and is **finitely bi-Lipschitz** if

$$0 \;<\; l(x,f) \;\leq\; L(x,f) \;<\; \infty \tag{10.38}$$

holds for all $x \in \Omega$; see (10.1) and (10.2) for the definitions of l and L.

Lemma 10.6. *Let $f: \Omega \to \mathbb{R}^n$ be a finitely Lipschitz mapping, and let $k = 1, \ldots, n$. Then $H^k(f(E)) = 0$ whenever $E \subset \Omega$ with $H^k(E) = 0$.*

Proof. First we prove the statement for Lipschitz mappings $f: A \to \mathbb{R}^n$ given on arbitrary sets $A \subset \mathbb{R}^n$, i.e., when there is $L > 0$ such that

$$|f(x) - f(y)| \leq L\,|x-y|, \quad \forall\, x,y \in A.$$

By Kirszbraun's theorem, such an f can be extended to a Lipschitz mapping on $\mathbb{R}^n$ with the same L; see either [150] or point 2.10.43 in [55].

Let $E \subseteq A$ and $H^k(E) = 0$. Then, for any $\varepsilon > 0$, there is a countable collection of balls $B_l = B(x_l, r_l)$ with centers x_l and radii r_l covering E such that

$$\sum_l V_k r_l^k < \varepsilon,$$

where V_k is the volume of the unit ball in $\mathbb{R}^k$.

Note that $C_l^* \subset B_l^*$ for all l and $f(E) \subset \bigcup_l C_l^* \subset \bigcup_l B_l^*$, where $C_l^* = f(B_l)$ and $B_l^* = B(f(x_l), Lr_l)$. Hence,

$$H^k(f(E)) \leq \sum_l V_k (Lr_l)^k = L^k \sum_l V_k r_l^k < L^k \varepsilon.$$

Thus, $H^k(f(E)) = 0$ because $\varepsilon > 0$ is arbitrary.

Now, let $f: \Omega \to \mathbb{R}^n$ be finitely Lipschitz. Denote by A_i the set of all points $x \in \Omega$ such that

$$|f(x+h) - f(x)| \leq i|h|$$

whenever $|h| < 1/i$ and $x+h \in \Omega$. Note that $A_i \subset A_{i+1}$ and $\Omega = \bigcup_i A_i$.

Let $E \subset \Omega$ such that $H^k(E) = 0$, and let $E_i = E \cap A_i$. Then $H^k(E_i) = 0$ and hence $H^k(f(E_i)) = 0$ by the above arguments for Lipschitz mappings. Thus, by the countable subadditivity of the Hausdorff measure,

$$H^k(f(E)) \leq \sum_i H^k(f(E_i)) = 0\,.$$

$\square$

Note that if a mapping $f : \Omega \to \mathbb{R}^n$ is a homeomorphism, then it has the inverse mapping f^{-1}, for which $l(x, f^{-1}) = 1/L(x, f)$ and $L(x, f^{-1}) = 1/l(x, f)$. Applying Lemma 10.6 to the mapping f^{-1}, we obtain the following statement.

Lemma 10.7. *Let $f : \Omega \to \mathbb{R}^n$ be a homeomorphism such that $l(x, f) > 0$ for all $x \in \Omega$, and let $k = 1, \ldots, n$. Then $H^k(E) = 0$ whenever $E \subset \Omega$ with $H^k(f(E)) = 0$.*

Combining Lemmas 10.6 and 10.7, by the Rademacher-Stepanoff theorem (see, e.g., point 3.1.9 in [55]) and the definition of FAD, we obtain the next statement.

Theorem 10.11. *Every finitely bi-Lipschitz homeomorphism is a mapping with finite area distortion.*

Recall that a mapping $f : \Omega \to \mathbb{R}^n$ is called **open** if the image of each open set in Ω is an open set in $\mathbb{R}^n$. A mapping $f : \Omega \to \mathbb{R}^n$ is called **discrete** if the preimage $f^{-1}(y)$ of each point $y \in \mathbb{R}^n$ consists of isolated points. In these terms we are able to formulate the following generalization of Lemma 10.7.

Lemma 10.8. *Let $f : \Omega \to \mathbb{R}^n$ be a discrete open mapping such that $l(x, f) > 0$ for all $x \in \Omega$, and let $k = 1, \ldots, n$. Then $H^k(E) = 0$ whenever $E \subset \Omega$ with $H^k(f(E)) = 0$.*

Proof. Indeed, since f is a discrete open mapping by Arkhangel'skii's theorem, $\Omega = \bigcup_i X_i$, where the X_i are closed subsets of Ω such that the mappings $f_i = f|_{X_i}$ are homeomorphisms; see Theorem 3.7 in [12], p. 218. Without loss of generality, we may also assume that the X_i are compact. Note that by the conditions of the lemma mappings, $g_i = f_i^{-1} : Y_i \to X_i$ are finitely Lipschitz. Thus, applying Lemma 10.6 to g_i, we come to the conclusion of Lemma 10.8 by the countable subadditivity of the Hausdorff measure H^k. $\square$

Combining Lemmas 10.6 and 10.8, we obtain the following result.

Theorem 10.12. *Every finitely bi-Lipschitz discrete open mapping is a mapping with finite area distortion.*

Theorem 10.13. *Let $f : \Omega \to \mathbb{R}^n$ be a finitely bi-Lipschitz discrete open mapping. Then f is a hyper Q-mapping with*

$$Q(x,y) \;=\; (K_I(x,f), K_I(y,f^{-1})), \tag{10.39}$$

where

$$K_I(y,f^{-1}) \;=\; \sum_{z\in f^{-1}(y)} K_O(z,f). \tag{10.40}$$

Corollary 10.8. *Every finitely bi-Lipschitz homeomorphism $f:\Omega\to\mathbb{R}^n$ is a hyper Q-mapping with*

$$Q(x,y) \;=\; (K_I(x,f),K_O(f^{-1}(y),f)). \tag{10.41}$$

Corollary 10.9. *Every finitely bi-Lipschitz homeomorphism $f:\Omega\to\mathbb{R}^n$ is a Q-homeomorphism with*

$$Q(x) \;=\; K_I(x,f). \tag{10.42}$$

Finally, by Theorem 10.2, we obtain the following important conclusion.

Corollary 10.10. *Every finitely bi-Lipschitz homeomorphism $f:\Omega\to\mathbb{R}^n$ is a lower Q-homeomorphism with*

$$Q(x) \;=\; K_O(x,f). \tag{10.43}$$

Thus, the whole theory of lower Q-homeomorphisms developed above is applicable to the finitely bi-Lipschitz mappings with $Q(x)=K_O(x,f)$. The same is true with respect to the theories of Q-homeomorphisms with $Q(x)=K_I(x,f)$ and FLD mappings; see, e.g., [127, 128, 204–209], and Chapters 4-6 and 8 in this volume.

Remark 10.5. By Theorem 8.1, every homeomorphism f in $\mathbb{R}^n$, $n\geq 2$, of the Sobolev class $W^{1,n}_{\mathrm{loc}}$ with $f^{-1}\in W^{1,n}_{\mathrm{loc}}$ is of FLD (finite length distortion) and hence by Theorem 8.7; cf. also Theorem 6.1, is a super Q-homeomorphism with $Q(x)=K_I(x,f)$. In this connection, we had the conjecture that such homeomorphisms are also of FAD_{n-1}, finite area distortion in dimension $n-1$, and hence by Lemma 10.3, they are lower Q-homeomorphisms with $Q(x)=K_O(x,f)$.

Sergei Vodopyanov pointed to the fact that (similarly to Lemma 4.1 in the recent preprint [47] published at December 2007) it is easy proved that homeomorphisms of the class $W^{1,n-1}_{\mathrm{loc}}$ have the (N)-property with respect to area on a.e. spheres centered at a boundary point and, thus, (10.24) can be proved for them similarly to Lemma 10.3. Consequently, homeomorphisms f of the class $W^{1,n}_{\mathrm{loc}}$ with $f^{-1}\in W^{1,n}_{\mathrm{loc}}$ are lower Q-homeomorphisms with $Q(x)=K_O(x,f)$. Furthermore, Sergei Vodopyanov with his student are preparing a preprint where they have proved that homeomorphisms $f\in W^{1,n-1}_{\mathrm{loc}}$ have the (N)-property on a.e. hyperplane and they are going to prove that the latter is valid on a.e. hypersurface.

Thus, our conjecture is verified and the theories of lower Q-homeomorphisms as well as of finite area distortion are applicable to homeomorphisms $f\in W^{1,n}_{\mathrm{loc}}$ with $K_I(x,f)\in L^1_{\mathrm{loc}}$ and, in particular, with $K_O(x,f)\in L^{n-1}_{\mathrm{loc}}$; cf. Corollaries 6.4 and 6.5.

Chapter 11
On Ring Solutions of the Beltrami Equation

In this chapter we prove uniqueness and existence theorems for ring Q-homeomorphisms in the plane, extending earlier results on the existence and uniqueness of ACL solutions for the Beltrami equation. One of the conditions for uniqueness and existence is expressed in terms of the finite mean oscillation of majorants for the tangential dilatation. We also prove a generalization of the Lehto existence theorem.

The existence problem for degenerate Beltrami equations has been studied extensively; see, e.g., [31, 32, 48, 98, 134, 169, 189, 203, 220, 241, 271–280, 310]. A more detailed discussion of these results can be found in the survey [297]. Some of those and many other results can be derived from the generalization of the Lehto existence theorem (Theorem 11.10 here), first obtained in [277].

11.1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$, i.e., an open and connected subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The **Beltrami equation** is of the form

$$f_{\bar{z}} = \mu(z) \cdot f_z, \tag{11.1}$$

where $f_{\bar{z}} = \bar{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly. The function μ is called the **complex coefficient** and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{11.2}$$

the **maximal dilatation** or, in short the **dilatation**, of Eq. (11.1). If

$$\operatorname{ess\,sup} K_\mu(z) = \infty,$$

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_11,

then the Beltrami equation (11.1) is said to be **degenerate**. As we know, the Beltrami equation plays an important role in mapping theory. The main goal of this chapter is to present general principles that allow us to obtain a variety of conditions for the existence of homeomorphic ACL solutions in the degenerate case. Our existence theorems are proved by an approximation method.

Given a point z_0 in D, the **tangential dilatation** and the **radial dilatation** of (11.1) with respect to z_0 are respectively defined by

$$K_\mu^T(z,z_0) = \frac{\left|1-\frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2} \tag{11.3}$$

and

$$K_\mu^r(z,z_0) = \frac{1-|\mu(z)|^2}{\left|1+\frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}\,; \tag{11.4}$$

cf. [98, 189, 253]. Reasons for the names will be given in Section 11.3.

Note that if $f \in$ ACL, then f has partial derivatives f_x and f_y a.e. and, thus, by the well-known Gehring–Lehto theorem, every ACL homeomorphism $f : D \to \mathbb{C}$ is differentiable a.e.; see [80] or [190], p. 128. For a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$ is nonnegative a.e.; see [190], p. 10. In this case, the **complex dilatation** μ_f of f is the ratio $\mu(z) = f_{\bar z}/f_z$ if $f_z \neq 0$ and $\mu(z) = 0$ otherwise, and the **dilatation** $K_f(z)$ of f at z is $K_\mu(z)$; see (11.2). Note that $|\mu(z)| \le 1$ a.e. and $K_\mu(z) \ge 1$ a.e.

Given a measurable function $K : D \to [1,\infty]$, we say (cf. [1]) that a sense-preserving ACL homeomorphism $f : D \to \overline{\mathbb{C}}$ is $K(z)$**-quasiconformal**, abbr. $K(z)$-qc, if

$$K_f(z) \le K(z) \quad \text{a.e.} \tag{11.5}$$

An ACL homeomorphism $f : D \to \mathbb{C}$ is called a **ring solution** of the Beltrami equation (11.1) with complex coefficient μ if f satisfies (11.1) a.e., $f^{-1} \in W^{1,2}_{\text{loc}}$ and f is a ring Q-homeomorphism at every point $z_0 \in D$ with $Q_{z_0}(z) = K^T_\mu(z,z_0)$; see Section 6.1; cf. Section 10.3. We show that ring solutions exist for wide classes of the degenerate Beltrami equations.

The condition $f^{-1} \in W^{1,2}_{\text{loc}}$ in the definition of a ring solution implies that a.e. point z is a **regular point** for the mapping f, i.e., f is differentiable at z and $J_f(z) \neq 0$. Note that the condition $K_\mu \in L^1_{\text{loc}}$ is necessary for a homeomorphic ACL solution f of (11.1) to have the property $g = f^{-1} \in W^{1,2}_{\text{loc}}$ because this property implies that

$$\begin{aligned}\int_C K_\mu(z)\,dxdy \;&\le\; 4\int_C \frac{dxdy}{1-|\mu(z)|^2}\\ &=\; 4\iint_{f(C)} \frac{J_g(w)\,dudv}{1-|\mu(g(w))|^2} \;=\; 4\int_{f(C)} |\partial g|^2\,dudv \;<\; \infty\end{aligned}$$

for every compact set $C \subset D$.

Note that every homeomorphic ACL solution f of the Beltrami equation with $K_\mu \in L^1_{\mathrm{loc}}$ belongs to the class $W^{1,1}_{\mathrm{loc}}$, as in all of our theorems. Note also that if, in addition, $K_\mu \in L^p_{\mathrm{loc}}$, $p \in [1,\infty]$, then $f \in W^{1,s}_{\mathrm{loc}}$, where $s = 2p/(1+p) \in [1,2]$. Indeed,

$$|\partial f| + |\overline{\partial} f| \;=\; K_\mu^{1/2}(z) \cdot J_f^{1/2}(z),$$

and by Hölder's inequality, on every compact set $C \subset D$,

$$\begin{aligned} ||\overline{\partial} f||_s \;&\le\; ||\partial f||_s \;\le\; ||K_\mu^{1/2}||_p \cdot ||J_f^{1/2}||_2 \\ &= ||K_\mu||_q^{1/2} \cdot ||J_f||_1^{1/2} \;\le\; ||K_\mu||_q^{1/2} \cdot A(f(C))^{1/2} \end{aligned}$$

(see, e.g., [190], p. 131), where $A(f(C))$ is the area of the set $f(C)$ and $1/p + 1/2 = 1/s$ and $q = p/2$. Hence, $f \in W^{1,s}_{\mathrm{loc}}$; see, e.g., [215], p. 8.

In the classical case when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty$, every ACL homeomorphic solution f of the Beltrami equation (11.1) is in the class $W^{1,2}_{\mathrm{loc}}$ together with its inverse mapping f^{-1}, and hence f is a ring solution of (11.1) by Theorem 11.1. In the case $\|\mu\|_\infty = 1$ with $K_\mu \le Q \in \mathrm{BMO}$, again $f^{-1} \in W^{1,2}_{\mathrm{loc}}$ and f belongs to $W^{1,s}_{\mathrm{loc}}$ for all $1 \le s < 2$ but not necessarily to $W^{1,2}_{\mathrm{loc}}$; see examples in [274]. However, there is a variety of degenerate Beltrami equations for which ring solutions exist, as shown ahead.

11.2 Finite Mean Oscillation

Let D be a domain in the complex plane $\mathbb{C}$. Recall that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation at a point** $z_0 \in D$ if

$$d_\varphi(z_0) = \overline{\lim_{\varepsilon \to 0}} \; \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, dxdy \;<\; \infty, \tag{11.6}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{D(z_0,\varepsilon)} \varphi(z) \, dxdy \;<\; \infty \tag{11.7}$$

is the mean value of the function $\varphi(z)$ over the disk $D(z_0,\varepsilon)$. We call $d_\varphi(z_0)$ the **dispersion** of the function φ at point z_0. We say that a function $\varphi : D \to \mathbb{R}$ is of **finite mean oscillation in D**, abbr. $\varphi \in$ FMO(D) or simply $\varphi \in$ FMO, if φ has a finite dispersion at every point $z \in D$.

Remark 11.1. Note that if a function $\varphi : D \to \mathbb{R}$ is integrable over $D(z_0,\varepsilon_0) \subset D$, then

$$\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, dxdy \;\le\; 2 \cdot \overline{\varphi}_\varepsilon(z_0) \tag{11.8}$$

and the right-hand side in (11.8) is continuous of $\varepsilon \in (0,\varepsilon_0]$ by the absolute continuity of the integral. Thus, for every $\delta_0 \in (0,\varepsilon_0)$,

$$\sup_{\varepsilon\in[\delta_0,\varepsilon_0]} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z)-\overline{\varphi}_\varepsilon(z_0)|\, dxdy \;<\infty. \tag{11.9}$$

If (11.6) holds, then

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z)-\overline{\varphi}_\varepsilon(z_0)|\, dxdy \;<\infty. \tag{11.10}$$

The value in (11.10) is called the **maximal dispersion** of the function φ in the disk $D(z_0,\varepsilon_0)$.

Proposition 11.1. *If, for some collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \; \fint_{D(z_0,\varepsilon)} |\varphi(z)-\varphi_\varepsilon|\, dxdy \;<\infty, \tag{11.11}$$

then φ has finite mean oscillation at z_0.

Proof. Indeed, by the triangle inequality,

$$\begin{aligned} &\fint_{D(z_0,\varepsilon)} |\varphi(z)-\overline{\varphi}_\varepsilon(z_0)|\, dxdy \\ &\le \fint_{D(z_0,\varepsilon)} |\varphi(z)-\varphi_\varepsilon|\, dxdy \;+\; |\varphi_\varepsilon-\overline{\varphi}_\varepsilon(z_0)| \\ &\le 2\cdot \fint_{D(z_0,\varepsilon)} |\varphi(z)-\varphi_\varepsilon|\, dxdy\,. \end{aligned}$$

$\square$

Corollary 11.1. *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \; \fint_{D(z_0,\varepsilon)} |\varphi(z)|\, dxdy \;<\; \infty, \tag{11.12}$$

then φ has finite mean oscillation at z_0.

Remark 11.2. Clearly BMO $\subset$ FMO. The example given at the end of this chapter shows that the inclusion is proper. Note that the function $\varphi(z) = \log 1/|z|$ belongs to BMO in the unit disk $\mathbb{D}$ (see, e.g., [255], p. 5) and hence also to FMO. However, $\overline{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (11.12) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0.

A point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0} \; \fint_{D(z_0,\varepsilon)} |\varphi(z)-\varphi(z_0)|\, dxdy \;=\; 0. \tag{11.13}$$

It is known that, for every function $\varphi \in L^1(D)$, almost every point in D is a Lebesgue point.

Corollary 11.2. *Every function $\varphi : D \to \mathbb{R}$, that is locally integrable has a finite mean oscillation at almost every point in D.*

Ahead we use the notations $D(r) = D(0,r) = \{z \in \mathbb{C} : |z| < r\}$ and

$$A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z| < \varepsilon_0\}. \tag{11.14}$$

Lemma 11.1. *Let $D \subset \mathbb{C}$ be a domain such that $D(1/2) \subset D$, and let $\varphi : D \to \mathbb{R}$ be a nonnegative function. If φ is integrable in $D(1/2)$ and of* FMO *at* 0, *then*

$$\int\limits_{A(\varepsilon,1/2)} \frac{\varphi(z)\, dxdy}{\left(|z| \log_2 \frac{1}{|z|}\right)^2} \leq C \cdot \log_2 \log_2 \frac{1}{\varepsilon} \tag{11.15}$$

for $\varepsilon \in (0, 1/4)$, where

$$C = 4\pi\, [\varphi_0 + 6d_0], \tag{11.16}$$

φ_0 is the mean value of φ over the disk $D(1/2)$, and d_0 is the maximal dispersion of φ in $D(1/2)$.

Versions of this lemma were first established for BMO functions and $n = 2$ in [271] and [273] and then for FMO functions in [127] and [276]. An n-dimensional version of the lemma for BMO functions was established in [205].

Proof. Let $0 < \varepsilon < 1/4$, $\varepsilon_k = 2^{-k}$, $A_k = \{z \in D : \varepsilon_{k+1} \leq |z| < \varepsilon_k\}$, $D_k = D(\varepsilon_k)$, and φ_k the mean value of $\varphi(z)$ over D_k, $k = 1, 2 \ldots$. Choose a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$ and $\alpha(t) = (t \log 1/t)^{-2}$. Then $A(\varepsilon, 2^{-1}) \subset A(\varepsilon) = \cup_{k=1}^{N} A_k$and

$$\eta(\varepsilon) = \int\limits_{A(\varepsilon)} \varphi(z)\, \alpha(|z|)\, dxdy \ \leq\ |S_1| + S_2,$$

where

$$S_1(\varepsilon) \ = \ \sum_{k=1}^{N} \int\limits_{A_k} (\varphi(z) - \varphi_k)\, \alpha(|z|)\, dxdy,$$

$$S_2(\varepsilon) \ = \ \sum_{k=1}^{N} \varphi_k \int\limits_{A_k} \alpha(|z|)\, dxdy.$$

Since $A_k \subset D_k$, $|z|^{-2} \leq 4\pi/|D_k|$ for $z \in A_k$ and $\log 1/|z| > k$ in A_k, we obtain

$$|S_1| \leq 4\pi d_0 \sum_{k=1}^{N} \frac{1}{k^2} < 8\pi d_0$$

because

$$\sum_{k=2}^{\infty} \frac{1}{k^2} < \int_1^{\infty} \frac{dt}{t^2} = 1.$$

Now,

$$\int_{A_k} \alpha(|z|)\, dxdy \ \le \ \frac{1}{k^2} \int_{A_k} \frac{dxdy}{|z|^2} = \frac{2\pi}{k^2}.$$

Moreover,

$$\begin{aligned} |\varphi_k - \varphi_{k-1}| &= \frac{1}{|D_k|} \left| \int_{D_k} (\varphi(z) - \varphi_{k-1})\, dxdy \right| \\ &\le \frac{4}{|D_{k-1}|} \int_{D_{k-1}} |\varphi(z) - \varphi_{k-1}|\, dxdy \le 4d_0 \end{aligned}$$

and by the triangle inequality, for $k \ge 2$,

$$\varphi_k = |\varphi_k| \le \varphi_1 + \sum_{l=2}^{k} |\varphi_l - \varphi_{l-1}| \le \varphi_1 + 4kd_0 = \varphi_0 + 4kd_0\,.$$

Hence,

$$S_2 = |S_2| \ \le \ 2\pi \sum_{k=1}^{N} \frac{\varphi_k}{k^2} \ \le \ 4\pi\varphi_0 + 8\pi d_0 \sum_{k=1}^{N} \frac{1}{k}\ .$$

But

$$\sum_{k=2}^{N} \frac{1}{k} < \int_1^{N} \frac{dt}{t} = \log\ N < \log_2 N$$

and, for $\varepsilon < \varepsilon_N$,

$$N = \log_2 \frac{1}{\varepsilon_N} < \log_2 \frac{1}{\varepsilon}.$$

Consequently,

$$\sum_{k=1}^{N} \frac{1}{k} < 1 + \log_2 \log_2 \frac{1}{\varepsilon},$$

and, thus, for $\varepsilon \in (0, 1/4)$,

$$\eta(\varepsilon) \ \le \ 4\pi \left(2d_0 \ + \ \frac{4d_0 + \varphi_0}{\log_2 \log_2 \frac{1}{\varepsilon}} \right) \cdot \log_2 \log_2 \frac{1}{\varepsilon} \ \le \ C \cdot \log_2 \log_2 \frac{1}{\varepsilon},$$

as required. □

We complete this section by constructing a function $\varphi : \mathbb{C} \to \mathbb{R}$ that belongs to FMO but not to L^p_{loc} for any $p > 1$ and hence does not belong to $\mathrm{BMO}_{\mathrm{loc}}$.

Example. Fix $p > 1$. For $k = 1,2,\dots$, set $z_k = 2^{-k}$, $r_k = 2^{-pk^2}$, and $D_k = D(z_k, r_k)$. Define $\varphi(z) = \Sigma_{k=2}^{\infty}\varphi_k(z)$, where $\varphi_k(z) = 2^{2k^2}$ if $z \in D_k$ and 0 otherwise. Then φ is locally bounded in $\mathbb{C}\setminus\{0\}$ and hence belongs to $\mathrm{BMO}_{\mathrm{loc}}(\mathbb{C}\setminus\{0\})$ and therefore to $\mathrm{FMO}(\mathbb{C}\setminus\{0\})$. To show that φ is of FMO at $z = 0$, calculate

$$\int\limits_{D_k} \varphi_k(z)\, dxdy \;=\; \pi 2^{-2(p-1)k^2} \tag{11.17}$$

and, thus,

$$\overline{\lim_{\varepsilon\to 0}} \;⨍_{D(\varepsilon)} \varphi(z)\, dxdy \;<\; \infty. \tag{11.18}$$

Indeed, setting

$$K = K(\varepsilon) = \left[\log_2 \frac{1}{\varepsilon}\right] \le \log_2 \frac{1}{\varepsilon}, \tag{11.19}$$

where $[A]$ is the integral part of a number A, we have

$$J = ⨍_{D(\varepsilon)} \varphi(z)\, dxdy \le \sum_{k=K}^{\infty} 2^{-2(p-1)k^2}/2^{-2(K+1)}. \tag{11.20}$$

If $(p-1)K > 1$, i.e., $K > 1/(p-1)$, then

$$\sum_{k=K}^{\infty} 2^{-2(p-1)k^2} \le \sum_{k=K}^{\infty} 2^{-2k} = 2^{-2K}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = \frac{4}{3}\cdot 2^{-2K}, \tag{11.21}$$

and hence $J \le 16/3$. Corollary 11.1 yields $\varphi \in \mathrm{FMO}$.

Finally, note that

$$\int\limits_{D_k} \varphi_k^p(z)\, dxdy = \pi, \tag{11.22}$$

and hence $\varphi \notin L^p(U)$ in any neighborhood U of 0.

11.3 Ring Q-Homeomorphisms in the Plane

We first recall the definition of a ring Q-homeomorphism adopted to the plane $\mathbb{C}$.

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Delta(E,F,D)$ denotes the family of all paths $\gamma\colon [a,b] \to \overline{\mathbb{C}}$ that join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma(t) \in D$ for $a < t < b$. We set $\Delta(E,F) = \Delta(E,F,\overline{\mathbb{C}})$ if $D = \overline{\mathbb{C}}$. A **ring domain**, or shortly a **ring,** in $\overline{\mathbb{C}}$ is a doubly connected domain R in $\overline{\mathbb{C}}$. Let R be a ring in $\overline{\mathbb{C}}$. If C_1 and C_2 are the components of $\overline{\mathbb{C}}\setminus R$, we write $R = R(C_1,C_2)$. The 2-capacity [see (2.15)] and the modulus of the path family $\Gamma(C_1,C_2,R)$ coincide,

$$\operatorname{cap} R(C_1,C_2) \;=\; M(\Delta(C_1,C_2,R)); \tag{11.23}$$

see, e.g., [74, 77] and [71], Appendix A.1. Note also that

$$M(\Delta(C_1,C_2,R)) \;=\; M(\Delta(C_1,C_2)); \tag{11.24}$$

see, e.g., Theorem 11.3 in [316].

Let D be a domain in $\mathbb{C}$, $z_0 \in D$, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, and $Q : D(z_0,r_0) \to [0,\infty]$ a measurable function in the disk

$$D(z_0,r_0) = \{z \in \mathbb{C} : |z-z_0| < r_0\}. \tag{11.25}$$

Set

$$A(r_1,r_2,z_0) = \{z \in \mathbb{C} : r_1 < |z-z_0| < r_2\}, \tag{11.26}$$

$$C_i \;:\; = C(z_0,r_i) = \{z \in \mathbb{C} : |z-z_0| = r_i\}, \quad i = 1,2. \tag{11.27}$$

We say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a **ring Q-homeomorphism** at the point z_0 if

$$M(\Delta(fC_1, fC_2, fD)) \;\leq\; \int\limits_A Q(z) \cdot \eta^2(|z-z_0|)\; dxdy \tag{11.28}$$

for every annulus $A = A(r_1,r_2,z_0)$, $0 < r_1 < r_2 < r_0$, and for every measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\; dr \;=\; 1. \tag{11.29}$$

In this section we find conditions on f under which f is a ring Q-homeomorphism.

Now, let z be a regular point for a mapping $f : D \to \mathbb{C}$. Here we consider $f'(z)$ as a linear map of $\mathbb{R}^2$. Given $\omega \in \mathbb{C}$, $|\omega| = 1$, the **derivative in the direction ω** of the mapping f at the point z is

$$\partial_\omega f(z) \;=\; \lim_{t \to +0} \frac{f(z+t\cdot\omega) - f(z)}{t} \;=\; f'(z)\,\omega. \tag{11.30}$$

The **radial direction** at a point $z \in D$ with respect to the center $z_0 \in \mathbb{C}$, $z_0 \neq z$, is

$$\omega_0 \;=\; \omega_0(z,z_0) \;=\; \frac{z-z_0}{|z-z_0|}. \tag{11.31}$$

The **radial dilatation** of f at z with respect to z_0 is defined by

$$K^r(z,z_0,f) \;=\; \frac{|J_f(z)|}{|\partial_r^{z_0} f(z)|^2} \tag{11.32}$$

and the **tangential dilatation** by

$$K^T(z,z_0,f) \;=\; \frac{|\partial_T^{z_0} f(z)|^2}{|J_f(z)|}, \tag{11.33}$$

where $\partial_r^{z_0} f(z)$ is the derivative of f at z in direction ω_0 and $\partial_T^{z_0} f(z)$ in $\tau = i\omega_0$, that is,

$$\partial_r^{z_0} f(z) \;=\; f'(z)\,\omega_0\,, \qquad \partial_T^{z_0} f(z) \;=\; f'(z)\, i\omega_0\,,$$

respectively.

Note that if z is a regular point of f and $|\mu(z)| < 1$, $\mu(z) = f_{\bar z}/f_z$, then

$$K^r(z,z_0,f) \;=\; K_\mu^r(z,z_0) \tag{11.34}$$

and

$$K^T(z,z_0,f) \;=\; K_\mu^T(z,z_0), \tag{11.35}$$

where $K_\mu^r(z,z_0)$ and $K_\mu^T(z,z_0)$ are defined by (11.4) and (11.3), respectively. Indeed, equalities (11.34) and (11.35) follow directly from the computations

$$\begin{aligned}\partial_r f \;&=\; \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r} \;+\; \frac{\partial f}{\partial \bar z}\cdot\frac{\partial \bar z}{\partial r} \\ &=\; \frac{z-z_0}{|z-z_0|}\cdot f_z \;+\; \frac{\overline{z-z_0}}{|z-z_0|}\cdot f_{\bar z}\,,\end{aligned} \tag{11.36}$$

where $r = |z-z_0|$, and

$$\begin{aligned}\partial_T f &= \frac{1}{r}\left(\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \vartheta} + \frac{\partial f}{\partial \bar z}\cdot\frac{\partial \bar z}{\partial \vartheta}\right) \\ &= i\cdot\left(\frac{z-z_0}{|z-z_0|}\cdot f_z - \frac{\overline{z-z_0}}{|z-z_0|}\cdot f_{\bar z}\right),\end{aligned} \tag{11.37}$$

where $\vartheta = \arg(z-z_0)$ because $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$.

The **big radial dilatation** of f at z with respect to z_0 is defined by

$$K^R(z,z_0,f) = \frac{|J_f(z)|}{|\partial_R^{z_0} f(z)|^2}, \tag{11.38}$$

where

$$|\partial_R^{z_0} f(z)| \;=\; \min_{\omega\in\mathbb{C},|\omega|=1} \frac{|\partial_\omega f(z)|}{|\mathrm{Re}\,\omega\overline{\omega_0}|}\,. \tag{11.39}$$

Here $\mathrm{Re}\,\omega\overline{\omega_0}$ is the scalar product of vectors ω and ω_0. In other words, $\mathrm{Re}\,\omega\overline{\omega_0}$ is the projection of the vector ω onto the radial direction ω_0. Obviously, there is a unit vector ω_* such that

$$|\partial_R^{z_0} f(z)| \;=\; \frac{|\partial_{\omega_*} f(z)|}{|\mathrm{Re}\,\omega_*\overline{\omega_0}|}\,. \tag{11.40}$$

Clearly

$$|\partial_r^{z_0} f(z)| \;\geq\; |\partial_R^{z_0} f(z)| \;\geq\; \min_{\omega\in\mathbb{C},|\omega|=1} |\partial_\omega f(z)| \tag{11.41}$$

and, hence,

$$K^r(z,z_0,f) \;\leq\; K^R(z,z_0,f) \;\leq\; K_\mu(z); \tag{11.42}$$

the equalities hold in (11.42) if and only if the minimum in the right-hand side of (11.41) is realized at the radial direction $\omega = \omega_0$.

Note that $\partial_r^{z_0} f(z) \neq 0$, $|\partial_R^{z_0} f(z)| \neq 0$, and $\partial_T^{z_0} f(z) \neq 0$ at every regular point $z \neq z_0$ of f; see, e.g., Section 1.2.1 in [256]. In view of (11.33), (11.35), and (11.3), the following lemma shows that the big radial dilatation coincides with the tangential dilatation at every regular point.

Lemma 11.2. *Let $z \in D$ be a regular point of a mapping $f : D \to \mathbb{C}$ with the complex dilatation $\mu(z) = f_{\bar z}/f_z$ such that $|\mu(z)| < 1$. Then*

$$K^R(z,z_0,f) \;=\; \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}\,. \tag{11.43}$$

Proof. The derivative of f at the regular point z in a direction $\omega = e^{i\alpha}$ is $\partial_\omega f(z) = f_z + f_{\bar z}\cdot e^{-2i\alpha}$, in complex notation; see, e.g., [190], pp. 17 and 182. Consequently,

$$\begin{aligned} X := \frac{|\partial_R^{z_0} f(z)|^2}{|f_z|^2} &= \min_{\alpha\in[0,2\pi]} \frac{|\mu(z)+e^{2i\alpha}|^2}{\cos^2(\alpha-\vartheta)} = \min_{\beta\in[0,2\pi]} \frac{|\nu - e^{2i\beta}|^2}{\sin^2\beta} \\ &= \min_{\beta\in[0,2\pi]} \frac{1+|\nu|^2-2|\nu|\cos(\kappa-2\beta)}{\sin^2\beta} \\ &= \min_{t\in[-1,1]} \frac{1+|\nu|^2-2|\nu|\cdot[(1-2t^2)\cos\kappa \pm 2t(1-t^2)^{1/2}\sin\kappa]}{t^2}, \end{aligned}$$

where $t = \sin\beta$, $\beta = \alpha + \frac{\pi}{2} - \vartheta$, $\nu = \mu(z)e^{-2i\vartheta}$, and $\kappa = \arg\nu = \arg\mu - 2\vartheta$. Hence, $X = \min_{\tau\in[1,\infty]} \varphi_\pm(\tau)$, where

$$\begin{aligned} \varphi_\pm(\tau) \;&=\; b \,+\, a\,\tau \,\pm\, c\,(\tau-1)^{1/2}\,, \qquad \tau \;=\; \frac{1}{\sin^2\beta}\,, \\ a \;&=\; 1 \,+\, |\nu|^2 - 2|\nu|\,\cos\kappa\,, \quad b \;=\; 4\,|\nu|\cos\kappa\,, \quad c \;=\; 4\,|\nu|\sin\kappa\,. \end{aligned}$$

Since $\varphi_\pm'(\tau) = a \pm (\tau-1)^{-1/2}c/2$, the minimum is obtained for $\tau = 1 + c^2/4a^2$. Now $(\tau-1)^{1/2} = \mp c/2a$, and thus,

$$X \;=\; b + \left(a + \frac{1}{4}\frac{c^2}{a}\right) - \frac{1}{2}\frac{c^2}{a} \;=\; \frac{(1-|\nu|^2)^2}{1+|\nu|^2-2|\nu|\cos\kappa}.$$

This yields (11.43), as required. □

Prototypes of the following theorem can be found in [253], [189], and [98]. These results use $|\mu|$ and arg μ in modulus estimates.

Theorem 11.1. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of class $W^{1,2}_{\mathrm{loc}}$ such that $f^{-1} \in W^{1,2}_{\mathrm{loc}}$. Then, at every point $z_0 \in D$, the mapping f is a ring Q-homeomorphism with $Q(z) = K^T_\mu(z,z_0)$, where $\mu(z) = \mu_f(z)$.*

Proof. Fix $z_0 \in D$, let r_1 and r_2 be such that $0 < r_1 < r_2 < r_0 \le \operatorname{dist}(z_0, \partial D)$, and let $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$ and $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$. Set $\Gamma = \Delta(C_1, C_2, D)$ and denote by Γ_* the family of all rectifiable paths $\gamma_* \in f\Gamma$ such that f^{-1} is absolutely continuous on every closed subpath of γ_*. Then $M(f\Gamma) = M(\Gamma_*)$ by the Fuglede theorem (see [64] and [316]), because $f^{-1} \in \mathrm{ACL}^2$; see, e.g., [215], p. 8.

Fix $\gamma_* \in \Gamma_*$. Set $\gamma = f^{-1} \circ \gamma_*$ and denote by s and s_* the natural (length) parameters of γ and γ_*, respectively. Note that the correspondence $s_*(s)$ between the natural parameters of γ_* and γ is a strictly monotone function and we may assume that $s_*(s)$ is increasing. For $\gamma_* \in \Gamma_*$, the inverse function $s(s_*)$ has the (N)-property and $s_*(s)$ is differentiable a.e. as a monotone function. Thus, $ds_*/ds \ne 0$ a.e. on γ by [244]. Let s be such that $z = \gamma(s)$ is a regular point for f and suppose that γ is differentiable at s with $ds_*/ds \ne 0$. Set $r = |z - z_0|$ and let ω be a unit tangential vector to the path γ at the point $z = \gamma(s)$. Then

$$\left|\frac{dr}{ds_*}\right| = \frac{\frac{dr}{ds}}{\frac{ds_*}{ds}} = \frac{|\mathrm{Re}\,\omega\overline{\omega_0}|}{|\partial_\omega f(z)|} \le \frac{1}{|\partial^{z_0}_R f(z)|}, \tag{11.44}$$

where $|\partial^{z_0}_R f(z)|$ is defined by (11.39).

Now, let $\eta : (r_1, r_2) \to [0, \infty]$ be a measurable function such that

$$\int_{r_1}^{r_2} \eta(r)\, dr = 1. \tag{11.45}$$

By the Lusin theorem, there is a Borel function $\eta_* : (r_1, r_2) \to [0, \infty]$ such that $\eta_*(r) = \eta(r)$ a.e.; see, e.g., Section 2.3.5 in [55] and [281], p. 69. Set

$$\rho(z) = \eta_*(|z - z_0|)$$

in the annulus $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and $\rho(z) = 0$ outside A. Also set

$$\rho_*(w) = \{\rho / |\partial^{z_0}_R f|\} \circ f^{-1}(w)$$

if $z = f^{-1}(w)$ is a regular point of f, $\rho_*(w) = \infty$ at the rest points of $f(D)$, and $\rho_*(w) = 0$ outside $f(D)$. Then, by (11.44) and (11.45), for $\gamma_* \in \Gamma_*$,

$$\int_{\gamma_*} \rho_* ds_* \ge \int_{\gamma_*} \eta(r) \left|\frac{dr}{ds_*}\right| ds_* \ge \int_{\gamma_*} \eta(r)\, \frac{dr}{ds_*}\, ds_* = \int_{r_1}^{r_2} \eta(r)\, dr = 1$$

because the function $z=\gamma(s(s_*))$ is absolutely continuous and hence so is $r=|z-z_0|$ as a function of the parameter s_*. Consequently, ρ_* is admissible for all $\gamma_*\in\Gamma_*$.

By Proposition 4.1, f and f^{-1} are regular a.e. and have the (N)-property . Thus, by a change of variables (see, e.g., Theorem 8.1 and Proposition 8.3), we have in view of Lemma 11.2 that

$$M(f\Gamma)\leq\int\limits_{f(A)}\rho_*(w)^2dudv\ =\ \int\limits_{A}\rho(z)^2K_{\mu}^{T}(z,z_0)\,dxdy$$
$$=\int\limits_{A}K_{\mu}^{T}(z,z_0)\cdot\eta^2(|z-z_0|)\,dxdy,$$

i.e., f is a ring Q-homeomorphism with $Q(z)=K_{\mu}^{T}(z,z_0)$. □

If f is a plane $W^{1,2}_{\mathrm{loc}}$ homeomorphism with a locally integrable $K_f(z)$, then $f^{-1}\in W^{1,2}_{\mathrm{loc}}$; see, e.g., [111]. Hence, we obtain the following consequences of Theorem 11.1.

Corollary 11.3. *Let $f:D\to\mathbb{C}$ be a sense-preserving homeomorphism of class $W^{1,2}_{\mathrm{loc}}$ and suppose that $K_f(z)$ is integrable in a disk $D(z_0,r_0)\subset D$ for some $z_0\in D$ and $r_0>0$. Then f is a ring Q-homeomorphism at the point $z_0\in D$ with $Q(z)=K_{\mu}^{T}(z,z_0)$, where $\mu(z)=\mu_f(z)$.*

Corollary 11.4. *Let $f:D\to\mathbb{C}$ be a sense-preserving homeomorphism of class $W^{1,2}_{\mathrm{loc}}$ with $K_\mu\in L^1_{\mathrm{loc}}$. Then f is a ring Q-homeomorphism at every point $z_0\in D$ with $Q(z)=K_\mu(z)$, where $\mu(z)=\mu_f(z)$.*

We close this section with a convergence theorem that plays an important role in our scheme for deriving the existence theorems of the Beltrami equation.

Theorem 11.2. *Let $f_n:D\to\overline{\mathbb{C}}$, $n=1,2,\ldots$ be a sequence of ring Q-homeomorphisms at a point $z_0\in D$. If the f_n converge locally uniformly to a homeomorphism $f:D\to\overline{\mathbb{C}}$, then f is also a ring Q-homeomorphism at the point z_0.*

Indeed, the proof follows by Theorem A.12 from the uniform convergence of the corresponding rings.

11.4 Distortion Estimates

In this section we use again, cf. Section 7.3, the standard conventions $a/\infty=0$ for $a\neq\infty$ and $a/0=\infty$ if $a>0$ and $0\cdot\infty=0$; see, e.g., [280], p. 6.

For points $z,\zeta\in\overline{\mathbb{C}}$, the **spherical (chordal) distance** $s(z,\zeta)$ between z and ζ is given by

$$s(z,\zeta) \;=\; \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}(1+|\zeta|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \neq \zeta, \tag{11.46}$$

$$s(z,\infty) \;=\; \frac{1}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty.$$

Given a set $E \subset \mathbb{C}$, $\delta(E)$ denotes the **spherical diameter** of E, i.e.,

$$\delta(E) \;=\; \sup_{z_1,z_2\in E} s(z_1,z_2). \tag{11.47}$$

Lemma 11.3. *Let $f : D \to \mathbb{C}$ be a homeomorphism with $\delta(\overline{\mathbb{C}}\setminus f(D)) \geq \Delta > 0$ and let z_0 be a point in D, $\zeta \in D(z_0,r_0)$, $r_0 < \operatorname{dist}(z_0,\partial D)$, $C_0 = \{z \in \mathbb{C} : |z-z_0| = r_0\}$, and $C = \{z \in \mathbb{C} : |z-z_0| = |\zeta - z_0|\}$. Then*

$$s(f(\zeta),f(z_0)) \;\leq\; \frac{32}{\Delta}\cdot\exp\left(-\frac{2\pi}{M(\Delta(fC,fC_0,fD))}\right). \tag{11.48}$$

Proof. Let E denote the component of $\overline{\mathbb{C}}\setminus fA$ containing $f(z_0)$ and F the component containing ∞, where $A = \{z \in \mathbb{C} : |\zeta - z_0| < |z-z_0| < r_0\}$. By the known Gehring lemma,

$$\operatorname{cap} R(E,F) \;\geq\; \operatorname{cap} R_T\left(\frac{1}{\delta(E)\delta(F)}\right), \tag{11.49}$$

where $\delta(E)$ and $\delta(F)$ denote the spherical diameters of the continua E and F, respectively, and $R_T(t)$ is the Teichmüller ring

$$R_T(t) \;=\; \overline{\mathbb{C}}\setminus([-1,0]\cup[t,\infty]), \quad\; t \;>\; 1; \tag{11.50}$$

see, e.g., Corollary 7.37 in [328] or [71]. We also know, that

$$\operatorname{cap} R_T(t) = \frac{2\pi}{\log\Phi(t)}, \tag{11.51}$$

where the function Φ admits the good estimates

$$t+1 \leq \Phi(t) \leq 16\cdot(t+1) < 32\cdot t, \quad\; t>1, \tag{11.52}$$

see, e.g., either (7.19) and Lemma 7.22 in [328] or [71], pp. 225–226, Section A.1. Hence, inequality (11.49) implies that

$$\operatorname{cap} R(E,F) \;\geq\; \frac{2\pi}{\log\frac{32}{\delta(E)\delta(F)}}\,. \tag{11.53}$$

Thus,

$$\delta(E) \;\leq \frac{32}{\delta(F)}\;\exp\left(-\frac{2\pi}{\operatorname{cap} R(E,F)}\right), \tag{11.54}$$

which implies the desired statement. □

Lemma 11.4. *Let $f : D \to \mathbb{C}$ be a ring Q-homeomorphism at a point $z_0 \in D$ with $Q : D(z_0, r_0) \to [0,\infty]$, $r_0 \le \operatorname{dist}(z_0, \partial D)$. Suppose that $\psi_\varepsilon : [0,\infty] \to [0,\infty]$, $0 < \varepsilon < \varepsilon_0 < r_0$, is a one-parameter family of measurable functions such that*

$$0 \; < \; I(\varepsilon) \; = \; \int_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t)\, dt \; < \; \infty, \quad \varepsilon \in (0, \varepsilon_0). \tag{11.55}$$

Set $C = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$, $C_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$, and

$$A(\varepsilon) \; = \; A(\varepsilon, \varepsilon_0, z_0) \; = \; \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}. \tag{11.56}$$

Then

$$M(\Delta(fC, fC_0, fD)) \; \le \; \omega(\varepsilon), \tag{11.57}$$

where

$$\omega(\varepsilon) \; = \; \frac{1}{I^2(\varepsilon)} \int_{A(\varepsilon)} Q(z) \cdot \psi_\varepsilon^2(|z - z_0|)\, dxdy. \tag{11.58}$$

Proof. Formula (11.57) follows from the definition (11.28) of a ring homeomorphism if we set $\eta(r) = \psi_\varepsilon(r)/I(\varepsilon)$, $r \in (\varepsilon, \varepsilon_0)$. □

Using Lemma 11.4, we now desire a sharp capacity estimate for ring Q-homeomorphisms $f : D \to \mathbb{C}$ at a point $z_0 \in D$. This estimate depends only on Q and implies as a special case an inequality of Reich and Walczak in [253], which several authors have applied.

Lemma 11.5. *Let D be a domain in $\mathbb{C}$, z_0 a point in D, $r_0 \le \operatorname{dist}(z_0, \partial D)$, $Q : D(z_0, r_0) \to [0,\infty]$ a measurable function, and $q(r)$ the mean of $Q(z)$ over the circle $|z - z_0| = r$, r, r_0. For $0 < r_1 < r_2 < r_0$, set*

$$I \; = \; I(r_1, r_2) \; = \; \int_{r_1}^{r_2} \frac{dr}{rq(r)} \tag{11.59}$$

and $C_j = \{z \in \mathbb{C} : |z - z_0| = r_j\}$, $j = 1, 2$. Then

$$M(\Delta(fC_1, fC_2, fD)) \; \le \; \frac{2\pi}{I} \tag{11.60}$$

whenever $f : D \to \mathbb{C}$ is a ring Q-homeomorphism at z_0.

Proof. With no loss of generality, we may assume that $I \ne 0$ because otherwise (11.60) is trivial and that $I \ne \infty$ because otherwise we can replace $Q(z)$ by $Q(z) + \delta$ with arbitrarily small $\delta > 0$ and then pass to the limit as $\delta \to 0$ in (11.60). The condition $I \ne \infty$ implies, in particular, that $q(r) \ne 0$ a.e. in (r_1, r_2).

If $I \ne 0$ or ∞, we can choose in Lemma 11.4

$$\psi_\varepsilon(t) \equiv \psi(t) := \begin{cases} 1/[tq(t)]\,, & t \in (0, \varepsilon_0), \\ 0\,, & \text{otherwise}, \end{cases} \tag{11.61}$$

with $\varepsilon = r_1$ and $\varepsilon_0 = r_2$, and since

$$\int\limits_A Q(z) \cdot \psi^2(|z - z_0|)\, dxdy \;=\; 2\pi I, \tag{11.62}$$

where

$$A \;=\; A(r_1, r_2, z_0) \;=\; \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}, \tag{11.63}$$

we obtain (11.60). □

Corollary 11.5. *For every ring Q-homeomorphism* $f : D \to \mathbb{C}$ *at* $z_0 \in D$ *and* $0 < r_1 < r_2 < r_0$,

$$\int\limits_{r_1}^{r_2} \frac{dr}{rq(r)} \;<\; \infty, \tag{11.64}$$

where $q(r)$ *is the mean of* $Q(z)$ *over the circle* $|z - z_0| = r$.

Indeed, by (11.53) with $E = fC_1$, $F = fC_2$, $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$, and $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$

$$M(\Delta(fC_1, fC_2, fD)) \;\geq\; \frac{2\pi}{\log \frac{32}{\delta(fC_1)\delta(fC_2)}}\;. \tag{11.65}$$

The right-hand side in (11.65) should be positive because f is injective. Thus, Corollary 11.5 follows from (11.60) in Lemma 11.5.

Corollary 11.6. *Let* $f : D \to \mathbb{C}$ *be a* $W^{1,2}_{\mathrm{loc}}$ *homeomorphism in a domain* $D \subset \mathbb{C}$ *such that*

$$K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \;\in\; L^1_{\mathrm{loc}}(D), \tag{11.66}$$

where $\mu(z) = \mu_f(z)$. *Set*

$$q^T_{z_0}(r) \;=\; \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{|1 - e^{-2i\vartheta}\mu(z_0 + re^{i\vartheta})|^2}{1 - |\mu(z_0 + re^{i\vartheta})|^2}\; d\vartheta. \tag{11.67}$$

Then

$$\int\limits_{r_1}^{r_2} \frac{dr}{rq^T_{z_0}(r)} \;<\; \infty \tag{11.68}$$

for every $z_0 \in D$ *and* $0 < r_1 < r_2 < d_0$, *where* $d_0 = \mathrm{dist}\,(z_0, \partial D)$.

Corollary 11.6 follows from Corollaries 11.5 and 11.3 and from the definition of the tangential dilatation $K^T_\mu(z, z_0)$; see (11.3).

Corollary 11.7. *Let $f : D \to \mathbb{C}$ be a $W^{1,2}_{\mathrm{loc}}$ homeomorphism with $K_\mu(z) \in L^1_{\mathrm{loc}}$, where $\mu(z) = \mu_f(z)$. Then*

$$M(\Delta(fC_1, fC_2, fD)) \le \left[\int_{r_1}^{r_2} \frac{dr}{r \int_0^{2\pi} \frac{|1-e^{-2i\vartheta}\mu(z_0+re^{i\vartheta})|^2}{1-|\mu(z_0+re^{i\vartheta})|^2}\, d\vartheta} \right]^{-1} . \tag{11.69}$$

Indeed, by Corollary 11.3, f is a ring Q-homeomorphism at z_0 with $Q(z) = K^T_\mu(z, z_0)$. The tangential dilatation $K^T_\mu(z, z_0)$ is given by (11.3), and (11.69) thus follows from Lemma 11.5.

Remark 11.3. The inequality (11.69) was first derived by Reich and Walczak [253] for quasiconformal mappings and then by Lehto [189] for certain μ-homeomorphisms. Later it was applied by Brakalova and Jenkins [31] and Gutlyanskiǐ, Martio, Sugawa, and Vuorinen [98] to the study of degenerate Beltrami equations.

The following lemma shows that the estimate (11.60), which implies (11.69), cannot be improved in the class of all ring Q-homeomorphisms. Note that the additional condition (11.70), which appears in the following lemma, holds automatically for every ring Q-homeomorphism by Corollary 11.5.

Lemma 11.6. *Fix $0 < r_1 < r_2 < r_0$, let $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, and suppose that $Q : D(z_0, r_0) \to [0, \infty]$ is a measurable function such that*

$$c_0 \;=\; \int_{r_1}^{r_2} \frac{dr}{rq(r)} \;<\; \infty, \tag{11.70}$$

where $q(r)$ is the mean of $Q(z)$ over the circle $|z - z_0| = r$, and set

$$\eta_0(r) = \frac{1}{c_0 rq(r)}. \tag{11.71}$$

Then

$$\begin{aligned} \frac{2\pi}{c_0} \;&=\; \int_A Q(z) \cdot \eta_0^2(|z - z_0|)\, dxdy \\ &\le \int_A Q(z) \cdot \eta^2(|z - z_0|)\, dxdy \end{aligned} \tag{11.72}$$

for every function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\, dr \;=\; 1. \tag{11.73}$$

Proof. If $c_0 = 0$, then $q(r) = \infty$ for a.e. $r \in (r_1, r_2)$ and both sides in (11.72) are equal to ∞. Hence, we may assume below that $0 < c_0 < \infty$.

Now, by (11.70) and (11.73), $q(r) \neq 0$ and $\eta(r) \neq \infty$ a.e. in (r_1, r_2). Set $\alpha(r) = rq(r)\eta(r)$ and $w(r) = 1/rq(r)$. Then, by the standard conventions, $\eta(r) = \alpha(r)w(r)$ a.e. in (r_1, r_2) and

$$C := \int_A Q(z) \cdot \eta^2(|z - z_0|)\, dxdy = 2\pi \int_{r_1}^{r_2} \alpha^2(r) \cdot w(r)\, dr. \tag{11.74}$$

By Jensen's inequality with weights (see, e.g., Theorem 2.6.2 in [252]) applied to the convex function $\varphi(t) = t^2$ in the interval $\Omega = (r_1, r_2)$ with the probability measure

$$\nu(E) \;=\; \frac{1}{c_0} \int_E w(r)\, dr, \tag{11.75}$$

we obtain

$$\left(\fint \alpha^2(r) w(r)\, dr \right)^{1/2} \;\geq\; \fint \alpha(r) w(r)\, dr \;=\; \frac{1}{c_0}, \tag{11.76}$$

where we also used the fact that $\eta(r) = \alpha(r)w(r)$ satisfies (11.73). Thus,

$$C \;\geq\; \frac{2\pi}{c_0}, \tag{11.77}$$

and the proof is complete. $\Box$

Given a number $\Delta \in (0,1)$, a domain $D \subset \mathbb{C}$, a point $z_0 \in D$, a number $r_0 \leq \operatorname{dist}(z_0, \partial D)$, and a measurable function $Q : D(z_0, r_0) \to [0, \infty]$, let $\mathfrak{R}^{\mathfrak{d}}_{\mathfrak{Q}}$ denote the class of all ring Q-homeomorphisms $f : D \to \overline{\mathbb{C}}$ at z_0 such that

$$\delta(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta. \tag{11.78}$$

Next, we introduce the classes $\mathfrak{B}^{\mathfrak{d}}_{\mathfrak{Q}}$ and $\mathfrak{F}^{\mathfrak{d}}_{\mathfrak{Q}}$ of qc mappings. Let $\mathfrak{B}^{\mathfrak{d}}_{\mathfrak{Q}}$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (11.78) such that

$$K^T_\mu(z, z_0) = \frac{\left| 1 - \frac{\overline{z - z_0}}{z - z_0} \mu(z) \right|^2}{1 - |\mu(z)|^2} \leq Q(z) \text{ a.e. in } D(z_0, r_0), \tag{11.79}$$

where $\mu = \mu_f$. Similarly, let $\mathfrak{F}^{\mathfrak{d}}_{\mathfrak{Q}}$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (11.78) such that

$$K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \;\leq\; Q(z) \quad \text{a.e. in} \quad D(z_0, r_0). \tag{11.80}$$

Remark 11.4. By Corollary 11.3, the relations (11.42) and (11.35) give the inclusions

$$\mathfrak{F}_{\mathfrak{Q}}^{\mathfrak{d}} \subset \mathfrak{B}_{\mathfrak{Q}}^{\mathfrak{d}} \subset \mathfrak{R}_{\mathfrak{Q}}^{\mathfrak{d}} . \tag{11.81}$$

Combining Lemmas 11.4 and 11.3, we obtain the following distortion estimates in the class $\mathfrak{R}_{\mathfrak{Q}}^{\mathfrak{d}}$.

Corollary 11.8. *Let $f \in \mathfrak{R}_{\mathfrak{Q}}^{\mathfrak{d}}$, and let $\omega(\varepsilon)$ be as in Lemma 11.4. Then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{\omega(|\zeta - z_0|)}\right) \tag{11.82}$$

for all $\zeta \in D(z_0, \varepsilon_0)$.

Theorem 11.3. *Let $f \in \mathfrak{R}_{\mathfrak{Q}}^{\mathfrak{d}}$, and let $\psi : [0,\infty] \to [0,\infty]$ be a measurable function such that*

$$0 < \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0). \tag{11.83}$$

Suppose that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z) \cdot \psi^2(|z - z_0|)\; dxdy \leq C \cdot \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt \tag{11.84}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Then

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{C} \cdot \int\limits_{|\zeta-z_0|}^{\varepsilon_0} \psi(t)\, dt\right) \tag{11.85}$$

whenever $\zeta \in D(z_0, \varepsilon_0)$.

Choosing in Theorem 11.3 the function $\psi(t)$ as in (11.61), we obtain the following distortion theorem for ring Q-homeomorphisms.

Theorem 11.4. *Let D be a domain in $\mathbb{C}$, z_0 a point in D, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, $Q : D(z_0, r_0) \to [0,\infty]$ a measurable function, and $f \in \mathfrak{R}_{Q}^{\Delta}$. Then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot \exp\left(-\int\limits_{|\zeta-z_0|}^{r_0} \frac{dr}{rq(r)}\right) \tag{11.86}$$

for all $\zeta \in D(z_0, r_0)$, where $q(r)$ is the mean of $Q(z)$ over the circle $|z - z_0| = r$.

In the following theorem the estimate of distortion is expressed in terms of maximal dispersion; see (11.10).

Theorem 11.5. *Let $f \in \mathfrak{R}_Q^\Delta$ for $\Delta > 0$ and Q with finite mean oscillation at $z_0 \in D$. If Q is integrable over a disk $D(z_0, \varepsilon_0) \subset D$, then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot \left(\log \frac{2\varepsilon_0}{|\zeta - z_0|}\right)^{-\beta_0} \tag{11.87}$$

for every point $\zeta \in D(z_0, \varepsilon_0/2)$, where

$$\beta_0 = \frac{1}{2}\,[q_0 + 6d_0]^{-1}\,, \tag{11.88}$$

q_0 is the mean, and d_0 is the maximal dispersion of $Q(z)$ in $D(z_0, \varepsilon_0)$.

Proof. The mean and the dispersion of a function over disks are invariant under linear transformations $w = (z - z_0)/2\varepsilon_0$. Hence, (11.87) follows by Theorem 11.3 and Lemma 11.1. □

Another consequence of Lemma 11.4 (see Corollary 11.8) can be formulated in terms of the **logarithmic mean** of Q over an annulus $A(\varepsilon) = A(\varepsilon, \varepsilon_0, z_0) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$, which is defined by

$$M_{\log}^Q(\varepsilon) = \fint_{\varepsilon}^{\varepsilon_0} q(t)\, d\log t \ := \frac{1}{\log \varepsilon_0/\varepsilon} \int_{\varepsilon}^{\varepsilon_0} q(t)\, \frac{dt}{t}, \tag{11.89}$$

where $q(t)$ denotes the mean value of Q over the circle $|z - z_0| = t$. Choosing in expression (11.58) $\psi_\varepsilon(t) = 1/t$ for $0 < \varepsilon < \varepsilon_0$, and setting $\varepsilon = |\zeta - z_0|$, we have the following statement.

Corollary 11.9. *Let $Q : D(z_0, r_0) \to [0, \infty]$, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, be a measurable function, $\varepsilon_0 \in (0, r_0)$, and $\Delta > 0$. If $f \in \mathfrak{R}_Q^\Delta$, then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \left(\frac{|\zeta - z_0|}{\varepsilon_0}\right)^{1/M_{\log}^Q(|\zeta - z_0|)} \tag{11.90}$$

for all $\zeta \in D(z_0, \varepsilon_0)$.

Note that for $Q \equiv K \in [1, \infty)$, (11.90) reduces to the well-known distortion estimate for qc mappings

$$s(f(\zeta), f(z_0)) \leq C \cdot \left(\frac{|\zeta - z_0|}{\varepsilon_0}\right)^{1/K}. \tag{11.91}$$

The corollaries and theorems presented here show that Lemmas 11.3 and 11.4 are useful tools in deriving various distortion estimates for ring Q-homeomorphisms. These, in turn, are instrumental in the study of properties of ring Q-homeomorphisms and, in particular, of ring solutions of the Beltrami equation (11.1), where

$Q(z)$ can be either the maximal dilatation $K_\mu(z)$ or the tangential dilatation $K_\mu^T(z,z_0)$, which are defined in (11.2) and (11.3), respectively.

11.5 General Existence Lemma and Its Corollaries

The following lemma and corollary serve as the main tool in obtaining many criteria for the existence of ring solutions for the Beltrami equation. Theorem 11.6 establishes the existence of a ring solution when, at every point $z_0 \in D$, the tangential dilatation $K_\mu^T(z,z_0)$ is assumed to be dominated by a function of finite mean oscillation at z_0 in the variable z. Theorem 11.7 formulates the condition for existence in terms of the mean of the tangential dilatation over infinitesimal disks. Since the maximal dilatation dominates the tangential dilatation, these two results obviously imply similar existence theorems in terms of conditions on the maximal dilatation, Theorem 11.8, and Corollary 11.11. The criterion for the existence in Theorem 11.9 is formulated in terms of the logarithmic mean. The section is completed by a generalization of the Lehto theorem, Theorem 11.10, and its corollaries.

Lemma 11.7. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. Suppose that for every $z_0 \in D$, there exist $\varepsilon_0 \le \operatorname{dist}(z_0, \partial D)$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, such that*

$$0 \;<\; I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt \;<\; \infty, \tag{11.92}$$

and

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\, dxdy \;=\; o(I^2_{z_0}(\varepsilon)) \tag{11.93}$$

as $\varepsilon \to 0$. Then the Beltrami equation (11.1) has a ring solution f_μ.

Proof. Fix z_1 and z_2 in D. For $n \in \mathbb{N}$, define $\mu_n : D \to \mathbb{C}$ as $\mu_n(z) = \mu(z)$ if $|\mu(z)| \le 1 - 1/n$ and 0 otherwise. Let $f_n : D \to \mathbb{C}$ be a homeomorphic ACL solution of (11.1), with μ_n instead of μ, that fixes z_1 and z_2. Such an f_n exists by the well-known existence theorem in the nondegenerate case (see, e.g., [1,26,190]). By Theorem 11.1 and Corollary 11.8, in view of (11.93), the sequence f_n is equicontinuous. Hence, by the Arzela–Ascoli theorem (see, e.g., [50], p. 267, and [51], p. 382), it has a subsequence, denoted again by f_n, that converges locally uniformly to some nonconstant mapping f in D. Then, by Theorem 3.1 and Corollary 5.12 on a converquece in [274], f is $K(z)$-qc with $K(z) = K_\mu(z)$ and f satisfies (11.1) a.e. Thus, f is a homeomorphic ACL solution of (11.1). Moreover, by Theorems 11.1 and 11.2, f is a ring Q-homeomorphism [see (11.28)] with $Q(z) = K_\mu^T(z,z_0)$ at every point $z_0 \in D$.

Since the locally uniform convergence $f_n \to f$ of the sequence f_n is equivalent to the continuous convergence, i.e., $f_n(z_n) \to f(z_0)$ if $z_n \to z_0$ (see [Du], p. 268) and since f is injective, it follows that $g_n = f_n^{-1} \to f^{-1} = g$ continuously and hence locally uniformly. By a change of variables, which is permitted because f_n and g_n are in $W^{1,2}_{\mathrm{loc}}$, we obtain, that for large n,

$$\int\limits_{B} |\partial g_n|^2 dudv = \int\limits_{g_n(B)} \frac{dxdy}{1-|\mu_n(z)|^2} \leq \int\limits_{B^*} Q(z)dxdy < \infty, \tag{11.94}$$

where B^* and B are relatively compact domains in D and in $f(D)$, respectively, with $g(\bar{B}) \subset B^*$. Now (11.94) implies that the sequence g_n is bounded in $\mathrm{W}^{1,2}(B)$, and hence $f^{-1} \in \mathrm{W}^{1,2}_{\mathrm{loc}}(f(D))$; see, e.g., [256], p. 319. $\Box$

Remark 11.5. If f_μ is as in Lemma 11.7, then f_μ^{-1} is locally absolutely continuous and preserves nulls sets, and f_μ is regular a.e., i.e., differentiable with $J_{f_\mu}(z) > 0$ a.e. Indeed, the assertion about f_μ^{-1} follows from the fact that $f_\mu^{-1} \in \mathrm{W}^{1,2}_{\mathrm{loc}}$; see [190], pp. 131 and 150. An ACL mapping f_μ has a.e. partial derivatives and hence by [80] has a total differential a.e. Let E denote the set of points of D where f_μ is differentiable and $J_{f_\mu}(z) = 0$, and suppose that $|E| > 0$. Then $|f_\mu(E)| > 0$, since $E = f_\mu^{-1}(f_\mu(E))$ and f_μ^{-1} preserves null sets. Clearly, f_μ^{-1} is not differentiable at any point of $f_\mu(E)$, contradicting the fact that f_μ^{-1} is differentiable a.e.

Corollary 11.10. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}$, and let $\psi : (0,\infty) \to (0,\infty)$ be a measurable function such that for all $0 < t_1 < t_2 < \infty$,*

$$0 < \int\limits_{t_1}^{t_2} \psi(t)\, dt < \infty, \qquad \int\limits_{0}^{t_2} \psi(t)\, dt = \infty. \tag{11.95}$$

Suppose that for every $z_0 \in D$, there is $\varepsilon_0 < \mathrm{dist}\,(z_0, \partial D)$ such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z) \cdot \psi^2(|z-z_0|)\, dxdy \leq O\left(\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt\right) \tag{11.96}$$

as $\varepsilon \to 0$. Then (11.1) has a ring solution.

If we choose

$$\psi_{z_0,\varepsilon}(t) = \frac{1}{t \log \frac{1}{t}}, \tag{11.97}$$

then Lemma 11.7 yields the following theorem; see also Lemma 11.1.

Theorem 11.6. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} such that*

$$K^T_\mu(z, z_0) \leq Q_{z_0}(z) \quad a.e. \tag{11.98}$$

for some function $Q_{z_0}(z)$ of finite mean oscillation at the point z_0 in the variable z. Then the Beltrami equation (11.1) has a ring solution.

The following theorem is a consequence of Theorem 11.6 and Corollary 11.1.

Theorem 11.7. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}$. Suppose that at every $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{D(z_0,\varepsilon)} \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}\, dxdy \ < \ \infty\,. \tag{11.99}$$

Then the Beltrami equation (11.1) has a ring solution f_μ.

The following theorem is an important particular case of Theorem 11.6.

Theorem 11.8. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) \ = \ \frac{1+|\mu(z)|}{1-|\mu(z)|} \ \leq \ Q(z) \ \in \ \mathrm{FMO}. \tag{11.100}$$

Then the Beltrami equation (11.1) has a ring solution.

Since every ring solution is an ACL homeomorphic solution and every BMO function is in FMO, the theorem generalizes and strengthens earlier results in [271, 274] about the existence of ACL homeomorphic solutions of the Beltrami equation when the conditions involve majorants of bounded mean oscillation.

Corollary 11.11. *If*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{D(z_0,\varepsilon)} \frac{1+|\mu(z)|}{1-|\mu(z)|}\, dxdy \ < \ \infty \tag{11.101}$$

at every $z_0 \in D$, then (11.1) has a ring solution.

Applying Lemma 11.7 with $\psi(t) = 1/t$, we also have the following statement, which is formulated in terms of the logarithmic mean [see (11.89)] of $K^T_\mu(z,z_0)$ over the annuli $A(\varepsilon) = \{z \in \mathbb{C} : \varepsilon < |z-z_0| < \varepsilon_0\}$ for a fixed $\varepsilon_0 = \delta(z_0) \leq \operatorname{dist}(z_0, \partial D)$.

Theorem 11.9. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}$. If at every point $z_0 \in D$, the logarithmic mean of K^T_μ over $A(\varepsilon)$ does not converge to ∞ as $\varepsilon \to 0$, i.e.,*

$$\liminf_{\varepsilon\to 0} M_{log}^{K^T_\mu}(\varepsilon) \ < \ \infty, \tag{11.102}$$

then the Beltrami equation (11.1) has a ring solution.

Corollary 11.12. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}$. Denote by $q^T_{z_0}(t)$ the mean of $K^T_\mu(z,z_0)$ over the circle $C = \{z \in \mathbb{C} : |z-z_0| = t\}$. If*

$$\int_0^{\delta(z_0)} q_{z_0}^T(t)\,\frac{dt}{t} \;<\; \infty \tag{11.103}$$

at every point $z_0 \in D$ for some $\delta(z_0) > 0$, then (11.1) has a ring solution.

Lehto considers in [189] degenerate Beltrami equations in the special case where the **singular set** S_μ

$$S_\mu = \{z \in \mathbb{C} : \lim_{\varepsilon \to 0} \|K_\mu\|_{L^\infty(D(z,\varepsilon))} \;=\; \infty\} \tag{11.104}$$

of the complex coefficient μ in (11.1) is of measure zero, and he shows that if, for every $z_0 \in \mathbb{C}$ and every r_1 and $r_2 \in (0,\infty)$, the integral

$$\int_{r_1}^{r_2} \frac{dr}{r(1+q_{z_0}^T(r))}\,, \qquad r_2 > r_1, \tag{11.105}$$

is positive and tends to ∞ as either $r_1 \to 0$ or $r_2 \to \infty$, where

$$q_{z_0}^T(r) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{|1-e^{-2i\vartheta}\mu(z_0+re^{i\vartheta})|^2}{1-|\mu(z_0+re^{i\vartheta})|^2}\;d\vartheta, \tag{11.106}$$

then there exists a homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that is ACL in $\mathbb{C} \setminus S_\mu$ and satisfies (11.1) a.e. Note that the integrand in (11.67) is the tangential dilatation $K_\mu^T(z,z_0)$; see (11.3).

We now present an extension of Lehto's existence theorem that enables us to derive many other existence theorems, as shown in [277]. In this extension we prove the existence of a ring solution in a domain $D \subset \mathbb{C}$, which, by definition, is ACL in D and not only in $D \setminus S_\mu$. Note that, in the following theorem, the situation $S_\mu = D$ is possible. Note also that condition (11.48) in the following theorem is weaker than the condition in Lehto's existence theorem.

Theorem 11.10. *Let D be a domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}$. Suppose that at every point $z_0 \in D$*

$$\int_0^{\delta(z_0)} \frac{dr}{r q_{z_0}^T(r)} \;=\; \infty, \tag{11.107}$$

where $\delta(z_0) < \operatorname{dist}(z_0, \partial D)$ and $q_{z_0}^T(r)$ is the mean of $K_\mu^T(z,z_0)$ over $|z-z_0| = r$. Then the Beltrami equation (11.1) has a ring solution.

Proof. Theorem 11.10 follows from Lemma 11.7 by specially choosing the functional parameter

$$\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := \begin{cases} 1/[tq_{z_0}^T(t)], & t \in (0,\varepsilon_0), \\ 0\,, & \text{otherwise}, \end{cases} \tag{11.108}$$

where $\varepsilon_0 = \delta(z_0)$. □

Corollary 11.13. *If $K_\mu \in L^1_{\mathrm{loc}}$ and at every point $z_0 \in D$*

$$q_{z_0}^T(r) = O\left(\log \frac{1}{r}\right) \quad as \quad r \to 0, \tag{11.109}$$

then (11.1) has a ring solution.

Since $K_\mu^T(z,z_0) \le K_\mu(z)$, we obtain as a consequence of Theorem 11.10 the following result which is due to Miklyukov and Suvorov [220] for the case $K_\mu \in L^p_{\mathrm{loc}}$, $p > 1$.

Corollary 11.14. *If $K_\mu \in L^p_{\mathrm{loc}}$ for $p \ge 1$ and (11.107) holds for $K_\mu(z)$ instead of $K_\mu^T(z,z_0)$ for every point $z_0 \in D$, then (11.1) has a $W^{1,s}_{\mathrm{loc}}$ homeomorphic solution with $s = 2p/(p+1)$.*

11.6 Representation, Factorization and Uniqueness Theorems

In Section 11.5 we established a series of theorems on the existence of ring solutions f_μ for the Beltrami equation (11.1) for a variety of different conditions on the complex coefficient μ. We now show that, in each of these cases, f_μ generates all $W^{1,2}_{\mathrm{loc}}$ solutions by composition with analytic functions.

Lemma 11.8. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. Suppose that for every $z_0 \in D$ there exist $\varepsilon_0 = \delta(z_0) \le \operatorname{dist}(z_0, \partial D)$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, such that*

$$0 < I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty\,, \qquad \varepsilon \in (0,\varepsilon_0), \tag{11.110}$$

and

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0) \cdot \psi_{z_0,\varepsilon}^2(|z-z_0|)\, dxdy = o(I_{z_0}^2(\varepsilon)) \tag{11.111}$$

as $\varepsilon \to 0$ and let f_μ be a ring solution of (11.1). Then every $W^{1,2}_{\mathrm{loc}}$ solution g of (11.1) has the representation

$$g = h \circ f_\mu \tag{11.112}$$

for some holomorphic function h in $f_\mu(D)$.

Proof. Let $\varphi = f_\mu^{-1}$ and $h = g \circ \varphi$. Since $g \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ and $\varphi \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ it follows that $h \in \mathrm{W}^{1,1}_{\mathrm{loc}}(f(D))$; see, e.g., [190], p. 151. Thus, by Weyl's lemma; see, e.g., [1], p. 33, it suffices to show that $\overline{\partial} h = 0$ a.e. in $f_\mu(D)$. Let E denote the set of points z in D where either f_μ or g does not satisfy (11.1) or $J_{f_\mu} = 0$. A direct computation (cf. [1], p. 9) shows that $\overline{\partial} h = 0$ in $f_\mu(D) \setminus f_\mu(E)$. Moreover, $\varphi \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ admits the change of variables (see, e.g., [190], pp. 121, 128-130 and 150)

$$\int_{f_\mu(E)}\!\!\int |\partial\varphi|^2 \, dudv \;=\; \int_{f_\mu(E)}\!\!\int J_\varphi(w)\,\frac{dudv}{1-|\mu(\varphi(w))|^2} \;=\; \iint_E \frac{dxdy}{1-|\mu(z)|^2} \;=\; 0,$$

which implies that $|\partial\varphi| = 0$ a.e. on $f_\mu(E)$, and since a.e. $|\overline{\partial}\varphi| \le |\partial\varphi|$ and

$$\overline{\partial} h \;=\; \overline{\partial}\varphi \cdot \partial g \circ \varphi + \overline{\partial\varphi} \cdot \overline{\partial} g \circ \varphi,$$

it follows that $|\overline{\partial} h| = 0$ a.e. on $f_\mu(E)$, and thus $\overline{\partial} h = 0$ a.e. in $f_\mu(D)$. Consequently, h is holomorphic in $f_\mu(D)$ and (11.112) holds. □

Iwaniec and Sverak [137] showed that if $K_\mu \in L^1_{\mathrm{loc}}$, then every $\mathrm{W}^{1,2}_{\mathrm{loc}}$ solution g of (11.1) has the representation $g = h \circ f$ for some holomorphic function h and some homeomorphism f. The conditions in Lemma 11.8 are more restrictive, but the representation (11.112) is more specific and the proof is simpler.

Remark 11.6. Since all theorems on the existence of a ring solution f_μ in Section 11.5 are based on Lemma 11.7, where the conditions are as in Lemma 11.8, every $\mathrm{W}^{1,2}_{\mathrm{loc}}$ solution g of the Beltrami equation (11.1) in each of these theorems has the representation (11.112).

It is not clear, even if μ satisfies the conditions of Lemma 11.8, whether an ACL homeomorphic solution of (11.1) is unique up to a composition with a conformal mapping, namely whether, for any two ACL homeomorphic solutions f_1 and f_2 of (11.1), $f_2 \circ f_1^{-1}$ is conformal. By (11.112) in Lemma 11.8, the answer is affirmative if f_1 and f_2 are in $W^{1,2}_{\mathrm{loc}}$ and μ is as in Lemma 11.8; see Corollary 11.15. Another type of condition for the uniqueness of a homeomorphic ACL solution can be obtained by imposing some conditions on the "size" of the singular set of μ. This will be done in Lemma 11.9 and Theorem 11.11.

Corollary 11.15. *Suppose that μ satisfies the conditions of one of the existence theorems in Section 11.5. If f_1 and f_2 are homeomorphic $W^{1,2}_{\mathrm{loc}}$ solutions of (11.1), then $f_2 \circ f_1^{-1}$ is conformal.*

Iwaniec and Martin have constructed ACL solutions for the Beltrami equation that are not in $\mathrm{W}^{1,2}_{\mathrm{loc}}$ and not open and discrete and, thus, are not generated by a homeomorphic solution in the sense of (11.112); see, e.g., [134]. However, for discrete open solutions, it is easy to obtain the following proposition by Stoilow's theorem.

Proposition 11.2. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \in L^1_{\mathrm{loc}} . \tag{11.113}$$

Then every (continuous) discrete and open ACL *solution g of the Beltrami equation (11.1) has the representation $g = h \circ f$, where f is a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of (11.1) and h is a holomorphic function in $f(D)$.*

Remark 11.7. As a consequence of the proposition, we obtain that if $K_\mu \in L^1_{\mathrm{loc}}$, then the Beltrami equation (11.1) either has a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution or has no continuous, discrete, and open ACL solution. Note that for every $p \in [1,\infty)$, there are examples of measurable functions $\mu : \mathbb{C} \to \mathbb{C}$ such that $|\mu(z)| < 1$ a.e. and $K_\mu(z) \in L^p_{\mathrm{loc}}$ and for which the Beltrami equation (11.1) has no homeomorphic ACL solution; see, e.g., Proposition 6.3 in [274].

Let (X,d) be a metric space and let $H = \{h_x(r)\}_{x\in X}$ be a family of functions $h_x : (0,\rho_x) \to (0,\infty)$, $\rho_x > 0$, such that $h_x(r) \to 0$ as $r \to 0$. Let

$$L^\rho_H(X) = \inf \Sigma\, h_{x_k}(r_k), \tag{11.114}$$

where the infimum is taken over all finite collections of $x_k \in X$ and $r_k \in (0,\rho)$ such that the balls

$$B(x_k,r_k) = \{x \in X : d(x,x_k) < r_k\} \tag{11.115}$$

cover X. The limit

$$L_H(X) := \lim_{\rho\to 0} L^\rho_H(X) \tag{11.116}$$

exists. We call $L_H(X)$ the H**-length** of X. In the particular case where $h_x(r) = r$ for all $x \in X$ and $r > 0$, the H-length is the usual (Hausdorff) length of X.

Obviously, singular set S_μ of μ is closed relative to the domain D.

Lemma 11.9. *Let μ be as in Lemma 11.8 and let f_μ be a ring solution of (11.1). Suppose that the singular set S_μ is of H-length zero for $H = \{h_{z_0}(r)\}_{z_0\in S_\mu}$ with*

$$h_{z_0}(r) = \exp\left(-\frac{2\pi}{\omega_{z_0}(r)}\right), \qquad z_0 \in S_\mu,\ r \in (0,\delta(z_0)), \tag{11.117}$$

and

$$\omega_{z_0}(\varepsilon) = \frac{1}{I^2_{z_0}(\varepsilon)} \int\limits_{A(\varepsilon)} K^T_\mu(z,z_0)\cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\ dxdy. \tag{11.118}$$

Then every homeomorphic ACL *solution f of (11.1) has the representation $f = h \circ f_\mu$ for some conformal mapping h in $f_\mu(D)$.*

Proof. If $L_H(S_\mu) = 0$, then $S'_\mu = f_\mu(S_\mu)$ is of length zero by Lemma 11.4. Consequently, S'_μ does not locally disconnect $f(D)$ (see, e.g., [317]) and hence $G = D \setminus S_\mu$

is a domain. The homeomorphisms f and f_μ are locally quasiconformal in the domain G and hence $h = f \circ f_\mu^{-1}$ is conformal in the domain $f_\mu(D) \setminus S'_\mu$. Since S'_μ is of length zero, it is removable for h, i.e., h can be extended to a conformal mapping in $f_\mu(D)$ by the Painleve theorem; see, e.g., [24]. □

Theorem 11.11. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\rm loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} and a measurable function $Q_{z_0}(z) : U_{z_0} \to [0,\infty]$ such that*

$$K_\mu^T(z,z_0) \ \le \ Q_{z_0}(z) \quad \textit{a.e. in } U_{z_0} \tag{11.119}$$

and that, for some $\delta(z_0) > 0$,

$$\int\limits_0^{\delta(z_0)} \frac{dt}{tq_{z_0}(t)} \ = \ \infty, \tag{11.120}$$

where $q_{z_0}(t)$ is the mean of $Q_{z_0}(z)$ over the circle $|z - z_0| = t$. Let f_μ be a ring solution of (11.1).

If the singular set S_μ has H-length zero for $H = \{h_{z_0}(r)\}_{z_0 \in S_\mu}$, where

$$h_{z_0}(r) = \exp\left(- \int\limits_r^{\delta(z_0)} \frac{dt}{tq_{z_0}(t)} \right), \quad z_0 \in S_\mu, \ \ r \in (0, \delta(z_0)), \tag{11.121}$$

then every homeomorphic ACL *solution f of (11.1) has the representation $f = h \circ f_\mu$ for some conformal mapping h in $f_\mu(D)$.*

Proof. Theorem 11.11 follows from Lemma 11.9 with

$$\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := \begin{cases} 1/[tq_{z_0}(t)] \ , & t \in (0,\varepsilon_0) \ , \\ 0 \ , & \text{otherwise,} \end{cases} \tag{11.122}$$

where $\varepsilon_0 = \delta(z_0)$ because

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} Q(z) \cdot \psi^2_{z_0}(|z - z_0|) \ dxdy \ = \ 2\pi \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0}(t) \ dt. \tag{11.123}$$

□

Corollary 11.16. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\rm loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0}, where (11.119) holds with a function $Q_{z_0}(z)$ of finite mean oscillation at z_0 in the variable z. Suppose also that the singular set of S_μ is of H-length zero for $H = \{h_{z_0}(r)\}_{z_0 \in S_\mu}$,*

$$h_{z_0}(r) = \left(\log \frac{\delta(z_0)}{r} \right)^{-\beta(z_0)}, \quad z_0 \in S_\mu, \quad r \in (0, \delta(z_0)), \tag{11.124}$$

where $\delta(z_0) < \operatorname{dist}(z_0, \partial D)$*, and* $2\beta(z_0) = (q(z_0) + 6d(z_0))^{-1}$*,* $q(z_0)$ *is the mean value of* $Q_{z_0}(z)$ *over* $D(z_0, \delta(z_0)/2)$*, and* $d(z_0)$ *is the maximal dispersion of* $Q_{z_0}(z)$ *in* $D(z_0, \delta(z_0)/2)$*. Let* f_μ *be a ring solution of (11.1).*

Then every homeomorphic ACL *solution* f *of (11.1) has the representation* $f = h \circ f_\mu$ *for some conformal mapping* h *in* $f_\mu(D)$*.*

Corollary 11.16 follows immediately from Lemmas 11.9 and 11.1.

Remark 11.8. In view of Remark 11.1, if the condition

$$Q^*(z_0) := \overline{\lim_{\varepsilon \to 0}} \fint_{D(z_0,\varepsilon)} Q_{z_0}(z)\, dxdy \;<\; \infty \tag{11.125}$$

holds for all $z_0 \in D$, then one may take $\beta(z) = \gamma / Q^*(z)$ in (11.124) for any $\gamma < 1/26$.

Lemma 11.9 makes it possible to formulate the corresponding uniqueness theorem in the spirit of Theorem 11.11 for every existence theorem in Section 11.5.

11.7 Examples

By (11.42), we have

$$K^r(z, z_0, f) \;\le\; K^R(z, z_0, f). \tag{11.126}$$

By (11.34) and (11.35) we have

$$K^r_\mu(z, z_0) \;\le\; K^T_\mu(z, z_0). \tag{11.127}$$

One may ask whether $K^R(z, z_0, f)$ in Theorem 11.1 can be replaced by $K^r(z, z_0, f)$ and whether $K^T_\mu(z, z_0)$ can be replaced by $K^r_\mu(z, z_0)$ in the criteria for the existence problems for the Beltrami equation (11.1).

Every qc mapping $f : D \to \mathbb{C}$ is a ring Q-homeomorphism at each point $z_0 \in D$ with $Q(z) = K^R(z, z_0, f)$, $z \in D(z_0, \operatorname{dist}(z_0, \partial D))$. The following example shows that there are qc mappings f that are not ring Q-homeomorphisms with $Q(z) = K^r(z, z_0, f)$.

Example 1. Consider the quasiconformal automorphism $f : \mathbb{D} \to \mathbb{D}$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which, in the polar coordinates, has the form

$$f(re^{i\vartheta}) = \exp\left(i\vartheta - \int\limits_r^1 \frac{1+ik}{1-ik} \frac{dt}{t} \right), \quad z = re^{i\vartheta},\ k \in (0,1); \tag{11.128}$$

see, e.g., [99]. Then by applying the formula

$$\mu(re^{i\vartheta}) \;=\; \frac{f_{\bar z}}{f_z} \;=\; e^{2i\vartheta}\, \frac{rf_r + if_\vartheta}{rf_r - if_\vartheta}, \tag{11.129}$$

one obtains by straightforward calculations

$$\mu(re^{i\vartheta}) \;=\; ike^{2i\vartheta}. \tag{11.130}$$

By (11.128) we have

$$|f(re^{i\vartheta})| \;=\; \exp\left(-\mathrm{Re}\int_r^1 \frac{1+ik}{1-ik}\frac{dt}{t}\right) \;=\; r^{\frac{1-k^2}{1+k^2}}. \tag{11.131}$$

On the other hand, the big radial dilatation at z with respect to the center $z_0 = 0$ is the constant

$$K_0 := \; K_\mu^T(z,z_0,f) \;=\; \frac{|1+ik|^2}{1-k^2} \;=\; \frac{1+k^2}{1-k^2}; \tag{11.132}$$

hence, in view of (11.131),

$$|f(re^{i\vartheta})| \;=\; r^{\frac{1}{K_0}}. \tag{11.133}$$

The example shows that, in general, quasiconformal mappings are not ring Q-homeomorphisms with the radial dilatation as Q, i.e.,

$$Q(z) := \; K^r(z,z_0,f) \;=\; \frac{1-|\mu(z)|^2}{\left|1+\frac{\bar z}{z}\mu(z)\right|^2} \tag{11.134}$$

at $z_0 = 0$.

Indeed, let us assume that the given f is a ring Q-homeomorphism with $Q = K^r$. Then, by Theorem 11.4, we must have the estimate

$$\frac{|f(z)|}{(1+|f(z)|^2)^{1/2}} \;\le\; 32\cdot\exp\left(-\int_{|z|}^1 \frac{dr}{rq(r)}\right), \tag{11.135}$$

and hence

$$|f(z)| \;\le\; 64\cdot\exp\left(-\int_{|z|}^1 \frac{dr}{rq(r)}\right), \tag{11.136}$$

where $q(r)$ is the mean of $Q(z)$ over the circle $|z| = r$.

However, in the case of (11.130) and (11.134), we obtain

$$Q(z) \;=\; \frac{1-k^2}{1+k^2} \;=\; \frac{1}{K_0} \tag{11.137}$$

and, thus, (11.136) would imply

$$|f(re^{i\vartheta})| \;\le\; 64\cdot r^{K_0}, \tag{11.138}$$

contradicting (11.133) because $K_0 > 1$.

The next example shows that the existence criteria for the Beltrami equation cannot be formulated in terms of majorants for the radial dilatation $K^r_\mu(z,z_0)$ instead of the tangential dilatation $K^T_\mu(z,z_0)$.

Example 2. Consider the complex coefficient μ given in polar coordinates in unit disk $\mathbb{D}$:

$$\mu(re^{i\vartheta}) \;=\; e^{2i\vartheta}\,\frac{M(r)-1}{M(r)+1}, \tag{11.139}$$

where

$$M(r) \;=\; r^{1/2}+i. \tag{11.140}$$

By straightforward computations with (11.129), it easy to verify that the smooth mapping

$$f(re^{i\vartheta}) = \exp\left(i\vartheta - \int_r^1 M(t)\frac{dt}{t}\right) = e^{-2(1-r^{1/2})+i(\vartheta+\log r)} \tag{11.141}$$

satisfies the Beltrami equation in $\mathbb{D}\setminus\{0\}$ with the given μ; see, e.g., Proposition 6.4 in [274]. Note that the homeomorphism f maps the punctured unit disk onto the annulus $A=\{z\in\mathbb{C}: e^{-2}<|z|<1\}$ and, thus, f cannot be extended by continuity to the origin.

Let us assume that there is a homeomorphic ACL solution g of the Beltrami equation with the given μ in the whole disk $\mathbb{D}$. Then, by the Riemann theorem, we may in addition assume that $g(\mathbb{D})=\mathbb{D}$ and $g(0)=0$. However, both homeomorphisms f and g are locally quasiconformal in $\mathbb{D}\setminus\{0\}$ and hence by the uniqueness theorem for quasiconformal mappings, $f=h\circ g$, where h is a conformal mapping of $\mathbb{D}\setminus\{0\}$. As we know, isolated singularities are removable for conformal mappings, and hence f can be extended by continuity to the origin. The contradiction disproves the above assumption.

On the other hand, we have

$$K^r_\mu(z,0) \;=\; \frac{1-|\mu(z)|^2}{\left|1+\frac{\bar z}{z}\mu(z)\right|^2} \;=\; \frac{r^{1/2}}{1+r} \tag{11.142}$$

and, consequently,

$$K^r_\mu(z,0) \;\le\; Q(z) \;\equiv\; \frac{1}{2}, \qquad |z| \;<\; 1. \tag{11.143}$$

Simultaneously, $K^r_\mu(z,z_0)\le K_\mu(z)$ for all z and z_0 in D, and

$$K_\mu(z) \;=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \;=\; 2r^{-1/2} \tag{11.144}$$

is integrable in the unit disk and locally bounded in the punctured unit disk.

Thus, even under a constant majorant for the radial dilatation $K^r_\mu(z,0)$, it is impossible to guarantee the existence of a homeomorphic ACL solution for the Beltrami equation (11.1). Thus, the tangent dilatation is more useful in this respect.

Chapter 12
Homeomorphisms with Finite Mean Dilatations

In this chapter we describe the topological mappings with finite integral characteristics following in the main the paper [88], see also [86, 87, 89, 170]. We extend here the method of extremal lengths (moduli) to this great class of mappings and establish various differential and geometric properties of these mappings. Classes of mappings with integral constraints for dilatations are more preferable than classes with measure constraints because these latter are not closed; see, e.g., [245].

The study of the classes of the so-called mappings quasiconformal in the mean was started by Ahlfors and has a rich history, see e.g. [4, 22, 25, 90, 169, 170, 172–174, 177–184, 239–241, 262–264, 301, 302, 313, 343, 344]. These classes are closely connected to the mappings with the bounded Dirichlet integral; see, e.g., [191, 303, 304]. The chapter is devoted to aspects of the theory of mappings with finite integral dilatations related to the moduli method.

12.1 Introduction

In geometric function theory, the quasiconformal homeomorphisms form a natural intermediate class of mappings between the classes of bi-Lipschitz mappings and general homeomorphisms. Under K-quasiconformal mapping, the n-module of any path family can change by a factor of at most K. All properties of quasiconformal mappings can be obtained from this inequality.

In this chapter we consider the homeomorphisms whose dilatations are bounded in a certain integral sense. The resulting notion generalizes quasiconformal mappings, mappings quasiconformal in the mean, etc. The main approach for investigation relies on the method of p-moduli of path families and surface families and involves more general inequalities than quasi-invariance.

Let $A:\ \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection. The numbers

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_12,

$$K_I(A) = \frac{|\det A|}{l^n(A)}, \quad K_O(A) = \frac{L^n(A)}{|\det A|}, \quad H(A) = \frac{L(A)}{l(A)}$$

are called the **inner**, **outer**, and **linear** dilatations of A, respectively. Here

$$l(A) = \min_{|h|=1} |Ah|, \qquad L(A) = \max_{|h|=1} |Ah|$$

and $\det A$ is the determinant of A; see e.g., [316].

Obviously, all three dilatations are not less than 1. They have the following geometric interpretation. The image of the unit ball B^n under A is an ellipsoid $E(A)$. Let $B_I(A)$ and $B_O(A)$ be the inscribed and circumscribed balls of $E(A)$, respectively. Then

$$K_I(A) = \frac{mE(A)}{mB_I(A)}, \qquad K_O(A) = \frac{mB_O(A)}{mE(A)},$$

and $H(A)$ is the ratio of the greatest and the smallest semiaxes of $E(A)$. Here $mD = m_nD$ denotes the n-dimensional Lebesgue measure of a set D.

Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ be the semiaxes of $E(A)$. Then

$$L(A) = \lambda_n, \quad l(A) = \lambda_1, \qquad |\det A| = \lambda_1 \cdot \ldots \cdot \lambda_n,$$

and we can also write

$$K_I(A) = \frac{\lambda_2 \cdot \ldots \cdot \lambda_n}{\lambda_1^{n-1}}, \qquad K_O(A) = \frac{\lambda_n^{n-1}}{\lambda_1 \cdot \ldots \cdot \lambda_{n-1}}, \qquad H(A) = \frac{\lambda_n}{\lambda_1}.$$

If $n=2$, then $K_I(A) = K_O(A) = H(A)$. In the general case, we have the relations

$$\begin{aligned} H(A) &\le \min(K_I(A), K_O(A)) \le H^{n/2}(A) \\ &\le \max(K_I(A), K_O(A)) \le H^{n-1}(A). \end{aligned} \tag{12.1}$$

Let G and G^* be two bounded domains in $\mathbb{R}^n$, $n \ge 2$, and let a mapping $f:\ G \to G^*$ be differentiable at a point $x \in G$. This means that there exists a linear mapping $f'(x):\ \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x+h) = f(x) + f'(x)h + \omega(x,h)|h|,$$

where $\omega(x,h) \to 0$ as $h \to 0$.

We denote

$$K_I(x,f) = K_I(f'(x)), \qquad K_O(x,f) = K_O(f'(x)),$$

and

$$L(x,f) = L(f'(x)), \qquad l(x,f) = l(f'(x)), \qquad J(x,f) = J(f'(x)).$$

Proposition 12.1. *Let $f:\ G \to G^*$ be a K-quasiconformal homeomorphism. Then*

(i) f is ACL,

(ii) $f \in W^{1,n}_{\mathrm{loc}}(G)$,

(iii) for almost every $x \in G$,

$$K_I(x,f) \le K, \qquad K_O(x,f) \le K.$$

Gehring [75] proved that $f \in W^{1,p}_{\mathrm{loc}}(G)$ with some $p \in [n, n+c)$, where $c \le n/(q^{1/(n-1)} - 1)$. In the planar case, this fact was first discovered by Bojarski [27].

12.2 Mean Inner and Outer Dilatations

Define for the linear bijections $A: \mathbb{R}^n \to \mathbb{R}^n$ the quantities

$$H_{I,\alpha}(A) = \frac{|J(A)|}{l^{\alpha}(A)}, \qquad H_{O,\alpha}(A) = \frac{L^{\alpha}(A)}{|J(A)|},$$

assuming $\alpha \ge 1$.

Now consider the homeomorphisms $f: G \to \mathbb{R}^n$, which are differentiable a.e. in a given domain $G \subset \mathbb{R}^n$, and let α, β be two real numbers such that $1 \le \alpha < \beta < \infty$. Put

$$H_{I,\alpha}(x,f) = H_{I,\alpha}(f'(x)), \qquad H_{O,\beta}(x,f) = H_{O,\beta}(f'(x)),$$

and define the integrals

$$HI_{\alpha,\beta}(f) = \int_G H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x,f)\, dx\,, \qquad HO_{\alpha,\beta}(f) = \int_G H_{O,\beta}^{\frac{\alpha}{\beta-\alpha}}(x,f)\, dx. \quad (12.2)$$

We call these integrals the **inner** and **outer mean dilatations** of the mapping f in G, respectively.

These characteristics were first introduced in [180]; see also [86] and [314].

Now consider the quadruples of the fixed real numbers $\alpha, \beta, \gamma, \delta$ such that $1 \le \alpha < \beta < \infty$, $1 \le \gamma < \delta < \infty$.

For two given domains G and G^* in $\mathbb{R}^n$, we define the **class** $B(G, G^*)$ whose elements are the homeomorphic mappings $f: G \to G^*$ that satisfy

(i) f and f^{-1} are ACL-homeomorphisms,

(ii) f and f^{-1} are differentiable with the Jacobians $J(x,f) \ne 0$ and $J(y, f^{-1}) \ne 0$ a.e. in G and G^*, respectively,

(iii) the inner and outer mean dilatations $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ are finite.

We call $B(G,G^*)$ the **class of mappings with finite mean dilatations**. The following theorem describes some basic differential properties of mappings in the class $B(G,G^*)$.

Theorem 12.1. *Suppose that $\alpha, \beta, \gamma, \delta$ are fixed real numbers such that $n-1 \leq \alpha < \beta < \infty$, $n-1 \leq \gamma < \delta < \infty$. Then the mappings of $B(G,G^*)$ belong to the Sobolev class $W^{1,p}_{\text{loc}}(G)$, while $f^{-1} \in W^{1,q}_{\text{loc}}(G^*)$, with $p = \max\big(\gamma, \beta/(\beta-n+1)\big)$ and $q = \max\big(\alpha, \delta/(\delta-n+1)\big)$.*

Proof. The assumption of the theorem implies that both mappings f and f^{-1} satisfy the (N)-property in the domains G and G^*, respectively. This property allows us to apply the standard rule for a change of variables under integration. Then the Hölder inequality and condition (iii) yield

$$\int_G L^\gamma(x,f)\,dx = \int_G \left[\left(\frac{L^\delta(x,f)}{|J(x,f)|}\right)^{\frac{\gamma}{\delta-\gamma}}\right]^{\frac{\delta-\gamma}{\delta}} |J(x,f)|^{\frac{\gamma}{\delta}}\,dx$$

$$\leq HO_{\gamma,\delta}^{\frac{\delta-\gamma}{\delta}}(f)\big(mG^*\big)^{\frac{\gamma}{\delta}} < \infty,$$

$$\int_G L^{\frac{\beta}{\beta-n+1}}(x,f)\,dx \leq \int_G \left(\frac{|J(x,f)|}{l^{n-1}(x,f)}\right)^{\frac{\beta}{\beta-n+1}} dx = HI_{n-1,\beta}(f)$$

$$\leq HI_{\alpha,\beta}^{\frac{(\alpha-n+1)\beta}{\alpha(\beta-n+1)}}(f)\big(mG^*\big)^{\frac{(\beta-\alpha)(n-1)}{\alpha(\beta-n+1)}} < \infty.$$

Conditions (i)–(iii) state that the mappings $f \in B(G,G^*)$ satisfy

$$HI_{\alpha,\beta}(f^{-1}) = HO_{\alpha,\beta}(f), \qquad HO_{\alpha,\beta}(f^{-1}) = HI_{\alpha,\beta}(f).$$

This means that if f is a mapping from the class $B(G,G^*) = B(G,G^*,\alpha,\beta,\gamma,\delta)$, then the inverse mapping $f^{-1} \in B(G^*,G,\gamma,\delta,\alpha,\beta)$. This completes the proof of the theorem. □

The relations (12.1) show that in the classical case of quasiconformal mappings, their dilatations are simultaneously either finite or infinite. However, this is not true for the mean dilatations. The following example shows that each of the mean dilatations $HI_{\alpha,\beta}(f)$ and $H_{\gamma,\delta}(f)$ can be unbounded, independently of restrictions on any other dilatation.

Example. Let

$$G = \{x = (x_1,\dots,x_n) : 0 < x_k < 1, k = 1,\dots,n\}$$

and

$$g(x) = \left(x_1,\dots,x_{n-1},\ \frac{x_n^{1-c}}{1-c}\right), \qquad 0 < c < 1.$$

Then the image $g(G)$ is the domain

$$G^* = \{y = (y_1,\dots,y_n) : 0 < y_k < 1, k = 1,\dots,n-1, 0 < y_n < 1/(1-c)\}.$$

It is easily to verify that g is differentiable in G and

$$l(x,g) = 1, \qquad L(x,g) = J(x,g) = x_n^{-c} > 1.$$

Thus,

$$H_{I,\alpha}(x,g) = \frac{J(x,g)}{l^{\alpha}(x,g)} = x_n^{-c}, \quad H_{O,\delta}(x,g) = \frac{L^{\delta}(x,g)}{J(x,g)} = x_n^{-c(\delta-1)},$$

$$HI_{\alpha,\beta}(g) = \int_G H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x,g)\,dx = \int_0^1 x_n^{-\frac{c\beta}{\beta-\alpha}}\,dx_n,$$

$$HO_{\gamma,\delta}(g) = \int_G H_{O,\delta}^{\frac{\gamma}{\delta-\gamma}}(x,g)\,dx = \int_0^1 x_n^{-\frac{c(\delta-1)\gamma}{\delta-\gamma}}\,dx_n.$$

One concludes from this that

$$HI_{\alpha,\beta}(g) < \infty \Longleftrightarrow 0 < c < 1 - \alpha/\beta,$$

$$HI_{\alpha,\beta}(g) = \infty \Longleftrightarrow 1 - \alpha/\beta \le c < 1,$$

$$HO_{\gamma,\delta}(g) < \infty \Longleftrightarrow 0 < c < 1 - (\gamma-1)\delta/(\delta-1)\gamma,$$

$$HO_{\gamma,\delta}(g) = \infty \Longleftrightarrow 1 - (\gamma-1)\delta/(\delta-1)\gamma \le c < 1.$$

The above example also shows that the class $B(G,G^*)$ is much wider than the class of quasiconformal mappings. For example, the mapping g belongs to $B(G,G^*)$ if

$$c \le \min\{1 - \alpha/\beta, 1 - (\gamma-1)\delta/(\delta-1)\gamma\},$$

but g is not quasiconformal for any c.

We shall now use the notation for the classes $B(G,G^*) = B(G,G^*,\alpha,\beta,\gamma,\delta)$ and study how these classes depend on a variety of parameters $\alpha,\beta,\gamma,\delta$. The following theorem provides the monotonicity.

Theorem 12.2. *Let $\alpha,\beta,\gamma,\delta,r,s,t,u$ be fixed real numbers such that $1 \le r < \alpha < \beta < s < \infty$ and $1 \le t < \gamma < \delta < u < \infty$. Then*

(a) $B(G,G^*,\alpha,\beta,\gamma,\delta) \subset B(G,G^*,r,\beta,\gamma,\delta)$,

(b) $B(G,G^*,\alpha,\beta,\gamma,\delta) \subset B(G,G^*,\alpha,s,\gamma,\delta)$,

(c) $B(G,G^*,\alpha,\beta,\gamma,\delta) \subset B(G,G^*,\alpha,\beta,t,\delta)$,

(d) $B(G,G^*,\alpha,\beta,\gamma,\delta) \subset B(G,G^*,\alpha,\beta,\gamma,u)$.

Proof. Applying Hölder's inequality to the inner mean dilatation $HI_{r,\beta}(f)$, one obtains

$$HI_{r,\beta}(f) = \int_G H_{I,r}^{\frac{\beta}{\beta-r}}(x,f)\,dx = \int_G \left(\frac{|J(x,f)|}{l^r(x,f)}\right)^{\frac{\beta}{\beta-r}} dx$$

$$\leq \left(\int_G \left(\frac{|J(x,f)|}{l^{\alpha}(x,f)} \right)^{\frac{\beta}{\beta-\alpha}} dx \right)^{\frac{r(\beta-\alpha)}{\alpha(\beta-r)}} \left(\int_G |J(x,f)| dx \right)^{\frac{\beta(\alpha-r)}{\alpha(\beta-r)}}$$

$$= HI_{\alpha,\beta}^{\frac{r(\beta-\alpha)}{\alpha(\beta-r)}}(f) \left(mG^* \right)^{\frac{\beta(\alpha-r)}{\alpha(\beta-r)}}.$$

This implies the first statement of the theorem.

The same argument works for the second part of Theorem 12.2. Indeed,

$$HI_{\alpha,s}(f) = \int_G H_{I,\alpha}^{\frac{s}{s-\alpha}}(x,f)\, dx = \int_G \left(\frac{|J(x,f)|}{l^{\alpha}(x,f)} \right)^{\frac{s}{s-\alpha}} dx$$

$$\leq \left(\int_G \left(\frac{|J(x,f)|}{l^{\alpha}(x,f)} \right)^{\frac{\beta}{\beta-\alpha}} dx \right)^{\frac{s(\beta-\alpha)}{\beta(s-\alpha)}} \left(\int_G dx \right)^{\frac{\alpha(s-\beta)}{\beta(s-\alpha)}}$$

$$= HI_{\alpha,\beta}^{\frac{s(\beta-\alpha)}{\beta(s-\alpha)}}(f) \left(mG \right)^{\frac{\alpha(s-\beta)}{\beta(s-\alpha)}}.$$

The similar assertions also hold for the outer mean dilatation. □

Note that for $\alpha, \beta, \gamma, \delta \leq n$, the widest class $B(G, G^*, \alpha, \beta, \gamma, \delta)$ consists of mappings that are (p,q)-quasiconformal in the mean. This class will be considered in Section 12.6.

12.3 On Distortion of p-Moduli

Let $\mathscr{S}_k$ be a family of k-dimensional surfaces $\mathscr{S}$ in $\mathbb{R}^n$, $1 \leq k \leq n-1$, which are the continuous images of a closed domain $D_s \subset \mathbb{R}^k$. Recall that the **p-modulus** of $\mathscr{S}_k$ is defined as

$$M_p(\mathscr{S}_k) = \inf \int_{\mathbb{R}^n} \rho^p\, dx, \qquad p \geq k,$$

where the infinum is taken over all Borel measurable functions $\rho \geq 0$ with

$$\int_{\mathscr{S}} \rho^k\, d\sigma_k \geq 1$$

for every $\mathscr{S} \in \mathscr{S}_k$. We call such functions ρ **admissible** for the family $\mathscr{S}_k$.

The following proposition characterizes quasiconformality in terms of the p-moduli of k-dimensional surfaces; see [290].

Proposition 12.2. *A homeomorphism f of a domain $G \subseteq \overline{\mathbb{R}^n}$ is K-quasiconformal, $1 \leq K < \infty$, if, for each family $\mathscr{S}_k$ of k-dimensional surfaces in G, $1 \leq k \leq n-1$, the double inequality*

$$K^{\frac{k-n}{n-1}} M_n(\mathscr{S}_k) \le M_n(f(\mathscr{S}_k)) \le K^{\frac{n-k}{n-1}} M_n(\mathscr{S}_k) \tag{12.3}$$

holds.

For the mappings of our classes $B(G,G^*)$, the double inequality (12.3) is extended as follows.

Theorem 12.3. *Let $f:\ G \to G^*$ be a homeomorphism satisfying*

(i) f and f^{-1} are ACL*,*

(ii) f and f^{-1} are differentiable a.e. in G and G^*, respectively,

(iii) the Jacobians $J(x,f)$ and $J(y,f^{-1})$ do not vanish a.e. in G and G^*, respectively.

Then, for every quadruple of fixed values $\alpha,\beta,\gamma,\delta$ such that $k \le \alpha < \beta < \infty$, $k \le \gamma < \delta < \infty$ and for any ring domain $D \subset G$, the inequalities

$$M_\alpha^\beta(\mathscr{S}_k^*) \le HI_{\alpha,\beta}^{\beta-\alpha}(f) M_\beta^\alpha(\mathscr{S}_k) \tag{12.4}$$

and

$$M_\gamma^\delta(\mathscr{S}_k) \le HO_{\gamma,\delta}^{\delta-\gamma}(f) M_\delta^\gamma(\mathscr{S}_k^*) \tag{12.5}$$

hold; here $\mathscr{S}_k^* = f(\mathscr{S}_k)$.

Proof. Let $\mathscr{S}_k$ be a family of k-surfaces in G, and let ρ be an admissible function for $\mathscr{S}_k$. Denote by $\mu_k(x,f)$ the minimal distortion of k-dimensional measures at x under f, i.e.,

$$\mu_k(x,f) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_k.$$

Note that $\mu_k(x,f) \ge l^k(x,f)$ for a.e. $x \in G$. Define in $D^* = f(D)$ the function

$$\rho^*(y) = \frac{\rho(x)}{\left[\mu_k(x,f)\right]^{1/k}},$$

where $x = f^{-1}(y)$. It is easy to check that $\rho^*(y)$ is an admissible function for $\mathscr{S}_k^*$.

Since $d\sigma_k^* \ge \mu_k(x,f)d\sigma_k$, we have

$$\int_{\mathscr{S}^*} \rho^{*^k}(y)\, d\sigma_k^* \ge \int_{\mathscr{S}} \rho^k(x)\, d\sigma_k \ge 1$$

for every surface $\mathscr{S}^* \in \mathscr{S}_k^*$.

We conclude from (i)–(iii) that f and f^{-1} satisfy the (N)-property in G and G^*, respectively, and

$$H_{I,\alpha}(x,f) = H_{O,\alpha}(y,f^{-1})$$

for a.e. $x \in G$ and $y \in G^*$. Applying Hölder's inequality and the properties of f and f^{-1}, we obtain

$$\int_{D^*} \rho^{*^{\alpha}}(y)\,dy = \int_{D} \frac{\rho^{\alpha}(x)}{\left[\mu_k(x,f)\right]^{\alpha/k}} |J(x,f)|\,dx \le \int_{D} \rho^{\alpha}(x) \frac{|J(x,f)|}{l^{\alpha}(x,f)}\,dx$$

$$\le \left(\int_{D} \rho^{\beta}(x)\,dx\right)^{\alpha/\beta} \left(\int_{D} H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x,f)\,dx\right)^{(\beta-\alpha)/\beta}.$$

Taking the infima over all such $\rho(x)$ yields (12.4). Interchanging $f:\ G \to G^*$ and $f^{-1}:\ G^* \to G$ in (12.4), we obtain inequality (12.5). $\square$

12.4 Moduli of Surface Families Dominated by Set Functions

Recall that a ring domain $D \subset \mathbb{R}^n$ is a bounded domain whose complement consists of two components C_0 and C_1. Setting $F_0 = \partial C_0$ and $F_1 = \partial C_1$, we obtain two boundary components of D. One of C_0 and C_1 contains the point of infinity; for definiteness, let us assume that $\infty \in C_1$.

We say that a path γ **joins the boundary components in** D if γ lies in D except for its endpoints, one of which is contained in F_0 and the second in F_1. We say that a compact set Σ **separates the boundary components of** D if $\Sigma \subset D$ and if C_0 and C_1 are located in the different components of $\widehat{\mathbb{R}} \setminus \Sigma$. Denote by Γ_D the family of all locally rectifiable paths $\gamma \subset D$ that join the boundary components of D and by Σ_D the family of all compact piecewise smooth $(n-1)$-dimensional surfaces Σ that separate the boundary components of D.

For each quantity V associated with D such as a subset of D or a family of sets contained in D, we denote its image under f by V^*.

We now introduce new classes of homeomorphisms that depend on the numerical parameters $\alpha, \beta, \gamma, \delta$ and on certain set functions.

Let Φ be a finite nonnegative function in a domain $G \subseteq \mathbb{R}^n$ defined for all open subsets E of G and such that

$$\sum_{k=1}^{m} \Phi(E_k) \ \le\ \Phi(E)$$

for any finite collection $\{E_k\}_{k=1}^m$ of open mutually disjoint sets $E_k \subset E$. We denote the class of all such set functions Φ by $\mathscr{F}$.

We fix the numbers $\alpha, \beta, \gamma, \delta$ satisfying

$$n-1 \le \alpha < \beta < \infty, \qquad n-1 \le \gamma < \delta < \infty,$$

and assume that there exists a nonempty family of homeomorphisms $f:\ G \to G^*$ such that there exist two set functions $\Phi, \Psi \in \mathscr{F}$ not depending on f such that for each ring domain $D \subset G$, the inequalities

$$M^{\beta}_{\alpha}(\Sigma^*_D) \le \Phi^{\beta-\alpha}(D) M^{\alpha}_{\beta}(\Sigma_D) \tag{12.6}$$

and

$$M^{\delta}_{\gamma}(\Sigma_D) \le \Psi^{\delta-\gamma}(D) M^{\gamma}_{\delta}(\Sigma^*_D) \tag{12.7}$$

hold. The class of such homeomorphisms will be denoted by $\mathscr{M}\mathscr{S}(G,G^*)$ (in fact, it also depends on $\alpha,\beta,\gamma,\delta$).

We shall need the following theorem from [87].

Theorem 12.4. *Let*

$$n-1<\alpha<\beta<\infty \quad \textit{and} \quad n-1<\gamma<\delta<\infty.$$

Then every mapping $f \in \mathscr{M}\mathscr{S}(G,G^*)$ has the following properties:

(a) f is ACL *in G;*

(b) f^{-1} is ACL *in G^*;*

(c) $f \in W^{1,a}_{\mathrm{loc}}(G)$, $a=\beta/(\beta-n+1)$;

(d) $f^{-1} \in W^{1,b}_{\mathrm{loc}}(G^*)$, $b=\delta/(\delta-n+1)$.

It is not hard to see that if $\beta \le n$, then $\beta/(\beta-n+1) \ge n$ and if $\beta \ge n$, then $\beta/(\beta-n+1) \le n$.

Now we introduce the mapping class $\mathscr{M}\mathscr{J}(G,G^*)$ related to α-moduli of the families of joining paths. Fix the numbers p,q,s,t, which satisfy

$$1 \le p<q<\infty, \qquad 1 \le s<t<\infty.$$

Suppose that there exists a (nonempty) family of homeomorphisms $f: G \to G^*$ such that for every ring domain $D \subset G$,

$$M^t_s(\Gamma^*_D) \le \Theta^{t-s}(D) M^s_t(\Gamma_D) \tag{12.8}$$

and

$$M^q_p(\Gamma_D) \le \Pi^{q-p}(D) M^p_q(\Gamma^*_D), \tag{12.9}$$

where Θ and Π are two given set functions in $\mathscr{F}$ not depending on f.

Properties of the mappings satisfying inequalities (12.8) and (12.9) in the equivalent terms of p-capacity were investigated for $n-1<s<t \le n$ and $n-1<p<q \le n$ in [178] and for wider bounds $n-1<s<t<\infty$ and $n-1<p<q<\infty$ in [314].

It was proved in [178] that f and f^{-1} are ACL and belong to $W^{1,t/(t-n+1)}_{\mathrm{loc}}$ and $W^{1,q/(q-n+1)}_{\mathrm{loc}}$, respectively. In [314], inequality (12.9) was extended to the mappings of Carnot groups. Using this inequality, the authors have established various properties of such mappings; for example, if a given mapping belongs to $W^{1,p}_{\mathrm{loc}}$, then the inverse mapping belongs to $W^{1,q/(q-n+1)}_{\mathrm{loc}}$. The case $t = q = n$ has been explicitly studied in [170].

For the equality of the p-capacity of rings and the p-moduli of families of joining paths for their boundary components, we refer to [122]; see Section A.3. Other relations between the p-capacities and the α-moduli of families of separating sets have been obtained by Ziemer [340]; see Section A.6. Ziemer applied the condition

$$\int_{\mathscr{S}} \rho \, d\sigma_{n-1} \geq 1$$

for admissibility of ρ and, in particular, established that

$$M_p(\Gamma_D) = M^{1-p}_{\frac{p}{p-1}}(\Sigma_D).$$

Note that in our notations the latter means

$$M_p(\Gamma_D) = M^{1-p}_{\frac{p(n-1)}{p-1}}(\Sigma_D).$$

The following relations are crucial:

$$1 < p < n \quad \Longleftrightarrow \quad n < \frac{p(n-1)}{p-1} < \infty,$$

$$p = n \quad \Longleftrightarrow \quad \frac{p(n-1)}{p-1} = n,$$

$$n < p < \infty \quad \Longleftrightarrow \quad 1 < \frac{p(n-1)}{p-1} < n.$$

Now put

$$\alpha = \frac{q(n-1)}{q-1}, \quad \beta = \frac{p(n-1)}{p-1}, \quad \gamma = \frac{t(n-1)}{t-1}, \quad \delta = \frac{s(n-1)}{s-1}.$$

It is easy to verify that if $\Pi(D) = \Phi(D)$, then inequalities (12.6) and (12.9) are equivalent. The same is true for (12.7) and (12.8) when $\Theta(D) = \Psi(D)$.

The conclusions of this section result in the following theorem.

Theorem 12.5. *Let*

$$n-1 < \alpha < \beta \leq n \quad \textit{and} \quad n-1 < \gamma < \delta \leq n$$

or

$$n \leq \alpha < \beta < \frac{(n-1)^2}{n-2} \quad \textit{and} \quad n \leq \gamma < \delta < \frac{(n-1)^2}{n-2}.$$

Then every mapping $f \in \mathscr{MS}(G)$ admits the following properties:

(a') f is ACL *in G;*

(b') f^{-1} is ACL *in G^*;*

(c') $f \in W^{1,a}_{\mathrm{loc}}(G)$ with $a = \max(\gamma, \beta/(\beta - n + 1))$;

(d') $f^{-1} \in W^{1,b}_{\mathrm{loc}}(G^*)$ with $b = \max(\alpha, \delta/(\delta - n + 1))$.

Note that each of inequalities (12.6) and (12.7) separately provides properties (a') and (b'). Properties (c') and (d') yield that both mappings f and f^{-1} have the (N)-property; cf. [174].

12.5 Alternate Characterizations of Classical Mappings

Specifying the set functions Φ and Ψ, we obtain a new characterization of quasiconformality and of quasi-isometry.

Theorem 12.6. *A homeomorphism $f:\ G \to G^*$ is K-quasiconformal if and only if there exists a constant K, $1 \le K < \infty$, such that for any ring domain $D \subset G$, the inequalities*

$$M^n_p(\Sigma^*_D) \le K^{\frac{p}{n-1}} \left(mD^*\right)^{n-p} M^p_n(\Sigma_D) \tag{12.10}$$

and

$$M^n_q(\Sigma_D) \le K^{\frac{q}{n-1}} \left(mD\right)^{n-q} M^q_n(\Sigma^*_D) \tag{12.11}$$

hold for $n-1 < p < q \le n$ or the inequalities

$$M^p_n(\Sigma^*_D) \le K^{\frac{p}{n-1}} \left(mD\right)^{p-n} M^n_p(\Sigma_D) \tag{12.12}$$

and

$$M^q_n(\Sigma_D) \le K^{\frac{q}{n-1}} \left(mD^*\right)^{q-n} M^n_q(\Sigma^*_D) \tag{12.13}$$

hold for $n \le p < q < (n-1)^2/(n-2)$.

Sketch of the proof. The relations (12.12) and (12.13) follow in fact from the results of [179]. Thus, it remains only to verify inequalities (12.10) and (12.11). The necessity of these inequalities follows from (12.3) by applying Hölder's inequality to $(n-1)$-dimensional surfaces.

The proof of sufficiency will be accomplished in three stages. It follows the lines of [210] and [170].

First we prove that f is ACL. Let $\Theta(V) = mV$, and let Q be an n-dimensional interval in G. Then the set function Θ belongs to the class $\mathscr{F}$. Write $Q = Q_0 \times J$, where Q_0 is an $(n-1)$-dimensional interval in $\mathbb{R}^{n-1}$ and J is an open segment of the axis x_n.

Following the notations in [246], we write $\Theta(T,Q) = \Theta(T \times J)$. The function $\Theta(T,Q)$ also belongs to the class $\mathscr{F}$ for the Borel sets $T \subset Q_0$.

Fix $z \in Q_0$ so that $\overline{\Theta}'(z,Q) < \infty$, and let $\Delta_1, \ldots, \Delta_k$ be the disjoint closed subintervals of the segment $J_z = \{z\} \times J$. Put $C_{0,i} = \Delta_i + r\overline{B}^n$ and $A_i = \Delta_i + 2rB^n$, where $\overline{B}^n$ is the closure of $B^n = B^n(x,r)$. The positive number r is chosen such that the domains $D_i = A_i \setminus C_{o,i}$ are disjoint and $D_i \subset Q$. Using the estimates of p-moduli of Σ_D, one obtains

$$\sum_{i=1}^{k} d(A_i^*) \leq C_1 \left(\overline{\Theta}'(z,Q)\right)^{\frac{1}{n}} \left(\sum_{i=1}^{k} m_1 \Delta_i\right)^{\frac{n-1}{n}},$$

where the constant C_1 depends only on p,n, and K. Thus, f is ACL.

In the second step, we show that a given homeomorphism f belongs to $W^{1,n}_{\text{loc}}$. For $\tilde{x}, x \in G$, $\tilde{x} \neq x$, define

$$k(x) = \limsup_{\tilde{x} \to x} \frac{|f(\tilde{x}) - f(x)|}{|\tilde{x} - x|}.$$

Consider for a point $x \in D$ the spherical ring $D_r(x) = \{y : r < |x-y| < 2r\}$ choosing $r > 0$ such that $D_r(x) \subset G$. It follows from the estimate

$$k(x) \leq \limsup_{r \to 0} \frac{d(A_r^*)}{r} \leq C_2 \left(\overline{\Theta}'(x)\right)^{\frac{1}{n}}$$

that $k(x) < \infty$ a.e. in G. Here C_2 depends only on n and K. Now applying Stepanov's theorem [300], one concludes that f is differentiable a.e. in G. Moreover, for each Borel set $V \subset G$, we have

$$\int_V k^n(x)\,dx \leq C_2 \int_V \overline{\Theta}'(x)\,dx \leq C_2 mV < \infty.$$

Finally, from (12.10) and (12.11), we obtain

$$K_O(x,f) \leq K, \qquad K_I(x,f) \leq K,$$

respectively. This completes the proof of Theorem 12.6. □

Replacing (12.10)–(12.11) and (12.12)–(12.13) by suitable inequalities for α-moduli with $\alpha \neq n$ given below, one obtains quasi-isometry for the mapping.

Recall that a homeomorphism $f:\ G \to G^*$ is called **quasi-isometric** if, for any $x, z \in G$ and $y, t \in G^*$, the inequalities

$$\limsup_{z \to x} \frac{|f(x) - f(z)|}{|x - z|} \leq M, \qquad \limsup_{t \to y} \frac{|f^{-1}(y) - f^{-1}(t)|}{|y - t|} \leq M,$$

hold with a constant M depending only on G and G^*.

Theorem 12.7. *Let $f:\ G \to G^*$ be a homeomorphism. Then the following conditions are equivalent:*

1^0. *f is quasi-isometric;*

2^0. *for fixed real numbers $\alpha,\beta,\gamma,\delta$ such that*

$$n-1<\alpha<\beta<n \quad \textit{and} \quad n-1<\gamma<\delta<n$$

or

$$n<\alpha<\beta<(n-1)^2/(n-2) \quad \textit{and} \quad n<\gamma<\delta<(n-1)^2/(n-2),$$

there exists a constant K such that for any ring domain $D\subset G$, the inequalities

$$M_\alpha^\beta(\Sigma_D^*)\le K^\beta\left(mD\right)^{\beta-\alpha}M_\beta^\alpha(\Sigma_D)$$

and

$$M_\gamma^\delta(\Sigma_D)\le K^\delta\left(mD^*\right)^{\delta-\gamma}M_\delta^\gamma(\Sigma_D^*)$$

hold.

The quasi-invariance of p-moduli of path or surface families is a characteristic property of quasi-isometry; see, e.g., [91]. This quasi-invariance is also represented by a double inequality. The implication $1^0\Rightarrow 2^0$ follows from this inequality by applying Hölder's inequality. The inverse implication $2^0\Rightarrow 1^0$ is proved through estimates in [76].

12.6 Mappings (α,β)-Quasiconformal in the Mean

In this section we mention the results established in [170].

Here a **condenser** in $\mathbb{R}^n$ is a triple of sets $A=(F_0,F_1;G)$, where G is a domain in $\mathbb{R}^n$ and F_0 and $F_1\subset G$ are nonempty sets being closed with respect to G. Given $\beta\in[1,n]$, β-**capacity** $\operatorname{cap}_\beta(F_0,F_1;G)$ of the condenser $A=(F_0,F_1;G)$ is ∞ if $F_1\cap F_2\neq\varnothing$ and

$$\operatorname{cap}_\beta(F_0,F_1;G) \;=\; \inf\int\limits_G|\nabla\varphi(x)|^\beta\,dx \tag{12.14}$$

if $F_1\cap F_2=\varnothing$, where the infimum is taken over all functions $\varphi:G\to\mathbb{R}^1$ of the class ACL in G such that $\varphi(x)\le 0$ for $x\in F_0$ and $\varphi(x)\ge 1$ for $x\in F_1$.

Let D and Δ be domains in $\mathbb{R}^n$, $n\ge 2$. A homeomorphism $f:D\to\Delta$ belongs to the **class** $Q_p(D)$ **[**$q_p(D)$**]**, $1/(n-1)\le p\le\infty$, if there is a nonnegative subadditive bounded (absolutely continuous) function Φ_p of Borel sets in D such that the inequality

$$\operatorname{cap}_{\frac{pn}{p+1}}(f(F_0),f(F_1);f(G)) \;\le\; [\Phi_p(G\setminus(F_0\cup F_1))]^{\frac{1}{p+1}}\operatorname{cap}_n^{\frac{p}{p+1}}(F_0,F_1;G)$$

holds for each condenser $(F_0,F_1;G)$ in D.

The (N)-property by Lusin was established for homeomorphisms of the class $q_p(D)$ for all $1/(n-1) \le p \le \infty$ [it was also shown, for $1/(n-1) \le p \le n-1$, that mappings of the class $q_p(D)$ are not, generally speaking, ACL^n homeomorphisms]. Moreover, the following estimate for distortion of the Euclidean distance has been obtained.

Theorem 12.8. *Let f be a homeomorphism of the class $Q_p(D)$ with $p > n-1$ and let $F \subset D$ be a compact set. Then*

$$|f(a) - f(b)| \le C \ \ln^{-\frac{p(n-1)}{n}} \frac{1}{|a-b|} \tag{12.15}$$

for each pair of points a, $b \in F$ such that $|a-b| < \delta \le \min\{1, d^4(F, \partial D)\}$, where C depends only on p, n, and $\Phi_p(D)$. Here $d(F, \partial D)$ is the Euclidean distance between F and ∂D.

Moreover, estimate (12.15) is precise by the order in the class of $Q_p(D)$.

Let D and Δ be domains in $\mathbb{R}^n$. We say that a homeomorphism $f : D \to \Delta$ is (α, β)**-quasiconformal in the mean** if f and f^{-1} are ACL, differentiable a.e., and

$$\int_D K_O^{\beta}(x, f)\, dx \ < \ \infty \tag{12.16}$$

and

$$\int_D K_I^{\alpha}(x, f)\, |J(x, f)|\, dx \ < \ \infty. \tag{12.17}$$

The main results are the following two theorems.

Theorem 12.9. *Let $f : D \to \Delta$ be a homeomorphism and $\alpha > n-1$, $\beta > n-1$. Then the following statements are equivalent:*

1. a mapping $f : D \to \Delta$ is (α, β)-quasiconformal in the mean;

2. $f \in Q_\alpha(D)$ and $f^{-1} \in Q_\beta(\Delta)$;

3. $f \in q_\alpha(D)$ and $f^{-1} \in q_\beta(\Delta)$.

Theorem 12.10. *Let $f : D \to \Delta$ be a homeomorphism and let $1/(n-1) \le \alpha \le n-1$ and $1/(n-1) \le \beta \le n-1$. Then the following statements are equivalent:*

1. a mapping $f : D \to \Delta$ is (α, β)-quasiconformal in the mean;

2. $f \in q_\alpha(D)$ and $f^{-1} \in q_\beta(\Delta)$ are ACL *and differentiable a.e.*

Note also that Kruglikov and Paikov have obtained a series of results on the boundary correspondence under mappings (α, β)-quasiconformal in the mean, in terms of the so-called prime ends; see, e.g., [172]. However, the latter is outside the framework of this book.

12.7 Coefficients of Quasiconformality of Ring Domains

In this section we determine the mean dilatations of domains in $\mathbb{R}^n$ and establish the extremal mappings, which minimize the inner and outer mean dilatations for some ring domains; see [89].

Let G and G^* be two homeomorphic bounded domains in $\mathbb{R}^n$, $n \geq 2$. Recall that the **inner** and **outer coefficients of quasiconformality** of G with respect to G^*, $K_I(G,G^*)$ and $K_O(G,G^*)$, are the infima of the inner and outer dilatations $K_I(f)$ and $K_O(f)$ of homeomorphisms $f: \ G \to G^*$, respectively; see, e.g., [316]. These coefficients are known only for several simple domains in $\mathbb{R}^n$, $n \geq 3$; see [85].

A mapping $f_0: \ G \to G^*$ is called **extremal** for $K_I(G,G^*)$ or $K_O(G,G^*)$ if $K_I(f_0) = K_I(G,G^*)$ or $K_O(f_0) = K_O(G,G^*)$, respectively.

First, we introduce the more general quantities called the mean coefficients of quasiconformality for ring domains, and apply the method of p-moduli of k-dimensional surfaces for solving the corresponding extremal problems. This allows us to calculate the mean coefficients for spherical rings in $\mathbb{R}^n$.

Consider the ring domains D and D^* in $\mathbb{R}^n$ and homeomorphisms f from D onto D^*. For the fixed real numbers $\alpha, \beta, \gamma, \delta$ such that

$$k \leq \alpha < \beta < \infty \quad \text{and} \quad k \leq \gamma < \delta < \infty,$$

and for a mapping $f: D \to D^*$, we define the **inner** and **outer mean dilatations** $I_{\alpha,\beta}(f)$ and $O_{\gamma,\delta}(f)$ by

$$I_{\alpha,\beta}(f) = \left(\sup \frac{M_\alpha^\beta(\mathscr{S}_k^*)}{M_\beta^\alpha(\mathscr{S}_k)} \right)^{\frac{1}{\beta-\alpha}}, \quad O_{\gamma,\delta}(f) = \left(\sup \frac{M_\gamma^\delta(\mathscr{S}_k)}{M_\delta^\gamma(\mathscr{S}_k^*)} \right)^{\frac{1}{\delta-\gamma}}, \tag{12.18}$$

where the suprema are taken over all families $\mathscr{S}_k$ of k-dimensional surfaces in D such that the numerator and denominator in each above fraction cannot be equal to 0 or ∞ simultaneously. Here $\mathscr{S}_k^* = f(\mathscr{S}_k)$. Obviously,

$$I_{\alpha,\beta}(f^{-1}) \ = \ O_{\alpha,\beta}(f), \qquad O_{\alpha,\beta}(f^{-1}) \ = \ I_{\alpha,\beta}(f). \tag{12.19}$$

Theorem 12.3 gives the next relationship between the integral and the mean dilatations

$$I_{\alpha,\beta}(f) \ \leq \ HI_{\alpha,\beta}(f), \qquad O_{\gamma,\delta}(f) \ \leq \ HO_{\gamma,\delta}(f). \tag{12.20}$$

Let us also introduce the quantities

$$I_{\alpha,\beta}(D,D^*) \ = \ \inf_f I_{\alpha,\beta}(f), \qquad O_{\gamma,\delta}(D,D^*) \ = \ \inf_f O_{\gamma,\delta}(f),$$

where the infima are taken over all mappings of the class $B(D,D^*)$. We call these quantities the **inner** and **outer mean coefficients** of the ring domains D and D^*.

The mappings minimizing the mean coefficients are called **extremal** for the corresponding mean coefficients.

As in the case of $K_I(D,D^*)$ and $K_O(D,D^*)$, determining the mean characteristics $I_{\alpha,\beta}(D,D^*)$ and $O_{\gamma,\delta}(D,D^*)$ for the ring domains D and D^* is very complicated; cf. [85]. To obtain upper bounds for given ring domains D and D^*, it suffices to construct a suitable homeomorphism f of D onto D^* and calculate the dilatations of f by (12.2). Obtaining the lower bounds is much more difficult since it requires finding lower bounds for various dilatations of all homeomorphisms in the class $B(D,D^*)$. We accomplish this by estimating the distortion of certain families of k-dimensional surfaces under a fixed homeomorphism f and then finding suprema in (12.18).

We are concerned with the extremal mappings, which minimize $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ on the class $B(D,D^*)$ for the spherical rings. Denote by

$$D(r_0) = \{x \in \mathbb{R}^n : 0 < r_0 < |x| < 1\}$$

and

$$D(\rho_0) = \{y \in \mathbb{R}^n : 0 < \rho_0 < |y| < 1\}$$

the spherical rings. Let us calculate the mean coefficients $I_{\alpha,\beta}(D(r_0),D(\rho_0))$ and $O_{\gamma,\delta}(D(r_0),D(\rho_0))$ for the case $0 < \rho_0 \le r_0 < 1$.

For $n-1 < \alpha < \beta < \infty$, we obtain, by (12.18),

$$I_{\alpha,\beta}(D(r_0),D(\rho_0)) \ge \left(\frac{M_\alpha^\beta(\Sigma_D^*)}{M_\beta^\alpha(\Sigma_D)}\right)^{\frac{1}{\beta-\alpha}}.$$

Substituting the explicit expressions of $M_\alpha(\Sigma_D^*)$ and $M_\beta(\Sigma_D)$ given, for example, in [76] (with $\alpha \neq n$ and $\beta \neq n$), we get

$$I_{\alpha,\beta}(D(r_0),D(\rho_0)) \ge \omega_{n-1}\left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha}\right)^{\frac{\beta}{\beta-\alpha}}\left(\frac{n-\beta}{1-r_0^{n-\beta}}\right)^{\frac{\alpha}{\beta-\alpha}},$$

where ω_{n-1} denotes the $(n-1)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^n$.

Further, consider two spherical systems of coordinates (r,φ_i) and (ρ,ψ_i), $i = \overline{1,n-1}$, on $D(r_0)$ and $D(\rho_0)$, respectively. It is easy to see that the mapping

$$f_1 = \left(\rho = \left[1 + \frac{\rho_0^{n-\alpha}-1}{r_0^{n-\beta}-1}(r^{n-\beta}-1)\right]^{\frac{1}{n-\alpha}}, \psi_i = \varphi_i, 0 \le \varphi_i < 2\pi, r_0 < r < 1\right)$$

carries out $D(r_0)$ onto $D(\rho_0)$. From (12.2) we calculate

$$HI_{\alpha,\beta}(f_1) = \omega_{n-1}\left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha}\right)^{\frac{\beta}{\beta-\alpha}}\left(\frac{n-\beta}{1-r_0^{n-\beta}}\right)^{\frac{\alpha}{\beta-\alpha}}.$$

Thus, the mapping $f_1 : D(r_0) \to D(\rho_0)$ is extremal for the inner mean coefficient $I_{\alpha,\beta}(D(r_0), D(\rho_0))$, and

$$I_{\alpha,\beta}(D(r_0), D(\rho_0)) = \omega_{n-1} \left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha} \right)^{\frac{\beta}{\beta-\alpha}} \left(\frac{n-\beta}{1-r_0^{n-\beta}} \right)^{\frac{\alpha}{\beta-\alpha}}.$$

In the cases $\alpha = n$ and $\beta = n$, the corresponding inner mean characteristics are of the form

$$I_{n,\beta}(D(r_0), D(\rho_0)) = \omega_{n-1} \left(\ln \frac{1}{\rho_0} \right)^{\frac{\beta}{\beta-n}} \left(\frac{n-\beta}{1-r_0^{n-\beta}} \right)^{\frac{n}{\beta-n}}$$

and

$$I_{\alpha,n}(D(r_0), D(\rho_0)) = \omega_{n-1} \left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha} \right)^{\frac{n}{n-\alpha}} \left(\ln \frac{1}{r_0} \right)^{-\frac{\alpha}{n-\alpha}}.$$

In the same way, we obtain from (12.18) the following estimate for $1 < \gamma < \delta < \infty$:

$$O_{\gamma,\delta}(D(r_0), D(\rho_0)) \geq \left(\frac{M_\gamma^\delta(\Gamma_D)}{M_\delta^\gamma(\Gamma_D^*)} \right)^{\frac{1}{\delta-\gamma}}. \tag{12.21}$$

Substituting into the right-hand side of (12.20) the well-known expressions of $M_\gamma(\Gamma_D)$ and $M_\delta(\Gamma_D^*)$ (see [91]), when $\gamma \neq n$ and $\delta \neq n$, we have

$$O_{\gamma,\delta}(D(r_0), D(\rho_0)) \geq \omega_{n-1} \left(\frac{n-\gamma}{\gamma-1} \right)^{\frac{(\gamma-1)\delta}{\delta-\gamma}} \left(\frac{\delta-1}{n-\delta} \right)^{\frac{(\delta-1)\gamma}{\delta-\gamma}} \frac{\left(r_0^{\frac{\gamma-n}{\gamma-1}} - 1 \right)^{\frac{(1-\gamma)\delta}{\delta-\gamma}}}{\left(\rho_0^{\frac{\delta-n}{\delta-1}} - 1 \right)^{\frac{(1-\delta)\gamma}{\delta-\gamma}}}.$$

The mapping

$$f_2 = \left(\rho = \left[1 + \frac{\rho_0^{\frac{\delta-n}{\delta-1}} - 1}{r_0^{\frac{\gamma-n}{\gamma-1}} - 1} \left(r^{\frac{\gamma-n}{\gamma-1}} - 1 \right) \right]^{\frac{\delta-1}{\delta-n}}, \psi_i = \varphi_i, 0 \leq \varphi_i < 2\pi, r_0 < r < 1 \right)$$

also carries $D(r_0)$ onto $D(\rho_0)$. By (12.2),

$$HO_{\gamma,\delta}(f_2) = \omega_{n-1} \left(\frac{n-\gamma}{\gamma-1} \right)^{\frac{(\gamma-1)\delta}{\delta-\gamma}} \left(\frac{\delta-1}{n-\delta} \right)^{\frac{(\delta-1)\gamma}{\delta-\gamma}} \frac{\left(r_0^{\frac{\gamma-n}{\gamma-1}} - 1 \right)^{\frac{(1-\gamma)\delta}{\delta-\gamma}}}{\left(\rho_0^{\frac{\delta-n}{\delta-1}} - 1 \right)^{\frac{(1-\delta)\gamma}{\delta-\gamma}}}.$$

This yields that the mapping $f_2 : D(r_0) \to D(\rho_0)$ is extremal for the coefficient $O_{\gamma,\delta}(D(r_0), D(\rho_0))$ and

$$O_{\gamma,\delta}(D(r_0),D(\rho_0)) = \omega_{n-1}\left(\frac{n-\gamma}{\gamma-1}\right)^{\frac{(\gamma-1)\delta}{\delta-\gamma}}\left(\frac{\delta-1}{n-\delta}\right)^{\frac{(\delta-1)\gamma}{\delta-\gamma}}\frac{\left(r_0^{\frac{\gamma-n}{\gamma-1}}-1\right)^{\frac{(1-\gamma)\delta}{\delta-\gamma}}}{\left(\rho_0^{\frac{\delta-n}{\delta-1}}-1\right)^{\frac{(1-\delta)\gamma}{\delta-\gamma}}}.$$

In the case $\gamma = n$, the outer mean characteristic is

$$O_{n,\delta}(D(r_0),D(\rho_0)) = \omega_{n-1}\left(\frac{\delta-1}{n-\delta}\right)^{\frac{(\delta-1)\gamma}{\delta-\gamma}}\left(\ln\frac{1}{r_0}\right)^{\frac{(1-n)\delta}{\delta-n}}\left(\rho_0^{\frac{\delta-n}{\delta-1}}-1\right)^{\frac{(\delta-1)n}{\delta-n}}.$$

The case $\delta = n$ was studied in [90].

It follows from (12.19) that, in the case $0 < r_0 \le \rho_0 < 1$, the mapping f_2^{-1} is extremal for the inner mean coefficient. Then the mapping f_1^{-1} is extremal for the outer mean coefficient.

Now put

$$IM^*_{\alpha,\beta}(D,D^*) = \left(\frac{I_{\alpha,\beta}(D,D^*)}{mD^*}\right)^{\frac{\beta-\alpha}{\alpha}}, \qquad IM_{\alpha,\beta}(D,D^*) = \left(\frac{I_{\alpha,\beta}(D,D^*)}{mD}\right)^{\frac{\beta-\alpha}{\beta}},$$

$$OM_{\alpha,\beta}(D,D^*) = \left(\frac{O_{\alpha,\beta}(D,D^*)}{mD}\right)^{\frac{\beta-\alpha}{\alpha}}, \qquad OM^*_{\alpha,\beta}(D,D^*) = \left(\frac{O_{\alpha,\beta}(D,D^*)}{mD^*}\right)^{\frac{\beta-\alpha}{\beta}}.$$

Theorem 12.11. *(a) If s,α,β are real numbers such that $1 < s < \alpha < \beta < \infty$, then*

$$IM^*_{s,\beta}(D,D^*) \le IM^*_{\alpha,\beta}(D,D^*)$$

and

$$OM_{s,\beta}(D,D^*) \le OM_{\alpha,\beta}(D,D^*)\,.$$

(b) If α,β,t are real numbers such that $1 < \alpha < \beta < t < \infty$, then

$$IM_{\alpha,t}(D,D^*) \le IM_{\alpha,\beta}(D,D^*)$$

and

$$OM^*_{\alpha,t}(D,D^*) \le OM^*_{\alpha,\beta}(D,D^*)\,.$$

Proof. We will prove only the first inequalities in each part of the theorem. Applying Hölder's inequality to $IM^*_{s,\beta}(D,D^*)$, we have

$$IM^*_{s,\beta}(D,D^*) = \inf\frac{1}{(mD^*)^{\frac{\beta-s}{s}}}\left(\sup\frac{M_s^{\beta}(\mathscr{S}_k^*)}{M_{\beta}^{s}(\mathscr{S}_k)}\right)^{\frac{1}{s}}$$

$$\leq \inf \frac{1}{\left(mD^*\right)^{\frac{\beta-s}{s}}} \left(\sup \frac{\left(mD^*\right)^{\frac{(\alpha-s)\beta}{\alpha}} M_\alpha^{\frac{s\beta}{\alpha}}(\mathscr{S}_k^*)}{M_\beta^s(\mathscr{S}_k)} \right)^{\frac{1}{s}}$$

$$= \inf \frac{1}{\left(mD^*\right)^{\frac{\beta-\alpha}{\alpha}}} \left(\sup \frac{M_\alpha^\beta(\mathscr{S}_k^*)}{M_\beta^\alpha(\mathscr{S}_k)} \right)^{\frac{1}{\alpha}} = IM^*_{\alpha,\beta}(D,D^*).$$

Similarly, for part (b),

$$IM_{\alpha,t}(D,D^*) = \inf \frac{1}{\left(mD\right)^{\frac{t-\alpha}{t}}} \left(\sup \frac{M_\alpha^t(\mathscr{S}_k^*)}{M_t^\alpha(\mathscr{S}_k)} \right)^{\frac{1}{t}}$$

$$\leq \inf \frac{1}{\left(mD\right)^{\frac{t-\alpha}{t}}} \left(\sup \frac{\left(mD\right)^{\frac{(t-\beta)\alpha}{\beta}} M_\alpha^t(\mathscr{S}_k^*)}{M_\beta^{\frac{\alpha t}{\beta}}(\mathscr{S}_k)} \right)^{\frac{1}{t}}$$

$$= \inf \frac{1}{\left(mD\right)^{\frac{\beta-\alpha}{\alpha}}} \left(\sup \frac{M_\alpha^\beta(\mathscr{S}_k^*)}{M_\beta^\alpha(\mathscr{S}_k)} \right)^{\frac{1}{\beta}} = IM_{\alpha,\beta}(D,D^*),$$

which yields the desired relations. □

In particular, we also obtain equalities that provide a new approach to calculating the coefficients of quasiconformality $K_I(D(r_0),D(\rho_0))$ and $K_O(D(r_0),D(\rho_0))$; cf. [316]. For example, if $0 < \rho_0 < r_0 < 1$, then

$$K_I(D(r_0),D(\rho_0)) = \lim_{\alpha\to n} IM^*_{\alpha,n}(D(r_0),D(\rho_0))$$

$$= \lim_{\beta\to n} IM_{n,\beta}(D(r_0),D(\rho_0)) = \frac{\ln\frac{1}{\rho_0}}{\ln\frac{1}{r_0}},$$

$$K_O(D(r_0),D(\rho_0)) = \lim_{\alpha\to n} OM_{\alpha,n}(D(r_0),D(\rho_0))$$

$$= \lim_{\beta\to n} OM^*_{n,\beta}(D(r_0),D(\rho_0)) = \left(\frac{\ln\frac{1}{\rho_0}}{\ln\frac{1}{r_0}}\right)^{n-1}.$$

Chapter 13
On Mapping Theory in Metric Spaces

In this chapter we investigate the problem of extending the boundary and removability of singularities of quasiconformal mappings and their generalizations in arbitrary metric spaces with measures; see [266]. The results can be applied, in particular, to Riemannian manifolds, the Loewner spaces, and the groups by Carnot and Heisenberg.

Here we study properties of weakly flat spaces which are a far-reaching generalization of the recently introduced Loewner spaces (see, e.g., [21, 33, 107, 112, 312]), including, in particular, the well-known groups by Carnot and Heisenberg; see, e.g., [108, 109, 166, 167, 197, 199, 221, 238, 314] and [324–326]. On this basis, we create the theory of the boundary behavior and removable singularities for quasiconformal mappings and their generalizations, which can be applied to any of the mentioned classes of spaces. In particular, we prove a generalization and strengthening of the known Gehring–Martio theorem on homeomorphic extension to the boundary of quasiconformal mappings between quasiextremal distance domains in $\mathbb{R}^n$, $n \geq 2$; see [81]. The modulus techniques for metric spaces are developed, for instance, in [64, 107, 112, 201].

13.1 Introduction

Given a set S in (X,d) and $\alpha \in [0,\infty)$, H^α denotes the **α-dimensional Hausdorff measure** of S in (X,d), i.e.,

$$H^\alpha(S) \;=\; \sup_{\varepsilon>0} H^\alpha_\varepsilon(S)\,, \tag{13.1}$$

$$H^\alpha_\varepsilon(S) \;=\; \inf \sum_{i=1}^{\infty} \delta_i^\alpha\,, \tag{13.2}$$

where the infimum is taken over all countable collections of numbers $\delta_i \in (0,\varepsilon)$ such that some sets S_i in (X,d) with diameters δ_i cover S. Note that H^α is nonincreasing

O. Martio et al., *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-85588-2_13,

in the parameter α. The **Hausdorff dimension** of S is the only number $\alpha \in [0,\infty]$ such that $H^{\alpha'}(S) = 0$ for all $\alpha' > \alpha$ and $H^{\alpha''}(S) = \infty$ for all $\alpha'' < \alpha$.

Recall, for a given continuous path $\gamma\colon [a,b] \to X$ in a metric space (X,d), that its **length** is the supremum of the sums

$$\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))$$

over all partitions $a = t_0 \leq t_1 \leq \ldots \leq t_k = b$ of the interval $[a,b]$. The path γ is called **rectifiable** if its length is finite.

In what follows, (X,d,μ) denotes a space X with a metric d and a locally finite Borel measure μ. Given a family of paths Γ in X, a Borel function $\rho : X \to [0,\infty]$ is called **admissible** for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \rho\, ds \;\geq\; 1 \tag{13.3}$$

for all $\gamma \in \Gamma$.

An open set in X whose points can all be connected pairwise by continuous paths is called a **domain** in X. Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geq 1$ in spaces (X,d,μ) and (X',d',μ'), and let $Q : G \to [0,\infty]$ be a measurable function. We say that a homeomorphism $f : G \to G'$ is a **Q-homeomorphism** if

$$M(f\Gamma) \leq \int\limits_{G} Q(x) \cdot \rho^{\alpha}(x)\, d\mu(x) \tag{13.4}$$

for every family Γ of paths in G and every admissible function ρ for Γ.

The **modulus** of the path family Γ in G is given by the equality

$$M(\Gamma) \;=\; \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{G} \rho^{\alpha}(x)\, d\mu(x). \tag{13.5}$$

In the case of the path family $\Gamma' = f\Gamma$, we take the Hausdorff dimension α' of the domain G'.

A space (X,d,μ) is called **(Ahlfors) α-regular** if there is a constant $C \geq 1$ such that

$$C^{-1} r^{\alpha} \leq \mu(B_r) \leq C r^{\alpha} \tag{13.6}$$

for all balls B_r in X with the radius $r < \operatorname{diam} X$. As we know, α-regular spaces have Hausdorff dimension α; see, e.g., [106], p. 61. We say that a space (X,d,μ) is **(Ahlfors) regular** if it is (Ahlfors) α-regular for some $\alpha \in (1,\infty)$.

We will say that a space (X,d,μ) is **upper α-regular at a point** $x_0 \in X$ if there is a constant $C > 0$ such that

$$\mu(B(x_0,r)) \le Cr^{\alpha} \tag{13.7}$$

for the balls $B(x_0,r)$ centered at $x_0 \in X$ with all radii $r < r_0$ for some $r_0 > 0$. We will also say that a space (X,d,μ) is **upper α-regular** if condition (13.7) holds at every point $x_0 \in X$.

13.2 Connectedness in Topological Spaces

Let us give definitions of some topological notions and related remarks of a general character that will be useful in what follows. Let T be an arbitrary topological space. A **path in** T is a continuous mapping $\gamma : [a,b] \to T$. Later on, $|\gamma|$ denotes the locus $\gamma([a,b])$. If A, B, and C are sets in T, then $\Delta(A,B,C)$ denotes a collection of all paths γ joining A and B in C, i.e., $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$, $t \in (a,b)$.

Recall that a topological space is a **connected space** if it is impossible to split it into two nonempty open sets. Compact connected spaces are called **continua**. A topological space T is said to be **path-connected** if any two points x_1 and x_2 in T can be joined by a path $\gamma : [0,1] \to T$, $\gamma(0) = x_1$, and $\gamma(1) = x_2$. A **domain** in T is an open path-connected set in T. We say that a metric space T is a **rectifiable** if any two points x_1 and x_2 in T can be joined by a rectifiable path. In particular, we say that a domain G in T is a **rectifiable domain** if G with the induced topology is a rectifiable space. A domain G in a topological space T is called **locally connected at a point** $x_0 \in \partial G$ if, for every neighborhood U of the point x_0, there is a neighborhood $V \subseteq U$ such that $V \cap G$ is connected; see [186], p. 232. Similarly, we say that a domain G is **locally path connected (rectifiable) at a point** $x_0 \in \partial G$ if, for every neighborhood U of the point x_0, there is a neighborhood $V \subseteq U$ such that $V \cap G$ is path connected (rectifiable).

Proposition 13.1. *Let T be a topological (metric) space with a base of topology $\mathscr{B}$ consisting of path-connected (rectifiable) sets. Then an arbitrary open set Ω in T is connected if and only if Ω is path connected (rectifiable).*

Corollary 13.1. *An open set Ω in $\mathbb{R}^n$, $n \ge 2$, or in any manifold is connected if and only if Ω is path connected (rectifiable).*

Remark 13.1. Thus, if a domain G in $\mathbb{R}^n$, $n \ge 2$, is locally connected at a point $x_0 \in \partial G$, then it is also path connected at x_0. The same is true for manifolds. As we will show later on, the connectedness and path connectedness are equivalent for open sets in a wide class of the so-called weakly flat spaces, which include the known spaces by Loewner and, in particular, the well-known groups by Carnot and Heisenberg.

Proof of Proposition 13.1. First let Ω be path connected. If Ω is simultaneously not connected, then $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are open, nonempty, disjoint sets in T. Take $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ and connect them with a path $\gamma : [0,1] \to$

Ω, $\gamma(0)=x_1$, and $\gamma(1)=x_2$. Then the sets $\omega_1=\gamma^{-1}\Omega_1$ and $\omega_2=\gamma^{-1}\Omega_2$ are disjoint, nonempty, and open in $[0,1]$ by the continuity of γ. However, the last contradicts the connectedness of the segment $[0,1]$.

Now, let Ω be connected. Take an arbitrary point $x_0\in\Omega$ and denote by Ω_0 the set of all points x_* in Ω that can be connected with x_0 through a finite chain of sets $B_k\subset\Omega$ in the base $\mathscr{B}$, $k=1,\ldots,m$, such that $x_0\in B_1, x_*\in B_m$, and $B_k\cap B_{k+1}\neq\emptyset, k=1,\ldots,m-1$.

Note, first, that the set Ω_0 is open. Indeed, if a point $y_0\in\Omega_0$, then there is its neighborhood $B_0\subseteq\Omega$ in the base $\mathscr{B}$ and all points of this neighborhood belong to Ω_0. Second, the set Ω_0 is closed in Ω.

Indeed, assume that $\partial\Omega_0\cap\Omega\neq\emptyset$. Then, for every point $z_0\in\partial\Omega_0\cap\Omega$, there is a neighborhood $B_0\subseteq\Omega$ in the base $\mathscr{B}$, and in this neighborhood there is a point $x_*\in\Omega_0$ because $z_0\in\partial\Omega_0$. Thus, $z_0\in\Omega_0$ by the definition of the set Ω_0. However, Ω_0 is open and hence $\Omega_0\cap\partial\Omega_0=\emptyset$. The obtained contradiction disproves the above assumption.

Thus, Ω_0 is simultaneously open and closed in Ω and, consequently, being nonempty, it coincides with the set Ω in view of its connectivity. But, by the construction, Ω_0 is obviously path connected.

Finally, if the space T has a base of topology $\mathscr{B}$ consisting of rectifiable domains, then, covering any path γ in T by elements of this base, we are able to choose its finite subcovering leading to the construction of the corresponding rectifiable path. □

Proposition 13.2. *If a domain G in a metric space (X,d) is locally path connected (rectifiable) at a point $x_0\in\partial G$, then x_0 is accessible from G through a (locally rectifiable) path $\gamma:[0,1)\to X$, $\gamma([0,1))\subset G$, $\lim_{t\to1}\gamma(t)=x_0$.*

Proof. Choose a decreasing sequence of neighborhoods V_m of the point x_0 where $W_m=V_m\cap G$ are path connected (rectifiable) and $W_m\subset B(x_0,2^{-m})$ and also a sequence of the points $x_m\in W_m$, $m=1,2,\ldots$, and connect the points x_m and x_{m+1} pairwise with (rectifiable) paths γ_m in W_m. Uniting the paths γ_m, $m=1,2,\ldots$, and joining x_0 in the end, we obtain the desired (locally rectifiable) path to the point x_0 from G. □

A family of paths Γ_1 in T is said to be **minorized** by a family of paths Γ_2 in T, abbr. $\Gamma_1>\Gamma_2$, if, for every path $\gamma_1\in\Gamma_1$, there is a path $\gamma_2\in\Gamma_2$ such that γ_2 is a restriction of γ_1.

Proposition 13.3. *Let Ω be an open set in a topological space T. Then*

$$\Delta(\Omega,T\setminus\Omega,T)>\Delta(\Omega,\partial\Omega,\Omega).$$

Proof. Indeed, for an arbitrary path $\gamma:[a,b]\to T$ with $\gamma(a)\in\Omega$ and $\gamma(b)\in T\setminus\Omega$, by the continuity of γ, the preimage $\omega=\gamma^{-1}(\Omega)$ is an open set in $[a,b]$ including the point a. Similarly, the preimage $\omega = \gamma^{-1}(T \setminus \overline{\Omega})$ is also open in $[a,b]$. Thus, in view of the connectivity of the segment $[a,b]$, there is $c \in \gamma^{-1}(\partial\Omega)$ such that $\gamma([a,c)) \subset \Omega$. □

Proposition 13.4. *Let γ be a rectifiable path in a metric space (X,d) connecting points $x_1 \in \overline{B(x_0,r_1)}$ and $x_2 \in X \setminus B(x_0,r_2)$, where $0 < r_1 < r_2 < \infty$, and let $\rho : [0,\infty] \to [0,\infty]$ be a Borel function. Then*

$$\int_{\gamma} \rho(d(x,x_0))\, ds \geq \int_{r_1}^{r_2} \rho(r)\, dr.$$

Proof. Indeed, by the definition of the length of a path in a metric space $\gamma : [a,b] \to X$, the length of a segment of the path

$$s(t_1,t_2) \geq d(\gamma(t_1),\gamma(t_2))\,.$$

Moreover, by the triangle inequality,

$$d(x_0,\gamma(t_2)) \leq d(x_0,\gamma(t_1)) + d(\gamma(t_1),\gamma(t_2))$$

and

$$d(x_0,\gamma(t_1)) \leq d(x_0,\gamma(t_2)) + d(\gamma(t_1),\gamma(t_2))\,;$$

thus,

$$d(\gamma(t_1),\gamma(t_2)) \geq |d(x_0,\gamma(t_2)) - d(x_0,\gamma(t_1))|\,.$$

Consequently,

$$ds \geq |dr|\,,$$

where $r = d(x,x_0)$, $x = x(s)$. Finally, by the Darboux property of connected sets, the continuous function $d(x,x_0)$ takes all intermediate values on γ; see, e.g., [186]. Hence, the multiplicity of any value r in the interval (r_1,r_2) of the path is not less than 1 and the desired inequality follows. □

Proposition 13.5. *If Ω and Ω' are open sets in metric spaces (X,d) and (X',d'), respectively, and $f : \Omega \to \Omega'$ is a homeomorphism, then the cluster set of f at every point $x_0 \in \partial\Omega$,*

$$C(x_0,f) := \{\, x' \in X' : x' = \lim_{n\to\infty} f(x_n),\ x_n \to x_0,\ x_n \in \Omega \,\}\,,$$

belongs to the boundary of the set Ω'.

Proof. Indeed, assume that some point $y_0 \in C(x_0,f)$ is inside the domain Ω'. Then, by the definition of the cluster set, there is a sequence $x_n \to x_0$ as $n \to \infty$ such that $y_n = f(x_n) \to y_0$. In view of the continuity of the inverse mapping $g = f^{-1}$, we have $x_n = g(y_n) \to g(y_0) = x_* \in \Omega$. However, the convergent sequence x_n cannot have two limits $x_0 \in \partial\Omega$ and $x_* \in \Omega$ in view of the triangle inequality $d(x_*,x_0) \leq d(x_*,x_n) + d(x_n,x_0)$. □

13.3 On Weakly Flat and Strongly Accessible Boundaries

In this section G is a domain of a finite Hausdorff dimension $\alpha \geq 1$ in a space (X,d,μ) with a metric d and a locally finite Borel measure μ.

We say that the boundary of G is **weakly flat at a point** $x_0 \in \partial G$ if, for every number $P > 0$ and every neighborhood U of the point x_0, there is a neighborhood $V \subset U$ such that

$$M(\Delta(E,F;G)) \geq P \tag{13.8}$$

for all continua E and F in G intersecting ∂U and ∂V.

We also say that the boundary of the domain G is **strongly accessible at a point** $x_0 \in \partial G$, if, for every neighborhood U of the point x_0, there are a compact set $E \subset G$, a neighborhood $V \subset U$ of the point x_0, and a number $\delta > 0$ such that

$$M(\Delta(E,F;G)) \geq \delta$$

for every continuum F in G intersecting ∂U and ∂V.

Finally, we say that the boundary ∂G is **weakly flat** and **strongly accessible** if the corresponding properties hold at every point of the boundary.

Remark 13.2. In the definitions of the weakly flat and strongly accessible boundaries, we can restrict ourselves by a base of neighborhoods of a point x_0 and, in particular, we can take as the neighborhoods U and V of the point x_0 only small enough balls (open or closed) centered at the point x_0. Moreover, here we may restrict ourselves only by continua E and F in $\overline{U}$.

Proposition 13.6. *If the boundary ∂G is weakly flat at a point $x_0 \in \partial G$, then ∂G is strongly accessible at the point x_0.*

Proof. Let $P \in (0,\infty)$ and $U = B(x_0,r_0)$, where $0 < r_0 < d_0 = \sup_{x\in G} d(x,x_0)$. Then, by the condition, there is $r \in (0,r_0)$ such that inequality (13.8) holds for all continua E and F intersecting $\partial B(x_0,r_0)$ and $\partial B(x_0,r)$. By the path connectedness of G, there exist points $y_1 \in G \cap \partial B(x_0,r_0)$ and $y_2 \in G \cap \partial B(x_0,r)$. Choose as a compactum E an arbitrary path connecting the points y_1 and y_2 in G.

Then, for every continuum F in G intersecting ∂U and ∂V where $V = B(x_0,r)$, inequality (13.8) holds. □

Lemma 13.1. *Let G be a (rectifiable) domain in (X,d,μ). If ∂G is weakly flat at a point $x_0 \in \partial G$, then G is locally path connected (rectifiable) at x_0.*

Proof. Let us assume that the domain G is not locally (rectifiable) path connected at the point x_0. Then there are $r_0 \in (0,d_0)$, $d_0 = \sup_{x\in G} d(x,x_0)$ such that $\mu_0 := \mu(G \cap B(x_0,r_0)) < \infty$, and, for every neighborhood $V \subseteq U := B(x_0,r_0)$ of the point x_0, at least one of the following conditions holds:

1. $V \cap G$ has at least two (rectifiable) path-connected components K_1 and K_2 such that $x_0 \in \overline{K_1} \cap \overline{K_2}$;

2. $V \cap G$ has infinitely many (rectifiable) path-connected components $K_1, \ldots, K_m, \ldots$, such that $x_0 = \lim_{m\to\infty} x_m$ for some $x_m \in K_m$ and $x_0 \notin \overline{K_m}$ for all $m = 1, 2, \ldots$. Note that $\overline{K_m} \cap \partial V \neq \emptyset$ for all $m = 1, 2, \ldots$ in view of the path connectedness of G; see Proposition 13.3.

In particular, either point 1 or 2 holds for the neighborhood $V = U = B(x_0, r_0)$. Let $r_* \in (0, r_0)$. Then

$$M(\Delta(K_i^*, K_j^*; G)) \leq M_0 := \frac{\mu_0}{[2(r_0 - r_*)]^{\alpha}} < \infty,$$

where $K_i^* = K_i \cap \overline{B(x_0, r_*)}$ and $K_j^* = K_j \cap \overline{B(x_0, r_*)}$ for all $i \neq j$. Indeed, one of the admissible functions for the family Γ_{ij} of all rectifiable paths in $\Delta(K_i^*, K_j^*; G)$ is

$$\rho(x) = \begin{cases} \frac{1}{2(r_0 - r_*)} & \text{if } x \in B_0 \setminus \overline{B_*}, \\ 0 & \text{if } x \in X \setminus (B_0 \setminus \overline{B_*}), \end{cases}$$

where $B_0 = B(x_0, r_0)$ and $B_* = B(x_0, r_*)$ because the components K_i and K_j cannot be connected by a (rectifiable) path in $V = B(x_0, r_0)$ and every (rectifiable) path connecting K_i^* and K_j^* in G at least twice intersects the ring $B_0 \setminus \overline{B_*}$; see Proposition 13.4.

In view point 1 and 2, the above modulus estimate contradicts the condition of the weak flatness at the point x_0. Really, by the condition, for instance, there is $r \in (0, r_*)$ such that

$$M(\Delta(K_{i_0}^*, K_{j_0}^*; G)) \geq M_0 + 1$$

for every large enough pair i_0 and j_0, $i_0 \neq j_0$, because in the corresponding $K_{i_0}^*$ and $K_{j_0}^*$ with $d(x_0, x_{i_0})$ and $d(x_0, x_{j_0}) < r$, there exist paths intersecting $\partial B(x_0, r_*)$ and $\partial B(x_0, r)$; see Proposition 13.3.

Thus, the above assumption on the absence of the (rectifiable) path connectedness of G at the point x_0 was not true. □

Corollary 13.2. *A (rectifiable) domain with a weakly flat boundary is locally (rectifiable) path connected at every point of its boundary.*

13.4 On Finite Mean Oscillation With Respect to Measure

Let G be a domain in a space (X, d, μ). Similarly to [127] (cf. also [110]), we say that a function $\varphi : G \to \mathbb{R}$ has **finite mean oscillation at a point** $x_0 \in \overline{G}$, abbr. $\varphi \in \mathrm{FMO}(x_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \;\fint_{G(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, d\mu(x) < \infty, \tag{13.9}$$

where

$$\overline{\varphi}_\varepsilon = \fint_{G(x_0,\varepsilon)} \varphi(x) \, d\mu(x) = \frac{1}{\mu(G(x_0,\varepsilon))} \int\limits_{G(x_0,\varepsilon)} \varphi(x) \, d\mu(x)$$

is the mean value of the function $\varphi(x)$ over the set

$$G(x_0,\varepsilon) = \{x \in G : d(x,x_0) < \varepsilon\}$$

with respect to the measure μ. Here condition (13.9) includes the assumption that φ is integrable with respect to the measure μ over the set $G(x_0,\varepsilon)$ for some $\varepsilon > 0$.

Proposition 13.7. *If, for some numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{G(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \; d\mu(x) < \infty, \tag{13.10}$$

then $\varphi \in \mathrm{FMO}(x_0)$.

Proof. Indeed, by the triangle inequality,

$$\begin{aligned}
\fint_{G(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \; d\mu(x) &\le \fint_{G(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \; d\mu(x) + |\varphi_\varepsilon - \overline{\varphi}_\varepsilon| \\
&\le 2 \cdot \fint_{G(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \; d\mu(x).
\end{aligned}$$

□

Corollary 13.3. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{G(x_0,\varepsilon)} |\varphi(x)| \; d\mu(x) < \infty, \tag{13.11}$$

then $\varphi \in \mathrm{FMO}(x_0)$.

Variants of the following lemma were first proved for the BMO functions and inner points of a domain G in $\mathbb{R}^n$ under $n = 2$ and $n \ge 3$, respectively, in [271]–[275] and [204]– [209], and then for boundary points of G in $\mathbb{R}^n$, $n \ge 2$, with the condition on doubling of a measure and for the FMO functions in [127] and [276]–[280].

Lemma 13.2. *Let G be a domain in a space (X,d,μ) that is upper α-regular with $\alpha \ge 2$ at a point $x_0 \in \overline{G}$ and*

$$\mu(G \cap B(x_0,2r)) \le \gamma \cdot \log^{\alpha-2}\frac{1}{r} \cdot \mu(G \cap B(x_0,r)) \tag{13.12}$$

for all $r \in (0,r_0)$. Then, for every nonnegative function $\varphi : G \to \mathbb{R}$ of the class $\mathrm{FMO}(x_0)$,

$$\int\limits_{G\cap A(\varepsilon,\varepsilon_0)} \frac{\varphi(x)\, d\mu(x)}{\left(d(x,x_0)\log\frac{1}{d(x,x_0)}\right)^{\alpha}} = O\left(\log\log\frac{1}{\varepsilon}\right) \tag{13.13}$$

as $\varepsilon \to 0$ and some $\varepsilon_0 \in (0,\delta_0)$, where $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup_{x\in G} d(x,x_0)$,

$$A(\varepsilon,\varepsilon_0) = \{x \in X : \varepsilon < d(x,x_0) < \varepsilon_0\}.$$

Proof. Choose $\varepsilon_0 \in (0,\delta_0)$ such that the function φ is integrable in $G_0 = G \cap B_0$ with respect to the measure μ, where $B_0 = B(x_0,\varepsilon_0)$,

$$\delta = \sup_{r\in(0,\varepsilon_0)} \fint_{G(r)} |\varphi(x) - \overline{\varphi}_r|\, d\mu(x) \ < \infty,$$

$G(r) = G \cap B(r)$, and $B(r) = B(x_0,r) = \{x \in X : d(x,x_0) < r\}$. Further, let $\varepsilon < 2^{-1}\varepsilon_0$, $\varepsilon_k = 2^{-k}\varepsilon_0$, $A_k = \{x \in X : \varepsilon_{k+1} \le d(x,x_0) < \varepsilon_k\}$, $B_k = B(\varepsilon_k)$, and let φ_k be the mean value of the function $\varphi(x)$ in $G_k = G \cap B_k, k = 0,1,2\ldots$, with respect to the measure μ. Choose a natural number N such that $\varepsilon \in [\varepsilon_{N+1},\varepsilon_N)$ and denote $\varkappa(t) = (t\log_2 1/t)^{-\alpha}$. Then $G\cap A(\varepsilon,\varepsilon_0) \subset \Delta(\varepsilon) := \bigcup_{k=0}^{N} \Delta_k$, where $\Delta_k = G\cap A_k$ and

$$\eta(\varepsilon) = \int\limits_{\Delta(\varepsilon)} \varphi(x)\ \varkappa(d(x,x_0))\, d\mu(x) \ \le\ |S_1| \ +\ S_2,$$

$$S_1(\varepsilon) = \sum_{k=1}^{N} \int\limits_{\Delta_k} (\varphi(x) - \varphi_k)\ \varkappa(d(x,x_0))\, d\mu(x),$$

$$S_2(\varepsilon) = \sum_{k=1}^{N} \varphi_k \int\limits_{\Delta_k} \varkappa(d(x,x_0))\, d\mu(x).$$

Since $G_k \subset G(2d(x,x_0))$ for $x \in \Delta_k$, then, by condition (13.7), $\mu(G_k) \le \mu(G(2d(x,x_0))) \le C\cdot 2^\alpha \cdot d(x,x_0)^\alpha$, i.e. $1/d(x,x_0)^\alpha \le C\cdot 2^\alpha (1/\mu(G_k))$.

Moreover, $k^\alpha \le (\log_2(1/d(x,x_0))^\alpha$ for $x \in \Delta_k$ and, thus,

$$|S_1| \ \le\ \delta C\cdot 2^\alpha \sum_{k=1}^{N} \frac{1}{k^\alpha} \le\ 2\delta C\cdot 2^\alpha$$

because under $\alpha \ge 2$,

$$\sum_{k=2}^{\infty} \frac{1}{k^\alpha} < \int_1^\infty \frac{dt}{t^\alpha} = \frac{1}{\alpha-1} \le 1.$$

Further,

$$\begin{aligned}\int\limits_{\Delta_k} \varkappa(d(x,x_0))\, d\mu(x) \ &\le\ \frac{1}{k^\alpha}\int\limits_{A_k} \frac{d\mu(x)}{d(x,x_0)^\alpha} \\ &\le\ \frac{C\cdot 2^\alpha}{k^\alpha}\cdot \frac{\mu(G_k) - \mu(G_{k+1})}{\mu(G_k)} \le \frac{C2^\alpha}{k^\alpha}.\end{aligned}$$

Moreover, by condition (13.12),

$$\mu(G_{k-1}) = \mu(B(2\varepsilon_k)\cap G)) \le \gamma\cdot \log_2^{\alpha-2}\frac{1}{\varepsilon_k}\cdot \mu(G_k)$$

and, hence,

$$\begin{aligned}|\varphi_k - \varphi_{k-1}| \;&=\; \frac{1}{\mu(G_k)}\left|\int\limits_{G_k}(\varphi(x)-\varphi_{k-1})\,d\mu(x)\right| \\ &\le\; \frac{\gamma\cdot\log_2^{\alpha-2}\frac{1}{\varepsilon_k}}{\mu(G_{k-1})}\int\limits_{G_{k-1}}|(\varphi(x)-\varphi_{k-1})|\,d\mu(x) \;\le\; \delta\cdot\gamma\cdot\log_2^{\alpha-2}\frac{1}{\varepsilon_k}\end{aligned}$$

and, by decreasing ε_k,

$$\varphi_k \;=\; |\varphi_k| \;\le\; \varphi_1 \;+\; \sum_{l=1}^{k}|\varphi_l - \varphi_{l-1}| \;\le\; \varphi_1 \;+ k\delta\gamma\cdot\log_2^{\alpha-2}\frac{1}{\varepsilon_k}.$$

Consequently, because under $\alpha \ge 2$

$$\sum_{k=1}^{\infty}\frac{1}{k^\alpha} \le 1+\int\limits_1^{\infty}\frac{dt}{t^\alpha} \;=\; 1+\frac{1}{\alpha-1} \;\le\; 2\,,$$

we have the following estimates:

$$\begin{aligned}S_2 \;=\; |S_2| \;&\le\; C2^\alpha\sum_{k=1}^{N}\frac{\varphi_k}{k^\alpha} \le\; C2^\alpha\sum_{k=1}^{N}\frac{\varphi_1 \;+k\delta\gamma\cdot\log_2^{\alpha-2}\frac{1}{\varepsilon_k}}{k^\alpha} \\ &\le C2^\alpha\left(2\varphi_1+\delta\gamma\sum_{k=1}^{N}\frac{(k+\log_2\varepsilon_0^{-1})^{\alpha-2}}{k^{\alpha-1}}\right) \\ &= C2^\alpha\left(2\varphi_1+\delta\gamma\sum_{k=1}^{N}\frac{1}{k}\frac{(k+\log_2\varepsilon_0^{-1})^{\alpha-2}}{k^{\alpha-2}}\right) \\ &\le C2^\alpha\left(2\varphi_1+\delta\gamma(1+\log_2\varepsilon_0^{-1})^{\alpha-2}\sum_{k=1}^{N}\frac{1}{k}\right)\end{aligned}$$

and

$$\eta(\varepsilon) \le 2^{\alpha+1}C(\delta+\varphi_1)+2^\alpha C\delta\gamma(1+\log_2\varepsilon_0^{-1})^{\alpha-2}\sum_{k=1}^{N}\frac{1}{k}\,.$$

Since

$$\sum_{k=2}^{N}\frac{1}{k} < \int\limits_1^{N}\frac{dt}{t} = \log N < \log_2 N$$

and, for $\varepsilon_0 \in (0, 2^{-1})$ and $\varepsilon < \varepsilon_N$,

$$N \;<\; N \;+\; \log_2\left(\frac{1}{\varepsilon_0}\right) \;=\; \log_2\left(\frac{1}{\varepsilon_N}\right) \;<\; \log_2\left(\frac{1}{\varepsilon}\right),$$

then under $\varepsilon_0 \in (0,\delta_0)$, $\delta_0 = \min(e^{-e}, d_0)$, and $\varepsilon \to 0$,

$$\eta(\varepsilon) \le 2^{\alpha+1}C(\delta+\varphi_1) + 2^{\alpha}C\delta\gamma(1+\log_2 \varepsilon_0^{-1})^{\alpha-2}\left(1+\log_2\log_2\frac{1}{\varepsilon}\right)$$
$$= O\left(\log\log\frac{1}{\varepsilon}\right).$$

$\square$

Remark 13.3. Note that condition (13.12) is weaker than the condition on doubling of a measure,

$$\mu(G\cap B(x_0,2r)) \le \gamma\cdot\mu(G\cap B(x_0,r)) \qquad \forall\, r\in(0,r_0) \tag{13.14}$$

applied before it in the context of $\mathbb{R}^n, n\ge 2$, in [127]. Note also that condition (13.14) automatically holds in the inner points of the domain G if X is Ahlfors regular.

13.5 On Continuous Extension to Boundaries

In what follows, (X,d,μ) and (X',d',μ') are spaces with metrics d and d' and locally finite Borel measures μ and μ', and G and G' domains with finite Hausdorff dimensions α and $\alpha'\ge 1$ in (X,d) and (X',d'), respectively.

Lemma 13.3. *Let a domain G be locally path connected at a point $x_0\in\partial G$, let $\overline{G'}$ be compact, and let $f: G\to G'$ be a Q-homeomorphism such that $\partial G'$ is strongly accessible at least at one point of the cluster set*

$$C(x_0,f) = \{y\in X' :\ y = \lim_{k\to\infty} f(x_k),\ x_k\to x_0,\ x_k\in G\}, \tag{13.15}$$

$Q: G\to[0,\infty]$ is a measurable function satisfying the condition

$$\int\limits_{G(x_0,\varepsilon,\varepsilon_0)} Q(x)\cdot\psi^{\alpha}_{x_0,\varepsilon}(d(x,x_0))\,d\mu(x) \;=\; o(I^{\alpha}_{x_0}(\varepsilon)) \tag{13.16}$$

as $\varepsilon\to 0$, where

$$G(x_0,\varepsilon,\varepsilon_0) = \{x\in G: \varepsilon < d(x,x_0) < \varepsilon(x_0)\},$$

$\varepsilon(x_0)\in(0,d(x_0))$, $d(x_0)=\sup_{x\in G} d(x,x_0)$, and $\psi_{x_0,\varepsilon}(t)$ is a family of nonnegative measurable (by Lebesgue) functions on $(0,\infty)$ such that

$$0 < I_{x_0}(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{x_0,\varepsilon}(t)\,dt < \infty \qquad \forall\,\varepsilon\in(0,\varepsilon_0)\,. \tag{13.17}$$

Then f can be extended to the point x_0 by continuity in (X', d').

Proof. Let us show that the cluster set $E = C(x_0, f)$ is a singleton. Note that $E \neq \emptyset$ in view of the compactness of $\overline{G'}$; see, e.g., Remark 3 of Chapter 41 in [186]. By the condition of the lemma, $\partial G'$ is strongly accessible at a point $y_0 \in E$. Assume that there is one more point $y^* \in E$. Let $U = B(y_0, r_0)$, where $0 < r_0 < d(y_0, y^*)$.

In view of the local path connectedness of the domain G at the point x_0, there is a sequence of neighborhoods V_m of the point x_0 such that $G_m = G \cap V_m$ are domains and $d(V_m) \to 0$ as $m \to \infty$. Then there exist points y_m and $y_m^* \in F_m$ that are close enough to y_0 and y^*, respectively, for which $d'(y_0, y_m) < r_0$ and $d'(y_0, y_m^*) > r_0$ and that can be joined by paths C_m in the domains $F_m = fG_m$. By the construction,

$$C_m \cap \partial B(y_0, r_0) \neq \emptyset$$

in view of the connectedness of C_m.

By the condition of the strong accessibility, there are a compact set $C \subset G'$ and a number $\delta > 0$ such that

$$M(\Delta(C, C_m; G')) \geq \delta$$

for large m because $\operatorname{dist}(y_0, C_m) \to 0$ as $m \to \infty$. Note that $K = f^{-1}(C)$ is compact as a continuous image of a compact set. Thus, $\varepsilon_0 = \operatorname{dist}(x_0, K) > 0$.

Let Γ_ε be the family of all paths in G connecting the ball $B_\varepsilon = \{x \in X : d(x, x_0) < \varepsilon\}$, $\varepsilon \in (0, \varepsilon_0)$, with the compactum K. Let $\psi^*_{x_0,\varepsilon}$ be a Borel function such that $\psi^*_{x_0,\varepsilon}(t) = \psi_{x_0,\varepsilon}(t)$ for a.e. $t \in (0, \infty)$ that exists by the Lusin theorem; see, e.g., point 2.3.5 in [55].

Then the function

$$\rho_\varepsilon(x) = \begin{cases} \psi^*_{x_0,\varepsilon}(d(x, x_0))/I_{x_0}(\varepsilon), & x \in G(x_0, \varepsilon), \\ 0, & x \in X \backslash G(x_0, \varepsilon) \end{cases}$$

is admissible for Γ_ε by Proposition 13.4 and, consequently,

$$M(f\Gamma_\varepsilon) \leq \int\limits_G Q(x) \cdot \rho_\varepsilon^\alpha(x)\, d\mu(x).$$

Hence, $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (13.16).

On the other hand, for any $\varepsilon \in (0, \varepsilon_0)$, $G_m \subset B_\varepsilon$ for large m, hence $C_m \subset fB_\varepsilon$ for such m, and, thus,

$$M(f\Gamma_\varepsilon) \geq M(\Delta(C, C_m; G')).$$

The obtained contradiction disproves the above assumption that the cluster set E is not degenerated to a point. □

Corollary 13.4. *In particular, if*

$$\overline{\lim_{\varepsilon \to 0}} \int\limits_{\varepsilon < d(x, x_0) < \varepsilon_0} Q(x) \cdot \psi^\alpha(d(x, x_0))\, d\mu(x) < \infty, \tag{13.18}$$

where $\psi(t)$ *is a measurable function on* $(0,\infty)$ *such that*

$$0 < I(\varepsilon,\varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt \;<\; \infty \qquad \forall\, \varepsilon \in (0,\, \varepsilon_0)$$

and $I(\varepsilon,\varepsilon_0) \to \infty$ *as* $\varepsilon \to 0$*, then any* Q*-homeomorphism* $f: G \to G'$ *can be extended to the point* x_0 *by continuity in* (X',d')*.*

Here, we assume that the function Q is extended by zero outside G.

Remark 13.4. In other words, it suffices for the singular integral (13.18) to be convergent in the sense of the principal value at the point x_0 at least for one kernel ψ with a nonintegrable singularity at zero. Furthermore, as the lemma shows, it is even sufficient for the given integral to be divergent but with the controlled speed

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\psi^{\alpha}(d(x,x_0))\,d\mu(x) \;=\; o(I^{\alpha}(\varepsilon,\varepsilon_0)). \tag{13.19}$$

Choosing in Lemma 13.3 $\psi(t) \equiv 1/t$, we obtain the following theorem.

Theorem 13.1. *Let* G *be locally path connected at a point* $x_0 \in \partial G$*,* $\overline{G'}$ *compact, and* $\partial G'$ *strongly accessible. If a measurable function* $Q: G \to [0,\infty]$ *satisfies the condition*

$$\int\limits_{G(x_0,\varepsilon,\varepsilon_0)} \frac{Q(x)d\mu(x)}{d(x,x_0)^{\alpha}} = o\left(\left[\log\frac{1}{\varepsilon}\right]^{\alpha}\right) \tag{13.20}$$

as $\varepsilon \to 0$*, where* $G(x_0,\varepsilon,\varepsilon_0) = \{x \in G : \varepsilon < d(x,x_0) < \varepsilon_0\}$ *for* $\varepsilon_0 < d(x_0) = \sup_{x\in G} d(x,x_0)$*, then any* Q*-homeomorphism* $f: G \to G'$ *can be extended to* x_0 *by continuity in* (X',d')*.*

Corollary 13.5. *In particular, the conclusion of Theorem 13.1 is valid if the singular integral*

$$\int \frac{Q(x)d\mu(x)}{d(x,x_0)^{\alpha}} \tag{13.21}$$

is convergent at the point x_0 *in the sense of the principal value.*

Here, as in Corollary 13.4, we assume that Q is extended by zero outside G.

Combining Lemmas 13.2 and 13.3, choosing $\psi_{\varepsilon}(t) \equiv t\log(1/t)$, $t \in (0,\delta_0)$, we obtain the following theorem.

Theorem 13.2. *Let* X *be upper* α*-regular with* $\alpha \geq 2$ *at a point* $x_0 \in \partial G$*, where* G *is locally path connected and satisfies condition (13.12), and let* $\overline{G'}$ *be compact and* $\partial G'$ *strongly accessible. If* $Q \in \mathrm{FMO}(x_0)$*, then any* Q*-homeomorphism* $f: G \to G'$ *can be extended to the point* x_0 *by continuity in* (X',d')*.*

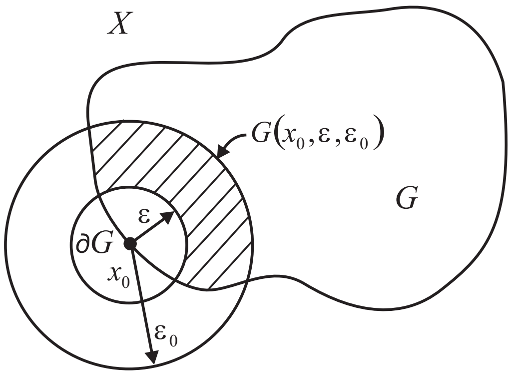

Figure 8

Finally, combining Theorem 13.2 and Corollary 13.3, we obtain the following statement.

Corollary 13.6. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{G(x_0,\varepsilon)} Q(x)\, d\mu(x) < \infty, \tag{13.22}$$

where $G(x_0,\varepsilon) = \{x \in G : d(x,x_0) < \varepsilon\}$*, then any* Q*-homeomorphism* $f : G \to G'$ *can be extended to the point* x_0 *by continuity in* (X',d')*.*

13.6 On Extending Inverse Mappings to Boundaries

As before, $C(x_0,f)$ denotes have the cluster set of the mapping f at a point $x_0 \in \partial G$; see (13.15).

Lemma 13.4. *Let* $f : G \to G'$ *be a* Q*-homeomorphism with* $Q \in L^1_\mu(G)$*. If the domain* G *is locally path connected at points* x_1 *and* $x_2 \in \partial G$*,* $x_1 \neq x_2$*, and* G' *has a weakly flat boundary, then* $C(x_1,f) \cap C(x_2,f) = \emptyset$*.*

Proof. Set $E_i = C(x_i,f)$, $i = 1,2$, and $\delta = d(x_1,x_2)$. Let us assume that $E_1 \cap E_2 \neq \emptyset$.

Since the domain G is locally path connected at the points x_1 and x_2, there exist neighborhoods U_1 and U_2 of the points x_1 and x_2, respectively, such that $W_1 = G \cap U_1$ and $W_2 = G \cap U_2$ are domains and $U_1 \subset B_1 = B(x_1,\delta/3)$ and $U_2 \subset B_2 = B(x_2,\delta/3)$. Then, by the triangle inequality, $\operatorname{dist}(W_1,W_2) \geq \delta/3$ and the function

$$\rho(x) = \begin{cases} \frac{3}{\delta} & \text{if } x \in G, \\ 0 & \text{if } x \in X \setminus G \end{cases}$$

is admissible for the path family $\Gamma = \Delta(W_1, W_2; G)$. Thus,

$$M(f\Gamma) \leq \int\limits_X Q(x)\,\rho^\alpha(x)\,d\mu(x) \leq \frac{3^\alpha}{\delta^\alpha}\int\limits_G Q(x)\,d\mu(x) < \infty$$

because $Q \in L^1_\mu(G)$.

The last estimate contradicts, however, the condition of the weak flatness (13.8) if there is a point $y_0 \in E_1 \cap E_2$. Indeed, then $y_0 \in \overline{fW_1} \cap \overline{fW_2}$ and in the domains $W_1^* = fW_1$ and $W_2^* = fW_2$ there exist paths intersecting any prescribed spheres $\partial B(y_0, r_0)$ and $\partial B(y_0, r_*)$ with small enough radii r_0 and r_*; see Proposition 13.3. Hence, the assumption that $E_1 \cap E_2 \neq \emptyset$ was not true. $\Box$

By Lemma 13.4, we obtain, in particular, the following conclusion.

Theorem 13.3. *Let G be locally path connected at all its boundary points and $\overline{G}$ compact, G' with a weakly flat boundary, and let $f: G \to G'$ be a Q-homeomorphism with $Q \in L^1_\mu(G)$. Then the inverse homeomorphism $g = f^{-1}: G' \to G$ admits a continuous extension $\overline{g}: \overline{G'} \to \overline{G}$.*

Remark 13.5. In fact, as is clear from the above proof (see also Proposition 13.5), it is sufficient in Lemma 13.4 and Theorem 13.3 as well as in all successive theorems to request instead of the condition $Q \in L^1_\mu(G)$ the integrability of Q in a neighborhood of ∂G assuming Q to be extended by zero outside G.

13.7 On Homeomorphic Extension to Boundaries

Combining the results of the previous sections, we obtain the following theorems.

Lemma 13.5. *Let G be locally path connected at its boundary, let G' have a weakly flat boundary, and $\overline{G}$, $\overline{G'}$ be compact. If a function $Q: G \to [0, \infty]$ of the class $L^1_\mu(G)$ satisfies condition (13.16) at every point $x_0 \in \partial G$, then any Q-homeomorphism $f: G \to G'$ is extended to a homeomorphism $\overline{f}: \overline{G} \to \overline{G'}$.*

Theorem 13.4. *Let G and G' have weakly flat boundaries, let $\overline{G}$ and $\overline{G'}$ be compact, and let $Q: G \to [0, \infty]$ be a function of the class $L^1_\mu(G)$ with*

$$\int\limits_{G(x_0,\varepsilon,\varepsilon_0)} \frac{Q(x)d\mu(x)}{d(x,x_0)^\alpha} = o\left(\left[\log\frac{1}{\varepsilon}\right]^\alpha\right) \tag{13.23}$$

at every point $x_0 \in \partial G$, where $G(x_0, \varepsilon, \varepsilon_0) = \{x \in G : \varepsilon < d(x, x_0) < \varepsilon_0\}$, $\varepsilon_0 = \varepsilon(x_0) < d(x_0) = \sup_{x \in G} d(x, x_0)$. Then any Q-homeomorphism $f: G \to G'$ admits an extension to a homeomorphism $\overline{f}: \overline{G} \to \overline{G'}$.

Corollary 13.7. *In particular, the conclusion of Theorem 13.4 holds if the singular integral*

$$\int \frac{Q(x)d\mu(x)}{d(x,x_0)^{\alpha}} \tag{13.24}$$

is convergent in the sense of the principal value at all boundary points.

As before, we assumed here that Q has been extended by zero outside G.

Theorem 13.5. *Let G be a domain in an upper α-regular space (X,d,μ), $\alpha \geq 2$, that is locally path connected and satisfies condition (13.12) at all boundary points, let G' be a domain with a weakly flat boundary in a space (X',d',μ'), and let $\overline{G}$ and $\overline{G'}$ be compact. If a function $Q : G \to [0,\infty]$ has finite mean oscillation at all boundary points, then any Q-homeomorphism $f : G \to G'$ can be extended to a homeomorphism $\overline{f} : \overline{G} \to \overline{G'}$.*

Corollary 13.8. *In particular, the conclusion of Theorem 13.5 holds if*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{G(x_0,\varepsilon)} Q(x)\, d\mu(x) < \infty \tag{13.25}$$

at all points $x_0 \in \partial G$, where $G(x_0,\varepsilon) = \{x \in G : d(x,x_0) < \varepsilon\}$.

Remark 13.6. If conditions of the type (13.16), (13.23), (13.24), (13.25) or finiteness of the mean oscillation hold only on a closed set $E \subset \partial G$, Q, extended by zero outside the domain G, is integrable in a neighborhood of E, $\overline{G}$ and $\overline{G'}$ are compact, G is locally connected at every point of E, and $\partial G'$ is weakly flat at all points of the cluster set

$$E' = C(E,f) = \{x' \in X' : \ x' = \lim_{k\to\infty} f(x_k),\ x_k \in G,\ x_k \to x_0 \in E\},$$

then the Q-homeomorphism $f : G \to G'$ admits a homeomorphic extension $\overline{f} : G \cup E \to G' \cup E'$.

13.8 On Moduli of Families of Paths Passing Through Point

In this section we establish conditions on a measure μ under which the modulus of a family of all paths in a space (X,d,μ) passing through a fixed point is zero.

Lemma 13.6. *Let the condition*

$$\int\limits_{A(x_0,r,R_0)} \psi^{\alpha}(d(x,x_0))\, d\mu(x) = o\left(\left[\int\limits_r^{R_0} \psi(t)\, dt\right]^{\alpha}\right) \tag{13.26}$$

hold as $r \to 0$, where

$$A(x_0,r,R_0) = \{x \in X : r < d(x,x_0) < R_0\}, \qquad R_0 \in (0,\infty),$$

and let $\psi(t)$ *be a nonnegative function on* $(0,\infty)$ *such that*

$$0 < \int\limits_r^{R_0} \psi(t)\, dt < \infty \qquad \forall\, r \in (0,R_0)\,.$$

Then the family of all paths in X *passing through the point* x_0 *has modulus zero.*

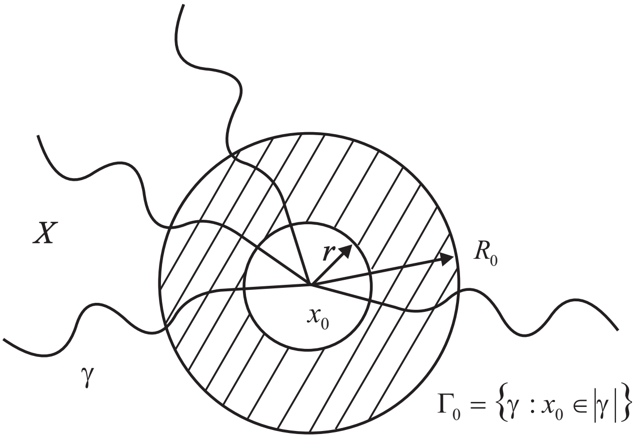

Figure 9

Remark 13.7. Condition (13.26) implies that under $r \to 0$,

$$\int\limits_{A(x_0,r,r_0)} \psi^{\alpha}(d(x,x_0))\, d\mu(x) = o\left(\left[\int\limits_r^{r_0} \psi(t)\, dt\right]^{\alpha}\right) \tag{13.27}$$

for all $r_0 \in (0,R_0)$.

Proof of Lemma 13.6. Let Γ be the family of all paths in X passing through the point x_0. Then all paths in X passing through the point x_0. Then $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$, where the Γ_k are the families of all paths in X passing through x_0 and intersecting the spheres $S_k = S(x_0, r_k)$ for some sequence such that $r_k \in (0,R_0)\,, r_k \to 0$, as $k \to \infty$.

However, $M(\Gamma_k) = 0$. Indeed, the function

$$\rho(x) = \begin{cases} \psi(d(x,x_0)) \left(\int\limits_r^{r_k} \psi(t)\, dt\right)^{-1} & \text{if } x \in A_k(r), \\ 0 & \text{if } x \in X \backslash A_k(r), \end{cases}$$

where $A_k(r) = A(x_0, r, r_k)$, is admissible for the family $\Gamma_k(r)$ of all paths intersecting the spheres S_k and $S(x_0, r)$, $r \in (0, r_k)$; see Proposition 13.4. Since $\Gamma_k > \Gamma_k(r)$, then

$$M(\Gamma_k) \leq M(\Gamma_k(r)) \leq \left(\int\limits_r^{r_k} \psi(t)\, dt \right)^{-\alpha} \int\limits_{A_k(r)} \psi^\alpha(d(x, x_0))\, d\mu(x)$$

and by condition (13.26) [cf. also (13.27)], it follows that $M(\Gamma_k) = 0$ because $r \in (0, r_k)$ is arbitrary.

Finally, from the subadditivity of the modulus, it follows that

$$M(\Gamma) \leq \sum_{k=1}^{\infty} M(\Gamma_k) = 0.$$

$\square$

Theorem 13.6. *For some $R_0 \in (0, \infty)$, under $r \to 0$, let*

$$\int\limits_{A(x_0, r, R_0)} \frac{d\mu(x)}{d^\alpha(x, x_0)} = o\left(\left[\log \frac{R_0}{r} \right]^\alpha \right). \tag{13.28}$$

Then the family of all paths in X passing through point x_0 has the modulus zero.

Remark 13.8. For $X = \mathbb{R}^n$, $n \geq 2$, and $R_0 \in (0, \infty)$,

$$\int\limits_{A(x_0, r, R_0)} \frac{dm(x)}{|x - x_0|^n} = \omega_{n-1} \log\left(\frac{R_0}{r} \right) = o\left(\left[\log \frac{R_0}{r} \right]^n \right), \tag{13.29}$$

where m denotes the Lebesgue measure and ω_{n-1} the area of the unit sphere in $\mathbb{R}^n$.

For spaces (X, d, μ) that are upper α-regular at the point x_0 with $\alpha > 1$,

$$\int\limits_{r < d(x_0, x) < R_0} \frac{d\mu(x)}{d(x, x_0)^\alpha} = O\left(\log \frac{R_0}{r} \right) \tag{13.30}$$

(see [107], cf. 54), and, thus, condition (13.28) also automatically holds in such spaces.

13.9 On Weakly Flat Spaces

Recall that a topological space T is said to be **locally (path) connected at a point** $x_0 \in T$ if, for every neighborhood U of the point x_0, there is a neighborhood $V \subseteq U$ of the point x_0 that is (path) connected; see [186], p. 232. We say that a space T is **(path) connected at a point** x_0 if, for every neighborhood U of the point x_0, there is a neighborhood $V \subseteq U$ of the point x_0 such that $V \setminus \{x_0\}$ is (path) connected. Note that (path) connectedness of a space T at a point x_0 implies its local (path) connectedness at the point x_0. The inverse conclusion is, generally speaking, not true.

Here (X,d,μ) is a space with metric d and locally finite Borel measure μ and with a finite Hausdorff dimension $\alpha \geq 1$.

We say that the path-connected space (X,d,μ) is **weakly flat at a point** $x_0 \in X$ if, for every neighborhood U of the point x_0 and every number $P > 0$, there is a neighborhood $V \subseteq U$ of x_0 such that

$$M(\Delta(E,F;X)) \geq P \tag{13.31}$$

for any continua E and F in X intersecting ∂V and ∂U.

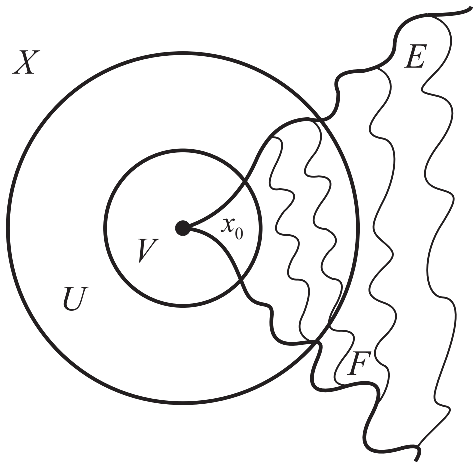

Figure 10

We also say that the path-connected space (X,d,μ) is **strongly connected at a point** $x_0 \in X$ if, for every neighborhood U of the point x_0, there are a neighborhood $V \subseteq U$ of x_0, a compact set E in X, and a number $\delta > 0$ such that

$$M(\Delta(E,F;X)) \geq \delta$$

for any continua F in X intersecting ∂V and ∂U.

Finally, we say that a space (X,d,μ) is **weakly flat** (**strongly connected**) if it is weakly flat (strongly connected) at every point.

Remark 13.9. In the definitions of weakly flat and strongly connected spaces, we may restrict ourselves by a base of neighborhoods of a point x_0 and, in particular, take as U and V only enough small balls (open or closed) centered at the point x_0.

Moreover, here we may restrict ourselves only by continua E and F in $\overline{U}$. It is also obvious that every domain in a weakly flat space is a weakly flat space.

The following statement is not so important and is proved similarly to Proposition 13.6; hence, we omit its proof here.

Proposition 13.8. *If a space (X,d,μ) is weakly flat at a point $x_0 \in X$, then X is strongly connected at the point x_0.*

In what follows, the following statement is much more important.

Lemma 13.7. *If a space (X,d,μ) is weakly flat at a point $x_0 \in X$, then (X,d,μ) is locally path connected at the point x_0.*

Proof. Let us assume that the space X is not locally path connected at the point x_0. Then there are $r_0 \in (0,d_0)$, $d_0 = \sup_{x\in X} d(x,x_0)$, such that $\mu_0 := \mu(\overline{B(x_0,r_0)}) < \infty$ and every neighborhood $V \subseteq U := \overline{B(x_0,r_0)}$ of the point x_0 has a path-connected component K_0 including x_0 and path-connected components $K_1,\dots,K_m,\dots$ that are different from K_0 such that $x_0 = \lim_{m\to\infty} x_m$ for some $x_m \in K_m$. Note that $\overline{K_m} \cap \partial V \neq \emptyset$ for all $m = 1,2,\dots$ in view of the path connectedness of X; see Proposition 13.3.

In particular, this is true for the neighborhood $V = U = \overline{B(x_0,r_0)}$. Let $r_* \in (0,r_0)$. Then, for all $i = 1,2,\dots$,

$$M(\Delta(K_i^*,K_0^*;G)) \le M_0 := \frac{\mu_0}{[2(r_0-r_*)]^\alpha} < \infty,$$

where $K_i^* = K_i \cap \overline{B(x_0,r_*)}$ and $K_0^* = K_0 \cap \overline{B(x_0,r_*)}$. Indeed, one of the admissible functions for the family Γ_i of all rectifiable curves in $\Delta(K_i^*,K_0^*;G)$ is

$$\rho(x) = \begin{cases} \frac{1}{2(r_0-r_*)}, & x \in B_0 \setminus \overline{B_*}, \\ 0, & x \in X \setminus (B_0 \setminus \overline{B_*}), \end{cases}$$

where $B_0 = B(x_0,r_0)$ and $B_* = B(x_0,r_*)$, because the components K_i and K_0 cannot be connected by a path in $V = \overline{B(x_0,r_0)}$ and every path connecting K_i^* and K_0^* at least twice intersects the ring $B_0 \setminus \overline{B_*}$,; see Proposition 13.4.

However, the above modulus estimate contradicts the condition of the weak flatness at the point x_0. Really, by this condition, for instance, there is $r \in (0,r_*)$ such that

$$M(\Delta(K_{i_0}^*,K_0^*;G)) \ge M_0 + 1$$

for every large enough $i_0 = 1,2,\dots$ because in the corresponding $K_{i_0}^*$ with $d(x_o,x_{i_0}) < r$ and K_0^* there exist paths intersecting $\partial B(x_0,r_*)$ and $\partial B(x_0,r)$; see Proposition 13.3.

Thus, the above assumption on the absence of the path connectedness of the space X at the point x_0 was not true. □

Combining Lemma 13.7 and Proposition 13.1, we obtain the following conclusion.

Corollary 13.9. *An open set Ω in a weakly flat space (X,d,μ) is path connected if and only if it is connected.*

Corollary 13.10. *A domain G in a weakly flat space (X,d,μ) is locally path connected at a point $x_0 \in \partial G$ if and only if G is locally connected at the point x_0.*

Combining Lemmas 13.6 and 13.7, we obtain the following result.

Theorem 13.7. *If a space (X,d,μ) is weakly flat at a point $x_0 \in X$ and condition (13.26), in particular, (13.28), holds, then (X,d,μ) is path connected at the point x_0.*

By Remark 13.8, we come to the following conclusion.

Corollary 13.11. *If a space X is weakly flat and upper α-regular at a point $x_0 \in X$ with $\alpha > 1$, then X is path connected at the point x_0.*

Remark 13.10. $\mathbb{R}^n$, $n \geq 2$, is a weakly flat space because

$$M(\Delta(E,F;\mathbb{R}^n)) \geq c_n \log \frac{R}{r} \tag{13.32}$$

for all continua E and F intersecting the boundaries of the balls $\mathbb{B}^n(R)$ and $\mathbb{B}^n(r)$; see, e.g., the subsection 10.12 in [316].

13.10 On Quasiextremal Distance Domains

Similarly to [81], we say that a domain G in (X,d,μ) is a **quasiextremal distance domain**, abbr. a QED **domain**, if

$$M(\Delta(E,F;X)) \leq K\, M(\Delta(E,F;G)) \tag{13.33}$$

for a finite number $K \geq 1$ and all continua E and F in G.

As is easy to see from the definitions, a QED domain G in a weakly flat space has a weakly flat boundary and, as a consequence, ∂G is strongly accessible and, moreover, G is locally path connected at all points of the boundary. Thus, all the above results on the extension of Q-homeomorphisms to the boundary hold for QED domains in weakly flat spaces. Let us review these results.

Lemma 13.8. *Let f be a Q-homeomorphism between* QED *domains G and G' in weakly flat spaces X and X', respectively, $\overline{G'}$ compact and at a point $x_0 \in \partial G$, let*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} Q(x)\, \psi^{\alpha}(d(x,x_0))\, d\mu(x) = o\left(\left[\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt\right]^{\alpha}\right) \tag{13.34}$$

as $\varepsilon \to 0$, where

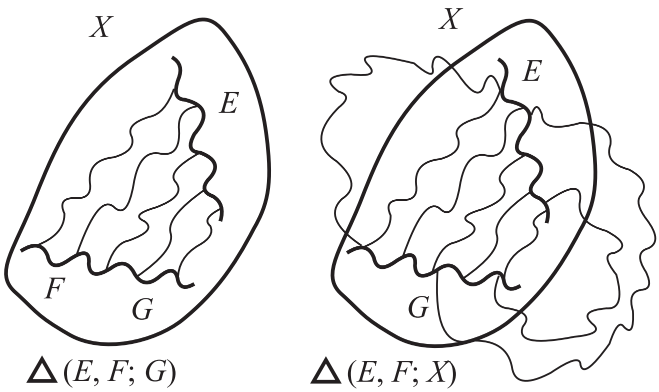

Figure 11

$$A(x_0, \varepsilon, \varepsilon_0) = \{x \in G : \varepsilon < d(x, x_0) < \varepsilon_0\},$$

and $\psi(t)$ is a nonnegative function on $(0, \infty)$ such that

$$0 < \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \qquad \forall \varepsilon \in (0, \varepsilon_0).$$

Then there is a limit of $f(x)$ as $x \to x_0$.

Corollary 13.12. *In particular, the limit of $f(x)$ as $x \to x_0$ exists if*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} Q(x)\, \psi^{\alpha}(d(x, x_0))\, d\mu(x) < \infty \tag{13.35}$$

and

$$\lim_{\varepsilon \to 0} \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt = \infty. \tag{13.36}$$

Theorem 13.8. *Let f be a Q-homeomorphism between* QED *domains G and G' in weakly flat spaces X and X', respectively, and let $\overline{G'}$ be compact. If, at a point $x_0 \in \partial G$,*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} \frac{Q(x)\, d\mu(x)}{d(x, x_0)^{\alpha}} = o\left(\left[\log \frac{\varepsilon_0}{\varepsilon}\right]^{\alpha}\right), \tag{13.37}$$

then f admits a continuous extension to the point x_0.

Corollary 13.13. *In particular, the conclusion of Theorem 13.8 holds if the singular integral*

$$\int \frac{Q(x)\, d\mu(x)}{d(x,x_0)^{\alpha}} \tag{13.38}$$

is convergent at x_0 in the sense of the principal value.

Here we assume that Q is extended by zero outside the domain G.

Lemma 13.9. *Let f be a Q-homeomorphism between* QED *domains G and G' in weakly flat spaces X and X', respectively, and let $\overline{G}$ be compact. If $Q \in L^1_\mu(G)$, then the inverse homeomorphism $g = f^{-1}$ admits a continuous extension $\overline{g} : \overline{G'} \to \overline{G}$.*

Theorem 13.9. *Let f be a Q-homeomorphism between* QED *domains G and G' in weakly flat spaces X and X' and let $\overline{G}$ and $\overline{G'}$ be compact. If $Q \in L^1_\mu(G)$ satisfies either (13.37) or (13.38) at every point $x_0 \in \partial G$, then f admits a homeomorphic extension $\overline{f} : \overline{G} \to \overline{G'}$.*

Theorem 13.10. *Let f be a Q-homeomorphism between* QED *domains G and G' in weakly flat spaces X and X', respectively, and let $\overline{G}$ and $\overline{G'}$ be compact. If the function $Q : X \to [0,\infty]$ has finite mean oscillation at a point $x_0 \in \partial G$,*

$$\mu(B(x_0,2r)) \leq \gamma \cdot \log^{\alpha-2} \frac{1}{r} \cdot \mu(B(x_0,r)) \qquad \forall\, r \in (0,r_0)\,, \tag{13.39}$$

and (X,d,μ) is upper α-regular with $\alpha \geq 2$ at x_0, then f admits a continuous extension to the point x_0. If the last two conditions hold at every point of ∂G, then f admits a homeomorphic extension to the boundary.

Remark 13.11. In the case of Ahlfors regular spaces, even the condition on doubling measure holds, which is stronger than condition (13.39); see Remark 13.3. In view of the compactness of $\overline{G}$, Q is integrable in a neighborhood of ∂G that follows from the condition of finite mean oscillation at all points of ∂G; see Remark 13.5. If Q is given only in a domain G, then it can be extended by zero outside G. In particular, to have $Q \in \mathrm{FMO}(x_0)$ for $x_0 \in \partial G$, it suffices to have the condition

$$\overline{\lim_{\varepsilon \to 0}} \; ⨍_{B(x_0,\varepsilon)} Q(x)\, d\mu(x) < \infty. \tag{13.40}$$

By [81], the QED domains coincide in the class of finitely connected plane domains with the so-called uniform domains introduced in [212]. The example in Section 3.8 shows that, even among simply connected plane domains, the class of domains with weakly flat boundaries is wider than the class of QED domains. The example is based on the fact that QED domains satisfy the condition on doubling measure (13.14) at every boundary point; see Lemma 2.13 in [81]. The example just shows that the property on doubling measure is, generally speaking, not valid for domains with weakly flat boundaries.

13.11 On Null Sets for Extremal Distance

We say that a closed set A in a space (X,d,μ) is a **null set for extremal distance**, abbr. NED **set**, if

$$M(\Delta(E,F;D)) = M(\Delta(E,F;D\setminus A)) \tag{13.41}$$

for any domain D in X and any continua E and F in D.

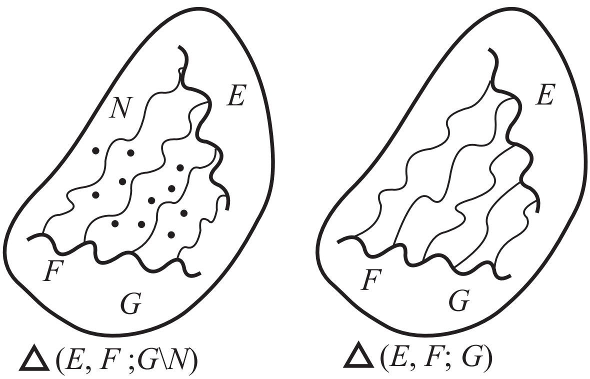

Figure 12

As in $\mathbb{R}^n, n \geq 2$, an NED set A in a weakly flat space X cannot have inner points and, moreover, they do not split the space X even locally, i.e., $G\setminus A$ has only one component of the path connectedness for any domain G in X. Thus, the complement of an NED set A in such an X is a very partial case of QED domains. Hence, NED sets in weakly flat spaces play the same role in the problems of removability of singular sets under quasiconformal mappings and their generalizations as in $\mathbb{R}^n$, $n \geq 2$.

Proposition 13.9. *Let A be an* NED *set in a weakly flat space (X,d,μ) that is not a singleton. Then*
1. A has no inner point,
2. $G\setminus A$ is a domain for every domain G in X.

Proof. (1) Let us assume that there is a point $x_0 \in A$ such that $B(x_0,r_0) \subseteq A$ for some $r_0 > 0$. Let $x_* \in X$, $x_* \neq x_0$, and γ be a path joining x_0 and x_* in X, $\gamma\colon [0,1] \to X$, $\gamma(0) = x_0$ and $\gamma(1) = x_*$. For small enough t, the continuum $C_t = \gamma([0,t])$ is in the ball $B(x_0,r_0)$ and, consequently, $\gamma([0,t]) \cap \overline{(X\setminus A)} = \emptyset$. Moreover, by Proposition 13.3, one can choose $t = t_0$ such that $C_{t_0} \setminus \{x_0\} \neq \emptyset$. Hence, setting $E = F = C_{t_0}$, we have $M(\Delta(E,F,X)) = \infty$ because the space X is weakly flat and, on the other hand, $M(\Delta(E,F;X\setminus A)) = 0$. The obtained contradiction disproves the above assumption.

(2) Denote by Ω_* one of the (path-) connected components of the open set $G\setminus A$; see Corollary 13.9. Let us assume that there is one more connected component of $G\setminus A$. Then $\Omega = G\setminus\overline{\Omega_*}\neq\emptyset$ and, considering G as a topological space T, and Ω as its (open) set, by Proposition 13.3 we have that there is a path $\gamma_0 : [0,1]\to G$ such that $\gamma_0([0,1))\subseteq\Omega$ and $x_0 := \gamma_0(1)\in\partial\Omega\cap\partial\Omega_*\cap G$. Note that the mutually complementany sets Ω and $\overline{\Omega_*}$ in the space G have a common boundary and $\partial\overline{\Omega_*}\subset\partial\Omega_*$. Let $x_*\in\Omega_*$ and $x_n\in\Omega_*$, $n=1,2,\dots,x_n\to x_0$ and γ_n be paths joining x_* and x_n in Ω_*. Then $M(\Delta(|\gamma_0|,|\gamma_n|;G))\to\infty$ as $n\to\infty$ because of the weak flatness of G by Remark 13.9, but $\Delta(|\gamma_0|,|\gamma_n|;G\setminus A))=\emptyset$ and, hence, $M(\Delta(|\gamma_0|,|\gamma_n|;G\setminus A))=0$.

The obtained contradiction disproves the above assumption that $G\setminus A$ has more than one connected component. □

Lemma 13.10. *Let X and X' be compact weakly flat spaces, let G be a domain in X, let $A\subset G$ be an* NED *set in X, and let f be a homeomorphism of $D=G\setminus A$ into X'. If the cluster set*

$$A' = C(A,f) = \{\, x'\in X' :\ x' = \lim_{k\to\infty} f(x_k),\ x_k\in D,\ \lim_{k\to\infty} x_k\in A\,\}$$

is an NED *set in X' and $D'=f(D)$, then $G'=D'\cup A'$ is a domain in X'. Moreover, there exist domains G^* in X with the properties $A\subset G^*$, $\overline{G^*}\subset G$, and $A'\cap A^*=\varnothing$, where $A^*=C(\partial G^*,f)$.*

Proof. First note that the NED set A is compact as a closed set in a compact space X and, hence, $\varepsilon_0=\operatorname{dist}(A,\partial G)>0$. Thus, A belongs to the open set

$$\Omega=\{x\in X : \operatorname{dist}(x,A)<\varepsilon\}$$

for any (fixed) $\varepsilon\in(0,\varepsilon_0)$ that itself is in G. Since A is compact, A is contained in a finite number of the connected components $\Omega_1,\dots,\Omega_m$ of Ω. Let x_0 be an arbitrary point of the domain G and let $x_k\in\Omega_k$, $k=1,\dots,m$. Then there exist paths $\gamma_k : [0,1]\to G$ with $\gamma_k(0)=x_0$ and $\gamma_k(1)=x_k$, $k=1,\dots,m$. Note that the set $C=\bigcup_{k=1}^{m}|\gamma_k|$ is compact and, hence, $\delta_0=\operatorname{dist}(C,\partial G)>0$.

Consider the open sets

$$G_\delta=\{x\in G : \operatorname{dist}(x,\partial G)>\delta\}.$$

By the triangle inequality, the set

$$C_0 = C\bigcup\left(\bigcup_{k=1}^{m}\Omega_k\right)$$

is contained in G_δ for any $\delta\in(0,d_0)$, where $d_0=\min(\varepsilon_0-\varepsilon,\delta_0)$. Furthermore, C_0 is contained in only one of the connected components G_δ^* of the set G_δ because the set C_0 is connected.

By the construction, $\overline{G_\delta^*} \subset G$, G_δ^* are domains in X and, consequently, they are weakly flat spaces. By Proposition 13.9, the sets $D_\delta = G_\delta^* \setminus A$ are domains with weakly flat boundaries A in the spaces G_δ^*, $\delta \in (0, d_0)$.

Let $f_\delta = f|_{D_\delta}$ and $g_\delta = (f_\delta)^{-1} : D'_\delta \to D_\delta$, where $D'_\delta = f_\delta(D_\delta)$. Then, as it follows by Proposition 13.5, we have the symmetry

$$A = C(A', g_\delta), \quad A' = C(A, f_\delta) \qquad \forall\, \delta \in (0, d_0)\,.$$

Note that ∂G_δ^* , $\delta \in (0, d_0)$, are compact subsets of the domain D and, consequently, $f \partial G_\delta^*$ are compact subsets of the domain $D' = f(D)$, which, by Proposition 13.5, do not intersect A'. Thus, $d_\delta = \mathrm{dist}(A', f\partial G_\delta^*) > 0$ for all $\delta \in (0, d_0)$. By Lemma 13.7, the space X' is locally path connected and hence, for every point $x_0 \in A'$, there is a domain $U \subset B(x_0, d_\delta)$ that is a neighborhood of x_0 and, by Proposition 13.9, $V = U \setminus A'$ is also a domain that is a subdomain of D' by the construction. Thus, $G' = D' \cup A'$ is a domain in X'. □

Finally, by Proposition 13.9 and Lemma 13.10, we obtain the following consequences for NED sets; see also Remarks 13.5 and 13.6.

Lemma 13.11. *Let X and X' be compact weakly flat spaces, G a domain in X, $A \subset G$ an* NED *set in X, and f a Q-homeomorphism of $D = G \setminus A$ into X' such that the cluster set $C(A, f)$ is an* NED *set in X'. If, at a point $x_0 \in A$, condition (13.34) holds, then f admits a continuous extension to the point x_0.*

Remark 13.12. In particular, f admits an extension to $x_0 \in A$ by continuity if at least one of the conditions (13.35)–(13.36), (13.37), (13.38), or (13.39) with $Q \in \mathrm{FMO}(x_0)$, (13.40) holds at the point.

Theorem 13.11. *Let X and X' be compact weakly flat spaces, G a domain in X, A an* NED *set in G, and f a Q-homeomorphism of $D = G \setminus A$ into X' such that the cluster set $A' = C(A, f)$ is an* NED *set in X'. If $Q \in L^1_\mu(G)$, then the inverse homeomorphism $g = f^{-1} : D' \to D$, $D' = f(D)$, admits a continuous extension $\overline{g} : G' \to G$, where $G' = D' \cup A'$.*

Remark 13.13. Thus, if $Q \in L^1_\mu(D)$ satisfies at least one of the conditions (13.35)–(13.39) with $Q \in \mathrm{FMO}(x_0)$, (13.40) at every point $x_0 \in A$, then any Q-homeomorphism f of the domain $D = G \setminus A$ into X' with NED sets A and $A' = C(A, f)$ admits a homeomorphic extension $\overline{f} : G \to G'$, where $G' = D' \cup A'$, $D' = f(D)$.

Theorem 13.12. *Let X and X' be compact weakly flat spaces, G a domain in X, $A \subset G$ an* NED *set in X, and f be a Q-homeomorphism of $D = G \setminus A$ into X' with an* NED *set $A' := C(A, f)$. If Q has finite mean oscillation and X is upper α–regular with $\alpha \geq 2$ at every point $x_0 \in A$, then f admits a homeomorphic extension $\overline{f} : G \to G'$, where $G' = D' \cup A'$ and $D' = f(D)$.*

13.12 On Continuous Extension to Isolated Singular Points

As before, here (X,d,μ) and (X',d',μ') are spaces with metrics d and d' and locally finite Borel measures μ and μ', and G and G' are domains in X and X' with finite Hausdorff dimensions α and $\alpha' \geq 1$, respectively.

Lemma 13.12. *Let a space X be path connected at a point $x_0 \in G$ that has a compact neighborhood, let X' be a compact weakly flat space, and let $f : G \setminus \{x_0\} \to G'$ be a Q-homeomorphism, where $Q : G \to [0,\infty]$ is a measurable function satisfying the condition*

$$\int\limits_{\varepsilon<d(x_0,x)<\varepsilon_0} Q(x) \cdot \psi^{\alpha}_{x_0,\varepsilon}(d(x,x_0))\, d\mu(x) = o(I^{\alpha}_{x_0}(\varepsilon)) \tag{13.42}$$

as $\varepsilon \to 0$, where $\varepsilon_0 < \operatorname{dist}(x_0, \partial G)$ and $\psi_{x_0,\varepsilon}(t)$ is a family of nonnegative (Lebesgue) measurable functions on $(0,\infty)$ such that

$$0 < I_{x_0}(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{x_0,\varepsilon}(t)\, dt < \infty, \quad \varepsilon \in (0,\varepsilon_0)\,. \tag{13.43}$$

Then f can be extended to the point x_0 by continuity in X'.

Proof. Let us show that the cluster set $E = C(x_0, f)$ is a singleton. The set E is contained in $\partial G'$ by Proposition 13.5. Moreover, E is a continuum because the domain G is connected at the point x_0. Indeed,

$$E = \limsup_{m\to\infty} f(G_m) = \bigcap_{m=1}^{\infty} \overline{f(G_m)},$$

where $G_m = G \cap U_m$ is a decreasing sequence of domains with neighborhoods U_m of the point x_0 and $d(G_m) \to 0$ as $m \to \infty$. Note that $\liminf_{m\to\infty} \overline{f(G_m)} = \liminf_{m\to\infty} f(G_m) \neq \emptyset$ in view of the compactness of X'; see, e.g., Remark 3, Section 41 in [186]. Consequently, $E \neq \emptyset$ is connected; see, e.g., I(9.12) in [334], p. 15. Moreover, E is closed by the construction and hence is compact as a closed subspace of the compact space X'; see, e.g., Theorem 2, IV, Section 41 in [30].

In view of the connectedness of G at the point x_0, there is a connected component G_* of the set $G \setminus \{x_0\} \cap B(x_0, r_0)$, $0 < r_0 < \operatorname{dist}(x_0, \partial G)$, containing $G \setminus \{x_0\} \cap B(x_0, r_*)$ for some $r_* \in (0, r_0)$. If $\partial G = \emptyset$, then here we set $\operatorname{dist}(x_0, \partial G) = \infty$. Since x_0 has a compact neighborhood, one may suppose that $\overline{B(x_0, r_0)}$ is compact.

Consider $G'_* = fG_*$. Let us show that the cluster set $E = C(x_0, f)$ is an isolated connected component of $\partial G'_*$. Indeed, $K = \partial G_* \setminus \{x_0\}$ is a compact set as a closed subset of the compact set $\overline{B(x_0, r_0)}$ and, consequently, $K_* = fK \subset G'$ is compact. On the other hand, the compact set E is contained in $\partial G'$, i.e., $E \cap K_* = \emptyset$. Thus, $\operatorname{dist}(E, K_*) > 0$. Finally, if $y_0 \in \partial G'_*$, then, by Proposition 13.5, $C(y_0, g) \subset \partial G_* = K \cup \{x_0\}$, where $g = f^{-1}|_{G'_*}$ and, consequently, either $y_0 \in E$ or $y_0 \in K_*$.

Let $z_0 \in G'_*$. Then, by Proposition 13.2, there is a path $\gamma_0 : [a,b) \to G_*$ from $\gamma_0(a) = f^{-1}(z_0)$ to $x_0 = \lim_{t\to b} \gamma_0(t)$ in G_*. Setting $\gamma'_0 = f\gamma_0 : [a,b) \to G'_*$, we have $\mathrm{dist}(\gamma'_0(t), E) \to 0$ as $t \to b$ by the definition of $E = C(x_0, f)$ in view of the compactness of the space X'. Set $C_* = \gamma'_0([a,b))$ and

$$\Gamma = \Delta(C_*, E, X').$$

Consider also the families of paths

$$\Gamma_0 = \Delta(C_*, E, G'_*)$$

and

$$\Gamma_* = \{\gamma \in \Gamma : \ |\gamma| \cap R \neq \emptyset\},$$

where

$$R = X' \setminus \{G'_* \cup E\}.$$

First, note that $M(\Gamma_0) = M(\widetilde{\Gamma})$, where $\widetilde{\Gamma} = \Gamma \setminus \Gamma_*$. Indeed, on the one hand, $\Gamma_0 \subset \widetilde{\Gamma}$ and hence $M(\Gamma_0) \leq M(\widetilde{\Gamma})$. On the other hand, $\Gamma_0 < \widetilde{\Gamma}$ by Proposition 13.3 and hence $M(\Gamma_0) \geq M(\widetilde{\Gamma})$; see, e.g., Theorem 1 in [64], Section, A.5. Second, note that

$$M(\Gamma_*) \leq M_* := \frac{\mu(X')}{(2\,\mathrm{dist}(C_* \cup E,\, \partial G'_* \setminus E))^{\alpha'}} < \infty$$

because $C_* \cup E$ and $\partial G'_* \setminus E$ are nonintersecting compact sets and $\mu'(X') < \infty$ in view of the compactness of X' and the local finiteness of the measure μ'.

Let us assume that the continuum E is not degenerate. Let $y_0 \in E$ be a limit point of $\gamma'_0(t)$ as $t \to b$ and $y_* \in E\,, y_* \neq y_0$. By the Darboux property of connected sets, $\partial B(y_0, r)$ intersecs C_* and E for all $r \in (0, r_0)$, where

$$r_0 = \min\{d'(y_0, \gamma_0(a)), d'(y_0, y_*)\}.$$

Consider continua $C(t) = \gamma_0([a,t])$, $t \in [a,b)$. Note that $\mathrm{dist}\,(C(t), E) \to 0$ as $t \to b$ by the construction. Thus,

$$M(\Delta(C(t), E, X')) \to \infty$$

as $t \to b$ because the space X' is weakly flat. Consequently, there is $t_0 \in [a,b)$ such that

$$M_0 := M(\Delta(C(t_0), E, X')) > M_*.$$

Recall that $\Gamma = \widetilde{\Gamma} \cup \Gamma_*$. We obtain by the monotonicity and subadditivity of the modulus

$$M_* < M_0 \leq M(\Gamma) \leq M(\widetilde{\Gamma}) + M(\Gamma_*) = M(\Gamma_0) + M(\Gamma_*) \leq M(\Gamma_0) + M_*.$$

Consequently,

$$M(\Gamma_0) > 0 .$$

However,

$$\Gamma_0 = \bigcup_{n=1}^{\infty} \Gamma_n,$$

where $\Gamma_n = \Delta(C(t_n), E, G'_*)$, $t_n \to b$ as $n \to \infty$, and by subadditivity of the modulus

$$M(\Gamma_0) \le \sum_{n=1}^{\infty} M(\Gamma_n) .$$

Thus, there is a continuum $C = C(t_n)$ such that

$$M(\Delta(C, E, G'_*)) > 0 .$$

Note that $C_0 = f^{-1}(C)$ is a compact set as a continuous image of a compact set. Thus, $\varepsilon_0 = \mathrm{dist}(x_0, C_0) > 0$. Let

$$\Gamma_\varepsilon = \Delta(C_0, B(x_0, \varepsilon), G_*), \qquad \varepsilon \in (0, \varepsilon_0) ,$$

and let $\psi^*_{x_0,\varepsilon}$ be a Borel function such that $\psi^*_{x_0,\varepsilon}(t) = \psi_{x_0,\varepsilon}(t)$ for a.e. $t \in (0, \infty)$, which there is in view of the Lusin theorem; see, e.g., Section 2.3.5 in [55].

Then, by Proposition 13.4, the function

$$\rho_\varepsilon(x) = \begin{cases} \psi^*_{x_0,\varepsilon}(d(x, x_0))/I(\varepsilon, \varepsilon_0), & x \in A(x_0, \varepsilon, \varepsilon_0), \\ 0, & x \in X \setminus A(x_0, \varepsilon, \varepsilon_0), \end{cases}$$

where

$$A(x_0, \varepsilon, \varepsilon_0) = \{x \in X : \varepsilon < d(x, x_0) < \varepsilon_0\} ,$$

is admissible for Γ_ε and, consequently,

$$M(f\Gamma_\varepsilon) \le \int_G Q(x) \cdot \rho_\varepsilon^\alpha(x) \, d\mu(x),$$

i.e., $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (13.42).

On the other hand,

$$M(f\Gamma_\varepsilon) \ge M(\Delta(C, E, G'_*)) > 0$$

because

$$\Delta(C_0, \{x_0\}, G_*) > \Gamma_\varepsilon$$

and

$$f^{-1}\Delta(C, E, G'_*) \subseteq \Delta(C_0, \{x_0\}, G_*)$$

for any $\varepsilon \in (0, \varepsilon_0)$ by Proposition 13.5 applied to the homeomorphism f^{-1} and $g = f^{-1}|_{G'_*}$, and $x'_0 \in E$, $x'_0 = \gamma(b)$, $\gamma \in \Delta(C, E, G'_*)$. The obtained contradiction disproves the assumption that E is not degenerate. $\Box$

Corollary 13.14. *In particular, if*

$$\overline{\lim_{\varepsilon \to 0}} \int\limits_{\varepsilon < d(x,x_0) < \varepsilon_0} Q(x) \cdot \psi^{\alpha}(d(x,x_0))\, d\mu(x) < \infty, \tag{13.44}$$

where $\psi(t)$ is a nonnegative measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty, \qquad \forall\, \varepsilon \in (0, \varepsilon_0),$$

and $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$, then any Q-homeomorphism $f : G \setminus \{x_0\} \to G' \subset X'$ is extended to the point x_0 by continuity in X'.

Remark 13.14. In other words, it suffices for the singular integral (13.44) to be convergent in the sense of the principal value at the point x_0 at least for one kernel ψ with a non-integrable singularity at zero. Furthermore, as Lemma 13.12 shows, it suffices for the given integral even to be divergent but with controlled speed:

$$\int\limits_{\varepsilon < d(x,x_0) < \varepsilon_0} Q(x) \cdot \psi^{\alpha}(d(x,x_0))\, d\mu(x) = o(I^{\alpha}(\varepsilon, \varepsilon_0)). \tag{13.45}$$

Choosing in Lemma 13.12 $\psi(t) \equiv 1/t$, we obtain the following theorem.

Theorem 13.13. *Let X and X' compact spaces, X path connected at a point $x_0 \in G$, and X' weakly flat. If a measurable function $Q : G \to [0,\infty]$ satisfies the condition*

$$\int\limits_{\varepsilon < d(x,x_0) < \varepsilon_0} \frac{Q(x)\, d\mu(x)}{d(x,x_0)^{\alpha}} = o\left(\left[\log \frac{1}{\varepsilon}\right]^{\alpha}\right) \tag{13.46}$$

as $\varepsilon \to 0$, where $\varepsilon_0 < \operatorname{dist}(x_0, \partial G)$, then any Q-homeomorphism $f : G \setminus \{x_0\} \to G'$ is extended by continuity to the point x_0.

Corollary 13.15. *In particular, the conclusion of Theorem 13.13 holds if the singular integral*

$$\int \frac{Q(x)\, d\mu(x)}{d(x,x_0)^{\alpha}} \tag{13.47}$$

is convergent in a neighborhood of the point in the sense of the principal value.

Combining Lemmas 13.2 and 13.12, and choosing $\psi_{\varepsilon}(t) \equiv t \log(1/t)$, $t \in (0, \delta_0)$, in the latter, we obtain the following theorem.

Theorem 13.14. *Let X and X' compact weakly flat spaces, G a domain in X that is upper α-regular with $\alpha \geq 2$ and path connected at a point $x_0 \in G$, and*

$$\mu(B(x_0,2r)) \leq \gamma \cdot \log^{\alpha-2}\frac{1}{r} \cdot \mu(B(x_0,r)) \tag{13.48}$$

for all $r \in (0,r_0)$. If $Q \in \mathrm{FMO}(x_0)$, then any Q-homeomorphism f of the domain $G \setminus \{x_0\}$ into X' is extended by continuity to the point x_0.

Combining Corollary 13.3 and Theorem 13.14, we obtain the following statement.

Corollary 13.16. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B(x_0,\varepsilon)} Q(x)\, d\mu(x) < \infty, \tag{13.49}$$

then any Q-homeomorphism $f : G \setminus \{x_0\} \to G' \subset X'$ is extended by continuity to the point x_0.

The following simple example shows that the above extension $\overline{f}$ of f to x_0 may be not a homeomorphism.

Example. Let $G = X$, where X is a space that coincides with a closed equilateral triangle T on one of the coordinate planes in $\mathbb{R}^3$ minus one of its vertices v. It is clear that X is not compact, although it is locally compact. Let us roll up the triangle T without any distortion in such a way that the vertex v will touch the center c of its opposite side. The obtained space X' is compact. Let $x_0 = c$. The above (rolling up) mapping $f : X \setminus \{x_0\} \to X' \setminus \{x_0\}$ is conformal if we take in X the usual Euclidean distance as the metric d and the usual area as the Borel measure μ and in X' set d' to be geodesic (thus, the path length is invariant under f) and $\mu'(B' \setminus \{x_0\}) = \mu(f^{-1}(B' \setminus \{x_0\}))$ for every Borel set in X' and $\mu'(\{x_0\}) = \mu(\{x_0\}) = 0$. By the construction, the mapping f can continuously be extended to x_0 and the extension $\overline{f}$ is injective, of course, but not a homeomorphism (the inverse mapping of $\overline{f}$ is not continuous).

Remark 13.15. By Proposition 13.5, the extension of f at the point x_0 is an injective mapping and, thus, a homeomorphism on any subdomain $G_* \subset\subset G$, i.e., if $\overline{G_*}$ is compact in G. The latter is, generally speaking, not true for the domain G, itself as the above example shows. However, this is true if, for instance, $G = X$ is compact; see, e.g., [186].

Moreover, if the family of all paths in X' (or only in G_*) passing through the point $y_0 = \overline{f}(x_0)$ has modulus zero (see Section 13.8), then the restriction of the mapping $g = \overline{f}|_{G_*}$ will be a Q-homeomorphism. For the Ahlfors regular spaces, this always holds; see Lemma 7.18 in [106]. Thus, an isolated singular point of a Q-homeomorphism in regular weakly flat spaces is locally removable under the conditions on Q enumerated above.

13.13 On Conformal and Quasiconformal Mappings

Finally, let us review some results for conformal and quasiconformal mappings which are direct consequences of the theory of Q-homeomorphisms in metric spaces with the measures developed above. Namely, as before, let (X,d,μ) and (X',d',μ') be spaces with metrics d and d' and locally finite Borel measures μ and μ', and with finite Hausdorff dimensions α and $\alpha' \geq 1$, respectively.

Similarly to the geometric definition by Väisäla in $\mathbb{R}^n$, $n \geq 2$ (cf. Chapter 1), given domains G and G' in (X,d,μ) and (X',d',μ'), respectively, we say that a homeomorphism $f : G \to G'$ is called K**-quasiconformal**, $K \in [1,\infty]$, if

$$K^{-1}M(\Gamma) \leq M(f\Gamma) \leq KM(\Gamma) \tag{13.50}$$

for every family Γ of paths in G. We say also that a homeomorphism $f : G \to G'$ is **quasiconformal** if f is K-quasiconformal for some $K \in [1,\infty)$, i.e., if the distortion of moduli of path families under the mapping f is bounded. In particular, we say that a homeomorphism $f : G \to G'$ is **conformal** if

$$M(f\Gamma) = M(\Gamma) \tag{13.51}$$

for any path families in G.

By Theorem 13.3, we obtain the following important conclusion.

Theorem 13.15. *Let G have a weakly flat boundary, let G' be locally path connected at all its boundary points, and let $\overline{G'}$ be compact. Then any quasiconformal mapping $f : G \to G'$ admits a continuous extension to the boundary $\overline{f} : \overline{G} \to \overline{G'}$.*

Combining Theorem 13.15 with Lemma 13.1, we come to the following statement.

Corollary 13.17. *If G and G' are domains with weakly flat boundaries and compact closures $\overline{G}$ and $\overline{G'}$, then any quasiconformal mapping $f : G \to G'$ admits a homeomorphic extension $\overline{f} : \overline{G} \to \overline{G'}$.*

Remark 13.16. In particular, the last conclusion holds for quasiconformal mappings between QED domains with compact closures in weakly flat spaces. Note that the closures of the domains are always compact in compact spaces. Recall also that locally compact spaces always admit the so-called one-point compactification; see, e.g., Section I.9.8. [30].

On the basic of Lemmas 13.1 and 13.10 and Theorem 13.15, we obtain the following theorem.

Theorem 13.16. *Let X and X' be compact weakly flat spaces, G a domain in X, $A \subset G$ an* NED *set and f a quasiconformal mapping of the domain $D = G \setminus A$ into X'. If the cluster set $A' = C(A,f)$ is also an* NED *set, then f admits a quasiconformal extension to G.*

By results in the previous section, single out also the following consequences on removability of isolated singularities.

Lemma 13.13. *Let X be path connected at a point $x_0 \in G$ with a c compact neighborhood, X' a compact weakly flat space, and $f : G \setminus \{x_0\} \to G'$ a quasiconformal mapping. If μ satisfies the condition*

$$\int\limits_{\varepsilon<d(x_0,x)<\varepsilon_0} \psi^{\alpha}(d(x,x_0))\,d\mu(x) = o(I^{\alpha}(\varepsilon,\varepsilon_0)) \tag{13.52}$$

as $\varepsilon \to 0$, where $\varepsilon_0 < \operatorname{dist}(x_0,\partial G)$, and $\psi(t)$ is a nonnegative (Lebesgue) measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon,\varepsilon_0) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt < \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0)\,,$$

then the mapping f is extended by continuity to the point x_0.

Theorem 13.17. *Let X be path connected at a point $x_0 \in G$ with a c compact neighborhood, X' a compact weakly flat space, and $f : G \setminus \{x_0\} \to G'$ a quasiconformal mapping. If μ satisfies the condition*

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} \frac{d\mu(x)}{d(x,x_0)^{\alpha}} = o\left(\left[\log\frac{1}{\varepsilon}\right]^{\alpha}\right) \tag{13.53}$$

as $\varepsilon \to 0$, where $\varepsilon_0 < \operatorname{dist}(x_0,\partial G)$, then the mapping f is extended by continuity to the point x_0.

Finally, in view of Remarks 13.3 and 13.8, we have the following important conclusion from Theorem 13.17.

Corollary 13.18. *Let X and X' be Ahlfors regular compact weakly flat spaces. Then any quasiconformal mapping $X \setminus \{x_0\}$ into X' is extended to a quasiconformal mapping of X into X'.*

Corollary 13.19. *Isolated singularities of quasiconformal mappings are locally removable in Ahlfors regular weakly flat spaces X and X' if, in addition, X is locally compact and X' is compact.*

Thus, the results of this umlaut chapter extend (and strengthen) the known theorems by F. Gehring, O. Martio, P. Näkki, U. Srebro, J. Väisäla, M. Vuorinen and others on quasiconformal mappings in $\mathbb{R}^n, n \geq 2$, to Q-homeomorphisms in metric spaces; cf. e.g. [81, 127, 128, 163, 204–209, 214, 224, 316, 329].

Appendix A
Moduli Theory

Here we have collected a series of classical results that not only have a great historical significance but also remain useful working tools in modern mapping theory, especially, in the framework of our book; see [64, 71, 122, 210, 293, 340]. This appendix can be considered together with Chapters 2 and 3 as a handbook in the theory of moduli. We attempted to keep author's styles of the given papers.

A.1 On Some Results by Gehring

After the well-known paper [5] by Ahlfors and Beurling, applications of the theory of moduli in the quasiconformal mapping theory began essentially with a theorem proved by Gehring in [77] that the conformal capacity of a space ring R is directly related to the modulus of a family of paths that join the boundary components of R. In this section we follow in the main [71].

Given a family Γ of nonconstant paths γ in $\overline{\mathbb{R}^n}$, we let adm Γ denote the family of Borel measurable functions $\rho : \mathbb{R}^n \to [0,\infty]$ such that

$$\int\limits_{\gamma} \rho \, ds \;\geq\; 1$$

for all locally rectifiable $\gamma \in \Gamma$. We call

$$M(\Gamma) \;=\; \inf_{\rho\in\mathrm{adm}\;\Gamma} \int\limits_{\mathbb{R}^n} \rho^n \, dm$$

and

$$\lambda(\Gamma) \;=\; M(\Gamma)^{\frac{1}{1-n}}$$

the **modulus** and **extremal length** of Γ, respectively.

The concept of extremal length first appeared in the article by Ahlfors and Beurling [5] containing applications to the theory of analytic functions of a complex variable. Later contributions to the theory of extremal length were made by Jenkins [139], Hersch [120, 121], and others. For our purpose it is preferable to operate with the modulus rather than the extremal length.

When Γ is a family of arcs, we may think of $M(\Gamma)$ as the conductance and $\lambda(\Gamma)$ as the resistance of a system of homogeneous wires. $M(\Gamma)$ is big when the wires are plentiful or short, small when the wires are few or long.

Theorem A.1. *$M(\Gamma)$ is an outer measure on the collections of path families Γ in $\overline{\mathbb{R}^n}$. That is,*

(a) $M(\emptyset) = 0,$

(b) $M(\Gamma_1) \leq M(\Gamma_2)$ *when* $\Gamma_1 \subset \Gamma_2,$

(c) $M(\bigcup_j \Gamma_j) \leq \sum_j M(\Gamma_j).$

Proof for (c). We may assume $M(\Gamma_j) < \infty$ for all j. Then, given $\varepsilon > 0$, we can choose for each j a $\rho_j \in \operatorname{adm} \Gamma_j$ such that

$$\int_{\mathbb{R}^n} \rho_j^n \, dm \leq M(\Gamma_j) + 2^{-j}\varepsilon.$$

Now set

$$\rho = \sup_j \rho_j$$

and

$$\Gamma = \bigcup_j \Gamma_j.$$

Then $\rho : \mathbb{R}^n \to [0,\infty]$ is Borel measurable. Moreover, if $\gamma \in \Gamma$ is locally rectifiable, then $\gamma \in \Gamma_j$ for some j,

$$\int_\gamma \rho \, ds \geq \int_\gamma \rho_j \, ds \geq 1,$$

and hence $\rho \in \operatorname{adm} \Gamma$. Thus,

$$M(\Gamma) \leq \int_{\mathbb{R}^n} \rho^n \, dm \leq \int_{\mathbb{R}^n} \sum_j \rho_j^n \, dm \leq \sum_j M(\Gamma_j) + \varepsilon.$$

□

Remark A.1. If we apply the Caratheodory criterion to the outer measure M to define the notion of a measurable path family, then we can show the following:

(a) Γ is measurable if $M(\Gamma) = 0$;

(b) Γ is not measurable if $0 < M(\Gamma) < \infty$;

(c) Γ may or may not be measurable if $M(\Gamma) = \infty$.

Theorem A.2. *If each path γ_1 of a family Γ_1 contains a subpath γ_2 of a family Γ_2, then $M(\Gamma_1) \le M(\Gamma_2)$.*

Proof. Choose $\rho \in \operatorname{adm} \Gamma_2$ and suppose $\gamma_1 \in \Gamma_1$ is locally rectifiable. Then

$$\int_{\gamma_1} \rho \, ds \ \ge \ \int_{\gamma_2} \rho \, ds,$$

where γ_2 is the subpath in Γ_2, and $\rho \in \operatorname{adm} \Gamma_1$. Thus,

$$M(\Gamma_1) \ \le \ \int_{\mathbb{R}^n} \rho^n \, dm$$

and taking the infimum over all such ρ yields

$$M(\Gamma_1) \ \le \ M(\Gamma_2).$$

□

Theorem A.3. *$M(\Gamma)$ is additive on path families in disjoint Borel sets. That is, if the E_j are disjoint Borel sets and the paths of Γ_j lie in E_j, then*

$$M\left(\bigcup_j \Gamma_j\right) \ = \ \sum_j M(\Gamma_j).$$

Proposition A.1. *If Γ is the family of paths γ joining two parallel faces with distance h of a rectangular parallelepiped of the $(n-1)$-dimensional area A in $\mathbb{R}^n$, $n \ge 2$, then*

$$M(\Gamma) \ = \ \frac{A}{h^{n-1}}.$$

Proof. Choose $\rho \in \operatorname{adm} \Gamma$ and let γ_y be the vertical segment from y in the base E. Then $\gamma_y \in \Gamma$ and

$$1 \ \le \ \left(\int_{\gamma} \rho \, ds\right)^n \ \le \ h^{n-1} \int_{\gamma_y} \rho^n \, ds.$$

This holds for all such y and, hence,

$$\int_{\mathbb{R}^n} \rho^n \, dm_n \ \ge \ \int_E \left(\int_{\gamma_y} \rho^n \, ds\right) dm_{n-1} \ \ge \ \frac{A}{h^{n-1}}.$$

Since ρ is arbitrary,

$$M(\Gamma) \ \ge \ \frac{A}{h^{n-1}}.$$

Next set $\rho = 1/h$ inside the parallelepiped and $\rho = 0$ otherwise. Then $\rho \in \operatorname{adm} \Gamma$ and

$$M(\Gamma) \le \int_{\mathbb{R}^n} \rho^n \, dm = \frac{A}{h^{n-1}}.$$

$\square$

Theorem A.4. *If all the paths in a path family Γ pass through a fixed point x_0, then $M(\Gamma) = 0$.*

Proof. Suppose first that $x_0 \neq \infty$ and for each j let Γ_j denote the subfamily of $\gamma \in \Gamma$ that intersects x_0 and $S(x_0, 1/j)$. Then each $\gamma \in \Gamma_j$ contains a subpath γ' in the family of all paths joining x_0 to $S(x_0, 1/j)$ in $B(x_0, 1/j)$. Hence, $M(\Gamma_j) = 0$ by Lemma 2.2. Since $\Gamma = \bigcup_j \Gamma_j$,

$$M(\Gamma) \le \sum_j M(\Gamma_j) = 0.$$

When $x_0 = \infty$, we argue as above with $S(x_0, 1/j)$ replaced by $S(0, j)$. $\square$

Theorem A.5. *If $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius transformation, then*

$$M(f\Gamma) = M(\Gamma)$$

for all path families Γ in $\overline{\mathbb{R}^n}$.

Proof. Choose $\rho' \in \operatorname{adm} f\Gamma$, set

$$\rho(x) = \rho' \circ f(x) |f'(x)|$$

for $x \in \mathbb{R}^n \setminus \{f^{-1}(\infty)\}$, and let Γ_0 be the family of $\gamma \in \Gamma$ that passes through $f^{-1}(\infty)$. Then

$$M(\Gamma) = M(\Gamma \setminus \Gamma_0), \qquad \rho \in \operatorname{adm}(\Gamma \setminus \Gamma_0)$$

and hence

$$M(\Gamma) \le \int_{\mathbb{R}^n} \rho^n \, dm = \int_{\mathbb{R}^n} (\rho' \circ f)^n |f'|^n \, dm$$
$$= \int_{\mathbb{R}^n} (\rho' \circ f)^n J(f) \, dm = \int_{\mathbb{R}^n} (\rho')^n \, dm.$$

Taking the infimum over every such ρ' gives $M(\Gamma) \le M(f\Gamma)$. The result follows by repeating the preceding argument with f replaced by f^{-1}. $\square$

Theorem A.6. *If f_j, $f : \overline{\mathbb{R}^n} \to [0, \infty]$ are Borel measurable and $f_j \to f$ in $L^n(\mathbb{R}^n)$, then there exist a subsequence $\{j_k\}$ and a path family Γ_0 with $M(\Gamma_0) = 0$ such that*

$$\lim_{k \to \infty} \int_\gamma |f_{j_k} - f| \, ds = 0$$

for all locally rectifiable paths γ, $\gamma \notin \Gamma_0$.

Proof. Choose a subsequence $\{j_k\}$ so that

$$\int_{\mathbb{R}^n} g_k^n \, dm \; < \; 2^{-(n+1)k}, \qquad\qquad g_k \; = \; |f_{j_k} - f|$$

and let Γ_0 be the family of all locally rectifiable γ in $\overline{\mathbb{R}^n}$ such that

$$\limsup_{k\to\infty} \int_\gamma g_k \, ds \; > \; 0.$$

We want to show that $M(\Gamma_0) = 0$.

Let Γ_k be the family of all locally rectifiable paths in $\mathbb{R}^n$ for which

$$\int_\gamma g_k \, ds \; \geq \; 2^{-k}.$$

Then $\rho = 2^k g_k \in \operatorname{adm} \Gamma_k$ and

$$M(\Gamma_k) \; \leq \; \int_{\mathbb{R}^n} \rho^n \, dm \; \leq \; 2^{nk} \int_{\mathbb{R}^n} g_k^n \, dm \; < \; 2^{-k}.$$

Now $\gamma \in \Gamma_0$ implies $\gamma \in \Gamma_k$ for infinitely many k. Thus, for each l,

$$\Gamma_0 \; \subset \; \bigcup_{k=l}^{\infty} \Gamma_k, \qquad M(\Gamma_0) \; \leq \; \sum_{k=l}^{\infty} M(\Gamma_k) \; < \; 2^{-l+1}$$

and hence $M(\Gamma_0) = 0$. □

Theorem A.7. *If $\{\Gamma_j\}$ is an increasing sequence of path families, then*

$$M\left(\bigcup_j \Gamma_j\right) \; = \; \lim_{j\to\infty} M(\Gamma_j).$$

Idea of proof. Let $\Gamma = \bigcup_j \Gamma_j$. Then by the monotonicity of the modulus,

$$M(\Gamma) \; \geq \; \lim_{j\to\infty} M(\Gamma_j).$$

For the reverse inequality, we may assume that the limit is finite. Then since $L(\mathbb{R}^n)$ is uniformly convex, we can choose $\rho_j \in \operatorname{adm} \Gamma_j$ so that $\rho_j \to \rho$ in $L^n(\mathbb{R}^n)$ and so that

$$\int_{\mathbb{R}^n} \rho^n \, dm \; = \; \lim_{j\to\infty} \int_{\mathbb{R}^n} \rho_j^n \, dm \; = \; \lim_{j\to\infty} M(\Gamma_j).$$

By Theorem A.6, there exist a subsequence $\{j_k\}$ and a family Γ_0 with $M(\Gamma_0) = 0$ such that

$$\int_\gamma \rho \, ds \;=\; \lim_{k\to\infty} \int_\gamma \rho_{j_k} \, ds$$

for all locally rectifiable $\gamma \in \Gamma \setminus \Gamma_0$. Since each such γ lies in Γ_{j_k} for large k,

$$\int_\gamma \rho \, ds \;\geq\; 1.$$

Thus, $\rho \in \operatorname{adm}(\Gamma \setminus \Gamma_0)$, and we conclude that

$$M(\Gamma) \;=\; M(\Gamma \setminus \Gamma_0) \;\leq\; \int_{\mathbb{R}^n} \rho^n \, dm \;=\; \lim_{j\to\infty} M(\Gamma_j).$$

□

Remark A.2. Since the Γ_j are increasing in Theorem A.7,

$$\Gamma \;=\; \bigcup_j \Gamma_j \;=\; \lim_{j\to\infty} \Gamma_j$$

in the set-theoretic sense and so we see that the conclusion of Theorem A.7 is a continuity property for the modulus. Unfortunately, no such result holds for decreasing families Γ_j with

$$\Gamma \;=\; \bigcap_j \Gamma_j \;=\; \lim_{j\to\infty} \Gamma_j.$$

A **condenser** is a domain $R \subset \overline{\mathbb{R}^n}$ whose complement is the union of two distinguished disjoint compact sets C_0 and C_1. For convenience, we write

$$R = R(C_0, C_1).$$

A **ring** is a condenser $R = R(C_0, C_1)$, where C_0 and C_1 are continua. We call C_0 and C_1 the **complementary components** of R.

Given a condenser $R = R(C_0, C_1)$ with $R \subset \mathbb{R}^n$, we let adm R denote the class of functions $u : \overline{\mathbb{R}^n} \to \mathbb{R}^1$ with the following properties:

1. u is continuous in $\overline{\mathbb{R}^n}$;
2. u has distribution derivatives in R;
3. $u = 0$ on C_0 and $u = 1$ on C_1.

Note that

$$u(x) \;=\; \min\left(1, \; \frac{h(x, C_0)}{h(C_1, C_0)}\right) \;\in\; \operatorname{adm} R$$

and hence adm $R \neq \emptyset$. We call

$$\operatorname{cap} R \;=\; \inf_{u \in \operatorname{adm} R} \int_R |\nabla u|^n \, dm,$$

$$\operatorname{mod} R = \left(\frac{\omega_{n-1}}{\operatorname{cap} R} \right)^{\frac{1}{n-1}}$$

the **conformal capacity** and **modulus** of R, respectively.

Example. If R is the ring in $\overline{\mathbb{R}^n}$ bounded by concentric spheres of radii a and b, $0 \le a < b \le +\infty$, then

$$\operatorname{cap} R = \omega_{n-1} \left(\log \frac{b}{a} \right)^{1-n}$$

$$\operatorname{mod} R = \log \frac{b}{a}.$$

Remark A.3. If $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius transformation, then $\operatorname{cap} fR = \operatorname{cap} R$ for all condensers R with $R, fR \subset \mathbb{R}^n$. Hence, we can use this fact to define $\operatorname{cap} R$ for all condensers in $\overline{\mathbb{R}^n}$ that contain ∞ as an interior point.

Given E, F, $G \subset \overline{\mathbb{R}^n}$, we let $\Delta(E,F;G)$ denote the family of all paths γ with

1. one endpoint in $\overline{E}$ and the other in $\overline{F}$,
2. interior in G.

Theorem A.8. *If* $R = R(C_0, C_1)$ *is a ring and* $\Gamma = \Delta(C_0, C_1; R)$*, then* $\operatorname{cap} R = M(\Gamma)$.

Outline of proof. By performing a preliminary Möbius transformation, we may assume that $\infty \in C_1$. Choose a locally Lipschitzian function $u \in \operatorname{adm} R$ and set

$$\rho(x) = \begin{cases} |\nabla u(x)| & \text{if } x \in R, \\ 0 & \text{if } x \in C_0 \cup C_1. \end{cases}$$

If $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ is a locally rectifiable path in Γ, then

$$\int_\gamma \rho \, ds \ \ge \ \left| \int_\gamma \nabla u \, ds \right| = |u(\gamma(b)) - u(\gamma(a))| \ \ge \ 1.$$

Hence, $\rho \in \operatorname{adm} \Gamma$ and

$$M(\Gamma) \le \int_R |\nabla u|^n \, dm.$$

By a smoothing argument, we can show that taking the infimum of the right-hand side over all such u gives $\operatorname{cap} R$. Thus,

$$M(\Gamma) \le \operatorname{cap} R.$$

Conversely, choose a bounded continuous $\rho \in \operatorname{adm} \Gamma$ and set

$$u(x) = \min \left(1, \inf_\gamma \int_\gamma \rho \, ds \right)$$

for $x \in R$, where the infimum is taken over all locally rectifiable γ joining C_0 to x in R. Then u has distribution derivatives and

$$\lim_{x \to C_0} u(x) = 0$$

and

$$\lim_{x \to C_1} u(x) = 1.$$

Hence, we can extend u to $\overline{\mathbb{R}^n}$ so that $u \in \operatorname{adm} R$. Then, since $|\nabla u| = \rho$ in R,

$$\operatorname{cap} R \leq \int_R \rho^n \, dm \leq \int_{\mathbb{R}^n} \rho^n \, dm.$$

Another smoothing argument shows the infimum over such ρ gives $M(\Gamma)$. Thus, $\operatorname{cap} R \leq M(\Gamma)$. □

Given a ray L from x_0 to ∞ and a compact set $E \subset \overline{\mathbb{R}^n}$, we define the **spherical symmetrization of** E **in** L as the set E^* satisfying the following conditions:

1. $x_0 \in E^*$ iff $x_0 \in E$;
2. $\infty \in E^*$ iff $\infty \in E$;
3. for $r \in (0,\infty)$, $E^* \cap S(x_0,r) \neq \emptyset$ iff $E \cap S(x_0,r) \neq \emptyset$, in which case $E^* \cap S(x_0,r)$ is a closed spherical cap centered on L with the same m_{n-1} measure $E \cap S(x_0,r)$.

We see that E^* is compact and that E^* is connected if E is also.

Theorem A.9. *If E^* is the spherical symmetrization of E in a ray L, then*

(a) $m_n(E^*) = m_n(E)$,

(b) $m_{n-1}(\partial E^*) = m_{n-1}(\partial E)$.

Outline of proof. To prove (a), we apply Fubini's theorem and obtain

$$m_n(E^*) = \int_0^\infty m_{n-1}(E^* \cap S(x_0,r))dr = \int_0^\infty m_{n-1}(E \cap S(x_0,r))\, dr = m_n(E).$$

For (b), assume first that E is a polyhedron. Then, for $r \in (0,\infty)$, the Brunn–Minkowski inequality implies that

$$E^*(r) = \{x : \operatorname{dist}(x,E^*) \leq r\} \subset \{x : \operatorname{dist}(x,E) \leq r\}^* = E(r)^*$$

and hence that

$$\begin{aligned} m_{n-1}(\partial E^*) &\leq \limsup_{r \to 0} \frac{m_n(E^*(r)) - m_n(E^*)}{2r} \\ &\leq \limsup_{r \to 0} \frac{m_n(E(r)) - m_n(E)}{2r} = m_{n-1}(\partial E). \end{aligned}$$

The general result follows by a limiting argument. □

Theorem A.10. *If $R = R(C_0, C_1)$ is a condenser and C_0^* and C_1^* are the spherical symmetrizations of C_0 and C_1 in opposite rays L_0 and L_1, then $R^* = R(C_0^*, C_1^*)$ is a condenser with*

$$\operatorname{cap} R^* \;=\; \operatorname{cap} R.$$

Idea of proof. Choose a locally Lipschitzian $u \in \operatorname{adm} R$ and define u^* so that $\{x : u^*(x) \le t\} = \{x : u(x) \le t\}^*$. Then $u^* \in \operatorname{adm} R^*$ and Theorem A.19 allows one to show that

$$\operatorname{cap} R^* \;\le\; \int_{\mathbb{R}^n} |\nabla u^*|^n \, dm \;\le\; \int_{\mathbb{R}^n} |\nabla u|^n \, dm.$$

Taking the infimum over all such u yields the result. □

Let $e_1, e_2, \ldots, e_n$ denote the basic vectors in $\mathbb{R}^n$. For $t \in (0,\infty)$, let $R_T(t)$ denote the ring domain in $\mathbb{R}^n$ whose complement consists of the ray from te_1 to ∞ and the segment from $-e_1$ to 0. $R_T(t)$ is called the **Teichmüller ring**. The following properties for its modulus can be established:

1. $\operatorname{mod} R_T(t) - \log(t+1)$ is nondecreasing in $(0,\infty)$;
2. $\lim_{t\to 0} \operatorname{mod} R_T(t) \;=\; 0$;
3. $\lim_{t\to\infty} (\operatorname{mod} R_T(t) - \log(t+1)) \;=\; \log \lambda_n \;<\; \infty$;
4. $\lambda_2 = 16$ and $\lim_{n\to\infty} \lambda_n^{1/n} = e^2$.

Thus,

5. $\operatorname{mod} R_T(t)$ is strictly increasing in $(0,\infty)$,
6. $\log(t+1) \;\le\; \operatorname{mod} R_T(t) \;\le\; \log \lambda_n (t+1)$.

Theorem A.11. *If $R = R(C_0, C_1)$ is a ring with a, $b \in C_0$ and c, $\infty \in C_1$, then*

$$\operatorname{mod} R \;\le\; \operatorname{mod} R_T \left(\frac{|c-a|}{|b-a|} \right).$$

Proof. By performing a preliminary similarity mapping, we may assume that $a = 0$, $b = -e_1$. Then the spherical symmetrizations C_0^*, C_1^* of C_0, C_1 in the negative and positive halves of the x_1-axis contain the complementary components of $R_T\,(|c-a|/|b-a|)$. Thus,

$$\operatorname{cap} R_T\,(|c-a|/|b-a|) \;\le\; \operatorname{cap} R^* \;\le \operatorname{cap} R,$$

as desired. □

Corollary A.1. *If $R = R(C_0, C_1)$ is a ring with a, $b \in C_0$ and c, $d \in C_1$, then*

$$\operatorname{mod} R \;\le\; \operatorname{mod} R_T \left(\frac{h(a,c)h(b,d)}{h(a,b)h(c,d)} \right).$$

Proof. By performing a preliminary chordal isometry, we may assume that $d = \infty$. Then

$$\frac{|c-a|}{|b-a|} = \frac{h(a,c)\sqrt{|c|^2+1}}{h(a,b)\sqrt{|b|^2+1}} = \frac{h(a,c)h(b,d)}{h(a,b)h(c,d)},$$

and we can apply Theorem A.11. □

Corollary A.2. *If $R = R(C_0, C_1)$ is a ring, then*

(a) $\operatorname{mod} R \;\leq\; \operatorname{mod} R_T\left(\frac{1}{h(C_0)h(C_1)}\right)$,

(b) $\operatorname{mod} R \;\leq\; \operatorname{mod} R_T\left(\frac{4h(C_0,C_1)}{h(C_0)h(C_1)}\right)$.

Proof. For (a), choose a, $b \in C_0$ and c, $d \in C_1$ so that

$$h(a,b) \;=\; h(C_0),$$

$$h(c,d) \;=\; h(C_1).$$

Then

$$\frac{h(a,c)h(b,d)}{h(a,b)h(c,d)} \;\leq\; \frac{1}{h(C_0)h(C_1)}$$

and we can apply Corollary A.1. For (b), choose $a \in C_0$ and $c \in C_1$ so that

$$h(a,c) \;=\; h(C_0,C_1).$$

Next pick $b \in C_0$ and $d \in C_1$ so that

$$h(a,b) \;\geq\; \frac{1}{2}\,h(C_0), \qquad h(c,d) \;\geq\; \frac{1}{2}\,h(C_1).$$

Then

$$\frac{h(a,c)h(b,d)}{h(a,b)h(c,d)} \;\leq\; \frac{4}{h(C_0)h(C_1)}$$

and we again apply Corollary A.1. □

We say that a sequence of sets E_j **converges uniformly to a set** E if, for each $\varepsilon > 0$, there exists a j_0 such that

$$\sup_{x \in E_j} h(x,E) \;<\; \varepsilon, \qquad \sup_{x \in E} h(x,E_j) \;<\; \varepsilon$$

for $j \geq j_0$.

Theorem A.12. *If the complementary components of a sequence of rings R_j converge uniformly to the corresponding complementary components of a ring R, then*

$$\operatorname{cap} R \;=\; \lim_{j\to\infty} \operatorname{cap} R_j.$$

Readers who wish to dispute this result should consult [78, 230] and [307].

A.2 The Inequalities by Martio–Rickman–Väisälä

In this section, following [210], we generalize the concept of a ring domain and give lower and upper estimates of its capacity.

Here a **condenser** is a pair $E = (A,C)$ where $A \subset \mathbb{R}^n$ is open and C is a non-empty compact set contained in A. E is a **ringlike condenser** if $B = A \setminus C$ is a ring, i.e., if B is a domain whose complement $\overline{\mathbb{R}^n} \setminus B$ has exactly two components where $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^n$. E is a **bounded condenser** if A is bounded. A condenser $E = (A,C)$ is said to be in a domain G if $A \subset G$.

The following lemma is immediate.

Lemma A.1. *If $f : G \to \mathbb{R}^n$ is open and $E = (A,C)$ is a condenser in G, then (fA, fC) is a condenser in fG.*

In the above situation we denote $fE = (fA, fC)$.

Let $E = (A,C)$ be a condenser. We set

$$\operatorname{cap} E = \operatorname{cap}(A,C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n \, dm$$

and call it the **capacity** of the condenser E . The set $W_0(E) = W_0(A,C)$ is the family of nonnegative functions $u : A \to R^1$ such that (1) $u \in C_0(A)$ (2) $u(x) \geq 1$ for $x \in C$, and (3) u is ACL. In the above formula

$$|\nabla u| = \left(\sum_{i=1}^{n} (\partial_i u)^2 \right)^{1/2}.$$

We mention some properties of the capacity of a condenser.

Lemma A.2. *If $E = (A,C)$ is a condenser, then*

$$\operatorname{cap} E = \inf_{u \in W_0^\infty(E)} \int_A |\nabla u|^n \, dm,$$

where $W_0^\infty(E) = W_0^\infty(A,C) = W_0(E) \cap C_0^\infty(A)$.

Proof. Obviously,

$$\operatorname{cap} E \leq \inf_{u \in W_0^\infty(E)} \int_A |\nabla u|^n \, dm.$$

The converse inequality is proved by a standard approximating argument. The construction involves first multiplying $u \in W_0(E)$ by $1+\varepsilon$, $\varepsilon > 0$, so that the resulting function is $\geq (1+\varepsilon)$ on C and then forming a smooth integral average; cf., e.g., [316], Section 27. The details may be omitted. □

Remark A.4. If $E = (A,C)$ is a ringlike condenser, then $\operatorname{cap} E = \operatorname{cap}(A \setminus C)$ in the sense of Gehring [66], p. 500. This is a direct consequence of n-dimensional versions of [66], Lemma 1, p. 501, and [66], Remark, p. 502.

Lemma A.3. *If $E = (A,C)$ is a condenser, then*

$$\operatorname{cap} E = \inf \operatorname{cap}\,(U,C)\,,$$

where the infimum is taken over all open sets U such that $\overline{U}$ is compact in A and $C \subset U$.

Proof. Obviously, $\operatorname{cap} E \le \operatorname{cap}(U,C)$ for all sets U of the above type; hence,

$$\operatorname{cap} E \le \inf \operatorname{cap}\,(U,\,C)\,.$$

Let $\varepsilon > 0$. Then there exists a function $u \in W_0(E)$ such that

$$\operatorname{cap} E > \int_A |\nabla u|^n \, dm - \varepsilon.$$

Since $\operatorname{spt} u$ is compact in A, there exists an open set U such that $\operatorname{spt} u \subset U$ and $\overline{U}$ is compact in A. Then $u \in W_0\,(U,C)$ and we obtain

$$\operatorname{cap}\,(U,C) \le \int_A |\nabla u|^n \, dm < \operatorname{cap}(A,C) + \varepsilon.$$

The lemma follows. □

Lemma A.4. *The inequality*

$$\operatorname{cap} E \le \frac{m(A)}{d\,(C,\partial A)^n}$$

holds for the capacity of a bounded condenser $E = (A,C)$.

Proof. Let $0 < \varepsilon < d(C,\partial A)^n$. There exists an open set U such that $C \subset U \subset \overline{U} \subset A$ and $d(C,\partial A)^n \le d(C,\partial U)^n + \varepsilon$. If we define $u(x) = d(x, CU)/d(C,\partial U)$, then $|u(x) - u(y)| \le |x-y|/d(C,\partial U)$ for all x, $y \in \mathbb{R}^n$. Thus, $u \in W_0(E)$ and $|\nabla u| \le 1/d(C,\partial U)$ a.e., which implies

$$\operatorname{cap} E \le \int_A d\,(C,\partial U)^{-n}\, dm = \frac{m(A)}{d\,(C,\partial U)^n} \le \frac{m(A)}{d\,(C,\partial A)^n - \varepsilon}.$$

Letting $\varepsilon \to 0$ gives the desired result. □

Lemma A.5. *Suppose that $E = (A,C)$ is a condenser such that C is connected. Then*

$$(\operatorname{cap} E)^{n-1} \ge b_n \frac{d(C)^n}{m(A)}\,,$$

where b_n is a positive constant that depends only on n.

Proof. By Lemma A.3 we may suppose that A is bounded. We may also assume that $d(C) = r > 0$ and that C contains the origin and the point re_n. Let $u \in W_0^\infty(E)$. For $0 < t < r$, we let $T(t)$ denote the hyperplane $x_n = t$. Using the method of [322], p. 9, we estimate the integral

$$\int_{T(t)} |\nabla u|^n \, dm_{n-1}.$$

Fix $z \in C \cap T(t)$. For $y \in \mathbb{S}^{n-2}$, let $R(y)$ be the supremum of all $t_0 > 0$ such that $z + ty \in A$ for $0 \le t < t_0$. Then

$$\int_0^{R(y)} |\nabla u(z+ty)| \, dt \ge u(z) - u(z + R(y)y) \ge 1$$

for all $y \in \mathbb{S}^{n-2}$. By Hölder's inequality, this implies

$$1 \le (n-1)^{n-1} R(y) \int_0^{R(y)} |\nabla u(z+ty)|^n t^{n-2} \, dt.$$

Integrating over $y \in \mathbb{S}^{n-2}$ yields

$$\begin{aligned}(n-1)^{1-n} \int_{\mathbb{S}^{n-2}} R^{-1} \, dm_{n-2} &\le \int_{\mathbb{S}^{n-2}} dm_{n-2}(y) \int_0^{R(y)} |\nabla u(z+ty)|^n t^{n-2} \, dt \qquad (A.1)\\ &\le \int_{T(t)} |\nabla u|^n \, dm_{n-1}.\end{aligned}$$

On the other hand, we obtain by Hölder's inequality

$$\begin{aligned}\omega_{n-2}^n = \left(\int_{\mathbb{S}^{n-2}} dm_{n-2} \right)^n &\le \int_{\mathbb{S}^{n-2}} R^{n-1} \, dm_{n-2} \left(\int_{\mathbb{S}^{n-2}} R^{-1} \, dm_{n-2} \right)^{n-1} \\ &\le (n-1) \, m_{n-1}(A \cap T(t)) \left(\int_{\mathbb{S}^{n-2}} R^{-1} \, dm_{n-2} \right)^{n-1}. \qquad (A.2)\end{aligned}$$

Setting $f(t) = m_{n-1}(A \cap T(t))$, we obtain from (A.1) and (A.2)

$$\int_{T(t)} |\nabla u|^n \, dm_{n-1} \ge (n-1)^{1-n-1/(n-1)} \, \omega_{n-2}^{n/(n-1)} f(t)^{1/(1-n)}.$$

Integrating over $0 < t < r$, we obtain

$$\int_A |\nabla u|^n \, dm \geq (n-1)^{1-n-1/(n-1)} \, \omega_{n-2}^{n/(n-1)} \int_0^r f(t)^{1/(1-n)} \, dt. \tag{A.3}$$

Hölder's inequality gives

$$\begin{aligned} r^n = \left(\int_0^r dt \right)^n &\leq \left(\int_0^r f(t) \, dt \right) \left(\int_0^r f(t)^{1/(1-n)} \, dt \right)^{n-1} \\ &\leq m(A) \left(\int_0^r f(t)^{1/(1-n)} \, dt \right)^{n-1}. \end{aligned}$$

By (A.3), this implies

$$\left(\int_A |\nabla u|^n \, dm \right)^{n-1} \geq (n-1)^{-2+2n-n^2} \, \omega_{n-2}^n \, \frac{r^n}{m(A)}.$$

Since this holds for every $u \in W_0^\infty(E)$, the lemma follows. □

A.3 The Hesse Equality

Let G be a domain in the compactified Euclidean n-space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, E and F disjoint, nonempty, compact sets in the closure of G. We associate two numbers with this geometric configuration as follows. Let $M_p(E,F,G)$ be the p-modulus (reciprocal of the p-extremal length) of the family of paths connecting E and F in G. Let $\mathrm{cap}_p(E,F,G)$ be the p-capacity of E and F relative to G, defined as the infimum of the numbers $\int_G |\nabla u(x)|^p \, dm(x)$ over all ACL functions u in G with boundary values 0 and 1 on E and F, respectively. Hesse has shown that $\mathrm{cap}_p(E,F,G) = M_p(E,F,G)$ whenever E and F do not intersect ∂G, which is the main goal of this section; see [122]. Hesse generalized Ziemer's result in [338], where he assumed that either E or F contains the complement of an open n-ball. He also obtained a continuity theorem for the p-modulus (Theorem A.16) and a theorem on the kinds of densities that can be used in computing the p-modulus (Theorem A.14).

For $n \geq 2$, $\overline{\mathbb{R}^n}$ denotes the one-point compactification of the Euclidean n-space $\mathbb{R}^n$: $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. All topological considerations here refer to the metric space $(\overline{\mathbb{R}^n}, q)$, where q is the chordal metric on $\overline{\mathbb{R}^n}$ defined by stereographic projection. If $A \subset \overline{\mathbb{R}^n}$, then $\overline{A}$ and ∂A denote the closure and boundary of A, respectively. If $b \in \overline{\mathbb{R}^n}$ and $B \subset \overline{\mathbb{R}^n}$, then $q(b,B)$ denotes the chordal distance of b from B. For $x \in \mathbb{R}^n$, $|x|$ denotes the usual Euclidean norm of x. $B^n(x,r)$ denotes the open n-ball with center

x and radius r. Set also $\mathbb{B}^n = B^n(0,1)$. For $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, $d(x,A)$ denotes the Euclidean distance of x from A. Lebesgue n-measure on $\mathbb{R}^n$ is denoted by m_n or by m if there is no chance for confusion. We let $\Omega_n = m_n(\mathbb{B}^n)$.

Let Γ be a collection of paths in $\overline{\mathbb{R}^n}$. We let adm$\,\Gamma$ denote the set of Borel functions $\rho : \mathbb{R}^n \to [0,\infty]$ satisfying the condition $\int_\gamma \rho \, ds \geq 1$ for every locally rectifiable $\gamma \in \Gamma$. adm$\,\Gamma$ is called the set of *admissible functions* for Γ. For $p \in (1,\infty)$, the *p-modulus of* Γ is defined as

$$M_p(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^p \, dm,$$

where the infimum is taken over all $\rho \in \operatorname{adm}\Gamma$. For the basic facts about the p-modulus, see [316], Chapter 1. The p-extremal length of Γ is defined as the reciprocal of the p-modulus of Γ.

Now, let G be a domain in $\overline{\mathbb{R}^n}$ and let E and F be compact, disjoint, nonempty sets in $\overline{G}$. Let $\Gamma(E,F,G)$ denote the set of all paths connecting E and F in G. More precisely, if $\gamma \in \Gamma(E,F,G)$, then $\gamma : I \to G$ is a continuous mapping, where I is an open interval and $\overline{\gamma(I)} \cap E$ and $\overline{\gamma(I)} \cap F$ are both nonempty. We write $M_p(E,F,G)$ for the p-modulus of $\Gamma(E,F,G)$. Let $\mathscr{A}(E,F,G)$ denote the set of real-valued functions u such that (1) u is continuous on $E \cup F \cup G$, (2) $u(x) = 0$ if $x \in E$ and $u(x) = 1$ if $x \in F$, and (3) u restricted to $G \setminus \{\infty\}$ is ACL. For the definition and basic facts about ACL functions; see [316], Chapter 3. Given $p \in (1,\infty)$, the *p-capacity of E and F relative to G* $\operatorname{cap}_p(E,F,G)$ is defined by

$$\operatorname{cap}_p(E,F,G) \; = \; \inf \int_G |\nabla u|^p \, dm,$$

where the infimum is taken over all $u \in \mathscr{A}(E,F,G)$.

The p-capacity has the following continuity property.

Theorem A.13. *Let $E_1 \supset E_2 \supset \ldots$ and $F_1 \supset F_2 \supset \ldots$ be disjoint sequences of nonempty compact sets in the closure of a domain G. Let $E = \bigcap_{i=1}^\infty E_i$, $F = \bigcap_{i=1}^\infty F_i$. Then*

$$\lim_{i \to \infty} \operatorname{cap}_p(E_i,F_i,G) \; = \; \operatorname{cap}_p(E,F,G).$$

Proof. Since $\mathscr{A}(E_i,F_i,G) \subset \mathscr{A}(E_{i+1},F_{i+1},G) \subset \mathscr{A}(E,F,G)$ for all i, it follows that $\operatorname{cap}_p(E_i,F_i,G)$ is monotone decreasing in i and, therefore, that

$$\lim_{i \to \infty} \operatorname{cap}_p(E_i,F_i,G) \; \geq \; \operatorname{cap}_p(E,F,G).$$

For the reverse inequality, choose $u \in \mathscr{A}(E,F,G)$ and $\varepsilon \in (0,1/2)$. Define $f : (-\infty,+\infty) \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \le \varepsilon, \\ (1-2\varepsilon)^{-1}(x-1+\varepsilon)+1 & \text{if } \varepsilon < x < 1-\varepsilon, \\ 1 & \text{if } x \ge 1-\varepsilon. \end{cases}$$

Let $u' = f \circ u$. Since f is Lipschitz continuous on $(-\infty, +\infty)$ with Lipschitz constant $(1-2\varepsilon)^{-1}$, it follows that u' is ACL on $G \setminus \{\infty\}$ and $|\nabla u'| \le (1-2\varepsilon)^{-1}|\nabla u|$ a.e. in G.

Let A and B be open sets in $\overline{\mathbb{R}^n}$ such that $\{x \in E \cup F \cup G : u(x) < \varepsilon\} = (E \cup F \cup G) \cap A$ and $\{x \in E \cup F \cup G : u(x) > 1-\varepsilon\} = (E \cup F \cup G) \cap B$. For large i, we have $E_i \subset A$ and $F_i \subset B$ and, for such i, we can extend u' continuously to $E_i \cup F_i \cup G$ by setting $u' = 0$ on $\partial G \cap (E_i \setminus E)$ and $u' = 1$ on $\partial G \cap (F_i \setminus F)$. Therefore, $u' \in \mathscr{A}(E_i, F_i, G)$ for large i. This implies that for large i we have

$$\mathrm{cap}_p(E_i, F_i, G) \le \int_G |\nabla u'|^p \, dm \le \frac{1}{(1-2\varepsilon)^p} \int_G |\nabla u|^p \, dm.$$

Hence,

$$\lim_{i \to \infty} \mathrm{cap}_p(E_i, F_i, G) \le \frac{1}{(1-2\varepsilon)^p} \int_G |\nabla u|^p \, dm.$$

Since $u \in \mathscr{A}(E, F, G)$ and $\varepsilon \in (0, 1/2)$ are arbitrary, we get the reverse inequality, as desired. □

Let Γ be a collection of paths in $\overline{\mathbb{R}^n}$. Let $\mathscr{B} \subset \mathrm{adm}\,\Gamma$. We say that $\mathscr{B}$ is p-complete if

$$M_p(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^p \, dm,$$

where the infimum is taken over all $\rho \in \mathscr{B}$.

Let $B \subset \mathrm{adm}\,\Gamma$ be the collection of $\rho \in \mathrm{adm}\,\Gamma$ such that ρ is lower semicontinuous. It follows from the Vitali–Caratheodory theorem (see, e.g., Theorem 2.24 in [261]) that B is p-complete for all $p \in (1, \infty)$.

Lemma A.6. *Let $\varphi : \mathbb{R}^n \to [0, \infty]$ be a Borel function in $\varphi \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, and let $r : \mathbb{R}^n \to [0, \infty]$ satisfy $|r(x_2) - r(x_1)| \le |x_2 - x_1|$ for all $x_1, x_2 \in \mathbb{R}^n$. Define $T_{\varphi,r} : \mathbb{R}^n \to [0, \infty]$ by*

$$T_{\varphi,r}(x) = \frac{1}{\Omega_n} \int_{B^n(1)} \varphi(x + r(x)y) \, dm(y).$$

Then $T_{\varphi,r}$ has the following properties.

(1) *If $r(x_0) > 0$, then*

$$T_{\varphi,r}(x_0) = \frac{1}{\Omega_n r(x_0)^n} \int_{B^n(x_0, r(x_0))} \varphi(y) \, dm(y) < \infty.$$

(2) *If φ is lower semicontinuous, then so is $T_{\varphi,r}$.*

(3) *If $r(x_0) > 0$, then $T_{\varphi,r}$ is continuous at x_0.*

(4) *If φ is finite and continuous on a domain G in $\mathbb{R}^n$ and $0 \le r(x) < d(x, \mathbb{R}^n \setminus G)$, then $T_{\varphi,r}$ is finite and continuous on G.*

(5) *$|T_{\varphi,r}(x) r(x)^{n/p}| \le C$ for some constant $C \in [0,\infty)$ and all $x \in \mathbb{R}^n$. The constant C depends on φ.*

(6) *Let $k = \sup |r(x_2) - r(x_1)||x_2 - x_1|^{-1}$, where the supremum is taken over all x_1, $x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$. Then $\|T_{\varphi,r}\|_p \le (1-k)^{-n/p}\|\varphi\|_p$, where $\| \, \|_p$ is the usual $L^p(\mathbb{R}^n)$-norm and the right-hand side of the inequality is infinite in case $k = 1$.*

Proof. Statement (1) follows from the change of variables $y' = x_0 + r(x_0)y$ and the Hölder inequality. To prove (2), let x_0 be an arbitrary point in $\mathbb{R}^n$ and $\{x_j\}_{j=1}^\infty$ a sequence of points in $\mathbb{R}^n$ tending to x_0. Fatou's lemma and the lower semicontinuity of φ imply

$$
\begin{aligned}
\liminf_{j\to\infty} T_{\varphi,r}(x_j) &= \liminf_{j\to\infty} \frac{1}{\Omega_n} \int\limits_{B^n(1)} \varphi(x_j + r(x_j)y) \, dm(y) \\
&\ge \frac{1}{\Omega_n} \int\limits_{B^n(1)} \liminf_{j\to\infty} \varphi(x_j + r(x_j)y) \, dm(y) \\
&\ge \frac{1}{\Omega_n} \int\limits_{B^n(1)} \varphi(x_0 + r(x_0)y) \, dm(y) \; = \; T_{\varphi,r}(x_0).
\end{aligned}
$$

This shows that $T_{\varphi,r}$ is lower semicontinuous.

To prove (3), we observe that since r is continuous, $r(x) > 0$ for all x in some neighborhood of x_0 and, therefore, by (1),

$$
T_{\varphi,r}(x) = \frac{1}{\Omega_n r(x)^n} \int\limits_{B^n(x,r(x))} \varphi(y) \, dm(y)
$$

for all x in some neighborhood of x_0. The right-hand side of the above formula is continuous in x and, therefore, $T_{\varphi,r}$ is continuous at x_0.

Now, we proceed to prove (4). We observe that if $x \in G$, then $x + r(x)y \in G$ for any $y \in \mathbb{R}^n$ with $|y| \le 1$. Fix $x_0 \in G$ and let B be a closed ball with center x_0 and lying in G. Then $B' = \{x' : x' = x + r(x)y, \; x \in B, \; |y| \le 1\}$ is a compact subset of G. Since φ is uniformly continuous on B', given $\varepsilon > 0$, there exists a $\delta 0$ such that $|\varphi(x_2') - \varphi(x_1')| < \varepsilon$ if $x_1', x_2' \in B'$ and $|x_2' - x_1'| < \delta$. Let $x_1 \in B$ with $|x_1 - x_0| < \delta/2$. Then $|(x_1 + r(x_1)y) - (x_0 + r(x_0)y)| < \delta$ for any $|y| \le 1$. Hence,

$$
|T_{\varphi,r}(x_1) - T_{\varphi,r}(x_0) \le \frac{1}{\Omega_n} \int\limits_{B^n(1)} |\varphi(x_1 + r(x_1)y) - \varphi(x_0 + r(x_0)y)| \, dm(y) < \varepsilon.
$$

Hence, $T_{\varphi,r}$ is continuous on G.

Next, to prove (5), we need only consider $x \in \mathbb{R}^n$ such that $r(x) > 0$. For such x, we have

$$T_{\varphi,r}(x) \;=\; \frac{1}{\Omega_n r(x)^n} \int\limits_{B^n(x,r(x))} \varphi(y)\, dm(y).$$

Applying Hölder's inequality with exponents p and $p/(p-1)$, we get

$$T_{\varphi,r}(x) \;\le\; \frac{1}{\Omega_n r(x)^n} \left[\, \int\limits_{B^n(x,r(x))} \varphi^p(y)\, dm(y) \right]^{1/p} \left[\Omega_n r(x)^n\right]^{(n-p)/p}.$$

Hence,

$$T_{\varphi,r}(x) r(x)^{n/p} \;\le\; C = \Omega_n^{-1/p} \left[\, \int\limits_{\mathbb{R}^n} \varphi^p\, dm \right]^{1/p} \;<\; \infty,$$

as desired.

Finally, we proceed to prove (6):

$$\|T_{\varphi,r}\|_p^p \;=\; \int\limits_{\mathbb{R}^n} T^p_{\varphi,r}(x)\, dm(x) \;=\; \int\limits_{\mathbb{R}^n} \left[\frac{1}{\Omega_n} \int\limits_{\mathbb{B}^n} \varphi(x + r(x)y)\, dm(y) \right]^p dm(x).$$

After applying Hölder's inequality to the inner integral and simplifying, we get

$$\|T_{\varphi,r}\|_p^p \;\le\; \frac{1}{\Omega_n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{B}^n} \varphi^p(x + r(x)y)\, dm(y)\, dm(x).$$

Interchanging the order of integration gives

$$\|T_{\varphi,r}\|_p^p \;\le\; \frac{1}{\Omega_n} \int\limits_{\mathbb{B}^n} \int\limits_{\mathbb{R}^n} \varphi^p(x + r(x)y)\, dm(x)\, dm(y). \tag{A.4}$$

Define, for $y \in \mathbb{B}^n$, $\theta_y : \mathbb{R}^n \to \mathbb{R}^n$ by $\theta_y(x) = x + r(x)y$. It easily follows that θ_y is injective and, hence, by a theorem in topology, $\theta_y(\mathbb{R}^n)$ is a domain. Since θ_y is Lipschitz continuous, it follows by Theorem 1 and Corollary 2 in [323] that the change-of-variables formula for multiple integrals holds with θ_y as the mapping function. Therefore,

$$\int\limits_{\theta_y(\mathbb{R}^n)} \varphi^p(x)\, dm(x) \;=\; \int\limits_{\mathbb{R}^n} \varphi^p \circ \theta_y(x)\, \mu_y'(x)\, dm(x), \tag{A.5}$$

where μ_y' is the volume derivative of the homeomorphism θ_y; see Definition 24.1 in [316]. Since

$$\mu_y'(x) = \lim_{r\to 0} \frac{m(\theta_y(\overline{B^n(x,r)}))}{\Omega_n r^n}$$

a.e. x, the estimates

$$m(\theta_y(\overline{B^n(x,r)})) \geq \Omega_n\{\inf_{|x'-x|=r} |\theta_y(x') - \theta_y(x)|\}^n$$

and

$$|\theta_y(x') - \theta_y(x)| \geq (1-k)\,|x'-x|$$

yield $\mu_y'(x) \geq (1-k)^n$ a.e. x in $\mathbb{R}^n$. This result and (A.4) and (A.5) give

$$\|T_{\varphi,r}\|_p^p \leq \frac{1}{\Omega_n(1-k)^n} \int_{\mathbb{B}^n}\int_{\mathbb{R}^n} \varphi^p(x)\, dm(x)\, dm(y) = (1-k)^{-n}\|\varphi\|_p^p,$$

as desired. □

For the remainder of this section, G will denote a domain in $\overline{\mathbb{R}^n}$, and E and F will be compact, disjoint, nonempty sets in $\overline{G}$. We write $\Gamma = \Gamma(E,F,G)$. We let $d:\mathbb{R}^n \to [0,\infty)$ be the function defined by $d(x) = d(x, ((\overline{\mathbb{R}^n}\setminus G)\cup E\cup F)\setminus\{\infty\})$ and let l.s.c.$(\mathbb{R}^n)$ be the extended real-valued, lower semicontinuos functions defined on $\mathbb{R}^n$.

Lemma A.7. *Let $\mathscr{A} \subset \operatorname{adm}\Gamma$ satisfying (1) $\rho \in$ l.s.c.$(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, (2) ρ is continuous on $G\setminus(E\cup F\cup\{\infty\})$, and (3) $\rho(x)\cdot d(x)^{n/p}$ is bounded above for $x\in\mathbb{R}^n$. Then $\mathscr{A}$ is a p-complete family.*

Proof. It suffices to prove that $M = \inf\int_{\mathbb{R}^n}\rho^n(x)\ dm(x) \leq M_p(\Gamma)$, where the infimum is taken over all $\rho\in\mathscr{A}$. Choose $\rho\in\operatorname{adm}\Gamma\cap L^p(\mathbb{R}^n)\cap$ l.s.c.$(\mathbb{R}^n)$. Let $\varepsilon\in(0,1)$ and let $g = T_{\rho,\varepsilon d}$. Suppose $\gamma\in\Gamma$ is locally rectifiable. We may assume, by parameterizing γ, that $\gamma:(a,b)\to G$, where a, $b\in[-\infty,+\infty]$, and that the length of $\gamma|[t_1,t_2]$ is t_2-t_1 for all t_1, $t_2\in(a,b)$. Note that γ restricted to closed subintervals of (a,b) is absolutely continuous.

Let $\gamma_y:(a,b)\to G$, $y\in\mathbb{B}^n$, be the path defined by $\gamma_y(t) = \gamma(t)+\varepsilon d(\gamma(t))y$. Choose $e\in\overline{\gamma(a,b)}\cap E$. Let $t_j\in(a,b)$, $j=1,2,\ldots$, be such that $\gamma(t_j)\to e$ as $j\to\infty$. If $e\neq\infty$, then, clearly, $\gamma_y(t_j)\to e$ as $j\to\infty$. If $e=\infty$, then, for fixed $t'\in(a,b)$, the triangle inequality and the fact that d is Lipschitz continuous with Lipschitz constant 1 imply $|\gamma_y(t_j)-\gamma_y(t')|\geq(1-\varepsilon)|\gamma(t_j)-\gamma(t')|$ and, therefore, $\gamma_y(t_j)\to\infty = e$ as $j\to\infty$. Hence, $\overline{\gamma_y(a,b)}\cap E\neq 0$. Similarly, $\overline{\gamma_y(a,b)}\cap F\neq 0$. Therefore, $\gamma_y\in\Gamma$. Also, γ_y restricted to closed subintervals of (a,b) is absolutely continuous. An easy estimate shows that $|\gamma_y'(t)|\leq 1+\varepsilon$ a.e. on (a,b).

We have

$$\int_\gamma g\, ds = \int_a^b g(\gamma(t))\, dt = \frac{1}{\Omega_n}\int_a^b\int_{\mathbb{B}^n}\rho(\gamma(t)+\varepsilon d(\gamma(t))y)\, dm(y)\, dt$$

$$= \frac{1}{\Omega_n} \int_{\mathbb{B}^n} \int_a^b \rho(\gamma_y(t)) |\gamma_y'(t)| |\gamma_y'(t)|^{-1} \, dt \, dm(y)$$

$$\geq \frac{1}{(1+\varepsilon)\Omega_n} \int_{\mathbb{B}^n} \int_{\gamma_y} \rho \, ds \, dm(y) \geq \frac{1}{1+\varepsilon}.$$

This result and Lemma A.6 show that $(1+\varepsilon)g \in \mathscr{A} \subset \operatorname{adm}\Gamma$. Hence,

$$M \leq (1+\varepsilon)^p \|g\|_p^p = (1+\varepsilon)^p \|T_{\rho,\varepsilon d}\|_p^p.$$

From Lemma A.6 (6), we get

$$M \leq \frac{(1+\varepsilon)^p}{(1-\varepsilon)^n} \int_{\mathbb{R}^n} \rho^p(x) \, dm(x).$$

Since $\varepsilon \in (0,1)$ and $\rho \in \operatorname{adm}\Gamma \cap L^p(\mathbb{R}^n) \cap \mathrm{l.s.c.}(\mathbb{R}^n)$ are arbitrary, we get $M \leq M_p(\Gamma)$, as desired. □

For $r \in (0,1)$, we define $E(r) = \{x \in \overline{\mathbb{R}^n} : q(x,E) \leq r\}$ and $F(r) = \{x \in \overline{\mathbb{R}^n} : q(x,F) \leq r\}$. Let $\rho : \mathbb{R}^n \to [0,\infty]$ be a Borel function. We define $L(\rho, r)$ as the infimum of the integrals $\int_\gamma \rho \, ds$, where γ is a locally rectifiable path in G connecting $E(r)$ and $F(r)$. Since $L(\rho, r)$ is non-decreasing for decreasing r, we can define

$$L(\rho) = \lim_{r \to 0} L(\rho, r).$$

Remark A.5. We observe that $L(\rho) \geq 1$ if and only if for every $\varepsilon \in (0,1)$ there exists a $\delta \in (0,1)$ such that $\int_\gamma \rho \, ds \geq 1 - \varepsilon$ for every locally rectifiable path γ in G connecting $E(r)$ and $F(r)$ with $r \leq \delta$.

Lemma A.8. *Suppose there exists a p-complete family $\mathscr{B}_0 \subset \operatorname{adm}\Gamma$ such that $L(\rho) \geq 1$ for every $\rho \in \mathscr{B}_0$. Then the family $\mathscr{B} \subset \operatorname{adm}\Gamma$ consisting of all $\rho \in \operatorname{adm}\Gamma$ such that* (1) $\rho \in \mathrm{l.s.c.}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *and* (2) ρ *is continuous on $G \setminus \{\infty\}$ is p-complete.*

Proof. Let $\mathscr{B}_1$ be the set of $\rho \in \operatorname{adm}\Gamma$ such that $\rho \in \mathrm{l.s.c.}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $L(\rho) \geq 1$. It follows from the Vitali–Caratheodory theorem that $\mathscr{B}_1$ is p-complete; see, e.g., Theorem 2.24 in [261].

Let $\rho \in \mathscr{B}_1$ and $\varepsilon \in (0,1)$. Let δ be as in Remark A.5 and choose $\delta' \in (0,1)$ such that if $x \in E \setminus \{\infty\}$ [resp., $F \setminus \{\infty\}$] and $y \in \mathbb{R}^n$, $|x-y| < \delta'$, then $y \in E(\delta)$ [resp., $F(\delta)$]. Let $r : \mathbb{R}^n \to [0,1]$ be defined by $r(x) = \varepsilon\delta' \min(1, d(x, \mathbb{R}^n \setminus G))$. Let $g = T_{\rho,r}$. Suppose $\gamma \in \Gamma$ is locally rectifiable and assume that $\gamma : (a,b) \to G$ is parameterized as in the proof of Lemma A.7. Let $\gamma_y : (a,b) \to G$, $y \in \mathbb{B}^n$, be the path defined by $\gamma_y(t) = \gamma(t) + r(\gamma(t))y$. It follows, using the same method as in the proof of Lemma A.7, that γ_y connects $E(\delta)$ and $F(\delta)$. A computation similar to the one in the proof of Lemma A.7 yields

$$\int\limits_{\gamma} g\, ds \;\geq\; \frac{1}{(1+\varepsilon)\Omega_n}\int\limits_{\mathbb{B}^n}\int\limits_{\gamma_y} \rho\, ds\, dm(y) \;\geq\; \frac{1-\varepsilon}{1+\varepsilon}.$$

The above and Lemma A.6 show that $(1+\varepsilon)(1-\varepsilon)^{-1}g \in \mathscr{B}$. Let $M = \inf \int_{\mathbb{R}^n} \rho^p(x)\, dm(x)$, where the infimum is taken over all $\rho \in \mathscr{B}$. Then, by Lemma A.6,

$$M \;\leq\; \frac{(1+\varepsilon)^p}{(1-\varepsilon)^p}\|g\|_p^p \;=\; \frac{(1+\varepsilon)^p}{(1-\varepsilon)^p}\|T_{\rho,r}\|_p^p \;\leq\; \frac{(1+\varepsilon)^p}{(1-\varepsilon)^p(1-\varepsilon)^n}\|\rho\|_p^p.$$

Since $\rho \in \mathscr{B}_1$ and $\varepsilon \in (0,1)$ are arbitrary and $\mathscr{B}_1$ is p-complete, it follows from the above that $M \leq M_p(\Gamma)$. This completes the proof since the reverse inequality is trivial. □

Lemma A.9. *Suppose $(E \cup F) \cap \partial G = \emptyset$. Let $\rho : \mathbb{R}^n \to [0,\infty]$ be a Borel function and let $\rho|G \setminus (E \cup F \cup \{\infty\})$ be finite-valued and continuous. Let $\varepsilon \in (0,\infty)$. Then there exists a locally rectifiable path $\gamma \in \Gamma$ such that*

$$\int\limits_{\gamma} \rho\, ds \;\leq\; L(\rho) + \varepsilon.$$

Proof. We may assume that $L(\rho) < \infty$. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon/8$. Let $\{r_k\}_{k=1}^{\infty}$ be a strictly monotone decreasing sequence of positive numbers such that (1) $\lim_{k\to\infty} r_k = 0$ and (2) $E(r_k) \cap F(r_k) = \emptyset$. $E(r_k)$, $F(r_k) \subset G$, and $\infty \notin \partial E(r_k)$, $\partial F(r_k)$ for $k = 1,2,\ldots$. It follows that $\partial E(r_k) \cap E = \emptyset$, $\partial F(r_k) \cap F = \emptyset$ for $k = 1,2,\ldots$. Let Γ_k be the paths in G connecting $E(r_k)$ and $F(r_k)$, $k = 1,2,\ldots$. Choose $\gamma_k \in \Gamma_k$ such that γ_k is locally rectifiable and

$$\int\limits_{\gamma_k} \rho\, ds \;\leq\; L(\rho, r_k) + \varepsilon/2 \leq L(\rho) + \frac{\varepsilon}{2}. \tag{A.6}$$

Let x_{kj} [resp., y_{kj}], defined for $j < k$, be the last [resp., first] point of γ_k in $E(r_j)$ [resp., $F(r_j)$]. We have $x_{kj} \in \partial E(r_j)$ and $y_{kj} \in \partial F(r_j)$. By considering successive subsequences and then a diagonal sequence and then relabeling the sequences, we may assume that $x_{kj} \to x_j \in \partial E(r_j)$ and $y_{kj} \to y_j \in \partial F(r_j)$ as $k \to \infty$. Let $V_j \subset G \setminus (E \cup F \cup \{\infty\})$ [resp., $W_j \subset G \setminus (E \cup F \cup \{\infty\})$] be an open Euclidean ball with the center x_j [resp., y_j] such that $\int \rho\, ds \;<\; \varepsilon_j$, where the integral is taken over any line segment lying in V_j [resp., W_j], $j = 1,2,\ldots$. This can be done since ρ is continuous on $G \setminus (E \cup F \cup \{\infty\})$ and, hence, locally bounded there.

Let Ψ_j [resp., Φ_j] be the set of rectifiable paths $\alpha : [a,b] \to G$ such that $\alpha(a) \in V_j$ [resp., $\alpha(a) \in W_j$] and $\alpha(b) \in V_{j-1}$ [resp., $\alpha(b) \in W_{j-1}$], $j = 2,3,\ldots$. Let Λ be the set of rectifiable paths $\alpha : [a,b] \to G$ such that $\alpha(a) \in V_1$ and $\alpha(b) \in W_1$. For any positive integer k, there exists a path in the sequence $\{\gamma_i\}_{i=1}^{\infty}$, say $\gamma_{i(k)}$, such that $x_{i(k),j} \in V_j$ and $y_{i(k),j} \in W_j$ for $j = 1,2\ldots,k$. This implies that $\gamma_{i(k)}$ has distinct subpaths in $\Psi_2, \Psi_3, \ldots, \Psi_k$, $\Phi_2, \Phi_3, \ldots, \Phi_k$, Λ. Hence, for every positive integer k, we have, using Eq. (A.6),

$$\inf_{\gamma\in\Lambda}\int\limits_{\gamma}\rho\, ds+\sum_{j=2}^{k}\inf_{\gamma\in\Psi_j}\int\limits_{\gamma}\rho\, ds+\sum_{j=2}^{k}\inf_{\gamma\in\Phi_j}\int\limits_{\gamma}\rho\, ds \le \int\limits_{\gamma_i(k)}\rho\, ds \le L(\rho)+\frac{\varepsilon}{2}.$$

Since k is arbitrary, we get

$$\inf_{\gamma\in\Lambda}\int\limits_{\gamma}\rho\, ds+\sum_{j=2}^{k}\inf_{\gamma\in\Psi_j}\int\limits_{\gamma}\rho\, ds+\sum_{j=2}^{k}\inf_{\gamma\in\Phi_j}\int\limits_{\gamma}\rho\, ds \tag{A.7}$$

$$\le L(\rho,r_k) + \frac{\varepsilon}{2}.$$

Choose $\theta\in\Lambda$ such that

$$\int\limits_{\theta}\rho\, ds < \inf_{\gamma\in\Lambda}\int\limits_{\gamma}\rho\, ds+\varepsilon_1. \tag{A.8}$$

Choose $\tau_j\in\Psi_j$, $\sigma_j\in\Phi_j$, $j=2,3,\dots$, such that

$$\int\limits_{\tau_j}\rho\, ds < \inf_{\gamma\in\Psi_j}\int\limits_{\gamma}\rho\, ds+\varepsilon_j \tag{A.9}$$

and

$$\int\limits_{\sigma_j}\rho\, ds < \inf_{\gamma\in\Phi_j}\int\limits_{\gamma}\rho\, ds+\varepsilon_j. \tag{A.10}$$

Let α_j [resp., β_j] be the line segment in V_j [resp., W_j] connecting the endpoints of τ_j and τ_{j+1} [resp., σ_j and σ_{j+1}], $j=2,3,\dots$. Let α_1 [resp., β_1] be the line segment in V_1 [resp., W_1] connecting the endpoints of τ_2 and θ [resp., σ_2 and θ]. We have

$$\int\limits_{\alpha_j}\rho\, ds < \varepsilon_j, \quad \int\limits_{\beta_j}\rho\, ds < \varepsilon_j, \quad j=1,2,\dots. \tag{A.11}$$

Let $\gamma\in\Gamma$ be the locally rectifiable path $\gamma=\dots\tau_3\alpha_2\tau_2\alpha_1\theta\beta_1\sigma_2\beta_2\sigma_3\dots$. Finally, by Eqs. (A.7)–(A.11),

$$\int\limits_{\gamma}\rho\, ds = \sum_{j=1}^{\infty}\int\limits_{\alpha_j}\rho\, ds + \sum_{j=1}^{\infty}\int\limits_{\beta_j}\rho\, ds + \int\limits_{\theta}\rho\, ds + \sum_{j=2}^{\infty}\int\limits_{\tau_j}\rho\, ds + \sum_{j=2}^{\infty}\int\limits_{\sigma_j}\rho\, ds$$

$$\le \sum_{j=1}^{\infty}\varepsilon_j + \sum_{j=1}^{\infty}\varepsilon_j + \sum_{j=2}^{\infty}\varepsilon_j + \sum_{j=2}^{\infty}\varepsilon_j + L(\rho) + \varepsilon,$$

as desired. □

Lemma A.10. *Suppose* $(E\cup F)\cap\partial G=\emptyset$. *Let* $\mathscr{B}\subset \operatorname{adm}\Gamma$ *be the set of* $\rho\in\operatorname{adm}\Gamma$ *such that* (1) $\rho\in$ 1.s.c.$(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ *and* (2) ρ *is continuous on* $G\setminus\{\infty\}$. *Then* $\mathscr{B}$ *is p-complete.*

Proof. Lemma A.9 shows that $L(\rho) \geq 1$ for every ρ in the p-complete family $\mathscr{A}$ defined in Lemma A.7. Hence, this family $\mathscr{A}$ satisfies the hypotheses of Lemma A.8. Therefore, $\mathscr{B}$ is p-complete. □

Theorem A.14. *Suppose* $(E \cup F) \cap \partial G = \emptyset$. *Let* $\mathscr{B} \subset \operatorname{adm} \Gamma$ *be the set of* $\rho \in \operatorname{adm} \Gamma$ *such that* (1) $\rho \in$ 1.s.c.$(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *and* (2) ρ *is continuous on* $G \setminus \{\infty\}$, (3) $\rho(x) \cdot d(x)^{n/p}$ *is bounded above for* $x \in \mathbb{R}^n$, *and* (4) $L(\rho) \geq 1$. *Then* $\mathscr{B}$ *is a p-complete family.*

Proof. Choose ρ in the p-complete family $\mathscr{B}$ of Lemma A.10 and let $\varepsilon \in (0,1)$. Let $g = T_{\rho,\varepsilon d}$. It follows exactly as in the proof of Lemma A.7 that $\int_\gamma g\, ds \geq (1+\varepsilon)^{-1}$ for every locally rectifiable path $\gamma \in \Gamma$. An application of Lemmas A.6 and A.9 shows that $(1+\varepsilon)g \in \mathscr{B}$. Let $M = \inf \int_{\mathbb{R}^n} \rho^p(x)\, dm(x)$, where the infimum is taken over all $\rho \in \mathscr{B}$. We have, by Lemma A.6,

$$M \;\leq\; (1+\varepsilon)^p \, \|g\|_p^p \;\leq\; \frac{(1+\varepsilon)^p}{(1-\varepsilon)^n} \, \|\rho\|_p^p \;=\; \frac{(1+\varepsilon)^p}{(1-\varepsilon)^n} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x).$$

Since $\rho \in \mathscr{B}$ and $\varepsilon \in (0,1)$ are arbitrary and $\mathscr{B}$ is p-complete, it follows that $M \leq M_p(\Gamma)$. Since the reverse inequality is trivial, the proof is complete. □

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a rectifiable path in $\mathbb{R}^n$, $\gamma_0 : [0,L] \to \mathbb{R}^n$ the arc length parameterization of γ, and f an ACL function defined in a neighborhood of $\gamma([a,b]) = \gamma_0([0,L])$. We say f is absolutely continuous on the path γ if

$$\int\limits_0^t \nabla f \cdot \frac{d\gamma_0}{dt}\, dt \;=\; f \circ \gamma_0(t) - f \circ \gamma_0(0)$$

for all $t \in [0,L]$. The integrand is the inner product of $d\gamma_0/dt$ and $\nabla f =$ the gradient of f. We use the convention that $\partial f/\partial x_i = 0$ at points x where $\partial f/\partial x_i$ is not defined. The above definition differs slightly from Definition 5.2 in [316] in that we require a little more than the absolute continuity of $f \circ \gamma_0$.

Lemma A.11. $M_p(\Gamma) \;\leq\; \operatorname{cap}_p(E,F,G)$.

Proof. Prove $u \in \mathscr{A}(E,F,G) \cap L^p(G)$. Let Γ_0 be the locally rectifiable paths $\gamma \in \Gamma$ for which u is absolutely continuous on every rectifiable subpath of γ. Define $\rho : \mathbb{R}^n \to [0,\infty]$ by

$$\rho(x) \;=\; \begin{cases} |\nabla u(x)| & \text{if } \; x \in G \setminus \{\infty\}, \\ 0 & \text{if } \; x \in \mathbb{R}^n \setminus G. \end{cases}$$

Suppose $\gamma \in \Gamma_0$ and $\gamma : (a,b) \to G$ is parameterized as in the proof of Lemma A.7. If $a < t_1 < t_2 < b$, then

$$\begin{aligned} \int\limits_\gamma \rho\, ds \;&=\; \int\limits_a^b \rho \circ \gamma(t)\, dt \;\geq\; \int\limits_{t_1}^{t_2} |\nabla u(\gamma(t))|\, dt \;\geq\; \left| \int\limits_{t_1}^{t_2} \nabla u(\gamma(t)) \cdot \frac{d\gamma}{dt}\, dt \right| \\ &=\; |u \circ \gamma(t_2) - u \circ \gamma(t_1)|. \end{aligned}$$

Since t_1 and t_2 are arbitrary, the above implies $\int_\gamma \rho\ ds \geq 1$. Hence, $\rho \in \operatorname{adm} \Gamma_0$. Therefore,

$$M_p(\Gamma_0) \leq \int_{\mathbb{R}^n} \rho^p(x)\, dm(x) = \int_G |\nabla u(x)|^p\, dm(x).$$

By a theorem of Fuglede (see Theorem 28.2 in [316]), we have $M_p(\Gamma) = M_p(\Gamma_0)$. Therefore,

$$M_p(\Gamma) \leq \int_G |\nabla u(x)|^p\, dm(x).$$

Since $u \in \mathscr{A}(E,F,G) \cap L^p(G)$ is arbitrary, we get the desired result. $\square$

Lemma A.12. *Let U be a domain in $\mathbb{R}^n$, let $g : U \to [0,\infty)$ be continuous, and suppose that K is a nonempty, bounded, compact set with $K \subset U$. Define $f : U \to [0,\infty)$ by $f(x) = \inf \int_\beta g\, ds$, where the infimum is taken over all rectifiable paths $\beta : [a,b] \to U$ with $\beta(a) \in K$ and $\beta(b) = x$. Then,* (1) *if the closed line segment $[x_1,x_2]$ lies in U, then*

$$|f(x_2) - f(x_1)| \leq \max_{x\in[x_1,x_2]} g(x)|x_2 - x_1| \tag{A.12}$$

and (2) *if $f : U \to [0,\infty)$ satisfies (A.12), then f is differentiable a.e. in U and $|\nabla f(x)| \leq g(x)$ a.e. in U.*

Proof. Let β be a rectifiable path connecting K and x_1. Then

$$f(x_2) \leq \int_\beta g\, ds + \int_{[x_1,x_2]} g\, ds \leq \int_\beta g\, ds + \max_{x\in[x_1,x_2]} g(x)|x_2 - x_1|.$$

Since β is arbitrary, we get

$$f(x_2) \leq f(x_1) + \max_{x\in[x_1,x_2]} g(x)|x_2 - x_1|.$$

In a similar way, we get

$$f(x_1) \leq f(x_2) + \max_{x\in[x_1,x_2]} g(x)|x_2 - x_1|.$$

This proves Eq. (A.12).

If f satisfies Eq. (A.12), then f is locally Lipschitz continuous in U and, therefore, by the theorem of Rademacher and Stepanov [316], f is differentiable a.e. in U. Suppose now that $x_0 \in U$ is a point of differentiability of f. Then $f(x_0+h) - f(x_0) = \nabla f(x_0)\cdot h + |h|\varepsilon(h)$ where $h \in \mathbb{R}^n$ and $\lim \varepsilon(h) = 0$ as $h \to 0$. For small $t \in (0,1)$, let $h = t\nabla f(x_0)/|\nabla f(x_0)|$. Substituting in the above formula gives $||\nabla f(x_0)| + \varepsilon(h)| \leq \max_{x\in[x_0,x_0+h]} g(x)$. If we let $t \to 0$, we get $|\nabla f(x_0)| \leq g(x_0)$, as desired. $\square$

Theorem A.15. *Suppose $(E \cap F) \cap \partial G = \emptyset$. Then $M_p(\Gamma) = \operatorname{cap}_p(E,F,G)$.*

Proof. It suffices, by Lemma A.11, to prove

$$\operatorname{cap}_p(E,F,G) \;\le\; M_p(\Gamma). \tag{A.13}$$

We assume, without any loss of generality, that E is bounded and we let $\mathscr{B} \subset \operatorname{adm}\Gamma$ be as in Theorem A.14. The proof is divided into four cases.

Case 1. Suppose $\infty \notin G$. Let $\rho \in \mathscr{B}$ and define $u : G \to [0,\infty)$ by $u(x) = \min(1, \inf\int_\beta \rho\, ds)$, where the infimum is taken over all rectifiable paths β in G connecting E and x. It follows, using Lemma A.11, that $u \in \mathscr{A}(E,F,G)$ and $|\nabla u| \le rho$ a.e. in G. Therefore,

$$\operatorname{cap}_p(E,F,G) \;\le\; \int_G |\nabla u|^p\, dm \;\le\; \int_{\mathbb{R}^n} \rho^p\, ds.$$

Since $\rho \in \mathscr{B}$ is arbitrary and $\mathscr{B}$ is p-complete, we get Eq. (A.13).

Case 2. Suppose $\infty \in G$ and $\infty \in F$. Choose $\rho \in \mathscr{B}$ and $\varepsilon \in (0,1)$. Since $L(\rho) \ge 1$, we can choose a small $r \in (0,1)$ so $\int_\gamma \rho\, ds \ge 1-\varepsilon$ for every locally rectifiable path γ in G connecting $E(r)$ and $F(r)$. Define $u : G \setminus \{\infty\} \to [0,\infty)$ by $u(x) = \min(1, (1-\varepsilon)^{-1} \inf\int_\beta \rho\, ds)$, where the infimum is taken over all rectifiable paths β in G connecting $E(r)$ and x. Since u is identically 1 in a deleted neighborhood of ∞, we see that u extends continuously to all of G. It follows, using Lemma A.12, that $u \in \mathscr{A}(E,F,G)$ and $|\nabla u| \le (1-\varepsilon)^{-1}\rho$ a.e. in G. Therefore,

$$\operatorname{cap}_p(E,F,G) \;\le\; \int_G |\nabla u|^p\, dm \;\le\; (1-\varepsilon)^{-p} \int_{\mathbb{R}^n} \rho^p\, dm.$$

Since $\rho \in \mathscr{B}$ and $\varepsilon \in (0,1)$ are arbitrary and $\mathscr{B}$ is p-complete, we get Eq. (A.13).

Case 3. Suppose $\infty \in G$, $\infty \notin F$, and $1 < p < n$. Choose $\rho \in \mathscr{B}$. Since $((\overline{\mathbb{R}^n} \setminus G) \cup E \cup F) \setminus \{\infty\}$ lies inside some ball, it follows that $|x| \le \text{constant} \cdot d(x)$ for large $|x|$. Therefore,

$$\rho(x) \;\le\; C|x|^{-n/p} \tag{A.14}$$

for some constant $C \in (0,\infty)$ and all large $|x|$, say $|x| > r_0$. Define $v : G \setminus \{\infty\} \to [0,\infty)$ by $v(x) = \inf\int_\beta \rho\, ds$, where the infimum is taken over all rectifiable paths β connecting E and x. We proceed to show that $v(\infty)$ can be defined continuously. Set $v(\infty) = \inf\int_\beta \rho\, ds$, where the infimum is taken over all continuous β such that $\beta : [a,b] \to G$ with $\beta(a) \in E$, $\beta(b) = \infty$ and $\beta|[a,t]$ is rectifiable for all $t \in [a,b)$. Choose any $x_0 \in \mathbb{R}^n$ so that the path $[x_0,\infty]$ lies in G, where $[x_0,\infty](t) = tx_0$, $t \in (1,\infty)$. Let γ be any rectifiable path in G connecting E and x_0. Let β the path obtained by connecting the paths γ and (x_0,∞). Then

$$v(\infty) \;\le\; \int_\beta \rho\, ds \;=\; \int_\gamma \rho\, ds + \int_{[x_0,\infty]} \rho\, ds.$$

Clearly, $\int_\gamma \rho\, ds$ is finite and $\int_{[x_0,\infty]} \rho\, ds$ is finite by the estimate (A.14) and the fact that $1 < n/p$. Hence, $v(\infty)$ is finite. Choose $r \in (r_0,\infty)$ large enough so that the complement in $\overline{\mathbb{R}^n}$ of $\overline{B^n(0,r)}$ lies in G and $E \subset B^n(0,r)$. Let $x_0 \in G \setminus \{\infty\}$ and $|x_0| > r$.

Suppose β is a rectifiable path in G connecting E and x_0. We have

$$v(\infty) \le \int_\beta \rho\, ds + \int_{[x_0,\infty]} \rho\, ds \le \int_\beta \rho\, ds + C\int_r^\infty t^{-n/p}\, dt.$$

Since the above is true for all such β, we get

$$v(\infty) - v(x_0) \le c\int_r^\infty t^{-n/p}\, dt. \tag{A.15}$$

Suppose now that β is a path connecting E and ∞ and is of the type used in defining $v(\infty)$. Let τ be a path that is part of a great circle on the sphere $\{x \in \mathbb{R}^n : |x| = |x_0|\}$ and that connects x_0 and y_0 at some point on the path β. Let β_1 be a subpath of β connecting E and y_0. We have

$$v(x_0) \le \int_{\beta_1} \rho\, ds + \int_\tau \rho\, ds \le \int_\beta \rho\, ds + \int_\tau \rho\, ds.$$

Also,

$$\int_\tau \rho\, ds \le \frac{C}{|x_0|^{n/p}} \cdot \text{length}(\tau) \le 2\pi C|x_0|^{1-n/p}.$$

Hence,

$$v(x_0) \le \int_\beta \rho\, ds + 2\pi C|x_0|^{1-n/p} \le \int_\beta \rho\, ds + 2\pi C r^{1-n/p}.$$

Since the above is true for all β connecting E and ∞, we have

$$v(x_0) - v(\infty) \le 2\pi C r^{1-n/p}. \tag{A.16}$$

Relations (A.16) show that v is continuous at ∞.

Define $u : G \to [0,\infty)$ by $u(x) = \min(1, v(x))$. Then it follows, using Lemma A.12, that $u \in \mathscr{A}(E,F,G)$ and $|\nabla u| \le \rho$ a.e. in G. Therefore,

$$\mathrm{cap}_p(E,F,G) \le \int_G |\nabla u|^p\, dm \le \int_{\mathbb{R}^n} \rho^p\, dm.$$

Since $\rho \in \mathscr{B}$ is arbitrary and $\mathscr{B}$ is p-complete, we get Eq. (A.13).

Case 4. Suppose $\infty \in G$, $\infty \notin F$, and $p \ge n$. Define $\theta : \mathbb{R}^n \to [0,1]$ by

$$\theta(x) = \begin{cases} 1/e & \text{if } |x| \le e, \\ 1/(|x| \log |x|) & \text{if } |x| > e. \end{cases}$$

It is straightforward to verify that $\theta \in L^p(\mathbb{R}^n)$ and $\int_0^\infty \theta(|x|)\, d|x| = \infty$. Choose $\rho \in \mathscr{B}$ and $\varepsilon \in (0,1)$. Let $\rho' = \rho + \varepsilon\theta$. Define $u : G \setminus \{\infty\} \to [0,\infty)$ by $u(x) = \min(1, \inf \int_\beta \rho'\, ds)$, where the infimum is taken over all rectifiable β in G connecting E and x. Choose $r \in (0,\infty)$ so that $E \subset B^n(0,r)$. If $|x_0| > r$ and β connects E and x_0, then

$$\int_\beta \rho'\, ds \ \ge\ \varepsilon \int_\beta \theta\, ds \ge\ \varepsilon \int_r^{|x|} \theta(|x|)\, d|x|.$$

It follows that if $|x_0|$ is large, then $\int_\beta \rho'\, ds \ \ge\ 1$. Therefore, u extends continuously to $u : G \to [0,\infty)$. Using Lemma A.12, we get $u \in \mathscr{A}(E,F,G)$ and $|\nabla u| \le \rho'$ a.e. in G. Hence,

$$\operatorname{cap}_p(E,F,G) \ \le\ \int_G |\nabla u|^p\, dm \ \le\ \int_{\mathbb{R}^n} (\rho + \varepsilon\theta)^p\, dm.$$

Since $\rho \in \mathscr{B}$ and $\varepsilon \in (0,1)$ are arbitrary and $\mathscr{B}$ is p-complete, we get Eq. (A.13). □

We use the previous theorem to prove a continuity theorem for the modulus.

Theorem A.16. *Suppose $E_1 \supset E_2 \supset \ldots$ and $F_1 \supset F_2 \supset \ldots$ are disjoint sequences of nonempty compact sets in a domain G. Then*

$$\lim_{i\to\infty} M_p(E_i, F_i, G) \ =\ M_p\left(\bigcap_{i=1}^\infty E_i,\ \bigcap_{i=1}^\infty F_i, G\right).$$

Proof. The theorem follows immediately from Theorems A.15 and A.13. □

The reader may wish to compare the proof of Eq. (A.13) with Ziemer's proof in [338]. Ziemer defines a function u derived from a density ρ in a way that is similar to the one in this section. Ziemer's technique will not work for the situation considered in this section since the "limiting path" of Lemma 3.3 in [338] need not necessarily lie in G. The present proof "works" because there is a p-complete family of densities ρ with $L(\rho) \ge 1$.

A.4 The Shlyk Equality

This section extends the result from Section A.3 to arbitrary condensers in the closure of a domain in $\mathbb{R}^n$, $n \ge 2$; see [293], cf. with arguments in Chapter 2 and in the papers [10, 37, 198].

Let us introduce some notations: $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is the one-point compactification of the Euclidean space $\mathbb{R}^n$, $n \ge 2$; $G \subset \mathbb{R}^n$ is an open set in the topology of the space $\overline{\mathbb{R}^n}$; F_0 and F_1 are nonempty, disjoint, closed sets in $\overline{G}$; $m_p\,(F_0,F_1,G)$ is the p-modulus of the family $\Delta(F_0,F_1,G)$ of all rectifiable paths connecting F_0 and F_1 in G; see [122] and [305]; here

$$m_p\,(F_0,F_1,G) = \inf \int\limits_G \rho^p\; dx,$$

where the infimum is taken over the class $\operatorname{adm}\Delta(F_0,F_1,G)$ of all Borel functions $\rho : G \to [0,\infty]$ satisfying the condition

$$\int\limits_\gamma \rho\; ds \geq 1$$

for every $\gamma \in \Delta\,(F_0,F_1,G)\,$; $C_p\,(F_0,F_1,G)$ is the p-capacity of the condenser $(F_0,F_1,G)\,$, i.e.,

$$C_p\,(F_0,F_1,G) = \inf \int\limits_G |\nabla u|^p\; dx,$$

where the infimum is taken over the class of all functions $u \in C^\infty\,(G) \cap L^1_p(G)$ (see [215]), which are equal to 1 and 0 on neighborhoods of the F_1 and F_1, respectively; $d(\cdot,\cdot)$ is the Euclidean distance; L_n is the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

Hesse has shown in [122] that $m_p\,(F_0,F_1,G) = C_p\,(F_0,F_1,G)\,$, where F_0 and F_1 are compact subsets of G, G is a domain, and the monotonicity of $m_p\,(F_0,F_1,G)$ is obtained as a consequence of this fact. Note that the connectivity of G in the Hesse proof is not an essential restriction. Later on, M. Ohtsuka in [228] raised the question of the monotonicity of $m_p\,(F_0,F_1,G)$ for arbitrary condensers $(F_0,F_1,G)\,$.

The equality $m_p\,(F_0,F_1,G) = C_p\,(F_0,F_1,G)$ for arbitrary condensers is proved in Theorem A.17 and answers the Ohtsuka question; see Corollary A.4.

Theorem A.17. *The equality* $m_p\,(F_0,F_1,G) = C_p\,(F_0,F_1,G)$ *holds for* $p > 1$.

Proof. The inequality $m_p\,(F_0,F_1,G) \leq C_p\,(F_0,F_1,G) < \infty$ is known; see [122]. We need to show that $m_p\,(F_0,F_1,G) \geq C_p\,(F_0,F_1,G)\,$. Since

$$m_p\,(F_0,F_1,G) \;=\; m_p(\partial F_0,\partial F_1,G\setminus(F_0\cup F_1))$$

and

$$C_p\,(F_0,F_1,G) \;=\; C_p\,(\partial F_0,\partial F_1,G\setminus(F_0\cup F_1))\,,$$

in what follows we assume that F_0, $F_1 \subset \partial G$. Consider the case when G is a bounded open set and, consequently, F_0 and F_1 are compact sets in $\mathbb{R}^n$. Then, given $\varepsilon \in (0,1)$, there is the cut-off function $\rho_1 \in \operatorname{adm}\Delta\,(F_0,F_1,G)$ such that

$$\left|\, m_p\,(F_0,F_1,G) - \int\limits_G \rho_1^p dx \,\right| \;<\; \frac{\varepsilon}{4}.$$

Corresponding to [122] is a function $\rho_2 \in \operatorname{adm} \Delta\,(F_0, F_1, G)$ that is continuous and satisfies the condition

$$\left| \int\limits_G \rho_2^p \, dx - \int\limits_G \rho_1^p dx \right| < \frac{\varepsilon}{4}.$$

Select $m \in \mathbb{N}$ such that, for

$$\rho_3 = \max\{\rho_2, 1/m\}, \tag{A.17}$$

the inequality

$$\left| \int\limits_G \rho_2^p \, dx - \int\limits_G \rho_3^p \, dx \right| < \frac{\varepsilon}{4}$$

holds. The function ρ_3 is continuous in G and belongs to the class $\operatorname{adm} \Delta\,(F_0, F, G)$.

In view of the inequality $d(F_0, F_1) > 0$, there is a compact set ω such that $\overline{\mathbb{R}^n} \setminus \omega = A_0 \cup B_0$, where A_0 and B_0 are open disjoint sets and $F_0 \subset A_0$ and $F_1 \subset B_0$; see [292]. Using a covering of ω by balls, we may assume that ω is of the $\mathscr{L}_n$-measure zero.

Similarly, let us construct open sets A_k and B_k, $k = 1, 2, \ldots$, such that $A_0 \Supset A_k \Supset A_{k+1}$, $B_0 \Supset B_k \Supset B_{k+1}$; $\mathscr{L}_n\,(\partial A_k \cup \partial B_k) = 0$, $\bigcap_{k=1}^{\infty} A_k = F_0$, $\bigcap_{k=1}^{\infty} B_k = F_1$. Set $\beta_1 = d\,(\partial A_1, \omega)$, $\eta_1 = d\,(\partial B_1, \omega)$, $\beta_k = d\,(\partial A_{k-1}, \partial A_k)$, and $\eta_k = d\,(\partial B_{k-1}, \partial B_k)$ for $k \geq 2$. It is clear that $\beta_k > 0$ and $\eta_k > 0$ as $k = 1, 2, \ldots$. Let us give a decreasing sequence $\varepsilon_k \in (0, 1)$ such that

$$\sum_{k=1}^{\infty} 2^{p+1} \varepsilon_k < \frac{\varepsilon}{4}, \qquad m \cdot \varepsilon_k < \min\,(\beta_k, \eta_k)\,. \tag{A.18}$$

Let us enclose $\partial G \setminus (F_0 \cup F_1) \neq \emptyset$ into the union U of a sequence (possibly, finite) of open balls such that the set $\mathbb{R}^n \setminus (A_k \cup B_k)$ intersects at most a finite number of its elements and $U \cap (F_0 \cup F_1) = \emptyset$, $\int_{V_k \cap W_k} \rho_3^p \, dx < \varepsilon_k^{p+1}$, where $V_k = (U \cap G) \cap \left(\overline{A_{k-1}} \setminus \overline{A_k}\right)$ and $W_k = (U \cap G) \cap \left(\overline{B_{k-1}} \setminus \overline{B_k}\right)$, $k = 1, 2, \ldots$.

If $\partial G \setminus (F_0 \cup F_1) = \emptyset$, we set $U = 0$. Then, under $U \neq \emptyset$ and $\partial G \cap \overline{A_{k-1} \setminus A_k} \neq \emptyset$, we have $d\left(\partial G \cap \overline{A_{k-1} \setminus A_k}, \partial U\right) > 0$. Similarly, under $U \neq \emptyset$ and $\partial G \cap \overline{B_{k-1} \setminus B_k} \neq \emptyset$, we have $d\left(\partial G \cap \overline{B_{k-1} \setminus B_k}, \partial U\right) > 0$. Thus, if

$$\rho_4 = \begin{cases} \rho_3/\varepsilon_{2k+1}, & x \in V_{2k+1} \cup W_{2k+1},\ k = 1, 2, \ldots, \\ \rho_3/\varepsilon_{2k}, & x \in V_{2k} \cup W_{2k},\ k = 1, 2, \ldots, \\ 0, & x \in G \setminus U, \end{cases}$$

then $\rho_5 = \rho_3 + \rho_4 \in \operatorname{adm} \Delta\,(F_0, F_1, G)$ and $|\rho_5^p - m_p\,(F_0, F_1, G)| < \varepsilon$. Set $F_{0,j} = \partial A_j \cap \overline{G}$, $F_{1,j} = \partial B_j \cap \overline{G}$ and $G_j = G \setminus \overline{A_j \cup B_j}$, $j = 1, 2, \ldots$. Then $\int_\gamma \rho_5 \, ds > 1 - \varepsilon$ for $j = j_0$ and all $\gamma \in \Gamma_j = \Delta\left(F_{0,j}, F_{1,j}, G_j\right)$.

We may assume that the family $\Delta(F_0, F_1, G)$ is not empty, because otherwise it is obvious that $m_p(F_0, F_1, G) = 0 = C_p(F_0, F_1, G)$. This implies the condition $\Gamma_j \neq \emptyset$, $j \geq 1$.

Indeed, let us assume the contrary. Without loss of generality, we may consider that, for every $j \in \mathbb{N}$, there is a path $\gamma_j \in \Gamma_j$ such that

$$\int\limits_{\gamma_j} \rho_5 \, ds \geq 1 - \varepsilon. \tag{A.19}$$

Since $\rho_5 \geq 1/m$ in G, in view of Eq. (A.19), the length $s(\gamma_j)$ of the path γ_j is at most $m(1-\varepsilon)$. Hence, $\overline{\gamma_j} = \gamma_j \cup \{a_j, b_j\}$, where $a_j \in F_{0,j}$, $b_j \in F_{1,j}$. By the Mazurkiewicz–Yanishevski theorem (see [186], p. 200), the continuum $\overline{\gamma_j}$ contains a continuum λ joining the points a_j and b_j and which is irreducible between them. It is clear that λ is a continuum of a finite one-dimensional Hausdorff measure and, consequently, is a rectifiable path; see [138], p. 180. In view of the Hahn–Mazurkiewicz–Sierpinski theorem [186], λ is a locally connected continuum. Applying Moore's theorem, (see [186], p. 262), we conclude that $\lambda \setminus \{a_j, b_j\}$ is a simple rectifiable path from Γ_j and $\int_\lambda \rho_5 \, ds \leq 1 - \varepsilon$. Therefore, we further assume that γ_j is a simple path.

Let $f_j : (0, s_j) \to \gamma_j$ be a natural parameterization of the same path γ_j, where $s_j = s(\gamma_j)$. Then $|f_j(s') - f_j(s'')| \leq |s' - s''|$ for all $s', s'' \in (0, s_j)$, $j \in \mathbb{N}$, and $|f_j'(s)| = 1$ a.e. in $(0, s_j)$ with respect to the measure $\mathscr{L}_1$. Without loss of generality, we may consider that $s_j \to s_0 < \infty$ as $j \to \infty$ and, by the Ascoli theorem, f_j converges uniformly to a function f on compact subsets of $(0, s_0)$, satisfying the Lipschitz condition with the constant 1. By construction, $\gamma = f((0, s_0))$ joins compact sets F_0 and F_1 and is contained in $\overline{G}$. Moreover, $|f'(s)| \leq 1$ a.e. on $(0, s_0)$. Let $\tau = (\gamma \setminus U) \cap G$. Since ρ_3 is continuous in G and $\rho_3 = \rho_5$ in $G \setminus U$, we have

$$1 - \varepsilon \geq \lim_{j \to \infty} \int\limits_e \rho_3(f_j(s)) \, ds = \int\limits_e \rho_3(f(s)) \, ds = \int\limits_e \rho_5(f(s)) \, ds$$

for every compactum $\tau' \subset \tau$, where $e = f^{-1}(\tau') \subset (0, s_0)$. Hence,

$$\int\limits_\tau \rho_5 \, ds \leq 1 - \varepsilon. \tag{A.20}$$

In particular, if $U = \emptyset$, then $\tau \supset \tilde{\gamma} \in \Delta(F_0, F_1, G)$ and $\int_{\tilde{\gamma}} \rho_5 \, ds \geq 1$, which, in view of Eq. (A.20), contradicts the above assumption. Hence, later on, we may consider $U \neq \emptyset$. Let $V_1 \neq \emptyset$. Set $\tau_1(V) = \left(\gamma \cap \overline{V_1}\right) \cap \left(U \cap \overline{G}\right)$. Since γ joins F_0 and F_1, one of the connected components of $\gamma \cap \left(A_0 \setminus \overline{A_1}\right)$ joins the compact sets ∂A_0 and ∂A_1. Denote this component by $\varkappa$. Then $\varkappa \setminus \tau_1(V) \neq \emptyset$. Indeed, let us assume that $\varkappa \setminus \tau_1(V) = \emptyset$ and let $(\alpha, \beta) \subset (0, s_0)$ be one of those intervals for which $f((\alpha, \beta)) = \varkappa$. Choosing δ in $(0, (\beta - \alpha)/2)$, for large j, we obtain

$$\varkappa_j = f_j([\alpha + \delta, \beta - \delta]) \subset V_1 \cap (U \cap G).$$

Since $1-\varepsilon \geq \int_{\gamma_j}\rho_5\ ds \geq \int_{\varkappa_j}\rho_5\ ds$, we have $\int_{\varkappa_j}\rho_4\ ds < 1$ and, consequently, $\int_{\varkappa_j}\rho_3\ ds < \varepsilon_1$. Hence, $s(\varkappa_j) < m\cdot\varepsilon_1$. Passing here to the limit as $j\to\infty$ and $\delta\to 0$, we have $\beta_1 = d\left(\partial A_0, \partial A_1\right) \leq s(\varkappa) < m\cdot\varepsilon_1$, which contradicts the choice of ε_k in Eq. (A.18) under $k=1$. Consequently, $\varkappa\setminus\tau_1\left(V\right)\neq\emptyset$. A similar property can be established for $\gamma\cap\left(B_0\setminus\overline{B_1}\right)$, $\gamma\cap\left(A_1\setminus\overline{A_2}\right)$, $\gamma\cap\left(B_1\setminus\overline{B_2}\right),\dots$. Then each connected component of $\tau_1(V)$ belongs in an open arc included in $\left(\overline{G}\cap U\right)\cap\overline{V_1\cup W_1\cup V_2}$ and either the endpoints of these arcs simultaneously lie in one of the sets $(\partial U\cap G\cap\partial V_1)$, $(\partial U\cap G\cap\partial W_1)$, $(\partial U\cap G\cap\partial V_2)$, or one of endpoints belongs to $(\partial U\cap G\cap\partial V_1)$ and the other one to one of the sets $(\partial U\cap G\cap\partial W_1)$, $(\partial U\cap G\cap\partial V_2)$. Let us enumerate these arcs as a (possibly finite) sequence $\tau_{1,l}(V)$, $l=1,2,\dots$, and set $\widehat{\tau_1}(V)=\bigcup_l \tau_{1,l}$. Let us parameterize $g_l : q_{1,l}\to\tau_{1,l}$ of the arc $\tau_{1,l}(V)$ by restricting f to the corresponding interval $q_{1,l}\subset(0,s_0)$. Since ρ_5 is a bounded function, in a small neighborhood of the compact $(\partial V_1\bigcup\partial W_1\bigcup\partial V_2)\cap(\partial U\cap G)$, the mappings f_j, f are Lipschitzian, $f_j\to f$ on $\overline{q}_{1,l}$ as $j\to\infty$, and we can find numbers n_1 and n_2 and segments $[c_l,d_l]\subset q_{1,l}$, $l=\overline{1,n_1}$, such that

1. if $\sigma_l = f_{n_2}\left([c_l,d_l]\right)$, then $\bigcup_{l=1}^{n_1}\sigma_l\subset\overline{V_1\cup W_1\cup V_2}\cap(U\cap G)$,
2. the set $e=f\left(\bigcup_{l\geq n_1} q_{1,l}\right)$ lies in a small neighborhood of the compactum

$$(\partial V_1\cup\partial W_1\cup\partial V_2)\cap(\partial U\cap G)$$

and

$$\int\limits_e \rho_5\ ds < \varepsilon_1, \tag{A.21}$$

3. if we join the point $f_{n_2}(c_l)$ by a line segment with the "left" endpoint of the arc $f\left(q_{1,l}\right)$, the point $f_{n_2}(d_l)$ by a line segment $e_{2,l}$ with the "right" endpoint of the arc $f\left(q_{1,l}\right)$, $e_{1,l}\cup e_{2,l}\subset G$, $l=\overline{1,n_1}$, then

$$\sum_{l=1}^{n_1}\left(\int\limits_{e_{1,l}}^{n_1}\rho_5\ ds+\int\limits_{e_{2,l}}^{n_1}\rho_5\ ds\right)<\varepsilon_1. \tag{A.22}$$

Note that

$$\sum_{l=1}^{n_1}\int\limits_{\sigma_l}\rho_4\ ds \leq \int\limits_{\gamma_{n_2}}\rho_5\ ds \leq 1-\varepsilon.$$

Hence,

$$\sum_{l=1}^{n_1}\int\limits_{\sigma_l}\rho_3 ds = \sum_{l=1}^{n_1}\left(\int\limits_{\sigma_l\cap\overline{V_1\cup W_1}}\rho_3 ds+\int\limits_{\sigma_l\cap V_2}\rho_3 ds\right)<\varepsilon_1+\varepsilon_2. \tag{A.23}$$

Replacing the arcs $\tau_{1,l}$ in $\widehat{\tau_1}(V)$ by arcs σ_l and by the line segments $e_{1,l}, e_{2,l}\dots$, $l=\overline{1,n_1}$, we obtain instead of $\widehat{\tau_1}(V)$ a set $\widetilde{\tau_1}(V)\subset G$ such that, in view of Eqs.

(A.21)–(A.23),

$$\int\limits_{\widetilde{\tau}_1(V)} \rho_3 \, ds < 3\varepsilon_1 + \varepsilon_2.$$

In the case $V_1 = \emptyset$, we set $\widetilde{\tau}_1(V) = \emptyset$.

Similarly, taking the sets $W_1, V_2, W_2, \ldots$ instead of V_1, we construct, for $\tau_1(W) = (\gamma \cap \overline{W_1}) \cap (U \cap \overline{G})$, $\tau_2(V) = (\gamma \cap \overline{V_2}) \cap (U \cap \overline{G})$, $\tau_2(W) = (\gamma \cap \overline{W_2}) \cap (U \cap \overline{G}), \ldots,$ the corresponding sets $\widetilde{\tau}_1(W), \widetilde{\tau}_2(V), \widetilde{\tau}_2(W), \ldots$ in G such that

$$\int\limits_{\widetilde{\tau}_1(W)} \rho_3 \, ds < 3\varepsilon_1 + \varepsilon_2, \qquad \int\limits_{\widetilde{\tau}_2(V)} \rho_3 \, ds < 3\varepsilon_2 + \varepsilon_1 + \varepsilon_3,$$
$$\int\limits_{\widetilde{\tau}_1(W)} \rho_3 \, ds < 3\varepsilon_2 + \varepsilon_1 + \varepsilon_3, \ldots. \tag{A.24}$$

Combining Eqs. (A.21)–(A.24), we come to the estimate

$$\sum_{k \geq 1} \int\limits_{\widetilde{\tau}_k(V) \cup \widetilde{\tau}_k(W)} \rho_3 \, ds < \frac{5}{8}\varepsilon.$$

Since $\widetilde{\gamma} = \tau \cup \left(\bigcap_{k \geq 1} (\widetilde{\tau}_k(V)) \cup (\widetilde{\tau}_k(W))\right) \in \Delta(F_0, F_1, G)$, we have $\int_{\widetilde{\gamma}} \rho_3 \, ds \geq 1$ and $1 - (5/8)\varepsilon < \int_\tau \rho_5 \, ds$, which contradicts Eq. (A.20). Consequently, there is a j_0 such that under $j \geq j_0$,

$$\int\limits_\gamma \rho_5 \, ds > 1 - \varepsilon$$

for all $\gamma \in \Gamma_j$. Hence, the function

$$\rho_6 = \begin{cases} \rho_5 / (1 - \varepsilon), & x \in G_j, \\ 0, & x \notin G_j, \end{cases}$$

belongs to $\operatorname{adm} \Delta(F_0, F_1, G_j \cup A_j \cup B_j)$, $j \geq j_0$. Using this and the equality $m_p(F_0, F_1, G_j \cup A_j \cup B_j) = C_p(F_0, F_1, G_j \cup A_j \cup B_j)$ from [122], we obtain the following relations:

$$\int\limits_G \rho_6^p \, dx = m_p(F_0, F_1, G) + o(1) \geq m_p(F_0, F_1, G_j \cup A_j \cup B_j)$$

$$= C_p(F_0, F_1, G_j \cup A_j \cup B_j) \geq C_p(F_0, F_1, G),$$

where $o(1) \to 0$ as $\varepsilon \to 0$ and $j \geq j_0$. Passing to the limit as $\varepsilon \to 0$, we get the conclusion of the theorem: $m_p(F_0, F_1, G) = C_p(F_0, F_1, G)$.

Note that the equality

$$m_p(F_0, F_1, G_j \cup A_j \cup B_j) = C_p(F_0, F_1, G_j \cup A_j \cup B_j)$$

can be established by repeating the previous arguments (replacing G by $G \cup A_j \cup B_j$) and constructing by the function ρ_6 an admissible function u in the capacity problem $C_p(F_0, F_1, G \cup A_j \cup B_j)$ such that $\int_{G \cup A_j \cup B_j} |\nabla u|^p \le \int_G \rho_6^p \, dx$; see, e.g., [122] and [305].

Now, let G be unbounded and open. Then we need the next modifications in the previous proof. Instead of $\rho_3 = \max\{\rho_2, 1/m\}$ in Eq. (A.17), we set $\rho_3 = \max\{\rho_2, g\}$, where g is a positive continuous function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} g^p \, dx < \varepsilon/4$. We correspondingly request for ε_k in Eq. (A.18) the following conditions:

$$\varepsilon_k < \min_{\overline{V_k} \cup \overline{W_k}} g(x) \min(\beta_k, \eta_k) \quad [\text{where} \min_{\overline{V_k} \cup \overline{W_k}} g(x) \stackrel{\text{def}}{=} \infty \text{ for } \overline{V}_k \cup \overline{W}_k = \emptyset],$$

if $\infty \in F_0 \cup F_1$, $k = 1, 2, \ldots$, and $\infty \in \left(A_0 \setminus \overline{A_1}\right) \cup \left(B_0 \setminus \overline{B_1}\right)$,

$$\varepsilon_k < \min_{\overline{V_k} \cup \overline{W_k}} g(x) \min(\beta_k, \eta_k) \quad [\text{where} \min_{\overline{V_k} \cup \overline{W_k}} g(x) \stackrel{\text{def}}{=} \infty \text{ for } \overline{V}_k \cup \overline{W}_k = \emptyset]$$

if $\infty \notin F_0 \cup F_1$, $k = 2, 3, \ldots$. Moreover, if $\infty \notin F_0 \cup F_1$, it is necessary to add to the set U the exterior of a closed ball centered at 0 with a large radius. Finally, in the definition of limit path γ, instead of $\rho_5 \ge 1/m$, it is necessary to use the estimate $\rho_5 \ge \min_K > 0$, where K is an arbitrary given compactum in $\mathbb{R}^n$. Using the Cantor diagonal process makes it possible to find subsequence $\{f_{j_k}\}$ converging locally uniformly to f. The function f determines either a bounded rectifiable path γ, joining F_0 and F_1 or a path γ joining F_0 and $\infty \in F_1$ or a path γ joining ∞ and F_0 and F_1. Moreover, $\gamma \subset \overline{G}$, $f_{j_k} \to f$ on every compactum $e \subset f^{-1}(\gamma \setminus \{\infty\})$, and f_j, f are Lipschitzian functions on e with the constant 1.

Similar to the above scheme, replacing γ by the corresponding $\widetilde{\gamma}$, we obtain the desired conclusion for unbounded G. The proof is complete. □

Corollary A.3. *Let E be an* NC_p*-set in G; see [92]. Then*

$$m_p(F_0, F_1, G) = m_p(F_0, F_1, G \setminus E)$$

for $p \in (1, \infty)$.

Proof. The statement follows from the equality $C_p(F_0, F_1, G) = C_p(F_0, F_1, G \setminus E)$. □

Corollary A.4. *Let $F_{0,1} \supset F_{0,2} \supset \ldots$ and $F_{1,1} \supset F_{1,2} \supset \ldots$ be disjoint sequences of nonempty closed sets in $\overline{G}$ and $\bigcap_{j=1}^{\infty} F_{i,j} = F_i$, $i = 0, 1$. Then $m_p(F_0, F_1, G) = \lim_{j \to \infty} m_p(F_{0,j}, F_{1,j}, G)$ for $p \in (1, \infty)$.*

Proof. The last relation follows from the similar equality for the p-capacity: $C_p(F_0, F_1, G) = \lim_{j \to \infty} C_p(F_{0,j}, F_{1,j}, G)$; see [122], Section A.3. □

A.5 The Moduli by Fuglede

Fuglede [64] considered measures in a fixed abstract set X. "A measure in X" means a countably additive σ-finite set function with nonnegative values (the value $+\infty$ being admitted) defined on a σ-field of subsets of X. The completion of a measure μ is denoted $\bar{\mu}$. The domain of $\bar{\mu}$ consists of all sets $E \subset X$ such that $A \subset E \subset B$ for suitable A and B from the domain of μ with $\mu(B \setminus A) = 0$; then $\bar{\mu} = \mu(A) = \mu(B)$.

One such measure in X will be kept fixed throughout the present section. This basic measure will be denoted by m and its domain of definition by $\mathscr{M}$. It is assumed that $X \in \mathscr{M}$. By the applications described in the other chapters, X will be the Euclidean n-dimensional space $\mathbb{R}^n$, $\mathscr{M}$ the system of Borel subsets of $\mathbb{R}^n$, m the n-dimensional Borel measure, and hence $\overline{m}$ the n-dimensional Lebesgue measure. We shall now consider other measures, or rather systems (= sets) of other measures, in relation to this fixed measure m. We denote by $\mathbf{M}$ the system of all measures μ in X whose domains contain the domain $\mathscr{M}$ of m. Given system $\mathbf{E}$ of measures $\mu \in \mathbf{M}$, a nonnegative m-measurable function ρ defined in X is called admissible for E, written $\rho \in \text{adm } E$, if

$$\int\limits_X \rho \, d\mu \ \geq \ 1$$

for every $\mu \in \mathbf{E}$. The modulus M_p, $0 < p < \infty$, is now defined as follows:

$$M_p(E) = \inf_{\rho \in \text{adm } E} \int\limits_X \rho^p \, dm,$$

interpreted as $+\infty$ if adm $E = \emptyset$ (it is possible only if E contains the measure $\mu \equiv 0$). As a partial motivation for this definition, we may mention that the measure $m(E)$ of an arbitrary set $E \in \mathscr{M}$ equals the minimum of $\int_X \rho(x)^p \, dm(x)$ when f ranges over all nonnegative m-measurable functions such that $\rho(x) \geq 1$ everywhere in E. A minimizing function f is the characteristic function χ_E for E. This analogy expresses, by the way, an actual connection between the measure m and the module M_p in the special case where the system $\mathbf{E}$ consists of "Dirac measures." [With any $x \in X$ is associated the Dirac measure χ_x defined by $\chi_x(A) = \chi_A(x) = 1$ or 0 depending on whether or not A contains x]. If $\mathbf{E}$ denotes a system of such measures χ_x, obtained by taking for x the points of some given set $E \in \mathbf{M}$, then it follows easily that $M_p(E) = m(E)$. Returning to general systems of measures, we shall establish a few simple properties of M_p.

Theorem A.18. *The module M_p is monotone and countably subadditive:*

(a) $M_p(\boldsymbol{E}) \leq M_p(\boldsymbol{E}')$ *if* $\boldsymbol{E} \subset \boldsymbol{E}'$;

(b) $M_p(\boldsymbol{E}) \leq \sum_i M_p(\boldsymbol{E}_i)$ *if* $\boldsymbol{E} = \bigcup_i \boldsymbol{E}_i$.

Proof. The monotonicity of M_p follows at once from the fact that $\rho \wedge \mathbf{E}'$ implies $\rho \wedge \mathbf{E}$ if $\mathbf{E} \subset \mathbf{E}'$. The subadditivity may be proved as follows. If $\rho(x) = \sup_i \rho_i(x)$, where each ρ_i is a nonnegative m-measurable function defined in X, then ρ is likewise such a function, and

$$\int_X \rho^p \, dm \;\le\; \sum_i \int_X \rho_i^p \, dm.$$

To see this, we define, for an arbitrary index n,

$$g_n(x) = \max\{\rho_1(x), \ldots, \rho_n(x)\},$$
$$X_i = \{x \in X : \rho_i(x) = g_n(x)\}.$$

Then g_n is m-measurable, $X_i \in \mathscr{M}$, and $X = \bigcup_{i=1}^n X_i$. Hence,

$$\int_X g_n^p \, dm \;\le\; \sum_{i=1}^n \int_{X_i} g_n^p \, dm \;=\; \sum_{i=1}^n \int_{X_i} \rho_i^p \, dm \;\le\; \sum_{i=1}^\infty \int_X \rho_i^p \, dm.$$

The desired inequality now follows for $n \to \infty$ since $g_n(x) \to f(x)$ monotonically, and hence $\int_X g_n^p \, dm \;\to\; \int_X \rho^p \, dm$. Next, let $\rho_i \in \operatorname{adm} \mathbf{E}_i$ and

$$\int_X \rho_i^p \, dm \;\le\; M_p(\mathbf{E}_i) + \frac{\varepsilon}{2^i}.$$

Then $\rho \in \operatorname{adm} \mathbf{E}$, and

$$M_p(\mathbf{E}) \;\le\; \int_X \rho^p \, dm \;\le\; \sum_{i=1}^\infty \int_X \rho_i^p \, dm \;\le\; \sum_{i=1}^\infty M_p(\mathbf{E}_i) + \varepsilon.$$

□

Remark A.6. If, in particular, the systems $\mathbf{E}_i$ are "separate" in the sense that there exist mutually disjoint sets $S_i \in \mathscr{M}$ such that $\mu(X \setminus S_i) = 0$ when $\mu \in \mathbf{E}_i$, then the sign of equality holds in Theorem A.18(b). In fact, if $\rho \in \operatorname{adm} \mathbf{E}$, and hence ρ is admissible for each $\mathbf{E}_i$, and if we define functions ρ_i by $\rho_i(x) = \rho(x)$ or $= 0$ dependins on whether $x \in S_i$ or $x \notin S_i$, then $\rho_i \in \operatorname{adm} \mathbf{E}_i$. Hence,

$$\int_{S_i} \rho^p \, dm \;=\; \int_X \rho_i^p \, dm \;\ge\; M_p(\mathbf{E}_i)$$

and, consequently,

$$\int_X \rho^p \, dm \;\ge\; \sum_i \int_{S_i} \rho^p \, dm \;\ge\; \sum_i M_p(\mathbf{E}_i),$$

which implies

$$M_p(\mathbf{E}) \;\ge\; \sum_i M_p(\mathbf{E}_i).$$

The concept of the extremal length, $\lambda_p = 1/M_p$, was introduced by A. Beurling. We shall say that a system $\mathbf{E}$ of measures $\mu \in \mathbf{M}$ is **minorized** by a system $\mathbf{E}'$ of such measures if there corresponds to any $\mu \in \mathbf{E}$ a measure $\mu' \in \mathbf{E}'$ such that $\mu \geq \mu'$, i.e., $\mu(A) \geq \mu'(A)$ for every point set $A \in \mathscr{M}$.

Theorem A.19. (c) *If $\boldsymbol{E}$ is minorized by $\boldsymbol{E}'$, then $\lambda_p(\boldsymbol{E}) \geq \lambda(\boldsymbol{E}')$.*

(d) *If the systems $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots$ are separate, and if each $\mathbf{E}_i$ is minorized by a system $\boldsymbol{E}$, then*

$$\lambda_p(\boldsymbol{E})^{-1} \;\geq\; \sum_i \lambda_p(\boldsymbol{E}_i)^{-1},$$

i.e.,

$$M_p(\boldsymbol{E}) \;\geq\; \sum_i M_p(\boldsymbol{E}_i).$$

(e) *If $p > 1$, the systems $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots$ are separate, and a system $\boldsymbol{E}$ is minorized by each $\boldsymbol{E}_i$, then*

$$\lambda_p(\boldsymbol{E})^{\frac{1}{p-1}} \;\geq\; \sum_i \lambda_p(\boldsymbol{E}_i)^{\frac{1}{p-1}}.$$

Proof. Statement (c) is easily verified. Statement (d) contains the above remark as a special case and is proved exactly like it. In particular, the sign of equality holds if $\mathbf{E} = \bigcup_i \mathbf{E}_i$, where the $\mathbf{E}_i$ are separate (but otherwise arbitrary) systems. By the proof of (e), it is convenient to express the definition of the "extremal length" λ_p in the following form:

$$\lambda_p(\mathbf{E}) = \sup_\rho \, L_\rho(\mathbf{E})^p, \qquad \rho \geq 0, \qquad \rho \in L^p(m), \qquad \int \rho^p dm = 1,$$

where

$$L_\rho(\mathbf{E}) \;=\; \inf_{\mu \in \mathbf{E}} \int \rho \, d\mu.$$

If $\lambda_p(\mathbf{E}_i) = 0$ for some i, the corresponding term may be neglected. If $\lambda_p(\mathbf{E}_i) = +\infty$ for some i, it follows from (c) that $\lambda_p(\mathbf{E}) = +\infty$. Thus, we may assume that $0 < \lambda_p(\mathbf{E}_i) < +\infty$ for every i and also that $0 < \lambda_p(\mathbf{E}) < +\infty$. To any given number $\varepsilon_i > 0$ corresponds a function $\rho_i \geq 0, \rho_i \in L^p(m)$, such that

$$\int \rho_i^p \, dm \;=\; 1$$

and

$$L_{\rho_i} \;>\; \lambda_p(\mathbf{E}_i)^{1/p} - \varepsilon_i.$$

Choosing disjoint sets S_i so that $\mu(X \setminus S_i) = 0$ when $\mu \in \mathbf{E}_i$, we may assume, moreover, that $\rho_i = 0$ in $X \setminus S_i$. Define $\rho(x) = \sum_i t_i \rho_i(x)$, where $t_i \geq 0$, and $\sum_i t_i^p = 1$. It follows that

$$\int \rho^p \, dm \;=\; \sum_i t_i^p \int \rho_i^p \, dm \;=\; \sum_i t_i^p \;=\; 1.$$

Hence,

$$\lambda_p(\mathbf{E}) \geq L_\rho(\mathbf{E})^p.$$

Let $\mu \in \mathbf{E}$. By assumption, there are measures $\mu_i \in \mathbf{E}_i$ such that $\mu \geq \mu_i$. Consequently,

$$\begin{aligned}\int \rho f \, d\mu &= \sum_i t_i \int \rho_i \, d\mu \geq \sum_i t_i \int \rho_i \, d\mu_i \\ &\geq \sum_i t_i L_{\rho_i}(\mathbf{E}_i) \geq \sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p} - \sum_i t_i \varepsilon_i.\end{aligned}$$

It follows that

$$L_\rho(\mathbf{E}) \geq \sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p} - \sum_i t_i \varepsilon_i$$

and, hence, that

$$\lambda_p(\mathbf{E}) \geq \left(\sum_i \lambda_p(\mathbf{E}_i)^{1/p}\right)^p.$$

In Hölder's inequality,

$$\left(\sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p}\right)^p \leq \sum_i t_i^p \left(\sum_i \lambda_p(\mathbf{E}_i)^{1/(p-1)}\right)^{p-1},$$

the sign of equality holds if, and only if, the numbers t_i^p are proportional to the numbers $\lambda_p(\mathbf{E}_i)^{1/(p-1)}$. This optimal choice of the multipliers t_i leads to the desired inequality.

The sign of equality holds in (e) if, in particular, $\mathbf{E} = \sum_i \mathbf{E}_i$, where the $\mathbf{E}_i$ are separate (but otherwise arbitrary) systems. In fact,

$$L_\rho(\mathbf{E}) \leq \sum_i L_\rho(\mathbf{E}_i)$$

for arbitrary $\rho \geq 0$, $\rho \in L^p(m)$, since $\mu_i \in \mathbf{E}_i$ implies $\sum_i \mu_i \in \mathbf{E}$. Defining $t_i = \{\int_{S_i} \rho^p \, dm\}^{1/p}$, and $\rho_i(x) = t_i^{-1}\rho(x)$ or $= 0$ depending on whether or not x to belongs S_i, we have $\rho \geq \sum_i t_i \rho_i$, and

$$\int \rho^p \, dm \geq \sum_i t_i^p. \tag{A.25}$$

On the other hand, $L_\rho(\mathbf{E}_i) = t_i L_{\rho_i}(\mathbf{E}_i)$, and hence

$$L_\rho(\mathbf{E}) \leq \sum_i L_\rho(\mathbf{E}_i) = \sum_i t_i L_{\rho_i}(\mathbf{E}_i) \leq \sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p}.$$

Applying Hölder's inequality as above, we obtain

$$L_\rho(\mathbf{E})^p \leq \sum_i t_i^p \left(\sum_i \lambda_p(\mathbf{E}_i)^{1/(p-1)}\right)^{p-1}. \tag{A.26}$$

Combining Eqs. (A.25) and (A.26), we arrive at the desired inequality:

$$\lambda_p(\mathbf{E}) \le \left(\sum_i \lambda_p(\mathbf{E}_i)^{1/(p-1)}\right)^{p-1}.$$

□

A system $\mathbf{E} \subset \mathbf{M}$ will be called **exceptional of order** p, abbr. p**-exc**, if $M_p(\mathbf{E}) = 0$.

The well-known fact concerning point sets $E \subset X$ that $\bar{m}(E) = 0$ if, and only if, there exists a function $\rho \in L^p(m)$, $\rho \ge 0$, such that $\rho(x) = +\infty$ for every $x \in E$ (the value of $p > 0$ being irrelevant) may be generalized as follows:

Theorem A.20. *A system $\boldsymbol{E} \subset \boldsymbol{M}$ is p-exc if, and only if, there exists a function $\rho \in L^p(m)$, $\rho \ge 0$, such that*

$$\int_X \rho \, d\mu = +\infty \qquad \forall \mu \in \boldsymbol{E}.$$

Proof. If ρ has these properties, then $n^{-1}\rho \in \operatorname{adm} \mathbf{E}$ for every $n = 1, 2, \ldots$; and $\int (n^{-1}\rho)^p \, dm = n^{-p} \int \rho^p \, dm \to 0$ as $n \to \infty$; hence, $M_p(\mathbf{E}) = 0$. Conversely, let $M_p(\mathbf{E}) = 0$ and choose a sequence of functions $\rho_n \in \operatorname{adm} E$ so that $\int \rho_n^p \, dm < 4^{-n}$. Writing

$$\rho(x) = \left\{\sum_n 2^n \rho_n(x)^p\right\}^{1/p},$$

we infer that

$$\int \rho^p \, dm = \sum_n 2^n \int \rho_n^p \, dm < \infty;$$

on the other hand,

$$\int \rho \, d\mu \ge \int 2^{n/p} \rho_n \, d\mu \ge 2^{n/p}$$

for every $\mu \in \mathbf{E}$ and every $n = 1, 2, \ldots$, and hence $\int \rho \, d\mu = +\infty$. □

A proposition concerning measures μ, which belong to some specified system $\mathbf{E} \subset \mathbf{M}$, is said to hold for **almost every** $\mu \in$ **E of order** p**,** abbr. p**-a.e.** $\mu \in \mathbf{E}$, if the system of all measures $\mu \in \mathbf{E}$ for which the proposition fails to hold is exceptional of order p. This amounts to the existence of a function $\rho \in L^p(m)$, $\rho \ge 0$, such that the proposition holds for every $\mu \in \mathbf{E}$ for which $\int \rho \, d\mu < \infty$.

Theorem A.21. (a) *Any subsystem of a p-*exc *system is p-*exc.

(b) *The union of a finite or denumerable family of p-*exc *systems is p-*exc .

(c) *If $p > q$, then every p-*exc *system of finite measures is q-*exc .

(d) *If $E \subset X$ and $\bar{m}(E) = 0$, then $\bar{\mu}(E) = 0$ for p-*a.e. $\mu \in \boldsymbol{M}$.

(e) *If $\rho \in L^p(\bar{m})$, then $\rho \in L^1(\bar{\mu})$ for p-*a.e. $\mu \in \boldsymbol{M}$.

(f) If a sequence of functions $\rho_i \in L^p(\bar{m})$ converges in the mean of order p with respect to $\bar{m}$ to some function ρ, i.e.,

$$\lim_{i\to\infty} \int_X |\rho_i - \rho|^p \, d\bar{m} \;=\; 0,$$

then there is a subsequence of indices i_ν tending to ∞ with the property that, for p-a.e. $\mu \in \boldsymbol{M}$, ρ_{i_ν} converges to ρ in the mean of order 1 with respect to $\bar{\mu}$:

$$\lim_{\nu\to\infty} \int_X |\rho_{i_\nu} - \rho| \, d\bar{\mu} \;=\; 0,$$

for p-a.e. $\mu \in \boldsymbol{M}$. Statements $(d), (e)$, and (f) remain valid if $\bar{m}$ and $\bar{\mu}$ are replaced by m and μ, respectively.

Proof. Statements (a) and (b) are contained in Theorem A.18. To prove (c), let $\mathbf{E}$ denote a p-exc system of finite measures $\mu \in \mathbf{M}$, and let $\rho \in L^p(m)$, $\rho \geq 0$, be chosen so that $\int \rho d\mu = +\infty$ for every $\mu \in \mathbf{E}$. Now, $\rho^{p/q} \in L^q(m)$, and an application of Hölder's inequality shows that $\int \rho^{p/q} d\mu = +\infty$ when $\mu \in \mathbf{E}$ since $\mu(X) < \infty$ and $p/q > 1$. In fact,

$$+\infty = \int \rho \, d\mu \;\leq\; \left(\int \rho^{p/q} \, d\mu\right)^{q/p} \mu(X)^{1-q/p}.$$

As to statements (d), (e), and (f), we begin by proving the corresponding statements in which $\bar{m}$ and $\bar{\mu}$ are replaced by m and μ, respectively. The statement corresponding to (e) is then contained in Theorem A.20, while (d) may be proved as follows. Let $E \in \mathscr{M}$, $m(E) = 0$, and $\rho(x) = +\infty$ for $x \in E$, $\rho(x) = 0$ for $x \notin E$. Then ρ belongs to $L^p(m)$ and

$$\int \rho \, d\mu \;=\; (+\infty) \cdot \mu(E) \;=\; +\infty$$

for every μ such that $\mu(E) > 0$.

As to (f), we choose an increasing sequence of integers i_ν so that

$$\int_X |\rho_{i_\nu}(x) - \rho(x)|^p \, dm(x) \;<\; 2^{-p\nu-\nu},$$

and we write $g_\nu(x) = |\rho_{i_\nu}(x) - \rho(x)|$. Introducing the systems

$$\mathbf{A}_\nu = \{\mu \in \mathbf{M} : \int g_\nu \, d\mu \;>\; 2^{-\nu}\},$$
$$\mathbf{B}_n = \bigcup_{\nu > n} \mathbf{A}_\nu,$$

and

$$\mathbf{E} = \bigcap_n \mathbf{B}_n,$$

we have $2^\nu g_\nu \in \operatorname{adm} \mathbf{A}_\nu$ and hence

$$M_p(\mathbf{A}_\nu) \le \int (2^\nu g_\nu)^p \, dm = 2^{p\nu} \int g_\nu^p \, dm < 2^{-\nu}.$$

This implies, in view of Theorem A.18, that

$$M_p(E) \le M_p(\mathbf{B}_n) \le \sum_{\nu > n} M_p(\mathbf{A}_\nu) < 2^{-n}.$$

Consequently, $M_p(E) = 0$. To every $\mu \in \mathbf{M} \setminus \mathbf{E}$ corresponds an index n such that $\mu \notin \mathbf{B}_n$, i.e., $\int |\rho_{i_\nu} - \rho| \, d\mu = \int g_\nu \, d\mu \le 2^{-\nu}$ for every $\nu > n$. Hence, $\lim_{\nu \to \infty} \int |\rho_{i_\nu} - \rho| \, d\mu = 0$.

It remains to reduce the original statements (d), (e), and (f) to the above corresponding statements in which $\bar{m}$ and $\bar{\mu}$ were replaced by m and μ, respectively. As to (d), let $E \subset X$ and assume that $\bar{m}(E) = 0$. There exists a set $E^* \in \mathscr{M}$ such that $m(E^*) = 0$ and $E^* \supset E$. The system of all measures μ such that $\bar{\mu}(E) > 0$ is, therefore, contained in the p-exc system of all measures μ such that $\mu(E^*) > 0$. As to (e), the function ρ may be replaced by an equivalent m-measurable function ρ^*. Applying (d) to the set $E = \{x : \rho(x) \neq \rho^*(x)\}$, we infer that $\bar{\mu} = 0$ for p-a.e. μ, in particular,

$$\int |\rho| \, d\bar{\mu} = \int |\rho^*| \, d\bar{\mu} = \int |\rho^*| \, d\mu < \infty$$

for p-a.e. μ. Statement (f) may be treated in a similar manner, and the proof is complete. □

Remark A.7. Simple examples show that the infimum in the definition of $M_p(\mathbf{E})$ is not necessarily attained by any function $\rho \in \operatorname{adm} \mathbf{E}$. However, the following theorem exists for any order $p > 1$ and any system $\mathbf{E}$ of measures $\mu \neq 0$, $\mu \in \mathbf{M}$: *There exists a function $\rho \ge 0$ such that $\int_X \rho^p dm = M_p(E)$ and $\int_X \rho \, d\mu \ge 1$ for p-a.e. $\mu \in \boldsymbol{E}$.* [The former property of ρ obviously depends only on the m-equivalence class of ρ, and so does the latter by virtue of Theorem A.21 (d).]

The existence of ρ is clear if $M_p(E) = +\infty$; and if $M_p(E) < +\infty$, it is a consequence of the well-known facts that the Banach space $L^p(m)$ is uniformly convex when $p > 1$ and that any convex, closed, and nonempty subset of a uniformly convex Banach space contains a unique vector with minimal norm (cf., e.g., [223], p. 7). For any system $\mathbf{E}$ of measures $\mu \neq 0$, $\mu \in \mathbf{M}$, the set of all functions $\rho \in L^p(m)$, $\rho \ge 0$, such that $\int_X \rho \, d\mu \ge 1$ for p-a.e. $\mu \in \mathbf{E}$, is convex and nonempty, and it is closed in $L^p(m)$ by virtue of Theorem A.21(f). From the uniqueness of the minimal vector, it follows that the minimal function ρ is uniquely determined up to a set of measure $m = 0$. Simple examples show that the restriction $p > 1$ is essential for the existence as well as the uniqueness of ρ.

A.6 The Ziemer Equality

Gehring in [77] showed that the conformal capacity is related to the modulus of a family of surfaces that separate the boundary components of R. Gehring assumes that the separating surfaces are sufficiently smooth; Krivov [168] establishes a similar result under the assumption that the extremal metric is well behaved. Under similar assumptions, other authors have dealt with the modulus of separating surfaces; cf. [64, 121, 290].

The purpose of this section is to eliminate the need for these assumptions; see [340]. Thus, we consider the case of two disjoint compact sets C_0, C_1 contained in the closure of bounded, open, connected set G. It is proved that the conformal capacity C of C_0, C_1, relative to G, is related to the $n/(n-1)$-dimensional module M of all closed sets that separate C_0 from C_1 in the closure of G by

$$C^{-\frac{1}{n-1}} = M. \tag{A.27}$$

This is accomplished by using a technique of Gehring's Lemma 3 in [66], which eliminates all assumptions concerning the behavior of the extremal metric. Then, a surface-theoretic approximation theorem, first developed in [60] (8.23), permits the consideration of arbitrary closed separating sets.

Below E^n is Euclidean n-space and L_n n-dimensional Lebesgue measure. Hausdorff k-dimensional measure in E^n is denoted by H^k (see, e.g., [57]), and in this section, only H^1 and H^{n-1} are used. If $A \subset E^n$, then ∂A means the boundary of A and for $x \in E^n$, $\operatorname{dist}(x,A)$ is the distance from x to A. More generally, $\operatorname{dist}(A,B)$ denotes the distance between the sets A and B, $B(x,r)$ stands for the open n-ball of radius r and centered at x. If A is an L_n-measurable subset of E^n, then $L^p(A)$ denotes those functions f for which $|f|^p$ is L_n-integrable over A and $\|f\|_p$ is its L^p-norm.

A real-valued function u defined on an open set $U \subset E^n$ is said to be **absolutely continuous in the sense of Tonelli ACT** on U if it is ACT on every interval $I \subset U$; cf. [281], p. 169. Thus, the gradient of $u, \nabla u$, exists at L_n-almost every point in U. The following co-area formula, which was proved in [342], will be used frequently throughout this section.

Theorem A.22. *If A is an L_n-measurable subset of U and u is* ACT *on U, then*

$$\int_A |\nabla u(x)|\, dL_n(x) \;=\; \int_{-\infty}^{+\infty} H^{n-1}(u^{-1}(s)\cap A)\, dL_1(x). \tag{A.28}$$

Therefore,

$$\int_U f(x)|\nabla u(x)|\, dL_n(x) \;=\; \int_{-\infty}^{+\infty} \int_{u^{-1}(s)} f(x)\, dH^{n-1}(x)\, dL_1(s) \tag{A.29}$$

whenever $f \in L^1(U)$.

The following will be standard notation throughout. G is an open, bounded, connected set in E_n and C_0, C_1 are disjoint compact sets in the closure of G. We will let $R = G \setminus (C_0 \cup C_1)$ and $R^* = R \cup C_0 \cup C_1$. The **conformal capacity of** C_0, C_1 **relative to the closure of** G is defined as

$$C[G, C_0, C_1] = \inf \int_R |\nabla u|^n \, dL_n, \tag{A.30}$$

where the infimum is taken over all functions u that are continuous on R^*, are ACT on R, and assume boundary values 1 on C_1 and 0 on C_0. Such functions are called **admissible** for $C[G, C_0, C_1]$. Sometimes we will write C for $C[G, C_0, C_1]$.

If $C_0 \cup C_1 \subset G$ and if $C_0 \cup C_1$ consists of only a finite number of nondegenerate components, then the arguments of [73] can be applied with only slight modifications to prove that the infimum in Eq. (A.30) is attained by a unique admissible function u that is ACT in G. By using the methods of Chapter III in [64], one can prove the existence of an extremal for more general situations. This extremal function u satisfies the variational condition

$$\int_R |\nabla u|^{n-2} \nabla u \nabla w \, dL_n \; = \; 0 \tag{A.31}$$

for any function w that is ACT on G, assumes boundary value 0 on $C_0 \cup C_1$, and for which $|\nabla w| \in L^n(R)$.

The following notion, which was first introduced in [56], p. 48 , is used later on. An L_n-measurable set $E \subset E^n$ has the unit vector $n(x)$ as **exterior normal to** E **at** x if, letting

$$\begin{aligned}
B(x,r) &= \{y : |y-x| < r\}, \\
B_+(x,r) &= B(x,r) \cap \{y : (y-x)n(x) \geq 0\}, \\
B_-(x,r) &= B(x,r) \cap \{y : (y-x)n(x) \leq 0\}, \\
\alpha &= L_n[B(x,1)],
\end{aligned}$$

we have

$$2 \lim_{r \to 0} L_n[B_-(x,r) \cap E] / \alpha r^n = 1 \tag{A.32}$$

and

$$2 \lim_{r \to 0} L_n[B_+(x,r) \cap E] / \alpha r^n = 0. \tag{A.33}$$

The set of points x for which $n(x)$ exists is called the **reduced boundary of** E and denoted by $\beta(E)$. Obviously, $\beta(E) \subset \partial E$. The importance of the exterior normal is seen in the following general version of the Gauss–Green theorem [49, 58].

Theorem A.23. *If E is a bounded L_n-measurable set with $H^{n-1}[\beta(E)] < \infty$, then*

$$\int_E \operatorname{div} f(x) \, dL_n(x) \; = \; \int_{\beta(E)} f(x) \cdot n(x) \, dH^{n-1}(x) \tag{A.34}$$

whenever $f : E^n \to E^n$ is continuously differentiable.

This theorem enables us to regard a bounded, open set $U \subset E^n$ with

$$H^{n-1}(\partial U) \,<\, \infty \tag{A.35}$$

as an integral current $\bmod\ 2$ (or integral class), i.e., integral current T with coefficients in the group of integers $\bmod\ 2$; see [341] or [63]. Thus, if φ is a differential n-form of class C^∞, then

$$T(\varphi) \,=\, \int\limits_U \varphi \, dL_n. \tag{A.36}$$

The boundary of T, ∂T, is defined as $\partial T(\omega) = T(d\omega)$ whenever ω is an $n-1$ form and $d\omega$ is its exterior derivative. Now $\beta(U)$ is a Hausdorff $(n-1)$-rectifiable set and, therefore, Theorem A.23 allows us to identify $\beta(U)$ with ∂T; see [341]. Thus, the support of ∂T is the closure of $\beta(U)$, cl $\beta(U) \subset \partial U$, the mass of T is $M(T) = L_n(U)$, and $M(\partial T) = H^{n-1}[\beta(U)]$; see [341].

In view of this identification, the following theorem is an immediate consequence of (6.2) in [341] although the original version was given by (8.23) in [60]. An open set is called a **convex cell** if it can be expressed as the finite intersection of open half-spaces and an n**-dimensional polyhedral set** is the union of a finite number of convex cells.

Theorem A.24. *If $U \subset E^n$ is a bounded, open set with $H^{n-1}(\partial U) < \infty$, then there is a sequence of n-dimensional polyhedral sets P_i such that*

(i) $P_i \subset \{x : \delta(x,U) < i^{-1}\}$,

(ii) $L_n(P_i) \to L_n(U)$ as $i \to \infty$,

(iii) $H^{n-1}(\partial P_i) \to H^{n-1}[\beta(U)]$ as $i \to \infty$.

Remark A.8. Moreover, by employing an argument similar to that of (5.6) and (5.7) in [63], it can be arranged that

(iv) $\partial P_i \subset \{x : \delta[x, \beta U] < i^{-1}\}$.

Of course, one could apply (5.6) and p. 170 in [63] directly after verifying that the boundary of U in Fleming's sense is $\beta(U)$. Finally, defining the measure μ_i as the restriction of H^{n-1} to ∂P_i and μ as the restriction of H^{n-1} to $\beta(U)$, (iii) above and (5.7) in [341] or (4.2) in [63] imply that

(v) $\mu_i \to \mu$ weakly as $i \to \infty$, i.e.,

$$\int f \, d\mu_i \to \int f \, d\mu \tag{A.37}$$

for every continuous f.

Instead of dealing with extremal length, we prefer to work with the module as developed in [64]; see the previous section.

Let M be the class of all Borel measures on E^n. With an arbitrary system $E \subset M$ of measures is associated a class of nonnegative Borel functions f subject to the condition

$$\int_{E^n} \rho \, d\mu \geq 1 \qquad \forall \mu \in E. \tag{A.38}$$

We will write $\rho \in \text{adm}\, \mu$ if Eq. (A.38) is satisfied for every $\mu \in E$. For $0 < p < \infty$, the **module of** E, $M_p(E)$, is defined as

$$M_p(E) = \inf_{\rho \in \text{adm}\, \mu} \int_{E^n} \rho^p \, dL_n. \tag{A.39}$$

A statement concerning measures $\mu \in M$ is said to hold for p-a.e. μ if the statement fails to hold for only a set of measures E_0 with $M_p(E_0) = 0$.

For the applications in this section, the measures μ are complete (in fact, they are the restrictions of Hausdorff measure to compact sets) and for such measures, we have the following.

Theorem A.25. *If $p \geq 2$, $E_1 \subset E_2 \subset \ldots$ are sets of complete measures, and $E = \cup E_i$, then*

$$M_p(E) = \lim_{t \to \infty} M_p(E_i). \tag{A.40}$$

Proof. In view of (i) above, observe that the limit exists and is dominated by $M_p(E)$. Therefore, only the case where the limit is finite needs to be considered.

For each i, let ρ_i be the Borel function associated with E_i as in (vii) above. Clarkson's inequality [43] states, for any i and j, that

$$\int_{E^n} \left| \frac{\rho_i + \rho_j}{2} \right|^p dL_n + \int_{E^n} \left| \frac{\rho_i - \rho_j}{2} \right|^p dL_n \leq 2^{-1} \int_{E^n} |\rho_i|^p \, dL_n + 2^{-1} \int_{E^n} |\rho_j|^p \, dL_n. \tag{A.41}$$

If $i > j$, then $2^{-1}(\rho_i + \rho_j) \in \text{adm}\, \mu$ for M_p-a.e. $\mu \in E_j$. Therefore, because of Eq. (A.41),

$$\int_{E^n} \left| \frac{\rho_i - \rho_j}{2} \right|^p dL_n \leq 2^{-1} M_p(E_i) - 2^{-1} M_p(E_j). \tag{A.42}$$

The right side of Eq. (A.42) tends to zero as $i, j \to \infty$ with $i > j$ and, therefore, there is a nonnegative function ρ such that $\|\rho_i - \rho\|_p \to 0$. Thus, from the above properties of module [especially (f) in Theorem A.21], we have that $\rho \in \text{adm}\, \mu$ for M_p-a.e. $\mu \in E$. This implies that

$$M_p(E) \leq \int_{E^n} \rho^p \, dL_n = \lim_{t \to \infty} M_p(E_i),$$

which is all that is required to prove. $\square$

Below we establish the relationship between conformal capacity and arbitrary closed separating sets G, R^*, R, C_1 and C_0.

We say that a **set $\sigma \subset E^n$ separates C_0 from C_1 in R^*** if $\sigma \cap R$ is closed in R and if there are disjoint sets A and B that are open in $R^* \setminus \sigma$ such that $R^* \setminus \sigma = A \cup B$, $C_0 \subset A$, and $C_1 \subset B$. Let Σ denote the class of all sets that separate C_0 from C_1 in R^*. With every $\sigma \in \Sigma$, associate a complete measure μ in the following way: For every H^{n-1}-measurable set $A \subset E^n$, define

$$\mu(A) \;=\; H^{n-1}(A \cap \sigma \cap R). \tag{A.43}$$

From the properties of Hausdorff measure, it is clear that the Borel sets in E^n are μ-measurable and, therefore, the module of Σ can be as defined as in Eq. (A.39). Thus, for $n' = n/(n-1)$,

$$M_{n'}(\Sigma) \;=\; \inf_{\rho \in \mathrm{adm}\ \Sigma} \int_{E^n} \rho^{n'}\, dL_n, \tag{A.44}$$

where $\rho \in \mathrm{adm}\ \Sigma$ means that ρ is a nonnegative Borel function on E^n such that

$$\int_{\sigma \cap R} \rho\, dH^{n-1} \;\geq\; 1 \qquad \forall\, \sigma \in \Sigma. \tag{A.45}$$

Remark A.9. As for (2.3) in [322], one can show that if Σ' denotes those $\sigma \in \Sigma$ for which $H^{n-1}(\sigma \cap R) = \infty$, then $M_{n'}(\Sigma') = 0$.

Lemma A.13. *Let u be an admissible function for $C[G, C_0, C_1]$ and let $S \subset [0,1]$ be an L_1-measurable set. If $\Sigma(S) = \{u^{-1}(s) : s \in S\}$ and $M_{n'}[\Sigma(S)] = 0$, then $L_1(S) = 0$.*

Proof. Since $M_{n'}[\Sigma(S)] = 0$, Theorem A.20 provides a nonnegative Borel function $\rho \in L^{n'}$ such that

$$\int_{u^{-1}(s) \cap R} \rho\, dH^{n-1} \;=\; \infty$$

for every $s \in S$.

However, Hölder's inequality and the co-area formula (A.29) imply that

$$\infty \;>\; \int_R \rho |\nabla u|\, dL_n \;\geq\; \int_0^1 \int_{u^{-1}(s) \cap R} \rho(x)\, dH^{n-1}(x)\, dL_1(s)$$

and, therefore, $L_1(S) = 0$. $\square$

Theorem A.26. $M_{n'}(\Sigma) \;\geq\; C[G, C_0, C_1]^{-1/n-1}$.

Proof. Choose $\varepsilon > 0$ and let ρ be any function for which $\rho \in \mathrm{adm}\ \Sigma$. Let u be an admissible function for $C = C[G, C_0, C_1]$ such that

$$\int_R |\nabla u|^n\, dL_n \;<\; C + \varepsilon.$$

R^* is connected since G is, and it is therefore clear that $u^{-1}(s) \in \Sigma$ for all $0 < s < 1$. Hence, by Hölder's inequality and Theorem A.18, we have

$$\left(\int_{E^n} \rho^{n'}\, dL_n\right)^{1/n'} (C+\varepsilon)^{1/n} \geq \left(\int_{E^n} \rho^{n'}\, dL_n\right)^{1/n'} \left(\int_R |\nabla u|^n\, dL_n\right)^{1/n}$$
$$\geq \int_R \rho |\nabla u|\, dL_n \;\geq\; \int_0^1 \int_{u^{-1}(s)} \rho(x)\, dH^{n-1}(x)\, dL_1(s) \;\geq\; 1.$$

Since ε is arbitrary,

$$\int_R \rho^{n'}\, dL_n \;\geq\; C^{-1/(n-1)},$$

which is also true for the infimum over all $\rho \in \operatorname{adm} \Sigma$, and thus the result follows. □

The opposite inequality will be established by a sequence of approximations. *We will begin by first assuming that $C_0 \cup C_1$ consists only of a finite number of nondegenerate continua and that $C_0 \cup C_1 \subset G$. We will also assume initially that $H^{n-1}(\partial G) < \infty$.*

Let V be an open connected set such that $G \supset cl\ V \supset V \supset C_0 \cup C_1$ and let v be the extremal function for $C = C[G, C_0, C_1]$; see Eq. (A.30). Since v satisfies the variational condition (A.31), the proof of the following lemma is very similar to that of Lemma 3 in [77] and will not be given here. It is possible to obtain a stronger result, but the following is sufficient for our purposes.

Lemma A.14. *Let π be the boundary of an n-dimensional polyhedral set P (P not necessarily contained in V) such that $C_0 \subset P$ and $C_1 \subset E^n \setminus cl\ P$. Then π separates C_0 from C_1 in V and*

$$\int_{\pi(b)} |\nabla v|^{n-1}\, dL_n \;\geq\; 2bC[V, C_0, C_1]$$

whenever $0 < b < \operatorname{dist}(\pi, C_0 \cup C_1)$ and where $\pi(b) = \{x : \operatorname{dist}(x, \pi) < b\}$.

Remark A.10. In Lemmas A.15 and A.16, the integral average ρ_r of $|\nabla v|^{n-1}$ will be used. Thus, defining $\nabla v = 0$ on the complement of V, for each $r > 0$, we have

$$\rho_r(x) \;=\; \alpha(r)^{-1} \int_{B(x,r)} |\nabla v(y)|^{n-1}\, dL_n(y),$$

where $\alpha(r) = L_n[B(x,r)]$. It is well known that ρ_r is continuous and that

$$\rho_r \to |\nabla v|^{n-1} \quad L_n\text{-a.e.}$$

as $r \to 0$.

Also, by a result of K. T. Smith [294] and Lebesgue's dominated convergence theorem, $\|\rho_r\|_{n'} \to \||\nabla v|^{n-1}\|_{n'}$ as $r \to 0$ and, consequently, $\|\rho_r - |\nabla v|^{n-1}\|_{n'} \to 0$.

Lemma A.15. *With π as in Lemma A.14,*

$$\int_{\pi} \rho_r \, dH^{n-1} \geq C[V, C_0, C_1]$$

whenever $r < \operatorname{dist}(\pi, C_0 \cup C_1)$.

Proof. Choose $b > 0$, $r > 0$ so that $b + r < \operatorname{dist}(\pi, C_0 \cup C_1)$. If π_y denotes the translation of π through the vector y, then Fubini's theorem and Lemma A.14 imply

$$\begin{aligned} &\int_{\pi(b)} \rho_r(x) \, dL_n(x) \\ &\qquad = \alpha(r)^{-1} \int_{B(0,r)} \int_{\pi(b)} |\nabla v(x+y)|^{n-1} \, dL_n(x) \, dL_n(y) \\ &\qquad = \alpha(r)^{-1} \int_{B(0,r)} \int_{\pi_y(b)} |\nabla v(x)|^{n-1} \, dL_n(x) \, dL_n(y) \geq 2bC[V, C_0, C_1] \end{aligned} \tag{A.46}$$

since π_y satisfies the condition of Lemma A.14. In addition to this, if $d(x) = \operatorname{dist}(x, \pi)$, then $|\nabla d| = 1$ L_n-a.e. on $E^n \setminus \pi$ (3) of 4.8 in [59]; therefore, Theorem A.18 gives

$$\int_{\pi(b)} \rho_r(x) \, dL_n(x) = \int_0^b \int_{d^{-1}(s)} \rho_r(x) \, dH^{n-1} \, dL_1(s). \tag{A.47}$$

Let $F(s)$ denote the inner integral on the right. Since ρ_r is continuous on E^n and π is the boundary of a polyhedral set, it is clear that

$$\lim_{s \to 0} F(s) = 2 \int_{\pi} \rho_r \, dH^{n-1}.$$

Hence, from Eqs. (A.46) and (A.47) we have

$$C[V, C_0, C_1] \leq \lim_{b \to 0} (2b)^{-1} \int_0^b F(s) \, dL_1(s) = \int_{\pi} \rho_r \, dH^{n-1}.$$

□

Lemma A.16. *If Σ is the class of sets that separate C_0 from C_1 in G, then*

$$\int_{\sigma \cap R} |\nabla v|^{n-1} \, dH^{n-1} \geq C[V, C_0, C_1]$$

for M_n-a.e. $\sigma \in \Sigma$.

Proof. Select $\sigma \in \Sigma$ and let U be that part of a partition of $G \setminus \sigma$ that contains C_0. Since $H^{n-1}(\partial G) < \infty$ by assumption, by appealing to Remark A.9, we can take $H^{n-1}(\partial U) < \infty$. Recall that $G \supset \mathrm{cl}\ V$ and that $\nabla v = 0$ on the complement of V. Hence, we can choose r_0 so small that the support of ρ_{r_0} is contained in G (and therefore for all $r \leq r_0$) and $r_0 \leq \mathrm{dist}(\partial U, C_0, C_1)$.

By applying Theorem A.11 to the set U, we obtain a sequence of n-dimensional polyhedral sets P_i. Let $\pi_i = \partial P_i$. From properties (i), (ii), and (iv) of Theorem A.11, it is clear that eventually $C_0 \subset P_i$ and $C_1 \subset E^n \setminus (\mathrm{cl}\, P_i)$. Thus, Lemma A.15 applies to π_i for all large i. Now, $\beta(U) \subset \partial U \subset \partial G \cup \sigma$ and since the support of ρ_r is contained in G for all $r \leq r_0$, it is clear that

$$\int_{\sigma} \rho_r \, dH^{n-1} \geq \int_{\beta(U)} \rho_r \, dH^{n-1} \quad \forall\, r \leq r_0. \tag{A.48}$$

Since ρ_r is continuous, (v) of Theorem A.11 and Lemma A.15 imply

$$\int_{\beta(U)} \rho_r \, dH^{n-1} = \lim_{i \to \infty} \int_{\pi_i} \rho_r \, dH^{n-1} \geq C[V, C_0, C_1] \quad \forall\, r \leq r_0.$$

Thus, from Eq. (A.48) we have

$$\int_{\sigma} \rho_r \, dH^{n-1} \geq C[V, C_0, C_1] \quad \forall\, r \leq r_0. \tag{A.49}$$

In light of the fact that $\|\rho_r - |\nabla v|^{n-1}\|_{n'} \to 0$ as $r \to 0$ (see Remark A.10), the result follows from Theorem A.21(e) and (f) and (A.49). $\square$

Lemma A.17. *Let* $C_0 \cup C_1$ *consist only of a finite number of nondegenerate continua, let* $C_0 \cup C_1 \subset G$ *and* $H^{n-1}(\partial G < \infty)$, *and let* u *be the extremal function for* $C[G, C_0, C_1]$. *Then for* $M_{n'}$-a.e. $\sigma \in \Sigma$,

$$\int_{\sigma \cap R} |\nabla u|^{n-1} \, dH^{n-1} \geq C[G, C_0, C_1] = C.$$

Proof. Let V_i be a sequence of open connected sets such that

$$G \supset V_{i+1} \supset V_{i+1} \supset \mathrm{cl}\ V_i \supset V_i \supset C_0 \cup C_1$$

and $G = \cup V_i$. Let v_i be the extremal function for $C[V_i, C_0, C_1] = C_i$.

We will first show that $C_i \to C$ as $i \to \infty$. Recall that $C < \infty$. If $i > j$, then the restriction of v_i to V_j is an admissible function for V_j and, thus, so is $2^{-1}(v_1 + v_2)$. As in the proof of Theorem A.25, an application of Clarkson's inequality [43] gives

$$\int\limits_R \left| \frac{\nabla v_i - \nabla v_j}{2} \right| dL_n \leq \frac{1}{2} C_i - \frac{1}{2} C_j \qquad \forall\, i > j. \tag{A.50}$$

Since C_i is a monotonically increasing sequence bounded above by C, Eq. (A.50) implies the existence of a vector-valued function ρ such that

$$\int\limits_R |\nabla v_i - \rho|^n \, dL_n \to 0 \tag{A.51}$$

as $i \to \infty$.

In fact, since $C_0 \cup C_1$ consists only of a finite number of nondegenerate continua, an argument similar to that of [73], p. 362, shows that there is an admissible function u' such that $\nabla u' = \rho$ L_n-a.e. on R. Thus, Eq. (A.51) shows that

$$\lim_{i \to \infty} C_i = C. \tag{A.52}$$

This also implies that u' is the extremal function for C, i.e., $u' = u$.

Since $\| |\nabla v_i| - |\nabla u| \|_n \to 0$ as $i \to \infty$, there is a subsequence of $|\nabla v_i|$ that will still be denoted by $|\nabla v_i|$ such that $|\nabla v_i| \to |\nabla u|$ L_n-a.e. and therefore that $|\nabla v_i|^{n-1} \to |\nabla u|^{n-1}$ L_n-a.e. on R. This fact, along with

$$\| |\nabla v_i|^{n-1} \|_{n'} \to \| |\nabla u|^{n-1} \|_{n'},$$

leads to

$$\| |\nabla v_i|^{n-1} - |\nabla u|^{n-1} \|_{n\prime} \to 0$$

as $i \to \infty$. Thus, by Theorem A.21(e), we have for another subsequence

$$\lim_{i \to \infty} \int\limits_{\sigma \cap R} |\nabla v_i|^{n-1} \, dH^{n-1} = \int\limits_{\sigma \cap R} |\nabla u|^{n-1} \, dH^{n-1} \tag{A.53}$$

for $M_{n'}$-a.e. $\sigma \in \Sigma$.

Lemma A.16 states that for each i,

$$\int\limits_{\sigma \cap R} |\nabla v_i|^{n-1} \, dH^{n-1} \geq C_1$$

for $M_{n'}$-a.e. $\sigma \in \Sigma$, and, therefore, the result follows from Eqs. (A.52) and (A.53), and Theorem A.18(b). □

Theorem A.27. *If G is a bounded, open, connected set, $C_0 \cup C_1$ consists only of a finite number of nondegenerate continua, and $C_0 \cup C_1 \subset G$, then*

$$M_{n'} = C[G, C_0, C_1]^{-1/n-1}.$$

Proof. If it were the case that $H^{n-1}(\partial G) < \infty$, then the result would follow immediately from Theorem A.26 and Lemma A.17: If we let $\rho = C^{-1} |\nabla u|^{n-1}$, where u

is the extremal for C, then by Lemma A.17, there is $\Sigma_0 \subset \Sigma$ such that $\rho \in \text{adm}\ \Sigma_0$ and $M_{n'}(\Sigma_0) = M_{n'}(\Sigma)$. Hence, by Theorem A.26,

$$C^{-1/n-1} \le M_{n'}(\Sigma) \le \int_R \rho^{n'}\, dL_n = C^{-1/(n-1)}.$$

In order to eliminate the assumption $H^{n-1}(\partial G) < \infty$, select a sequence of open, connected sets $V_1 \subset V_2 \subset \ldots$ such that $C_0 \cup C_1 \subset V_1$, $H^{n-1}(\partial G) < \infty$, and $G = \cup V_i$. As in the proof of Lemma A.17, let v_i be the extremal for $C[V_i, C_0, C_1] = C_i$ and again we have

$$C_i \to C, \quad \| |\nabla v_i|^{n-1} - |\nabla u|^{n-1} \|_{n\prime} \to 0 \tag{A.54}$$

as $i \to \infty$. Thus, for a subsequence,

$$\lim_{i\to\infty} \int_{\sigma\cap R} |\nabla v_i|^{n-1}\, dH^{n-1} = \int_{\sigma\cap R} |\nabla u|^{n-1}\, dH^{n-1} \tag{A.55}$$

for $M_{n'}$-a.e. $\sigma \in \Sigma$.

For each i, every $\sigma \in \Sigma$ separates C_0 from C_1 in V_i, and, thus, applying Lemma A.17 with V_i replacing G, we have

$$\int_{\sigma\cap R} |\nabla v_i|^{n-1}\, dH^{n-1} \ge \int_{\sigma\cap v_i} |\nabla v_i|^{n-1}\, dH^{n-1} \ge C_i$$

for $M_{n'}$-a.e. $\sigma \in \Sigma$. (Observe that Lemma A.17, with V_i replacing G, applies only to Σ_i, which are those sets that separate C_0 from C_1 in V_i. However, a class in Σ that is $M_{n'}$-zero relative to Σ_i is $M_{n'}$-zero relative to Σ.) Hence, in view of Eqs. (A.54) and (A.55),

$$\int_{\sigma\cap R} |\nabla u|^{n-1}\, dH^{n-1} \ge C \tag{A.56}$$

for $M_{n'}$-a.e. $\sigma \in \Sigma$, which, as we have seen from the above, is sufficient to establish the theorem. □

Corollary A.5. *With the hypotheses of Theorem A.27, and if u is the extremal for $C[G, C_0, C_1]$, then*

(*i*) $0 \le u(x) \le 1$ *for all* $x \in G$,

(*ii*) $\int_{\sigma\cap R} |\nabla u|^{n-1}\, dH^{n-1} \ge C[G, C_0, C_1]$ *for* $M_{n'}$*-a.e.* $\sigma \in \Sigma$,

(*iii*) $\int_{u^{-1}(s)} |\nabla u|^{n-1}\, dH^{n-1} = C[G, C_0, C_1]$ *for* L_1*-a.e.* $s \in [0, 1]$.

Proof. By truncating u at levels 1 and 0 if necessary, a new admissible function would be formed whose gradient would be bounded above by the gradient of u. However, the extremal is unique and, thus, (i) follows. (ii) is just a restatement of Eq. (A.56).

In order to prove (iii), let

$$F(s) = \int_{u^{-1}(s)} |\nabla u|^{n-1} \, dH^{n-1}$$

for L_1-a.e. s, and observe that since G is connected, $u^{-1}(s) \in \Sigma$ whenever $0 < s < 1$. Thus, (ii) above and Lemma A.13 imply that $F(s) \geq C$ for L_1-a.e. $s \in [0,1]$. However, (i) and an application of Lemma A.18 give

$$C = \int_R |\nabla u|^n \, dL_n = \int_0^1 F(s) \, dL_1(s),$$

which implies that $F(s) = C$ for L_1-a.e. $s \in [0,1]$. □

Remark A.11. The following observation has some interest in view of Theorem 2 of [73]: *In addition to the hypotheses of Theorem A.5, assume that $H^{n-1}(C_0) = 0$. Then there is a point $x_0 \in C_0$ such that*

$$\limsup_{x \to x_0} |\nabla u(x)| = \infty.$$

If this were not the case, then, since C_0 is compact, there would be a constant $K > 0$ and an open set U such that $G \setminus C_1 \supset \operatorname{cl} U \supset U \supset C_0$ and $|\nabla u|^{n-1} < K\, L_n$-a.e. on U. Choose $\varepsilon > 0$. Since $H^{n-1}(C_0) = 0$, C_0 can be covered by a countable number of open n-balls B_i such that

$$\cup B_i \subset U \tag{A.57}$$

and

$$\sum H^{n-1}(\partial B_i) < \varepsilon K^{-1}. \tag{A.58}$$

Since C_0 is compact, a finite number of the B_i will cover C_0, say $B_1, B_2, \ldots, B_k$. According to Theorem A.20, there is a nonnegative Borel function $\rho \in L^{n'}(R)$ such that (ii) of Corollary A.5 fails to hold for only those $\sigma \in \Sigma$ for which

$$\int_{\sigma \cap R} \rho \, dH^{n-1} = \infty.$$

By employing Theorem A.18, we can replace each n-ball B_i, $i = 1, 2, \ldots, k$, by a slightly larger one B_i' such that

$$\int_{S_i'} \rho \, dH^{n-1} < \infty,$$

where $S_i' = \partial B_i'$, $|\nabla u|^{n-1} < K\, H^{n-1}$-a.e. on S_i', and Eq. (A.57) still holds. Now let $\sigma = \partial(\cup B_i')$. Then $\sigma \in \Sigma$ and (ii) of Corollary A.5 imply that $C[G, C_0, C_1] < \varepsilon$, which means that it is zero since ε is arbitrary. This would mean that $\nabla u = 0\, L_n$-a.e. on G. That is, since G is connected and u is ACT on G, u would be constant, a contradiction. □

In the following theorem, we will consider the general case of two disjoint compact sets C_0 and C_1 that are contained in the closure of an open, bounded, connected set G.

Theorem A.28. $M_{n'}(\Sigma) \;=\; C[G, C_0, C_1]^{-1/(n-1)}$.

Proof. In view of Theorem A.26, we may assume that $C = C[G, C_0, C_1] \neq 0$. For each positive integer i, let

$$K_0(i) \;=\; cl\,\{x : \operatorname{dist}(x, C_0) < (2i)^{-1}\}$$

and

$$H_0(i) \;=\; \{x : \operatorname{dist}(x, C_0) < i^{-1}\}.$$

Define $K_1(i)$ and $H_1(i)$ similarly and let $G_i = G \cup H_0(i) \cup H_1(i)$. Since G is connected, it is clear that G_i is open and connected, and notice that $K_0(i) \cup K_1(i)$ consists only of a finite number of nondegenerate continua. We will consider only those i for which $K_0(i)$ and $K_1(i)$ are disjoint. Thus, let v_i be the extremal function for $C_i = C[G_i, K_0(i), K_1(i)]$ and observe that if $i > j$, then the restriction of v_i to G_i is an admissible function for C_i. Finally, let Σ_i be those sets σ that separate $K_0(i)$ from $K_1(i)$ in G_i and subject to the condition that $\sigma \cap [H_0(i) \cup H_1(i)] = 0$. The purpose for this requirement is that now an $M_{n'}$-null class in Σ_i is also $M_{n'}$-null in Σ. It is clear that $\Sigma_1 \subset \Sigma_2 \subset \ldots$, $\Sigma_i \subset \Sigma$ for all i, and

$$\bigcup_{i=1}^{\infty} \Sigma_i \;=\; \Sigma. \tag{A.59}$$

Since C_i is a monotonically decreasing sequence bounded below by C, we can employ again part of the argument of Theorem A.17 to find a vector-valued function ρ such that

$$\lim_{i\to\infty} \int_{E^n} |\nabla v_i - \rho|^n \, dL_n \;=\; 0$$

and, therefore, for a subsequence,

$$\lim_{i\to\infty} \| |\nabla v_i|^{n-1} - |\rho|^{n-1} \|_{n'} \;=\; 0. \tag{A.60}$$

Hence, if $L = \lim_{i\to\infty} C_i$, $g_i = C_i^{-1} |\nabla v_i|^{n-1}$, and $g = L^{-1} |f|^{n-1}$, then Theorem A.21(f) provides another subsequence such that

$$\lim_{i\to\infty} \int_{\sigma\cap R} g_i \, dH^{n-1} \;=\; \int_{\sigma\cap R} g \, dH^{n-1} \tag{A.61}$$

for $M_{n'}$-a.e. $\sigma \in \Sigma$.

Now, by employing Corollary A.5 with G, C_0, C_1 replaced by G_i, $K_0(i)$, $K_1(i)$, respectively, we have for each i,

$$\int_{\sigma\cap R} g_i \, dH^{n-1} \geq 1 \tag{A.62}$$

for $M_{n'}$-a.e. $\sigma \in \Sigma_i$. Therefore, by the Fuglede theorem, Eqs. (A.59) and (A.61) imply that

$$\int_{\sigma\cap R} g \, dH^{n-1} \geq 1 \tag{A.63}$$

for $M_{n'}$-a.e. $\sigma \in \Sigma$. Since v_i is the extremal for C_i, (ii) of Corollary A.5 and Theorem A.26 show that, for each i,

$$\int_{E^n} (g_i)^{n'} \, dL_n = C_i^{-1/(n-1)}.$$

Thus, with Eqs. (A.60), (A.62), and (A.63), we have

$$C^{-1/n-1} \geq \lim_{i\to\infty} C_i^{-1/n-1} = \lim_{i\to\infty} \int_{E^n} (g_i)^{n'} \, dL_n = \int_{E^n} g^{n'} \, dL_n \geq M(\Sigma).$$

Theorem A.26 gives the opposite inequality and, thus, the proof is complete. □

We will conclude with a result concerning null sets for conformal capacity.

Theorem A.29. *Suppose C_0 and C_1 are disjoint compact sets in the closure of G. If $E \subset G \setminus (C_0 \cup C_1)$ is a compact set with $H^{n-1}(E) = 0$, then*

$$C[G, C_0, C_1] = C[G \setminus E, C_0, C_1].$$

Proof. The topological dimension of E is no more than $n-2$ since $H^{n-1}(E) = 0$ and, therefore, $G \setminus E$ is connected. Thus, the right-hand side of the equality has meaning. Clearly,

$$C[G, C_0, C_1] \geq C[G \setminus E, C_0, C_1]. \tag{A.64}$$

The opposite inequality will be established by considering separating sets. Let Σ^* be those sets that separate C_0 from C_1 in

$$[(G \setminus E) \setminus (C_0 \cup C_1)] \cup [C_0 \cup C_1]$$

and let Σ be those that separate C_0 from C_1 in R^*. In light of Eq. (A.64) and Theorem A.28, it is sufficient to prove that

$$M_{n'}(\Sigma) \geq M_{n'}(\Sigma^*). \tag{A.65}$$

To this end, let ρ be a function for which $\rho \in \operatorname{adm} \Sigma$. In order to establish Eq. (A.65), we need only show that $\rho \in \operatorname{adm} \Sigma^*$. Choose $\sigma^* \in \Sigma^*$ and notice that $\sigma^* \cup E \in \Sigma$. Thus,

$$\int_{\sigma\cup E} \rho \, dH^{n-1} \geq 1$$

and since $H^{n-1}(E) = 0$,

$$\int_{\sigma^*} \rho \, dH^{n-1} \geq 1.$$

This shows that $\rho \in \text{adm } \Sigma^*$ and, consequently, proves the theorem. $\square$

When G is compactified in E^n, Bagby showed that $C[G, C_0, C_1] = M_n(\Gamma)$, where Γ is the family of all arcs that meet both C_0 and C_1 (Ph.D. thesis, Harvard univ. 1966). By using Theorem A.28, [64], and [332], one can show that this result is valid when G is an open, bounded, connected set and $C_0 \cup C_1 \subset G$. [Moreover, if $C_0 \cup C_1 \subset \text{cl } G$, the result is also valid if certain conditions are imposed on the tangential behavior of $(\partial G) \cap (C_0 \cup C_1)$.] Thus, if Γ^* is the family of arcs that join C_0 to C_1 in $G \setminus E$, then Theorem A.29 implies

$$M_n(\Gamma^*) = M_n(\Gamma).$$

This result was obtained by Väisälä [317] in the case where C_0 and C_1 are nondegenerate continua.

Appendix B
BMO Functions by John–Nirenberg

A real-valued function u in a domain D in $\mathbb{R}^n$ is said to be of **bounded mean oscillation** in D, $u \in \mathrm{BMO}(D)$, if $u \in \mathrm{L}^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(x) - u_B| \, dm(x) \; < \; \infty, \tag{B.1}$$

where the supremum is taken over all balls B in D,

$$u_B \; = \; \frac{1}{|B|} \int\limits_B u(x) \, dm(x),$$

and m denotes the Lebesgue measure in $\mathbb{R}^n$. We say that $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D. We will write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is.

If $u \in \mathrm{BMO}$ and c is a constant, then $u + c \in \mathrm{BMO}$ and $\|u\|_* = \|u + c\|_*$. Thus, the space of BMO functions modulo constants with the norm given by Eq. (B.1) is a Banach space. Obviously, $\mathrm{L}^\infty \subset \mathrm{BMO}$. Fefferman and Stein [61] showed that BMO can be characterized as the dual space of the Hardy space H^1. BMO has become an important concept in harmonic analysis, partial differential equations, and related areas. John and Nirenberg have established the following fundamental lemma, which plays an important role in the theory of BMO-qc mappings; see [140], cf. also [110].

Lemma B.1. *If u is a nonconstant function in* $\mathrm{BMO}(D)$*, then*

$$|\{x \in B : |u(x) - u_B| > t\}| \leq a e^{-\frac{b}{\|u\|_*} \cdot t} \cdot |B| \tag{B.2}$$

for every ball B in D and all $t > 0$, where a and b are absolute positive constants that do not depend on B and u. Conversely, if $u \in L^1_{\mathrm{loc}}$, and if for every ball B in D and for all $t > 0$,

$$|\{x \in B : |u(x) - u_B| > t\}| < a e^{-bt} |B| \tag{B.3}$$

for some positive constants a and b, then $u \in \mathrm{BMO}(D)$.

We need the following lemma, which follows from Lemma B.1.

Lemma B.2. *If u is a nonconstant function in* $\mathrm{BMO}(D)$*, then*

$$|\{x \in B : |u(x)| > \tau\}| \leq Ae^{-\beta\tau} \cdot |B| \tag{B.4}$$

for every ball B in D and all $\tau > |u_B|$*, where*

$$\beta = b/\|u\|_* \,, \qquad A = ae^{b|u_B|/\|u\|_*}, \tag{B.5}$$

and the constants a and b are as in Lemma B.1.

Proof. For $t > 0$, let $\tau = t + |u_B|$, $D_1 = \{x \in B : |u(x)| > \tau\}$, and $D_2 = \{x \in B : |u(x) - u_B| > t\}$. Then, by the triangle inequality, $D_1 \subset D_2$ and, hence, by Eq. (B.2),

$$|D_1| \leq |D_2| \leq ae^{b|u_B|/\|u\|_*} \cdot e^{-\tau b/\|u\|_*} \cdot |B|,$$

which implies Eq. (B.4) with A and β as in Eq. (B.5). $\square$

Corollary B.1. $\mathrm{BMO} \subset L^p_{\mathrm{loc}}$ *for all* $p \in [1, \infty)$.

Indeed, for $u \in \mathrm{BMO}$, $u \neq \mathrm{const}$, by Lemma B.2,

$$\int_B |u(x)|^p \, dx \;\leq\; |B| \, \{|u_B|^p \; + A \int_{|u_B|}^{\infty} t^p e^{-\beta t} \, dt\} \;<\; \infty.$$

We need also several facts about BMO functions given in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ and their relations to BMO functions on $\mathbb{R}^n$.

Let S^n be the unit sphere in $\mathbb{R}^{n+1}$,

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

The space $\mathbb{R}^n$ may be identified with the hyperplane $x_{n+1} = 0$ in $\mathbb{R}^{n+1}$. This is done with the aid of the stereographic projection P of S^n onto $\overline{\mathbb{R}^n}$, which is given by

$$y = P(x) = \frac{(x_1, \ldots, x_n)}{1 - x_{n+1}}$$

for $x = (x_1, \ldots, x_n, x_{n+1}) \in S^n \setminus (0, \ldots, 0, 1)$. The point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ corresponds to $\infty \in \overline{\mathbb{R}^n}$. Note that the element of the spherical area

$$d\sigma \;=\; \left(\frac{2}{1 + |y|^2}\right)^n dm(y)$$

is invariant with respect to the rotations of the sphere S^n.

A real-valued measurable function u in a domain $D \subset \overline{\mathbb{R}^n}$ is said to be in $\mathrm{BMO}(D)$ if u is locally integrable with respect to the spherical area and

$$\|u\|_{*\sigma} = \sup_B \frac{1}{\sigma(B)} \int_B |u - u_B| \, d\sigma < \infty, \tag{B.6}$$

where the supremum is taken over all spherical balls B in D and

$$u_B = \frac{1}{\sigma(B)} \int_B u \, d\sigma. \tag{B.7}$$

The first of the following two lemmas enables us to decide whether a function u in a domain $D \subset \overline{\mathbb{R}^n}$ belongs to $\mathrm{BMO}(D)$ (in the spherical sense) by considering the restriction u_0 of u to $D_0 = D \setminus \{\infty\}$; see [255], p. 7. The second lemma is a consequence of Lemma B.2.

Lemma B.3. $u \in \mathrm{BMO}(D)$ *iff* $u_0 \in \mathrm{BMO}(D_0)$. *Furthermore,*

$$c^{-1}\|u_0\|_* \leq \|u\|_{*\sigma} \leq c\|u_0\|_*\,, \tag{B.8}$$

where c is an absolute constant.

Lemma B.4. *If either* $u \in \mathrm{BMO}(D)$ *or* $u_0 \in \mathrm{BMO}(D_0)$, *then, for* $\tau > \gamma$,

$$\sigma\{x \in B : |u(x)| > \tau\} \leq \alpha e^{-\beta\tau} \tag{B.9}$$

for every spherical ball B in D, where the constants α, β, *and* γ *depend on B as well as on the function u.*

Proof. If $\overline{B} \in \mathbb{R}^n$, then we have Eq. (B.4) by Lemma B.2, and, since $\sigma(E) \leq 4|E|$ for every measurable set $E \subset \mathbb{C}$, Eq. (B.9) follows. If $\infty \in \overline{B}$, then for a suitable rotation R of S^n, ∞ is exterior to $\overline{B'}, B' = R(B)$, and the assertion follows by Lemma B.3 and the validity of Eq. (B.9) with B' and $\hat{u} = u \circ R^{-1}$ instead of B and u. Now, in view of the invariance of the spherical area with respect to rotations, by Lemmas B.2 and B.3, we have Eq. (B.9). $\square$

The following lemma holds for BMO functions and cannot be extended to $\mathrm{BMO}_{\mathrm{loc}}$ functions; see, e.g., [255], p. 5.

Proposition B.1. *Let E be a discrete set in a domain D,* $D \subset \mathbb{R}^n$, *and let u be a function in* $\mathrm{BMO}(D \setminus E)$. *Then any extension* $\hat{u}$ *of u to D is in* $\mathrm{BMO}(D)$ *and* $\|u\|_* = \|\hat{u}\|_*$.

The following lemma is a special case of a theorem by Reimann on the characterization of qc maps in $\mathbb{R}^n$, $n \geq 2$, in terms of the induced isomorphism on BMO; see [254], p. 266.

Proposition B.2. *If f is a K-*qc *map of a domain D in $\mathbb{R}^n$ onto a domain D' and $u \in$* BMO(D')*, then $v = u \circ f$ belongs to* BMO(D) *and*

$$\|v\|_* \leq c\|u\|_*\,,$$

where c is a constant that depends only on K.

We say that a Jordan surface E in $\overline{\mathbb{R}^n}$ is a K-**quasisphere** if $E = f(S^{n-1})$ for some K-qc map $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$. The next statement is due to Jones; see [143].

Proposition B.3. *Let D be a Jordan domain such that ∂D is a K-quasisphere, and let u be a function in* BMO(D). *Then u has an extension $\hat{u}$ to $\mathbb{R}^n$ that belongs to* BMO$(\mathbb{R}^n)$ *and*

$$\|\hat{u}\|_* \leq c\|u\|_*\,,$$

where c depends only on K.

The following proposition (see, e.g., [255], p. 8), which concerns a symmetric extension of BMO functions, will be needed in studying the reflection principle and boundary behavior of BMO-qc. It should be noted that this proposition cannot be extended to $\mathrm{BMO}_{\mathrm{loc}}$ functions.

Proposition B.4. *If u belongs to the class* BMO *in the unit ball $\mathbb{B}^n$ and $\hat{u}$ is an extension of u to $\mathbb{R}^n$ that satisfies the symmetry condition*

$$\hat{u}(t) = \begin{cases} u(x) & \text{if } x \in \mathbb{B}^n, \\ u(x/|x|^2) & \text{if } x \in \mathbb{R}^n \backslash \mathbb{B}^n, \end{cases}$$

then $\hat{u} \in$ BMO$(\mathbb{R}^n)$ *and $\|u\|_* = \|\hat{u}\|_*$.*

Remark B.1. Given a domain $D, D \subset \mathbb{R}^n$, there is a nonnegative real-valued function u in D such that $u(x) \leq Q(x)$ a.e. for some $Q(x)$ in BMO(D) and $u \notin$ BMO(D). For $D = \mathbb{R}^2$, one can take, for instance, $Q(x,y) = 1 + |\log|y||, (x,y) \in \mathbb{R}^2$, and $u(x,y) = Q(x,y)$ if $y > 0$ and $u(x,y) = 1$ if $y \leq 0$.

Acknowledgments. The second author's research was partially supported by the (Ukrainian) State Foundation of Fundamental Investigations (SFFI) (Grant nos. 01.07/00241 and F25.1/055). He would like to thank the Department of Mathematics of Helsinki University, Finland, Technion, Haifa, and the Holon Institute of Technology, Israel, Institute of Mathematics of Polish Academy of Sciences, Warsaw. The third author's is research was partially supported by a grant from the Israel Science Foundation (Grant no. 198/00) and by the Technion Fund for the Promotion of Research.The fourth author was partially supported by a grant from the Israel Science Foundation (Grant no. 198/00).

The authors thank Professor Anatoly Golberg for his survey of results in the theory of mappings with finite mean dilatations related to the moduli method and for his careful reading the text. The authors thank Professors Denis Kovtonyuk, Ruslan Salimov, Evgeny Sevostyanov and also Andrei Ignat'ev for their essential contribution to the book. Moreover, we would like to thank Professor Sergei Vodopyanov for his useful remarks on further applications of the developed theory to mappings in Sobolev's classes. The authors are very grateful to reviewers whose remarks made it possible to improve the text of the monograph. Finally, we would like to thank our assistant Ol'ga Sapunova for her great technical help under the preparation of the monograph.

References

1. Ahlfors L.: Lectures on Quasiconformal Mappings. Van Nostrand, New York (1966).
2. Ahlfors L.: Quasiconformal deformations and mappings in $\mathbb{R}^n$. J. Anal. Math. **30**, 74–97 (1976).
3. Ahlfors L.: On a class of quasiconformal mappings, Österreich. Acad. Wiss. Math.-Naturw. K1. S. BII **185**, 5–10 (1976).
4. Ahlfors L.: On quasiconformal mappings. J. Anal. Math. **3**, 1–58 (1953/54).
5. Ahlfors L., Beurling A.: Conformal invariants and function-theoretic null-sets. Acta Math. **83**, 101–129 (1950).
6. Ahlfors L., Beurling A.: The boundary correspondence under quasiconformal mappings. Acta Math. **96**, 125–142 (1956).
7. Andreian Cazacu C.: On the length-area dilatation. Complex Var. Theory Appl. **50**(7–11), 765–776 (2005).
8. Andreian Cazacu C.: Some formulae on the extremal length in n-dimensional case. Proc. Rom.-Finn. Sem. on Teichmüller Spaces and Quasiconformal Mappings (Brazov, 1969), pp. 87–102, Publ. House of Acad. Soc. Rep. Romania, Bucharest (1971).
9. Andreian Cazacu C., Stanciu V.: BMO-mappings in the plane. Topics in analysis and its applications. NATO Sci. Ser. II Math.Phys. Chem. **147**, 11–30 (2004).
10. Aikava H., Ohtsuka M.: Extremal length of vector measures. Ann. Acad. Sci. Fenn. Math. **24**(1), 61–88 (1999).
11. Alexandroff P.: Dimensionstheorie. Math. Ann. **106**, 161–238 (1932).
12. Arkhangel'skii A.V.: Open and near open mappings. Connections between spaces. Trans. Moscow Math. Soc. **15**, 204–250 (1966).
13. Arkhangel'skii A.V., Ponomarev V.J.: Fundamentals of general topology problems and exercises. Nauka, Moscow (1974) [in Russian].
14. Aseev V.V.: A certain module property. Dokl. Akad. Nauk. SSSR **200**, 513–514 (1971) [in Russian].
15. Aseev V., Sychev A.: On the removable sets of spatially quasiconformal mappings. Sibirsk. Mat. $\check{Z}$. **15**, 1213–1227 (1974) [in Russian].
16. Astala K.: Area distortion of quasiconformal mappings. Acta Math. **173**, 37–60 (1994).
17. Astala K.: A remark on quasiconformal mappings and BMO-functions. Michigan Math. J. **30**, 209–212 (1983).
18. Astala K., Gehring F.W.: Injectivity, the BMO norm and the universal Teichmüller space. J. Anal. Math. **46**, 16–57 (1986).
19. Astala K., Iwaniec T., Koskela P., Martin G.: Mappings of BMO-bounded distortion. Math. Ann. **317**, 703–726 (2000).
20. Beckenbach E.F., Bellman R. Inequalities. Springer-Verlag, New York (1965).
21. Balogh Z., Koskela P.: Quasiconformality, quasisymmetry, and removability in Loewner spaces, with an appendix by Jussi Väisälä. Duke Math. J. **101**(3), 554–577 (2000).

22. Belinskii P.P.: General properties of quasiconformal mappings. Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk (1974) [in Russian].
23. Bellman R.: Introduction to Matrix Analysis. McGraw-Hill, New York (1970).
24. Besicovitch A.S.: On sufficient conditions for a function to be analytic. Proc. London Math. Soc. **32**, 1–9 (1932).
25. Biluta P.A.: Extremal problems for mappings quasiconformal in the mean. Sib. Mat. Zh. **6**, 717–726 (1965) [in Russian].
26. Bishop C., Gutlyanskii V., Martio O., Vuorinen M.: On conformal dilatation in space. IJMMS **22**, 1397–1420 (2003).
27. Bojarski B.: Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients. Mat. Sb. **43**(85), 451–503 (1957).
28. Bojarski B., Iwaniec T.: Analytic foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$. Ann. Acad. Sci. Fenn. Ser. A1. Math. **8**, 257–324 (1983).
29. Borsuk K.: Uber Schnitte der n-dimensionalen Euklidischen Räume. Math. Ann. **106**, 239–248 (1932).
30. Bourbaki N.: Functions of a real variable. Nauka, Moscow (1965) [in Russian].
31. Brakalova M.A., Jenkins J.A.: On solutions of the Beltrami equation. J. Anal. Math. **76**, 67–92 (1998).
32. Brakalova M.A., Jenkins J.A.: On solutions of the Beltrami equation. II. Publ. Inst. Math. (Beograd) (N.S.) **75(89)**, 3–8 (2004).
33. Brania A., Yang S.: Domains with controlled modulus and quasiconformal mappings. Nonlinear Stud. **9**(1), 57–73 (2002).
34. Brezis H., Nirenberg L.: Degree theory and BMO. I. Compact manifolds without boundaries. Selecta Math. (N.S.) **1**(2), 197–263 (1995).
35. Busemann H.: Length-preserving mappings. Pacific J. Math. **14**, 457–477 (1964).
36. Caraman P. n-dimensional quasiconformal mappings, Haessner Publishing, Newfoundland, NJ (1974).
37. Caraman P.: Relations between p-capacity and p-module, I, II. Rev. Roumaine Math. Pures Appl. **39**(6), 509–553, 555–577 (1994).
38. Cheeger J.: Differentiability of Lipschitz functions on metric spaces. Geom. Funct. Anal., 428–517 (1999).
39. Chernavskii A.V.: Finite-to-one mappings of manifolds. Transl. Amer. Math. Soc. **(2)100**, 253–267 (1964) [translated from Russian: Mat. Sb. **65(107)**, 357–369 (1964)].
40. Chernavskii A.V.: Addendum to the paper "Finite-to-one mappings of manifolds." Transl. Amer. Math. Soc. **(2) 100**, 269–270 (1964) [translated from Russian: Mat. Sb. **66(108)**, 471–472 (1965)].
41. Chiarenza F., Frasca M., Longo P.: $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. Trans. Amer. Math. Soc. **336**(2), 841–853 (1993).
42. Cianchi A., Pick L.: Sobolev embeddings into BMO, VMO, and L_∞. Ark. Mat. **36**(2), 317–340 (1998).
43. Clarkson J.A.: Uniformly convex spaces. Trans. Amer. Math. Soc. **40**, 396–414 (1936).
44. Cristea M.: A generalization of Sard's theorem and a Jacobian condition for interiority. Demonstratio Math. **21**(2), 399–405 (1988).
45. Cristea M.: Open discrete open mappings having local ACL^n inverses. Preprint, Inst. Math., Romanian Acad. **6** (2008).
46. Cristea M.: Local homeomorphisms having local ACL^n inverses. Complex Var. Elliptic Equ. **53**(1), 77–99 (2008).
47. Csörnyei M., Hencl S., Maly J.: Homeomorphisms in the Sobolev space $W^{1,n-1}$. Charles Univ. in Prague, Prerpint MATH–KMA **252**, 15 pp. (2007).
48. David G.: Solutions de l'equation de Beltrami avec $\|\mu\|_\infty = 1$. Ann. Acad. Sci. Fenn. Ser. A1. Math. **13**, 25–70 (1988).
49. DeGiorgi E.: Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. Ricerche Mat. **4**, 95–113 (1955) [in Italian].

50. Dugundji J.: Topology. Allyn and Bacon, Boston (1966).
51. Dunford N., Schwartz J.T.: Linear Operators, Part I: General Theory. Interscience Publishers, New York (1957).
52. Duren P.L.: Univalent Functions. Springer-Verlag, Berlin (1983).
53. Evans L.C., Gapiery R.F.: Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton, FL (1992).
54. Faraco D., Koskela P., Zhong X.: Mappings of finite distortion: The degree of regularity. Adv. Math. **190**(2), 300–318 (2005).
55. Federer H.: Geometric Measure Theory. Springer-Verlag, Berlin (1969).
56. Federer F.: The Gauss–Green theorem. Trans. Amer. Math. Soc. **58**, 44–76 (1945).
57. Federer F.: The (φ, K) rectifiable subsets of n space. Trans. Amer. Math. Soc. **62**, 114–192 (1947).
58. Federer F.: A note on the Gauss–Green theorem. Proc. Amer. Math. Soc. **9**, 447–451 (1958).
59. Federer F.: Curvature measures. Trans. Amer. Math. Soc. **93**, 418–491 (1959).
60. Federer F., Fleming W.H.: Normal and integral currents. Ann. Math. **72**, 458–520 (1960).
61. Fefferman C., Stein E.M.: H^p spaces of several variables. Acta Math. **129**, 137–193 (1972).
62. Fonseca I., Gangbo W.: Degree Theory in Analysis and Applications. Clarendon Press, Oxford (1995).
63. Fleming W.H.: Flat chains over a finite coefficient group. Trans. Amer. Math. Soc. **121**, 160–186 (1966).
64. Fuglede B.: Extremal length and functional completion. Acta Math. **98**, 171–219 (1957).
65. Gehring F.W.: The definitions and exceptional sets for quasiconformal mappings. Ann. Acad. Sci. Fenn. Math. Ser. A1 **281**, 1–38 (1960).
66. Gehring F.W.: Symmetrization of rings in space. Trans. Amer. Math. Soc. **101**, 499–519 (1961).
67. Gehring F.W.: Extension of quasiconformal mappings in three space. J. Anal. Math. **14**, 171–182 (1965).
68. Gehring F.W.: Univalent functions and the Schwarzian derivative. Comment. Math. Helv. **52**, 561–572 (1977).
69. Gehring F.W.: Injectivity of local quasi-isometries. Comment. Math. Helv. **57**, 202–220 (1982).
70. Gehring F.W.: Characteristic Properties of Quasidisk. Les Presses de l'Université de Montréal, Montréal (1982).
71. Gehring F.W.: Quasiconformal Mappings in Complex Analysis and Its Applications, vol. 2. International Atomic Energy Agency, Vienna (1976).
72. Gehring F.W.: Uniform domains and ubiquitous quasidisk. Jahresber. Deutsch. Math.-Verein **89**, 88–103 (1987).
73. Gehring F.W.: Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc. **103**, 353–393 (1962).
74. Gehring F.W.: Extremal length definitions for the conformal capacity of rings in space. Comment. Math. Helv. **36**, 42–46 (1961).
75. Gehring F.W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. **130**, 265–277 (1973).
76. Gehring F.W.: Lipschitz mappings and the p-capacity of rings in n-space. Advances in the Theory of Riemann Surfaces, Princeton University Press, pp. 175–193 (1971).
77. Gehring F.W.: Extremal length definitions for the conformal capacity in space. Michigan Math. J. **9**, 137–150 (1962).
78. Gehring F.W.: A remark on the moduli of rings. Comment Math. Helv. **36**, 42–46 (1961).
79. Gehring F.W., Iwaniec T.: The limit of mappings with finite distortion, Ann. Acad. Sci. Fenn. Math. **24**, 253–264 (1999).
80. Gehring F.W., Lehto O.: On the total differentiability of functions of a complex variable. Ann. Acad. Sci. Fenn. A1. Math. **272**, 9 (1959).
81. Gehring F.W., Martio O.: Quasiextremal distance domains and extension of quasiconformal mappings. J. Anal. Math. **45**, 181–206 (1985).

82. Gehring F.W., Martio O.: Lipschitz classes and quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. **10**, 203–219 (1985).
83. Gehring F.W., Osgood B.G.: Uniform domains and the quasihyperbolic metric. J. Anal. Math. **36**, 50–74 (1979).
84. Gehring F.W., Palka B.P.: Quasiconformally homogeneous domains. J. Anal. Math. **30**, 172–199 (1976).
85. Gehring F.W., Väisälä J.: The coefficients of quasiconformality of domains in space. Acta Math. **114**, 1–70 (1965).
86. Golberg A.: Some classes of plane topological mappings with generalized first derivatives. Ukrain. Math. J. **44**(8), 1114–1116 (1992) [in Russian]; translation in Ukrain. Math. J. **44**(8), 1114–1116 (1992).
87. Golberg A.: Distortions of p-moduli of separating sets under mappings with finite integral dilatations. Bull. Soc. Sci. Lettres Lódź Sér. Rech. Déform. **40**, 41–51 (2003).
88. Golberg A.: Homeomorphisms with finite mean dilatations. Contemporary Math. **382**, 177–186 (2005).
89. Golberg A.: Coefficients of quasiconformality of ring domains. Proc. Inst. Math. NAS Ukraine **1**(3), 77–86 (2004).
90. Golberg A., Kud'yavin V.S.: Mean coefficients of quasiconformality of pair of domains. Ukrain. Mat. Zh. **43**(12), 1709-1712 (1991) [in Russian]; translation in Ukrain. Math. J. **43**, 1594–1597 (1991).
91. Gol'dshtein V.M., Reshetnyak Yu.G.: Quasiconformal Mappings and Sobolev Spaces. Kluwer Academic Publishers, New York (1990).
92. Gol'dshtein V.M., Vodop'yanov S.K.: A test of the removability of sets for L_p^1 spaces of quasiconformal and quasi-isomorphic mappings. Sib. Math. J. **18**(1), 48–68 (1977) [in Russian].
93. Gol'dstein V.M., Vodop'janov S.K.: Prolongement des fonctions de classe L_p' et applications quasiconformes. C. R. Acad. Sci. Paris **290**, 453–456 (1980).
94. Gotoh Y.: On global integrability of BMO functions on general domains. J. Anal. Math. **75**, 67–84 (1998).
95. Gotoh Y.: BMO extension theorem for relatively uniform domains. J. Math. Kyoto Univ. **33**, 171–193 (1993).
96. Grötzsch H.: Eine Bemerkung zum Koebeschen Kreisnormierungsprinzip. Berichteüber die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leibzig **87**, 319–324 (1936).
97. Gutlyanskiǐ V.Y., Martio O., Ryazanov V.I., Vuorinen M.: On convergence theorems for space quasiregular mappings. Forum Math. **10**, 353–375 (1998).
98. Gutlyanskiǐ V., Martio O., Sugawa T., Vuorinen M.: On the degenerate Beltrami equation. Trans. Amer. Math. Soc. **357**(3), 875–900 (2005).
99. Gutlyanskiǐ V.Y., Ryazanov V.I.: On the theory of local behavior for quasiconformal mappings. Izv. Ross. Akad. Nauk. Ser. Math. **59**(3), 31–58 (1995) [in Russian].
100. Gutlyanskiǐ V.Y., Ryazanov V.I.: On boundary correspondence under quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. **21**, 167–178 (1996).
101. Hajlasz P.: Sobolev spaces on an arbitrary metric space. Potential Anal. **5**, 403–415 (1996).
102. Hajlasz P., Koskela P.: Sobolev Met Poincaré. Memoirs Amer. Math. Soc. **145** (2000).
103. Hajlasz P., Martio O.: Traces of Sobolev functions on fractal type sets and characterization of extension domains. J. Funct. Anal. **143**, 221–246 (1997).
104. Halmos P.R.: Measure Theory. Nostrand, Princeton, N.J. (1952).
105. Hardy G.H., Littlewood J.E., Polya G.: Inequalities. Cambridge Univ. Press, Cambridge (1934).
106. Heinonen J.: Quasiconformal mappings onto John domains. Rev. Mat. Iberoamericana **5**, 97–123 (1989).
107. Heinonen J.: Lectures of Analysis on Metric Spaces. Springer-Verlag, New York (2000).
108. Heinonen J.: A capacity estimate on Carnot groups. Bull. Sci. Math. **119**(1), 475–484 (1995).
109. Heinonen J., Holopainen I.: Quasiregular mappings on Carnot groups. J. Geom. Anal. **7**(1), 109–148 (1997).

110. Heinonen J., Kilpelainen T., Martio O.: Nonlinear Potential Theory of Degenerate Elliptic Equations. Clarendon Press, Oxford univ. Press (1993).
111. Heinonen J., Koskela P.: Sobolev mappings with integrable dilatations. Arch. Rational Mech. Anal. **125**, 81–97 (1993).
112. Heinonen J., Koskela P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1–41 (1998).
113. Heinonen J., Semmes S.: Thirty-three years or no questions about mappings, measures and metrics. Conform. Geom. Dyn. **1**, 1–12 (1997).
114. Hencl S., Koskela P.: Mappings with finite distortion: discreteness and openness for quasi–light mappings, Ann. Inst. H. Poincaré Anal. Non Lineare **22**(3), 331–342 (2005).
115. Hencl S., Koskela P., Onninen J.: A note on extremal mappings of finite distortion. Math. Res. Lett. **12**(2-3), 231–237 (2005).
116. Hencl S., Maly J.: Mappings with finite distortion: Hausdorff measure of zero sets. Math. Ann. **324**(3), 451–464 (2002).
117. Herron D.A., Koskela P.: Locally uniform domains and quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. **20**, 187–206 (1995).
118. Herron D.A., Koskela P.: Quasiextremal distance domains and conformal mappings onto circle domains. Complex Variables Theory Appl. **15**, 167–179 (1990).
119. Herron D.A., Koskela P.: Mappings with finite distortion: Gauge dimension of generalized quasicircles. Illinois J. Math. **47**(4), 1243–1259 (2003).
120. Hersch J.: Longeurs extremales, mesure harmonique et distance hyperbolique. C. R. Acad. Sci. Paris **235**, 569–571 (1952).
121. Hersch J.: “Longeurs extremales” dans l’espace, resistance electrique et capacite. C. R. Acad. Sci. Paris **238**, 1639–1641 (1954).
122. Hesse J.: A p-extremal length and p-capacity equality. Ark. Mat. **13**, 131–144 (1975).
123. Hocking J.G., Young G.S.: Topology. Addison-Wesley, London (1961).
124. Holopainen I., Pankka P.: Mappings of finite distortion: Global homeomorphism theorem. Ann. Acad. Sci. Fenn. Math. **29**(1), 59–80 (2004).
125. Horn R.A., Johnson C.R.: Topics in Matrix Analysis. Nauka, Moscow (1992) [in Russian].
126. Hurewicz W., Wallman H.: Dimension Theory. Princeton Univ. Press, Princeton, NJ (1948).
127. Ignat’ev A., Ryazanov V.: Finite mean oscillation in the mapping theory. Ukr. Mat. Vis. **2**(3), 395–417, 443 (2005) [in Russian]; translation in Ukr. Math. Bull. **2**(3), 403–424, 443 (2005).
128. Ignat’ev A., Ryazanov V.: To the theory of the boundary behavior of space mappings. Ukr. Mat. Vis. **3**(2), 199–211, 301 (2006) [in Russian]; translation in Ukr. Math. Bull. **3**(2), 189–201 (2006).
129. Iwaniec T., Kopiecki R.: Stability in the differential equations for quasiregular mappings. Lecture Notes in Math. **798**, 203–214 (1980).
130. Iwaniec T., Koskela P., Martin G.: Mappings of BMO-distortion and Beltrami-type operators. J. Anal. Math. **8**, 337–381 (2002).
131. Iwaniec T., Koskela P., Martin G., Sbordone C.: Mappings of finite distortion: $L^n\log^\alpha L$-integrability. J. London Math. Soc. (**2)67**, 123–136 (2003).
132. Iwaniec T., Koskela P., Onninen J.: Mappings of finite distortion: Monotonicity and continuity. Invent. Math. **144**(3), 507–531 (2001).
133. Iwaniec T., Koskela P., Onninen J.: Mappings of finite distortion: Compactness. Ann. Acad. Sci. Fenn. Math. **27**(2), 391–417 (2002).
134. Iwaniec T., Martin G.: Geometrical Function Theory and Nonlinear Analysis. Clarendon Press, Oxford Univ. Press (2001).
135. Iwaniec T., Martin G.: What’s new for the Beltrami equation? Geometric analysis and applications (Canberra, 2000), 132–148, Proc. Centre Math. Appl. Austral. Nat. Univ., **39**, Austral. Nat. Univ., Canberra (2001).
136. Iwaniec T., Sbordone C.: Riesz transforms and elliptic PDEs with VMO coefficients. J. Anal. Math. **74**, 183–212 (1998).
137. Iwaniec T., Sverak V.: On mappings with integrable dilatation. Proc. Amer. Math. Soc. **118**, 181–188 (1993).

138. Ivanov L.D.: Variations of Sets and Functions, Nauka, Moscow (1975).
139. Jenkins J.A.: Some results related to extremal length. Ann. Math. Stud. **30**, 87–94 (1953).
140. John F., Nirenberg L.: On functions of bounded mean oscillation. Commun. Pure Appl. Math. **14**, 415–426 (1961).
141. John F.: Rotation and strain. Commun. Pure Appl. Math. **14**, 391–413 (1961).
142. Jones P.M.: Quasiconformal mappings and extendability of functions in Sobolev spaces. Acta Math. **147**, 71–88 (1981).
143. Jones P.M.: Extension theorems for BMO. Indiana Univ. Math. J. **29**, 41–66 (1980).
144. Jurchescu M.: Modulus of a boundary component. Pacific J. Math. **8**, 791–809 (1958).
145. Kallunki S.: Mappings of finite distortion: The metric definition. Dissertation, Univ. Jyvaskyla, Jyvaskyla. Ann. Acad. Sci. Fenn. Math. Diss. **131**, p. 33 (2002).
146. Kallunki S., Martio O.: Linear dilatations and ACL homeomorphisms. Proc. Amer. Math. Soc. **130**(4), 1073–1078 (2002).
147. Kauhanen J., Koskela P., Maly J.: Mappings of finite distortion: Discreteness and openness. Arch. Rat. Mech. Anal. **160**, 135–151 (2001).
148. Kauhanen J., Koskela P., Maly J.: Mappings of finite distortion: Condition N. Michigan Math. J. **49**, 169–181 (2001).
149. Kauhanen J., Koskela P., Maly J., Onninen J., Zhong X.: Mappings of finite distortion: Sharp Orlich-conditions. Rev. Mat. Iberoamericana **19** (3), 857–872 (2003).
150. Kirszbraun M.D.: Uber die zusammenziehende und Lipschitzsche Transformationen. Fund. Math. J. **22**, 77–108 (1934).
151. Koskela P., MacManus P.: Quasiconformal mappings and Sobolev spaces. Stud. Math. **131**, 1–17 (1998).
152. Koskela P., Maly J.: Mappings of finite distortion: The zero set of the Jacobian. J. Eur. Math. Soc. **5**(2), 95–105 (2003).
153. Koskela P., Onninen J.: Mappings on finite distortion: The sharp modulus of continuity. Trans. Amer. Math. Soc. **355**, 1905–1920 (2003).
154. Koskela P., Onninen J.: Mappings of finite distortion: Capacity and modulus inequalities. J. Reine Agnew. Math. **599**, 1–26 (2006).
155. Koskela P., Onninen J., Rajala K.: Mappings of finite distortion: Injectivity radius of a local homeomorphism. Future trends in geometrical function theory, 169–174, Rep. Univ. Jyvaskyla Dep. Math. Stat., **92**, Univ. Jyvaskyla, Jyvaskyla (2003).
156. Koskela P., Rajala K.: Mappings of finite distortion: Removable singularities. Israel J. Math. **136**, 269–283 (2003).
157. Kovtonyuk D.: On finitely bi-Lipschitz mappings. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine **10**, 79–87 (2005) [in Russian].
158. Kovtonyuk D.: On finitely Lipschitz mappings. Reports of NAS of Ukraine **10**, 7–11 (2005) [in Russian].
159. Kovtonyuk D., Ryazanov V.: On mappings with finite hyperarea distortion. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine **9**, 102–111 (2004) [in Russian].
160. Kovtonyuk D., Ryazanov V.: On mappings with finite hyperarea distortion. Reports of NAS of Ukraine **4**, 18–22 (2005) [in Russian].
161. Kovtonyuk D., Ryazanov V.: On lower Q-homeomorphisms. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine **11**, 105–119 (2005) [in Russian].
162. Kovtonyuk D., Ryazanov V.: On homeomorphic extension to the boundary of lower Q-homeomorphisms. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine **12**, 100–105 (2006) [in Russian].
163. Kovtonyuk D., Ryazanov V.: On boundaries of space domains. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine **13**, 110–120 (2006) [in Russian].
164. Kovtonyuk D., Ryazanov V.: On the theory of lower Q-homeomorphisms. Ukr. Mat. Visn. **5**(2), 159–184 (2008) [in Russian]; translated in Ukrainian Math. Bull. by AMS.
165. Kovtonyuk D., Ryazanov V.: On the theory of mappings with finite area distortion. J. Anal. Math., **104**, 291–306 (2008).
166. Koranyi A., Reimann H.: Quasiconformal mappings on the Heisenberg group. Invent. Math. **80**, 309–338 (1985).

167. Koranyi A., Reimann H.: Foundations for the theory of quasiconformal mappings on the Heisenberg group. Adv. Math. **111**, 1–87 (1995).
168. Krivov V.V.: Some properties of moduli in space. Dokl. Akad. Nauk. SSSR **154**, 510–513 (1964).
169. Kruglikov V.I.: The existence and uniqueness of mappings that are quasiconformal in the mean. In: Metric Questions of the Theory of Functions and Mappings, pp. 123–147. Naukova Dumka, Kiev (1973) [in Russian].
170. Kruglikov V.I.: Capacities of condensors and quasiconformal in the mean mappings in space. Mat. Sb. **130**(2), 185–206 (1986) [in Russian].
171. Kruglikov V.I., Miklyukov V.M.: On classes of plane topological mappings with generalized derivatives. In: Metric Questions of the Theory of Functions and Mappings, pp. 102–122. Naukova dumka, Kiev (1973) [in Russian].
172. Kruglikov V.I., Paikov V.I.: Capacities and prime ends of an n-dimensional domain. Dokl. Akad. Nauk Ukrain. SSR Ser. A. **5**, 10–13, 84 (1987) [in Russian].
173. Krushkal' S.L.: On mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR **157**(3), 517–519 (1964) [in Russian].
174. Krushkal' S.L.: On the absolute integrability and differentiability of some classes of mappings of many-dimensional domains. Sib. Mat. Zh. **6**(3), 692–696 (1965) [in Russian].
175. Krushkal' S.L.: On mappings ε-quasiconformal in the mean. Sib. Mat. Zh. **8**(4), 798–806 (1967) [in Russian].
176. Krushkal, Samuil L.: Quasiconformal mappings and Riemann surfaces. John Wiley & Sons, New York (1979).
177. Krushkal' S.L., Kühnau R.: Quasiconformal mappings, new methods and applications. Nauka, Novosibirsk (1984) [in Russian].
178. Kud'yavin V.S.: A characteristic property of a class of n-dimensional homeomorphisms. Dokl. Akad. Nauk Ukrain. SSR ser. A. **3**, 7–9 (1990) [in Russian].
179. Kud'yavin V.S.: Quasiconformal mappings and α-moduli of families of curves. Dokl. Akad. Nauk Ukrain., **7**, 11–13 (1992) [in Russian].
180. Kud'yavin V.S.: Estimation of the distortion of distances under mappings quasiconformal in the mean. Dinam. Sploshn. Sred., **52**, 168–171 (1981) [in Russian].
181. Kud'yavin V.S.: Local boundary properties of mappings quasiconformal in the mean. In: Collection of Scientific Works, Institute of Mathematics, Siberian Branch of the Academy of Sciences of the USSR, pp. 168–171. Novosibirsk (1981) [in Russian].
182. Kud'yavin V.S.: Behavior of a class of mappings quasiconformal in the mean at an isolated singular point. Dokl. Akad. Nauk SSSR **277**(5), 1056–1058 (1984) [in Russian].
183. Kühnau R.: Über Extremalprobleme bei im Mittel quasiconformen Abbildungen. Lecture Notes in Math. **1013**, 113–124 (1983) [in German].
184. Kühnau R.: Canonical conformal and quasiconformal mappings. Identities. Kernel functions. In: Handbook of Complex Analysis, Geometry Function Theory, vol. 2, pp. 131–163. Elsevier, Amsterdam (2005).
185. Kuratowski K.: Topology, vol. 1. Academic Press, New York (1968).
186. Kuratowski K.: Topology, vol. 2. Academic Press, New York (1968).
187. Lavrentiev M.A.: Variational method in boundary-value problems for systems of equations of elliptic type. AN SSSR, Moscow (1962) [in Russian].
188. Lavrentiev M. A.: One differential property of homeomorphic mappings of three-dimensional domains. Dokl. Akad. Nauk SSSR. **20**, 241–242 (1938).
189. Lehto O.: Homeomorphisms with a prescribed dilatation. Lecture Notes in Math. **118**, 58–73 (1968).
190. Lehto O., Virtanen K.: Quasiconformal Mappings in the Plane. Springer-Verlag, New York (1973).
191. Lelong-Ferrand J.: Representation conforme et transformations à integrale de Dirichlet bornée. Gauthier-Villars, Paris (1955).
192. Loewner C.: On conformal capacity in space. J. Math. Mech. **8**, 411–414 (1959).
193. Maly J., Martio O.: Lusin's condition (N) and mappings of the class $W^{1,n}$. J. Reine Agnew. Math. **485**, 19–36 (1995).

194. Maly J., Ziemer W.P.: Fine Regularity of Solutions of Elliptic Partial Differential Equations. Math. Surveys and Monographs **51**, AMS (1997).
195. Manfredi J.J., Villamor E.: Mappings with integrable dilatation in higher dimensions. Bull. Amer. Math. Soc. **32**(2), 235–240 (1995).
196. Manfredi J.J., Villamor E.: An extension of Reshetnyak's theorem. Indiana Univ. Math. J. **47**(3), 1131–1145 (1998).
197. Margulis G.A., Mostow G.D.: The differential of quasi-conformal mapping of a Carnot-Caratheodory space. Geom. Func. Anal. **5**(2), 402–433 (1995).
198. Markina I.: Coincidence of p-module of a family of curves and p-capacity on Carnot group. Rev. Mat. Iberoamericana **19**(1), 143–160 (2003).
199. Markina I., Vodop'yanov S.: On value distribution for quasimeromorphic mappings on $\mathbb{H}$-type Carnot groups. Bull. Sci. Math. **130**(6), 467–523 (2006).
200. Martin G.J.: Quasiconformal and bi-Lipschitz homeomorphisms, uniform domains and the quasihyperbolic metric. Trans. Amer. Math. Soc. **292**, 169–191 (1985).
201. Martio O.: Modern tools in the theory of quasiconformal maps. Texts Math. Ser. B, **27**. Univ. Coimbra, Dept. Mat., Coimbra, pp. 1–43 (2000).
202. Martio O.: Definitions for uniform domains. Ann. Acad. Sci. Fenn. Ser. A1. Math. **5**, 197–205 (1980).
203. Martio O., Miklyukov V.: On existence and uniqueness of degenerate Beltrami equation. Complex Var. Theory Appl. **49**(7–9), 647–656 (2004).
204. Martio O., Ryazanov V., Srebro U., Yakubov E.: To the theory of Q-homeomorphisms. Dokl. Akad. Nauk Rossii **381**(1), 20–22 (2001) [in Russian].
205. Martio O., Ryazanov V., Srebro U., Yakubov E.: BMO-quasiconformal mappings and Q-homeomorphisms in space. Reports Dept. Math. Univ. Helsinki **288**, 1–24 (2001).
206. Martio O., Ryazanov V., Srebro U., Yakubov E.: On boundary behavior of Q-homeomorphisms. Reports Dept. Math. Univ. Helsinki **318**, 1–12 (2002).
207. Martio O., Ryazanov V., Srebro U., Yakubov E.: Mappings with finite length distortion. J. d'Anal. Math. **93**, 215–236 (2004).
208. Martio O., Ryazanov V., Srebro U., Yakubov E.: Q-homeomorphisms. Contemp. Math. **364**, 193–203 (2004).
209. Martio O., Ryazanov V., Srebro U., Yakubov E.: On Q-homeomorphisms. Ann. Acad. Sci. Fenn. **30**, 49–69 (2005).
210. Martio O., Rickman S., Väisälä J.: Definitions for quasiregular mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. **448**, 40 (1969).
211. Martio O., Ryazanov V., Vuorinen M.: BMO and injectivity of space quasiregular mappings. Math. Nachr. **205**, 149–161 (1999).
212. Martio O., Sarvas J.: Injectivity theorems in plane and space. Ann. Acad. Sci. Fenn. Ser. A1 Math. **4**, 384–401 (1978/1979).
213. Martio O., Väisälä J.: Elliptic equations and maps of bounded metric distortion. Math. Ann. **282**, 423–443 (1988).
214. Martio O., Vuorinen M.: Whitney cubes, p-capacity and Minkowski content. Expo. Math. **5**, 17–40 (1987).
215. Maz'ya V.: Sobolev Classes. Springer-Verlag, Berlin (1985).
216. Maz'ya V.G., Poborchi S.V.: Differentiable Functions on Bad Domains. World Science, Singapore (1997).
217. Mayer V.: Quasiregular analogues of critically finite rational functions with parabolic orbifold. J. d'Anal. Math. **75**, 105–119 (1998).
218. McShane E.J.: Extension of range of functions. Bull. Amer. Math. Soc. **40**, 837–842 (1934).
219. Mikljukov V.M.: Removable singularities of quasiconformal mappings in space. Dokl. Akad. Nauk SSSR **188**, 525–527 (1969) [in Russian].
220. Miklyukov V.M., Suvorov G.D.: On existence and uniqueness of quasiconformal mappings with unbounded characteristics. In: Investigations in the Theory of Functions of Complex Variables and its Applications, Inst. Math., Kiev, (1972) [in Russian].
221. Mitchell J.: On Carnot–Caratheodory metrics. J. Differ. Geom. **21**, 35-45 (1985).

222. Müller S.: Higher integrability of determinants and weak convergence in L^1. J. Reine Agnew. Math. **412**, 20–34 (1990).
223. Nagy B.S.: Spektraldarstellung linearer Transformationen des Hilbertschen Raumes. Ergebnisse d. Math **5** (1942).
224. Näkki R.: Boundary behavior of quasiconformal mappings in n-space. Ann. Acad. Sci. Fenn. Ser. A1. Math. **484**, 1–50 (1970).
225. Näkki R.: Extension of Loewner's capacity theorem. Trans. Amer. Math. Soc. **180**, 229–236 (1973).
226. Näkki R., Väisälä J.: John disks. Expo Math. **9**(1), 3–43 (1991).
227. Newman M.H.A.: Elements of the topology of plane sets of points. Cambridge Univ. Press, Cambridge (1954).
228. Ohtsuka M.: Extremal Length and Precise Functions. Gakkotosho Co., Tokyo (2003).
229. Ohtsuka M.: On weighted extremal length of families of curves. Complex Analysis, Birkhäuser, pp. 191–200 (1988).
230. Ohtsuka M.: Limiting property of extremal length, problems in the theory of quasiconformal mappings. Lecture Notes in Math., **1039**, 466–467 (1983).
231. Onninen J.: Mappings of finite distortion: future directions and problems. The p-harmonic equation and recent advances in analysis, Amer. Math. Soc., Contemp. Math., **370**, 199–207, Providence, RI (2005).
232. Onninen J.: Mappings of finite distortion: Minors of the differential matrix. Calc. Var. Partial Differ. Eq. **21**(4), 335–348 (2004).
233. Onninen J.: Mappings of finite distortion: Continuity. Dissertation, Univ. Jyvaskyla, Jyvaskyla, 24 (2002).
234. Osgood B.G.: Univalence criteria in multiply-connected domains. Trans. Amer. Math. Soc. **260**, 459–473 (1980).
235. Ostrowski A.: Solutions of equations and systems of equations. 2nd ed., Academic Press, New York (1966).
236. Palagachev D.K.: Quasilinear elliptic equations with VMO coefficients. Trans. Amer. Math. Soc. **347**(7), 2481–2493 (1995).
237. Pankka P.: Mappings of finite distortion and weighted parabolicity. Future trends in geometrical function theory, 175–182, Rep. Univ. Jyvaskyla Dep. Math. Stat., **92**, Univ. Jyvaskyla, Jyvaskyla (2003).
238. Pansu P.: Metriques de Carnot–Caratheodory et quasiisometries des espaces symetriques de rang un. Ann. Math. **119**, 1–60 (1989).
239. Perovich M.: Isolated singularity of the mean quasiconformal mappings. Lect. Notes in Math. **743**, 212–214 (1979).
240. Perovich M.: Global homeomorphism of mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR **230**(4), 781–784 (1976) [in Russian].
241. Pesin I.N.: Mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR **187**(4), 740–742 (1969) [in Russian].
242. Poletskii E.A.: The modulus method for non-homeomorphic quasiconformal mappings. Mat. Sb. **83**(125), 261–272 (1970).
243. Ponomarev S.P.: On the N-property of homeomorphisms of the class W_p^1. Sibirsk. Mat. Zh. **28**(2), 140–148 (1987) [in Russian].
244. Ponomarev S.P.: The N^{-1}-property of mappings, and Lusin's (N) condition. Mat. Zametki **58**, 411–418 (1995) [in Russian]; transl. in Math. Notes **58** (1995), 960–965 (1996).
245. Potyemkin V., Ryazanov V.: On noncompactness of the David classes. Ann. Acad. Sci. Fenn. Ser. A1. Math. **23**(1), 191–204 (1998).
246. Rado T., Reichelderfer P.V.: Continuous Transformations in Analysis. Springer-Verlag, Berlin (1955).
247. Ragusa M.A.: Elliptic boundary value problem in vanishing mean oscillation hypothesis. Comment. Math. Univ. Carolin. **40**(4), 651–663 (1999).
248. Rajala K.: Mappings of finite distortion: the Rickman–Picard theorem for mappings of finite lower order. J. Anal. Math. **94**, 235–248 (2004).

249. Rajala K.: Mappings of finite distortion: removability of Cantor sets. Ann. Acad. Sci. Fenn. Math. **29**(2), 269–281 (2004).
250. Rajala K.: Mappings of finite distortion: Removable singularities for locally homeomorphic mappings. Proc. Amer. Math. Soc. **132**(11), 3251–3258 (2004) (electronic).
251. Rajala K.: Mappings of finite distortion: Removable singularities. Dissertation, Univ. Jyvaskyla, Jyvaskyla, p. 74 (2003).
252. Ransford T.: Potential Theory in the Complex Plane. Cambridge, Univ. Press, Cambridge (1995).
253. Reich E., Walczak H.: On the behavior of quasiconformal mappings at a point. Trans. Amer. Math. Soc. **117**, 338–351 (1965).
254. Reimann H.M.: Functions of bounded mean oscillation and quasiconformal mappings. Comment. Math. Helv. **49**, 260–276 (1974).
255. Reimann H.M., Rychener T.: Funktionen Beschränkter Mittlerer Oscillation. Lecture Notes in Math. **487** (1975).
256. Reshetnyak Yu.G.: Space mappings with bounded distortion. Transl. Math. Monographs **73**, AMS (1989).
257. Reshetnyak Yu.G.: Some geometric properties of functions and mappings with generalized derivatives. Sibirsk. Mat. Zh. **7**, 886–919 (1966) [in Russian].
258. Reshetnyak Yu.G.: Generalized derivatives and differentiability almost everywhere. Mat. Sb. (N.S.) **75**(117), 323–334 (1968) [in Russian].
259. Reshetnyak Yu.G.: Liouville's conformal mapping theorem under minimal regularity hypotheses. Sibirsk. Math. Zh. **8**, 835–840 (1967) [in Russian]; transl. in Siberian Math. J. **8** (1967).
260. Rickman S.: Quasiregular Mappings. Springer-Verlag, Berlin (1993).
261. Rudin W.: Real and Complex Analysis, 3rd ed., McGraw-Hill, New York (1987).
262. Ryazanov V.I.: On compactification of classes with integral restrictions on the Lavrent'ev characteristics. Sibirsk. Mat. Zh. **33**(1), 87–104 (1992) [in Russian]; translation in Siberian Math. J. **33**(1), 70–86 (1992).
263. Ryazanov V.I.: On quasiconformal mappings with measure restrictions. Ukrain. Mat. Zh. **45**(7), 1009–1019 (1993) [in Russian]; translation in Ukrain Math. J. **45**(7), 1121–1133 (1993).
264. Ryazanov V.I.: On mappings that are quasiconformal in the mean. Sibirsk. Mat. Zh. **37**(2), 378–388 (1996) [in Russian]; translation in Siberian Math. J. **37**(2), 325–334 (1996).
265. Ryazanov V.I.: Some questions of convergence and compactness for quasiconformal mappings. Amer. Math. Soc. Transl. **131**(2), 7–19 (1986).
266. Ryazanov V., Salimov R.: Weakly flat spaces and bondaries in the mapping theory. Ukr. Mat. Vis. **4**(2), 199–234 (2007) [in Russian]; translation in Ukrain. Math. Bull. **4**(2), 199–233 (2007).
267. Ryazanov V., Sevost'yanov E.: On normal families of space mappings. Reports of NAS of Ukraine **3**, 19–24 (2004) [in Russian].
268. Ryazanov V., Sevost'yanov E.: On normal families of Q-homeomorphisms. Proc. Inst. Appl. Math. & Mech. NAS Ukraine **9**, 161–176 (2004) [in Russian].
269. Ryazanov V., Sevost'yanov E.: Equicontinuons classes of ring Q-homeomorphisms. Sibirsk. Math. Zh. **48**(6), 1361–1376 (2007) [in Russian].
270. Ryazanov V., Sevost'yanov E.: To the theory of ring Q-homeomorphisms. Israel J. Math. (to appear).
271. Ryazanov V., Srebro U., Yakubov E.: BMO-quasiconformal mappings. Reports Dept. Math. Univ. Helsinki **155**, 1–22 (1997).
272. Ryazanov V., Srebro U., Yakubov E.: To the theory of BMO-quasiregular mappings. Dokl. Akad. Nauk Rossii **369**(1), 13–15 (1999) [in Russian].
273. Ryazanov V., Srebro U., Yakubov E.: BMO-quasiconformal mappings. J. d'Anal. Math. **83**, 1–20 (2001).
274. Ryazanov V., Srebro U., Yakubov E.: Plane mappings with dilatation dominated by functions of bounded mean oscillation. Sib. Adv. Math. **11**(2), 94–130 (2001).

275. Ryazanov V., Srebro U., Yakubov E.: Degenerate Beltrami equation and radial Q-homeomorphisms. Preprint, Department of Mathematics, University of Helsinki, **369** (2003).
276. Ryazanov V., Srebro U., Yakubov E.: Finite mean oscillation and the Beltrami equation. Israel J. Math. **153**, 247–266 (2006).
277. Ryazanov V., Srebro U., Yakubov E.: On ring solutions of Beltrami equation. J. d'Anal. Math. **96**, 117–150 (2005).
278. Ryazanov V., Srebro U., Yakubov E.: Beltrami equation and FMO functions. Contemp. Math. **382**, 357–364 (2005).
279. Ryazanov V., Srebro U., Yakubov E.: On the theory of the Beltrami equation. Ukrain. Math. J. **58**(11), 1571–1583 (2006).
280. Ryazanov V., Srebro U., Yakubov E.: The Beltrami equation and ring homeomorphisms. Ukrain. Math. Bull. **4**(1), 79–115 (2007). [in Russian]
281. Saks S.: Theory of the Integral. Dover, New York (1964).
282. Salimov R.R.: ACL and differentiability of Q-homeomorphisms. Ann. Acad. Sci. Fenn. Ser. A1. Math. **33**, 295–301 (2008).
283. Sarason D.: Functions of vanishing mean oscillation. Trans. Amer. Math. Soc. **207**, 391–405 (1975).
284. Sastry S.: Boundary behavior of BMO-qc automorphisms. Israel J. Math. **129**, 373–380 (2002).
285. Seittenranta P.: Linear dilatation of quasiconformal maps in space. Duke Math. J. **91**, 1–16 (1998).
286. Serrin J.: On the differentiability of functions of several variables. Arch. Rational Mech. Anal. **7**, 359–372 (1961).
287. Sevost'yanov E.A.: On convergence theorems of mappings with finite length distortion. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine **10**, 173–183 (2005) [in Russian].
288. Sevost'yanov E.A.: To the convergence theory of space mappings of Sobolev class. Reports of NAS of Ukraine **10**, 40–41 (2005) [in Russian].
289. Sevost'yanov E.A.: Convergence theory and quasilinear Beltrami equation. Proc. Inst. of Appl. Math. & Mech. **12**, 128–139 (2006) [in Russian].
290. Shabat B.V.: The modulus method in space. Soviet Math. Dokl. **130**, 1210–1213 (1960).
291. Shanmugalingam N.: Newtonian spaces: An extension of Sobolev spaces to metric measure spaces. Rev. Math. Iberoamericana **16**, 243–279 (2000).
292. Shlyk V.A.: The capacity of a condenser and the modulus of a family of separating surfaces. Notes Sci. Sem. LOMI/Math Inst. Steklov, Leningrad, **185**, 168–182 (1990) [in Russian].
293. Shlyk V.A.: On the equality between p-capacity and p-modulus Sibirsk. Mat. Zh. **34**(6), 216–221 (1993) [in Russian]; transl. in Siberian Math. J. **34**(6), 1196–1200 (1993).
294. Smith K.T.: A generalization of an inequality of Hardy and Littlewood. Canad. J. Math. **8**, 157–170 (1956).
295. Smith W.S., Stegenga D.A.: Exponential integrability of the quasihyperbolic metric on Hölder domains. Ann. Acad. Sci. Fenn. Ser. A1. Math. **16**, 345–360 (1991).
296. Srebro U., Yakubov E.: Branched folded maps and alternating Beltrami equations. J. d'Anal. Math. **70**, 65–90 (1996).
297. Srebro U., Yakubov E.: The Beltrami equation. In: Handbook of Complex Analysis, Geometry Function Theory, vol. 2, pp. 555–597. Elsevier, Amsterdam (2005).
298. Stanciu V.: Normal families of BMO_{loc}-quasiregular mappings. Complex Var. Theory Appl. **49**(10), 681–688 (2004).
299. Staples S.G.: L^p-averaging domains and the Poincaré inequality. Ann. Acad. Sci. Fenn. Ser. A1. Math. **14**, 103–127 (1989).
300. Stepanov V.V.: Sur les conditions de l'existence de la differentielle total. Mat. Sb. **30**, 487–489 (1924) [in Russian].
301. Strugov Y.F.: Compactness of the classes of mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR **243**(4), 859–861 (1978) [in Russian].
302. Strugov Y.F., Sychov A.V.: On different classes of mappings quasiconformal in the mean. Vest. PANI, **7**, 14–19 (2002) [in Russian].

303. Suvorov G.D.: Generalized principle of the length and area in the mapping theory. Naukova Dumka, Kiev (1985) [in Russian].
304. Suvorov G.D.: The metric theory of prime ends and boundary properties of plane mappings with bounded Dirichlet integrals. Naukova Dumka, Kiev (1981) [in Russian].
305. Sychov A.V.: Moduli and space quasiconformal mappings. Nauka, Novosibirsk (1983) [in Russian].
306. Tamrazov P.M.: Moduli and extremal metrics in nonorientable and twisted Riemannian manifolds. Ukrain. Mat. Zh. **50**(10), 1388–1398 (1998) [in Russian]; transl. Ukrain. Math. J. **50**(10), 1586–1598 (1998).
307. Tamrazov P.M.: Continuity of certain conformal invariants. Ukrain. Mat. Zh. **18**(6), 78–84 (1966) [in Russian].
308. Titus C.J., Young G.S.: The extension of interiority with some applications. Trans. Amer. Math. Soc. **103**, 329–340 (1962).
309. Tukia P.: Hausdorff dimension and quasisymmetric mappings. Math. Scand. **65**, 152–160 (1989).
310. Tukia P.: Compactness properties of μ-homeomorphisms. Ann. Acad. Sci. Fenn. Ser. A1. Math. **16**, 47–69 (1991).
311. Tukia P., Väisälä J.: Quasisymmetric embeddings of metric spaces. Ann. Acad. Sci. Fenn. Ser. A1. Math. **5**, 97–114 (1980).
312. Tyson J.T.: Metric and geometric quasiconformality in Ahlfors regular Loewner spaces. Conform. Geom. Dyn. **5**, 21–73 (2001) (electronic).
313. Uhlov A., Vodopyanov S.: Mappings associated with weighted Sobolev spaces. Complex Anal. Dynam. Sys. III. Contemp. Math. **455**, 363–382 (2008).
314. Ukhlov A., Vodopyanov S.: Sobolev spaces and (P,Q)-quasiconformal mappings of Carnot groups. Siberian Math. J. **39**, 665–682 (1998).
315. Vasil'ev, A.: Moduli of families of curves for conformal and quasiconformal mappings. Lecture Notes in Math. **1788**, Springer-Verlag, Berlin (2002).
316. Väisälä J.: Lectures on n-dimensional quasiconformal mappings. Lecture Notes in Math. **229**, Springer-Verlag, Berlin (1971).
317. Väisälä J.: On the null-sets for extremal distances. Ann. Acad. Sci. Fenn. Ser. A1. Math. **322**, 1–12 (1962).
318. Väisälä J.: Uniform domains. Tohoku Math. J. **40**, 101–118 (1988).
319. Väisälä J.: Invariants for quasisymmetric, quasi-Möbius and bi-Lipschitz mappings. J. Anal. Math. **50**, 201–223 (1988).
320. Väisälä J.: Relatively and inner uniform domains. Conform. Geom. Dyn. **2**, 56–88 (1998).
321. Väisälä J.: Questions on quasiconformal maps in quasiconformal mappings and analysis. (Ann Arbor, 1995), pp. 369–374. Springer-Verlag (1998).
322. Väisälä J.: On quasiconformal mappings in space. Ann. Acad. Sci. Fenn. Ser. A1. Math. **298**, 1–36 (1961).
323. Varberg D.E.: Change of variables in multiple integrals. Amer. Math. Monthly **78**, 42–45 (1971).
324. Vodopyanov S.: Mappings with bounded distortion and with finite distortion on Carnot groups. Sibirsk. Mat. Zh. **40**(4), 764–804 (1999) [in Russian]; transl. in Siberian Math. J. **40**(4), 644–677 (1999).
325. Vodopyanov S.: Geometry of Carnot–Caratheodory spaces and differentiability of mappings. Contemp. Math. **424**, 247–301 (2007).
326. Vodopyanov S.: Foundations of the theory of mappings with bounded distortion on Carnot groups. Contemp. Math. **424**, 303–344 (2007).
327. VolkovyskiĭL.I.: Quasiconformal mappings. Lvov. Univ., Lvov (1954).
328. Vuorinen M.: Conformal geometry and quasiregular mappings. Lecture Notes in Math. **1319**, Springer-Verlag, Berlin (1988).
329. Vuorinen M.: Exceptional sets and boundary behavior of quasiregular mappings in n-space. Ann. Acad. Sci. Fenn. Ser. A1. Math. Dissertattiones **11**, 44 (1976).

330. Vuorinen M.: Lower bounds for the moduli of path families with applications to nontangential limits of quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. **2**, 279–291 (1979).
331. Walker M.F.: Linearly locally connected sets and quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. **11**.
332. Wallin H.: A connection between α-capacity and L^p-classes of differentiable functions. Ark. Mat. **5**, 331–341 (1963–1965).
333. Whitney H.: On totally differentiable and smooth functions. Pacific J. Math. **1**, 143–159 (1951); **1**, 77–86 (1986).
334. Whyburn G.T.: Analytic Topology. Amer. Math. Soc., Providence, RI (1942).
335. Wilder R.L.: Topology of Manifolds. AMS, New York (1949).
336. Yan B.: On the weak limit of mappings with finite distortion. Proc. Amer. Math. Soc. **128**(11), 3335–3340 (2000).
337. Young W.H.: Zur Lehre der nicht abgeschlossenen Punktmengen. Ber. Verh. Sachs. Akad. Leipzig **55**, 287–293 (1903).
338. Ziemer W.P.: Extremal length and p-capacity. Michigan Math. J. **16**, 43–51 (1969).
339. Ziemer W.P.: Weakly differentiable functions. Springer-Verlag, Berlin (1989).
340. Ziemer W.P.: Extremal length and conformal capacity. Trans. Amer. Math. Soc. **126**(3), 460–473 (1967).
341. Ziemer W.P.: Integral currents $\bmod 2$. Trans. Amer. Math. Soc. **105**, 496–524 (1962).
342. Ziemer W.P.: Some lower bounds for Lebesgue area. Pacific J. Math. **19**, 381–390 (1966).
343. Zorich V.A.: Admissible order of growth of the characteristic of quasiconformality in the Lavrent'ev theory. Dokl. Akad. Nauk SSSR **181** (1968) [in Russian].
344. Zorich V.A.: Isolated singularities of mappings with bounded distortion. Mat. Sb. **81**, 634–638 (1970) [in Russian].

Index