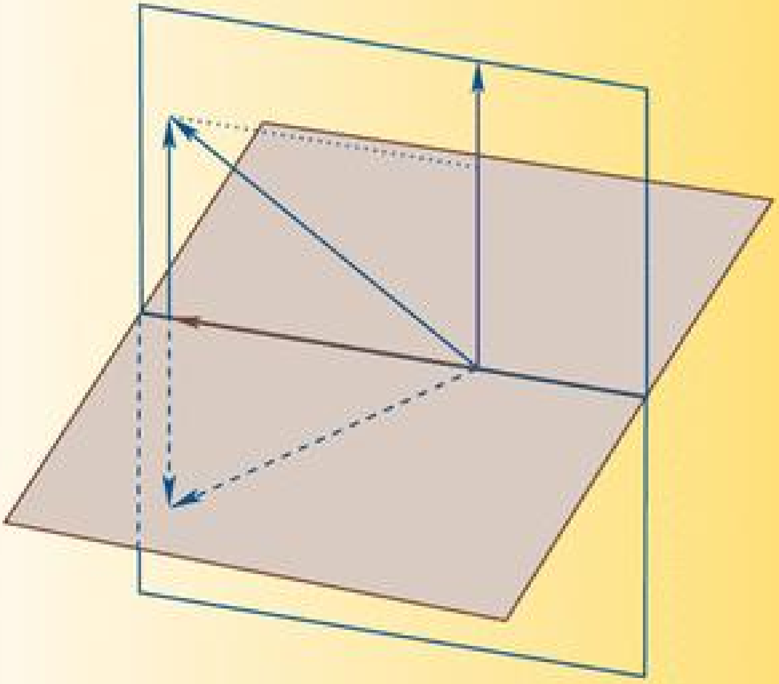
Thomas S. Shores

APPLIED LINEAR ALGEBRA AND MATRIX ANALYSIS



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(continued after index)

Thomas S. Shores

Applied Linear Algebra and Matrix Analysis

Springer

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To my wife, Muriel

Preface

This book is about matrix and linear algebra, and their applications. For many students the tools of matrix and linear algebra will be as fundamental in their professional work as the tools of calculus; thus it is important to ensure that students appreciate the utility and beauty of these subjects as well as the mechanics. To this end, applied mathematics and mathematical modeling ought to have an important role in an introductory treatment of linear algebra. In this way students see that concepts of matrix and linear algebra make concrete problems workable.

In this book we weave significant motivating examples into the fabric of the text. I hope that instructors will not omit this material; that would be a missed opportunity for linear algebra! The text has a strong orientation toward numerical computation and applied mathematics, which means that matrix analysis plays a central role. All three of the basic components of linear algebra — theory, computation, and applications — receive their due. The proper balance of these components gives students the tools they need as well as the motivation to acquire these tools. Another feature of this text is an emphasis on linear algebra as an experimental science; this emphasis is found in certain examples, computer exercises, and projects. Contemporary mathematical software make ideal "labs" for mathematical experimentation. Nonetheless, this text is independent of specific hardware and software platforms. Applications and ideas should take center stage, not software.

This book is designed for an introductory course in matrix and linear algebra. Here are some of its main goals:

- To provide a balanced blend of applications, theory, and computation that emphasizes their interdependence.
- To assist those who wish to incorporate mathematical experimentation through computer technology into the class. Each chapter has computer exercises sprinkled throughout and an optional section on computational notes. Students should use the locally available tools to carry out the ex-

periments suggested in the project and use the word processing capabilities of their computer system to create reports of results.
- To help students to express their thoughts clearly. Requiring written reports is one vehicle for teaching good expression of mathematical ideas.
- To encourage cooperative learning. Mathematics educators are becoming increasingly appreciative of this powerful mode of learning. Team projects and reports are excellent vehicles for cooperative learning.
- To promote individual learning by providing a complete and readable text. I hope that readers will find the text worthy of being a permanent part of their reference library, particularly for the basic linear algebra needed in the applied mathematical sciences.

An outline of the book is as follows: Chapter 1 contains a thorough development of Gaussian elimination. It would be nice to assume that the student is familiar with complex numbers, but experience has shown that this material is frequently long forgotten by many. Complex numbers and the basic language of sets are reviewed early on in Chapter 1. Basic properties of matrix and determinant algebra are developed in Chapter 2. Special types of matrices, such as elementary and symmetric, are also introduced. About determinants: some instructors prefer not to spend too much time on them, so I have divided the treatment into two sections, the second of which is marked as optional and not used in the rest of the text. Chapter 3 begins with the "standard" Euclidean vector spaces, both real and complex. These provide motivation for the more sophisticated ideas of abstract vector space, subspace, and basis, which are introduced largely in the context of the standard spaces. Chapter 4 introduces geometrical aspects of standard vector spaces such as norm, dot product, and angle. Chapter 5 introduces eigenvalues and eigenvectors. General norm and inner product concepts for abstract vector spaces are examined in Chapter 6. Each section concludes with a set of exercises and problems.

Each chapter contains a few more "optional" topics, which are independent of the nonoptional sections. Of course, one instructor's optional is another's mandatory. Optional sections cover tensor products, linear operators, operator norms, the Schur triangularization theorem, and the singular value decomposition. In addition, each chapter has an optional section of computational notes and projects. I employ the convention of marking sections and subsections that I consider optional with an asterisk.

There is more than enough material in this book for a one-semester course. Tastes vary, so there is ample material in the text to accommodate different interests. One could increase emphasis on any one of the theoretical, applied, or computational aspects of linear algebra by the appropriate selection of syllabus topics. The text is well suited to a course with a three-hour lecture and lab component, but computer-related material is not mandatory. Every instructor has her/his own idea about how much time to spend on proofs, how much on examples, which sections to skip, etc.; so the amount of material covered will vary considerably. Instructors may mix and match any of the optional sections according to their own interests, since these sections are largely independent of each other. While it would be very time-consuming to cover them all, every instructor ought to use some part of this material. The unstarred sections form the core of the book; most of this material should be covered. There are 27 unstarred sections and 10 optional sections. I hope the optional sections come in enough flavors to please any pure, applied, or computational palate.

Of course, no one size fits all, so I will suggest two examples of how one might use this text for a three-hour one-semester course. Such a course will typically meet three times a week for fifteen weeks, for a total of 45 classes. The material of most of the unstarred sections can be covered at a rate of about one and one-half class periods per section. Thus, the core material could be covered in about 40 class periods. This leaves time for extra sections and in-class exams. In a two-semester course or a course of more than three hours, one could expect to cover most, if not all, of the text.

If the instructor prefers a course that emphasizes the standard Euclidean spaces, and moves at a more leisurely pace, then the core material of the first five chapters of the text are sufficient. This approach reduces the number of unstarred sections to be covered from 27 to 23.

I employ the following taxonomy for the reader tasks presented in this text. *Exercises* constitute the usual learning activities for basic skills; these come in pairs, and solutions to the odd-numbered exercises are given in an appendix. More advanced conceptual or computational exercises that ask for explanations or examples are termed *problems*, and solutions for problems are not given, but hints are supplied for those problems marked with an asterisk. Some of these exercises and problems are computer-related. As with pencil-and-paper exercises, these are learning activities for basic skills. The difference is that some computing equipment (ranging from a programmable scientific calculator to a workstation) is required to complete such exercises and problems. At the next level are *projects.* These assignments involve ideas that extend the standard text material, possibly some numerical experimentation and some written exposition in the form of brief project papers. These are analogous to lab projects in the physical sciences. Finally, at the top level are *reports.* These require a more detailed exposition of ideas, considerable experimentation — possibly open ended in scope — and a carefully written report document. Reports are comparable to "scientific term papers." They approximate the kind of activity that many students will be involved in throughout their professional lives. I have included some of my favorite examples of all of these activities in this textbook. Exercises that require computing tools contain a statement to that effect. Perhaps projects and reports I have included will provide templates for instructors who wish to build their own project/report materials. In my own classes I expect projects to be prepared with text processing software to which my students have access in a mathematics computer lab.

About numbering: exercises and problems are numbered consecutively in each section. All other numbered items (sections, theorems, definitions, etc.) are numbered consecutively in each chapter and are prefixed by the chapter number in which the item occurs.

Projects and reports are well suited for team efforts. Instructors should provide background materials to help the students through local system-dependent issues. When I assign a project, I usually make available a Maple, Matlab, or Mathematica notebook that amounts to a brief background lecture on the subject of the project and contains some of the key commands students will need to carry out the project. This helps students focus more on the mathematics of the project rather than computer issues. Most of the computational computer tools that would be helpful in this course fall into three categories and are available for many operating systems:

- Graphing calculators with built-in matrix algebra capabilities such as the HP 48, or the TI 89 and 92.
- Computer algebra systems (CAS) such as Maple, Mathematica, and Macsyma. These software products are fairly rich in linear algebra capabilities. They prefer symbolic calculations and exact arithmetic, but can be coerced to do floating-point calculations.
- Matrix algebra systems (MAS) such as Matlab, Octave, and Scilab. These software products are specifically designed to do matrix calculations in floating-point arithmetic and have the most complete set of matrix commands of all categories.

In a few cases I include in this text software-specific information for some projects for purpose of illustration. This is not to be construed as an endorsement or requirement of any particular software or computer. Projects may be carried out with different software tools and computer platforms. Each system has its own strengths. In various semesters I have obtained excellent results with all these platforms. Students are open to all sorts of technology in mathematics. This openness, together with the availability of inexpensive high-technology tools, has changed how and what we teach in linear algebra.

I would like to thank my colleagues whose encouragement has helped me complete this project, particularly David Logan. I would also like to thank my wife, Muriel Shores, for her valuable help in proofreading and editing the text, and Dr. David Taylor, whose careful reading resulted in many helpful comments and corrections. Finally, I would like to thank the outstanding staff at Springer, particularly Mark Spencer, Louise Farkas, and David Kramer, for their support in bringing this project to completion.

I continue to develop a linear algebra home page of material such as project notebooks, supplementary exercises, errata sheet, etc., for instructors and students using this text. This site can be reached at

`http://www.math.unl.edu/~tshores1/mylinalg.html`

Suggestions, corrections, or comments are welcome. These may be sent to me at `tshores1@math.unl.edu`.

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1

LINEAR SYSTEMS OF EQUATIONS

The two central problems about which much of the theory of linear algebra revolves are the problem of finding all solutions to a linear system and that of finding an eigensystem for a square matrix. The latter problem will not be encountered until Chapter 4; it requires some background development and even the motivation for this problem is fairly sophisticated. By contrast, the former problem is easy to understand and motivate. As a matter of fact, simple cases of this problem are a part of most high-school algebra backgrounds. We will address the problem of when a linear system has a solution and how to solve such a system for all of its solutions. Examples of linear systems appear in nearly every scientific discipline; we touch on a few in this chapter.

1.1 Some Examples

Here are a few elementary examples of linear systems:

Example 1.1. For what values of the unknowns x and y are the following equations satisfied?

$$\begin{aligned} x + 2y &= 5 \\ 4x + y &= 6. \end{aligned}$$

Solution. The first way that we were taught to solve this problem was the geometrical approach: every equation of the form $ax+by+c=0$ represents the graph of a straight line. Thus, each equation above represents a line. We need only graph each of the lines, then look for the point where these lines intersect, to find the unique solution to the graph (see Figure 1.1). Of course, the two equations may represent the same line, in which case there are infinitely many solutions, or distinct parallel lines, in which case there are no solutions. These could be viewed as exceptional or "degenerate" cases. Normally, we expect the solution to be unique, which it is in this example.

We also learned how to solve such an equation algebraically: in the present case we may use either equation to solve for one variable, say x, and substitute the result into the other equation to obtain an equation that is easily solved for y. For example, the first equation above yields $x = 5 - 2y$ and substitution into the second yields $4(5 - 2y) + y = 6$, i.e., $-7y = -14$, so that $y = 2$. Now substitute 2 for y in the first equation and obtain that $x = 5 - 2(2) = 1$. □

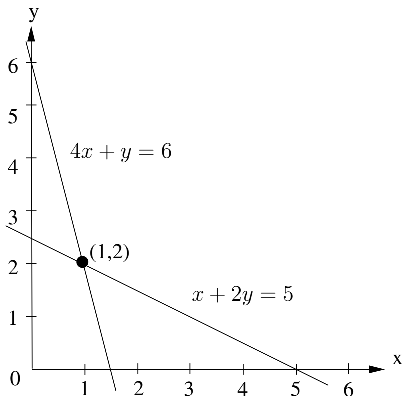


Fig. 1.1. Graphical solution to Example 1.1.

Example 1.2. For what values of the unknowns x, y, and z are the following equations satisfied?

$$\begin{aligned} x + y + z &= 4 \\ 2x + 2y + 5z &= 11 \\ 4x + 6y + 8z &= 24. \end{aligned}$$

Solution. The geometrical approach becomes impractical as a means of obtaining an explicit solution to our problem: graphing in three dimensions on a flat sheet of paper doesn't lead to very accurate answers! The solution to this problem can be discerned roughly in Figure 1.2. Nonetheless, the geometrical approach gives us a qualitative idea of what to expect without actually solving the system of equations.

With reference to our system of three equations in three unknowns, the first fact to take note of is that each of the three equations is an instance of the general equation $ax + by + cz + d = 0$. Now we know from analytical geometry that the graph of this equation is a plane in three dimensions. In general, two planes will intersect in a line, though there are exceptional cases of the two planes represented being identical or distinct and parallel. Similarly, three planes will intersect in a plane, line, point, or nothing. Hence, we know that the above system of three equations has a solution set that is either a plane, line, point, or the empty set.

Which outcome occurs with our system of equations? Figure 1.2 suggests a single point, but we need the algebraic point of view to help us calculate the solution. The matter of dealing with three equations and three unknowns is a bit trickier than the problem of two equations and unknowns. Just as with two equations and unknowns, the key idea is still to use one equation to solve for one unknown. In this problem, subtract 2 times the first equation from the second and 4 times the first equation from the third to obtain the system

$$\begin{aligned} 3z &= 3 \\ 2y + 4z &= 8, \end{aligned}$$

which is easily solved to obtain $z = 1$ and $y = 2$. Now substitute back into the first equation $x + y + z = 4$ and obtain $x = 1$. □

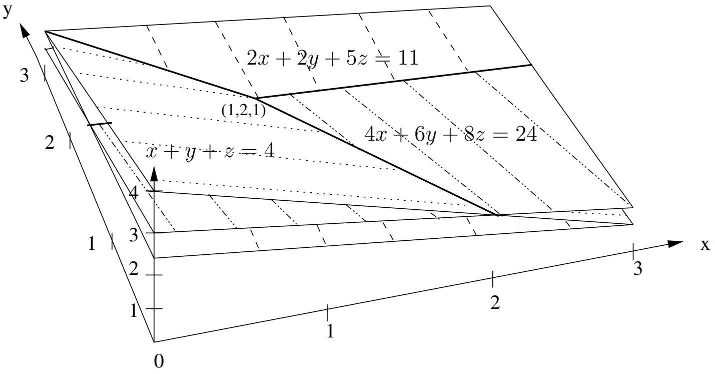


Fig. 1.2. Graphical solution to Example 1.2.

Some Key Notation

Here is a formal statement of the kind of equation that we want to study in this chapter. This formulation gives us the notation for dealing with the general problem later on.

Definition 1.1. A *linear equation* in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

Linear Equation

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where the coefficients $a_1, a_2, \ldots, a_n$ and term b of the right-hand side are given constants.

Of course, there are many interesting and useful nonlinear equations, such as $ax^2 + bx + c = 0$, or $x^2 + y^2 = 1$. But our focus is on systems that consist solely of linear equations. In fact, our next definition gives a fancy way of describing a general linear system.

Linear System

Definition 1.2. A *linear system* of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ is a list of m equations of the form

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n & = & b_i \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n & = & b_m . \end{array} \tag{1.1}$$

Row and Column Index

Notice how the coefficients are indexed: in the ith row the coefficient of the jth variable, x_j, is the number a_{ij}, and the right-hand side of the ith equation is b_i. This systematic way of describing the system will come in handy later, when we introduce the matrix concept. About indices: it would be safer — but less convenient — to write $a_{i,j}$ instead of a_{ij}, since ij could be construed to be a single symbol. In those rare situations where confusion is possible, e.g., numeric indices greater than 9, we will separate row and column number with a comma.

* Examples of Modeling Problems

It is easy to get the impression that linear algebra is only about the simple kinds of problems such as the preceding examples. So why develop a whole subject? We shall consider a few examples whose solutions are not so apparent as those of the previous two examples. The point of this chapter, as well as that of Chapters 2 and 3, is to develop algebraic and geometrical methodologies that are powerful enough to handle problems like these.

Diffusion Processes

Consider a diffusion process arising from the flow of heat through a homogeneous material. A basic physical observation is that heat is directly proportional to temperature. In a wide range of problems this hypothesis is true, and we shall assume that we are modeling such a problem. Thus, we can measure the amount of heat at a point by measuring temperature since they differ by a known constant of proportionality. To fix ideas, suppose we have a rod of material of unit length, say, situated on the x-axis, on $0 \le x \le 1$. Suppose further that the rod is laterally insulated, but has a known internal heat source that doesn't change with time. When sufficient time passes, the temperature of the rod at each point will settle down to "steady-state" values, dependent only on position x. Say the heat source is described by a function $f(x)$, $0 \le x \le 1$, which gives the additional temperature contribution per unit length per unit time due to the heat source at the point x. Also suppose that the left and right ends of the rod are held at fixed temperatures y_{left} and y_{right}, respectively.

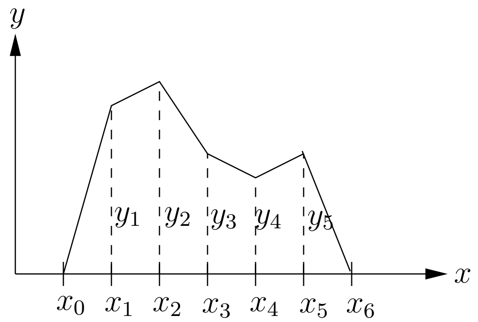


Fig. 1.3. Discrete approximation to temperature function ($n = 5$).

How can we model a steady state? Imagine that the continuous rod of uniform material is divided up into a finite number of equally spaced points, called nodes, namely $x_0 = 0, x_1, x_2, \ldots, x_{n+1} = 1$, and that all the heat is concentrated at these points. Assume that the nodes are a distance h apart. Since spacing is equal, the relation between h and n is $h = 1/(n+1)$. Let the temperature function be $y(x)$ and let $y_i = y(x_i)$. Approximate $y(x)$ in between nodes by connecting adjacent points (x_i, y_i) with a line segment. (See Figure 1.3 for a graph of the resulting approximation to $y(x)$.) We know that at the end nodes the temperature is specified: $y(x_0) = y_{\text{left}}$ and $y(x_{n+1}) = y_{\text{right}}$. By examining the process at each interior node, we can obtain the following linear equation for each interior node index $i = 1, 2, \ldots, n$ involving a constant k called the conductivity of the material. A derivation of these equations is given in Section 1.5, following two related project descriptions:

$$k\frac{-y_{i-1} + 2y_i - y_{i+1}}{h^2} = f(x_i)$$

or

$$-y_{i-1} + 2y_i - y_{i+1} = \frac{h^2}{k} f(x_i). \tag{1.2}$$

Example 1.3. Suppose we have a rod of material of conductivity $k = 1$ and situated on the x-axis, for $0 \leq x \leq 1$. Suppose further that the rod is laterally insulated, but has a known internal heat source and that both the left and right ends of the rod are held at 0 degrees Fahrenheit. What are the steady-state equations approximately for this problem?

Solution. Follow the notation of the discussion preceding this example. Notice that in this case $x_i = ih$. Remember that y_0 and y_{n+1} are known to be 0, so the terms y_0 and y_{n+1} disappear. Thus we have from equation (1.2) that there are n equations in the unknowns y_i, $i = 1, 2, \ldots, n$.

It is reasonable to expect that the smaller h is, the more accurately y_i will approximate $y(x_i)$. This is indeed the case. But consider what we are confronted with when we take $n = 5$, i.e., $h = 1/(5+1) = 1/6$, which is hardly a small value of h. The system of five equations in five unknowns becomes

$$\begin{array}{rrrrrl}
2y_1 & -y_2 & & & & = f\left(1/6\right)/36 \\
-y_1 & +2y_2 & -y_3 & & & = f\left(2/6\right)/36 \\
 & -y_2 & +2y_3 & -y_4 & & = f\left(3/6\right)/36 \\
 & & -y_3 & +2y_4 & -y_5 & = f\left(4/6\right)/36 \\
 & & & -y_4 & +2y_5 & = f\left(5/6\right)/36.
\end{array}$$

This problem is already about as large as we might want to work by hand, if not larger. The basic ideas of solving systems like this are the same as in Examples 1.1 and 1.2. For very small h, say $h = .01$ and hence $n = 99$, we clearly would need some help from a computer or calculator. □

Leontief Input–Output Models

Here is a simple model of an open economy consisting of three sectors that supply each other and consumers. Suppose the three sectors are (E)nergy, (M)aterials, and (S)ervices and suppose that the demands of a sector are proportional to its output. This is reasonable; if, for example, the materials sector doubled its output, one would expect its needs for energy, material, and services to likewise double. We require that the system be in equilibrium in the sense that total output of the sector E should equal the amounts consumed by all sectors and consumers.

Example 1.4. Given the following input–output table of demand constants of proportionality and consumer (D)emand (a fixed quantity) for the output of each sector, express the equilibrium of the system as a system of equations.

		Consumed by			
		E	M	S	D
	E	0.2	0.3	0.1	2
Produced by	M	0.1	0.3	0.2	1
	S	0.4	0.2	0.1	3

Solution. Let x, y, z be the total outputs of the sectors E, M, and S respectively. Consider how we balance the total supply and demand for energy. The total output (supply) is x units. The demands from the three sectors E, M, and S are, according to the table data, $0.2x$, $0.3y$, and $0.1z$, respectively. Further, consumers demand 2 units of energy. In equation form,

$$x = 0.2x + 0.3y + 0.1z + 2.$$

Likewise we can balance the input/output of the sectors M and S to arrive at a system of three equations in three unknowns:

$$\begin{aligned} x &= 0.2x + 0.3y + 0.1z + 2 \\ y &= 0.1x + 0.3y + 0.2z + 1 \\ z &= 0.4x + 0.2y + 0.1z + 3. \end{aligned}$$

The questions that interest economists are whether this system has solutions, and if so, what they are. □

Next, consider the situation of a closed economic system, that is, one in which everything produced by the sectors of the system is consumed by those sectors.

Example 1.5. An administrative unit has four divisions serving the internal needs of the unit, labeled (A)ccounting, (M)aintenance, (S)upplies, and (T)raining. Each unit produces the "commodity" its name suggests, and charges the other divisions for its services. The input–output table of demand rates is given by the following table. Express the equilibrium of this system as a system of equations.

		Consumed by			
		A	M	S	T
	A	0.2	0.3	0.3	0.2
Produced by	M	0.1	0.2	0.2	0.1
	S	0.4	0.2	0.2	0.2
	T	0.4	0.1	0.3	0.2

Solution. Let x, y, z, w be the total outputs of the sectors A, M, S, and T, respectively. The analysis proceeds along the lines of the previous example and results in the system

$$\begin{aligned} x &= 0.2x + 0.3y + 0.3z + 0.2w \\ y &= 0.1x + 0.2y + 0.2z + 0.1w \\ z &= 0.4x + 0.2y + 0.2z + 0.2w \\ w &= 0.4x + 0.1y + 0.3z + 0.2w. \end{aligned}$$

There is an obvious, but useless, solution to this system. One hopes for nontrivial solutions that are meaningful in the sense that each variable takes on a nonnegative value. □

Note 1.1. In some of the exercises and projects in this text you will find references to "your computer system." This may be a scientific calculator that is required for the course or a computer system for which you are given an account. This textbook does not depend on any particular system, but certain exercises require a computational device. The abbreviation "MAS" stands for a matrix algebra system like Matlab, Scilab, or Octave. The shorthand "CAS" stands for a computer algebra system like Maple, Mathematica, or MathCad. A few of the projects are too large for most calculators and will require a CAS or MAS.

1.1 Exercises and Problems

Exercise 1. Solve the following systems algebraically.

(a) $\begin{aligned} x + 2y &= \ \ 1 \\ 3x - y &= -4 \end{aligned}$ (b) $\begin{aligned} x - y + 2z &= 6 \\ 2x - z &= 3 \\ y + 2z &= 0 \end{aligned}$ (c) $\begin{aligned} x - y \ &= 1 \\ 2x - y &= 3 \\ x + y \ &= 3 \end{aligned}$

Exercise 2. Solve the following systems algebraically.

(a) $\begin{aligned} x - y &= -3 \\ x + y &= \ \ 1 \end{aligned}$ (b) $\begin{aligned} x - y + 2z &= \ \ 0 \\ x - z &= -2 \\ z &= \ \ 0 \end{aligned}$ (c) $\begin{aligned} x + 2y &= 1 \\ 2x - y &= 2 \\ x + y \ &= 2 \end{aligned}$

Exercise 3. Determine whether each of the following systems of equations is linear. If so, put it in standard format.

(a) $\begin{aligned} x + 2 \ &= y + z \\ 3x - y &= 4 \end{aligned}$ (b) $\begin{aligned} xy + 2 &= 1 \\ 2x - 6 &= y \end{aligned}$ (c) $\begin{aligned} x + 2y &= -2y \\ 2x \ \ &= y \\ 2 \ \ \ &= x + y \end{aligned}$

Exercise 4. Determine whether each of the following systems of equations is linear. If so, put it in standard format.

(a) $\begin{aligned} x + 2 &= \ 1 \\ x + 3 &= y^2 \end{aligned}$ (b) $\begin{aligned} x + 2z &= y \\ 3x - y &= y \end{aligned}$ (c) $\begin{aligned} x + y &= -3y \\ 2x &= \ \ xy \end{aligned}$

Exercise 5. Express the following systems of equations in the notation of the definition of linear systems by specifying the numbers m, n, a_{ij}, and b_i.

(a) $\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ x_2 &= 1 \\ -x_1 + x_3 &= 1 \end{aligned}$ (b) $\begin{aligned} x_1 - 3x_2 &= 1 \\ x_2 &= 5 \end{aligned}$

Exercise 6. Express the following systems of equations in the notation of the definition of linear systems by specifying the numbers m, n, a_{ij}, and b_i.

(a) $\begin{aligned} x_1 - x_2 \ &= 1 \\ 2x_1 - x_2 &= 3 \\ x_2 + x_1 \ &= 3 \end{aligned}$ (b) $\begin{aligned} -2x_1 + x_3 &= 1 \\ x_2 - x_3 &= 5 \end{aligned}$

Exercise 7. Write out the linear system that results from Example 1.3 if we take $n = 4$ and $f(x) = 3y(x)$.

Exercise 8. Write out the linear system that results from Example 1.5 if we take $n = 3$ and $f(x) = xy(x) + x^2$.

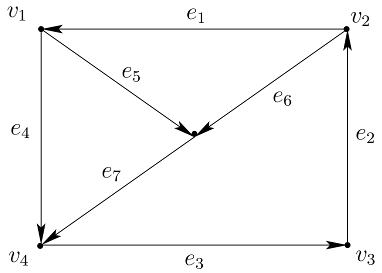
Exercise 9. Suppose that in the input–output model of Example 1.4 each producer charges a unit price for its commodity, say p_1, p_2, p_3, and that the EMS columns of the table represent the fraction of each producer commodity needed by the consumer to produce one unit of its own commodity. Derive equations for prices that achieve equilibrium, that is, equations that say that the price received for a unit item equals the cost of producing it.

Exercise 10. Suppose that in the input–output model of Example 1.5 each producer charges a unit price for its commodity, say p_1, p_2, p_3, p_4 and that the columns of the table represent fraction of each producer commodity needed by the consumer to produce one unit of its own commodity. Derive equilibrium equations for these prices.

Problem 11. Use a symbolic calculator or CAS to solve the systems of Examples 1.4 and 1.5. Comment on your solutions. Are they sensible?

Problem 12. A polynomial $y = a_0 + a_1 x + a_2 x^2$ is required to interpolate a function $f(x)$ at $x = 1, 2, 3$, where $f(1) = 1$, $f(2) = 1$, and $f(3) = 2$. Express these three conditions as a linear system of three equations in the unknowns a_0, a_1, a_2. What kind of general system would result from interpolating $f(x)$ with a polynomial at points $x = 1, 2, \ldots, n$ where $f(x)$ is known?

*Problem 13. The topology of a certain network is indicated by the following graph, where five vertices (labeled v_j) represent locations of hardware units that receive and transmit data along connection edges (labeled e_j) to other units in the direction of the arrows. Suppose the system is in a steady state and that the data flow along each edge e_j is the nonnegative quantity x_j. The single law that these flows must obey is this: net flow in equals net flow out at each of the five vertices (like Kirchhoff's first law in electrical circuits). Write out a system of linear equations satisfied by variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7$.



Problem 14. Use your calculator, CAS, or MAS to solve the system of Example 1.3 with conductivity $k = 1$ and internal heat source $f(x) = x$ and graph the approximate solution by connecting the nodes (x_j, y_j) as in Figure 1.3.

1.2 Notation and a Review of Numbers

The Language of Sets

The language of sets pervades all of mathematics. It provides a convenient shorthand for expressing mathematical statements. Loosely speaking, a set can be defined as a collection of objects, called the *members* of the set. This definition will suffice for us. We use some shorthand to indicate certain relationships between sets and elements. Usually, sets will be designated by uppercase letters such as A, B, etc., and elements will be designated by lowercase letters such as a, b, etc. As usual, set A is a *subset* of set B if every element of A is an element of B, and a *proper* subset if it is a subset but not equal to B. Two sets A and B are said to be *equal* if they have exactly the same elements. Some shorthand:

Set Symbols

$\emptyset$ denotes the empty set, i.e., the set with no members.
$a \in A$ means "a is a member of the set A."
$A = B$ means "the set A is equal to the set B."
$A \subseteq B$ means "A is a subset of B."
$A \subset B$ means "A is a proper subset of B."

There are two ways in which we may define a set: we may *list* its elements, such as in the definition $A = \{0, 1, 2, 3\}$, or specify them by *rule* such as in the definition $A = \{x \mid x \text{ is an integer and } 0 \leq x \leq 3\}$. (Read this as "$A$ is the set of x such that x is an integer and $0 \leq x \leq 3$.") With this notation we can give formal definitions of set intersections and unions:

Set Union and Intersection

Definition 1.3. Let A and B be sets. Then the *intersection* of A and B is defined to be the set $A \cap B = \{x \mid x \in A \text{ or } x \in B\}$. The *union* of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (inclusive or, which means that $x \in A$ or $x \in B$ or both.) The *difference* of A and B is the set $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

Example 1.6. Let $A = \{0, 1, 3\}$ and $B = \{0, 1, 2, 4\}$. Then

$$\begin{aligned} A \cup \emptyset &= A, \\ A \cap \emptyset &= \emptyset, \\ A \cup B &= \{0, 1, 2, 3, 4\}, \\ A \cap B &= \{0, 1\}, \\ A - B &= \{3\}. \end{aligned}$$

About Numbers

One could spend a whole course fully developing the properties of number systems. We won't do that, of course, but we will review some of the basic sets of numbers, and assume that the reader is familiar with properties of numbers we have not mentioned here. At the start of it all is the kind of numbers that everyone knows something about: the *natural* or *counting* numbers. This is the set

Natural Numbers

$$\mathbb{N} = \{1, 2, \ldots\}.$$

One could view most subsequent expansions of the concept of number as a matter of rising to the challenge of solving new equations. For example, we cannot solve the equation

$$x + m = n, \quad m, n \in \mathbb{N},$$

for the unknown x without introducing subtraction and extending the notion of natural number that of *integer*. The set of integers is denoted by

Integers

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}.$$

Next, we cannot solve the equation

$$ax = b, \; a, b \in \mathbb{Z},$$

for the unknown x without introducing division and extending the notion of integer to that of *rational number*. The set of rationals is denoted by

Rational Numbers

$$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}.$$

Rational-number arithmetic has some characteristics that distinguish it from integer arithmetic. The main difference is that nonzero rational numbers have multiplicative inverses: the multiplicative inverse of a/b is b/a. Such a number system is called a *field* of numbers. In a nutshell, a *field of numbers* is a system of objects, called numbers, together with operations of addition, subtraction, multiplication, and division that satisfy the usual arithmetic laws; in particular, it must be possible to subtract any number from any other and divide any number by a nonzero number to obtain another such number. The associative, commutative, identity, and inverse laws must hold for each of addition and multiplication; and the distributive law must hold for multiplication over addition. The rationals form a field of numbers; the integers don't since division by nonzero integers is not always possible if we restrict our numbers to integers.

The jump from rational to real numbers cannot be entirely explained by algebra, although algebra offers some insight as to why the number system still needs to be extended. An equation like

$$x^2 = 2$$

does not have a rational solution, since $\sqrt{2}$ is irrational. (Story has it that this is lethal knowledge, in that followers of a Pythagorean cult claim that the gods threw overboard from a ship one of their followers who was unfortunate enough to discover that fact.) There is also the problem of numbers like π and the mathematical constant e which do not satisfy any polynomial equation. The heart of the problem is that if we consider only rationals on a number line, there are many "holes" that are filled by numbers like π and $\sqrt{2}$. Filling in these holes leads us to the set $\mathbb{R}$ of real numbers, which are in one-to-one correspondence with the points on a number line. We won't give an exact definition of the set of real numbers. Recall that every real number admits a (possibly infinite) decimal representation, such as $1/3 = 0.333\ldots$ or $\pi = 3.14159\ldots$. This provides us with a loose definition: real numbers are numbers

Real Numbers

that can be expressed by a decimal representation, i.e., limits of finite decimal expansions.

There is one more problem to overcome. How do we solve a system like

$$x^2 + 1 = 0$$

Complex Numbers

over the reals? The answer is we can't: if x is real, then $x^2 \geq 0$, so $x^2 + 1 > 0$. We need to extend our number system one more time, and this leads to the set $\mathbb{C}$ of *complex* numbers. We define i to be a quantity such that $\mathrm{i}^2 = -1$ and

$$\mathbb{C} = \{a + b\mathrm{i} \mid a, b \in \mathbb{R}\} .$$

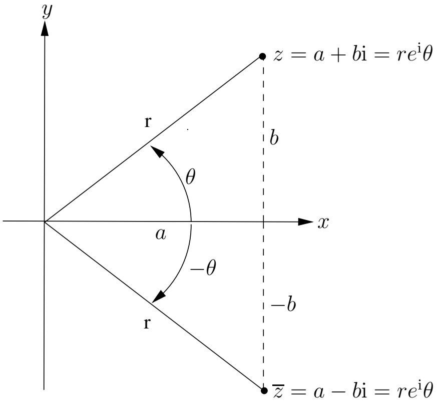

y
$z = a + b\mathrm{i} = re^{\mathrm{i}}\theta$
r
b
θ
x
a
$-\theta$
$-b$
r
$\overline{z} = a - b\mathrm{i} = re^{\mathrm{i}}\theta$

Fig. 1.4. Standard and polar coordinates in the complex plane.

Standard Form

Real and Imaginary Parts

If the complex number $z = a + b\mathrm{i}$ is given, then we say that the form $a + b\mathrm{i}$ is the *standard form* of z. In this case the real part of z is $\Re(z) = a$ and the imaginary part is defined as $\Im(z) = b$. (Notice that the imaginary part of z is a real number: it is the real coefficient of i.) Two complex numbers are *equal* precisely when they have the same real part and the same imaginary part. All of this could be put on a more formal basis by initially defining complex numbers to be ordered pairs of real numbers. We will not do so, but the fact that complex numbers behave like ordered pairs of real numbers leads to an important geometrical insight: complex numbers can be identified with points in the plane. Instead of an x- and y-axis, one lays out a *real* and an *imaginary* axis (which are still usually labeled with x and y) and plots complex numbers $a + b\mathrm{i}$ as in Figure 1.4. This results in the so-called *complex plane.*

Absolute Value

Arithmetic in $\mathbb{C}$ is carried out using the usual laws of arithmetic for $\mathbb{R}$ and the algebraic identity $\mathrm{i}^2 = -1$ to reduce the result to standard form. In addition, there are several more useful ideas about complex numbers that we will need. The *length,* or *absolute value,* of a complex number $z = a + b\mathrm{i}$ is defined as the nonnegative real number $|z| = \sqrt{a^2 + b^2}$, which is exactly the length of z viewed as a plane vector. The *complex conjugate* of z is defined as $\overline{z} = a - b\mathrm{i}$ (see Figure 1.4). Thus we have the following laws of complex arithmetic:

Laws of Complex Arithmetic

$$\boxed{\begin{array}{rcl} (a+b\mathrm{i}) + (c+d\mathrm{i}) &=& (a+c) + (b+d)\mathrm{i} \\ (a+b\mathrm{i}) \cdot (c+d\mathrm{i}) &=& (ac-bd) + (ad+bc)\mathrm{i} \\ \overline{a+b\mathrm{i}} &=& a - b\mathrm{i} \\ |a+b\mathrm{i}| &=& \sqrt{a^2+b^2} \end{array}}$$

In particular, notice that complex addition is exactly like the vector addition of plane vectors, that is, it is coordinatewise. Complex multiplication does not admit such a simple interpretation.

Example 1.7. Let $z_1 = 2 + 4\mathrm{i}$ and $z_2 = 1 - 3\mathrm{i}$. Compute $z_1 - 3z_2$.

Solution. We have that

$$z_1 - 3z_2 = (2+4\mathrm{i}) - 3(1-3\mathrm{i}) = 2 + 4\mathrm{i} - 3 + 9\mathrm{i} = -1 + 13\mathrm{i}. \qquad \square$$

Here are some easily checked and very useful facts about absolute value and complex conjugation:

Laws of Conjugation and Absolute Value

$$\boxed{\begin{array}{rcl} |z_1 z_2| &=& |z_1|\,|z_2| \\ |z_1 + z_2| &\le& |z_1| + |z_2| \\ |z|^2 &=& z\overline{z} \\ \overline{z_1 + z_2} &=& \overline{z_1} + \overline{z_2} \\ \overline{z_1 z_2} &=& \overline{z_1}\,\overline{z_2} \\ \frac{z_1}{z_2} &=& \frac{z_1 \overline{z_2}}{|z_2|^2} \end{array}}$$

Example 1.8. Let $z_1 = 2 + 4\mathrm{i}$ and $z_2 = 1 - 3\mathrm{i}$. Verify that $|z_1 z_2| = |z_1|\,|z_2|$.

Solution. First calculate that $z_1\, z_2 = (2+4\mathrm{i})\,(1-3\mathrm{i}) = (2+12) + (4-6)\,\mathrm{i}$, so that $|z_1\, z_2| = \sqrt{14^2 + (-2)^2} = \sqrt{200}$, while $|z_1| = \sqrt{2^2+4^2} = \sqrt{20}$ and $|z_2| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$. It follows that $|z_1 z_2| = \sqrt{10}\sqrt{20} = |z_1|\,|z_2|$. $\square$

Example 1.9. Verify that the product of conjugates is the conjugate of the product.

Solution. This is just the last fact in the preceding list. Let $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ be in standard form, so that $\overline{z}_1 = x_1 - \mathrm{i}y_1$ and $\overline{z}_2 = x_2 - \mathrm{i}y_2$. We calculate

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1),$$

so that

$$\overline{z_1 z_2} = (x_1 x_2 - y_1 y_2) - \mathrm{i}(x_1 y_2 + x_2 y_1).$$

Also,

$$\overline{z}_1\,\overline{z}_2 = (x_1 - \mathrm{i}y_1)(x_2 - \mathrm{i}y_2) = (x_1x_2 - y_1y_2) - \mathrm{i}(x_1y_2 - x_2y_1) = \overline{z_1z_2}. \quad \square$$

The complex number $z = \mathrm{i}$ solves the equation $z^2 + 1 = 0$ (no surprise here: it was invented expressly for that purpose). The big surprise is that once we have the complex numbers in hand, we have a number system so complete that we can solve *any* polynomial equation in it. We won't offer a proof of this fact ; it's very nontrivial. Suffice it to say that nineteenth-century mathematicians considered this fact so fundamental that they dubbed it the "Fundamental Theorem of Algebra," a terminology we adopt.

Fundamental Theorem of Algebra

Theorem 1.1. Let $p(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a nonconstant polynomial in the variable z with complex coefficients $a_0, \ldots, a_n$. Then the polynomial equation $p(z) = 0$ has a solution in the field $\mathbb{C}$ of complex numbers.

Note that the fundamental theorem doesn't tell us how to find a root of a polynomial, only that it can be done. As a matter of fact, there are no general formulas for the roots of a polynomial of degree greater than four, which means that we have to resort to numerical approximations in most practical cases.

In vector space theory the numbers in use are sometimes called *scalars*, and we will use this term. Unless otherwise stated or suggested by the presence of i, the field of scalars in which we do arithmetic is assumed to be the field of real numbers. However, we shall see later, when we study eigensystems, that even if we are interested only in real scalars, complex numbers have a way of turning up quite naturally.

Let's do a few more examples of complex-number manipulation.

Example 1.10. Solve the linear equation $(1 - 2\mathrm{i})\,z = (2 + 4\mathrm{i})$ for the complex variable z. Also compute the complex conjugate and absolute value of the solution.

Solution. The solution requires that we put the complex number $z = (2 + 4\mathrm{i})/(1 - 2\mathrm{i})$ in standard form. Proceed as follows: multiply both numerator and denominator by $(\overline{1 - 2\mathrm{i}}) = 1 + 2\mathrm{i}$ to obtain that

$$z = \frac{2 + 4\mathrm{i}}{1 - 2\mathrm{i}} = \frac{(2 + 4\mathrm{i})(1 + 2\mathrm{i})}{(1 - 2\mathrm{i})(1 + 2\mathrm{i})} = \frac{2 - 8 + (4 + 4)\mathrm{i}}{1 + 4} = \frac{-6}{5} + \frac{8}{5}\mathrm{i}.$$

Next we see that

$$\overline{z} = \overline{\frac{-6}{5} + \frac{8}{5}\mathrm{i}} = -\frac{6}{5} - \frac{8}{5}\mathrm{i}$$

and

$$|z| = \left|\frac{1}{5}(-6 + 8\mathrm{i})\right| = \frac{1}{5}\,|(-6 + 8\mathrm{i})| = \frac{1}{5}\sqrt{(-6)^2 + 8^2} = \frac{10}{5} = 2. \quad \square$$

Practical Complex Arithmetic

We conclude this section with a discussion of the more advanced aspects of complex arithmetic. This material will not be needed until Chapter 4. Recall from basic algebra the so-called *roots theorem*: the linear polynomial $z-a$ is a factor of a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ if and only if a is a *root* of the polynomial, i.e., $f(a) = 0$. If we team this fact up with the Fundamental Theorem of Algebra, we see an interesting fact about factoring polynomials over $\mathbb{C}$: every polynomial can be completely factored into a product of linear polynomials of the form $z - a$ times a constant. The numbers a that occur are exactly the roots of $f(z)$. Of course, these roots could be repeated roots, as in the case of $f(z) = 3z^2 - 6z + 3 = 3(z-1)^2$. But how can we use the Fundamental Theorem of Algebra in a practical way to find the roots of a polynomial? Unfortunately, the usual proofs of the Fundamental Theorem of Algebra don't offer a clue, because they are *nonconstructive*, i.e., they prove that solutions must exist, but do not show how to explicitly construct such a solution. Usually, we have to resort to numerical methods to get approximate solutions, such as the Newton's method used in calculus. For now, we will settle on a few ad hoc methods for solving some important special cases.

First-degree equations offer little difficulty: the solution to $az = b$ is $z = b/a$, as usual. There is one detail to attend to: what complex number is represented by the expression b/a? We saw how to handle this by the trick of "rationalizing" the denominator in Example 1.10.

Quadratic equations are also simple enough: use the quadratic formula, which says that the solutions to $az^2 + bz + c = 0$ are given by

Quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

One little catch: what does the square root of a complex number mean? What we are really asking is this: how do we solve the equation $z^2 = d$ for z, where d is a complex number? Let's try for a little more: how do we solve $z^n = d$ for all possible solutions z, where d is a given complex number? In a few cases, such an equation is quite easy to solve. We know, for example, that $z = \pm \mathrm{i}$ are solutions to $z^2 = -1$, so these are all the solutions. Similarly, one can check by hand that $\pm 1, \pm \mathrm{i}$ are all solutions to $z^4 = 1$. Consequently, $z^4 - 1 = (z-1)(z+1)(z-\mathrm{i})(z+\mathrm{i})$. Roots of the equation $z^n = 1$ are sometimes called the nth roots of unity. Thus the 4th roots of unity are ± 1 and $\pm \mathrm{i}$. But what about something like $z^3 = 1 + \mathrm{i}$?

The key to answering this question is another form of a complex number $z = a + b\mathrm{i}$. In reference to Figure 1.4 we can write $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}$, where θ is a real number, r is a nonnegative real, and $\mathrm{e}^{\mathrm{i}\theta}$ is *defined* by the following expression, which is called *Euler's formula*:

Definition 1.4. $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta.$

Polar form

Notice that $|\mathrm{e}^{\mathrm{i}\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, so that $|r\mathrm{e}^{\mathrm{i}\theta}| = |r||\mathrm{e}^{\mathrm{i}\theta}| = r$, provided r is nonnegative. The expression $r\mathrm{e}^{\mathrm{i}\theta}$ with $r = |z|$ and the angle θ measured counterclockwise in radians is called the *polar form* of z. The number θ is sometimes called an *argument* of z. It is important to notice that θ is not unique. If the angle θ_0 works for the complex number z, then so does $\theta = \theta_0 + 2\pi k$, for any integer k, since $\sin\theta$ and $\cos\theta$ are periodic of period 2π. It follows that a complex number may have more than one polar form. For example, $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{e}^{\mathrm{i}5\pi/2}$ (here $r = 1$). In fact, the most general polar expression for i is $\mathrm{i} = \mathrm{e}^{\mathrm{i}(\pi/2+2k\pi)}$, where k is an arbitrary integer.

Example 1.11. Find the possible polar forms of $1 + \mathrm{i}$.

Solution. Draw a picture of the number $1 + \mathrm{i}$ as in Figure 1.5. We see that the angle $\theta_0 = \pi/4$ works fine as a measure of the angle from the positive x-axis to the radial line from the origin to z. Moreover, the absolute value of z is $\sqrt{1+1} = \sqrt{2}$. Hence, a polar form for z is $z = \sqrt{2}e^{\mathrm{i}\pi/4}$. However, we can adjust the angle θ_0 by any multiple of 2π, a full rotation, and get a polar form for z. So the most general polar form for z is $z = \sqrt{2}e^{\mathrm{i}(\pi/4+2k\pi)}$, where k is any integer. □

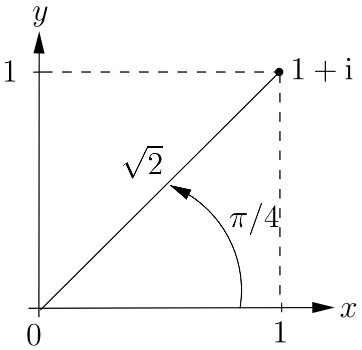


Fig. 1.5: Polar form of $1 + \mathrm{i}$.

As the notation suggests, polar forms obey the laws of exponents. A simple application of the laws for the sine and cosine of a sum of angles shows that for angles θ and ψ we have the identity

$$\mathrm{e}^{\mathrm{i}(\theta+\psi)} = \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\psi}.$$

By using this formula n times, we obtain that $\mathrm{e}^{\mathrm{i}n\theta} = \left(\mathrm{e}^{\mathrm{i}\theta}\right)^n$, which can also be expressed as *de Moivre's Formula:*

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

Now for solving $z^n = d$: First, find the general polar form of d, say $d = a\mathrm{e}^{\mathrm{i}(\theta_0+2k\pi)}$, where θ_0 is the so-called *principal angle* for d, i.e., $0 \le \theta_0 < 2\pi$, and $a = |d|$. Next, write $z = r\mathrm{e}^{\mathrm{i}\theta}$, so that the equation to be solved becomes

$$r^n\mathrm{e}^{\mathrm{i}n\theta} = a\mathrm{e}^{\mathrm{i}(\theta_0+2k\pi)}.$$

Taking absolute values of both sides yields that $r^n = a$, whence we obtain the unique value of $r = \sqrt[n]{a} = \sqrt[n]{|d|}$. What about θ? The most general form for $n\theta$ is

$$n\theta = \theta_0 + 2k\pi.$$

Hence we obtain that

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}.$$

Notice that the values of $\mathrm{e}^{\mathrm{i}2k\pi/n}$ start repeating themselves as k passes a multiple of n, since $\mathrm{e}^{\mathrm{i}2\pi} = \mathrm{e}^0 = 1$. Therefore, one gets exactly n distinct values for $\mathrm{e}^{\mathrm{i}\theta}$, namely

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}, \quad k = 0, \ldots, n-1.$$

These points are equally spaced around the unit circle in the complex plane, starting with the point $\mathrm{e}^{\mathrm{i}\theta_0}$. Thus we have obtained n distinct solutions to the equation $z^n = d$, namely

$$\boxed{z = a^{1/n} e^{\mathrm{i}(\theta_0/n + 2k\pi/n)}, \quad k = 0, \ldots, n-1, \text{ where } d = a\mathrm{e}^{\mathrm{i}\theta_0}}$$

General solution to $z^n = d$

Example 1.12. Solve the equation $z^3 = 1 + \mathrm{i}$ for the unknown z.

Solution. The solution goes as follows: We have seen that $1 + \mathrm{i}$ has a polar form

$$1 + \mathrm{i} = \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}.$$

Then according to the previous formula, the three solutions to our cubic are

$$z = (\sqrt{2})^{1/3}\mathrm{e}^{\mathrm{i}(\pi/4 + 2k\pi)/3} = 2^{1/6}\mathrm{e}^{\mathrm{i}(1+8k)\pi/12}, \quad k = 0, 1, 2.$$

See Figure 1.6 for a graph of these complex roots. □

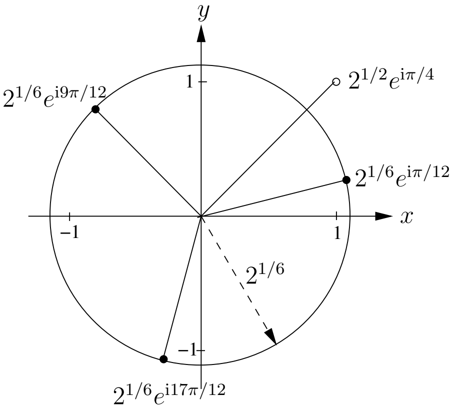

y
1
$2^{1/2}e^{\mathrm{i}\pi/4}$
$2^{1/6}e^{\mathrm{i}9\pi/12}$
$2^{1/6}e^{\mathrm{i}\pi/12}$
-1
1
x
$2^{1/6}$
-1
$2^{1/6}e^{\mathrm{i}17\pi/12}$

Fig. 1.6. Roots of $z^3 = 1 + \mathrm{i}$.

We conclude with a little practice with square roots and the quadratic formula. In regard to square roots, notice that the expression $w = \sqrt{d}$ is ambiguous. With a positive real number d this means the positive root of the equation $w^2 = d$. But when d is complex (or even negative), it no longer makes sense to talk about "positive" and "negative" roots of $w^2 = d$. In this case we simply interpret $\sqrt{d}$ to be one of the roots of $w^2 = d$.

Example 1.13. Compute $\sqrt{-4}$ and $\sqrt{\mathrm{i}}$.

Solution. Observe that $-4 = 4 \cdot (-1)$. It is reasonable to expect the laws of exponents to continue to hold, so we should have $(-4)^{1/2} = 4^{1/2} \cdot (-1)^{1/2}$. Now we know that $\mathrm{i}^2 = -1$, so we can take $\mathrm{i} = (-1)^{1/2}$ and obtain that $\sqrt{-4} = (-4)^{1/2} = 2\mathrm{i}$. Let's check it: $(2\mathrm{i})^2 = 4\mathrm{i}^2 = -4$.

We have to be a bit more careful with $\sqrt{\mathrm{i}}$. We'll just borrow the idea of the formula for solving $z^n = d$. First, put i in polar form as $\mathrm{i} = 1 \cdot \mathrm{e}^{\mathrm{i}\pi/2}$. Now raise each side to the $1/2$ power to obtain

$$\begin{aligned}\sqrt{\mathrm{i}} = \mathrm{i}^{1/2} &= 1^{1/2} \cdot (\mathrm{e}^{\mathrm{i}\pi/2})^{1/2} \\ &= 1 \cdot \mathrm{e}^{\mathrm{i}\pi/4} = \cos(\pi/4) + \mathrm{i}\sin(\pi/4) \\ &= \frac{1}{\sqrt{2}}(1 + \mathrm{i}).\end{aligned}$$

A quick check confirms that $((1 + \mathrm{i})/\sqrt{2})^2 = 2\mathrm{i}/2 = \mathrm{i}$. □

Example 1.14. Solve the equation $z^2 + z + 1 = 0$.

Solution. According to the quadratic formula, the answer is

$$z = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = -1 \pm \mathrm{i}\frac{\sqrt{3}}{2}.$$ □

Example 1.15. Solve $z^2 + z + 1 + \mathrm{i} = 0$ and factor this polynomial.

Solution. This time we obtain from the quadratic formula that

$$z = \frac{-1 \pm \sqrt{1 - 4(1 + \mathrm{i})}}{2} = \frac{-1 \pm \sqrt{-(3 + 4\mathrm{i})}}{2}.$$

What is interesting about this problem is that we don't know the polar angle θ for $z = -(3 + 4i)$. Fortunately, we don't have to. We know that $\sin\theta = -4/5$ and $\cos\theta = -3/5$. We also have the standard half angle formulas from trigonometry to help us:

$$\cos^2\theta/2 = \frac{1 + \cos\theta}{2} = \frac{1}{5} \quad \text{and} \quad \sin^2\theta/2 = \frac{1 - \cos\theta}{2} = \frac{4}{5}.$$

Since θ is in the third quadrant of the complex plane, $\theta/2$ is in the second, so

$$\cos\theta/2 = \frac{-1}{\sqrt{5}} \quad \text{and} \quad \sin\theta/2 = \frac{2}{\sqrt{5}}.$$

Now notice that $|-(3+4\mathrm{i})| = 5$. It follows that a square root of $-(3+4\mathrm{i})$ is given by

$$s = \sqrt{5}\left(\frac{-1}{\sqrt{5}} + \frac{2}{\sqrt{5}}\mathrm{i}\right) = -1 + 2\mathrm{i}.$$

Check that $s^2 = -(3+4\mathrm{i})$, so the two roots to our quadratic equation are given by

$$z = \frac{-1 \pm (-1+2\mathrm{i})}{2} = -1+\mathrm{i},\ -\mathrm{i}.$$

In particular, we see that $z^2 + z + 1 + \mathrm{i} = (z+1-\mathrm{i})(z+\mathrm{i})$. □

1.2 Exercises and Problems

In the following exercises, z is a complex number, and answers should be expressed in standard form if possible.

Exercise 1. Determine the following sets, given that $A = \{x \,|\, x \in \mathbb{R} \text{ and } x^2 < 3\}$ and $B = \{x \,|\, x \in \mathbb{Z} \text{ and } x > -1\}$:
(a) $A \cap B$ (b) $B - A$ (c) $\mathbb{Z} - B$ (d) $\mathbb{N} \cup B$ (e) $\mathbb{R} \cap A$

Exercise 2. Given that $C = \{x \,|\, x \in \mathbb{Z} \text{ and } x^2 > 4\}$ and $D = \{x \,|\, x \in \mathbb{Z} \text{ and } x > -1\}$, determine the following sets:
(a) $C \cup D$ (b) $D - C$ (c) $D \cap \emptyset$ (d) $\mathbb{R} \cup D$

Exercise 3. Put the following complex numbers into polar form and sketch them in the complex plane:
(a) $-\mathrm{i}$ (b) $1+\mathrm{i}$ (c) $-1+\mathrm{i}\sqrt{3}$ (d) 1 (e) $2-2\mathrm{i}$ (f) $2\mathrm{i}$ (g) π

Exercise 4. Put the following complex numbers into polar form and sketch them in the complex plane:
(a) $3+\mathrm{i}$ (b) i (c) $1+\mathrm{i}\sqrt{3}$ (d) -1 (e) $3-\mathrm{i}$ (f) $-\pi$ (g) e^{π}

Exercise 5. Calculate the following:
(a) $(4+2\mathrm{i})-(3-6\mathrm{i})$ (b) $(2+4\mathrm{i})(3-\mathrm{i})$ (c) $\dfrac{2+\mathrm{i}}{2-\mathrm{i}}$ (d) $\dfrac{1-2\mathrm{i}}{1+2\mathrm{i}}$ (e) $\overline{7(6-\mathrm{i})}$

Exercise 6. Calculate the following:
(a) $|2+4\mathrm{i}|$ (b) $-7\mathrm{i}^2+6\mathrm{i}^3$ (c) $(3+4\mathrm{i})(7-6\mathrm{i})$ (d) $\overline{\mathrm{i}(1-\mathrm{i})}$

Exercise 7. Solve the following systems for the unknown z:
(a) $(2+\mathrm{i})z = 4-2\mathrm{i}$ (b) $z^4 = -16$ (c) $\dfrac{z+1}{z} = 2$ (d) $(z+1)(z^2+1) = 0$

Exercise 8. Solve the equations for the unknown z:
(a) $(2+\mathrm{i})z = 1$ (b) $-\mathrm{i}z = 2z + 5$ (c) $\Im(z) = 2\Re(z) + 1$ (d) $\overline{z} = z$

Exercise 9. Find the polar and standard form of the complex numbers:
(a) $\dfrac{1}{1-\mathrm{i}}$ (b) $-2e^{\mathrm{i}\pi/3}$ (c) $\mathrm{i}\left(\mathrm{i}+\sqrt{3}\right)$ (d) $-\mathrm{i}/2$ (e) $\mathrm{i}e^{\pi/4}$

Exercise 10. Find the polar and standard form of the complex numbers:
(a) $(2+4\mathrm{i})(3-\mathrm{i})$ (b) $(2+4\mathrm{i})(3-\mathrm{i})$ (c) $1/\mathrm{i}$ (d) $-1+\mathrm{i}$ (e) $\mathrm{i}e^{\pi/4}$

Exercise 11. Find all solutions to the following equations:
(a) $z^2+z+3=0$ (b) $z^2-1=\mathrm{i}z$ (c) $z^2-2z+\mathrm{i}=0$ (d) $z^2+4=0$

Exercise 12. Find the solutions to the following equations:
(a) $z^3=1$ (b) $z^3=-8$ (c) $(z-1)^3=-1$ (d) $z^4+z^2+1=0$

Exercise 13. Describe and sketch the set of complex numbers z such that
(a) $|z|=2$ (b) $|z+1|=|z-1|$ (c) $|z-2|<1$
Hint: It's easier to work with absolute value squared.

Exercise 14. What is the set of complex numbers z such that
(a) $|z+1|=2$ (b) $|z+3|=|z-1|$ (c) $|z-2|>2$
Sketch these sets in the complex plane.

Exercise 15. Let $z_1 = 2+4\mathrm{i}$ and $z_2 = 1-3\mathrm{i}$. Verify for this z_1 and z_2 that $\overline{z_1}+\overline{z_2}=\overline{z_1+z_2}$.

Exercise 16. Let $z_1 = 2+3\mathrm{i}$ and $z_2 = 2-3\mathrm{i}$. Verify for this z_1 and z_2 that $\overline{z_1 z_2}=\overline{z_1}\,\overline{z_2}$.

Exercise 17. Find the roots of the polynomial $p(z)=z^2-2z+2$ and use this to factor the polynomial. Verify the factorization by expanding it.

Exercise 18. Show that $1+\mathrm{i}, 1-\mathrm{i}$, and 2 are roots of the polynomial $p(z) = z^3-4z^2+6z-4$ and use this to factor the polynomial.

Problem 19. Write out the values of i^k in standard form for integers $k = -1,0,1,2,3,4$ and deduce a formula for i^k consistent with these values.

Problem 20. Verify that for any two complex numbers, the sum of the conjugates is the conjugate of the sum.

*Problem 21. Use the notation of Example 1.9 to show that $|z_1 z_2| = |z_1|\,|z_2|$.

Problem 22. Use the definitions of exponentials along with the sum of angles formulas for $\sin(\theta+\psi)$ and $\cos(\theta+\psi)$ to verify the law of addition of exponents: $e^{\mathrm{i}(\theta+\psi)} = e^{\mathrm{i}\theta}e^{\mathrm{i}\psi}$.

Problem 23. Use a computer or calculator to find all roots to the polynomial equation $z^5+z+1=0$. How many roots (counting multiplicities) should this equation have? How many of these roots can you find with your system?

*Problem 24. Show that if w is a root of the polynomial $p(z)$, that is, $p(w)=0$, where $p(z)$ has real coefficients, then $\overline{w}$ is also a root of $p(z)$.

1.3 Gaussian Elimination: Basic Ideas

We return now to the main theme of this chapter, which is the systematic solution of linear systems, as defined in equation (1.1) of Section 1.1. The principal methodology is the method of *Gaussian elimination* and its variants, which we introduce by way of a few simple examples. The idea of this process is to reduce a system of equations by certain legitimate and reversible algebraic operations (called "elementary operations") to a form in which we can easily see what the solutions to the system are, if there are any. Specifically, we want to get the system in a form where the first equation involves all the variables, the second equation involve all but the first, and so forth. Then it will be simple to solve for each variable one at a time, starting with the last equation, which will involve only the last variable. In a nutshell, this is Gaussian elimination.

One more matter that will have an effect on our description of solutions to a linear system is that of the number system in use. As we noted earlier, it is customary in linear algebra to refer to numbers as "scalars." The two basic choices of scalar fields are the real number system and the complex number system. Unless complex numbers occur explicitly in a linear system, we will assume that the scalars to be used in finding a solution come from the field of real numbers. Such will be the case for most of the problems in this chapter.

An Example and Some Shorthand

Example 1.16. Solve the simple system

$$\begin{aligned} 2x - y &= 1 \\ 4x + 4y &= 20. \end{aligned} \tag{1.3}$$

Solution. First, let's switch the equations to obtain

$$\begin{aligned} 4x + 4y &= 20 \\ 2x - y &= 1. \end{aligned} \tag{1.4}$$

Next, multiply the first equation by 1/4 to obtain

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 1. \end{aligned} \tag{1.5}$$

Now, multiply a copy of the first equation by -2 and add it to the second. We can do this easily if we take care to combine like terms as we go. In particular, the resulting x term in the new second equation will be $-2x + 2x = 0$, the y term will be $-2y - y = -3y$, and the constant term on the right-hand side will be $-2 \cdot 5 + 1 = -9$. Thus we obtain

$$\begin{aligned} x + y &= 5 \\ 0x - 3y &= -9. \end{aligned} \tag{1.6}$$

This completes the first phase of Gaussian elimination, which is called "forward solving." Note that we have put the system in a form in which only the first equation involves the first variable and only the first and second involve the second variable. The second phase of Gaussian elimination is called "back solving," and it works like it sounds. Use the last equation to solve for the last variable, then work backward, solving for the remaining variables in reverse order. In our case, the second equation is used to solve for y simply by dividing by -3 to obtain that

$$y = \frac{-9}{-3} = 3.$$

Now that we know what y is, we can use the first equation to solve for x, and we obtain

$$x = 5 - y = 5 - 3 = 2.$$ □

The preceding example may seem like too much work for such a simple system. We could easily scratch out the solution in much less space. But what if the system is larger, say 4 equations in 4 unknowns, or more? How do we proceed then? It pays to have a systematic strategy and notation. We also had an ulterior motive in the way we solved this system. All of the operations we will ever need to solve a linear system were illustrated in the preceding example: switching equations, multiplying equations by nonzero scalars, and adding a multiple of one equation to another.

Before proceeding to another example, let's work on the notation a bit. Take a closer look at the system of equations (1.3). As long as we write numbers down systematically, there is no need to write out all the equal signs or plus signs. Isn't every bit of information that we require contained in the following table of numbers?

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 4 & 20 \end{bmatrix}.$$

Of course, we have to remember that each row of numbers represents an equation, the first two columns of numbers are coefficients of x and y, respectively, and the third column consists of terms on right-hand side. So we could embellish the table with a few reminders in an extra top row:

$$\begin{array}{c} \begin{array}{ccc} x & y & = \text{r.h.s.} \end{array} \\ \begin{bmatrix} 2 & -1 & 1 \\ 4 & 4 & 20 \end{bmatrix} \end{array}$$

With a little practice, we will find that the reminders are usually unnecessary, so we dispense with them. Rectangular tables of numbers are very useful in representing a system of equations. Such a table is one of the basic objects studied in this text. As such, it warrants a formal definition.

Matrices and Vectors

Definition 1.5. A *matrix* is a rectangular array of numbers. If a matrix has m rows and n columns, then the *size* of the matrix is said to be $m \times n$. If the matrix is $1 \times n$ or $m \times 1$, it is called a *vector*. If $m = n$, then it is called a *square matrix of order n.* Finally, the number that occurs in the ith row and jth column is called the (i, j)th *entry* of the matrix.

The objects we have just defined are basic "quantities" of linear algebra and matrix analysis, along with scalar quantities. Although every vector is itself a matrix, we want to single vectors out when they are identified as such. Therefore, we will follow a standard typographical convention: matrices are usually designated by capital letters, while vectors are usually designated by boldface lowercase letters. In a few cases these conventions are not followed, but the meaning of the symbols should be clear from context.

We shall need to refer to parts of a matrix. As indicated above, the location of each entry of a matrix is determined by the index of the row and column it occupies.

The statement "$A = [a_{ij}]$" means that A is a matrix whose (i, j)th entry is denoted by a_{ij}. Generally, the size of A will be clear from context. If we want to indicate that A is an $m \times n$ matrix, we write

$$A = [a_{ij}]_{m,n} .$$

Order of Square Matrix

Similarly, the statement "$\mathbf{b} = [b_i]$" means that b is a n-vector whose ith entry is denoted by b_i. In case the type of the vector (row or column) is not clear from context, the default is a column vector. Many of the matrices we encounter will be *square*, that is, $n \times n$. In this case we say that n is the *order* of the matrix. Another term that we will use frequently is the following.

Leading Entry

Definition 1.6. The *leading entry* of a row vector is the first nonzero element of that vector. If all entries are zero, the vector has no leading entry.

The equations of (1.3) have several matrices associated with them. First is the full matrix that describes the system, which we call the *augmented matrix* of the system. In our previous example, this is the 2 × 3 matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 4 & 20 \end{bmatrix} .$$

Note, for example, that we would say that the $(1, 1)$th entry of this matrix is 2, which is also the leading entry of the first row, and the $(2, 3)$th entry is 20. Next, there is the submatrix consisting of coefficients of the variables. This is called the *coefficient matrix* of the system, and in our case it is the 2 × 2 matrix

$$\begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix} .$$

Finally, there is the single column matrix of right-hand-side constants, which we call the right-hand-side vector. In our example, it is the 2 × 1 vector

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix} .$$

Coefficient Matrix

How can we describe the matrices of the general linear system of equation (1.1)? First, there is the $m \times n$ coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notice that the way we subscripted entries of this matrix is really very descriptive: the first index indicates the row position of the entry, and the second, the column position of the entry. Next, there is the $m \times 1$ right-hand-side vector of constants

Right-Hand-Side Vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}.$$

Finally, stack this matrix and vector along side each other (we use a vertical bar below to separate the two symbols) to obtain the $m \times (n+1)$ augmented matrix

Augmented Matrix

$$\widetilde{A} = [A \mid \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

The Elementary Row Operations

Here is more notation that we will find extremely handy in the sequel. This notation is related to the operations that we performed on the preceding example. Now that we have the matrix notation, we could just as well perform these operations on each row of the augmented matrix, since a row corresponds to an equation in the original system. Three types of operations were used. We shall catalog these and give them names, so that we can document our work in solving a system of equations in a concise way. Here are the three elementary operations we shall use, described in terms of their action on rows of a matrix; an entirely equivalent description applies to the equations of the linear system whose augmented matrix is the matrix below.

- E_{ij}: This is shorthand for the elementary operation of *switching the ith and jth rows* of the matrix. For instance, in Example 1.16 we moved from equation (1.3) to equation (1.4) by using the elementary operation E_{12}.
- $E_i(c)$: This is shorthand for the elementary operation of *multiplying the ith row by the nonzero constant c*. For instance, we moved from equation (1.4) to (1.5) by using the elementary operation $E_1(1/4)$.
- $E_{ij}(d)$: This is shorthand for the elementary operation of *adding d times the jth row to the ith row.* (Read the symbols from right to left to get the right order.) For instance, we moved from equation (1.5) to equation (1.6) by using the elementary operation $E_{21}(-2)$.

Notation for Elementary Operations

Now let's put it all together. The whole forward-solving phase of Example 1.16 could be described concisely with the notation we have developed:

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 4 & 20 \end{bmatrix} \xrightarrow[E_{12}]{} \begin{bmatrix} 4 & 4 & 20 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[E_1(1/4)]{} \begin{bmatrix} 1 & 1 & 5 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[E_{21}(-2)]{} \begin{bmatrix} 1 & 1 & 5 \\ 0 & -3 & -9 \end{bmatrix}.$$

This is a big improvement over our first description of the solution. There is still the job of back solving, which is the second phase of Gaussian elimination. When doing hand calculations, we're right back to writing out a bunch of extra symbols again, which is exactly what we set out to avoid by using matrix notation.

Gauss–Jordan Elimination

Here's a better way to do the second phase by hand: stick with the augmented matrix. Starting with the last nonzero row, convert the leading entry (this means the first nonzero entry in the row) to a 1 by an elementary operation, and then use elementary operations to convert all entries above this 1 entry to 0's. Now work backward, row by row, up to the first row. At this point we can read off the solution to the system. Let's see how it works with Example 1.16. Here are the details using our shorthand for elementary operations:

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & -3 & -9 \end{bmatrix} \xrightarrow[E_2(-1/3)]{} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

All we have to do is remember the function of each column in order to read off the answer from this last matrix. The underlying system that is represented is

$$\begin{aligned} 1 \cdot x + 0 \cdot y &= 2 \\ 0 \cdot x + 1 \cdot y &= 3. \end{aligned}$$

This is, of course, the answer we found earlier: $x = 2$, $y = 3$.

The method of combining forward and back solving into elementary operations on the augmented matrix has a name: it is called *Gauss–Jordan elimination*, and it is the method of choice for solving many linear systems. Let's see how it works on an example from Section 1.1.

Example 1.17. Solve the following system by Gauss–Jordan elimination:

$$\begin{aligned} x + y + z \ &= \ 4 \\ 2x + 2y + 5z &= 11 \\ 4x + 6y + 8z &= 24 \end{aligned}$$

Solution. First form the augmented matrix of the system, the 3×4 matrix

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 5 & 11 \\ 4 & 6 & 8 & 24 \end{bmatrix}.$$

Now forward solve:

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 5 & 11 \\ 4 & 6 & 8 & 24 \end{bmatrix} \xrightarrow[E_{21}(-2)]{} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 3 & 3 \\ 4 & 6 & 8 & 24 \end{bmatrix} \xrightarrow[E_{31}(-4)]{} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 4 & 8 \end{bmatrix} \xrightarrow[E_{23}]{} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 3 & 3 \end{bmatrix}.$$

Notice, by the way, that the row switch of the third step is essential. Otherwise, we cannot use the second equation to solve for the second variable, y. Next back solve:

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow[E_{3}(1/3)]{} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[E_{23}(-4)]{} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[E_{13}(-1)]{} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[E_{2}(1/2)]{} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

At this point we read off the solution to the system: $x = 1$, $y = 2$, $z = 1$. □

Systems with Nonunique Solutions

Pivots

Next, we consider an example that will pose a new kind of difficulty, namely, that of infinitely many solutions. Here is some handy terminology. An entry of a matrix used to zero out entries above or below it by means of elementary row operations is called a *pivot*.

The entries that we use in Gaussian or Gauss–Jordan elimination for pivots are always leading entries in the row that they occupy. For the sake of emphasis, in the next few examples we will put a circle around the pivot entries as they occur.

Example 1.18. Solve for the variables x, y, and z in the system

$$\begin{aligned} x + \ \ y + \ \ z \ &= \ 2 \\ 2x + 2y + 4z &= \ 8 \\ z \ &= \ 2. \end{aligned}$$

.

Solution. Here the augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now proceed to use Gaussian elimination on the matrix:

$$\begin{bmatrix} \textcircled{1} & 1 & 1 & 2 \\ 2 & 2 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[E_{21}(-2)]{} \begin{bmatrix} \textcircled{1} & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

What do we do next? Neither the second nor the third row corresponds to equations that involve the variable y. Switching the second and third equations won't help, either. Here is the point of view that we adopt in applying Gaussian elimination to this system: The first equation has already been "used up" and is reserved for eventually solving for x. We now restrict our attention to the "unused" second and third equations. Perform the following operations to do Gauss–Jordan elimination on the system:

$$\begin{bmatrix} \textcircled{1} & 1 & 1 & 2 \\ 0 & 0 & \textcircled{2} & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[E_{2}(1/2)]{} \begin{bmatrix} \textcircled{1} & 1 & 1 & 2 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[E_{32}(-1)]{} \begin{bmatrix} \textcircled{1} & 1 & 1 & 2 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} \textcircled{1} & 1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

How do we interpret this result? We take the point of view that the first row represents an equation to be used in solving for x since the leading entry of the row is in the column of coefficients of x. Similarly, the second row represents an equation to be used in solving for z, since the leading entry of that row is in the column of coefficients of z. What about y? Notice that the third equation represented by this matrix is simply $0 = 0$, which carries no information. The point is that there is not enough information in the system to solve for the variable y, even though we started with three distinct equations. Somehow, they contained redundant information. Therefore, we take the point of view that y is not to be solved for; it is a *free* variable in the sense that we can assign it any value whatsoever and obtain a legitimate solution to the system. On the other hand, the variables x and z are *bound* in the sense that they will be solved for in terms of constants and free variables. The equations represented by the last matrix above are

Free and Bound Variables

$$\begin{aligned} x + y &= 0 \\ z &= 2 \\ 0 &= 0. \end{aligned}$$

Use the first equation to solve for x and the second to solve for z to obtain the *general form* of a solution to the system:

$$\begin{aligned} x &= -y \\ z &= 2 \\ y & \text{ is free.} \end{aligned}$$ □

In the preceding example y can take on any scalar value. For example, $x = 0$, $z = 2$, $y = 0$ is a solution to the original system (check this). Likewise, $x = -5$, $z = 2$, $y = 5$ is a solution to the system. Clearly, we have an infinite number of solutions to the system, thanks to the appearance of free variables. Up to this point, the linear systems we have considered had unique solutions, so every variable was solved for, and hence bound. Another point to note, incidentally, is that the scalar field we choose to work with has an effect on our answer. The default is that y is allowed to take on any *real* value from $\mathbb{R}$. But if, for some reason, we choose to work with the complex numbers as our scalars, then y would be allowed to take on any *complex* value from $\mathbb{C}$. In this case, another solution to the system would be given by $x = -3 - \mathrm{i}$, $z = 2$, $y = 3 + \mathrm{i}$, for example.

To summarize, once we have completed Gauss–Jordan elimination on an augmented matrix, we can immediately spot the free and bound variables of the system: the column of a bound variable will have a pivot in it, while the column of a free variable will not. Another example will illustrate the point.

Example 1.19. Suppose the augmented matrix of a linear system of three equations involving variables x, y, z, w becomes, after applying suitable elementary row operations,

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Describe the general solution to the system.

Solution. We solve this problem by observing that the first and third columns have pivots in them, which the second and fourth do not. The fifth column represents the right-hand side. Put our little reminder labels in the matrix, and we obtain

$$\left[\begin{array}{ccccc} x & y & z & w & \text{rhs} \\ \textcircled{1} & 2 & 0 & -1 & 2 \\ 0 & 0 & \textcircled{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, x and z are bound variables, while y and w are free. The two nontrivial equations that are represented by this matrix are

$$\begin{aligned} x + 2y - w &= 2 \\ z + 3w &= 0. \end{aligned}$$

Use the first to solve for x and the second to solve for z to obtain the general solution

$$\begin{aligned} x &= 2 - 2y + w \\ z &= -3w \\ y, w &\text{ are free.} \end{aligned} \qquad \square$$

We have seen so far that a linear system may have exactly one solution or infinitely many. Actually, there is only one more possibility, which is illustrated by the following example.

Example 1.20. Solve the linear system

$$\begin{aligned} x + y &= 1 \\ 2x + y &= 2 \\ 3x + 2y &= 5. \end{aligned}$$

Solution. We extract the augmented matrix and proceed with Gauss–Jordan elimination. This time we'll save a little space by writing more than one elementary operation between matrices. It is understood that they are done in order, starting with the top one. This is a very efficient way of doing hand calculations and minimizing the amount of rewriting of matrices as we go:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow[\substack{E_{21}(-2) \\ E_{31}(-3)}]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[E_{32}(-1)]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Stop everything! We aren't done with Gauss–Jordan elimination yet, since we've only done the forward-solving portion. But something strange is going on here. Notice that the third row of the last matrix above stands for the equation $0x+0y = 2$, i.e., $0 = 2$. This is impossible. What this matrix is telling us is that the original system has no solution, i.e., it is *inconsistent.* A system can be identified as inconsistent as soon as one encounters a leading entry in the column of constant terms. For this always means that an equation of the form $0 =$ nonzero constant has been formed from the system by legitimate algebraic operations. Thus, one need proceed no further. The system has no solutions. □

Definition 1.7. A system of equations is *consistent* if it has at least one solution. Otherwise it is called *inconsistent.*

Consistent Systems

Our last example is one involving complex numbers explicitly.

Example 1.21. Solve the following system of equations:

$$\begin{aligned} x + y &= 4 \\ (-1 + \mathrm{i})x + y &= -1. \end{aligned}$$

Solution. The procedure is the same, no matter what the field of scalars is. Of course, the arithmetic is a bit harder. Gauss–Jordan elimination yields

$$\begin{bmatrix} 1 & 1 & 4 \\ -1+\mathrm{i} & 1 & -1 \end{bmatrix} \xrightarrow[E_{21}(1-\mathrm{i})]{} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2-\mathrm{i} & 3-4\mathrm{i} \end{bmatrix}$$

$$\xrightarrow[E_2(1/(2-\mathrm{i}))]{} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2-\mathrm{i} \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & 0 & 2+\mathrm{i} \\ 0 & 1 & 2-\mathrm{i} \end{bmatrix}.$$

Here we used the fact that

$$\frac{3-4\mathrm{i}}{2-\mathrm{i}} = \frac{(3-4\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})} = \frac{10-5\mathrm{i}}{5} = 2-\mathrm{i}.$$

Thus, we see that the system has the unique solution

$$\begin{aligned} x &= 2+\mathrm{i} \\ y &= 2-\mathrm{i}. \end{aligned}$$

□

1.3 Exercises and Problems

Exercise 1. For each of the following matrices identify the size and the (i, j)th entry for all relevant indices i and j:

(a) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (d) $[1+\mathrm{i}]$

Exercise 2. For each of the following matrices identify the size and the (i, j)th entry for all relevant indices i and j:

(a) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ \mathrm{i} \end{bmatrix}$

Exercise 3. Exhibit the augmented matrix of each system and give its size. Then use Gaussian elimination and back solving to find the general solution to the systems.

(a) $\begin{aligned} 2x+3y &= 7 \\ x+2y &= -2 \end{aligned}$ (b) $\begin{aligned} 3x_1+6x_2-x_3 &= -4 \\ -2x_1-4x_2+x_3 &= 3 \\ x_3 &= 1 \end{aligned}$ (c) $\begin{aligned} x_1+x_2 &= -2 \\ 5x_1+2x_2 &= 5 \\ x_1+2x_2 &= -7 \end{aligned}$

Exercise 4. Use Gaussian elimination and back solving to find the general solution to the systems.

(a) $\begin{aligned} x+3y &= 7 \\ x+2y &= 1 \end{aligned}$ (b) $\begin{aligned} 2x_1+6x_2 &= 2 \\ -2x_1+x_2 &= 1 \end{aligned}$ (c) $\begin{aligned} x_1+x_2 &= -2 \\ 5x_1+2x_2 &= 5 \\ x_1+2x_2 &= -7 \end{aligned}$

Exercise 5. Use Gauss–Jordan elimination to find the general solution to the systems. Show the elementary operations you use.

$$\text{(a)}\quad \begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x_3 + x_4 &= 1 \\ -2x_1 - 4x_2 + x_3 &= 0 \\ 3x_1 + 6x_2 + x_4 &= 0 \end{aligned} \qquad \text{(c)}\quad \begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 2x_1 + 5x_2 + 9x_3 &= 1 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$$

$$\text{(d)}\quad \begin{aligned} x_1 - x_2 &= \mathrm{i} \\ 2x_1 + x_2 &= 3 + \mathrm{i} \end{aligned} \qquad \text{(e)}\quad \begin{aligned} x_1 + x_2 + x_3 - x_4 &= 0 \\ -2x_1 - 4x_2 + x_3 &= 0 \\ x_1 + 6x_2 - x_3 + x_4 &= 0 \end{aligned}$$

Exercise 6. Use Gauss–Jordan elimination to find the general solution to the systems.

$$\text{(a)}\quad \begin{aligned} x_1 + x_2 + x_4 &= 1 \\ 2x_1 + 2x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_4 &= 2 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x_3 + x_4 &= 0 \\ -2x_1 - 4x_2 + x_3 &= 0 \\ -x_3 + x_4 &= 0 \end{aligned} \qquad \text{(c)}\quad \begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 2x_1 + 5x_2 + 9x_3 &= 1 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$$

$$\text{(d)}\quad \begin{aligned} 2x_1 + x_2 + 7x_3 &= -1 \\ 3x_1 + 2x_2 - 2x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - 2x_4 &= 4 \end{aligned} \qquad \text{(e)}\quad \begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + x_2 &= \mathrm{i} \\ 2x_1 + 2x_2 + \mathrm{i}x_3 &= 4 \end{aligned}$$

Exercise 7. Each of the following matrices results from applying Gauss–Jordan elimination to the augmented matrix of a linear system. In each case, write out the general solution to the system or indicate that it is inconsistent.

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 8. Write out the general solution to the system with the following augmented matrix or indicate that it is inconsistent.

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 9. Use any method to find the solution to each of the following systems. Here, b_1, b_2 are constants and x_1, x_2 are the unknowns.

$$\text{(a)}\quad \begin{aligned} x_1 - x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x_1 - x_2 &= b_1 \\ 2x_1 - 2x_2 &= b_2 \end{aligned} \qquad \text{(c)}\quad \begin{aligned} \mathrm{i}x_1 - x_2 &= b_1 \\ 2x_1 + 2x_2 &= b_2 \end{aligned}$$

Exercise 10. Apply the operations used in Exercise 5 (a), (c) in the same order to the right-hand-side vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. What does this tell you about each system's consistency?

Exercise 11. Find the general solution for the system of equations in Exercise 9 of Section 1. Is there a solution to this problem with nonnegative entries?

Exercise 12. Find the general solution for the system of equations in Exercise 10 of Section 1. Is there meaningful solution to this problem?

Exercise 13. Solve the three systems

$$\text{(a)}\ \begin{aligned} x_1 - x_2 &= 1 \\ x_1 - 2x_2 &= 0 \end{aligned} \qquad \text{(b)}\ \begin{aligned} x_1 - x_2 &= 0 \\ x_1 - 2x_2 &= 1 \end{aligned} \qquad \text{(c)}\ \begin{aligned} x_1 - x_2 &= 2 \\ x_1 - 2x_2 &= 3 \end{aligned}$$

by using a single augmented matrix that has all three right-hand sides in it.

Exercise 14. Set up a single augmented matrix for the three systems

$$\text{(a)}\ \begin{aligned} x_1 + x_2 &= 1 \\ x_2 + 2x_3 &= 0 \\ 2x_2 + x_3 &= 0 \end{aligned} \qquad \text{(b)}\ \begin{aligned} x_1 + x_2 &= 0 \\ x_2 + 2x_3 &= 0 \\ 2x_2 + x_3 &= 0 \end{aligned} \qquad \text{(c)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_2 + 2x_3 &= 3 \\ 2x_2 + x_3 &= 3 \end{aligned}$$

and use it to solve the three systems simultaneously.

Exercise 15. Show that the following nonlinear systems become linear if we view the unknowns as $1/x$, $1/y$, and $1/z$ rather than x, y, and z. Use this to find the solution sets of the nonlinear systems. (You must also account for the possibilities that one of x, y, z is zero.)

$$\text{(a)}\ \begin{aligned} 2x - y + 3xy &= 0 \\ 4x + 2y - xy &= 0 \end{aligned} \qquad \text{(b)}\ \begin{aligned} yz + 3xz - xy &= 0 \\ yz + 2xy &= 0 \end{aligned}$$

Exercise 16. Show that the following nonlinear systems become linear if we make the right choice of unknowns from x, y, z, $1/x$, $1/y$, and $1/z$ rather than x, y, and z. Use this to find the solution sets of these nonlinear systems.

$$\text{(a)}\ \begin{aligned} 3x - xy &= 1 \\ 4x - xy &= 2 \end{aligned} \qquad \text{(b)}\ \begin{aligned} 2xy &= 1 \\ y + z - 3yz &= 0 \\ xz - 2z &= -1 \end{aligned}$$

*Problem 17. Suppose that the input–output table of Example 1.5 is modified so that all entries are nonnegative, but the sum of the entries in each row is smaller than 1. Show that the only solution to the problem with nonnegative values is the solution with all variables equal to zero.

Problem 18. Use a CAS, MAS, or other software to solve the system of Example 1.3 with $f(x) = \sin(\pi x)$. Graph this approximation along with the true solution, which is $y(x) = \sin(\pi x)/\pi^2$.

*Problem 19. Suppose the function $f(x)$ is to be interpolated at three interpolating points x_0, x_1, x_2 by a quadratic polynomial $p(x) = a + bx + cx^2$, that is, $f(x_i) = p(x_i), i = 0, 1, 2$. As in Exercise 12 of Section 1.1, this leads to a system of three linear equations in the three unknowns a, b, c.
(a) Solve these equations in the case that $f(x) = e^x$, $0 \leq x \leq 1$, and $x_j = 0, \frac{1}{2}, 1$.
(b) Plot the error function $f(x) - p(x)$ and estimate the largest value of the error function (in absolute value).
(c) Use trial and error to find three points x_1, x_2, x_3 on the interval $0 \leq x \leq 1$ for which the resulting interpolating quadratic gives an error function with a largest absolute error that is less than half of that found in part (b).

Problem 20. Solve the network system of Problem 13 of Section 1 and exhibit all physically meaningful solutions.

Problem 21. Suppose one wants to solve the integral equation $\int_0^1 e^{st}x(s)ds = 1 + t^2$ for the unknown function $x(t)$. If we only want to approximate the values of $x(t)$ at $x = 0, \frac{1}{2}, 1$, derive and solve a system of equations for these three values by evaluating the integral equation at $t = 0, \frac{1}{2}, 1$, and using the trapezoidal method to approximate the integrals with the values of $x(s)$, $s = 0, \frac{1}{2}, 1$.

1.4 Gaussian Elimination: General Procedure

The preceding section introduced Gaussian elimination and Gauss–Jordan elimination at a practical level. In this section we will see why these methods work and what they really mean in matrix terms. Then we will find conditions of a very general nature under which a linear system has either no, one, or infinitely many solutions. A key idea that comes out of this section is the notion of the *rank* of a matrix.

Equivalent Systems

The first question to be considered is this: how is it that Gaussian elimination or Gauss–Jordan elimination gives us *every* solution of the system we begin with and *only* solutions to that system? To see that linear systems are special, consider the following nonlinear system of equations.

Example 1.22. Solve for the real roots of the system

$$\begin{aligned} x + y &= 2 \\ \sqrt{x} &= y. \end{aligned}$$

Solution. Let's follow the Gauss–Jordan elimination philosophy of using one equation to solve for one unknown. The first equation enables us to solve for y to get $y = 2 - x$. Substitute this into the second equation to obtain $\sqrt{x} = 2 - x$. Then square both sides to obtain $x = (2 - x)^2$, or

$$0 = x^2 - 5x + 4 = (x - 1)(x - 4).$$

Now $x = 1$ leads to $y = 1$, a solution to the system. But $x = 4$ gives $y = -2$, which is not a solution to the system since $\sqrt{x}$ cannot be negative. □

What went wrong in this example is that the squaring step, which does not correspond to any elementary operation, introduced extraneous solutions to the system. Is Gaussian or Gauss–Jordan elimination safe from this kind of difficulty? The answer lies in examining the kinds of operations we perform with these methods. First, we need some terminology. Up to this point we have always described a solution to a linear system in terms of a list of equations. For general problems this is a bit of a nuisance. Since we are using the matrix/vector notation, we may as well go all the way and use it to concisely describe solutions as well. We will use column vectors to define solutions as follows.

Solution Vector

Definition 1.8. A *solution vector* for the general linear system given by equation (1.1) is a vector

$$\mathbf{x} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

such that the resulting equations are satisfied for these choices of the variables. The set of all such solutions is called the *solution set* of the linear system, and two linear systems are said to be *equivalent* if they have the same solution set.

Tuple Convention

We will want to make frequent reference to vectors without having to display them in the text. Of course, for $1 \times n$ row vectors this is no problem. To save space in referring to column vectors, we shall adopt the convention that a column vector will also be denoted by a tuple with the same entries. The n-tuple $(x_1, x_2, \ldots, x_n)$ is a shorthand for the $n \times 1$ column vector $\mathbf{x}$ with entries $x_1, x_2, \ldots, x_n$. For example, we can write $(1, 3, 2)$ in place of $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

Example 1.23. Describe the solution sets of all the examples worked out in the previous section.

Solution. Here is the solution set to Example 1.16. It is the singleton set

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} = \{(2, 3)\}.$$

The solution set for Example 1.17 is $S = \{(1,2,1)\}$; remember that we can designate column vectors by tuples if we wish.

For Example 1.18 the solution set requires some fancier set notation, since it is an infinite set. Here it is:

$$S = \left\{ \begin{bmatrix} -y \\ y \\ 2 \end{bmatrix} \mid y \in \mathbb{R} \right\} = \{(-y, y, 2) \mid y \in \mathbb{R}\}.$$

Example 1.20 is an inconsistent system, so has no solutions. Hence its solution set is $S = \emptyset$. Finally, the solution set for Example 1.21 is the singleton set $S = \{(2 + \mathrm{i}, 2 - \mathrm{i})\}$. □

A key question about Gaussian elimination and equivalent systems: what happens to a system if we change it by performing one elementary row operation? After all, Gaussian and Gauss–Jordan elimination amount to a sequence of elementary row operations applied to the augmented matrix of a given linear system. Answer: nothing happens to the solution set!

Theorem 1.2. Suppose a linear system has augmented matrix $\widetilde{A}$ upon which an elementary row operation is applied to yield a new augmented matrix $\widetilde{B}$ corresponding to a new linear system. Then these two linear systems are equivalent, i.e., have the same solution set.

Proof. If we replace the variables in the system corresponding to $\widetilde{A}$ by the values of a solution, the resulting equations will be satisfied. Now perform the elementary operation in question on this system of equations to obtain that the equations for the system corresponding to the augmented matrix $\widetilde{B}$ are also satisfied. Thus, every solution to the old system is also a solution to the new system resulting from performing an elementary operation. For the converse, it is sufficient for us to show that the old system can be obtained from the new one by another elementary operation. In other words, we need to show that the effect of any elementary operation can be undone by another elementary operation. This will show that every solution to the new system is also a solution to the old system. If E represents an elementary operation, then the operation that undoes it could reasonably be designated as E^{-1}, since the effect of the inverse operation is rather like canceling a number by multiplying by its inverse. Let us examine each elementary operation in turn.

- E_{ij}: The elementary operation of switching the ith and jth rows of the matrix. Notice that the effect of this operation is undone by performing the same operation, E_{ij}, again. This switches the rows back. Symbolically we write $E_{ij}^{-1} = E_{ij}$.
- $E_i(c)$: The elementary operation of multiplying the ith row by the nonzero constant c. This elementary operation is undone by performing the elementary operation $E_i(1/c)$; in other words, by multiplying the ith row by the nonzero constant $1/c$. We write $E_i(c)^{-1} = E_i(1/c)$.

Inverse Elementary Operations

- $E_{ij}(d)$: The elementary operation of adding d times the jth row to the ith row. This operation is undone by adding $-d$ times the jth row to the ith row. We write $E_{ij}(d)^{-1} = E_{ij}(-d)$.

Thus, in all cases the effects of an elementary operation can be undone by applying another elementary operation of the same type, which is what we wanted to show. □

The inverse notation we used here doesn't do much for us yet. In Chapter 2 this notation will take on an entirely new and richer meaning.

The Reduced Row Echelon Form

Theorem 1.2 tells us that the methods of Gaussian and Gauss–Jordan elimination do not alter the solution set we are interested in finding. Our next objective is to describe the end result of these methods in a precise way. That is, we want to give a careful definition of the form of the matrix that these methods lead us to, starting with the augmented matrix of the original system. Recall that the *leading entry* of a row is the first nonzero entry of that row. (So a row of zeros has no leading entry.)

Reduced Row Form

Definition 1.9. A matrix R is said to be in *reduced row form* if:

(1) The nonzero rows of R precede the zero rows.
(2) The column numbers of the leading entries of the nonzero rows, say rows $1, 2, \ldots, r$, form an increasing sequence of numbers $c_1 < c_2 < \cdots < c_r$.

Reduced Row Echelon Form

The matrix R is said to be in *reduced row echelon form* if in addition to the above:

(3) Each leading entry is a 1.
(4) Each leading entry has only zeros above it.

Example 1.24. Consider the following matrices (whose leading entries are enclosed in a circle). Which are in reduced row form? Reduced row echelon form?

$$\text{(a)} \quad \begin{bmatrix} \textcircled{1} & 2 \\ 0 & \textcircled{3} \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{3} \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} 0 & 0 & 0 \\ \textcircled{1} & 0 & 0 \end{bmatrix}$$

$$\text{(d)} \quad \begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(e)} \quad \begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & \textcircled{1} & 0 \end{bmatrix}$$

Solution. Checking through (1)–(2), we see that (a), (b), and (d) fulfill all the conditions for reduced row matrices. But (c) fails, since a zero row precedes the nonzero ones; matrix (e) fails to be in reduced row form because the column numbers of the leading entries do not form an increasing sequence. Matrices

(a) and (b) don't satisfy (3), so matrix (d) is the only one that satisfies (3)–(4). Hence, it is the only matrix in the list in reduced row echelon form. □

We can now describe the goal of Gaussian elimination as follows: use elementary row operations to reduce the augmented matrix of a linear system to reduced row form; then back solve the resulting system. On the other hand, the goal of Gauss–Jordan elimination is to use elementary operations to reduce the augmented matrix of a linear system to reduced row echelon form. From this form one can read off the solution(s) to the system.

Is it always possible to reduce a matrix to a reduced row form or row echelon form? If so, to how many such forms? These are important questions. If we take the matrix in question to be the augmented matrix of a linear system, what we are really asking becomes, does Gaussian elimination always work on a linear system? If so, does it lead us to answers that have the same form? Notice how the last question was phrased. We know that the solution set of a linear system is unaffected by elementary row operations. Therefore, the solution sets we obtain will always be the same with either method, *as sets*. But couldn't the form that describes this set change? For instance, in Example 1.18 we obtained a form for the general solution that involved one free variable, y, and two bound variables x and z. Is it possible that by a different sequence of elementary operations we could have reduced to a form where there were two free variables and only one bound variable? This would be a rather different form, even though it might lead to the same solution set.

Certainly, matrices can be transformed by elementary row operations to different reduced row forms, as the following simple example shows:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow[E_2(1/2)]{} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1/2 \end{bmatrix}.$$

Every matrix of this example is already in reduced row form. The last matrix is also in reduced row echelon form. Yet all three of these matrices can be obtained from each other by elementary row operations. It is significant that only one of the three matrices is in reduced row echelon form. As a matter of fact, any matrix can be reduced by elementary row operations to *one and only one* reduced row echelon form, which we can call *the* reduced row echelon form of the given matrix. The example above shows that the matrix A has as its reduced row echelon form the matrix $E = \left[\begin{smallmatrix} 1 & 0 & 5 \\ 0 & 1 & -1/2 \end{smallmatrix}\right]$. Our assertions are justified by the following fundamental theorem about matrices.

Uniqueness of Reduced Row Echelon Form

Theorem 1.3. Every matrix can be reduced by a sequence of elementary row operations to one and only one reduced row echelon form.

Proof. First we show that every $m \times n$ matrix A can be reduced to some reduced row echelon form. Here is the algorithm we have been using: given that the first s columns of A are in reduced row echelon form with r nonzero rows and that $r < m$ and $s < n$, find the smallest column number j such that $a_{ij} \neq 0$ and $i > r$, $j > s$. If none is found, A is already in reduced row echelon form. Otherwise, interchange rows i and $r+1$, then use elementary row operations to convert $a_{r+1,j}$ to 1 and to zero out the entries above and below this one. Now set $s = j$ and increment r by one. Continue this procedure until $r = m$ or $s = n$. This must occur at some point since both r and s increase with each step, and when it occurs, the resulting matrix is in reduced row echelon form.

Next, we prove uniqueness. Suppose that some matrix could be reduced to two distinct reduced row echelon forms. We show this is impossible. If it were possible, we could find such an $m \times n$ matrix $\widetilde{A}$ with the fewest possible columns n; that is, the theorem is true for every matrix with fewer columns. Then $n > 1$, since a single column matrix can be reduced to only one reduced row echelon form, namely either the 0 column or a column with first entry 1 and the other entries 0. We are assuming that $\widetilde{A}$ can be reduced to two different reduced row echelon forms, say R_1 and R_2, with $R_1 \neq R_2$. Write $\widetilde{A} = [A \mid \mathbf{b}]$, so that we can think of $\widetilde{A}$ as the augmented matrix of a linear system (1.1). Now for $i = 1, 2$ write each R_i as $R_i = [L_i \mid \mathbf{b}_i]$, where $\mathbf{b}_i$ is the last column of the $m \times n$ matrix R_i, and L_i is the $m \times (n-1)$ matrix formed from the first $n-1$ columns of R_i. Each L_i satisfies the definition of reduced row echelon form, since each R_i is in reduced row echelon form. Also, each L_i results from performing elementary row operations on the matrix A, which has only $n-1$ columns. By the minimum columns hypothesis, we have that $L_1 = L_2$. There are two possibilities to consider.

Case 1: The last column b_i of either R_i has a leading entry in it. Then the system of equations represented by $\widetilde{A}$ is inconsistent. It follows that both columns $\mathbf{b}_i$ have a leading entry in them, which must be a 1 in the first row whose portion in L_i consists of zeros, and the entries above and below this leading entry must be 0. Since $L_1 = L_2$, it follows that $\mathbf{b}_1 = \mathbf{b}_2$, and thus $R_1 = R_2$, a contradiction. So this case can't occur.

Case 2: Each b_i has no leading entry in it. Then the system of equations represented by $\widetilde{A}$ is consistent. Both augmented matrices have the same basic and free variables since $L_1 = L_2$. Hence we obtain the same solution with either augmented matrix by setting the free variables of the system equal to 0. When we do so, the bound variables are uniquely determined: the first equation says that the first bound variable equals the first entry in the right-hand-side vector since all other variables will either be zero or have zero coefficient in the first equation of the system. Similarly, the second says that the second bound variable equals the second entry in the right-hand-side vector, and so forth. Whether we use R_1 or R_2 to solve the system, we obtain the same result, since we can manipulate one such solution into the other by elementary row operations. Therefore, $\mathbf{b}_1 = \mathbf{b}_2$ and thus $R_1 = R_2$, a contradiction again. Hence, there can be no counterexample to the theorem, which completes the proof. □

The following consequence of the preceding theorem is a fact that we will find helpful in Chapter 2.

Corollary 1.1. Let the matrix B be obtained from the matrix A by performing a sequence of elementary row operations on A. Then B and A have the same reduced row echelon form.

Proof. We can obtain the reduced row echelon form of B in the following manner: First perform the elementary operations on B that undo the ones originally performed on A to get B. The matrix A results from these operations. Now perform whatever elementary row operations are needed to reduce A to its reduced row echelon form. Since B can be reduced to one and only one reduced row echelon form, the reduced row echelon forms of A and B coincide, which is what we wanted to show. □

Rank and Nullity of a Matrix

Now that we have Theorem 1.3 in hand, we can introduce the notion of *rank* of a matrix, for it says that A has exactly one reduced row echelon form.

Rank of Matrix

Definition 1.10. The *rank* of a matrix A is the number of nonzero rows of the reduced row echelon form of A. This number is written as $\operatorname{rank} A$.

There are other ways to describe the rank of a matrix. For example, rank can also be defined as the number of nonzero rows in any reduced row form of a matrix. One has to check that any two reduced row forms have the same number of nonzero rows. Rank can also be defined as the number of columns of the reduced row echelon form with leading entries in them, since each leading entry of a reduced row echelon form occupies a unique column. We can count up the other columns as well.

Nullity

Definition 1.11. The *nullity* of a matrix A is the number of columns of the reduced row echelon form of A that do not contain a leading entry. This number is written as $\operatorname{null} A$.

In the case that A is the coefficient matrix of a linear system, we can interpret the rank of A as the number of bound variables of the system and the nullity of A as the number of free variables of the system. One has to be a little careful about this idea of rank. Consider the following example.

Example 1.25. Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 3 & 2 \end{bmatrix}.$$

Solution. We know that the rank is at most 3 by the definition of rank. Elementary row operations give

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 3 & 2 \end{bmatrix} \xrightarrow[E_{21}(-2)]{} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 3 & 3 & 2 \end{bmatrix} \xrightarrow[E_{31}(-3)]{} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow[\substack{E_{32}(4) \\ E_{12}(-2)}]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the reduced row echelon form of A at the far right we see that the rank of A is 2, that is, $\operatorname{rank} A = 2$. Since only one column does not contain a pivot, we see that the nullity of A is 1, that is, $\operatorname{null} A = 1$. □

The point that the previous example makes is that one cannot determine the rank of a matrix by counting nonzero rows of the original matrix.

Caution: Remember that the rank of A is the number of nonzero rows in one of its reduced row forms, and *not* the number of nonzero rows of A itself.

The rank of a matrix is a nonnegative number, but it *could* be 0! This happens if the matrix has only zero entries, so that it has no nonzero rows. In this case, the nullity of the matrix is as large as possible, namely the number of columns of the matrix. Here are some simple limits on the size of $\operatorname{rank} A$ and $\operatorname{null} A$. We need a notation that occurs frequently throughout the text, so we explain it first.

Max and Min

Definition 1.12. Given a list of real numbers $a_1, a_2, \ldots, a_m$, the smallest number in the list is $\min\{a_1, a_2, \ldots, a_m\}$, and $\max\{a_1, a_2, \ldots, a_m\}$ is the largest number in the list.

Theorem 1.4. Let A be an $m \times n$ matrix. Then

(1) $0 \le \operatorname{rank} A \le \min\{m, n\}$.
(2) $\operatorname{rank} A + \operatorname{null} A = n$.

Proof. By definition, $\operatorname{rank} A$ is the number of nonzero rows of the reduced row echelon form of A, which is itself an $m \times n$ matrix. There can be no more leading entries than rows; hence $\operatorname{rank} A \le m$. Also, each leading entry of a matrix in reduced row echelon form is the unique nonzero entry in its column. So there can be no more leading entries than columns n. Since $\operatorname{rank} A$ is less than or equal to both m and n, it is less than or equal to their minimum, which is the first inequality. The number of pivot columns is $\operatorname{rank} A$ and the number of nonpivot columns is $\operatorname{null} A$. The sum of these numbers is n. □

Full Column Rank

In words, item (1) of Theorem 1.4 says that the rank of a matrix cannot exceed the number of rows or columns of the matrix. If the rank of a matrix equals its column number we say that the matrix has *full column rank*. Similarly, a matrix has *full row rank* if its rank equals the row number of the matrix. For example, matrix A of Example 1.25 is 3×3 of rank 2. Since this rank is smaller than 3, A does not have full column or row rank. Here is an application of the rank concept to systems.

Consistency in Terms of Rank

Theorem 1.5. The general linear system 1.1 with $m \times n$ coefficient matrix A, right-hand-side vector $\mathbf{b}$, and augmented matrix $\widetilde{A} = [A \mid \mathbf{b}]$ is consistent if and only if $\operatorname{rank} A = \operatorname{rank} \widetilde{A}$, in which case either

(1) $\operatorname{rank} A = n$, in which case the system has a unique solution, or
(2) $\operatorname{rank} A < n$, in which case the system has infinitely many solutions.

Proof. We can reduce $\widetilde{A}$ to reduced row echelon form by first doing the elementary operations that reduce the A part of the matrix to reduced row echelon form, then attending to the last column. Hence, it is always the case that $\operatorname{rank} A \leq \operatorname{rank} \widetilde{A}$. The only way to get strict inequality is to have a leading entry in the last column, which means that some equation in the equivalent system corresponding to the reduced augmented matrix is $0 = 1$, which implies that the system is inconsistent. On the other hand, we have already seen (in the proof of Theorem 1.3, for example) that if the last column does not contain a leading entry, then the system is consistent. This establishes the first statement of the theorem.

Now suppose that $\operatorname{rank} A = \operatorname{rank} \widetilde{A}$, so that the system is consistent. By Theorem 1.4, $\operatorname{rank} A \leq n$, so that either $\operatorname{rank} A < n$ or $\operatorname{rank} A = n$. The number of variables of the system is n. Also, the number of leading entries (equivalently, pivots) of the reduced row form of $\widetilde{A}$, which is $\operatorname{rank} A$, is equal to the number of bound variables; the remaining $n - \operatorname{rank} A$ variables are the free variables of the system. Thus, to say that $\operatorname{rank} A = n$ is to say that no variables are free; that is, solving the system leads to a unique solution. And to say that $\operatorname{rank} A < n$ is to say that there is at least one free variable, in which case the system has infinitely many solutions. □

Here is an example of how this theorem can be put to work. It confirms our intuition that if a system does not have "enough" equations, then it can't have a unique solution.

Corollary 1.2. If a consistent linear system of equations has more unknowns than equations, then the system has infinitely many solutions.

Proof. In the notation of the previous theorem, the hypothesis simply means that $m < n$. But we know from Theorem 1.4 that $\operatorname{rank} A \leq \min\{m, n\}$. Thus $\operatorname{rank} A < n$ and the last part of Theorem 1.5 yields the desired result. □

Of course, there is still the question of when a system is consistent. In general, there isn't an easy way to see when this is so outside of explicitly solving the system. However, in special cases there is an easy answer. One such important special case is given by the following definition.

Homogeneous Systems

Definition 1.13. The general linear system (1.1) with $m \times n$ coefficient matrix A and right-hand-side vector $\mathbf{b}$ is said to be *homogeneous* if the entries of $\mathbf{b}$ are all zero. Otherwise, the system is said to be *inhomogeneous.*

Trivial Solution

The nice feature of homogeneous systems is that they are always consistent! In fact, it is easy to exhibit a specific solution to the system, namely, take the value of all the variables to be zero. For obvious reasons this solution is called the *trivial* solution to the system. Thus, the previous corollary implies that a homogeneous linear system with fewer equations than unknowns must have infinitely many solutions. Of course, if we want to find all the solutions, we will have to do the work of Gauss–Jordan elimination. However, we acquire a small notational convenience in dealing with homogeneous systems. Notice that the right-hand side of zeros is never changed by an elementary row operation. So why bother writing out the augmented matrix of such a system? It suffices to perform elementary operations on the coefficient matrix alone. In the end, the right-hand side is still a column of zeros.

Example 1.26. Solve and describe the solution set of the homogeneous system

$$\begin{aligned} x_1 + x_2 + x_4 &= 0 \\ x_1 + x_2 + 2x_3 &= 0 \\ x_1 + x_2 &= 0. \end{aligned}$$

Solution. In this case we perform only row operations on the coefficient matrix to obtain

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[E_{31}(-1)]{E_{21}(-1)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[E_3(-1)]{E_2(1/2)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{13}(-1)]{E_{23}(1/2)} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One has to be a little careful here: the leading entry in the last column does not indicate that the system is inconsistent, since we deleted the right-hand-side column. Had we carried it along in the calculations above, we would have obtained

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

which is the matrix of a consistent system. We see from the reduced row echelon form of the coefficient matrix that x_2 is free and the other variables are bound. The general solution is

$$\begin{aligned} x_1 &= -x_2 \\ x_3 &= 0 \\ x_4 &= 0 \\ x_2 & \text{ is free.} \end{aligned}$$

Finally, the solution set S of this system can be described as

$$S = \{(-x_2, x_2, 0, 0) \mid x_2 \in \mathbb{R}\}.$$

□

1.4 Exercises and Problems

Exercise 1. Circle leading entries and determine which of the following matrices are in reduced row form or reduced row echelon form.

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (h) $[1\ 3]$

Exercise 2. Circle leading entries and determine which of the following matrices can be put into reduced row echelon form with at most one elementary operation.

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Exercise 3. The rank of the following matrices can be determined by inspection. Inspect these matrices and specify their rank.

(a) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Exercise 4. Inspect these matrices and specify their rank without pencil and paper calculation.

(a) $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Exercise 5. Show that the elementary operations you use to find the reduced row echelon form of the following matrices. Give the rank and nullity of each matrix.

(a) $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 3 & 4 \\ 2 & 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 9 & 2 \\ -3 & 0 & 6 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 4 & 2 \\ 4 & 9 & 3 \\ 2 & 3 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 2 & 5 & 6 \\ 1 & 1 & -2 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Exercise 6. Compute a reduced row form that can be reached in a minimum number of steps and the reduced row echelon forms of the following matrices. Given that the matrices are augmented matrices for a linear system, write out the general solution to the system.

(a) $\begin{bmatrix} 0 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 & 2 \\ -3 & 1 & 6 & -5 \\ 3 & 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 2 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

Exercise 7. Find the rank of the augmented and coefficient matrix of the following linear systems and the solution sets to the following systems. Are these systems equivalent?

(a)
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 2x_1 + x_2 - 2x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - 2x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_3 + x_4 &= 0 \\ -2x_1 - 4x_2 &= 0 \\ 3x_1 + 6x_2 - x_3 + x_4 &= 0 \end{aligned}$$

Exercise 8. Show that the following systems are equivalent and find a sequence of elementary operations that transforms the augmented matrix of (a) into that of (b).

(a)
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 2x_1 + x_2 - 2x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - 2x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 4x_1 + 3x_2 + 2x_3 - 4x_4 &= 5 \\ 7x_1 + 6x_2 + 5x_3 - 7x_4 &= 11 \end{aligned}$$

Exercise 9. Find upper and lower bounds on the rank of the 4 × 3 matrix A, given that some system with coefficient matrix A has infinitely many solutions.

Exercise 10. Find upper and lower bounds on the rank of matrix A, given that A has four rows and some system of equations with coefficient matrix A has a unique solution.

Exercise 11. For what values of c are the following systems inconsistent, with unique solution or with infinitely many solutions?

(a)
$$\begin{aligned} x_2 + cx_3 &= 0 \\ x_1 - cx_2 &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 - x_1 &= c \\ x_1 + 3x_2 + x_3 &= 1 \\ 3x_1 + 7x_2 - x_3 &= 4 \end{aligned}$$

(c)
$$\begin{aligned} cx_1 + x_2 + x_3 &= 2 \\ x_1 + cx_2 + x_3 &= 2 \\ x_1 + x_2 + cx_3 &= 2 \end{aligned}$$

Exercise 12. Consider the system

$$\begin{aligned} ax + by &= c \\ bx + cy &= d \end{aligned}$$

in the unknowns x, y, where $a \neq 0$. Use the reduced row echelon form to determine conditions on the other constants such that the system has no, one, or infinitely many solutions.

Exercise 13. Consider the system

$$\begin{aligned} x_1 + 2x_2 &= a \\ x_1 + x_2 + x_3 - x_4 &= b \\ 2x_3 + 2x_4 &= c \end{aligned}$$

in the unknowns x_1, x_2, x_3, x_4. Solve this system by reducing the augmented matrix to reduced row echelon form. This system will have solutions for any right-hand side. Justify this fact in terms of rank.

Exercise 14. Give a rank condition for a homogeneous system that is equivalent to the system having a unique solution. Justify your answer.

Exercise 15. Fill in the blanks:

(a) If A is a 3×7 matrix then the rank of A is at most ____________.
(b) Equivalent systems have the same ____________.
(c) The inverse of the elementary operation $E_{23}(5)$ is ____________.
(d) The rank of a nonzero 3×3 matrix with all entries equal is ____________.

Exercise 16. Fill in the blanks:

(a) If A is a 4×8 matrix, then the nullity of A is larger than ____________.
(b) The rank of a nonzero 4×3 matrix with constant entries in each column is ____________.
(c) An example of a matrix with nullity 1 and rank 2 is ____________.
(d) The size of the matrix $\begin{bmatrix} 0 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ is ____________.

*Problem 17. Answer True/False and explain your answers:

(a) If a linear system is inconsistent, then the rank of the augmented matrix exceeds the number of unknowns.
(b) Any homogeneous linear system is consistent.
(c) A system of 3 linear equations in 4 unknowns has infinitely many solutions.
(d) Every matrix can be reduced to only one matrix in reduced row form.
(e) Any homogeneous linear system with more equations than unknowns has a nontrivial solution.

Problem 18. Show that a system of linear equations has a unique solution if and only if every column, except the last one, of the reduced row echelon form of the augmented matrix has a pivot entry in it.

Problem 19. Prove or disprove by example: if two linear systems are equivalent, then they must have the same size augmented matrix.

*Problem 20. Use Theorem 1.3 to show that any two reduced row forms for a matrix A must have the same number of nonzero rows.

Problem 21. Suppose that the matrix C can be written in the augmented form $C = [A \,|\, B]$, where the matrix B may have more than one column. Prove that $\operatorname{rank} C \leq \operatorname{rank} A + \operatorname{rank} B$.

1.5 *Computational Notes and Projects

Roundoff Errors

In many practical problems, calculations are not exact. There are several reasons for this unfortunate fact. For one, scientific calculators are by their very nature only finite-precision machines. That is, only a fixed number of significant digits of the numbers we are calculating may be used in any given calculation. For instance, verify this simple arithmetic fact on a calculator or computational software such as Matlab (but excluding computer algebra systems such as Derive, Maple, and Mathematica — since symbolic calculation is the default on these systems, they will give the correct answer):

$$\left(\left(\frac{2}{3}+100\right)-100\right)-\frac{2}{3}=0.$$

In many cases this calculation will not yield 0. The problem is that if, for example, a calculator uses 6-digit accuracy, then $\frac{2}{3}$ is calculated as 0.666667, which is really incorrect. Even if arithmetic calculations were exact, the data that form the basis of our calculations are often derived from scientific measurements that themselves will almost certainly be in error. Starting with erroneous data and doing an exact calculation can be as bad as starting with exact data and doing an inexact calculation. In fact, in a certain sense they are equivalent to each other. Error resulting from truncating data for storage or finite-precision arithmetic calculations is called *roundoff error.*

Roundoff Error

We will not give an elaborate treatment of roundoff error. A thorough analysis can be found in the Golub and Van Loan text [9] of the bibliography, a text that is considered a standard reference work. The subject of this book, numerical linear algebra, is a part of an entire field of applied mathematics known as numerical analysis. The text [13] is an excellent introductory treatment of this subject. We will consider this question: Could roundoff error be a significant problem in Gaussian elimination? It isn't at all clear that there is a problem. After all, even in the above example, the final error is relatively small. Is it possible that with all the arithmetic performed in Gaussian elimination the errors pile up and become large? The answer is yes. With the advent of computers came a heightened interest in these questions. In the early 1950s numerical analysts intensified efforts to determine whether Gaussian elimination can reliably solve larger linear systems. In fact, we don't really have to look at complicated examples to realize that there are potential difficulties. Consider the following example.

Example 1.27. Let ϵ be a number so small that our calculator yields $1+\epsilon=1$. This equation appears a bit odd, but from the calculator's point of view it may be perfectly correct; if, for example, our calculator performs 6-digit decimal arithmetic, then $\epsilon=10^{-6}$ will do nicely. Notice that with such a calculator, $1+1/\epsilon=(\epsilon+1)/\epsilon=1/\epsilon$. Now solve the linear system

$$\begin{aligned} \epsilon x_1 + x_2 &= 1 \\ x_1 - x_2 &= 0. \end{aligned} \tag{1.7}$$

Solution. Let's solve this system by Gauss–Jordan elimination with our calculator to obtain

$$\begin{bmatrix} \epsilon & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \overrightarrow{E_{21}\left(-\frac{1}{\epsilon}\right)} \begin{bmatrix} \epsilon & 1 & 1 \\ 0 & \frac{1}{\epsilon} & -\frac{1}{\epsilon} \end{bmatrix} \overrightarrow{E_2(\epsilon)} \begin{bmatrix} \epsilon & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \overrightarrow{E_{12}(-1)} \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \overrightarrow{E_1\left(\frac{1}{\epsilon}\right)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus we obtain the calculated solution $x_1 = 0$, $x_2 = 1$. This answer is spectacularly bad! If $\epsilon = 10^{-6}$ as above, then the correct answer is

$$x_1 = x_2 = \frac{1}{1+\epsilon} = 0.99999909999990 \ldots .$$

Our calculated answer is not even good to one digit. So we see that there can be serious problems with Gaussian or Gauss–Jordan elimination on finite-precision machines. □

Pivoting Strategies

It turns out that information that would be significant for x_1 in the first equation is lost in the truncated arithmetic that says that $1+1/\epsilon = 1/\epsilon$. There is a fix for problems such as this, namely a technique called *partial pivoting.* The idea is fairly simple: Do not choose the next available column entry for a pivot. Rather, search down the column in question for the largest entry (in absolute value). Then switch rows, if necessary, and use this entry as a pivot. For instance, in the preceding example, we would not pivot off the ϵ entry of the first column. Since the entry of the second row, first column, is larger in absolute value, we would switch rows and then do the usual Gaussian elimination step. Here is what we would get (remember that with our calculator $1 + \epsilon = 1$):

$$\begin{bmatrix} \epsilon & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \overrightarrow{E_{21}} \begin{bmatrix} 1 & -1 & 0 \\ \epsilon & 1 & 1 \end{bmatrix} \overrightarrow{E_{21}(-\epsilon)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \overrightarrow{E_{12}(1)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Now we get the quite acceptable answer $x_1 = x_2 = 1$.

But partial pivoting is not a panacea for numerical problems. In fact, it can be easily defeated. Multiply the second equation by ϵ^2, and we get a system for which partial pivoting still picks the wrong pivot. Here the problem is a matter of scale. It can be cured by dividing each row by the largest entry of the row before beginning the Gaussian elimination process. This procedure is known as *row scaling.* The combination of row scaling and partial pivoting overcomes many of the numerical problems of Gaussian and Gauss–Jordan elimination (but not all!). There is a more drastic procedure, known as *complete pivoting.* In this procedure one searches all the unused rows (excluding the right-hand sides) for the largest entry, then uses it as a pivot for Gaussian elimination. The columns used in this procedure do not move in that left-to-right fashion we are used to seeing in system solving. It can be shown rigorously that the error of roundoff propagates in a predictable and controlled fashion with complete pivoting; in contrast, we do not really have a satisfactory explanation as to why row scaling and partial pivoting work well. Yet in most cases they do reasonably well. Since this combination involves much less calculation than complete pivoting, it is the method of choice for many problems.

There are deeper reasons for numerical problems in solving some systems than the one the preceding example illustrates. One difficulty has to do with the "sensitivity" of the coefficient matrix to small changes. That is, in some systems, small changes in the coefficient matrix lead to dramatic changes in the *exact* answer. The practical effect of roundoff error can be shown to be equivalent to introducing small changes in the coefficient matrix and obtaining an exact answer to the perturbed (changed) system. There is no cure for these difficulties, short of computing in higher precision. A classical example of this type of problem, the Hilbert matrix, is discussed in one of the projects below. We will attempt to quantify this "sensitivity" in Chapter 6.

Computational Efficiency of Gaussian Elimination

How much work is it to solve a linear system and how does the amount of work grow with the dimensions of the system? The first thing we need is a unit of work. In computer science one of the principal units of work in numerical computation is a *flop* (floating point operation), namely a single $+, -, \times$, or $\div$. For example, we say that the amount of work in computing $e + \pi$ or $e \times \pi$ is one flop, while the work in calculating $e + 3 \times \pi$ is two flops. The following example is extremely useful.

Flop

Example 1.28. How many flops does it cost to add a multiple of one row to another, as in Gaussian elimination, given that rows have n elements?

Solution. A little experimentation with an example or two shows that the answer should be $2n$. Here is a justification of that count. Say that row $\mathbf{a}$ is to be multiplied by the scalar α, and added to the row $\mathbf{b}$. Designate the row $a = [a_i]$ and the row $b = [b_i]$. We have n entries to worry about. Consider a typical one, say the ith one. The ith entry of $\mathbf{b}$, namely b_i, will be replaced by the quantity $b_i + \alpha a_i$. The amount of work in this calculation is two flops. Since there are n entries to compute, the total work is $2n$ flops. □

Our goal is to determine the expense of solving a system by Gauss–Jordan elimination. For the sake of simplicity, let's assume that the system under consideration has n equations in n unknowns and the coefficient matrix has rank n. This ensures that we will have a pivot in every row of the matrix. We won't count row exchanges either, since they don't involve any flops. (This may not be realistic on a fast computer, since memory fetches and stores may not take significantly less time than a floating-point operation.) Now consider the expense of clearing out the entries under the first pivot. A picture of the augmented matrix looks something like this, where an $\times$ is an entry that may not be 0 and an $\textcircled{\times}$ is a nonzero pivot entry:

$$\begin{bmatrix} \enclose{circle}{\times} & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ \times & \times & \cdots & \times \end{bmatrix} \xrightarrow[\text{el. ops.}]{} \! \underset{n-1}{} \begin{bmatrix} \enclose{circle}{\times} & \times & \cdots & \times \\ 0 & \enclose{circle}{\times} & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \times & \cdots & \times \end{bmatrix}.$$

Each elementary operation will involve adding a multiple of the first row, starting with the *second* entry, since we don't need to do arithmetic in the first column — we know what goes there — to the $n-1$ subsequent rows. By the preceding example, each of these elementary operations will cost $2n$ flops. Add 1 flop for the cost of determining the multiplier to obtain $2n+1$. So the total cost of zeroing out the first column is $(n-1)(2n+1)$ flops. Now examine the lower unfinished block in the above figure. Notice that it's as though we were starting over with the row and column dimensions reduced by 1. Therefore, the total cost of the next phase is $(n-2)(2(n-1)+1)$ flops. Continue in this fashion, and we obtain a count of

$$0+\sum_{j=2}^{n}(j-1)(2j+1)=\sum_{j=1}^{n}(j-1)(2j+1)=\sum_{j=1}^{n}2j^2-j-1$$

flops. Recall the identities for sums of consecutive integers and their squares:

$$\sum_{j=1}^{n} j=\frac{n(n+1)}{2} \quad\text{and}\quad \sum_{j=1}^{n} j^2=\frac{n(n+1)(2n+1)}{6}.$$

Thus we have a total flop count of

$$\sum_{j=1}^{n}2j^2-3j+1=2\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}-n=\frac{2n^3}{3}+\frac{n^2}{2}-\frac{7n}{6}.$$

This is the cost of forward solving. Now let's simplify our answer a bit more. For large n we have that n^3 is much larger than n or n^2 (e.g., for $n=10$ compare 1000 to 10 or 100). Hence, we ignore the lower-degree terms and arrive at a simple approximation to the number of flops required to forward solve a linear system of n equations in n unknowns using Gauss–Jordan elimination. There remains the matter of back solving. We leave as an exercise to show that the total work of back solving is quadratic in n. Therefore the "leading-order" approximation that we found for forward solving remains unchanged. Hence we have the following estimate of the *complexity* of Gaussian elimination.

Theorem 1.6. The number of flops required to solve a linear system of n equations in n unknowns using Gaussian or Gauss–Jordan elimination without row exchanges is *approximately* $2n^3/3$.

Thus, for example, the work of forward solving a system of 21 equations in 21 unknowns is approximately $2 \cdot 21^3/3 = 6174$ flops. Exact answer: 6374.

Project Topics

In this section we give a few samples of project material. These projects provide an opportunity to explore a subject in a greater depth than exercises permit. Your instructor will define her/his own expectations for projects. Also, the computing platform used for the projects will vary. We cannot discuss every platform in this text, so we will give a few examples of implementation notes that an instructor might supply.

About writing project/reports: Here are a few suggestions.

- *Know your audience.* Usually, you may assume that your report will be read by your supervisors, who are technical people such as yourself. Therefore, you should write a brief statement of the problem and discussion of methodology. In practice, reports assume physical laws and assumptions without further justification, but in real life you would be expected to offer some explanation of physical principles used in your model.
- *Structure your paper.* Stream of consciousness doesn't work here. Have in mind a target length for your paper. Don't clutter your work with long lists of numbers and try to keep the length at a minimum rather than maximum. Generally, a discourse should have three parts: beginning, middle, and end. Roughly, a beginning should consist of introductory material. In the middle you develop the ideas described or theses proposed in the introduction, and in the end you summarize your work and tie up loose ends.
- *Pay attention to appearance and neatness*, but don't be overly concerned about your writing style. Remember that "simpler is better." Prefer short and straightforward sentences to convoluted ones. Use a vocabulary with which you are comfortable. Use a spell-checker if one is available.
- *Pay attention to format.* A given project/report assignment may be supplied with a report template by your instructor or carry explicit instructions about format, intended audience, etc. Read and follow these instructions carefully.
- *Acknowledge your sources.* Use every available resource, of course. In particular, we all know that the worldwide web is a gold mine of information (and disinformation!). Utilize it and other resources fully, but give appropriate references and credits, just as you would with a textbook source.

Of course, rules about paper writing are not set in concrete. Also, a part can be quite short; for example, an introduction might only be a paragraph or two. Here is a sample skeleton for a report (perhaps rather more elaborate than you need): 1. Introduction (title page, summary, and conclusions); 2. Main sections (problem statement, assumptions, methodology, results, conclusions); 3. Appendices (such as mathematical analysis, graphs, possible extensions, etc.), and References.

Project: Heat Flow I

Problem Description: You are working for the firm Universal Dynamics on a project that has a number of components. You have been assigned the analysis of a component that is similar to a laterally insulated rod. The problem: part of the specs for the rod dictate that *no point of the rod should stay at temperatures above* 60 *degrees Celsius for a long period of time.* You must decide whether any of the materials listed below are acceptable for making the rod and write a report on your findings. You may assume that the rod is of unit length. Suppose further that internal heat sources come from a position-dependent function $f(x)$, $0 \leq x \leq 1$, and that heat is also generated at each point in amounts proportional to the temperature at the point. Also suppose that the left and right ends of the rod are held at 0 and 50 degrees Celsius, respectively. When sufficient time passes, the temperature of the rod at each point will settle down to "steady-state" values, dependent only on position x. These are the temperatures you are interested in. Refer to the discussion in Section 1.1 for the details of the descriptive equations that result from discretizing the problem into finitely many nodes. Here k is the *thermal conductivity* of the rod, which is a property associated with the material used to make the rod. For your problem take the source term to be $f(x) = 200\cos(x^2)$. Here are the conductivity constants for the materials under consideration for the rod. Which of these materials (if any) are acceptable?

Platinum: $k = .17$	Aluminum: $k = .50$
Pure iron: $k = .19$	Gold: $k = .75$
Zinc: $k = .30$	Silver: $k = 1.00$

Procedure: For the solution of the problem, formulate a discrete approximation to the BVP just as in Example 1.3. Choose an integer n and divide the interval [0, 1] into $n+1$ equal subintervals with endpoints $0 = x_0, x_1, \ldots, x_{n+1} = 1$. Then the width of each subinterval is $h = 1/(n+1)$. Next let u_i be your approximation to $u(x_i)$ and proceed as in Example 1.3 . There results a linear system of n equations in the n unknowns $u_1, u_2, \ldots, u_n$. For this problem divide the rod into 4 equally sized subintervals and take $n = 3$. Use the largest u_i as an estimate of the highest temperature at any point in the rod. Now double the number of subintervals and see whether your values for u change appreciably at a given value of x. If they do, you may want to repeat this procedure until you obtain numbers that you judge to be satisfactory.

Implementation Notes (for users of Mathematica): Set up the coefficient matrix A and right-hand side b for the system. Both the coefficient matrix and the right-hand side can be set up using the `Table` command of Mathematica. For **b**, the command `100* h^2*Table[Cos[(i h)^ 2,{i,n}]/k` will generate **b**, except for the last coordinate. Use the command `b[[14]] = b[[14]] + 50` to add $u(1)$ to the right-hand side of the system and get the correct **b**. For A, the command `Table[Switch[i-j,1,-1,0,2,-1,-1,_,0],{i,n},{j,n}]` will generate a matrix of the desired form. (Use the Mathematica on-line help for all commands you want to know more about.) For floating-point numbers, you want to simulate ordinary floating-point calculations on Mathematica. You will get some symbolic expressions that you don't want, e.g., for **b**. To turn **b** into floating-point approximation, use the command `b = N[b]`. The `N[ ]` function turns the symbolic values of b into numbers, with a precision of about 16 digits if no precision is specified. For solving linear systems use the command `u = LinearSolve[a,b]`, which will solve the system with coefficient matrix a and right-hand side b, and store the result in u. About vectors: Mathematica does not distinguish between row vectors and column vectors unless you insist on it. Hardcopy: You can get hardcopy from Mathematica. Be sure to make a presentable solution for the project. You should describe the form of the system you solved and summarize your results. This shouldn't be a tome (don't simply print out a transcript of your session), nor should it be a list of numbers.

Project: Heat Flow II

Problem Description: You are given a laterally insulated rod of a homogeneous material whose conductivity properties are unknown. The rod is laid out on the x-axis, $0 \leq x \leq 1$. A current is run through the rod, which results in a heat source of 10 units of heat (per unit length) at each point along the rod. The rod is held at zero temperature at each end. After a time, the temperatures in the rod settle down to a steady state. A single measurement is taken at $x = 0.3$, which results in a temperature reading of approximately 11 units. Based on this information, determine the best estimate you can for the true value of the conductivity constant k of the material. Also try to guess a formula for the shape of the temperature function on the interval $[0, 1]$ that results when this value of the conductivity is used.

Methodology: You should use the model that is presented in Section 1.1. This will result in a linear system, which Maple can solve. One way to proceed is simply to use trial and error until you think you've hit on the right value of k, that is, the one that gives a value of approximately 11 units at $x = 0.3$. Then plot the resulting approximate function doing a dot-to-dot on the node values. You should give some thought to step size h.

Output: Return your results in the form of an annotated Maple notebook, which should have the name of the team members at the top of the file and an explanation of your solution in text cells interspersed between input cells that the user can happily click his/her way through. This explanation should be intelligible to your fellow students.

Comments: This project introduces you to a very interesting area of mathematics called "inverse theory." The idea is, rather than proceeding from problem (the governing equations for temperature values) to solution (temperature values), you are given the "solution," namely the measured solution value at a point, and are to determine from this information the "problem," that is, the conductivity coefficient that is needed to define the governing equations.

Derivation of the Diffusion Equations for Steady-State Flow

We follow the notation that has already been developed, except that the values y_i will refer to quantity of heat rather than temperature (this will yield equations for temperature, since heat is a constant times temperature). The explanation requires one more experimentally observed law known as *Fourier's heat law.* For a one-dimensional flow, it says that the flow of heat per unit length from one point to another is proportional to the negative rate of change in temperature with respect to (directed) distance from the one point to the other. The positive constant of proportionality k is known as the *conductivity* of the material. In addition, we interpret the heat created at node x_i to be $hf(x_i)$, since f measures heat created per unit length. Thus, at node x_i the net flows per unit length from the left node x_{i-1} to x_i and from x_i to the right node x_{i+1} are given by

$$\text{left flow } = -k\frac{y_{i-1}-y_i}{h}, \quad \text{right flow} = -k\frac{y_i - y_{i+1}}{h}.$$

In order to balance heat flowing through the ith node with heat created at node x_j per unit length at this node, we should have

$$\text{left flow} + \text{right flow} = -k\frac{y_{i-1}-y_i}{h} - k\frac{y_{i+1}-y_i}{h} = hf\left(x_i\right).$$

In other words,

$$k\frac{-y_{i-1}+2y_i-y_{i+1}}{h^2} = f\left(x_i\right), \text{ or } \quad -y_{i-1}+2y_i-y_{i+1} = \frac{h^2}{k}f\left(x_i\right).$$

Project: The Accuracy of Gaussian Elimination

Problem Description: This project is concerned with determining the accuracy of Gaussian elimination as applied to two linear systems, one of which is known to be difficult to solve numerically. Both of these systems will be square (equal number of unknowns and equations) and have a unique solution. Also, both of these systems are to be solved for various sizes, namely $n = 4, 8, 12, 16$. In order to get a handle on the error, our main interest, we shall start with a known answer. The answer shall consist in setting all variables equal to 1. So it is the solution vector $(1, 1, \ldots, 1)$. The coefficient matrix shall be one of two types:

(1) A Hilbert matrix, i.e., an $n \times n$ matrix given by the formula $H_n = \left[\frac{1}{i+j-1}\right]$.

(2) An $n \times n$ matrix with random entries between 0 and 1.

The right-hand-side vector $\mathbf{b}$ is uniquely determined by the coefficient matrix and solution. In fact, the entries of b are easy to obtain: simply add up all the entries in the ith row of the coefficient matrix to obtain the ith entry of $\mathbf{b}$.

The problem is to measure the error of Gaussian elimination. This is done by finding the largest (in absolute value) difference between the computed value of each variable and actual value, which in all cases is 1. Discuss your results and draw conclusions from your experiments.

Implementation Notes (for users of Maple): Maple has a built-in procedure for defining a Hilbert matrix A of size n, as in the command `A := hilbert(n);`. Before executing this command (and most other linear algebra commands), you must load the linear algebra package by the command `with(linalg);`. A vector of 1's of size n can also be constructed by the single command `x := vector(n,1);`. To multiply this matrix and vector together use the command `evalm(A &* x);` . There is a feature that all computer algebra systems have: they do exact arithmetic whenever possible. Since we are trying to gauge the effects of finite-precision calculations, we don't want exact answers (such as 425688/532110), but rather, finite-precision floating-point answers (such as 0.8). Therefore, it would be a good idea at some point to force the quantities in question to be finite-precision numbers by encapsulating their definitions in an evaluate-as floating-point command, e.g., `evalf(evalm(A &* x));`. This will force the CAS to do finite-precision arithmetic.

1.5 Exercises and Problems

Problem 1. Carry out the calculation $((\frac{2}{3}+100)-100)-\frac{2}{3}$ on a scientific calculator. Do you get the correct answer?

Problem 2. Use Gaussian elimination with partial pivoting and calculations with four significant digits to solve the system (1.7) with $\epsilon = 10^{-4}$. How many digits of accuracy does your answer contain?

*Problem 3. Enter the matrix A given below into a computer algebra system and use the available commands to compute (a) the rank of A and (b) the reduced row echelon form of A. (For example, in Maple the relevant commands are `rref(A)` and `rank(A)`.) Now convert A into its floating-point form and execute the same commands. Do you get the same answers? If not, which is correct?

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 6 & 18 & -15 & -6 & 12 & -9 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

*Problem 4. Show that the flop count for back solving an $n \times n$ system is quadratic in n.

Problem 5. Compare the strategy of Gauss–Jordan elimination by using each pivot to zero out all entries above and below before proceeding to the next pivot to the forward solve/back solve strategy. Which is computationally more expensive? Illustrate both strategies with the matrix $\left[\begin{smallmatrix} 3 & 1 & 9 & 2 \\ -3 & 0 & 6 & -5 \\ 6 & 1 & 3 & 0 \end{smallmatrix}\right]$.

2

MATRIX ALGEBRA

In Chapter 1 we used matrices and vectors as simple storage devices. In this chapter matrices and vectors take on a life of their own. We develop the arithmetic of matrices and vectors. Much of what we do is motivated by a desire to extend the ideas of ordinary arithmetic to matrices. Our notational style of writing a matrix in the form $A = [a_{ij}]$ hints that a matrix could be treated like a single number. What if we could manipulate equations with matrix and vector quantities in the same way that we do equations with scalars? We shall see that this is a useful idea. Matrix arithmetic gives us new powers for formulating and solving practical problems. In this chapter we will use it to find effective methods for solving linear and nonlinear systems, solve problems of graph theory and analyze an important modeling tool of applied mathematics called a Markov chain.

2.1 Matrix Addition and Scalar Multiplication

To begin our discussion of arithmetic we consider the matter of equality of matrices. Suppose that A and B represent two matrices. When do we declare them to be equal? The answer is, of course, if they represent the same matrix! Thus we expect that all the usual laws of equalities will hold (e.g., equals may be substituted for equals) and in fact, they do. There are times, however, when we need to prove that two symbolic matrices are equal. For this purpose, we need something a little more precise. So we have the following definition, which includes vectors as a special case of matrices.

Equality of Matrices

Definition 2.1. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if these matrices have the same size, and for each index pair (i, j), $a_{ij} = b_{ij}$, that is, corresponding entries of A and B are equal.

Example 2.1. Which of the following matrices are equal, if any?

(a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1-1 & 1+1 \end{bmatrix}$

Solution. The answer is that only (c) and (d) have any chance of being equal, since they are the only matrices in the list with the same size (2 × 2). As a matter of fact, an entry-by-entry check verifies that they really are equal. □

Matrix Addition and Subtraction

How should we define addition or subtraction of matrices? We take a clue from elementary two- and three-dimensional vectors, such as the type we would encounter in geometry or calculus. There, in order to add two vectors, one condition has to hold: the vectors have to be the same size. If they are the same size, we simply add the vectors coordinate by coordinate to obtain a new vector of the same size. That is precisely what the following definition does.

Matrix Addition and Subtraction

Definition 2.2. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. Then the *sum* of the matrices, denoted by $A + B$, is the $m \times n$ matrix defined by the formula

$$A + B = [a_{ij} + b_{ij}]\,.$$

The *negative* of the matrix A, denoted by $-A$, is defined by the formula

$$-A = [-a_{ij}]\,.$$

Finally, the *difference* of A and B, denoted by $A - B$, is defined by the formula

$$A - B = [a_{ij} - b_{ij}]\,.$$

Notice that matrices must be the same size before we attempt to add them. We say that two such matrices or vectors are *conformable for addition.*

Example 2.2. Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

Find $A + B$, $A - B$, and $-A$.

Solution. Here we see that

$$A + B = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 3-3 & 1+2 & 0+1 \\ -2+1 & 0+4 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}.$$

Likewise,

$$A - B = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 3--3 & 1-2 & 0-1 \\ -2-1 & 0-4 & 1-0 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -1 \\ -3 & -4 & 1 \end{bmatrix}.$$

The negative of A is even simpler:

$$-A = \begin{bmatrix} -3 & -1 & -0 \\ --2 & -0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}. \qquad \square$$

Scalar Multiplication

The next arithmetic concept we want to explore is that of scalar multiplication. Once again, we take a clue from the elementary vectors, where the idea behind scalar multiplication is simply to "scale" a vector a certain amount by multiplying each of its coordinates by that amount. That is what the following definition says.

Definition 2.3. Let $A = [a_{ij}]$ be an $m \times n$ matrix and c a scalar. Then the *product* of the scalar c with the matrix A, denoted by cA, is defined by the formula

$$cA = [ca_{ij}] .$$

Scalar Multiplication

Recall that the default scalars are real numbers, but they could also be complex numbers.

Example 2.3. Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad c = 3.$$

Find cA, $0A$, and $-1A$.

Solution. Here we see that

$$cA = 3\begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 3 & 3\cdot 1 & 3\cdot 0 \\ 3\cdot -2 & 3\cdot 0 & 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 0 \\ -6 & 0 & 3 \end{bmatrix},$$

while

$$0A = 0\begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$(-1)\,A = (-1)\begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = -A.$$ □

Linear Combinations

Now that we have a notion of scalar multiplication and addition, we can blend these two ideas to yield a very fundamental notion in linear algebra, that of a *linear combination.*

Definition 2.4. A *linear combination* of the matrices $A_1, A_2, \ldots, A_n$ is an expression of the form

Linear Combinations

$$c_1A_1 + c_2A_2 + \cdots + c_nA_n$$

where $c_1, c_2, \ldots, c_n$ are scalars and $A_1, A_2, \ldots, A_n$ are matrices all of the same size.

Example 2.4. Given that

$$A_1 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

compute the linear combination $-2A_1 + 3A_2 - 2A_3$.

Solution. The solution is that

$$\begin{aligned} -2A_1 + 3A_2 - 2A_3 &= -2 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \cdot 2 + 3 \cdot 2 - 2 \cdot 1 \\ -2 \cdot 6 + 3 \cdot 4 - 2 \cdot 0 \\ -2 \cdot 4 + 3 \cdot 2 - 2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \qquad \square$$

It seems like too much work to write out objects such as the vector $(0, 0, 0)$ that occurred in the last equation; after all, we know that all the entries are all 0. So we make the following notational convention for convenience. A *zero matrix* is a matrix whose every entry is 0. We shall denote such matrices by the symbol 0.

Zero Matrix

Caution: This convention makes the symbol 0 ambiguous, but the meaning of the symbol will be clear from context, and the convenience gained is worth the potential ambiguity. For example, the equation of the preceding example is stated very simply as $-2A_1 + 3A_2 - 2A_3 = 0$, where we understand from context that 0 has to mean the 3×1 column vector of zeros. If we use boldface for vectors, we will also then use boldface for the vector zero, so some distinction is regained.

Example 2.5. Suppose that a linear combination of matrices satisfies the identity $-2A_1 + 3A_2 - 2A_3 = 0$, as in the preceding example. Use this fact to express A_1 in terms of A_2 and A_3.

Solution. To solve this example, just forget that the quantities A_1, A_2, A_3 are anything special and use ordinary algebra. First, add $-3A_2 + 2A_3$ to both sides to obtain

$$-2A_1 + 3A_2 - 2A_3 - 3A_2 + 2A_3 = -3A_2 + 2A_3,$$

so that

$$-2A_1 = -3A_2 + 2A_3,$$

and multiplying both sides by the scalar $-\frac{1}{2}$ yields the identity

$$A_1 = \frac{-1}{2}(-2A_1) = \frac{-1}{2}(-3A_2 + 2A_3) = \frac{3}{2}A_2 - A_3. \qquad \square$$

The linear combination idea has a really useful application to linear systems, namely, it gives us another way to express the solution set of a linear system that clearly identifies the role of free variables. The following example illustrates this point.

Example 2.6. Suppose that a linear system in the unknowns x_1, x_2, x_3, x_4 has general solution $(x_2 + 3x_4, x_2, 2x_2 - x_4, x_4)$, where the variables x_2, x_4 are free. Describe the solution set of this linear system in terms of linear combinations with free variables as coefficients.

Solution. The trick here is to use only the parts of the general solution involving x_2 for one vector and the parts involving x_4 as the other vectors in such a way that these vectors add up to the general solution. In our case we have

$$\begin{bmatrix} x_2 + 3x_4 \\ x_2 \\ 2x_2 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 2x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_4 \\ 0 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Now simply define vectors $A_1 = (1, 1, 2, 0)$, $A_2 = (3, 0, -1, 1)$, and we see that since x_2 and x_4 are arbitrary, the solution set is

$$S = \{x_2 A_1 + x_4 A_2 \mid x_2, x_4 \in \mathbb{R}\}.$$

In other words, the solution set to the system is the set of all possible linear combinations of the vectors A_1 and A_2. □

The idea of solution sets as linear combinations is an important one that we will return to in later chapters. You might notice that once we have the general form of a solution vector we can see that there is an easier way to determine the constant vectors A_1 and A_2. Simply set $x_2 = 1$ and the other free variable(s) equal to zero—in this case just x_4—to get the solution vector A_1, and set $x_4 = 1$ and $x_2 = 0$ to get the solution vector A_2.

Laws of Arithmetic

The last example brings up an important point: to what extent can we rely on the ordinary laws of arithmetic and algebra in our calculations with matrices and vectors? For matrix *multiplication* there are some surprises. On the other hand, the laws for addition and scalar multiplication are pretty much what we would expect them to be. Here are the laws with their customary names. These same names can apply to more than one operation. For instance, there is a closure law for addition and one for scalar multiplication as well.

Laws of Matrix Addition and Scalar Multiplication

Let A, B, C be matrices of the same size $m \times n$, 0 the $m \times n$ zero matrix, and c and d scalars.

(1) (Closure Law) $A + B$ is an $m \times n$ matrix.
(2) (Associative Law) $(A + B) + C = A + (B + C)$
(3) (Commutative Law) $A + B = B + A$
(4) (Identity Law) $A + 0 = A$
(5) (Inverse Law) $A + (-A) = 0$
(6) (Closure Law) cA is an $m \times n$ matrix.
(7) (Associative Law) $c(dA) = (cd)A$
(8) (Distributive Law) $(c + d)A = cA + dA$
(9) (Distributive Law) $c(A + B) = cA + cB$
(10) (Monoidal Law) $1A = A$

It is fairly straightforward to prove from definitions that these laws are valid. The verifications all follow a similar pattern, which we illustrate by verifying the commutative law for addition: let $A = [a_{ij}]$ and $B = [b_{ij}]$ be given $m \times n$ matrices. Then we have that

$$\begin{aligned} A + B &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \\ &= \ B + A, \end{aligned}$$

where the first and third equalities come from the definition of matrix addition, and the second equality follows from the fact that for all indices i and j, $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ by the commutative law for addition of scalars.

2.1 Exercises and Problems

Exercise 1. Calculate the following where possible.
(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (b) $2\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (c) $2\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$
(d) $a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 2 & -2 \end{bmatrix} + 2\begin{bmatrix} 3 & 1 & 0 \\ 5 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (f) $x\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

Exercise 2. Calculate the following where possible.
(a) $8\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ (b) $-\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & 3 \end{bmatrix} + (-4)\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$
(d) $4\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 & 0 \\ -3 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$ (e) $2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + u\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Exercise 3. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, and compute the following, where possible.
(a) $A + 3B$ (b) $2A - 3C$ (c) $A - C$ (d) $6B + C$ (e) $2C - 3(A - 2C)$

Exercise 4. With A, B, C as in Exercise 3, solve for the unknown matrix X in the equations

(a) $X + 3A = C$ (b) $A - 3X = 3C$ (c) $2X + \begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix} = B.$

Exercise 5. Write the following vectors as a linear combination of constant vectors with scalar coefficients x, y, or z.

(a) $\begin{bmatrix} x + 2y \\ 2x - z \end{bmatrix}$ (b) $\begin{bmatrix} x - y \\ 2x + 3y \end{bmatrix}$ (c) $\begin{bmatrix} 3x + 2y \\ -z \\ x + y + 5z \end{bmatrix}$ (d) $\begin{bmatrix} x - 3y \\ 4x + z \\ 2y - z \end{bmatrix}$

Exercise 6. Write the following vectors as a linear combination of constant vectors with scalar coefficients x, y, z, or w.

(a) $\begin{bmatrix} 3x + y \\ x + y + z \end{bmatrix}$ (b) $\begin{bmatrix} 3x + 2y - w \\ w - z \\ x + y - 2w \end{bmatrix}$ (c) $\begin{bmatrix} x + 3y \\ 2y - x \end{bmatrix}$ (d) $\begin{bmatrix} x - 2y \\ 4x + z \\ 3w - z \end{bmatrix}$

Exercise 7. Find scalars a, b, c such that

$$\begin{bmatrix} c & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a - b & c + 2 \\ a + b & a - b \end{bmatrix}.$$

Exercise 8. Find scalars a, b, c, d such that

$$\begin{bmatrix} d & 2a \\ 2d & a \end{bmatrix} = \begin{bmatrix} a - b & b + c \\ a + b & c - b + 1 \end{bmatrix}.$$

Exercise 9. Express the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as a linear combination of the four matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Exercise 10. Express the matrix $D = \begin{bmatrix} 3 & 3 \\ 1 & -3 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$.

Exercise 11. Verify that the associative law and commutative laws for addition hold for

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Exercise 12. Verify that both distributive laws for addition hold for $c = 2$, $d = -3$, and A, B, and C as in Exercise 11.

Problem 13. Show by examples that it is false that for arbitrary matrices A and B, and constant c,
(a) $\operatorname{rank}(cA) = \operatorname{rank} A$ (b) $\operatorname{rank}(A + B) \geq \operatorname{rank} A + \operatorname{rank} B$.

Problem 14. Prove that the associative law for addition of matrices holds.

Problem 15. Prove that both distributive laws hold.

*Problem 16. Prove that if A and B are matrices such that $2A - 4B = 0$ and $A + 2B = I$, then $A = \frac{1}{2}I$.

Problem 17. Prove the following assertions for $m \times n$ matrices A and B by using the laws of matrix addition and scalar multiplication. Clearly specify each law that you use.
(a) If $A = -A$, then $A = 0$.
(b) If $cA = 0$ for some scalar c, then either $c = 0$ or $A = 0$.
(c) If $B = cB$ for some scalar $c \neq 1$, then $B = 0$.

2.2 Matrix Multiplication

Matrix multiplication is somewhat more subtle than matrix addition and scalar multiplication. Of course, we could define matrix multiplication to be a coordinatewise operation, just as addition is (there is such a thing, called Hadamard multiplication). But our motivation is not merely to make definitions, but rather to make *useful* definitions for basic problems.

Definition of Multiplication

To motivate the definition, let us consider a single linear equation

$$2x - 3y + 4z = 5.$$

We will find it handy to think of the left-hand side of the equation as a "product" of the coefficient matrix $[2, -3, 4]$ and the column matrix of unknowns $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$. Thus, we have that the product of this row and column is

$$[2, -3, 4] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [2x - 3y + 4z].$$

Notice that we have made the result of the product into a 1×1 matrix. This introduces us to a permanent abuse of notation that is almost always used in linear algebra: we don't distinguish between the scalar a and the 1×1 matrix $[a]$, though technically perhaps we should. In the same spirit, we make the following definition.

Row Column Product

Definition 2.5. The *product* of the $1 \times n$ row $[a_1, a_2, \ldots, a_n]$ with the $n \times 1$ column $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is defined to be the 1×1 matrix $[a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]$.

It is this row-column product strategy that guides us to the general definition. Notice how the column number of the first matrix had to match the row number of the second, and that this number disappears in the size of the resulting product. This is exactly what happens in general.

Matrix Product

Definition 2.6. Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [b_{ij}]$ a $p \times n$ matrix. Then the *product* of the matrices A and B, denoted by AB, is the $m \times n$ matrix whose (i, j)th entry, for $1 \leq i \leq m$ and $1 \leq j \leq n$, is the entry of the product of the ith row of A and the jth column of B; more specifically, the (i, j)th entry of AB is

$$a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj}.$$

Notice that, in contrast to the case of addition, two matrices may be of different sizes when we can multiply them together. If A is $m \times p$ and B is $p \times n$, we say that A and B are *conformable* for multiplication. It is also worth noticing that if A and B are square *and of the same size,* then the products AB and BA are always defined.

Some Illustrative Examples

Let's check our understanding with a few examples.

Example 2.7. Compute, if possible, the products AB of the following pairs of matrices A, B.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Solution. In part (a) A is 2×3 and B is 3×2. First check conformability for multiplication. Stack these dimensions alongside each other and see that the 3's match; now "cancel" the matching middle 3's to obtain that the dimension of the product is $2 \times \cancel{3}\ \cancel{3} \times 2 = 2 \times 2$. To obtain, for example, the $(1, 2)$th entry of the product matrix, multiply the first row of A and second column of B to obtain

$$[1, 2, 1] \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = [1 \cdot (-2) + 2 \cdot 1 + 1 \cdot 1] = [1] \, .$$

The full product calculation looks like this:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 2\cdot 0 + 1\cdot 2 & 1\cdot(-2) + 2\cdot 1 + 1\cdot 1 \\ 2\cdot 4 + 3\cdot 0 + (-1)\cdot 2 & 2\cdot(-2) + 3\cdot 1 + (-1)\cdot 1 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 1 \\ 6 & -2 \end{bmatrix}.$$

A size check of part (b) reveals a mismatch between the column number of the first matrix (3) and the row number (2) of the second matrix. Thus these matrices are *not conformable* for multiplication in the specified order. Hence

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

is undefined.

In part (c) a size check shows that the product has size $2 \times \cancel{1}\ \cancel{1} \times 2 = 2 \times 2$. The calculation gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 & 0\cdot 2 \\ 0\cdot 1 & 0\cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For part (d) the size check shows gives $1 \times \cancel{2}\ \cancel{2} \times 1 = 1 \times 1$. Hence the product exists and is 1×1. The calculation gives

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = [1\cdot 0 + 2\cdot 0] = [0].$$

Something very interesting comes out of parts (c) and (d). Notice that AB and BA are *not* the same matrices—never mind that their entries are all 0's—the important point is that these matrices are not even the same size! Thus a very familiar law of arithmetic, the commutativity of multiplication, has just fallen by the wayside.

Matrix Multiplication Not Commutative or Cancellative

Things work well in (e), where the size check gives $2 \times \cancel{2}\ \cancel{2} \times 3 = 2 \times 3$ as the size of the product. As a matter of fact, this is a rather interesting calculation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 0\cdot 2 & 1\cdot 2 + 0\cdot 3 & 1\cdot 1 + 0\cdot(-1) \\ 0\cdot 1 + 1\cdot 2 & 0\cdot 2 + 1\cdot 3 & 0\cdot 1 + 1\cdot(-1) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}.$$

Notice that we end up with the second matrix in the product. This is similar to the arithmetic fact that $1\cdot x = x$ for a given real number x. So the matrix on the left acted like a multiplicative identity. We'll see that this is no accident.

Finally, for the calculation in (f), notice that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 1\cdot -1 & 1\cdot 1 + 1\cdot -1 \\ 1\cdot 1 + 1\cdot -1 & 1\cdot 1 + 1\cdot -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

There's something very curious here, too. Notice that two nonzero matrices of the same size multiplied together to give a zero matrix. This kind of thing never happens in ordinary arithmetic, where the cancellation law assures that if $a \cdot b = 0$ then $a = 0$ or $b = 0$. □

The calculation in (e) inspires some more notation. The left-hand matrix of this product has a very important property. It acts like a "1" for matrix multiplication. So it deserves its own name. A matrix of the form

Identity Matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & 0 & & \\ \vdots & & \ddots & & \\ & & & 1 & 0 \\ 0 & & & 0 & 1 \end{bmatrix} = [\delta_{ij}]$$

is called an $n \times n$ *identity matrix*. The (i, j)th entry of I_n is designated by the Kronecker symbol δ_{ij}, which is 1 if $i = j$ and 0 otherwise. If n is clear from context, we simply write I in place of I_n.

Kronecker Symbol

So we see in the previous example that the left-hand matrix of part (e) is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Linear Systems as a Matrix Product

Let's have another look at a system we examined in Chapter 1. We'll change the names of the variables from x, y, z to x_1, x_2, x_3 in anticipation of a notation that will work with any number of variables.

Example 2.8. Express the following linear system as a matrix product:

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 5x_3 &= 11 \\ 4x_1 + 6x_2 + 8x_3 &= 24 \end{aligned}$$

Solution. Recall how we defined multiplication of a row vector and column vector at the beginning of this section. We use that as our inspiration. Define

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 11 \\ 24 \end{bmatrix}, \quad \text{and } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix}.$$

Of course, A is just the coefficient matrix of the system and b is the right-hand-side vector, which we have seen several times before. But now these take on a new significance. Notice that if we take the first row of A and multiply it by $\mathbf{x}$ we get the left-hand side of the first equation of our system. Likewise for the second and third rows. Therefore, we may write in the language of matrices that

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 24 \end{bmatrix} = \mathbf{b}.$$

Thus the system is represented very succinctly as $A\mathbf{x} = \mathbf{b}$. □

Once we understand this example, it is easy to see that the general abstract system that we examined in Section 1.1 can just as easily be abbreviated. Now we have a new way of looking at a system of equations: it is just like a simple first-degree equation in one variable. Of course, the catch is that the symbols $A, \mathbf{x}, \mathbf{b}$ now represent an $m \times n$ matrix, and $n \times 1$ and $m \times 1$ vectors, respectively. In spite of this, the matrix multiplication idea is very appealing. For instance, it might inspire us to ask whether we could somehow solve the system $A\mathbf{x} = \mathbf{b}$ by multiplying both sides of the equation by some kind of matrix "$1/A$" so as to cancel the A and get

$$(1/A)A\mathbf{x} = I\mathbf{x} = \mathbf{x} = (1/A)\mathbf{b}.$$

We'll follow up on this idea in Section 2.5.

Here is another perspective on matrix–vector multiplication that gives a powerful way of thinking about such multiplications.

Example 2.9. Interpret the matrix product of Example 2.8 as a linear combination of column vectors.

Solution. Examine the system of this example and we see that the column $(1, 2, 4)$ appears to be multiplied by x_1. Similarly, the column $(1, 2, 6)$ is multiplied by x_2 and the column $(1, 5, 8)$ by x_3. Hence, if we use the same right-hand-side column $(4, 11, 24)$ as before, we obtain that this column can be expressed as a linear combination of column vectors, namely

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 24 \end{bmatrix}. \qquad \square$$

Matrix-Vector Multiplication

We could write the equation of the previous example very succinctly as follows: let A have columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, so that $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, and let $\mathbf{x} = (x_1, x_2, x_3)$. Then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3.$$

This formula extends to general matrix–vector multiplication. It is extremely useful in interpreting such products, so we will elevate its status to that of a theorem worth remembering.

Theorem 2.1. Let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ be an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Laws of Arithmetic

We have already seen that the laws of matrix arithmetic may not be quite the same as the ordinary arithmetic laws that we are used to. Nonetheless, as long as we don't assume a cancellation law or a commutative law for multiplication, things are pretty much what one might expect.

Laws of Matrix Multiplication

Let A, B, C be matrices of the appropriate sizes so that the following multiplications make sense, I a suitably sized identity matrix, and c and d scalars.

(1) (Closure Law) The product AB is a matrix.
(2) (Associative Law) $(AB)C = A(BC)$
(3) (Identity Law) $AI = A$ and $IB = B$
(4) (Associative Law for Scalars) $c(AB) = (cA)B = A(cB)$
(5) (Distributive Law) $(A + B)C = AC + BC$
(6) (Distributive Law) $A(B + C) = AB + AC$

One can formally verify these laws by working through the definitions. For example, to verify the first half of the identity law, let $A = [a_{ij}]$ be an $m \times n$ matrix, so that $I = [\delta_{ij}]$ has to be I_n in order for the product AI to make sense. Now we see from the formal definition of matrix multiplication that

$$AI = \left[\sum_{k=1}^{n} a_{ik}\delta_{kj}\right] = [a_{ij} \cdot 1] = A.$$

The middle equality follows from the fact that δ_{kj} is 0 unless $k = j$. Thus the sum collapses to a single term. A similar calculation verifies the other laws.

We end our discussion of matrix multiplication with a familiar-looking notation that will prove to be extremely handy in the sequel. This notation applies only to *square* matrices. Let A be a square $n \times n$ matrix and k a nonnegative integer. Then we define the *kth power* of A to be

Exponent Notation

$$A^k = \begin{cases} I_n & \text{if } k = 0, \\ \underbrace{A \cdot A \cdots A}_{k \quad \text{times}} & \text{if } k > 0. \end{cases}$$

As a simple consequence of this definition we have the standard exponent laws.

Laws of Exponents

For nonnegative integers i, j and square matrix A:

(1) $A^{i+j} = A^i \cdot A^j$
(2) $A^{ij} = (A^i)^j$

Notice that the law $(AB)^i = A^i B^i$ is missing. It won't work with matrices. Why not? The following example illustrates a very useful application of the exponent notation.

Example 2.10. Let $f(x) = 1 - 2x + 3x^2$ be a polynomial function. Use the definition of matrix powers to derive a sensible interpretation of $f(A)$, where A is a square matrix. Evaluate $f\left(\left[\begin{smallmatrix} 2 & -1 \\ 0 & 1 \end{smallmatrix}\right]\right)$ explicitly with this interpretation.

Solution. Let's take a closer look at the polynomial expression

$$f(x) = 1 - 2x + 3x^2 = 1x^0 - 2x + 3x^2.$$

Once we've rewritten the polynomial in this form, we recall that $A^0 = I$ and that other matrix powers make sense since A is square, so the interpretation is easy:

$$f(A) = A^0 - 2A^1 + 3A^2 = I - 2A + 3A^2.$$

In particular, for a 2×2 matrix we take $A = I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and obtain

$$\begin{aligned} f\left(\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\right) &= I - 2\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}^2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 12 & -9 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -7 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

□

2.2 Exercises and Problems

Exercise 1. Carry out these calculations or indicate they are impossible, given that $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 + i \\ 0 & -1 \end{bmatrix}$.

(a) $\mathbf{b}C\mathbf{a}$ (b) $\mathbf{ab}$ (c) $C\mathbf{b}$ (d) $(\mathbf{a}C)\,\mathbf{b}$ (e) $C\mathbf{a}$ (f) $C(\mathbf{ab})$ (g) $\mathbf{ba}$ (h) $C(\mathbf{a} + \mathbf{b})$

Exercise 2. For each pair of matrices A, B, calculate the product AB or indicate that the product is undefined.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 8 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 0 \\ 4 & 3 & 2 \end{bmatrix}, \begin{bmatrix} -5 & 4 & -2 \\ -2 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} -5 & 4 & -2 \\ -2 & 3 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 & 4 \\ -2 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Exercise 3. Express these systems of equations in the notation of matrix multiplication and as a linear combination of vectors as in Example 2.8.

$$\text{(a)}\ \begin{aligned} x_1 - 2x_2 + 4x_3 &= 3 \\ x_2 - x_3 &= 2 \\ -x_1 + 4x_4 &= 1 \end{aligned} \quad \text{(b)}\ \begin{aligned} x - y - 3z &= 3 \\ 2x + 2y + 4z &= 10 \\ -x + z &= 3 \end{aligned} \quad \text{(c)}\ \begin{aligned} x - 3y + 1 &= 0 \\ 2y &= 0 \\ -x + 3y &= 0 \end{aligned}$$

Exercise 4. Express these systems of equations in the notation of matrix multiplication and as a linear combination of vectors as in Example 2.8.

(a) $\begin{aligned} x_1 + x_3 &= -1 \\ x_2 + x_3 &= 0 \\ x_1 + x_3 &= 1 \end{aligned}$ (b) $\begin{aligned} x - y - 3z &= 1 \\ z &= 0 \\ -x + y &= 3 \end{aligned}$ (c) $\begin{aligned} x - 4y &= 0 \\ 2y &= 0 \\ -x + 3y &= 0 \end{aligned}$

Exercise 5. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 3 & -2 \\ 4 & 2 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{X} = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$. Find the coefficient matrix of the linear system $\mathbf{X}A\mathbf{b} + A\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ in the variables x, y, z.

Exercise 6. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Find the coefficient matrix of the linear system $AX - XA = I_2$ in the variables x, y, z, w.

Exercise 7. Let $\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (0, 1, 1)$, and $\mathbf{w} = (1, 3, 1)$. Write each of the following expressions as single matrix product.

(a) $2\mathbf{u} - 4\mathbf{v} - 3\mathbf{w}$ (b) $\mathbf{w} - \mathbf{v} + 2\mathrm{i}\mathbf{u}$ (c) $x_1\mathbf{u} - 3x_2\mathbf{v} + x_3\mathbf{w}$

Exercise 8. Express the following matrix products as linear combinations of vectors.

(a) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 + \mathrm{i} \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

Exercise 9. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, $f(x) = 1 + x + x^2$, $g(x) = 1 - x$, and $h(x) = 1 - x^3$. Verify that $f(A)\, g(A) = h(A)$.

Exercise 10. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$. Compute $f(A)$ and $f(B)$, where $f(x) = 2x^3 + 3x - 5$.

Exercise 11. Find all possible products of two matrices from among the following:

$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

Exercise 12. Find all possible products of three matrices from among the following:

$A = \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$ $C = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ $E = \begin{bmatrix} -2 & 4 \end{bmatrix}$

Exercise 13. A square matrix A is said to be *nilpotent* if there is a positive integer k such that $A^k = 0$. Determine which of the following matrices are nilpotent. (You may assume that if A is $n \times n$ nilpotent, then $A^n = 0$.)

(a) $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 & -4 \\ -1 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & -2 & -1 \end{bmatrix}$

Exercise 14. A square matrix A is *idempotent* if $A^2 = A$. Determine which of the following matrices are idempotent.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

Exercise 15. Show by example that a sum of nilpotent matrices need not be nilpotent.

Exercise 16. Show by example that a product of idempotent matrices need not be idempotent.

Exercise 17. Verify that the product $\mathbf{uv}$, where $\mathbf{u} = (1, 0, 2)$ and $\mathbf{v} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$, is a rank-one matrix.

Exercise 18. Verify that the product $\mathbf{uv} + \mathbf{wu}$, where $\mathbf{u} = (1, 0, 2)$, $\mathbf{v} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$, and $\mathbf{w} = (1, 0, 1)$, is a matrix of rank at most two.

Exercise 19. Verify that both associative laws of multiplication hold for

$$c = 4, \qquad A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1+\mathrm{i} & 1 \\ 1 & 2 \end{bmatrix}.$$

Exercise 20. Verify that both distributive laws of multiplication hold for

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1+\mathrm{i} & 1 \\ 1 & 2 \end{bmatrix}.$$

Problem 21. Find examples of 2×2 matrices A and B that fulfill each of the following conditions.

(a) $(AB)^2 \neq A^2B^2$ (b) $AB \neq BA$

Problem 22. Find examples of nonzero 2×2 matrices A, B, and C that fulfill each of the following conditions.

(a) $A^2 = 0,\ B^2 = 0$ (b) $(AB)^2 \neq 0$

*Problem 23. Show that if A is a 2×2 matrix such that $AB = BA$ for every 2×2 matrix B, then A is a multiple of I_2.

Problem 24. Prove that the associative law for scalars is valid.

Problem 25. Prove that both distributive laws for matrix multiplication are valid.

Problem 26. Show that if A is a square matrix such that $A^{k+1} = 0$, then

$$(I - A)\left(I + A + A^2 + \cdots + A^k\right) = I.$$

*Problem 27. Show that if two matrices A and B of the same size have the property that $A\mathbf{b} = B\mathbf{b}$ for every column vector $\mathbf{b}$ of the correct size for multiplication, then $A = B$.

2.3 Applications of Matrix Arithmetic

We next examine a few more applications of the matrix multiplication idea that should reinforce the importance of this idea and provide us with some interpretations of matrix multiplication.

Matrix Multiplication as Function

The function idea is basic to mathematics. Recall that a *function* f is a rule of correspondence that assigns to each argument x in a set called its domain, a unique value $y = f(x)$ from a set called its target. Each branch of mathematics has its own special functions; for example, in calculus differentiable functions $f(x)$ are fundamental. Linear algebra also has its special functions. Suppose that $T(\mathbf{u})$ represents a function whose arguments $\mathbf{u}$ and values $\mathbf{v} = T(\mathbf{u})$ are vectors.

We say that the function T is *linear* if T preserves linear combinations, that is, for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T, and scalars c, d, we have that $c\mathbf{u} + d\mathbf{v}$ is in the domain of T and

Linear Functions

$$T\left(c\mathbf{u} + d\mathbf{v}\right) = cT\left(\mathbf{u}\right) + dT\left(\mathbf{v}\right).$$

Example 2.11. Show that the function T, whose domain is the set of 2×1 vectors and definition is given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x,$$

is a linear function.

Solution. Let (x, y) and (z, w) be two elements in the domain of T and c, d any two scalars. Now compute

$$\begin{aligned} T\left(c\begin{bmatrix} x \\ y \end{bmatrix} + d\begin{bmatrix} z \\ w \end{bmatrix}\right) &= T\left(\begin{bmatrix} cx \\ cy \end{bmatrix} + \begin{bmatrix} dz \\ dw \end{bmatrix}\right) = T\left(\begin{bmatrix} cx+dz \\ cy+dw \end{bmatrix}\right) \\ &= cx + dz = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + dT\left(\begin{bmatrix} z \\ w \end{bmatrix}\right). \end{aligned}$$

Thus, T satisfies the definition of linear function. □

One can check that the function T just defined can be expressed as a matrix multiplication, namely, $T\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. This example gives yet another reason for defining matrix multiplication in the way that we do. Here is a general definition for these kinds of functions (also known as linear transformations or linear operators).

Matrix Operator

Definition 2.7. Let A be an $m \times n$ matrix. The function T_A that maps $n \times 1$ vectors to $m \times 1$ vectors according to the formula

$$T_A(\mathbf{u}) = A\mathbf{u}$$

is called the *linear function (operator or transformation)* associated with the matrix A or simply a *matrix operator.*

Let's verify that this function T actually is linear. Use the definition of T_A along with the distributive law of multiplication and associative law for scalars to obtain that

$$\begin{aligned} T_A(c\mathbf{u} + d\mathbf{v}) &= A(c\mathbf{u} + d\mathbf{v}) \\ &= A(c\mathbf{u}) + A(d\mathbf{v}) \\ &= c(A\mathbf{u}) + d(A\mathbf{v}) \\ &= cT_A(\mathbf{u}) + dT_A(\mathbf{v}). \end{aligned}$$

Thus multiplication of vectors by a fixed matrix A is a linear function. Notice that this result contains Example 2.11 as a special case.

Function Composition Notation

Recall that the composition of functions f and g is the function $f \circ g$ whose definition is $(f \circ g)(x) = f(g(x))$ for all x in the domain of g.

Example 2.12. Use the associative law of matrix multiplication to show that the composition of matrix multiplication functions corresponds to the matrix product.

Solution. For all vectors $\mathbf{u}$ and for suitably sized matrices A, B, we have by the associative law that $A(B\mathbf{u}) = (AB)\mathbf{u}$. In function terms, this means that $T_A(T_B(\mathbf{u})) = T_{AB}(\mathbf{u})$. Since this is true for all arguments $\mathbf{u}$, it follows that $T_A \circ T_B = T_{AB}$, which is what we were to show. □

We will have more to say about linear functions in Chapters 3 and 6, where they will go by the name of linear operators. Here is an example that gives another slant on why the "linear" in "linear function."

Example 2.13. Describe the action of the matrix operator T_A on the x-axis and y-axis, where $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

Solution. A typical element of the x-axis has the form $\mathbf{v} = (x, 0)$. Thus we have that $T(\mathbf{v}) = T((x, 0))$. Now calculate

$$T(\mathbf{v}) = T_A((x,0)) = A\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 2x \\ 4x \end{bmatrix} = x \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Thus the x-axis is mapped to all multiples of the vector $(2, 4)$. Set $t = 2x$, and we see that $x(2, 4) = (t, 2t)$. Hence, these are simply points on the line given by $x = t$, $y = 2t$. Equivalently, this is the line $y = 2x$. Similarly, one checks that the y-axis is mapped to the line $y = 2x$ as well. □

Example 2.14. Let L be set of points (x, y) defined by the equation $y = x+1$ and let $T_A(L) = \{T(((x, y)) \mid (x, y) \in L\}$, where $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. Describe and sketch these sets in the plane.

Solution. Of course, the set L is just the straight line defined by the linear equation $y = x + 1$. To see what $T_A(L)$ looks like, write a typical element of L in the form $(x, x + 1)$. Now calculate

$$T_A((x, x+1)) = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 3x+1 \\ 6x+2 \end{bmatrix}.$$

Next make the substitution $t = 3x + 1$, and we see that a typical element of $T_A(L)$ has the form $(t, 2t)$, where t is any real number. We recognize these points as exactly the points on the line $y = 2x$. Thus, the function T_A maps the line $y = x + 1$ to the line $y = 2x$. Figure 2.1 illustrates this mapping as well as the fact that T_A maps the line segment from $\left(\frac{-1}{3}, \frac{2}{3}\right)$ to $\left(\frac{1}{6}, \frac{7}{6}\right)$ on L to the line segment from $(0, 0)$ to $\left(\frac{3}{2}, 3\right)$ on $T_A(L)$. □

Graphics specialists and game programmers have a special interest in *real-time rendering*, the discipline concerned with algorithms that create synthetic images fast enough that the viewer can interact with a virtual environment. For a comprehensive treatment of this subject, consult the text [2]. A number of fundamental matrix-defined operators are used in real-time rendering, where they are called *transforms*. Here are a few examples of such operators. A *scaling operator* is effected by multiplying each coordinate of a point by a fixed (positive) scale factor. A *shearing operator* is effected by adding a constant shear factor times one coordinate to another coordinate of the point. A *rotation operator* is effected by rotating each point a fixed angle θ in the counterclockwise direction about the origin.

Real-Time Rendering

Scaling and Shearing Graphics Transforms

Example 2.15. Let the scaling operator S on points in two dimensions have scale factors of $\frac{3}{2}$ in the x-direction and $\frac{1}{2}$ in the y-direction. Let the shearing

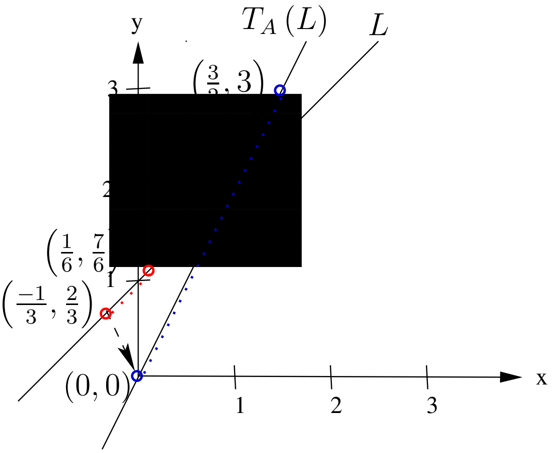

y
$T_A(L)$
L
$\left(\frac{3}{2}, 3\right)$
3
2
$\left(\frac{1}{6}, \frac{7}{6}\right)$
1
$\left(\frac{-1}{3}, \frac{2}{3}\right)$
$(0,0)$
1
2
3
x

Fig. 2.1. Action of T_A on line L given by $y = x + 1$, points on L, and the segment between them.

operator H on these points have a shear factor of $\frac{1}{2}$ by the y-coordinate on the x-coordinate. Express these operators as matrix operators and graph their action on four unit squares situated diagonally from the origin.

Solution. First consider the scaling operator. The point (x, y) will be transformed into the point $\left(\frac{3}{2}x, \frac{1}{2}y\right)$. Observe that

$$S((x,y)) = \begin{bmatrix} \frac{3}{2}x \\ \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T_A((x,y)),$$

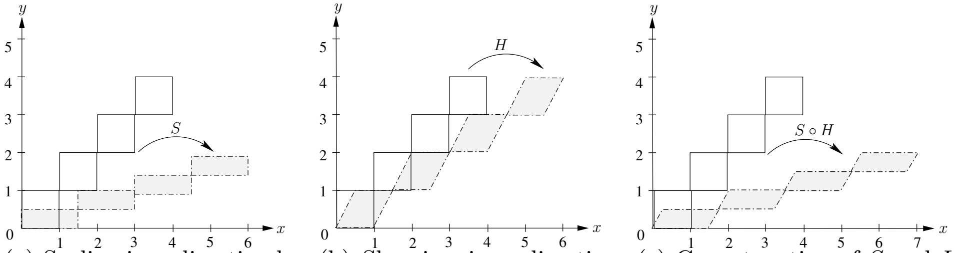
where $A = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & frac12 \end{bmatrix}$. Similarly, the shearing operator transforms the point (x, y) into the point $\left(x + \frac{1}{2}y, y\right)$. Thus we have

$$H((x,y)) = \begin{bmatrix} x + \frac{1}{2}y \\ y \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T_B((x,y)),$$

where $B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$. The action of these operators on four unit squares is illustrated in Figure 2.2. □

Example 2.16. Express the concatenation $S \circ H$ of the scaling operator S and shearing operator H of Example 2.15 as a matrix operator and graph the action of the concatenation on four unit squares situated diagonally from the origin.

Solution. From Example 2.15 we have that $S = T_A$, where $A = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, and $H = T_B$, where $B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$. From Example 2.12 we know that function composition corresponds to matrix multiplication, that is,



(a) Scaling in x-direction by $\frac{3}{2}$, y-direction by 1

(b) Shearing in x-direction by y, shear factor $\frac{1}{2}$

(c) Concatenation of S and H

Fig. 2.2. Action of scaling operator, shearing operator, and concatenation.

$$\begin{aligned}
S \circ H\left((x,y)\right) &= T_A \circ T_B\left((x,y)\right) = T_{AB}\left((x,y)\right) \\
&= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T_C\left((x,y)\right),
\end{aligned}$$

where $C = AB = \begin{bmatrix} \frac{3}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$. The action of $S \circ H$ on four unit squares is illustrated in Figure 2.2. □

Example 2.17. Describe the rotation operator (about the origin) for the plane.

Solution. Consult Figure 2.3. Observe that if the point (x,y) is given by $(r\cos\phi, r\sin\phi)$ in polar coordinates, then the rotated point (x',y') has coordinates $(r\cos(\theta+\phi), r\sin(\theta+\phi))$. Now use the double-angle formula for angles and obtain that

$$\begin{aligned}
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ r\sin\theta\cos\phi + r\cos\theta\sin\phi \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\end{aligned}$$

Now define the *rotation matrix* $R(\theta)$ by

Rotation Matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

It follows that $(x',y') = T_{R(\theta)}\left((x,y)\right)$. □

Discrete Dynamical Systems

Discrete dynamical systems are an extremely useful modeling tool in a wide variety of disciplines. Here is the definition of such a system.

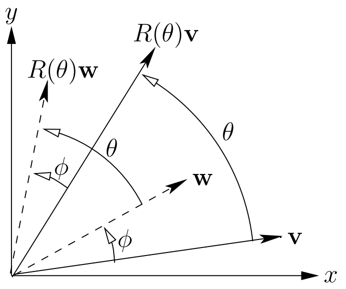


Fig. 2.3. Action of rotation matrix $R(\theta)$ on vectors $\mathbf{v}$ and $\mathbf{w}$.

Discrete Dynamical System

Definition 2.8. A *discrete linear dynamical system* is a sequence of vectors $\mathbf{x}^{(k)}$, $k = 0, 1, \ldots$, called *states*, which is defined by an initial vector $\mathbf{x}^{(0)}$ and by the rule

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}, \qquad k = 0, 1, \ldots,$$

where A is a given fixed square matrix, called the *transition matrix* of the system.

A Markov chain is a certain type of discrete dynamical system. Here is an example.

Example 2.18. Suppose two toothpaste companies compete for customers in a fixed market in which each customer uses either Brand A or Brand B. Suppose also that a market analysis shows that the buying habits of the customers fit the following pattern in the quarters that were analyzed: each quarter (three-month period), 30% of A users will switch to B, while the rest stay with A. Moreover, 40% of B users will switch to A in a given quarter, while the remaining B users will stay with B. If we *assume* that this pattern does not vary from quarter to quarter, we have an example of what is called a Markov *chain model.* Express the data of this model in matrix–vector language.

Solution. Notice that if a_0 and b_0 are the fractions of the customers using A and B, respectively, in a given quarter, a_1 and b_1 the fractions of customers using A and B in the next quarter, then our hypotheses say that

$$\begin{aligned} a_1 &= 0.7a_0 + 0.4b_0 \\ b_1 &= 0.3a_0 + 0.6b_0. \end{aligned}$$

We could figure out what happens in the quarter after this by replacing the indices 1 and 0 by 2 and 1, respectively, in the preceding formula. In general, we replace the indices $1, 0$ by $k, k+1$, to obtain

$$\begin{aligned} a_{k+1} &= 0.7a_k + 0.4b_k \\ b_{k+1} &= 0.3a_k + 0.6b_k. \end{aligned}$$

We express this system in matrix form as follows: let

$$\mathbf{x}^{(k)} = \begin{bmatrix} a_k \\ b_k \end{bmatrix} \text{ and } A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$$

Then the system may be expressed in the matrix form

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}. \qquad \square$$

The state vectors $\mathbf{x}^{(k)}$ of the preceding example have the following property: Seach coordinate is nonnegative and all the coordinates sum to 1. Such a vector is called a *probability distribution vector*. Also, the matrix A has the property that each of its columns is a probability distribution vector. Such a square matrix is called a *stochastic* matrix. In these terms we now give a precise definition of a Markov chain.

Probability Distribution Vector
Stochastic Matrix

Definition 2.9. A *Markov chain* is a discrete dynamical system whose initial state $\mathbf{x}^{(0)}$ is a probability distribution vector and whose transition matrix A is stochastic, that is, each column of A is a probability distribution vector.

Markov Chain

Let us return to Example 2.18. The state vectors and transition matrices

$$\mathbf{x}^{(k)} = \begin{bmatrix} a_k \\ b_k \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

should play an important role. And indeed they do, for in light of our interpretation of a linear system as a matrix product, we see that the two equations of Example 2.18 can be written simply as $\mathbf{x}^{(1)} = Ax^{(0)}$. A little more calculation shows that

$$\mathbf{x}^{(2)} = A\mathbf{x}^{(1)} = A \cdot (A\mathbf{x}^{(0)}) = A^2\mathbf{x}^{(0)}$$

and in general,

$$\mathbf{x}^{(k)} = A\mathbf{x}^{(k-1)} = A^2\mathbf{x}^{(k-2)} = \cdots = A^k\mathbf{x}^{(0)}.$$

In fact, this is true of any discrete dynamical system, and we record this as a *key fact:*

For any positive integer k and discrete dynamical system with transition matrix A and initial state $\mathbf{x}^{(0)}$, the k-th state is given by

$$\mathbf{x}^{(k)} = A^k\mathbf{x}^{(0)}.$$

Computing DDS States

Now we really have a very good handle on the Markov chain problem. Consider the following instance of our example.

Example 2.19. In the notation of Example 2.18 suppose that initially Brand A has all the customers (i.e., Brand B is just entering the market). What are the market shares 2 quarters later? 20 quarters? Answer the same questions if initially Brand B has all the customers.

Solution. To say that initially Brand A has all the customers is to say that the initial state vector is $\mathbf{x}^{(0)} = (1, 0)$. Now do the arithmetic to find $\mathbf{x}^{(2)}$:

$$\begin{aligned}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \mathbf{x}^{(2)} = A^2\mathbf{x}^{(0)} = & \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \left(\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} & \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} .61 \\ .39 \end{bmatrix}.\end{aligned}$$

Thus, Brand A will have 61% of the market and Brand B will have 39% of the market in the second quarter. We did not try to do the next calculation by hand, but rather used a computer to get the approximate answer:

$$\mathbf{x}^{(20)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .57143 \\ .42857 \end{bmatrix}.$$

Thus, after 20 quarters, Brand A's share will have fallen to about 57% of the market and Brand B's share will have risen to about 43%. Now consider what happens if the initial scenario is completely different, i.e., $\mathbf{x}^{(0)} = (0, 1)$. We compute by hand to find that

$$\begin{aligned}\mathbf{x}^{(2)} = & \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \left(\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ = & \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} .52 \\ .48 \end{bmatrix}.\end{aligned}$$

Then we use a computer to find that

$$\mathbf{x}^{(20)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}^{20} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .57143 \\ .42857 \end{bmatrix}.$$

Surprise! For $k = 20$ we get the same answer as we did with a completely different initial condition. Coincidence? We will return to this example again in Chapters 3 and 5, where concepts introduced therein will cast new light on this model (no, it isn't a coincidence). Another curious feature of these state vectors: each one is a probability distribution vector. This is no coincidence either (see Problem 18). □

Structured Population Model

Another important type of model is a so-called *structured population model.* In such a model a population of organisms is divided into a finite number of disjoint states, such as age by year or weight by pound, so that the entire population is described by a state vector that represents the population at discrete times that occur at a constant period such as every day or year. A comprehensive development of this concept can be found in Hal Caswell's text [4]. Here is an example.

Example 2.20. A certain insect has three life stages: egg, juvenile, and adult. A population is observed in a certain environment to have the following properties in a two-day time span: 20% of the eggs will not survive, and 60% will move to the juvenile stage. In the same time-span 10% of the juveniles will not survive, and 60% will move to the adult stage, while 80% of the adults will survive. Also, in the same time-span adults will product about 0.25 eggs per adult. Assume that initially, there are 10, 8, and 6 eggs, juveniles, and adults (measured in thousands), respectively. Model this population as a discrete dynamical system and use it to compute the population total in 2, 10, and 100 days.

Solution. We start time at day 0 and the kth stage is day $2k$. Here the time period is two days and a state vector has the form $\mathbf{x}^{(k)} = (a_k, b_k, c_k)$, where a_k is the number of eggs, b_k the number of juveniles, and c_k the number of adults (all in thousands) on day $2k$. We are given that $\mathbf{x}^{(0)} = (10, 8, 6)$. Furthermore, the transition matrix has the form

$$A = \begin{bmatrix} 0.2 & 0 & 0.25 \\ 0.6 & 0.3 & 0 \\ 0 & 0.6 & 0.8 \end{bmatrix}.$$

The first column says that 20% of the eggs will remain eggs over one time period, 60% will progress to juveniles, and the rest do not survive. The second column says that juveniles produce no offspring, 30% will remain juveniles, 60% will become adults, and the rest do not survive. The third column says that .25 eggs results from one adult, no adult becomes a juvenile, and 80% survive. Now do the arithmetic to find the state $\mathbf{x}^{(1)}$ on day 2:

$$\mathbf{x}^{(1)} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = A^1\mathbf{x}^{(0)} = \begin{bmatrix} 0.2 & 0 & 0.25 \\ 0.6 & 0.3 & 0 \\ 0 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 8.4 \\ 9.6 \end{bmatrix}.$$

For the remaining calculations we use a computer (you should check these results with your own calculator or computer) to obtain approximate answers (we use $\approx$ for approximate equality)

$$\mathbf{x}^{(10)} = \begin{bmatrix} a_{10} \\ b_{10} \\ c_{10} \end{bmatrix} = A^{10}\mathbf{x}^{(0)} \approx \begin{bmatrix} 3.33 \\ 2.97 \\ 10.3 \end{bmatrix},$$

$$\mathbf{x}^{(100)} = \begin{bmatrix} a_{100} \\ b_{100} \\ c_{100} \end{bmatrix} = A^{100}\mathbf{x}^{(0)} \approx \begin{bmatrix} 0.284 \\ 0.253 \\ 0.877 \end{bmatrix}.$$

It appears that the population is declining with time. □

Calculating Power of Graph Vertices

Example 2.21. *(Dominance Directed Graphs)* You have incomplete data about four teams who have played each other in matches. Each match produces a winner and a loser, with no score attached. Identify the teams by labels $1, 2, 3,$ and 4. We could describe a match by a pair of numbers (i, j), where team i played and defeated team j (no ties allowed). Here are the given data:

$$\{(1,2),(1,4),(3,1),(2,3),(4,2)\}\,.$$

Give a reasonable graphical representation of these data.

Solution. We can draw a picture of all the data that we are given by representing each team as a point called a "vertex" and each match by connecting two points with an arrow, called a "directed edge," which points from the winner toward the loser in the match. See Figure 2.4. □

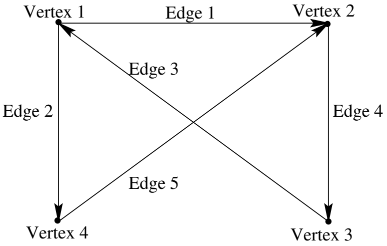


Fig. 2.4. Data from Example 2.21.

Consider the following question relating to Example 2.21. Given this incomplete data about the teams, how would we determine a ranking of each team in some sensible way? In order to answer this question, we are going to introduce some concepts from graph theory that are useful modeling tools for many problems.

The data of Figure 2.4 is an example of a *directed graph*, a modeling tool that can be defined as follows.

Directed Graph

Definition 2.10. A *directed graph* (digraph for short) is a set V whose elements are called *vertices*, together with a set or list (to allow for repeated edges) E of ordered pairs with coordinates in V, whose elements are called *(directed) edges.*

Walk

Another useful idea for us is the following: a *walk* in the digraph G is a sequence of digraph edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)$ that goes from vertex v_0 to vertex v_m. The *length* of the walk is m.

Here is an interpretation of "power" that has proved to be useful in many situations. The *power* of a vertex in a digraph is the number of walks of length 1 or 2 originating at the vertex. In our example, the power of vertex 1 is 4. Why only walks of length 1 or 2? One good reason is that walks of length 3 introduce the possibility of *loops*, i.e., walks that "loop around" to the same point. It isn't very informative to find out that team 1 beat team 2 beat team 3 beat team 1.

The digraph of Example 2.21 has no edges from a vertex to itself (so-called self-loops), and for a pair of distinct vertices, at most one edge connecting the two vertices. In other words, a team doesn't play itself and plays another team at most once. Such a digraph is called a *dominance-directed graph*. Although the notion of power of a point is defined for any digraph, it makes the most sense for dominance-directed graphs.

Dominance Directed Graph

Example 2.22. Find the power of each vertex in the graph of Example 2.21 and use this information to rank the teams.

Solution. In this example we could find the power of all points by inspection of Figure 2.4. Let's do it: simple counting gives that the power of vertex 1 is 4, the power of vertex 3 is 3, and the power of vertices 2 and 4 is 2. Consequently, teams 2 and 4 are tied for last place, team 3 is in second place, and team 1 is first. □

One can imagine situations (like describing the structure of the communications network pictured in Figure 2.5) in which the edges shouldn't really have a direction, since connections are bidirectional. For such situations a more natural tool is the concept of a *graph*, which can be defined as follows: a *graph* is a set V, whose elements are called *vertices*, together with a set or list (to allow for repeated edges) E of *unordered* pairs with coordinates in V, called *edges*.

Graph

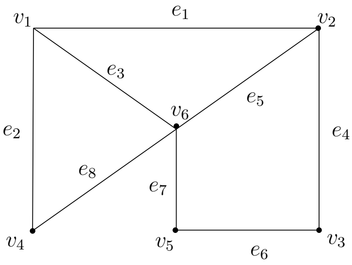


Fig. 2.5. A communications network graph.

Just as with digraphs, we define a *walk* in the graph G as a sequence of digraph edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)$ that goes from vertex v_0 to vertex v_m. The *length* of the walk is m. For example, the graph of Figure 2.5 has vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, with $e_1 = (v_1, v_2)$, etc, as in the figure. Also, the sequence e_1, e_4, e_6 is a walk from vertex v_1 to v_5 of length 3. As with digraphs, we can define the power of a vertex in any graph as the number of walks of length at most 2 originating at the vertex.

A practical question: how could we write a computer program to compute powers? More generally, how can we compute the total number of walks of a certain length? Here is a key to the answer: all the information about our graph (or digraph) can be stored in its *adjacency matrix*. In general, this is defined to be a square matrix whose rows and columns are indexed by the vertices of the graph and whose (i,j)th entry is the number of edges going from vertex i to vertex j (it is 0 if there are none). Here we understand that a directed edge of a digraph must start at i and end at j, while no such restriction applies to the edges of a graph.

Adjacency Matrix

Just for the record, if we designate the adjacency matrix of the digraph of Figure 2.4 by A and the adjacency matrix of the graph of Figure 2.5 by B, then

$$A = \begin{bmatrix} 0&1&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0&1&0&1&0&1\\ 1&0&1&0&0&1\\ 0&1&0&0&1&0\\ 1&0&0&0&0&1\\ 0&0&1&0&0&1\\ 1&1&0&1&1&0 \end{bmatrix}.$$

Notice that we could reconstruct the entire digraph or graph from this matrix. Also notice that in the adjacency matrix for a graph, an edge gets accounted for twice, since it can be thought of as proceeding from one vertex to the other, or from the other to the one.

For a general graph with n vertices and adjacency matrix $A = [a_{ij}]$, we can use this matrix to compute powers of vertices without seeing a picture of the graph. To count up the walks of length 1 emanating from vertex i, simply add up the elements of the ith row of A. Now what about the paths of length 2? Observe that there is an edge from i to k and then from k to j precisely when the product $a_{ik}a_{kj}$ is equal to 1. Otherwise, one of the factors will be 0 and therefore the product is 0. So the number of paths of length 2 from vertex i to vertex j is the familiar sum

$$a_{i1}a_{1j} + a_{i2}a_{2j} + \cdots + a_{in}a_{nj}.$$

This is just the (i,j)th entry of the matrix A^2. A similar argument shows the following fact:

Vertex Power

Theorem 2.2. *If A is the adjacency matrix of the graph G, then the (i,j)th entry of A^r gives the number of walks of length r starting at vertex i and ending at vertex j.*

Since the power of vertex i is the number of all paths of length 1 or 2 emanating from vertex i, we have the following key fact:

Theorem 2.3. *If A is the adjacency matrix of the digraph G, then the power of the ith vertex is the sum of all entries in the ith row of the matrix $A + A^2$.*

Example 2.23. Use the preceding facts to calculate the powers of all the vertices in the digraph of Example 2.21.

Solution. Using the matrix A above we calculate that

$$A + A^2 = \begin{bmatrix} 0&1&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix} + \begin{bmatrix} 0&1&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix} = \begin{bmatrix} 0&2&1&1\\ 1&0&1&0\\ 1&1&0&1\\ 0&1&1&0 \end{bmatrix}.$$

An easy way to sum each row is to multiply $A + A^2$ on the right by a column of 1's, but in this case we see immediately that the power of vertex 1 is 4, the power of vertex 3 is 3, and the power of vertices 2 and 4 is 2, which is consistent with what we observed earlier by inspection of the graph. □

Difference Equations

The idea of a difference equation has numerous applications in mathematics and computer science. In the latter field, these equations often go by the name of "recurrence relations." They can be used for a variety of applications ranging from population modeling to analysis of complexity of algorithms. We will introduce them by way of a simple financial model.

Example 2.24. Suppose that you invest in a contractual fund where you must invest in the funds for three years before you can receive any return on your investment (with a positive first-year investment). Thereafter, you are vested in the fund and may remove your money at any time. While you are vested in the fund, annual returns are calculated as follows: money that was in the fund one year ago earns nothing, while money that was in the fund two years ago earns 6% of its value and money that was in the fund three years ago earns 12% of its value. Find an equation that describes your investment's growth.

Solution. Let a_k be the amount of your investment in the kth year. The numbers a_0, a_1, a_2 represent your investments for the first three years (we're counting from 0). Consider the third year amount a_3. According to your contract, your total funds in the third year will be

$$a_3 = a_2 + 0.06a_1 + 0.12a_0.$$

Now it's easy to write out a general formula for a_{k+3} in terms of the preceding three terms, using the same line of thought, namely

$$a_{k+3} = a_{k+2} + 0.06a_{k+1} + 0.12a_k, \quad k = 0, 1, 2, \ldots. \tag{2.1}$$

This is the desired formula. □

In general, a *homogeneous linear difference equation* (or *recurrence relation*) of order m in the variables $a_0, a_1, \ldots$ is an equation of the form

Homogeneous Linear Difference Equation

$$a_{k+m} + c_{m-1}a_{k+m-1} + \cdots + c_1a_{k+1} + c_0a_k = 0, \quad k = 0, 1, 2, \ldots.$$

Notice that such an equation cannot determine the numbers $a_0, a_1, \ldots, a_{k-1}$. These values have to be initially specified, just as in our fund example. Notice that in our fund example, we have to bring all terms of equation (2.1) to the left-hand side to obtain the difference equation form

$$a_{k+3} - a_{k+2} - 0.06a_{k+1} - 0.12a_k = 0.$$

Now we see that $c_2 = -1$, $c_1 = -0.06$, and $c_0 = -0.12$.

There are many ways to solve difference equations. We are not going to give a complete solution to this problem at this point; we postpone this issue to Chapter 5, where we introduce eigenvalues and eigenvectors. However, we can now show how to turn a difference equation as given above into a matrix equation. Consider our fund example. The secret is to identify the right vector variables. To this end, define an indexed vector $\mathbf{x}_k$ by the formula

$$\mathbf{x}_k = \begin{bmatrix} a_{k+2} \\ a_{k+1} \\ a_k \end{bmatrix}, \qquad k = 0, 1, 2, \ldots.$$

Thus

$$\mathbf{x}_{k+1} = \begin{bmatrix} a_{k+3} \\ a_{k+2} \\ a_{k+1} \end{bmatrix},$$

from which it is easy to check that since $a_{k+3} = a_{k+2} + 0.06a_{k+1} + 0.12a_k$, we have

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1 & 0.06 & 0.12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}_k = A\mathbf{x}_k.$$

This is the matrix form we seek. It appears to have a lot in common with the Markov chains examined earlier in this section, in that we pass from one "state vector" to another by multiplication by a fixed "transition matrix" A.

2.3 Exercises and Problems

Exercise 1. Determine the effect of the matrix operator T_A on the x-axis, y-axis, and the points $(\pm 1, \pm 1)$, where A is one of the following.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Exercise 2. Determine the effect of the matrix operator T_A on the x-axis, y-axis, and the points $(\pm 1, \pm 1)$, where A is one of the following. Plot the images of the squares with corners $(\pm 1, \pm 1)$.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

Exercise 3. Express the following functions, if linear, as matrix operators.
(a) $T((x_1, x_2)) = (x_1 + x_2, 2x_1, 4x_2 - x_1)$ (b) $T((x_1, x_2)) = (x_1 + x_2, 2x_1x_2)$
(c) $T((x_1, x_2, x_3)) = (2x_3, -x_1)$ (d) $T((x_1, x_2, x_3)) = (x_2 - x_1, x_3, x_2 + x_3)$

Exercise 4. Express the following functions, if linear, as matrix operators.
(a) $T((x_1, x_2, x_3)) = x_1 - x_3 + 2x_2$ (b) $T((x_1, x_2)) = (|x_1|, 2x_2, x_1 + 3x_2)$
(c) $T((x_1, x_2)) = (x_1, 2x_1, -x_1)$ (d) $T((x_1, x_2, x_3)) = (-x_3, x_1, 4x_2)$

Exercise 5. A linear operator on $\mathbb{R}^2$ is defined by first applying a scaling operator with scale factors of 2 in the x-direction and 4 in the y-direction, followed by a counterclockwise rotation about the origin of $\pi/6$ radians. Express this operator and the operator that results from reversing the order of the scaling and rotation as matrix operators.

Exercise 6. A linear operator on $\mathbb{R}^2$ is defined by first applying a shear in the x-direction with a shear factor of 3 followed by a clockwise rotation about the origin of $\pi/4$ radians. Express this operator and the operator that results from reversing the order of the shear and rotation as matrix operators.

Exercise 7. A *fixed-point* of a linear operator T_A is a vector $\mathbf{x}$ such that $T_A(\mathbf{x}) = \mathbf{x}$. Find all fixed points, if any, of the linear operators in Exercise 3.

Exercise 8. Find all fixed points, if any, of the linear operators in Exercise 4.

Exercise 9. Given transition matrices for discrete dynamical systems

$$\text{(a)} \begin{bmatrix} .1 & .3 & 0 \\ 0 & .4 & 1 \\ .9 & .3 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} .5 & .3 & 0 \\ 0 & .4 & 0 \\ .5 & .3 & 1 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 0 & 0 & 0.9 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0.1 \end{bmatrix}$$

and initial state vector $\mathbf{x}^{(0)} = \frac{1}{2}(1, 1, 0)$, calculate the first and second state vector for each system and determine whether it is a Markov chain.

Exercise 10. For each of the dynamical systems of Exercise 9, determine by calculation whether the system tends to a limiting steady-state vector. If so, what is it?

Exercise 11. A digraph G has vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(2, 1), (1, 5), (2, 5), (5, 4), (4, 2), (4, 3), (3, 2)\}$. Sketch a picture of the graph G and find its adjacency matrix. Use this to find the power of each vertex of the graph and determine whether this graph is dominance-directed.

Exercise 12. A digraph has the following adjacency matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Sketch a picture of this digraph and compute the total number of walks in the digraph of length at most 3.

Exercise 13. Convert these difference equations into matrix–vector form.
(a) $2a_{k+3} + 3a_{k+2} - 4a_{k+1} + 5a_k = 0$ (b) $a_{k+2} - a_{k+1} + 2a_k = 1$

Exercise 14. Convert these difference equations into matrix–vector form.
(a) $2a_{k+3} + 2a_{k+1} - 3a_k = 0$ (b) $a_{k+2} + a_{k+1} - 2a_k = 3$

*Problem 15. Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a real 2×2 matrix, then the matrix multiplication function maps a line through the origin onto a line through the origin or a point.

Problem 16. Show how the transition matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for a Markov chain can be described using only two variables.

*Problem 17. Use the definition of matrix multiplication function to show that if $T_A = T_B$, then $A = B$.

Problem 18. Show that if the state vector $\mathbf{x}^{(k)} = (a_k, b_k, c_k)$ in a Markov chain is a probability distribution vector, then so is $\mathbf{x}^{(k+1)}$.

Problem 19. Suppose that in Example 2.24 you invest \$1,000 initially (the zeroth year) and no further amounts. Make a table of the value of your investment for years 0 to 12. Also include a column that calculates the annual interest rate that your investment is earning each year, based on the current and previous year's values. What conclusions do you draw? You will need a computer or calculator for this exercise.

2.4 Special Matrices and Transposes

There are certain types of matrices that are so important that they have acquired names of their own. We are going to introduce some of these in this section, as well as one more matrix operation that has proved to be a very practical tool in matrix analysis, namely the operation of transposing a matrix.

Elementary Matrices and Gaussian Elimination

We are going to show a new way to execute the elementary row operations used in Gaussian elimination. Recall the shorthand we used:

- E_{ij}: The elementary operation of *switching the ith and jth rows* of the matrix.
- $E_i(c)$: The elementary operation of *multiplying the ith row by the nonzero constant c*.

- $E_{ij}(d)$: The elementary operation of *adding d times the jth row to the ith row.*

From now on we will use the very same symbols to represent matrices. The size of the matrix will depend on the context of our discussion, so the notation is ambiguous, but it is still very useful.

Elementary Matrix

An *elementary matrix* of size n is obtained by performing the corresponding elementary row operation on the identity matrix I_n. We denote the resulting matrix by the same symbol as the corresponding row operation.

Example 2.25. Describe the following elementary matrices of size $n = 3$:
(a) $E_{13}(-4)$ (b) $E_{21}(3)$ (c) E_{23} (d) $E_1(\frac{1}{2})$

Solution. We start with

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For part (a) we add -4 times the 3rd row of I_3 to its first row to obtain

$$E_{13}(-4) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For part (b) add 3 times the first row of I_3 to its second row to obtain

$$E_{21}(3) = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For part (c) interchange the second and third rows of I_3 to obtain that

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Finally, for part (d) we multiply the first row of I_3 by $\frac{1}{2}$ to obtain

$$E_1\left(\frac{1}{2}\right) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

What good are these matrices? One can see that the following fact is true:

Theorem 2.4. Let $C = BA$ be a product of two matrices and perform an elementary row operation on C. Then the same result is obtained if one performs the same elementary operation on the matrix B and multiplies the result by A on the right.

We won't give a formal proof of this statement, but it isn't hard to see why it is true. For example, suppose one interchanges two rows, say the ith and jth, of $C = BA$ to obtain a new matrix D. How do we get the ith or jth row of C? Answer: multiply the corresponding row of B by the matrix A. Therefore, we would obtain D by interchanging the ith and jth rows of B and multiplying the result by the matrix A, which is exactly what the theorem says. Similar arguments apply to the other elementary operations.

Now take $B = I$, and we see from the definition of elementary matrix and Theorem 2.4 that the following is true.

Corollary 2.1. If an elementary row operation is performed on a matrix A to obtain a matrix A', then $A' = EA$, where E is the elementary matrix corresponding to the elementary row operation performed.

The meaning of this corollary is that we accomplish an elementary row operation by multiplying by the corresponding elementary matrix on the left. Of course, we don't need elementary matrices to accomplish row operations; but they give us another perspective on row operations.

Elementary Operations as Matrix Multiplication

Example 2.26. Express these calculations of Example 1.16 in matrix product form:

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 4 & 20 \end{bmatrix} \xrightarrow[E_{12}]{} \begin{bmatrix} 4 & 4 & 20 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[E_1(1/4)]{} \begin{bmatrix} 1 & 1 & 5 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[E_{21}(-2)]{} \begin{bmatrix} 1 & 1 & 5 \\ 0 & -3 & -9 \end{bmatrix} \xrightarrow[E_2(-1/3)]{} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

Solution. One point to observe: the order of elementary operations. We compose the elementary matrices on the left in the same order that the operations are done. Thus we may state the above calculations in the concise form

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = E_{12}(-1)\, E_2(-1/3)\, E_{21}(-2)\, E_1(1/4)\, E_{12} \begin{bmatrix} 2 & -1 & 1 \\ 4 & 4 & 20 \end{bmatrix}. \qquad \square$$

It is important to read the preceding line carefully and understand how it follows from the long form above. This conversion of row operations to matrix multiplication will prove to be very practical in the next section.

Some Matrices with Simple Structure

Certain types of matrices have already turned up in our discussions. For example, the identity matrices are particularly easy to deal with. Another example is the reduced row echelon form. So let us classify some simple matrices and attach names to them. The simplest conceivable matrices are zero matrices. We have already seen that they are important in matrix addition arithmetic. What's next? For square matrices, we have the following definitions, in ascending order of complexity.

Definition 2.11. Let $A = [a_{ij}]$ be a square $n \times n$ matrix. Then A is

Simple Structure Matrices

- *Scalar* if $a_{ij} = 0$ and $a_{ii} = a_{jj}$ for all $i \neq j$. (Equivalently: $A = cI_n$ for some scalar c, which explains the term “scalar.”)
- *Diagonal* if $a_{ij} = 0$ for all $i \neq j$. (Equivalently: off-diagonal entries of A are 0.)
- *(Upper) triangular* if $a_{ij} = 0$ for all $i > j$. (Equivalently: subdiagonal entries of A are 0.)
- *(Lower) triangular* if $a_{ij} = 0$ for all $i < j$. (Equivalently: superdiagonal entries of A are 0.)
- *Triangular* if the matrix is upper or lower triangular.
- *Strictly triangular* if it is triangular and the diagonal entries are also zero.
- *Tridiagonal* if $a_{ij} = 0$ when $j > i + 1$ or $j < i - 1$. (Equivalently: entries off the main diagonal, first subdiagonal, and first superdiagonal are zero.)

The index conditions that we use above have simple interpretations. For example, the entry a_{ij} with $i > j$ is located further down than over, since the row number is larger than the column number. Hence, it resides in the “lower triangle” of the matrix. Similarly, the entry a_{ij} with $i < j$ resides in the “upper triangle.” Entries a_{ij} with $i = j$ reside along the main diagonal of the matrix. See Figure 2.6 for a picture of these triangular regions of the matrix.

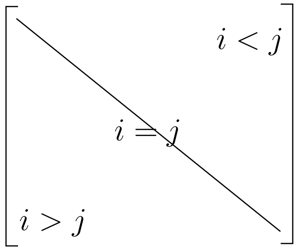


Fig. 2.6: Matrix regions.

Example 2.27. Classify the following matrices (elementary matrices are understood to be 3×3) in the terminology of Definition 2.11.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $E_{21}(3)$ (g) $E_2(-3)$ (h) $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

Solution. Notice that (a) is not scalar, since diagonal entries differ from each other, but it is a diagonal matrix, since the off-diagonal entries are all 0. On the other hand, the matrix of (b) is really just $2I_3$, so this matrix is a scalar matrix. Matrix (c) has all terms below the main diagonal equal to 0, so this matrix is triangular and, specifically, upper triangular. Similarly, matrix (d) is lower triangular. Matrix (e) is clearly upper triangular, but it is also strictly upper triangular since the diagonal terms themselves are 0. Finally, we have

$$E_{21}(3) = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2(-3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that $E_{21}(3)$ is (lower) triangular and $E_2(-3)$ is a diagonal matrix. Matrix (h) comes from Example 1.3, where we saw that an approximation to a certain diffusion problem led to matrices of that form. Moreover, if we want more accurate solutions to the original problem, we would need to solve systems with a similar, but larger, coefficient matrix. This matrix is clearly tridiagonal. In fact, note that the matrices of (a), (b), (f), and (g) also can be classified as tridiagonal. □

Block Matrices

Another type of matrix that occurs frequently enough to be discussed is a *block matrix*. Actually, we already used the idea of blocks when we described the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ as the matrix $\widetilde{A} = [A \,|\, \mathbf{b}]$. We say that $\widetilde{A}$ has the *block*, or *partitioned*, form $[A, \mathbf{b}]$. What we are really doing is partitioning the matrix $\widetilde{A}$ by inserting a vertical line between elements. There is no reason we couldn't partition by inserting more vertical lines or horizontal lines as well, and this partitioning leads to the blocks. The main point to bear in mind when using the block notation is that the blocks must be correctly sized so that the resulting matrix makes sense. The main virtue of the block form that results from partitioning is that for purposes of matrix addition or multiplication, we can treat the blocks rather like scalars, provided the addition or multiplication that results makes sense. We will use this idea from time to time without fanfare. One could go through a formal description of partitioning and proofs; we won't. Rather, we'll show how this idea can be used by example.

Block Notation

Example 2.28. Use block multiplication to simplify the following multiplication:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution. The blocking we want to use makes the column numbers of the blocks on the left match the row numbers of the blocks on the right and looks like this:

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

We see that these submatrices are built from zero matrices and these blocks:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now we can work this product out by interpreting it as

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & C \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} A \cdot 0 + 0 \cdot 0 & A \cdot C + 0 \cdot I_2 \\ 0 \cdot 0 + B \cdot 0 & 0 \cdot C + B \cdot I_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \qquad \square$$

For another (important!) example of block arithmetic, examine Example 2.9 and the discussion following it. There we view a matrix as blocked into its respective columns, and a column vector as blocked into its rows, to obtain

$$A\mathbf{x} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3.$$

Transpose of a Matrix

Sometimes we prefer to work with a different form of a given matrix that contains the same information. Transposes are operations that allow us to do that. The idea is simple: interchange rows and columns. It turns out that for complex matrices, there is an analogue that is not quite the same thing as transposing, though it yields the same result when applied to real matrices. This analogue is called the conjugate (Hermitian) transpose. Here are the appropriate definitions.

Transpose and Conjugate Matrices

Definition 2.12. Let $A = [a_{ij}]$ be an $m \times n$ matrix with (possibly) complex entries. Then the *transpose* of A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A, so that the (i, j)th entry of A^T is a_{ji}. The *conjugate* of A is the matrix $\overline{A} = \left[\overline{a_{ij}}\right]$. Finally, the *conjugate (Hermitian) transpose* of A is the matrix $A^* = \overline{A}^T$.

Notice that in the case of a real matrix (that is, a matrix with real entries) A there is no difference between transpose and conjugate transpose, since in this case $A = \overline{A}$. Consider these examples.

Example 2.29. Compute the transpose and conjugate transpose of the following matrices:

$$\text{(a)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 1 & 1 + \mathrm{i} \\ 0 & 2\mathrm{i} \end{bmatrix}.$$

Solution. For matrix (a) we have

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

Notice, by the way, how the dimensions of a transpose get switched from the original.

For matrix (b) we have

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^* = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix},$$

and for matrix (c) we have

$$\begin{bmatrix} 1 & 1+\mathrm{i} \\ 0 & 2\mathrm{i} \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 1-\mathrm{i} & -2\mathrm{i} \end{bmatrix}, \quad \begin{bmatrix} 1 & 1+\mathrm{i} \\ 0 & 2\mathrm{i} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 1+\mathrm{i} & 2\mathrm{i} \end{bmatrix}.$$

In this case, transpose and conjugate transpose are not the same. □

Even when dealing with vectors alone, the transpose notation is handy. For example, there is a bit of terminology that comes from tensor analysis (a branch of higher linear algebra used in many fields including differential geometry, engineering mechanics, and relativity) that can be expressed very concisely with transposes:

Inner and Outer Products

Definition 2.13. Let $\mathbf{u}$ and $\mathbf{v}$ be column vectors of the same size, say $n \times 1$. Then the *inner product* of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity $\mathbf{u}^T\mathbf{v}$, and the *outer product* of $\mathbf{u}$ and $\mathbf{v}$ is the $n \times n$ matrix $\mathbf{u}\mathbf{v}^T$.

Example 2.30. Compute the inner and outer products of the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.$$

Solution. Here we have the inner product

$$\mathbf{u}^T\mathbf{v} = [2, -1, 1] \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = 2 \cdot 3 + (-1)4 + 1 \cdot 1 = 3,$$

while the outer product is

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} [3, 4, 1] = \begin{bmatrix} 2 \cdot 3 & 2 \cdot 4 & 2 \cdot 1 \\ -1 \cdot 3 & -1 \cdot 4 & -1 \cdot 1 \\ 1 \cdot 3 & 1 \cdot 4 & 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 2 \\ -3 & -4 & -1 \\ 3 & 4 & 1 \end{bmatrix}. \quad \square$$

Here are a few basic laws relating transposes to other matrix arithmetic that we have learned. These laws remain correct if transpose is replaced by conjugate transpose, with one exception: $(cA)^* = \bar{c}A^*$.

Laws of Matrix Transpose

Let A and B be matrices of the appropriate sizes so that the following operations make sense, and c a scalar.

(1) $(A + B)^T = A^T + B^T$
(2) $(AB)^T = B^T A^T$
(3) $(cA)^T = cA^T$
(4) $(A^T)^T = A$

These laws are easily verified directly from definition. For example, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then we have that $(A+B)^T$ is the $n \times m$ matrix given by

$$\begin{aligned}(A+B)^T &= [a_{ij}+b_{ij}]^T = [a_{ji}+b_{ji}] \\ &= [a_{ji}] + [b_{ji}] \\ &= A^T + B^T.\end{aligned}$$

The other laws are proved similarly.

We will require explicit formulas for transposes of the elementary matrices in some later calculations. Notice that the matrix $E_{ij}(c)$ is a matrix with 1's on the diagonal and 0's elsewhere, except that the (i,j)th entry is c. Therefore, transposing switches the entry c to the (j,i)th position and leaves all other entries unchanged. Hence $E_{ij}(c)^T = E_{ji}(c)$. With similar calculations we have these facts:

Transposes of Elementary Matrices

- $E_{ij}^T = E_{ij}$
- $E_i(c)^T = E_i(c)$
- $E_{ij}(c)^T = E_{ji}(c)$

Elementary Column Operations

These formulas have an interesting application. Up to this point we have considered only elementary row operations. However, there are situations in which elementary *column* operations on the columns of a matrix are useful. If we want to use such operations, do we have to start over, reinvent elementary column matrices, and so forth? The answer is no and the following example gives an indication of why the transpose idea is useful. This example shows how to do column operations in the language of matrix arithmetic. Here's the basic idea: suppose we want to do an elementary column operation on a matrix A corresponding to elementary row operation E to get a new matrix B from A. To do this, turn the columns of A into rows, do the row operation, and then transpose the result back to get the matrix B that we want. In algebraic terms

$$B = \left(EA^T\right)^T = \left(A^T\right)^T E^T = AE^T.$$

So all we have to do to perform an elementary column operation is multiply by the transpose of the corresponding elementary row matrix on the right. Thus we see that the transposes of elementary row matrices could reasonably be called *elementary column matrices.*

Elementary Column Matrix

Example 2.31. Let A be a given matrix. Suppose that we wish to express the result B of swapping the second and third columns of A, followed by adding -2 times the first column to the second, as a product of matrices. How can this be done? Illustrate the procedure with the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Solution. Apply the preceding remark twice to obtain that

$$B = AE_{23}^T E_{21}(-2)^T = AE_{23}E_{12}(-2).$$

Thus we have

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as a matrix product. □

A very important type of special matrix is one that is invariant under the operation of transposing. These matrices turn up naturally in applied mathematics. They have some very remarkable properties that we will study in Chapters 4, 5, and 6.

Symmetric and Hermitian Matrices

Definition 2.14. The matrix A is said to be *symmetric* if $A^T = A$ and *Hermitian* if $A^* = A$. (Equivalently, $a_{ij} = a_{ji}$ and $a_{ij} = \overline{a_{ji}}$, for all i, j, respectively.)

From the laws of transposing elementary matrices above we see right away that E_{ij} and $E_i(c)$ supply us with examples of symmetric matrices. Here are a few more.

Example 2.32. Are the following matrices symmetric or Hermitian?

$$\text{(a)} \begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 1 & 1+\mathrm{i} \\ 1+\mathrm{i} & 2\mathrm{i} \end{bmatrix}$$

Solution. For matrix (a) we have

$$\begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2 \end{bmatrix}^* = \begin{bmatrix} 1 & \overline{1+\mathrm{i}} \\ \overline{1-\mathrm{i}} & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2 \end{bmatrix}.$$

Hence this matrix is Hermitian. However, it is *not* symmetric since the $(1,2)$th and $(2,1)$th entries differ. Matrix (b) is easily seen to be symmetric by inspection. Matrix (c) is symmetric since the $(1,2)$th and $(2,1)$th entries agree, but it is not conjugate Hermitian since

$$\begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2\mathrm{i} \end{bmatrix}^* = \begin{bmatrix} 1 & \overline{1+\mathrm{i}} \\ \overline{1-\mathrm{i}} & \overline{2\mathrm{i}} \end{bmatrix}^T = \begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & -2\mathrm{i} \end{bmatrix},$$

and this last matrix is clearly not equal to matrix (c). □

Example 2.33. Consider the quadratic form (this means a homogeneous second-degree polynomial in the variables)

$$Q(x, y, z) = x^2 + 2y^2 + z^2 + 2xy + yz + 3xz.$$

Express this function in terms of matrix products and transposes.

Solution. Write the quadratic form as

$$x(x+2y+3z)+y(2y+z)+z^2 = \begin{bmatrix} x\ y\ z \end{bmatrix} \begin{bmatrix} x+2y+3z \\ 2y+z \\ z \end{bmatrix}$$
$$= \begin{bmatrix} x\ y\ z \end{bmatrix} \begin{bmatrix} 1\ 2\ 3 \\ 0\ 2\ 1 \\ 0\ 0\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{x}^T A\mathbf{x},$$

where

$$\mathbf{x} = (x,y,z) \text{ and } A = \begin{bmatrix} 1\ 2\ 3 \\ 0\ 2\ 1 \\ 0\ 0\ 1 \end{bmatrix}. \qquad \square$$

Rank of the Matrix Transpose

A basic question is how the rank of a matrix transpose (or Hermitian transpose) is connected to the rank of the matrix. There is a nice answer. We will focus on transposes. First we need the following theorem.

Theorem 2.5. Let A, B be matrices such that the product AB is defined. Then

$$\operatorname{rank} AB \leq \operatorname{rank} A.$$

Proof. Let E be a product of elementary matrices such that $EA = R$, where R is the reduced row echelon form of A. If $\operatorname{rank} A = r$, then the first r rows of R have leading entries of 1, while the remaining rows are zero rows. Also, we saw in Chapter 1 that elementary row operations do not change the rank of a matrix, since according to Corollary 1.1 they do not change the reduced row echelon form of a matrix. Therefore,

$$\operatorname{rank} AB = \operatorname{rank} E(AB) = \operatorname{rank}(EA)B = \operatorname{rank} RB.$$

Now the matrix RB has the same number of rows as R, and the first r of these rows may or may not be nonzero, but the remaining rows must be zero rows, since they result from multiplying columns of B by the zero rows of R. If we perform elementary row operations to reduce RB to its reduced row echelon form we will possibly introduce more zero rows than R has. Consequently, $\operatorname{rank} RB \leq r = \operatorname{rank} A$, which completes the proof. □

Theorem 2.6. For any matrix A,

Rank Invariant Under Transpose

$$\operatorname{rank} A = \operatorname{rank} A^T.$$

Proof. As in the previous theorem, let E be a product of elementary matrices such that $EA = R$, where R is the reduced row echelon form of A. If $\operatorname{rank} A = r$, then the first r rows of R have leading entries of 1 whose column numbers form an increasing sequence, while the remaining rows are zero rows. Therefore, $R^T = A^T E^T$ is a matrix whose columns have leading entries of 1 and whose row numbers form an increasing sequence. Use elementary row operations to clear out the nonzero entries below each column with a leading 1 to obtain a matrix whose rank is equal to the number of such leading entries, i.e., equal to r. Thus, $\operatorname{rank} R^T = r$.

From Theorem 2.5 we have that $\operatorname{rank} A^T E^T \leq \operatorname{rank} A^T$. It follows that

$$\operatorname{rank} A = \operatorname{rank} R^T = \operatorname{rank} A^T E^T \leq \operatorname{rank} A^T.$$

If we substitute the matrix A^T for the matrix A in this inequality, we obtain that

$$\operatorname{rank} A^T \leq \operatorname{rank}(A^T)^T = \operatorname{rank} A.$$

It follows from these two inequalities that $\operatorname{rank} A = \operatorname{rank} A^T$, which is what we wanted to show. □

It is instructive to see how a specific example might work out in the preceding proof. For example, R might look like this, where an x designates an arbitrary entry:

$$R = \begin{bmatrix} 1 & 0 & x & 0 & x \\ 0 & 1 & x & 0 & x \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that R^T would given by

$$R^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & x & x & 0 \end{bmatrix}.$$

Thus if we use elementary row operations to zero out the entries below a column pivot, all entries to the right and below this pivot are unaffected by these operations. Now start with the leftmost column and proceed to the right, zeroing out all entries under each column pivot. The result is a matrix that looks like

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now swap rows to move the zero rows to the bottom if necessary, and we see that the reduced row echelon form of R^T has exactly as many nonzero rows as did R, that is, r nonzero rows.

A first application of this important fact is to give a fuller picture of the rank of a product of matrices than that given by Theorem 2.5:

Rank of Matrix Product

Corollary 2.2. If the product AB is defined, then

$$\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

Proof. We know from Theorem 2.5 that

$$\operatorname{rank} AB \leq \operatorname{rank} A \text{ and } \operatorname{rank} B^T A^T \leq \operatorname{rank} B^T.$$

Since $B^T A^T = (AB)^T$, Theorem 2.6 tells us that

$$\operatorname{rank} B^T A^T = \operatorname{rank} AB \text{ and } \operatorname{rank} B^T = \operatorname{rank} B.$$

Put all this together, and we have

$$\operatorname{rank} AB = \operatorname{rank} B^T A^T \leq \operatorname{rank} B^T = \operatorname{rank} B.$$

It follows that $\operatorname{rank} AB$ is at most the smaller of $\operatorname{rank} A$ and $\operatorname{rank} B$, which is what the corollary asserts. □

2.4 Exercises and Problems

Exercise 1. Convert the following 3×3 elementary operations to matrix form and convert matrices to elementary operation form.

(a) $E_{23}(3)$ (b) E_{13} (c) $E_3(2)$ (d) $E_{23}^T(-1)$

(e) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Exercise 2. Convert the following 4×4 elementary operations to matrix form and convert matrices to elementary operation form.

(a) E_{24}^T (b) $E_4(-1)$ (c) $E_3^T(2)$ (d) $E_{14}(-1)$

(e) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$

Exercise 3. Describe the effect of multiplying the 3×3 matrix A by each matrix in Exercise 1 on the left.

Exercise 4. Describe the effect of multiplying the 4×4 matrix A by each matrix in Exercise 2 on the right.

Exercise 5. Compute the reduced row echelon form of the following matrices and express each form as a product of elementary matrices and the original matrix.

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ (3) $\begin{bmatrix} 0 & 1+\mathrm{i} & \mathrm{i} \\ 1 & 0 & -2 \end{bmatrix}$

Exercise 6. Compute the reduced row echelon form of the following matrices and express each form as a product of elementary matrices and the original matrix.

(a) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1+\mathrm{i} \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 & 0 & 2 \\ 1 & 1 & -4 & 3 \end{bmatrix}$

Exercise 7. Identify the minimal list of simple structure descriptions that apply to these matrices (e.g., if “upper triangular,” omit “triangular.”)

(a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 4 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) I_3 (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

Exercise 8. Identify the minimal list of simple structure descriptions that apply to these matrices.

(a) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Exercise 9. Identify the appropriate blocking and calculate the matrix product AB using block multiplication, where

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 4 & 1 & 2 & 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & 3 & 2 \end{bmatrix},$$

and as many submatrices that form scalar matrices or zero matrices are blocked out as possible.

Exercise 10. Confirm that sizes are correct for block multiplication and calculate the matrix product AB, where $A = \begin{bmatrix} R & 0 \\ S & T \end{bmatrix}$, $B = \begin{bmatrix} U \\ V \end{bmatrix}$, $R = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $T = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$, and $V = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$.

Exercise 11. Express the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 2 & 4 & 2 \end{bmatrix}$ as an outer product of two vectors.

Exercise 12. Express the rank-two matrix $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ as the sum of two outer products of vectors.

Exercise 13. Compute the transpose and conjugate transpose of the following matrices and determine which are symmetric or Hermitian.

(a) $\begin{bmatrix} 1 & -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

Exercise 14. Determine which of the following matrices are symmetric or Hermitian.

(a) $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} \mathrm{i} & 1 \\ -1 & \mathrm{i} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Exercise 15. Answer True/False.
(a) $E_{ij}(c)^T = E_{ji}(c)$.
(b) Every elementary matrix is symmetric.
(c) The rank of the matrix A may differ from the rank of A^T.
(d) Every real diagonal matrix is Hermitian.
(e) For matrix A and scalar c, $(cA)^* = cA^*$.

Exercise 16. Answer True/False and give reasons.
(a) For matrices A, B, $(AB)^T = B^T A^T$.
(b) Every diagonal matrix is symmetric.
(c) $\operatorname{rank}(AB) = \min\{\operatorname{rank} A, \operatorname{rank} B\}$.
(d) Every diagonal matrix is Hermitian.
(e) Every tridiagonal matrix is symmetric.

Exercise 17. Express the quadratic form $Q(x, y, z) = 2x^2 + y^2 + z^2 + 2xy + 4yz - 6xz$ in the matrix form $\mathbf{x}^T A \mathbf{x}$, where A has as few nonzero entries as possible.

Exercise 18. Express the quadratic form $Q(x, y, z) = x^2 + y^2 - z^2 + 4yz - 6xz$ in the matrix form $\mathbf{x}^T A \mathbf{x}$, where A is a lower triangular matrix.

Exercise 19. Let $A = \begin{bmatrix} -2 & 1 - 2\mathrm{i} \\ 0 & 3 \end{bmatrix}$ and verify that both $A^* A$ and AA^* are Hermitian.

Exercise 20. A square matrix A is called *normal* if $A^* A = AA^*$. Determine which of the following matrices are normal.

(a) $\begin{bmatrix} 2 & \mathrm{i} \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & \mathrm{i} \\ 1 & 2 + \mathrm{i} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

Problem 21. Show that a triangular and symmetric matrix is a diagonal matrix.

*Problem 22. Let A and C be square matrices and suppose that the matrix $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is in block form. Show that for some matrix D, $M^2 = \begin{bmatrix} A^2 & D \\ 0 & C^2 \end{bmatrix}$.

Problem 23. Show that if $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in block form, then $\operatorname{rank} C = \operatorname{rank} A + \operatorname{rank} B$.

Problem 24. Prove from the definition that $(A^T)^T = A$.

Problem 25. Let A be an $m \times n$ matrix. Show that both A^*A and AA^* are Hermitian.

Problem 26. Use Corollary 2.2 to prove that the outer product of any two vectors is either a rank-one matrix or zero.

Problem 27. Let A be a square real matrix. Show the following.
(a) The matrix $B = \frac{1}{2}\left(A + A^T\right)$ is symmetric.
(b) The matrix $C = \frac{1}{2}\left(A - A^T\right)$ is skew-symmetric (a matrix C is *skew-symmetric* if $C^T = -C$.)
(c) The matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.
(d) With B and C as in parts (a) and (b), show that for any vector $\mathbf{x}$ of conformable size, $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T B\mathbf{x}$.
(e) Express $A = \begin{bmatrix} 2 & 2 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

Problem 28. Find all 2×2 idempotent upper triangular matrices A (idempotent means $A^2 = A$).

*Problem 29. Let D be a diagonal matrix with distinct entries on the diagonal and B any other matrix of the same size. Show that $DB = BD$ if and only if B is diagonal.

Problem 30. Show that an $n \times n$ strictly upper triangular matrix N is nilpotent. (It might help to see what happens in a 2×2 and a 3×3 case first.)

Problem 31. Use Problem 27 to show that every quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ defined by matrix A can be defined by a symmetric matrix $B = (A + A^T)/2$ as well. Apply this result to the matrix of Example 2.33.

*Problem 32. Suppose that $A = B + C$, where B is a symmetric matrix and C is a skew-symmetric matrix. Show that $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$.

2.5 Matrix Inverses

Definitions

We have seen that if we could make sense of "$1/A$," then we could write the solution to the linear system $A\mathbf{x} = \mathbf{b}$ as simply $\mathbf{x} = (1/A)\mathbf{b}$. We are going to tackle this problem now. First, we need a definition of the object that we are trying to uncover. Notice that "inverses" could work only on one side. For example,

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = [1] = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

which suggests that both $\begin{bmatrix} 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \end{bmatrix}$ should qualify as left inverses of the matrix $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix} \right]]$, since multiplication on the left by them results in a 1×1 identity matrix. As a matter of fact, right and left inverses are studied and do have applications. But they have some unusual properties such as nonuniqueness. We are going to focus on a type of inverse that is closer to the familiar inverses in fields of numbers, namely, *two-sided* inverses. These make sense only for square matrices, so the nonsquare example above is ruled out.

Invertible Matrix

Definition 2.15. Let A be a square matrix. Then a *(two-sided) inverse* for A is a square matrix B of the same size as A such that $AB = I = BA$. If such a B exists, then the matrix A is said to be *invertible.*

Nonsingular Matrix

Of course, any nonsquare matrix is noninvertible. Square matrices are classified as either "*singular*," i.e., noninvertible, or "*nonsingular*," i.e., invertible. Since we will mostly be concerned with two-sided inverses, the unqualified term "inverse" will be understood to mean a "two-sided inverse." Notice that this definition is actually symmetric in A and B. In other words, if B is an inverse for A, then A is an inverse for B.

Examples of Inverses

Example 2.34. Show that $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is an inverse for $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

Solution. All we have to do is check the definition. But remember that there are *two* multiplications to confirm. (We'll show later that this isn't necessary, but right now we are working strictly from the definition.) We have

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 - 1 \cdot 1 & 2 \cdot 1 - 1 \cdot 2 \\ -1 \cdot 1 + 1 \cdot 1 & -1 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and similarly

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot (-1) & 1 \cdot (-1) + 1 \cdot 1 \\ 1 \cdot 2 + 2 \cdot (-1) & 1 \cdot (-1) + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Therefore the definition for inverse is satisfied, so that A and B work as inverses to each other. □

Example 2.35. Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ cannot have an inverse.

Solution. How do we get our hands on a "noninverse"? We try an indirect approach. If A had an inverse B, then we could always find a solution to the linear system $A\mathbf{x} = \mathbf{b}$ by multiplying each side on the left by B to obtain that $BA\mathbf{x} = I\mathbf{x} = \mathbf{x} = B\mathbf{b}$, *no matter what right-hand-side vector* $\mathbf{b}$ we used. Yet it is easy to come up with right-hand-side vectors for which the system has no solution. For example, try $\mathbf{b} = (1, 2)$. Since the resulting system is clearly inconsistent, there cannot be an inverse matrix B, which is what we wanted to show. □

The moral of this last example is that it is not enough for every entry of a matrix to be nonzero for the matrix itself to be invertible. Our next example contains a gold mine of invertible matrices, namely any elementary matrix we can construct.

Example 2.36. Find formulas for inverses of all the elementary matrices.

Solution. Recall from Corollary 2.1 that left multiplication by an elementary matrix is the same as performing the corresponding elementary row operation. Furthermore, from the discussion following Theorem 1.2 we see the following:

Elementary Matrix Inverses

- E_{ij}: The elementary operation of switching the ith and jth rows is undone by applying E_{ij}. Hence
$$E_{ij}E_{ij} = E_{ij}E_{ij}I = I,$$
so that E_{ij} works as its own inverse. (This is rather like -1, since $(-1) \cdot (-1) = 1$.)
- $E_i(c)$: The elementary operation of multiplying the ith row by the nonzero constant c, is undone by applying $E_i(1/c)$. Hence
$$E_i(1/c)E_i(c) = E_i(1/c)E_i(c)I = I.$$
- $E_{ij}(d)$: The elementary operation of adding d times the jth row to the ith row is undone by applying $E_{ij}(-d)$. Hence
$$E_{ij}(-d)E_{ij}(d) = E_{ij}(-d)E_{ij}(d)I = I.$$ □

More examples of invertible matrices:

Diagonal Matrix Inverse

Example 2.37. Show that if D is a diagonal matrix with nonzero diagonal entries, then D is invertible.

Solution. Suppose that
$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

Diagonal Matrix Shorthand

For a convenient shorthand, we write $D = \operatorname{diag}\{d_1, d_2, \ldots, d_n\}$. It is easily checked that if $E = \operatorname{diag}\{e_1, e_2, \ldots, e_n\}$, then

$$DE = \operatorname{diag}\{d_1e_1, d_2e_2, \ldots, d_ne_n\} = \operatorname{diag}\{e_1d_1, e_2d_2, \ldots, e_nd_n\} = ED.$$

Therefore, if $d_i \neq 0$, for $i = 1, \ldots, n$, then

$$\operatorname{diag}\{d_1, d_2, \ldots, d_n\} \operatorname{diag}\{1/d_1, 1/d_2, \ldots, 1/d_n\} = \operatorname{diag}\{1, 1, \ldots, 1\} = I_n,$$

which shows that $\operatorname{diag}\{1/d_1, 1/d_2, \ldots, 1/d_n\}$ is an inverse of D. □

Laws of Inverses

Here are some of the basic laws of inverse calculations.

Laws of Matrix Inverses

Let A, B, C be matrices of the appropriate sizes so that the following multiplications make sense, I a suitably sized identity matrix, and c a nonzero scalar. Then

(1) (Uniqueness) If the matrix A is invertible, then it has only one inverse, which is denoted by A^{-1}.
(2) (Double Inverse) If A is invertible, then $\left(A^{-1}\right)^{-1} = A$.
(3) (2/3 Rule) If any two of the three matrices A, B, and AB are invertible, then so is the third, and moreover, $(AB)^{-1} = B^{-1}A^{-1}$.
(4) If A is invertible, then $(cA)^{-1} = (1/c)A^{-1}$.
(5) (Inverse/Transpose) If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$ and $(A^*)^{-1} = (A^{-1})^*$.
(6) (Cancellation) Suppose A is invertible. If $AB = AC$ or $BA = CA$, then $B = C$.

Notes: Observe that the 2/3 rule reverses order when taking the inverse of a product. This should remind you of the operation of transposing a product. A common mistake is to forget to reverse the order. Secondly, notice that the cancellation law restores something that appeared to be lost when we first discussed matrices. Yes, we can cancel a common factor from both sides of an equation, but (1) the factor must be on the same side and (2) the factor must be an invertible matrix.

Verification of Laws: Suppose that both B and C work as inverses to the matrix A. We will show that these matrices must be identical. The associative and identity laws of matrices yield

$$B = BI = B(AC) = (BA)C = IC = C.$$

Matrix Inverse Notation

Henceforth, we shall write A^{-1} for the unique (two-sided) inverse of the square matrix A, provided of course that there is an inverse at all (remember that existence of inverses is not a sure thing).

The double inverse law is a matter of examining the definition of inverse:

$$AA^{-1} = I = A^{-1}A$$

shows that A is an inverse matrix for A^{-1}. Hence, $(A^{-1})^{-1} = A$.

Now suppose that A and B are both invertible and of the same size. Using the laws of matrix arithmetic, we see that

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and that

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

In other words, the matrix $B^{-1}A^{-1}$ works as an inverse for the matrix AB, which is what we wanted to show. We leave the remaining cases of the 2/3 rule as an exercise.

Suppose that c is nonzero and perform the calculation

$$(cA)(1/c)A^{-1} = (c/c)AA^{-1} = 1 \cdot I = I.$$

A similar calculation on the other side shows that $(cA)^{-1} = (1/c)A^{-1}$.

Next, apply the transpose operator to the definition of inverse (equation (2.15)) and use the law of transpose products to obtain that

$$(A^{-1})^T A^T = I^T = I = A^T (A^{-1})^T.$$

This shows that the definition of inverse is satisfied for $(A^{-1})^T$ relative to A^T, that is, that $(A^T)^{-1} = (A^{-1})^T$, which is the inverse/transpose law. The same argument works with conjugate transpose in place of transpose.

Finally, if A is invertible and $AB = AC$, then multiply both sides of this equation on the left by A^{-1} to obtain that

$$A^{-1}(AB) = (A^{-1}A)B = B = A^{-1}(AC) = (A^{-1}A)C = C,$$

which is the cancellation that we want. □

Negative Matrix Power

We can now extend the power notation to negative exponents. Let A be an invertible matrix and k a positive integer. Then we write

$$A^{-k} = A^{-1}A^{-1} \cdots A^{-1},$$

where the product is taken over k terms.

The laws of exponents that we saw earlier can now be expressed for arbitrary integers, *provided* that A is invertible. Here is an example of how we can use the various laws of arithmetic and inverses to carry out an inverse calculation.

Example 2.38. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that $(I - A)^3 = 0$ and use this to find A^{-1}.

Solution. First we calculate that

$$(I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and check that

$$\begin{aligned} (I - A)^3 &= \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Next we do some symbolic algebra, using the laws of matrix arithmetic:

$$0 = (I - A)^3 = (I - A)(I^2 - 2A + A^2) = I - 3A + 3A^2 - A^3.$$

Subtract all terms involving A from both sides to obtain that

$$3A - 3A^2 + A^3 = A \cdot 3I - 3A^2 + A^3 = A(3I - 3A + A^2) = I.$$

Since $A(3I - 3A + A^2) = (3I - 3A + A^2)A$, we see from definition of inverse that

$$A^{-1} = 3I - 3A + A^2 = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Notice that in the preceding example we were careful not to leave a “3” behind when we factored out A from $3A$. The reason is that $3+3A+A^2$ makes no sense as a sum, since one term is a scalar and the other two are matrices.

Rank and Inverse Calculation

Although we can calculate a few examples of inverses such as the last example, we really need a general procedure. So let's get right to the heart of the matter. How can we find the inverse of a matrix, or decide that none exists? Actually, we already have done all the hard work necessary to understand computing inverses. The secret is in the notions of reduced row echelon form and rank. (Remember, we use elementary row operations to reduce a matrix to its reduced row echelon form. Once we have done so, the rank of the matrix is simply the number of nonzero rows in the reduced row echelon form.) Let's recall the results of Example 2.24:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = E_{12}(-1)E_2(-1/3)E_{21}(-2)E_1(1/4)E_{12} \begin{bmatrix} 2 & -1 & 1 \\ 4 & 4 & 20 \end{bmatrix}.$$

Now remove the last column from each of the matrices at the right of each side and we have this result:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_{12}(-1)E_2(-1/3)E_{21}(-2)E_1(1/4)E_{12} \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix}.$$

This suggests that if $A = \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix}$, then

$$A^{-1} = E_{12}(-1)E_2(-1/3)E_{21}(-2)E_1(1/4)E_{12}.$$

To prove this, we argue in the general case as follows: let A be an $n \times n$ matrix and suppose that by a succession of elementary row operations $E_1, E_2, \ldots, E_k$, we reduce A to its reduced row echelon form R, which happens to be I. In the language of matrix multiplication, what we have obtained is

$$I = E_k E_{k-1} \cdots E_1 A.$$

Now let $B = E_k E_{k-1} \cdots E_1$. By repeated application of the 2/3 rule, we see that a product of any number of invertible matrices is invertible. Since each elementary matrix is invertible, it follows that B is. Multiply both sides of the equation $I = BA$ by B^{-1} to obtain that $B^{-1}I = B^{-1} = B^{-1}BA = A$. Therefore, A is the inverse of the matrix B, hence is itself invertible.

Superaugmented Matrix

Here's a practical trick for computing this product of elementary matrices on the fly: form what we term the *superaugmented matrix* $[A \mid I]$. If we perform the elementary operation E on the superaugmented matrix, we have the same result as

$$E[A \mid I] = [EA \mid EI] = [EA \mid E].$$

So the matrix occupied by the I part of the superaugmented matrix is just the product of the elementary matrices that we have used so far. Now continue applying elementary row operations until the part of the matrix originally occupied by A is reduced to the reduced row echelon form of A. We end up with this schematic picture of our calculations:

$$\left[A \mid I \right] \xrightarrow[E_1, E_2, \ldots, E_k]{} \left[R \mid B \right],$$

where R is the reduced row echelon form of A and $B = E_k E_{k-1} \cdots E_1$ is the product of the various elementary matrices we used, composed in the correct order of usage. We can summarize this discussion with the following algorithm:

Inverse Algorithm

Given an $n \times n$ matrix A, to compute A^{-1}:

(1) Form the superaugmented matrix $\widetilde{A} = [A \mid I_n]$.

(2) Reduce the first n columns of $\widetilde{A}$ to reduced row echelon form by performing elementary operations on the matrix $\widetilde{A}$ resulting in the matrix $[R \mid B]$.

(3) If $R = I_n$ then set $A^{-1} = B$, otherwise, A is singular and A^{-1} does not exist.

Example 2.39. Use the inverse algorithm to compute the inverse of Example 2.8,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. Notice that this matrix is already upper triangular. Therefore, as in Gaussian elimination, it is a bit more efficient to start with the bottom pivot and clear out entries above in reverse order. So we compute

$$[A \mid I_3] = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{23}(-1)]{} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{1,2}(-2)]{} \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We conclude that A is indeed invertible and

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

There is a simple formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Set $D = ad - bc$. It is easy to verify that if $D \neq 0$, then

Two by Two Inverse

$$A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 2.40. Use the 2×2 inverse formula to find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$, and verify that the same answer results if we use the inverse algorithm.

Solution. First we apply the inverse algorithm:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[E_{21}(-1)]{} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix} \xrightarrow[E3(1/3)]{} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/3 & 1/3 \end{bmatrix}$$
$$\xrightarrow[E_{12}(1)]{} \begin{bmatrix} 1 & 0 & 2/3 & 1/3 \\ 0 & 1 & -1/3 & 1/3 \end{bmatrix}.$$

Thus we have found that
$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} = \tfrac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$
To apply the inverse formula, calculate $D = 1 \cdot 2 - 1 \cdot (-1) = 3$. Swap diagonal entries of A, negate the off-diagonal entries, and divide by D to get the same result as in the preceding equation for the inverse. $\square$

The formula of the preceding example is well worth memorizing, since we will frequently need to find the inverse of a 2×2 matrix. Notice that in order for it to make sense, we have to have D nonzero. The number D is called the *determinant* of the matrix A. We will have more to say about this number in the next section. It is fairly easy to see why A must have $D \neq 0$ in order for its inverse to exist if we look ahead to the next theorem. Notice in the above elementary operation calculations that if $D = 0$ then elementary operations on A lead to a matrix with a row of zeros. Therefore, the rank of A will be smaller than 2. Here is a summary of our current knowledge of the invertibility of a square matrix.

Conditions for Invertibility

Theorem 2.7. The following are equivalent conditions on the square $n \times n$ matrix A:

(1) The matrix A is invertible.
(2) There is a square matrix B such that $BA = I$.
(3) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every right-hand-side vector $\mathbf{b}$.
(4) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for some right-hand-side vector $\mathbf{b}$.
(5) The linear system $A\mathbf{x} = 0$ has only the trivial solution.
(6) $\operatorname{rank} A = n$.
(7) The reduced row echelon form of A is I_n.
(8) The matrix A is a product of elementary matrices.
(9) There is a square matrix B such that $AB = I$.

Proof. The method of proof is to show that each of conditions (1)–(7) implies the next, and that condition (8) implies (1). This connects (1)–(8) in a circle, so that any one condition will imply any other and therefore all are equivalent to each other. Finally, we show that (9) is equivalent to (1)–(8). Here is our chain of reasoning:

(1) implies (2): Assume A is invertible. Then the choice $B = A^{-1}$ certainly satisfies condition (2).

(2) implies (3): Assume (2) is true. Given a system $A\mathbf{x} = \mathbf{b}$, we can multiply both sides on the left by B to get that $B\mathbf{b} = BA\mathbf{x} = I\mathbf{x} = \mathbf{x}$. So there is only one solution, if any. On the other hand, if the system were inconsistent then we would have $\operatorname{rank} A < n$. By Corollary 2.2, $\operatorname{rank} BA < n$, contradicting the fact that $\operatorname{rank} I_n = n$. Hence, there is a solution, which proves (3).

(3) implies (4): This statement is obvious.

(4) implies (5): Assume (4) is true. Say the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_0$. If the system $A\mathbf{x} = 0$ had a nontrivial solution, say $\mathbf{z}$, then we could add $\mathbf{z}$ to $\mathbf{x}_0$ to obtain a different solution $\mathbf{x}_0 + \mathbf{z}$ of the system $A\mathbf{x} = \mathbf{b}$ (check: $A(\mathbf{z} + \mathbf{x}_0) = A\mathbf{z} + A\mathbf{x}_0 = 0 + \mathbf{b} = \mathbf{b}$). This is impossible since (4) is true, so (5) follows.

(5) implies (6): Assume (5) is true. We know from Theorem 1.5 that the consistent system $A\mathbf{x} = 0$ has a unique solution precisely when the rank of A is n. Hence (6) must be true.

(6) implies (7): Assume (6) is true. The reduced row echelon form of A is the same size as A, that is, $n \times n$, and must have a row pivot entry 1 in every row. Also, the pivot entry must be the only nonzero entry in its column. This exactly describes the matrix I_n, so that (7) is true.

(7) implies (8): Assume (7) is true. We know that the matrix A is reduced to its reduced row echelon form by applying a sequence of elementary operations, or what amounts to the same thing, by multiplying the matrix A on the left by elementary matrices $E_1, E_2, \ldots, E_k$, say. Then $E_1 E_2 \cdots E_k A = I$. But we know from Example 2.36 that each elementary matrix is invertible and that their inverses are themselves elementary matrices. By successive multiplications on the left we obtain that $A = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} I$, showing that A is a product of elementary matrices, which is condition (8).

(8) implies (1): Assume (8) is true. Repeated application of the 2/3 rule shows that the product of any number of invertible matrices is itself invertible. Since elementary matrices are invertible, condition (1) must be true.

(9) is equivalent to (1): Assume (1) is true. Then A is invertible and the choice $B = A^{-1}$ certainly satisfies condition (9). Conversely, if (9) is true, then $I^T = I = (AB)^T = B^T A^T$, so that A^T satisfies (2), which is equivalent to (1). However, we already know that if a matrix is invertible, so is its transpose (Law (5) of Matrix Inverses), so $\left(A^T\right)^T = A$ is also invertible, which is condition (1). □

Notice that Theorem 2.7 relieves us of the responsibility of checking that a square one-sided inverse of a square matrix is a two-sided inverse: this is now automatic in view of conditions (2) and (9). Another interesting consequence of this theorem that has been found to be useful is an either/or statement, so it will always have something to say about any square linear system. This type of statement is sometimes called a *Fredholm alternative.* Many theorems go by this name, and we'll state another one in Chapter 5. Notice that a matrix is not invertible if and only if one of the conditions of the theorem fails. Certainly it is true that a square matrix is either invertible or not invertible. That's all the Fredholm alternative really says, but it uses the equivalent conditions (3) and (5) of Theorem 2.7 to say it in a different way:

Fredholm Alternative

Corollary 2.3. Given a square linear system $A\mathbf{x} = \mathbf{b}$, either the system has a unique solution for every right-hand-side vector $\mathbf{b}$ or there is a nonzero solution $\mathbf{x} = \mathbf{x}_0$ to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

We conclude this section with an application to the problem of solving nonlinear equations. Although we focus on two equations in two unknowns, the same ideas can be extended to any number of equations in as many unknowns.

Recall from calculus that we could solve the one-variable equation $f(x) = 0$ for a solution point x_1 at which $f(x_1) = 0$ from a given "nearby" point x_0 by setting $dx = x_1 - x_0$, and assuming that the change in f is

$$\begin{aligned}\Delta f &= f(x_1) - f(x_0) = 0 - f(x_0)\\ &\approx df = f'(x_0)\,dx = f'(x_0)(x_1 - x_0).\end{aligned}$$

Now solve for x_1 in the equation $-f(x_0) = f'(x_0)(x_1 - x_0)$ and get the equation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Replace 1 by $n+1$ and 0 by n to obtain the famous Newton formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2.2}$$

The idea is to start with x_0, use the formula to get x_1 and if $f(x_1)$ is not close enough to 0, then repeat the calculation with x_1 in place of x_0, and so forth until a satisfactory value of $x = x_n$ is reached. How does this relate to a two-variable problem? We illustrate the basic idea in two variables.

Newton's Method for Systems

Example 2.41. Describe concisely an algorithm analogous to Newton's method in one variable to solve the two-variable problem

$$\begin{aligned}x^2 + \sin(\pi xy) &= 1\\ x + y^2 + e^{x+y} &= 3.\end{aligned}$$

Solution. Our problem can be written as a system of two (nonlinear) equations in two unknowns, namely

$$\begin{aligned}f(x,y) &= x^2 + \sin(\pi xy) - 1 = 0\\ g(x,y) &= x + y^2 + e^{x+y} - 3 = 0.\end{aligned}$$

Now we can pull the same trick with differentials as in the one-variable problem by setting $dx = x_1 - x_0$, $dy = y_1 - y_0$, where $f(x_1, y_1) = 0$, approximating the change in both f and g by total differentials, and recalling the definition of these total differentials in terms of partial derivatives. This leads to the system

$$\begin{aligned}f_x(x_0,y_0)\;dx + f_y(x_0,y_0)\;dy &= -f((x_0,y_0))\\ g_x(x_0,y_0)\;dx + g_y(x_0,y_0)\;dy &= -g((x_0,y_0)).\end{aligned}$$

Next, write everything in vector style, say

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix}, \quad \mathbf{x}^{(0)} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Now we can write the *vector* differentials in the forms

$$d\mathbf{F} = \begin{bmatrix} df \\ dg \end{bmatrix} \quad \text{and} \quad d\mathbf{x} = \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} x_1 - x_0 \\ y_1 - x_0 \end{bmatrix} = \mathbf{x}^{(1)} - \mathbf{x}^{(0)}.$$

The original Newton equations now look like a matrix multiplication involving $d\mathbf{x}$, $\mathbf{F}$, and a matrix of derivatives of $\mathbf{F}$, namely the so-called Jacobian matrix

$$J_{\mathbf{F}}(x_0, y_0) = \begin{bmatrix} f_x((x_0, y_0)) & f_y((x_0, y_0)) \\ g_x((x_0, y_0)) & g_y((x_0, y_0)) \end{bmatrix}.$$

Specifically, we see from the definition of matrix multiplication that the Newton equations are equivalent to the vector equations

$$d\mathbf{F} = J_{\mathbf{F}}(\mathbf{x}_0)\, d\mathbf{x} = -\mathbf{F}\left(\mathbf{x}^{(0)}\right).$$

If the Jacobian matrix is invertible, then

$$\mathbf{x}^{(1)} - \mathbf{x}^{(0)} = J_{\mathbf{F}}\left(\mathbf{x}^{(0)}\right)^{-1} \mathbf{F}\left(\mathbf{x}^{(0)}\right),$$

whence by adding $\mathbf{x}_0$ to both sides we see that

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - J_{\mathbf{F}}\left(\mathbf{x}^{(0)}\right)^{-1} \mathbf{F}\left(\mathbf{x}^{(0)}\right).$$

Now replace 1 by $n+1$ and 0 by n to obtain the famous Newton formula in vector form:

Newton's Formula in Vector Form

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - J_{\mathbf{F}}\left(\mathbf{x}^{(n)}\right)^{-1} \mathbf{F}\left(\mathbf{x}^{(n)}\right).$$

This beautiful analogy to the Newton formula of (2.2)needs the language and algebra of vectors and matrices. One can now calculate the Jacobian for our particular$\mathbf{F}\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right)$ and apply this formula. We leave the details as an exercise.
□

2.5 Exercises and Problems

Exercise 1. Find the inverse or show that it does not exist.

(a) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & i \\ 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Exercise 2. Find the inverse or show that it does not exist.

(a) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 10 \\ 9 & 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$ (e) $\begin{bmatrix} i+1 & 0 \\ 1 & i \end{bmatrix}$

Exercise 3. Express the following systems in matrix form and solve by inverting the coefficient matrix of the system.

(a) $\begin{aligned} 2x + 3y &= 7 \\ x + 2y &= -2 \end{aligned}$ (b) $\begin{aligned} 3x_1 + 6x_2 - x_3 &= -4 \\ -2x_1 + x_2 + x_3 &= 3 \\ x_3 &= 1 \end{aligned}$ (c) $\begin{aligned} x_1 + x_2 &= -2 \\ 5x_1 + 2x_2 &= 5 \end{aligned}$

Exercise 4. Solve the following systems by matrix inversion.

(a) $\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_2 + x_3 &= 1 \\ x_2 - x_3 &= 1 \end{aligned}$ (b) $\begin{aligned} x_1 + 6x_2 - x_3 &= 4 \\ x_1 + x_2 &= 0 \\ x_2 &= 1 \end{aligned}$ (c) $\begin{aligned} x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 11 \end{aligned}$

Exercise 5. Express inverses of the following matrices as products of elementary matrices using the notation of elementary matrices.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 & 0 \\ \mathrm{i} & 3 \end{bmatrix}$

Exercise 6. Show that the following matrices are invertible by expressing them as products of elementary matrices.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 3 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Exercise 7. Find $A^{-1}C$ if $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 2 & 5 & -6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 2 & 0 & -6 & 0 \end{bmatrix}$.

Exercise 8. Solve $AX = B$ for X, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & -1 & 1 & 1 \end{bmatrix}$.

Exercise 9. Verify the matrix law $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ with $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$.

Exercise 10. Verify the matrix law $(A^*)^{-1} = \left(A^{-1}\right)^*$ with $A = \begin{bmatrix} 2 & 1 - 2\mathrm{i} \\ 0 & \mathrm{i} \end{bmatrix}$.

Exercise 11. Verify the matrix law $(AB)^{-1} = B^{-1}A^{-1}$ in the case that $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & 1 \\ 2 & 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercise 12. Verify the matrix law $(cA)^{-1} = (1/c)\, A^{-1}$ in the case that $A = \begin{bmatrix} 1 & 2 - \mathrm{i} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $c = 2 + \mathrm{i}$.

Exercise 13. Determine for what values of k the following matrices are invertible and find the inverse in that case.

(a) $\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ k & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & k \end{bmatrix}$

Exercise 14. Determine the inverses for the following matrices in terms of the parameter c and conditions on c for which the matrix has an inverse.

(a) $\begin{bmatrix} 1 & 2 \\ c & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & c+1 \\ 0 & 1 & 1 \\ 0 & 0 & c \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & c+\mathrm{i} \\ 0 & -1 & 0 \\ 0 & c & c \end{bmatrix}$

Exercise 15. Give a 2×2 example showing that the sum of invertible matrices need not be invertible.

Exercise 16. Give a 2×2 example that the sum of singular matrices need not be singular.

Exercise 17. Problem 26 of Section 2.2 yields a formula for the inverse of the matrix $I-N$, where N is nilpotent, namely, $(I-N)^{-1} = I+N+N^2+\cdots+N^k$. Apply this formula to matrices (a) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Exercise 18. If a matrix can be written as $A = D(I-N)$, where D is diagonal with nonzero entries and N is nilpotent, then $A^{-1} = (I-N)^{-1} D^{-1}$. Use this fact and the formulas of Exercise 17 and Example 2.37 to find inverses of the matrices (a) $\begin{bmatrix} 2 & 2 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$ and (b) $\begin{bmatrix} 2 & 0 \\ \mathrm{i} & 3 \end{bmatrix}$.

Exercise 19. Solve the nonlinear system of equations of Example 2.41 by using nine iterations of the vector Newton formula (2.5), starting with the initial guess $\mathbf{x}^{(0)} = (0,1)$. Evaluate $F\left(\mathbf{x}^{(9)}\right)$.

Exercise 20. Find the minimum value of the function $F(x,y) = \left(x^2+y+1\right)^2 + x^4 + y^4$ by using the Newton method to find critical points of the function $F(x,y)$, i.e., points where $f(x,y) = F_x(x,y) = 0$ and $g(x,y) = F_y(x,y) = 0$.

*Problem 21. Show from the definition that if a square matrix A satisfies the equation $A^3 - 2A + 3I = 0$, then the matrix A must be invertible.

Problem 22. Verify directly from the definition of inverse that the two by two inverse formula gives the inverse of a 2×2 matrix.

Problem 23. Assume that the product of invertible matrices is invertible and deduce that if A and B are invertible matrices of the same size and both B and AB are invertible, then so is A.

*Problem 24. Let A be an invertible matrix. Show that if the product of matrices AB is defined, then $\operatorname{rank}(AB) = \operatorname{rank}(B)$, and if BA is defined, then $\operatorname{rank}(BA) = \operatorname{rank}(B)$.

Problem 25. Prove that if $D = ABC$, where A, C, and D are invertible matrices, then B is invertible.

Problem 26. Given that $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in block form with A and B square, show that C is invertible if and only if A and B are, in which case $C^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$.

Problem 27. Let T be an upper triangular matrix, say $T = D + M$, where D is diagonal and M is strictly upper triangular.
(a) Show that if D is invertible, then $T = D(I - N)$, where $N = D^{-1}M$ is strictly upper triangular.
(b) Assume that D is invertible and use part (a) and Exercise 26 to obtain a formula for T^{-1} involving D and N.

Problem 28. Show that if the product of matrices BA is defined and A is invertible, then $\operatorname{rank}(BA) = \operatorname{rank}(B)$.

*Problem 29. Given the matrix $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where the blocks A and C are invertible matrices, find a formula for M^{-1} in terms of A, B, and C.

2.6 Basic Properties of Determinants

What Are They?

Many students have already had some experience with determinants and may have used them to solve square systems of equations. Why have we waited until now to introduce them? In point of fact, they are not really the best tool for solving systems. That distinction goes to Gaussian elimination. Were it not for the *theoretical* usefulness of determinants they might be consigned to a footnote in introductory linear algebra texts as a historical artifact of linear algebra.

To motivate determinants, consider Example 2.40. Something remarkable happened in that example. Not only were we able to find a formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, but we were able to compute a single number $D = ad - bc$ that told us whether A was invertible. The condition of noninvertibility, namely that $D = 0$, has a very simple interpretation: this happens exactly when one row of A is a multiple of the other, since the example showed that this is when elementary operations use the first row to zero out the second row. Can we extend this idea? Is there a single number that will tell us whether there are dependencies among the rows of the square matrix

A that cause its rank to be smaller than its row size? The answer is yes. This is exactly what determinants were invented for. The concept of determinant is subtle and not intuitive, and researchers had to accumulate a large body of experience before they were able to formulate a “correct” definition for this number. There are alternative definitions of determinants, but the following will suit our purposes. It is sometimes referred to as “expansion down the first column.”

Determinant

Definition 2.16. The *determinant* of a square $n \times n$ matrix $A = [a_{ij}]$ is the scalar quantity $\det A$ defined recursively as follows: if $n = 1$ then $\det A = a_{11}$; otherwise, we suppose that determinants are defined for all square matrices of size less than n and specify that

$$\begin{aligned}\det A &= \sum_{k=1}^{n} a_{k1}(-1)^{k+1} M_{k1}(A) \\ &= a_{11}M_{11}(A) - a_{21}M_{21}(A) + \cdots + (-1)^{n+1} a_{n1} M_{n1}(A),\end{aligned}$$

where $M_{ij}(A)$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and jth column of A.

Caution: The determinant of a matrix A is a scalar number. It is *not* a matrix quantity.

Example 2.42. Describe the quantities $M_{21}(A)$ and $M_{22}\,(A)$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Solution. If we erase the second row and first column of A we obtain something like

$$\begin{bmatrix} & 1 & 0 \\ & & \\ & 1 & 2 \end{bmatrix}.$$

Now collapse the remaining entries together to obtain the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Therefore

$$M_{21}(A) = \det \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Similarly, erase the second row and column of A to obtain

$$\begin{bmatrix} 2 & & 0 \\ & & \\ 0 & & 2 \end{bmatrix}.$$

Now collapse the remaining entries together to obtain

$$M_{22}(A) = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \qquad \square$$

Now how do we calculate these determinants? Part (b) of the next example answers the question.

Example 2.43. Use the definition to compute the determinants of the following matrices.

$$\text{(a) } [-4] \qquad \text{(b) } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{(c) } \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution. (a) From the first part of the definition we see that

$$\det[-4] = -4.$$

For (b) we set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and use the formula of the definition to obtain that

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a_{11}M_{11}(A) - a_{21}M_{21}(A) = a \det[d] - c \det[b] = ad - cb.$$

This calculation gives a handy formula for the determinant of a 2 × 2 matrix. For (c) use the definition to obtain that

$$\begin{aligned} \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} &= 2 \det \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + 0 \det \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \\ &= 2(1 \cdot 2 - 1 \cdot (-1)) - 1(1 \cdot 2 - 1 \cdot 0) + 0(1 \cdot (-1) - 1 \cdot 0) \\ &= 2 \cdot 3 - 1 \cdot 2 + 0 \cdot (-1) \\ &= 4. \end{aligned}$$

A point worth observing here is that we didn't really have to calculate the determinant of any matrix if it is multiplied by a zero. Hence, the more zeros our matrix has, the easier we expect the determinant calculation to be! □

Another common symbol for $\det A$ is $|A|$, which is also written with respect to the elements of A by suppressing matrix brackets:

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

This notation invites a certain oddity, if not abuse, of language: we sometimes refer to things like the "second row" or "$(2, 3)$th element" or the "size" of the determinant. Yet the determinant is only a number and in that sense doesn't really have rows or entries or a size. Rather, it is the underlying matrix whose determinant is being calculated that has these properties. So be careful of this notation; we plan to use it frequently because it's handy, but you should bear in mind that determinants and matrices are *not* the same thing! Another reason that this notation can be tricky is the case of a one-dimensional matrix, say $A = [a_{11}]$. Here it is definitely *not* a good idea to forget the brackets, since we already understand $|a_{11}|$ to be the absolute value of the scalar a_{11}, a nonnegative number. In the 1×1 case use $|[a_{11}]|$ for the determinant, which is just the number a_{11} and may be positive or negative.

Minors and Cofactors

The number $M_{ij}(A)$ is called the (i, j)th *minor* of the matrix A. If we collect the sign term in the definition of determinant together with the minor we obtain the (i, j)th *cofactor* $A_{ij} = (-1)^{i+j} M(A)$ of the matrix A. In the terminology of cofactors,

$$\det A = \sum_{k=1}^{n} a_{k1} A_{k1}.$$

Laws of Determinants

Our primary goal here is to show that determinants have the magical property we promised: a matrix is singular exactly when its determinant is 0. Along the way we will examine some useful properties of determinants. There is a lot of clever algebra that can be done here; we will try to keep matters straightforward (if that's possible with determinants). In order to focus on the main ideas, we place most of the proofs of key facts in the last section for optional reading. Also, a concise summary of the basic determinantal laws is given at the end of this section. Unless otherwise stated, we assume throughout this section that matrices are square, and that $A = [a_{ij}]$ is an $n \times n$ matrix.

For starters, let's observe that it's very easy to calculate the determinant of upper triangular matrices. Let A be such a matrix. Then $a_{k1} = 0$ if $k > 1$, so

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$
$$= \cdots = a_{11} \cdot a_{22} \cdots a_{nn}.$$

Hence we have established our first determinantal law:

D1: If A is an upper triangular matrix, then the determinant of A is the product of all the diagonal elements of A.

Example 2.44. Compute $D = \begin{vmatrix} 4 & 4 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{vmatrix}$ and $|I_n| = \det I_n$.

Solution. By D1 we can do this at a glance: $D = 4 \cdot (-1) \cdot 2 \cdot 2 = -16$. Since I_n is diagonal, it is certainly upper triangular. Moreover, the entries down the diagonal of this matrix are 1's, so D1 implies that $|I_n| = 1$. □

Next, suppose that we notice a common factor of the scalar c in a row, say for convenience, the first one. How does this affect the determinantal calculation? In the case of a 1×1 determinant, we could simply factor it out of the original determinant. The general situation is covered by this law:

D2: If B is obtained from A by multiplying one row of A by the scalar c, then $\det B = c \cdot \det A$.

Here is a simple illustration:

Example 2.45. Compute $D = \begin{vmatrix} 5 & 0 & 10 \\ 5 & 5 & 5 \\ 0 & 0 & 2 \end{vmatrix}$.

Solution. Put another way, D2 says that scalars may be factored out of individual rows of a determinant. So use D2 on the first and second rows and then use the definition of determinant to obtain

$$\begin{vmatrix} 5 & 0 & 10 \\ 5 & 5 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 5 \cdot \begin{vmatrix} 1 & 0 & 2 \\ 5 & 5 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 5 \cdot 5 \cdot \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 25 \cdot \left(1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \right)$$
$$= 50.$$

One can easily check that this is the same answer we get by working the determinant directly from the definition. □

Next, suppose we interchange two rows of a determinant.

D3: If B is obtained from A by interchanging two rows of A, then $\det B = -\det A$.

Example 2.46. Use D3 to show the following handy fact: if a determinant has a repeated row, then it must be 0.

Solution. Suppose that the ith and jth rows of the matrix A are identical, and B is obtained by switching these two rows of A. Clearly $B = A$. Yet, according to D3, $\det B = -\det A$. It follows that $\det A = -\det A$, i.e., if we add $\det A$ to both sides, $2 \cdot \det A = 0$, so that $\det A = 0$, which is what we wanted to show. □

What happens to a determinant if we add a multiple of one row to another?

D4: If B is obtained from A by adding a multiple of one row of A to another row of A, then $\det B = \det A$.

Example 2.47. Compute $D = \begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix}$.

Solution. What D4 really says is that any elementary row operation $E_{ij}(c)$ can be applied to the matrix behind a determinant and the determinant will be unchanged. So in this case, add -1 times the first row to the second and $-\frac{1}{2}$ times the third row to the fourth, then apply D1 to obtain

$$\begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 & 1 \\ 0 & -5 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1/2 \end{vmatrix} = 1 \cdot (-5) \cdot 2 \cdot \frac{1}{2} = -5.$$ □

Example 2.48. Use D3 to show that a matrix with a row of zeros has zero determinant.

Solution. Suppose A has a row of zeros. Add any other row of the matrix A to this zero row to obtain a matrix B with repeated rows. □

We now have enough machinery to establish the most important property of determinants. First of all, we can restate laws D2–D4 in the language of elementary matrices as follows:

Determinant of Elementary Matrices

- D2: $\det(E_i(c)A) = c \cdot \det A$ (remember that for $E_i(c)$ to be an elementary matrix, $c \neq 0$).
- D3: $\det(E_{ij}A) = -\det A$.
- D4: $\det(E_{ij}(s)A) = \det A$.

Apply a sequence of elementary row operations on the $n \times n$ matrix A to reduce it to its reduced row echelon form R, or equivalently, multiply A on the left by elementary matrices $E_1, E_2, \ldots, E_k$ and obtain

$$R = E_1 E_2 \cdots E_k A.$$

Take the determinant of both sides to obtain

$$\det R = \det(E_1 E_2 \cdots E_k A) = \pm(\text{nonzero constant}) \cdot \det A.$$

Therefore, $\det A = 0$ precisely when $\det R = 0$. Now the reduced row echelon form of A is certainly upper triangular. In fact, it is guaranteed to have zeros on the diagonal, and therefore have zero determinant by D1, unless $\operatorname{rank} A = n$, in which case $R = I_n$. According to Theorem 2.7 this happens precisely when A is invertible. Thus:

D5: The matrix A is invertible if and only if $\det A \neq 0$.

Example 2.49. Determine whether the following matrices are invertible without actually finding the inverse.

$$\text{(a)}\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \qquad\qquad \text{(b)}\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution. Compute the determinants:

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2 \cdot 3 - 2 = 4,$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \cdot 1 - 1 \cdot 2 = 0.$$

Hence by D5, matrix (a) is invertible and matrix (b) is not invertible. □

There are two more surprising properties of determinants that we now discuss. Their proofs involve using determinantal properties of elementary matrices (see the next section for details).

D6: Given matrices A, B of the same size,

$$\det AB = \det A \det B.$$

Example 2.50. Verify D6 in the case that $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. How do $\det(A+B)$ and $\det A + \det B$ compare in this case?

Solution. We have easily that $\det A = 1$ and $\det B = 2$. Therefore, $\det A + \det B = 1 + 2 = 3$, while $\det A \cdot \det B = 1 \cdot 2 = 2$. On the other hand,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \\ A + B &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \end{aligned}$$

so that $\det AB = 2 \cdot 3 - 4 \cdot 1 = 2 = \det A \cdot \det B$, as expected. On the other hand, we have that $\det(A+B) = 3 \cdot 2 - 1 \cdot 1 = 5 \neq \det A + \det B$. □

This example raises a very important point.

Caution: In general, $\det A + \det B \neq \det(A+B)$, though there are occasional exceptions.

In other words, determinants do not distribute over sums. (It is true, however, that the determinant is additive in *one row at a time*. See the proof of D4 for details.)

Finally, we ask how $\det A^T$ compares to $\det A$. Simple cases suggest that there is no difference in determinant. This is exactly what happens in general.

D7: For all square matrices A, $\det A^T = \det A$.

Example 2.51. Compute $D = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & 2 & -2 & 0 \\ 1 & 0 & 1 & 2 \end{vmatrix}$.

Solution. By D7 and D1 we see immediately that $D = 4 \cdot 1 \cdot (-2) \cdot 2 = -16$. □

D7 is a very useful fact. Let's look at it from this point of view: transposing a matrix interchanges the rows and columns of the matrix. Therefore, everything that we have said about rows of determinants applies equally well to the columns, *including the definition of determinant itself!* Therefore, we could have given the definition of determinant in terms of expanding across the first row instead of down the first column and gotten the same answers. Likewise, we could perform elementary column operations instead of row operations and get the same results as D2–D4. Furthermore, the determinant of a lower triangular matrix is the product of its diagonal elements thanks to D7+D1. By interchanging rows or columns then expanding by first row or column, we see that the same effect is obtained by simply expanding the determinant down any column or across any row. We have to alternate signs starting with the sign $(-1)^{i+j}$ of the first term we use.

Now we can really put it all together and compute determinants to our heart's content with a good deal less effort than the original definition specified. We can use D1–D4 in particular to make a determinant calculation no worse than Gaussian elimination in the amount of work we have to do. We simply reduce a matrix to triangular form by elementary operations, then take the product of the diagonal terms.

Example 2.52. Calculate $D = \begin{vmatrix} 3 & 0 & 6 & 6 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{vmatrix}$.

Solution. We are going to do this calculation two ways. We may as well use the same elementary operation notation that we have employed in Gaussian elimination. The only difference is that we have equality instead of arrows, provided that we modify the value of the new determinant in accordance with the laws D1–D3. So here is the straightforward method:

$$D = 3 \begin{vmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{vmatrix} \underset{\substack{E_{21}(-1) \\ E_{31}(-2) \\ E_{41}(1)}}{=} 3 \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -4 & -3 \\ 0 & 2 & 2 & 2 \end{vmatrix} \underset{E_{24}}{=} -3 \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -4 & -3 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -24.$$

Here is another approach: let's expand the determinant down the second column, since it is mostly 0's. Remember that the sign in front of the first minor must be $(-1)^{1+2} = -1$. Also, the coefficients of the first three minors are 0, so we need only write down the last one in the second column:

$$D = +2 \begin{vmatrix} 3 & 6 & 6 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix}.$$

Expand down the second column again:

$$D = 2\left(-6\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix}\right) = 2(-6 \cdot (-1) + 2 \cdot (-9)) = -24. \qquad \square$$

An Inverse Formula

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We have already seen that we can expand the determinant of A down any column of A (see the discussion following Example 2.51). These expansions lead to cofactor formulas for each column number j:

$$\det A = \sum_{k=1}^{n} a_{kj} A_{kj} = \sum_{k=1}^{n} A_{kj} a_{kj}.$$

This formula resembles a matrix multiplication formula. Consider the slightly altered sum

$$\sum_{k=1}^{n} A_{ki} a_{kj} = A_{1i} a_{1j} + A_{2i} a_{2j} + \cdots + A_{ni} a_{nj}.$$

The key to understanding this expression is to realize that it is exactly what we would get if we replaced the ith column of the matrix A by its jth column and then computed the determinant of the resulting matrix by expansion down the ith column. But such a matrix has two equal columns and therefore has a zero determinant, which we can see by applying Example 2.46 to the transpose of the matrix and using D7. So this sum must be 0 if $i \neq j$. We can combine these two sums by means of the Kronecker delta ($\delta_{ij} = 1$ if $i = j$ and 0 otherwise) in the formula

Kronecker Delta

$$\sum_{k=1}^{n} A_{ki} a_{kj} = \delta_{ij} \det A.$$

In order to exploit this formula we make the following definitions:

Adjoint, Minor, and Cofactor Matrices

Definition 2.17. The *matrix of minors* of the $n \times n$ matrix $A = [a_{ij}]$ is the matrix $M(A) = [M_{ij}(A)]$ of the same size. The *matrix of cofactors* of A is the matrix $A_{\text{cof}} = [A_{ij}]$ of the same size. Finally, the *adjoint matrix* of A is the matrix $\operatorname{adj} A = A_{\text{cof}}^{T}$.

Example 2.53. Compute the determinant, minors, cofactors, and adjoint matrices for $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$ and compute $A \operatorname{adj} A$.

Solution. The determinant is easily seen to be 2. Now for the matrix of minors:

$$M(A) = \begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ -2 & -1 & 0 \end{bmatrix}.$$

To get the matrix of cofactors, simply overlay $M(A)$ with the following "checkerboard" of $+/-$'s $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ to obtain the matrix $A_{\text{cof}} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & -2 \\ -2 & 1 & 0 \end{bmatrix}$. Now transpose A_{cof} to obtain

$$\operatorname{adj} A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}.$$

We check that

$$A \operatorname{adj} A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (\det A) I_3. \qquad \square$$

Of course, the example simply confirms the formula that preceded it since this formula gives the (i, j)th entry of the product $(\operatorname{adj} A)A$. If we were to do determinants by row expansions, we would get a similar formula for the (i, j)th entry of $A \operatorname{adj} A$. We summarize this information in matrix notation as the following determinantal property:

D8: For a square matrix A,

Adjoint Formula

$$A \operatorname{adj} A = (\operatorname{adj} A)A = (\det A)I.$$

What does this have to do with inverses? We already know that A is invertible exactly when $\det A \neq 0$, so the answer is staring at us! Just divide the terms in D8 by $\det A$ to obtain an explicit formula for A^{-1}:

D9: For a square matrix A such that $\det A \neq 0$,

Inverse Formula

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Example 2.54. Compute the inverse of the matrix A of Example 2.53 by the inverse formula.

Solution. We already computed the adjoint matrix of A, and the determinant of A is just 2, so we have that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{2} \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}. \qquad \square$$

Example 2.55. Interpret the inverse formula in the case of the 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution. We have $M(A) = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$, $A_{\text{cof}} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ and $\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, so that the inverse formula becomes

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

As you might expect, this is exactly the same as the formula we obtained in Example 2.40. □

Cramer's Rule

Thanks to the inverse formula, we can now find an explicit formula for solving linear systems with a nonsingular coefficient matrix. Here's how we proceed. To solve $A\mathbf{x} = \mathbf{b}$ we multiply both sides on the left by A^{-1} to obtain that $\mathbf{x} = A^{-1}\mathbf{b}$. Now use the inverse formula to obtain

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det A} \operatorname{adj}(A)\mathbf{b}.$$

The explicit formula for the ith coordinate of $\mathbf{x}$ that comes from this fact is

$$x_i = \frac{1}{\det A} \sum_{j=1}^{n} A_{ji} b_j.$$

The summation term is exactly what we would obtain if we started with the determinant of the matrix B_i obtained from A by replacing the ith column of A by $\mathbf{b}$ and then expanding the determinant down the ith column. Therefore, we have arrived at the following rule:

Theorem 2.8. Let A be an invertible $n \times n$ matrix and $\mathbf{b}$ an $n \times 1$ column vector. Denote by B_i the matrix obtained from A by replacing the ith column of A by $\mathbf{b}$. Then the linear system $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where

Cramer's Rule

$$x_i = \frac{\det B_i}{\det A}, \quad i = 1, 2, \ldots, n.$$

Example 2.56. Use Cramer's rule to solve the system

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ 4x_1 + 4x_2 &= 20. \end{aligned}$$

Solution. The coefficient matrix and right-hand-side vectors are

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 20 \end{bmatrix},$$

so that

$$\det A = 8 - (-4) = 12,$$

and therefore

$$x_1 = \frac{\begin{vmatrix} 2 & 1 \\ 4 & 20 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 4 & 4 \end{vmatrix}} = \frac{36}{12} = 3 \quad \text{and} \quad x_2 = \frac{\begin{vmatrix} 1 & -1 \\ 20 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 4 & 4 \end{vmatrix}} = \frac{24}{12} = 2.$$ □

Summary of Determinantal Laws

Now that our list of the basic laws of determinants is complete, we record them in a concise summary.

Laws of Determinants

Let A, B be $n \times n$ matrices.

D1: If A is upper triangular, $\det A$ is the product of all the diagonal elements of A.
D2: $\det(E_i(c)A) = c \cdot \det A$.
D3: $\det(E_{ij}A) = -\det A$.
D4: $\det(E_{ij}(s)A) = \det A$.
D5: The matrix A is invertible if and only if $\det A \neq 0$.
D6: $\det AB = \det A \det B$.
D7: $\det A^T = \det A$.
D8: $A \operatorname{adj} A = (\operatorname{adj} A)A = (\det A)I$.
D9: If $\det A \neq 0$, then $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$.

2.6 Exercises and Problems

Exercise 1. Compute all cofactors for these matrices.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 - \mathrm{i} \\ 0 & 1 \end{bmatrix}$

Exercise 2. Compute all minors for these matrices.

(a) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & \mathrm{i}+1 \\ \mathrm{i} & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 3. Compute these determinants. Which of the matrices represented are invertible?

(a) $\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1+i \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 7 \\ -2 & 3 & 4 & 6 \end{vmatrix}$ (e) $\begin{vmatrix} -1 & -1 \\ 1 & 1-2\mathrm{i} \end{vmatrix}$

Exercise 4. Use determinants to determine which of these matrices are invertible.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & 3 & 4 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 1 & 1 & 2 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Exercise 5. Verify by calculation that determinantal law D7 holds for the following choices of A.

(a) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

Exercise 6. Let $A = B$ and verify by calculation that determinantal law D6 holds for the following choices of A.

(a) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 7 \end{bmatrix}$

Exercise 7. Use determinants to find conditions on the parameters in these matrices under which the matrices are invertible.

(a) $\begin{bmatrix} a & 1 \\ ab & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & c & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

Exercise 8. Find conditions on the parameters in these matrices under which the matrices are invertible.

(a) $\begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & a \\ 0 & 0 & -a & b \end{bmatrix}$ (b) $\begin{bmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ 3 & 1 & \lambda-1 \end{bmatrix}$ (c) $\lambda I_2 - \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix}$

Exercise 9. For each of the following matrices calculate the adjoint matrix and the product of the matrix and its adjoint.

$$\text{(a)} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Exercise 10. For each of the following matrices calculate the adjoint matrix and the product of the adjoint and the matrix.

$$\text{(a)} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Exercise 11. Find the inverse of following matrices by adjoints.

$$\text{(a)} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -1 \\ 1 & -3 & 1 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & \mathrm{i} \\ -2\mathrm{i} & 1 \end{bmatrix}$$

Exercise 12. For each of the following matrices, find the inverse by superaugmented matrices and by adjoints.

$$\text{(a)} \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & -4 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

Exercise 13. Use Cramer's Rule to solve the following systems.

$$\text{(a)} \begin{aligned} x - 3y &= 2 \\ 2x + y &= 11 \end{aligned} \qquad \text{(b)} \begin{aligned} 2x_1 + x_2 &= b_1 \\ 2x_1 - x_2 &= b_2 \end{aligned} \qquad \text{(c)} \begin{aligned} 3x_1 + x_3 &= 2 \\ 2x_1 + 2x_2 &= 1 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

Exercise 14. Use Cramer's Rule to solve the following systems.

$$\text{(a)} \begin{aligned} x + y + z &= 4 \\ 2x + 2y + 5z &= 11 \\ 4x + 6y + 8z &= 24 \end{aligned} \qquad \text{(b)} \begin{aligned} x_1 - 2x_2 &= 2 \\ 2x_1 - x_2 &= 4 \end{aligned} \qquad \text{(c)} \begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + 2x_2 &= 1 \\ x_1 - x_3 &= 2 \end{aligned}$$

Problem 15. Verify that $\begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix}$.

Problem 16. Confirm that the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is -1.

We can now assert without any further calculation that the inverse matrix of A has integer coefficients. Explain why in terms of laws of determinants.

*Problem 17. Let

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}.$$

(Such a matrix is called a *Vandermonde* matrix.) Express $\det V$ as a product of factors $(x_j - x_k)$.

Problem 18. Show by example that $\det A^* \neq \det A$ and prove that in general $\det A^* = \overline{\det A}$.

*Problem 19. Use a determinantal law to show that $\det(A)\det\left(A^{-1}\right) = 1$ if A is invertible.

Problem 20. Use the determinantal laws to show that any matrix with a row of zeros has zero determinant.

*Problem 21. If A is a 5×5 matrix, then in terms of $\det(A)$, what can we say about $\det(-2A)$? Explain and express a law about a general matrix cA, c a scalar, that contains your answer.

Problem 22. Let A be a skew-symmetric matrix, that is, $A^T = -A$. Show that if A has odd order n, i.e., A is $n \times n$, then A must be singular.

*Problem 23. Show that if

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

then $\det M = \det A \cdot \det C$.

*Problem 24. Let J_n be the $n \times n$ counteridentity, that is, J_n is a square matrix with ones along the counterdiagonal (the diagonal that starts in the lower left corner and ends in the upper right corner), and zeros elsewhere. Find a formula for $\det J_n$. (Hint: show that $J_n^2 = I_n$, which narrows down $\det J_n$.)

Problem 25. Show that the *companion matrix* of the polynomial $f(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + x^n$, which is defined to be

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix},$$

is invertible if and only if $c_0 \neq 0$.

Prove that if the matrix A is invertible, then $\det(A^T A) > 0$.

Problem 26. Suppose that the square matrix A is singular. Prove that if the system $A\mathbf{x} = \mathbf{b}$ is consistent, then $(\operatorname{adj} A)\,\mathbf{b} = \mathbf{0}$.

2.7 *Computational Notes and Projects

LU Factorization

Here is a problem: suppose we want to solve a nonsingular linear system $Ax = b$ repeatedly, with different choices of b. A perfect example of this kind of situation is the heat flow problem Example 1.3, where the right-hand side is determined by the heat source term $f(x)$. Suppose that we need to experiment with different source terms. What happens if we do straight Gaussian elimination or Gauss–Jordan elimination? Each time we carry out a complete calculation on the augmented matrix $\widetilde{A} = [A \,|\, b]$ we have to resolve the whole system. Yet, the main part of our work is the same: putting the part of $\widetilde{A}$ corresponding to the coefficient matrix A into reduced row echelon form. Changing the right-hand side has no effect on this work. What we want here is a way to somehow record our work on A, so that solving a new system involves very little additional work. This is exactly what the LU factorization is all about.

Definition 2.18. Let A be an $n \times n$ matrix. An LU factorization of A is a pair of $n \times n$ matrices L, U such that

LU Factorization

(1) L is lower triangular.
(2) U is upper triangular.
(3) $A = LU$.

Even if we could find such beasts, what is so wonderful about them? The answer is that *triangular* systems $A\mathbf{x} = \mathbf{b}$ are easy to solve. For example, if A is upper triangular, we learned that the smart thing to do was to use the last equation to solve for the last variable, then the next-to-last equation for the next-to-last variable, etc. This is the secret of Gaussian elimination! But lower triangular systems are just as simple: use the first equation to solve for the first variable, the second equation for the second variable, and so forth. Now suppose we want to solve $A\mathbf{x} = \mathbf{b}$ and we know that $A = LU$. The original system becomes $LU\mathbf{x} = \mathbf{b}$. Introduce an intermediate variable $\mathbf{y} = U\mathbf{x}$. Now perform these steps:

1. (Forward solve) Solve lower triangular system $L\mathbf{y} = \mathbf{b}$ for the variable $\mathbf{y}$.
2. (Back solve) Solve upper triangular system $U\mathbf{x} = \mathbf{y}$ for the variable $\mathbf{x}$.

This does it! Once we have the matrices L, U, we don't have to worry about right-hand sides, except for the small amount of work involved in solving two triangular systems. Notice, by the way, that since A is assumed nonsingular, we have that if $A = LU$, then $\det A = \det L \det U \neq 0$. Therefore, neither triangular matrix L or U can have zeros on its diagonal. Thus, the forward and back solve steps can always be carried out to give a unique solution.

Example 2.57. You are given that

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & -1 \\ 2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Use this fact to solve $Ax = b$, where $\mathbf{b} = [1, 0, 1]^T$ or $\mathbf{b} = [-1, 2, 1]^T$.

Solution. Set $\mathbf{x} = [x_1, x_2, x_3]^T$ and $\mathbf{y} = [y_1, y_2, y_3]^T$. For $\mathbf{b} = [1, 0, 1]^T$, forward solve

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

to get $y_1 = 1$, then $y_2 = 0 + 1y_1 = 1$, then $y_3 = 1 - 1y_1 - 2y_2 = -2$. Then back solve

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

to get $x_3 = -2/(-1) = 2$, then $x_2 = 1 + x_3 = 3$, then $x_1 = (1 - 1x_2)/2 = -1$.

For (b) forward solve

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

to get $y_1 = -1$, then $y_2 = 0 + 1y_1 = -1$, then $y_3 = 1 - 1y_1 - 2y_2 = 4$. Then back solve

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

to get $x_3 = 4/(-1) = -4$, then $x_2 = 1 + x_3 = -3$, then $x_1 = (1 - 1x_2)/2 = 2$. □

Notice how simple the previous example was, given the LU factorization. Now how do we find such a factorization? In general, a nonsingular matrix may not have such a factorization. A good example is the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ However, if Gaussian elimination can be performed on the matrix A *without row exchanges*, then such a factorization is really a by-product of Gaussian elimination. In this case let $[a_{ij}^{(k)}]$ be the matrix obtained from A after using the kth pivot to clear out entries below it (thus $A = [a_{ij}^{(0)}]$). Remember that in Gaussian elimination we need only two types of elementary operations, namely row exchanges and adding a multiple of one row to another. Furthermore, the only elementary operations of the latter type that we use are of this form: $E_{ij}(-a_{jj}^{(k)}/a_{ij}^{(k)})$, where $[a_{ij}^{(k)}]$ is the matrix obtained from A from the various elementary operations up to this point. The numbers $m_{ij} = -a_{jj}^{(k)}/a_{ij}^{(k)}$, where $i > j$, are sometimes called *multipliers*. In the way of notation, let us call a triangular matrix a *unit* triangular matrix if its diagonal entries are all 1's.

Multipliers

Theorem 2.9. If Gaussian elimination is used without row exchanges on the nonsingular matrix A, resulting in the upper triangular matrix U, and if L is the unit lower triangular matrix whose entries below the diagonal are the negatives of the multipliers m_{ij}, then $A = LU$.

Proof. The proof of this theorem amounts to noticing that the product of all the elementary operations that reduces A to U is a unit lower triangular matrix $\widetilde{L}$ with the multipliers m_{ij} in the appropriate positions. Thus $\widetilde{L}A = U$. To undo these operations, multiply by a matrix L with the negatives of the multipliers in the appropriate positions. This results in

$$L\widetilde{L}A = A = LU$$

as desired. □

The following example shows how one can write an efficient program to implement LU factorization. The idea is this: as we do Gaussian elimination, the U part of the factorization gradually appears in the upper parts of the transformed matrices $A^{(k)}$. Below the diagonal we replace nonzero entries with zeros, column by column. Instead of wasting this space, use it to store the negative of the multipliers in place of the element it zeros out. Of course, this storage part of the matrix should not be changed by subsequent elementary row operations. When we are finished with elimination, the diagonal and upper part of the resulting matrix is just U, and the strictly lower triangular part on the unit lower triangular matrix L is stored in the lower part of the matrix.

Example 2.58. Use the shorthand of the preceding discussion to compute an LU factorization for

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & -1 \\ 2 & 3 & -3 \end{bmatrix}.$$

Solution. Proceed as in Gaussian elimination, but store negative multipliers:

$$\begin{bmatrix} \textcircled{2} & 1 & 0 \\ -2 & 0 & -1 \\ 2 & 3 & -3 \end{bmatrix} \xrightarrow[E_{31}(-1)]{E_{21}(1)} \begin{bmatrix} 2 & 1 & 0 \\ -1 & \textcircled{1} & -1 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow[E_{32}(-2)]{} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}.$$

Now we read off the results from the last matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$ □

What can be said if row exchanges are required (for example, we might want to use a partial pivoting strategy)? Take the point of view that we could see our way to the end of Gaussian elimination and store the product P of all row-exchanging elementary operations that we use along the way. A product of such matrices is called a *permutation matrix*; such a matrix is invertible, since

Permutation Matrix

it is a product of invertible matrices. Thus if we apply the correct permutation matrix P to A we obtain a matrix for which Gaussian elimination will succeed without further row exchanges. Consequently, we have a theorem that applies to all nonsingular matrices. Notice that it does not limit the usefulness of LU factorization since the linear system $Ax = b$ is equivalent to the system $PAx = Pb$. The following theorem could be called the "PLU factorization theorem."

Theorem 2.10. If A is a nonsingular matrix, then there exists a permutation matrix P, upper triangular matrix U, and unit lower triangular matrix L such that $PA = LU$.

There are many other useful factorizations of matrices that numerical analysts have studied, e.g., LDU and Cholesky. We will stop at LU, but there is one last point we want to make. The amount of work in finding the LU factorization is the same as Gaussian elimination itself, which is approximately $2n^3/3$ flops (see Section 1.5). The additional work of back and forward solving is about $2n^2$ flops. So the dominant amount of work is done by computing the factorization rather than the back and forward solving stages.

Efficiency of Determinants and Cramer's Rule in Computation

Computational Efficiency of Determinants

The truth of the matter is that Cramer's Rule and adjoints are good only for small matrices and theoretical arguments. For if you evaluate determinants in a straightforward way from the definition, the work in doing so is about $n \cdot n!$ flops for an $n \times n$ system. (Recall that a "flop" in numerical linear algebra is a single addition or subtraction, or multiplication or division.) For example, it is not hard to show that the operation of adding a multiple of one row vector of length n to another requires $2n$ flops. This number $n \cdot n!$ is vast when compared to the number $2n^3/3$ flops required for Gaussian elimination, even with "small" n, say $n = 10$. In this case we have $2 \cdot 10^3/3 \approx 667$, while $10 \cdot 10! = 36,288,000$.

On the other hand, there is a clever way to evaluate determinants that requires much less work than the definition: use elementary row operations together with D2, D6, and the elementary operations that correspond to these rules to reduce the determinant to that of a triangular matrix. This requires about $2n^3/3$ flops. As a matter of fact, it is tantamount to Gaussian elimination. But to use Cramer's Rule, you will have to calculate $n+1$ determinants. So why bother with Cramer's Rule on larger problems when it still will take about n times as much work as Gaussian elimination? A similar remark applies to computing adjoints instead of using Gauss–Jordan elimination on the superaugmented matrix of A.

Proofs of Some of the Laws of Determinants

D2: If B is obtained from A by multiplying one row of A by the scalar c, then $\det B = c \cdot \det A$.

To keep the notation simple, assume that the first row is multiplied by c, the proof being similar for other rows. Suppose we have established this for all determinants of size less than n (this is really another "proof by induction," which is how most of the following determinantal properties are established). For an $n \times n$ determinant we have

$$\begin{aligned}\det B &= \begin{vmatrix} c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\ &= c \cdot a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \sum_{k=2}^{n} a_{k1}(-1)^{k+1} M_{k1}(B).\end{aligned}$$

But the minors $M_{k1}(B)$ all are smaller and have a common factor of c in the first row. Pull this factor out of every remaining term and we get that

$$\begin{vmatrix} c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Thus we have shown that property D2 holds for all matrices.

D3: If B is obtained from A by interchanging two rows of A, then $\det B = -\det A$.

To keep the notation simple, assume we switch the first and second rows. In the case of a 2×2 determinant, we get the negative of the original determinant (check this for yourself). Suppose we have established that the same is true for all matrices of size less than n. For an $n \times n$ determinant we have

$$\begin{aligned}\det B &= \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\ &= a_{21} M_{11}(B) - a_{11} M_{21}(B) + \sum_{k=3}^{n} a_{k1}(-1)^{k+1} M_{k1}(B) \\ &= a_{21} M_{21}(A) - a_{11} M_{11}(A) + \sum_{k=3}^{n} a_{k1}(-1)^{k+1} M_{k1}(B).\end{aligned}$$

But all the determinants in the summation sign come from a submatrix of A with the first and second rows interchanged. Since they are smaller than n, each is just the negative of the corresponding minor of A. Notice that the first two terms are just the first two terms in the determinantal expansion of A, except that they are out of order and have an extra minus sign. Factor this minus sign out of every term and we have obtained D3. □

D4: If B is obtained from A by adding a multiple of one row of A to another row of A, then $\det B = \det A$.

Actually, it's a little easier to answer a slightly more general question: what happens if we replace a row of a determinant by that row plus some other row vector $\mathbf{r}$ (not necessarily a row of the determinant)? Again, simply for convenience of notation, we assume that the row in question is the first. The same argument works for any other row. Some notation: let B be the matrix that we obtain from the $n \times n$ matrix A by adding the row vector $\mathbf{r} = [r_1, r_2, \ldots, r_n]$ to the first row and C the matrix obtained from A by replacing the first row by $\mathbf{r}$. The answer turns out to be that $|B| = |A| + |C|$. So we can say that the determinant function is "additive in each row." Let's see what happens in the one dimensional case:

$$|B| = |[a_{11} + r_1]| = a_{11} + r_1 = |[a_{11}]| + |[r_1]| = |A| + |C|.$$

Suppose we have established that the same is true for all matrices of size less than n and let A be $n \times n$. Then the minors $M_{k1}(B)$, with $k > 1$, are smaller than n, so the property holds for them. Hence we have

$$\begin{aligned}
\det B &= \begin{vmatrix} a_{11}+r_1 & a_{12}+r_2 & \cdots & a_{1n}+r_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&= (a_{11}+r_1)M_{11}(A) + \sum_{k=2}^{n} a_{k1}(-1)^{k+1}M_{k1}(B) \\
&= (a_{11}+r_1)M_{11}(A) + \sum_{k=2}^{n} a_{k1}(-1)^{k+1}(M_{k1}(A)+M_{k1}(C)) \\
&= \sum_{k=1}^{n} a_{k1}(-1)^{k+1}M_{k1}(A) + r_1 M_{11}(C) + \sum_{k=2}^{n} a_{k1}(-1)^{k+1}M_{k1}(C) \\
&= \det A + \det C.
\end{aligned}$$

Now what about adding a multiple of one row to another in a determinant? For notational convenience, suppose we add s times the second row to the first. In the notation of the previous paragraph,

$$\det B = \begin{vmatrix} a_{11}+s\cdot a_{21} & a_{12}+s\cdot a_{22} & \cdots & a_{1n}+s\cdot a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and

$$\det C = \begin{vmatrix} s \cdot a_{21} & s \cdot a_{22} & \cdots & s \cdot a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = s \cdot \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0,$$

where we applied D2 to pull the common factor s from the first row and the result of Example 2.46 to get the determinant with repeated rows to be 0. But $|B| = |A| + |C|$. Hence we have shown D4. □

D6: Given matrices A, B of the same size, $\det AB = \det A \det B$.

The key here is that we now know that determinant calculation is intimately connected with elementary matrices, rank, and the reduced row echelon form. First let's reinterpret D2–D4 still one more time. First of all take $A = I$ in the discussion of the previous paragraph, and we see that

- $\det E_i(c) = c$
- $\det E_{ij} = -1$
- $\det E_{ij}(s) = 1$

Therefore, D2–D4 can be restated (yet again) as

- D2: $\det(E_i(c)A) = \det E_i(c) \cdot \det A$ (here $c \neq 0$.)
- D3: $\det(E_{ij}A) = \det E_{ij} \cdot \det A$
- D4: $\det(E_{ij}(s) = \det E_{ij}(s) \cdot \det A$

In summary: For any elementary matrix E and arbitrary matrix A of the same size, $\det(EA) = \det(E)\det(A)$.

Now let's consider this question: how does $\det(AB)$ relate to $\det(A)$ and $\det(B)$? If A is not invertible, $\operatorname{rank} A < n$ by Theorem 2.7 and so $\operatorname{rank} AB < n$ by Corollary 2.2. Therefore, $\det(AB) = 0 = \det A \cdot \det B$ in this case. Next suppose that A is invertible. Express it as a product of elementary matrices, say $A = E_1 E_2 \cdots E_k$, and use our summary of D1–D3 to disassemble and reassemble the elementary factors:

$$\begin{aligned}\det(AB) &= \det(E_1 E_2 \cdots E_k B) = (\det E_1 \det E_2 \cdots \det E_k)\det B \\ &= \det(E_1 E_2 \cdots E_k)\det B = \det A \cdot \det B.\end{aligned}$$

Thus we have shown that **D6** holds. □

D7: For all square matrices A, $\det A^T = \det A$.

Recall these facts about elementary matrices:

- $\det E_{ij}^T = \det E_{ij}$
- $\det E_i(c)^T = \det E_i(c)$
- $\det E_{ij}(c)^T = \det E_{ji}(c) = 1 = \det E_{ij}(c)$

Therefore, transposing does not affect determinants of elementary matrices. Now for the general case observe that since A and A^T are transposes of each other, one is invertible if and only if the other is by the Transpose/Inverse law. In particular, if both are singular, then $\det A^T = 0 = \det A$. On the other hand, if both are nonsingular, then write A as a product of elementary matrices, say $A = E_1 E_2 \cdots E_k$, and obtain from the product law for transposes that $A^T = E_k^T E_{k-1}^T \dots E_1^T$, so by D6

$$\begin{aligned}\det A^T &= \det E_k^T \det E_{k-1}^T \cdots \det E_1^T = \det E_k \det E_{k-1} \cdots \det E_1 \\ &= \det E_1 \det E_2 \cdots \det E_k = \det A.\end{aligned}$$ □

Tensor Product of Matrices

How do we solve a system of equations in which the unknowns can be organized into a matrix X and the linear system in question is of the form

$$AX - XB = C, \tag{2.3}$$

Sylvester Equation

where A, B, C are given matrices? We call this equation the *Sylvester equation.* Such systems occur in a number of physical applications; for example, discretizing certain partial differential equations in order to solve them numerically can lead to such a system. Of course, we could simply expand each system laboriously. This direct approach offers us little insight as to the nature of the resulting system.

We are going to develop a powerful "bookkeeping" method that will rearrange the variables of Sylvester's equation automatically. The first basic idea needed here is that of the tensor product of two matrices, which is defined as follows:

Tensor Product

Definition 2.19. Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [b_{ij}]$ an $n \times q$ matrix. Then the *tensor product* of A and B is the $mn \times pq$ matrix $A \otimes B$ that can be expressed in block form as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1j}B & \cdots & a_{1p}B \\ a_{21}B & a_{22}B & \cdots & a_{2j}B & \cdots & a_{2p}B \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1}B & a_{i2}B & \cdots & a_{ij}B & \cdots & a_{ip}B \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mj}B & \cdots & a_{mp}B \end{bmatrix}.$$

Example 2.59. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Exhibit $A \otimes B$, $B \otimes A$, and $I_2 \otimes A$ and conclude that $A \otimes B \neq B \otimes A$.

Solution. From the definition,

$$A \otimes B = \begin{bmatrix} 1B & 3B \\ 2B & 1B \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ -1 & -3 \\ 8 & 4 \\ -2 & -1 \end{bmatrix}, \quad B \otimes A = \begin{bmatrix} 4A \\ -1A \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ -8 & -2 \\ -1 & -3 \\ -2 & -1 \end{bmatrix},$$

$$\text{and } I_2 \otimes A = \begin{bmatrix} 1A & 0A \\ 0A & 1A \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}. \qquad \square$$

The other ingredient that we need to solve equation (2.3) is an operator that turns matrices into vectors. It is defined as follows.

Definition 2.20. Let A be an $m \times n$ matrix. Then the $mn \times 1$ vector $\operatorname{vec} A$ is obtained from A by stacking the n columns of A vertically, with the first column at the top and the last column of A at the bottom.

Vec Operator

Example 2.60. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$. Compute $\operatorname{vec} A$.

Solution. There are two columns to stack, yielding $\operatorname{vec} A = [1, 2, 3, 1]^T$. □

The vec operator is linear ($\operatorname{vec}(aA + bB) = a \operatorname{vec} A + b \operatorname{vec} B$). We leave the proof, along proofs of the following simple tensor facts, to the reader.

Theorem 2.11. Let A, B, C, D be suitably sized matrices. Then

(1) $(A + B) \otimes C = A \otimes C + B \otimes C$
(2) $A \otimes (B + C) = A \otimes B + A \otimes C$
(3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
(4) $(A \otimes B)^T = A^T \otimes B^T$
(5) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
(6) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

The next theorem lays out the key bookkeeping between tensor products and the vec operator.

Theorem 2.12. If A, X, B are matrices conformable for multiplication, then

$$\operatorname{vec}(AXB) = \left(B^T \otimes A\right) \operatorname{vec} X.$$

Bookkeeping Theorem

Corollary 2.4. The following linear systems in the unknown X are equivalent.

(1) $A_1 X B_1 + A_2 X B_2 = C$
(2) $\left(\left(B_1^T \otimes A_1\right) + \left(B_2^T \otimes A_2\right)\right) \operatorname{vec} X = \operatorname{vec} C$

For Sylvester's equation, note that $AX - XB = IAX + (-I)XB$.

The following is a very basic application of the tensor product. Suppose we wish to model a two-dimensional heat diffusion process on a flat plate that occupies the unit square in the xy-plane. We proceed as we did in the one-dimensional process described in the introduction. To fix ideas, we assume that the heat source is described by a function $f(x,y), 0 \leq x \leq 1, 0 \leq y \leq 1$, and that the temperature is held at 0 at the boundary of the unit square. Also, the conductivity coefficient is assumed to be the constant k. Cover the square with a uniformly spaced set of grid points $(x_i, y_j), 0 \leq i,j \leq n+1$, called nodes, and assume that the spacing in each direction is a width $h = 1/(n+1)$. Also assume that the temperature function at the (i,j)th node is $u_{ij} = u(x_i, y_j)$ and that the source is $f_{ij} = f(x_i, y_j)$. Notice that the values of u on boundary grid points is set at 0. For example, $u_{01} = u_{20} = 0$. By balancing the heat flow in the horizontal and vertical directions, one arrives at a system of linear equations, one for each node, of the form

$$-u_{i-1,j} - u_{i+1,j} + 4u_{ij} - u_{i,j-1} - u_{i,j+1} = \frac{h^2}{k} f_{ij}, \quad i,j = 1,\ldots,n. \qquad (2.4)$$

Observe that values of boundary nodes are zero, so these are not unknowns, which is why the indexing of the equations starts at 1 instead of 0. There are exactly as many equations as unknown grid point values. Each equation has a "molecule" associated with it that is obtained by circling the nodes that occur in the equation and connecting these circles. A picture of a few nodes is given in Figure 2.7.

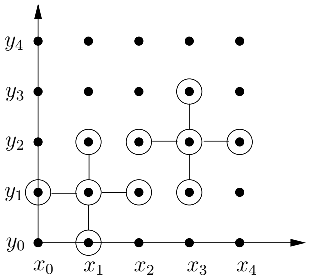


Fig. 2.7. Molecules for $(1,1)$th and $(3,2)$th grid points.

Example 2.61. Set up and solve a system of equations for the two-dimensional heat diffusion problem described above.

Solution. Equation (2.4) gives us a system of n^2 equations in the n^2 unknowns u_{ij}, $i,j = 1,2,\ldots,n$. Rewrite equation (2.4) in the form

$$(-u_{i-1,j} + 2u_{ij} - u_{i+1,j}) + (-u_{i,j-1} + 2u_{ij} - u_{i,j+1}) = \frac{h^2}{k} f_{ij}.$$

Now form the $n \times n$ matrices

$$T_n = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Set $U = [u_{ij}]$ and $F = [f_{ij}]$, and the system can be written in matrix form as

$$T_n U + U T_n = T_n U I_n + I_n U T_n = \frac{h^2}{k} F.$$

However, we can't as yet identify a coefficient matrix, which is where Corollary 2.4 comes in handy. Note that both I_n and T_n are symmetric and apply the corollary to obtain that the system has the form

$$(I_n \otimes T_n + T_n \otimes I_n) \operatorname{vec} U = \operatorname{vec} \frac{h^2}{k} F.$$

Now we have a coefficient matrix, and what's more, we have an automatic ordering of the doubly indexed variables u_{ij}, namely

$$u_{1,1}, u_{2,1}, \ldots, u_{n,1}, u_{1,2}, u_{2,2}, \ldots, u_{n,2}, \ldots, u_{1,n}, u_{2,n}, \ldots, u_{n,n}.$$

This is sometimes called the "row ordering," which refers to the rows of the nodes in Figure 2.7, and not the rows of the matrix U. □

Here is one more example of a problem in which tensor notation is an extremely helpful bookkeeper. This is a biological model that gives rise to an inverse theory problem. ("Here's the answer, what's the question?")

Example 2.62. Refer to Example 2.20, where a three-state insect (egg, juvenile, adult) is studied in stages spaced at intervals of two days. One might ask how the entries of the matrix were derived. Clearly, observation plays a role. Let us suppose that we have taken samples of the population at successive stages and recorded our estimates of the population state. Suppose we have estimates of states $\mathbf{x}^{(0)}$ through $\mathbf{x}^{(4)}$. How do we translate these observations into transition matrix entries?

Solution. We postulate that the correct transition matrix has the form

$$A = \begin{bmatrix} P_1 & 0 & F \\ G_1 & P_2 & 0 \\ 0 & G_2 & P_3 \end{bmatrix}.$$

Theoretically, we have the transition equation $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$ for $k = 0, 1, 2, 3$. Remember that this is an inverse problem, where the "answers," population states $\mathbf{x}^{(k)}$, are given, and the question "What are populations given A?" is unknown. We could simply write out each transition equation and express the results as linear equations in the unknown entries of A. However, this is laborious and not practical for problems involving many states or larger amounts of data.

Here is a better idea: assemble all of the transition equations into a single matrix equation by setting

$$M = \left[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}\right] = [m_{ij}] \quad \text{and} \quad N = \left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)}\right] = [n_{ij}]\,.$$

The entire ensemble of transition equations becomes $AM = N$ with M and N known matrices. Here A is 3×3 and both M, N are 3×4. Next, write the transition equation as $I_3 AM = N$ and invoke the bookkeeping theorem to obtain the system

$$\operatorname{vec}(I_3 AM) = \left(M^T \otimes I_3\right) \operatorname{vec} A = \operatorname{vec} N.$$

This is a system of 12 equations in 9 unknowns. We can simplify it a bit by deleting the third, fourth, and eighth entries of $\operatorname{vec} A$ and the same columns of the coefficient matrix, since we know that the variables a_{31}, a_{12}, and a_{23} are zero. We thus end up with a system of 12 equations in 6 unknowns, which will determine the nonzero entries of A. □

Project Topics

Project: LU Factorization

Write a program module that implements Theorem 2.10 using partial pivoting and implicit row exchanges. This means that space is allocated for the $n \times n$ matrix $A = [a[i, j]]$ and an array of row indices, say indx$[i]$. Initially, indx should consist of the integers $1, 2, \ldots, n$. Whenever two rows need to be exchanged, say the first and third, then the indices indx[1] and indx[3] are exchanged. References to array elements throughout the Gaussian elimination process should be indirect: refer to the $(1, 4)$th entry of A as the element $a\,[\text{indx}[1], 4]$. This method of reference has the same effect as physically exchanging rows, but without the work. It also has the appealing feature that we can design the algorithm as though no row exchanges have taken place provided we replace the direct reference $a[i, j]$ by the indirect reference $a[\text{indx}[i], j]$. The module should return the lower/upper matrix in the format of Example 2.58 as well as the permuted array indx$[i]$. Effectively, this index array tells the user what the permutation matrix P is.

Write an LU system solver module that uses the LU factorization to solve a general linear system. Also write a module that finds the inverse of an $n \times n$ matrix A by first using the LU factorization module, then making repeated use of the LU system solver to solve $A\mathbf{x}^{(i)} = \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith column of the identity. Then we will have

$$A^{-1} = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}].$$

Be sure to document and test your code and report on the results.

Project: Markov Chains

Refer to Example 2.18 and Section 2.3 for background. Three automobile insurance firms compete for a fixed market of customers. Annual premiums are sold to these customers. Label the companies A, B, and C. You work for Company A, and your team of market analysts has done a survey that draws the following conclusions: in each of the past three years, the number of A customers switching to B is 20%, and to C is 30%. The number of B customers switching to A is 20%, and to C is 20%. The number of C customers switching to A is 30%, and to B is 10%. Those who do not switch continue to use their current company's insurance for the next year. Model this market as a Markov chain. Display the transition matrix for the model. Illustrate the workings of the model by showing what it would predict as the market shares three years from now if currently A, B, and C owned equal shares of the market.

The next part of your problem is as follows: your team has tested two advertising campaigns in some smaller test markets and are confident that the first campaign will convince 20% of the B customers who would otherwise stay with B in a given year to switch to A. The second advertising campaign would convince 20% of the C customers who would otherwise stay with C in a given year to switch to A. Both campaigns have about equal costs and would not change other customers' habits. Make a recommendation, based on your experiments with various possible initial state vectors for the market. Will these campaigns actually improve your company's market share? If so, which one do you recommend? Write up your recommendation in the form of a report, with supporting evidence. It's a good idea to hedge on your bets a little by pointing out limitations to your model and claims, so devote a few sentences to those points.

It would be a plus to carry the analysis further (your manager might appreciate that). For instance, you could turn the additional market share from, say B customers, into a variable and plot the long-term gain for your company against this variable. A manager could use this data to decide whether it was worthwhile to attempt gaining more customers from B.

Project: Affine Transforms in Real-Time Rendering

Refer to the examples in Section 2.3 for background. Graphics specialists find it important to distinguish between vector objects and point objects in three-dimensional space. They simultaneously manipulate these two kinds of objects with invertible linear operators, which they term *transforms*. To this end, they use the following clever ruse: identify three-dimensional vectors in the usual way, that is, by their coordinates x_1, x_2, x_3. Do the same with three-dimensional points. To distinguish between the two, embed them in the set of 4×1 vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$, called *homogeneous vectors*, with the understanding that if $x_4 = 0$, then $\mathbf{x}$ represents a three-dimensional vector object, and if $x_4 \neq 0$, then the vector represents a three-dimensional point whose coordinates are $x_1/x_4, x_2/x_4, x_3/x_4$.

Homogeneous Vector

Transforms (invertible linear operators) have the general form

$$T_M(\mathbf{x}) = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Homogeneous and Affine Transforms

If $m_{44} = 1$ and the remaining entries of the last row and column are zero, the transform is called a *homogeneous transform.* If $m_{44} = 1$ and the remaining entries of the last row are zero, the transform is called *affine.* If the transform matrix M takes the block form $M = \begin{bmatrix} I_3 & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$, the transform T_M is called a *translation* by the vector $\mathbf{t}$. All other operators are called *nonaffine.*

In real-time rendering it is sometimes necessary to invert an affine transform. Computational efficiency is paramount in these calculations (after all, this is real time!). So your objective in this project is to design an algorithm that accomplishes this inversion with a minimum number of flops. Preface discussion of your algorithm with a description of affine transforms. Give a geometrical explanation of what homogeneous and translation transforms do to vectors and points. You might also find it helpful to show that every affine transform is the composition of a homogeneous and a translation transform. Illustrate the algorithm with a few examples. Finally, you might discuss the stability of your algorithm. Could it be a problem? If so, how would you remedy it? See the discussion of roundoff error in Section 1.5.

Project: Modeling with Directed Graphs I

Refer to Example 2.21 and Section 2.3 for background. As a social scientist you have studied the influence factors that relate seven coalition groups. For simplicity, we will label the groups as $1, 2, 3, 4, 5, 6, 7$. Based on empirical studies, you conclude that the influence factors can be well modeled by a dominance-directed graph with each group as a vertex. The meaning of the presence of an edge (i, j) in the graph is that coalition group i can dominate, i.e., swing coalition group j its way on a given political issue. The data you have gathered suggest that the appropriate edge set is the following:

$$E = \{(1,2),(1,3),(1,4),(1,7),(2,4),(2,6),(3,2),(3,5),(3,6),$$
$$(4,5),(4,7),(5,1),(5,6),(5,7),(6,1),(6,4),(7,2),(7,6)\}.$$

Do an analysis of this power structure. This should include a graph. (It might be a good idea to arrange the vertices in a circle and go from there.) It should also include a power rating of each coalition group. Now suppose you were an adviser to one of these coalition groups, and by currying certain favors, this group could gain influence over another coalition group (thereby adding an edge to the graph or reversing an existing edge of the graph). In each case, if you could pick the best group for your client to influence, which would that be? Explain your results in the context of matrix multiplication if you can.

2.7 Exercises and Problems

Exercise 1. Use LU factorization of $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 3 & -2 \\ 4 & 2 & -2 \end{bmatrix}$ to solve $A\mathbf{x} = \mathbf{b}$, where

(a) $\mathbf{b} = (6, -8, -4)$ (b) $\mathbf{b} = (2, -1, 2)$ (c) $\mathbf{b} = (1, 2, 4))$ (d) $\mathbf{b} = (1, 1, 1)$.

Exercise 2. Use PLU factorization of $A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 3 & -2 \\ 4 & 2 & -2 \end{bmatrix}$ to solve $A\mathbf{x} = \mathbf{b}$,

(a) $\mathbf{b} = (3, 1, 4)$ (b) $\mathbf{b} = (2, -1, 3)$ (c) $\mathbf{b} = (1, 2, 0))$ (d) $\mathbf{b} = (1, 0, 0)$.

Exercise 3. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. Calculate the following.

(a) $A \otimes B$ (b)$B \otimes A$ (c)$A^{-1} \otimes B^{-1}$ (d)$(A \otimes B)^{-1}$

Exercise 4. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}$. Calculate the following.

(a) $A \otimes B$ (b)$B \otimes A$ (c)$A^T \otimes B^T$ (d)$(A \otimes B)^T$

Exercise 5. With A and B as in Exercise 3, $C = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}$, and $X = [x_{ij}]$ a 3×2 matrix of unknowns, use tensor products to determine the coefficient matrix of the linear system $AX + XB = C$ in matrix–vector form.

Exercise 6. Use the matrix A and methodology of Example 2.62 with $\mathbf{x}^{(0)} = (1, 2, 3)$, $\mathbf{x}^{(0)} = (0.9, 1.2, 3.6)$, and $\mathbf{x}^{(0)} = (1, 1.1, 3.4)$ to express the resulting system of equations in the six unknown nonzero entries of A in matrix–vector form.

*Problem 7. Show that if A is a nonsingular matrix with a zero $(1, 1)$th entry, then A does not have an LU factorization.

Problem 8. Prove that if A is $n \times n$, then $\det(-A) = (-1)^n \det A$.

Problem 9. Let A and B be invertible matrices of the same size. Use determinantal law D9 to prove that $\operatorname{adj} A^{-1} = (\operatorname{adj} A)^{-1}$ and $\operatorname{adj}(AB) = \operatorname{adj} A \operatorname{adj} B$.

Problem 10. Verify parts 1 and 4 of Theorem 2.11.

Problem 11. Verify parts 5 and 6 of Theorem 2.11.

Problem 12. If heat is transported with a horizontal velocity v as well as diffused in Example 2.61, a new equation results at each node in the form

$$-u_{i-1,j} - u_{i+1,j} + 4u_{ij} - u_{i,j-1} - u_{i,j+1} - \frac{vh}{2k}\left(u_{i+1,j} - u_{i-1,j}\right) = \frac{h^2}{k} f_{ij}$$

for $i, j = 1, \ldots, n$. Vectorize the system and use tensor products to identify the coefficient matrix of this linear system.

*Problem 13. Prove the Bookkeeping Theorem (Theorem 2.12).

3

VECTOR SPACES

It is hard to overstate the importance of the idea of a vector space, a concept that has found application in mathematics, engineering, physics, chemistry, biology, the social sciences, and other areas. What we encounter is an abstraction of the idea of vector space that we studied in calculus or high school geometry. These "geometrical vectors" can easily be visualized. In this chapter, abstraction will come in two waves. The first wave, which could properly be called *generalization*, consists in generalizing the familiar ideas of geometrical vectors of calculus to vectors of size greater than three. The second wave consists in abstracting the vector idea to entirely different kinds of objects. Abstraction can sometimes be difficult. For some, the study of abstract ideas is its own reward. For others, the natural reaction is to expect some payoff for the extra effort required to master abstraction. In the case of vector spaces we are happy to report that both kinds of students will be satisfied: vector space theory really is a thing of beauty in itself and there is indeed a payoff for its study. It is a practical tool that enables us to understand phenomena that would otherwise escape our comprehension. Examples abound: the theory will be used in network analysis, for "best" solutions to an inconsistent system (least squares), for studying functions as systems of vectors, and to obtain new perspectives on our old friend $A\mathbf{x} = \mathbf{b}$.

3.1 Definitions and Basic Concepts

Generalization

We begin with the most concrete form of vector spaces, one that is closely in tune with what we learned when we were first introduced to two- and three-dimensional vectors using real numbers as scalars. However, we have seen that the complex numbers are a perfectly legitimate and useful field of numbers to work with. Therefore, our concept of a vector space must include the selection of a field of scalars. The requirements for such a field are that it have binary operations of addition and multiplication that satisfy the usual arithmetic laws: Both operations are closed, commutative, and associative; have identities and satisfy distributive laws. And there exist additive inverses and multiplicative inverses for nonzero elements. Although other fields are possible, for our purposes the only fields of scalars are $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. Unless there is some indication to the contrary, the field of scalars will be assumed to be the default, the real numbers $\mathbb{R}$.

A formal definition of vector space will come later. For now we describe a "vector space" over a field of scalars $\mathbb{F}$ as a nonempty set V of vectors of the same size, together with the binary operations of scalar multiplication and vector addition, subject to the following laws: for all vectors $\mathbf{u}, \mathbf{v} \in V$ and scalars $a \in \mathbb{F}$, (a) (Closure of vector addition) $\mathbf{u} + \mathbf{v} \in V$. (b) (Closure of scalar multiplication) $a\mathbf{v} \in V$. For vectors $\mathbf{u}, \mathbf{v}$, we define $-\mathbf{u} = (-1)\mathbf{u}$ and $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

Vector Negatives and Subtraction

Very simple examples are $\mathbb{R}^2$ and $\mathbb{R}^3$, which we discuss below. Another is any line through the origin in $\mathbb{R}^2$, which takes the form $V = \{c\,(x_0, y_0) \mid c \in \mathbb{R}\}$.

Geometrical Vectors

Geometrical vector spaces. We may have already seen the vector idea in geometry or calculus. In those contexts, a vector was supposed to represent a direction and a magnitude in two- or three-dimensional space, which is not the same thing as a point, that is, location in space. At first, one had to deal with these intuitive definitions until they could be turned into something more explicitly computational, namely the displacements of a vector in coordinate directions. This led to the following two vector spaces over the field of real numbers:

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\},$$
$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

The distinction between vector spaces and points becomes a little hazy here. Once we have set up a coordinate system, we can identify each point in two- or three-dimensional space with its coordinates, which we write in the form of a tuple, i.e., a vector. The arithmetic of these two vector spaces is just the standard coordinatewise vector addition and scalar multiplication. One can visualize the direction represented by a vector (x, y) by drawing an arrow, i.e., directed line segment, from the origin of the coordinate system to the point with coordinates (x, y). The magnitude of this vector is the length of the arrow, which is just $\sqrt{x^2 + y^2}$. The arrows that we draw only *represent* the vector we are thinking of. More than one arrow could represent the same vector as in Figure 3.1. The definitions of vector arithmetic could be represented geometrically too. For example, to get the sum of vectors $\mathbf{u}$ and $\mathbf{v}$, one places a representative of vector $\mathbf{u}$ in the plane, then places a representative of $\mathbf{v}$ whose tail is at the head of $\mathbf{v}$, and the vector $\mathbf{u} + \mathbf{v}$ is then represented by the third leg of this triangle, with base at the base of $\mathbf{u}$. To get a scalar multiple of a vector $\mathbf{w}$ one scales $\mathbf{w}$ in accordance with the coefficient. See

Figure 3.1. Though instructive, this version of vector addition is not practical for calculations.

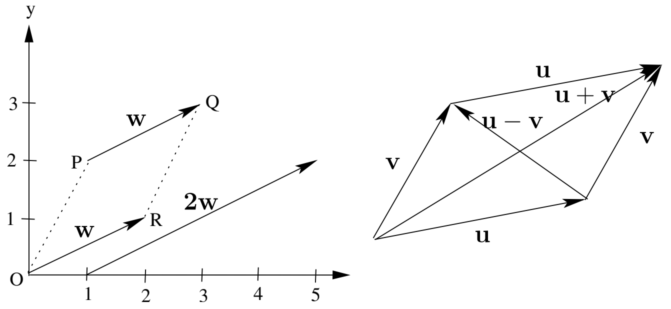


Fig. 3.1. Displacement vectors and graphical vector operations.

As a practical matter, it is also convenient to draw directed line segments connecting points; such a vector is called a *displacement vector.* For example, see Figure 3.1 for representatives of a displacement vector $\mathbf{w} = \overrightarrow{PQ}$ from the point P with coordinates $(1, 2)$ to the point Q with coordinates $(3, 3)$. One of the first nice outcomes of vector arithmetic is that this displacement vector can be deduced from a simple calculation,

Displacement and Position Vector

$$\mathbf{w} = (3, 3) - (1, 2) = (3 - 1, 3 - 2) = (2, 1)\,.$$

A displacement vector of the form $\mathbf{w} = \overrightarrow{OR}$, where O is the origin, is called a *position vector.*

Geometrical vector spaces look a lot like the object we studed in Chapter 2 with the tuple notation as a shorthand for column vectors. The arithmetic of $\mathbb{R}^2$ and $\mathbb{R}^3$ is the same as the standard arithmetic for column vectors. Now, even though we can't draw real geometrical pictures of vectors with four or more coordinates, we have seen that larger vectors are useful in our search for solutions of linear systems. So the question presents itself, why stop at three? The answer is that we won't! We will use the familiar pictures of $\mathbb{R}^2$ and $\mathbb{R}^3$ to guide our intuition about vectors in higher-dimensional spaces, which we now define.

Definition 3.1. Given a positive integer n, we define the *standard vector space of dimension n over the reals* to be the set of vectors

Standard Real Vector Space

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

together with the standard vector addition and scalar multiplication. (Recall that $(x_1, x_2, \ldots, x_n)$ is shorthand for the column vector $[x_1, x_2, \ldots, x_n]^T$.)

We see immediately from the definition that the required closure properties of vector addition and scalar multiplication hold, so these really are vector spaces in the sense defined above. The standard real vector spaces are often called the real Euclidean vector spaces once the notion of a norm (a notion of length covered in the next chapter) is attached to them.

Homogeneous vector spaces. Graphics specialists and others find it important to distinguish between geometrical vectors and points (locations) in three-dimensional space. They want to be able to simultaneously manipulate these two kinds of objects, in particular, to do vector arithmetic and operator manipulation that reduces to the ordinary vector arithmetic when applied to geometrical vectors.

Homogeneous Vectors and Points

Here's the idea that neatly does the trick: set up a coordinate system and identify geometrical vectors in the usual way, that is, by their coordinates x_1, x_2, x_3. Do the same with geometrical points. To distinguish between the two, embed them as vectors $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with the understanding that if $x_4 = 0$, then $\mathbf{x}$ represents a geometrical vector, and if $x_4 = 1$, then $\mathbf{x}$ represents a geometrical point. The vector $\mathbf{x}$ is called a *homogeneous vector* and $\mathbb{R}^4$ with the standard vector operations is called *homogeneous space*. If $x_4 \neq 0$, then the vector represents a point whose coordinates are $x_1/x_4, x_2/x_4, x_3/x_4$, and this point is said to be obtained from the vector $\mathbf{x}$ by *normalizing* the vector. Notice that the line through the origin that passes through the point $P = (x_1, x_2, x_3, 1)$ consists of vectors of the form (tx_1, tx_2, tx_3, t), where t is any real number. Conversely, any such nonzero vector is normalized $(tx_1/t, tx_2/t, tx_3/t, t/t) = P$. In this way, such lines through the origin correspond to points. (Readers who have seen projective spaces before may recognize this correspondence as identifying finite points in projective space with lines through the origin in $\mathbb{R}^4$. The ideas of homogeneous space actually originate in projective geometry.)

Now the standard vector arithmetic for $\mathbb{R}^4$ allows us to do arithmetic on geometrical vectors, for if $\mathbf{x} = (x_1, x_2, x_3, 0)$ and $\mathbf{y} = (y_1, y_2, y_3, 0)$ are such vectors, then as elements of $\mathbb{R}^4$ we have

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, x_2, x_3, 0) + (y_1, y_2, y_3, 0) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, 0)\,, \\ c\mathbf{x} &= c\,(x_1, x_2, x_3, 0) = (cx_1, cx_2, cx_3, 0)\,, \end{aligned}$$

which result in geometrical vectors.

Example 3.1. Interpret the result of adding a point and vector in homogeneous space.

Solution. Notice that we can't add two points and obtain a point without some extra normalization; however, addition of a point $\mathbf{x} = (x_1, x_2, x_3, 1)$ and vector $\mathbf{y} = (y_1, y_2, y_3, 0)$ yields

$$\mathbf{x} + \mathbf{y} = (x_1, x_2, x_3, 1) + (y_1, y_2, y_3, 0) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, 1)\,.$$

This has a rather elegant interpretation as the translation of the point $\mathbf{x}$ by the vector $\mathbf{y}$ to another point $\mathbf{x} + \mathbf{y}$. It reinforces the idea that geometrical vectors are simply displacements from one point to another. □

We can't draw pictures of $\mathbb{R}^4$, of course. But we can get an intuitive feeling for how homogenization works by moving down one dimension. Regard $\mathbb{R}^3$ as homogeneous space for the plane that consists of points $(x_1, x_2, 1)$. Figure 3.2 illustrates this idea.

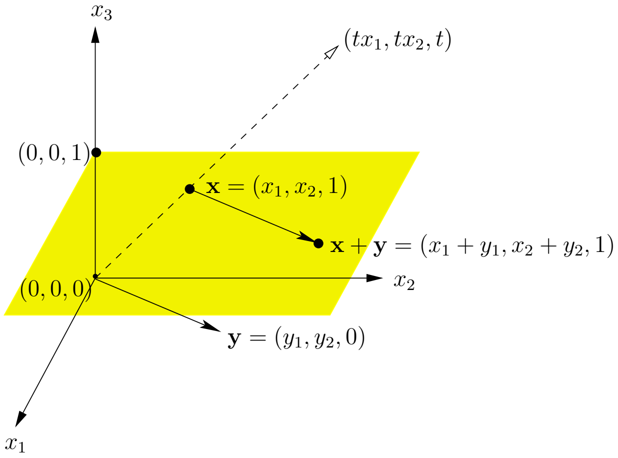

x_3
(tx_1, tx_2, t)
$(0,0,1)$
$\mathbf{x} = (x_1, x_2, 1)$
$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, 1)$
$(0,0,0)$
x_2
$\mathbf{y} = (y_1, y_2, 0)$
x_1

Fig. 3.2. Homogeneous space for planar points and vectors.

As in Chapter 2, we don't have to stop at the reals. For those situations in which we want to use complex numbers, we have the following vector spaces:

Definition 3.2. Given a positive integer n, we define the *standard vector space of dimension n over the complex numbers* to be the set of vectors

Standard Complex Vector Space

$$\mathbb{C}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{C}\}$$

together with the standard vector addition and scalar multiplication.

The standard complex vector spaces are also sometimes called Euclidean spaces. It's rather difficult to draw honest spatial pictures of complex vectors. The space $\mathbb{C}^1$ isn't too bad: complex numbers can be identified by points in the complex plane. What about $\mathbb{C}^2$? Where can we put $(1 + 2\mathrm{i}, 3 - \mathrm{i})$? It seems that we need four real coordinates, namely the real and imaginary parts of two independent complex numbers, to keep track of the point. This is too big to fit in real three-dimensional space, where we have only three independent coordinates. We don't let this technicality deter us. We can still draw fake vector pictures of elements of $\mathbb{C}^2$ to help our intuition, but do the algebra of vectors exactly from the definition.

Example 3.2. Find the displacement vector from the point P with coordinates $(1 + 2\mathrm{i}, 1 - 2\mathrm{i})$ to the point Q with coordinates $(3 + \mathrm{i}, 2\mathrm{i})$.

Solution. We compute

$$\begin{aligned}\overrightarrow{PQ} &= (3+\mathrm{i}, 2\mathrm{i}) - (1+2\mathrm{i}, 1-2\mathrm{i}) \\ &= (3+\mathrm{i}-(1+2\mathrm{i}), 2\mathrm{i}-((1-2\mathrm{i})) \\ &= (2-\mathrm{i}, -1+4\mathrm{i}).\end{aligned}$$ □

Abstraction

We can see hints of a problem with the coordinate way of thinking about geometrical vectors. Suppose the vector in question represents a force. In one set of coordinates the force might have coordinates $(1,0,1)$. In another, it could have coordinates $(0,1,1)$. Yet the the force doesn't change, only its representation. This suggests an idea: why not think about geometrical vectors as independent of any coordinate representation? From this perspective, geometrical vectors are really more abstract than the row or column vectors we have studied so far.

This line of thought leads us to consider an abstraction of our concept of vector space. First we have to identify the essential vector space properties, enough to make the resulting structure rich, but not so much that it is tied down to an overly specific form. We saw in Chapter 2 that many laws hold for the standard vector spaces. The essential laws were summarized in Section 2.1. These laws become the basis for our definition of an abstract vector space.

Abstract Vector Space

Definition 3.3. An *(abstract) vector space* is a nonempty set V of elements called vectors, together with operations of vector addition $(+)$ and scalar multiplication $(\,\cdot\,)$, such that the following laws hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $a, b \in \mathbb{F}$:

(1) (Closure of vector addition) $\mathbf{u}+\mathbf{v} \in V$.
(2) (Commutativity of addition) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.
(3) (Associativity of addition) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$.
(4) (Additive identity) There exists an element $\mathbf{0} \in V$ such that $\mathbf{u}+\mathbf{0} = \mathbf{u} = \mathbf{0}+\mathbf{u}$.
(5) (Additive inverse) There exists an element $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0} = (-\mathbf{u})+\mathbf{u}$.
(6) (Closure of scalar multiplication) $a \cdot \mathbf{u} \in V$.
(7) (Distributive law) $a \cdot (\mathbf{u}+\mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$.
(8) (Distributive law) $(a+b) \cdot \mathbf{u} = a \cdot \mathbf{u} + b \cdot \mathbf{u}$.
(9) (Associative law) $(ab) \cdot \mathbf{u} = a \cdot (b \cdot \mathbf{u})$.
(10) (Monoidal law) $1 \cdot \mathbf{u} = \mathbf{u}$.

About notation: just as in matrix arithmetic, for vectors $\mathbf{u}, \mathbf{v} \in V$, we understand that $\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v})$. We also suppress the dot $(\cdot)$ of scalar multiplication and usually write $a\mathbf{u}$ instead of $a \cdot \mathbf{u}$.

Examples of these abstract vector spaces are the standard spaces just introduced, and these will be our main focus in this section. Yet, if we squint a bit, we can see vector spaces everywhere. There are other, entirely nonstandard examples, that make the abstraction worthwhile. Here are just a few such examples. Our first example is closely related to the standard spaces, though strictly speaking it is not one of them. It blurs the distinction between matrices and vectors in Chapter 2, since it makes matrices into "vectors" in the abstract sense of the preceding definition.

Example 3.3. Let $\mathbb{R}^{m,n}$ denote the set of all $m \times n$ matrices with real entries. Show that this set, with the standard matrix addition and scalar multiplication, forms a vector space.

Matrices as Vector Space

Solution. We know that any two matrices of the same size can be added to yield a matrix of that size. Likewise, a scalar times a matrix yields a matrix of the same size. Thus the operations of matrix addition and scalar multiplication are closed. Indeed, these laws and all the other vector space laws are summarized in the laws of matrix addition and scalar multiplication of page 60. □

The next example is important in many areas of higher mathematics and is quite different from the standard vector spaces. Yet it is a perfectly legitimate vector space. All the same, at first it seems odd to think of functions as "vectors" even though this is meant in the abstract sense.

Example 3.4. Let $C[0,1]$ denote the set of all real-valued functions that are continuous on the interval $[0,1]$ and use the standard function addition and scalar multiplication for these functions. That is, for $f(x), g(x) \in C[0,1]$ and real number c, we define the functions $f+g$ and cf by

Function Space

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ (cf)(x) &= c(f(x)).\end{aligned}$$

Show that $C[0,1]$ with the given operations is a vector space.

Solution. We set $V = C[0,1]$ and check the vector space axioms for this V. For the rest of this example, we let f, g, h be arbitrary elements of V. We know from calculus that the sum of any two continuous functions is continuous and that any constant times a continuous function is also continuous. Therefore the closure of addition and that of scalar multiplication hold. Now for all x such that $0 \leq x \leq 1$, we have from the definition and the commutative law of real number addition that

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x).$$

Since this holds for all x, we conclude that $f+g = g+f$, which is the commutative law of vector addition. Similarly,

$$\begin{aligned}((f+g)+h)(x) &= (f+g)(x) + h(x) = (f(x)+g(x)) + h(x) \\ &= f(x) + (g(x)+h(x)) = (f+(g+h))(x).\end{aligned}$$

Since this holds for all x, we conclude that $(f+g)+h = f+(g+h)$, which is the associative law for addition of vectors.

Next, if 0 denotes the constant function with value 0, then for any $f \in V$ we have that for all $0 \leq x \leq 1$,

$$(f+0)(x) = f(x) + 0 = f(x).$$

(We don't write the zero element of this vector space in boldface because it's customary not to write functions in bold.) Since this is true for all x we have that $f + 0 = f$, which establishes the additive identity law. Also, we define $(-f)(x) = -(f(x))$ so that for all $0 \leq x \leq 1$,

$$(f + (-f))(x) = f(x) - f(x) = 0,$$

from which we see that $f + (-f) = 0$. The additive inverse law follows. For the distributive laws note that for real numbers a, b and continuous functions $f, g \in V$, we have that for all $0 \leq x \leq 1$,

$$a(f+g)(x) = a(f(x) + g(x)) = af(x) + ag(x) = (af + ag)(x),$$

which proves the first distributive law. For the second distributive law, note that for all $0 \leq x \leq 1$,

$$((a+b)g)(x) = (a+b)g(x) = ag(x) + bg(x) = (ag + bg)(x),$$

and the second distributive law follows. For the scalar associative law, observe that for all $0 \leq x \leq 1$,

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = (a(bf))(x),$$

so that $(ab)f = a(bf)$, as required. Finally, we see that

$$(1f)(x) = 1f(x) = f(x),$$

from which we have the monoidal law $1f = f$. Thus, $C\,[0,1]$ with the prescribed operations is a vector space. □

The preceding example could have just as well been $C\,[a,b]$, the set of all continuous functions on the interval $a \leq x \leq b$. Indeed, most of what we say about $C\,[0,1]$ is equally applicable to the more general space $C\,[a,b]$. We usually stick to the interval $0 \leq x \leq 1$ for simplicity. The next example is also based on the "functions as vectors" idea.

Example 3.5. One of the two sets $V = \{f\,(x) \in C\,[0,1] \mid f(1/2) = 0\}$ and $W = \{f\,(x) \in C\,[0,1] \mid f(1/2) = 1\}$, with the operations of function addition and scalar multiplication as in Example 3.4, forms a vector space over the reals, while the other does not. Determine which.

Solution. Notice that we don't have to check the commutativity of addition, associativity of addition, distributive laws, associative law, or monoidal law. The reason is that we already know from the previous example that these laws hold when the operations of the space $C[0,1]$ are applied to any elements of $C[0,1]$, whether they belong to V or W or not. So the only laws to be checked are the closure laws and the identity laws.

First let $f(x), g(x) \in V$ and let c be a scalar. By definition of the set V we have that $f(1/2) = 0$ and $g(1/2) = 0$. Add these equations together and we obtain

$$(f+g)(1/2) = f(1/2) + g(1/2) = 0 + 0 = 0.$$

It follows that V is closed under addition with these operations. Furthermore, if we multiply the identity $f(1/2) = 0$ by the real number c we obtain that

$$(cf)(1/2) = c \cdot f(1/2) = c \cdot 0 = 0.$$

It follows that V is closed under scalar multiplication. Now certainly the zero function belongs to V, since this function has value 0 at any argument. Therefore, V contains an additive identity element. Finally, we observe that the negative of a function $f(x) \in V$ is also an element of V, since

$$(-f)(1/2) = -1 \cdot f(1/2) = -1 \cdot 0 = 0.$$

Therefore, the set V with the given operations satisfies all the vector space laws and is an (abstract) vector space in its own right.

When we examine the set W in a similar fashion, we run into a roadblock at the closure of addition law. If $f(x), g(x) \in W$, then by definition of the set W we have that $f(1/2) = 1$ and $g(1/2) = 1$. Add these equations together and we obtain

$$(f+g)(1/2) = f(1/2) + g(1/2) = 1 + 1 = 2.$$

This means that $f+g$ is not in W, so the closure of addition fails. We need go no further. If only one of the vector space axioms fails, then we do not have a vector space. Hence, W with the given operations is not a vector space. □

There is a certain economy in this example. A number of laws did not need to be checked by virtue of the fact that the sets in question were subsets of existing vector spaces with the same vector operations. Here are two more examples that utilize this economy.

Example 3.6. Show that the set $\mathcal{P}_2$ of all polynomials of degree at most two with the standard function addition and scalar multiplication forms a vector space.

Polynomial Space

Solution. Certainly, polynomials are continuous functions on $[0,1]$ (actually, continuous everywhere, but a polynomial will be uniquely determined by its values on $[0,1]$). As in the preceding example, we don't have to check the commutativity of addition, associativity of addition, distributive laws, associative law, or monoidal law since we know that these laws hold for all continuous functions. Let $f, g \in \mathcal{P}_2$, say $f(x) = a_1 + b_1 x + c_1 x^2$ and $g(x) = a_2 + b_2 x + c_2 x^2$. Let c be any scalar. Then we have both

$$(f+g)(x) = f(x) + g(x) = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 \in \mathcal{P}_2$$

and

$$(cf)(x) = cf(x) = c\left(a_1 + b_1 x + c_1 x^2\right) = ca_1 + cb_1 x + cc_1 x^2 \in \mathcal{P}_2.$$

Hence $\mathcal{P}_2$ is closed under the operations of function addition and scalar multiplication. Furthermore, the zero function is a constant, hence a polynomial of degree at most two. Also, the negative of a polynomial of degree at most two is also a polynomial of degree at most two. So all of the laws for a vector space are satisfied and $\mathcal{P}_2$ is an (abstract) vector space. □

Example 3.7. Show that the set S_n of all $n \times n$ real symmetric matrices with the standard matrix addition and scalar multiplication form a vector space.

Solution. Just as in the preceding example, we don't have to check the commutativity of addition, associativity of addition, distributive laws, associative law, or monoidal law since we know that these laws hold for any matrices, symmetric or not. Now let $A, B \in S_n$. This means by definition that $A = A^T$ and $B = B^T$. Let c be any scalar. Then we have both

$$(A+B)^T = A^T + B^T = A + B$$

and

$$(cA)^T = cA^T = cA.$$

It follows that the set S_n is closed under the operations of matrix addition and scalar multiplication. Furthermore, the zero $n \times n$ matrix is clearly symmetric, so the set S_n has an additive identity element. Finally, $(-A)^T = -A^T = -A$, so each element of S_n has an additive inverse as well. Therefore, all of the laws for a vector space are satisfied, so S_n together with these operations is an (abstract) vector space. □

One of the virtues of abstraction is that it allows us to cover many cases with one statement. For example, there are many simple facts that are deducible from the vector space laws alone. With the standard vector spaces, these facts seem transparently clear. For abstract spaces, the situation is not quite so obvious. Here are a few examples of what can be deduced from the definition.

Example 3.8. Let $\mathbf{v} \in V$, a vector space, and $\mathbf{0}$ the vector zero. Deduce *from the vector space properties alone* that $0\mathbf{v} = \mathbf{0}$.

Solution. Certainly we have the scalar identity $0 + 0 = 0$. Multiply both sides on the right by the vector $\mathbf{v}$ to obtain that

$$(0+0)\mathbf{v} = 0\mathbf{v}.$$

Now use the distributive law to obtain

$$0\mathbf{v} + 0\mathbf{v} = 0\mathbf{v}.$$

Next add $-(0\mathbf{v})$ to both sides (remember, we don't know it's $\mathbf{0}$ yet), use the associative law of addition to regroup, and obtain that

$$0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v})) = 0\mathbf{v} + (-0\mathbf{v}).$$

Now use the additive inverse law to obtain that

$$0\mathbf{v} + \mathbf{0} = \mathbf{0}.$$

Finally, use the identity law to obtain

$$0\mathbf{v} = \mathbf{0},$$

which is what we wanted to show. □

Example 3.9. Show that the vector space V has only one zero element.

Solution. Suppose that both $\mathbf{0}$ and $\mathbf{0}_*$ act as zero elements in the vector space. Use the additive identity property of $\mathbf{0}$ to obtain that $\mathbf{0}_* + \mathbf{0} = \mathbf{0}_*$, while the additive identity property of $\mathbf{0}_*$ implies that $\mathbf{0} + \mathbf{0}_* = \mathbf{0}$. By the commutative law of addition, $\mathbf{0}_* + \mathbf{0} = \mathbf{0} + \mathbf{0}_*$. It follows that $\mathbf{0}_* = \mathbf{0}$, whence there can be only one zero element. □

There are several other such arithmetic facts that we want to identify, along with the one of this example. In the case of standard vectors, these facts are obvious, but for abstract vector spaces, they require a proof similar to the one we have just given. We leave these as exercises.

Laws of Vector Arithmetic

Let $\mathbf{v}$ be a vector in some vector space V and let c be any scalar. Then

(1) $0\mathbf{v} = \mathbf{0}$.
(2) $c\mathbf{0} = \mathbf{0}$.
(3) $(-c)\mathbf{v} = c(-\mathbf{v}) = -(c\mathbf{v})$.
(4) If $c\mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$ or $c = 0$.
(5) A vector space has only one zero element.
(6) Every vector has only one additive inverse.

Linear Operators

We were introduced in Section 2.3 to the idea of a linear function in the context of standard vectors. Now that we have a notion of an abstract vector space, we can examine linearity in this larger setting. We have seen that some of our "vectors" can themselves be functions, as in the case of the vector space $C[0,1]$ of continuous functions on the interval $[0,1]$. In order to avoid confusion in cases like this, we prefer to designate linear functions by the term *linear operator.* Other common terms for this object are *linear mapping* and *linear transformation.*

Before giving the definition of linear operator, let us recall some notation that is associated with functions in general. We identify a function f with the notation $f : D \to T$, where D and T are the *domain* and *target* of the function, respectively. This means that for each x in the domain D, the value $f(x)$ is a uniquely determined element in the target T. We want to emphasize at the outset that there is a difference here between the *target* of a function and its *range*. The *range* of the function f is defined as the subset of the target

Domain, Range and Target

$$\text{range}(f) = \{y \mid y = f(x) \text{ for some } x \in D\},$$

which is just the set of all possible values of $f(x)$. A function is said to be one-to-one if, whenever $f(x) = f(y)$, then $x = y$. Also, a function is said to be *onto* if the range of f equals its target. For example, we can define a function $f : \mathbb{R} \to \mathbb{R}$ by the formula $f(x) = x^2$. It follows from our specification of f that the target of f is understood to be $\mathbb{R}$, while the range of f is the set of nonnegative real numbers. Therefore, f is not onto. Moreover, $f(-1) = f(1)$ and $-1 \neq 1$, so f is not one-to-one either.

One-to-One and Onto Function

A function that maps elements of one vector space into another, say $f : V \to W$, is sometimes called an *operator* or *transformation.* One of the simplest mappings of a vector space V is the so-called *identity function* $\mathrm{id}_V : V \to V$ given by $\mathrm{id}_V(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in V$. Here domain, range, and target all agree. Of course, matters can become more complicated. For example, the operator $f : \mathbb{R}^2 \to \mathbb{R}^3$ might be given by the formula

Identity Function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}.$$

Notice in this example that the target of f is $\mathbb{R}^3$, which is not the same as the range of f, since elements in the range have nonnegative first and third coordinates. From the point of view of linear algebra, this function lacks the essential feature that makes it really interesting, namely linearity.

Definition 3.4. A function $T : V \to W$ from the vector space V into the space W over the same field of scalars is called a *linear operator (mapping, transformation)* if for all vectors $\mathbf{u}, \mathbf{v} \in V$ and scalars c, d, we have

Linear Operator

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

By taking $c = d = 1$ in the definition, we see that a linear function T is *additive,* that is, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Also, by taking $d = 0$ in the definition, we see that a linear function is *outative,* that is, $T(c\mathbf{u}) = cT(\mathbf{u})$. As a matter of fact, these two conditions imply the linearity property, and so are equivalent to it. We leave this fact as an exercise.

Additive and Outative Operator

By repeated application of the linearity definition, we can extend the linearity property to any linear combination of vectors, not just two terms. This means that for any scalars $c_1, c_2, \ldots, c_n$ and vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we have

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n).$$

Example 3.10. Determine whether $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear operator, where T is given by the formula
(a) $T((x, y)) = (x^2, xy, y^2)$ or (b) $T((x, y)) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution. If T is defined by (a) then we show by a simple example that T fails to be linear. Let us calculate

$$T((1, 0) + (0, 1)) = T((1, 1)) = (1, 1, 1),$$

while

$$T((1, 0)) + T((0, 1)) = (1, 0, 0) + (0, 0, 1) = (1, 0, 1).$$

These two are not equal, so T fails to satisfy the linearity property.

Next consider the operator T defined as in (b). Write

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix},$$

and we see that the action of T can be given as $T(\mathbf{v}) = A\mathbf{v}$. Now we have already seen in Section 2.3 that the operation of multiplication by a fixed matrix is a linear operator. □

Example 3.11. Let $\mathbf{t} = (t_1, t_2, t_3)$, $A = [a_{ij}]$ and $M = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$. Show that the linear operator $T_M : \mathbb{R}^4 \to \mathbb{R}^4$ mapping homogeneous space into itself maps points to points and geometrical vectors to vectors.

Solution. Let $\mathbf{x} = (x_1, x_2, x_3, x_4) = (\mathbf{v}, x_4)$ with $\mathbf{v} = (x_1, x_2, x_3)$ and use block arithmetic to obtain that

$$T_M(\mathbf{x}) = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ x_4 \end{bmatrix} = \begin{bmatrix} A\mathbf{v} + x_4\mathbf{t} \\ x_4 \end{bmatrix}.$$

Thus if $\mathbf{x}$ is a vector, which means $x_4 = 0$, then so is $T_M(\mathbf{x})$. Likewise, if $\mathbf{x}$ is a point, which means $x_4 = 1$, then so is $T_M(\mathbf{x})$. □

Recall that an operator $f : V \to W$ is said to be *invertible* if there is an operator $g : W \to V$ such that the composition of functions satisfies $f \circ g = \mathrm{id}_W$ and $g \circ f = \mathrm{id}_V$. In other words, $f(g(\mathbf{w})) = \mathbf{w}$ and $g(f(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{w} \in W$ and $\mathbf{v} \in V$. We write $g = f^{-1}$ and call f^{-1} the inverse of f. One can show that for any operator f, linear or not, being invertible is equivalent to being both one-to-one and onto.

Invertible Operator

Example 3.12. Show that if $f : V \to W$ is an invertible linear operator on vector spaces, then f^{-1} is also a linear operator.

Solution. We need to show that for $\mathbf{u}, \mathbf{v} \in W$, the linearity property $f^{-1}(c\mathbf{u} + d\mathbf{v}) = cf^{-1}(\mathbf{u}) + df^{-1}(\mathbf{v})$ is valid. Let $\mathbf{w} = cf^{-1}(\mathbf{u}) + df^{-1}(\mathbf{v})$. Apply the function f to both sides and use the linearity of f to obtain that

$$f(\mathbf{w}) = f\left(cf^{-1}(\mathbf{u}) + df^{-1}(\mathbf{v})\right) = cf\left(f^{-1}(\mathbf{u})\right) + df\left(f^{-1}(\mathbf{v})\right) = c\mathbf{u} + d\mathbf{v}.$$

Apply f^{-1} to obtain that $\mathbf{w} = f^{-1}(f(\mathbf{w})) = f^{-1}(c\mathbf{u} + d\mathbf{v})$, which proves the linearity property. □

In Chapter 2 the following useful fact was shown, which we now restate for standard real vector spaces. It is also valid for standard complex spaces.

Theorem 3.1. Let A be an $m \times n$ matrix and define an operator $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by the formula $T(\mathbf{v}) = A\mathbf{v}$, for all $\mathbf{v} \in \mathbb{R}^n$. Then T_A is a linear operator.

One can use this theorem and Example 3.12 to deduce the following fact, whose proof we leave as an exercise.

Corollary 3.1. Let A be an $n \times n$ matrix. The matrix operator T_A is invertible if and only if A is an invertible matrix.

Abstraction gives us a nice framework for certain key properties of mathematical objects, some of which we have seen before. For example, in calculus we were taught that differentiation has the "linearity property." Now we can express this assertion in a larger context: let V be the space of differentiable functions and define an operator T on V by the rule $T(f((x)) = f'(x)$. Then T is a linear operator on the space V.

3.1 Exercises and Problems

In Exercises 1–2 the x-axis points east, y-axis north, and z-axis upward.

Exercise 1. Express the following geometric vectors as elements of $\mathbb{R}^3$.
(a) The displacement vector from the origin to the point P with coordinates $-2, 3, 1$.
(b) The displacement vector from the point P with coordinates $2, 1, 3$ to a location 3 units north, 4 units east, and 6 units upward.

Exercise 2. Express the following geometric points and vectors as elements of homogeneous space $\mathbb{R}^4$.
(a) The vectors of Exercise 1.
(b) The point situated 2 units upward, 4 units west, and -5 units north of the point with coordinates $1, 2, 0$.

In Exercises 3–10 determine whether the given set and operations define a vector space. If not, indicate which laws fail.

Exercise 3. $V = \left\{ \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ with the standard matrix addition and scalar multiplication.

Exercise 4. $V = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R} \right\}$ with the standard matrix addition and scalar multiplication.

Exercise 5. $V = \{[a, b, \bar{a}] \mid a, b \in \mathbb{C}\}$ with the standard matrix addition and scalar multiplication. In this example the scalar field is the complex numbers.

Exercise 6. V consists of all continuous functions $f(x)$ on the interval $[0, 1]$ such that $f(0) = 0$ with the standard function addition and scalar multiplication (see Example 3.4).

Exercise 7. V consists of all quadratic polynomial functions $f(x) = ax^2 + bx + c, a \neq 0$ with the standard function addition and scalar multipication.

Exercise 8. V consists of all continuous functions $f(x)$ on the interval $[0, 1]$ such that $f(0) = f(1)$ with the standard function addition and scalar multipication.

Exercise 9. V is the set of complex vectors $(z_1, z_2, z_3, 0)$ in space $\mathbb{C}^4$ with the standard vector addition and scalar multiplication.

Exercise 10. V is the set of points $(z_1, z_2, z_3, 1)$, $z_1, z_2, z_3 \in \mathbb{C}$, with scalar multiplication and vector addition given by $c(x_1, x_2, x_3, 1) = (cx_1, cx_2, cx_3, 1)$ and $(x_1, x_2, x_3, 1) + (y_1, y_2, y_3, 1) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, 1)$.

Exercise 11. Determine which of the these formulas for $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear operator. If so, write the operator as a matrix multiplication and determine whether the target of T equals its range. Here $\mathbf{x} = (x, y, z)$ and $T(\mathbf{x})$ follows.
(a) $(x, x + 2y - 4z)$ (b) $(x + y, xy)$ (c) (y, y) (d) $x(0, y)$ (e) $(\sin y, \cos z)$

Exercise 12. Repeat Exercise 11 for the following formulas for $T : \mathbb{R}^3 \to \mathbb{R}^3$.
(a) $(-y, z, -x)$ (b) $(x, y, 1)$ (c) $(y - x + z, 2x + z, 3x - y - z)$ (d) $(x^2, 0, z^2)$

Exercise 13. Let $V = C[0, 1]$ and define an operator $T : V \to V$ by the following formulas for $T(f)$ as a function of the variable x. Which of these operators is linear? If so, is the target V of the operator equal to its range?
(a) $f(1)x^2$ (b) $f^2(x)$ (c) $2f(x)$ (d) $\int_0^x f(s)\, ds$

Exercise 14. Let $V = \mathbb{R}^{2,2}$ and define an operator T with domain V by the following formulas for $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)$. Which of these operators is linear?

(a) a_{22} (b) $\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ (c) $\det A$ (d) $[a_{11}a_{22}, 0]$

Exercise 15. Given an arbitrary vector space V, is the identity operator $\mathrm{id}_V : V \to V$ linear? Invertible? If so, specify its inverse.

Exercise 16. Given arbitrary vector spaces U and V over the same scalars, is the zero operator $0_{U,V} : U \to V$ given by $0_{U,V}(\mathbf{v}) = \mathbf{0}$ linear? Invertible? If so, specify its inverse.

Exercise 17. A transform of homogeneous space is given by $M = \begin{bmatrix} I_3 & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ with $\mathbf{t} = (2, -1, 3)$. Calculate and describe in words the action of T_M on the point $\mathbf{x} = (x_1, x_2, x_3, 1)$. Find the inverse of this transform.

Exercise 18. A transform of homogeneous space is given by $M = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ with $\mathbf{t} = (2, -1, 3)$ and $A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$. Calculate and describe in words the action of T_M on the point $\mathbf{x} = (x_1, x_2, x_3, 1)$. Find the inverse of this transform. (See Example 2.17 in Chapter 2.)

*Problem 19. Use the definition of vector space to prove the vector law of arithmetic (2): $c\mathbf{0} = \mathbf{0}$.

Problem 20. Use the definition of vector space to prove the vector law of arithmetic (3): $(-c)\mathbf{v} = c(-\mathbf{v}) = -(c\mathbf{v})$.

Problem 21. Use the definition of vector space to prove the vector law of arithmetic (4): If $c\mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$ or $c = 0$.

Problem 22. Let $\mathbf{u}, \mathbf{v} \in V$, where V is a vector space. Use the vector space laws to prove that the equation $\mathbf{x} + \mathbf{u} = \mathbf{v}$ has one and only one solution vector $\mathbf{x} \in V$, namely $\mathbf{x} = \mathbf{v} - \mathbf{u}$.

Problem 23. Let U and V be vector spaces over the same field of scalars and form the set $U \times V$ consisting of all ordered pairs $(\mathbf{u}, \mathbf{v})$ where $\mathbf{u} \in U$ and $\mathbf{v} \in V$. We can define an addition and scalar multiplication on these ordered pairs as follows:

$$(\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2),$$
$$c \cdot (\mathbf{u}_1, \mathbf{v}_1) = (c\mathbf{u}_1, c\mathbf{v}_1).$$

Verify that with these operations $U \times V$ becomes a vector space over the same field of scalars as U and V.

Problem 24. Show that for any vector space V, the identity function $\mathrm{id}_V : V \to V$ is a linear operator.

Problem 25. Let $T : \mathbb{R}^3 \to \mathcal{P}_2$ be defined by $T((a, b, c)) = a + bx + cx^2$. Show that T is a linear operator whose range is $\mathcal{P}_2$.

Problem 26. Prove the remark following Definition 3.4: if a function $T : V \to W$ between vector spaces V and W is additive and outative, then it is linear.

*Problem 27. Prove Corollary 3.1.

*Problem 28. Transforms of homogeneous space are given by
$M_1 = \begin{bmatrix} I_3 & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$, $\mathbf{t} = (t_1, t_2, t_3)$ and $M_2 = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$,
where A is an invertible 3×3 matrix. Show that the transform T_{M_1} (called a *translation transform*) and T_{M_2} (called a *homogeneous transform*) commute with each other, that is, $T_{M_1} \circ T_{M_2} = T_{M_2} \circ T_{M_1}$.

3.2 Subspaces

We now turn our attention to the concept of a *subspace*, which is a rich source for examples of vector spaces. It frequently happens that a certain vector space of interest is a subset of a larger, and possibly better understood, vector space, and that the vector operations are the same for both spaces. An example of this situation is given by the vector space V of Example 3.5, which is a subset of the larger vector space $C[0, 1]$ with both spaces sharing the same definitions of vector addition and scalar multiplication. Here is a precise formulation of the subspace idea.

Definition 3.5. A *subspace* of the vector space V is a subset W of V such that W, together with the binary operations it inherits from V, forms a vector space (over the same field of scalars as V) in its own right. Subspace

Given a subset W of the vector space V, we can apply the definition of vector space directly to the subset W to obtain the following very useful test.

Theorem 3.2. Let W be a subset of the vector space V. Then W is a subspace of V if and only if Subspace Test

(1) W contains the zero element of V.
(2) (Closure of addition) For all $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$.
(3) (Closure of scalar multiplication) For all $\mathbf{u} \in W$ and scalars c, $c\mathbf{u} \in W$.

Proof. Let W be a subspace of the vector space V. Then the closure of addition and scalar multiplication are automatically satisfied by the definition of vector space. For condition (1), we note that W must contain a zero element by definition of vector space. Let $\mathbf{0}_*$ be this element, so that $\mathbf{0}_* + \mathbf{0}_* = \mathbf{0}_*$. Add the negative of $\mathbf{0}_*$ (as an element of V) to both sides, cancel terms and we see that $\mathbf{0}_* = \mathbf{0}$, the zero of V. This shows that W satisfies condition (1).

Conversely, suppose that W is a subset of V satisfying the three conditions. Since the operations of W are those of the vector space V, and V is a vector space, most of the laws for W are automatic. Specifically, the laws of commutativity, associativity, distributivity, and the monoidal law hold for elements of W. The additive identity law follows from condition (1).

The only law that needs any work is the additive inverse law. Let $\mathbf{w} \in W$. By closure of scalar multiplication, $(-1)\mathbf{w}$ is in W. By the laws of vector arithmetic in the preceding section, this vector is simply $-\mathbf{w}$. This proves that every element of W has an additive inverse in W, which shows that W is a subspace of V. □

One notable point that comes out of the subspace test is that every subspace of V contains the zero vector. This is certainly not true of arbitrary subsets of V and serves to remind us that although every subspace is a subset of V, not every subset is a subspace. Confusing the two is a common mistake, so much so that we issue the following caution:

Caution: Every subspace of a vector space is a subset, but not every subset is a subspace.

Example 3.13. Which of the following subsets of the standard vector space $V = \mathbb{R}^3$ are subspaces of V?

(a) $W_1 = \{(x, y, z) \mid x - 2y + z = 0\}$ (b) $W_2 = \{(x, y, z) \mid x, y, z \text{ are positive}\}$
(c) $W_3 = \{(0, 0, 0)\}$ (d) $W_4 = \left\{(x, y, z) \mid x^2 - y = 0\right\}$

Solution. (a) Take $\mathbf{w} = (0, 0, 0)$ and obtain that

$$0 - 2 \cdot 0 + 0 = 0,$$

so that $\mathbf{w} \in W_1$. Next, check closure of W_1 under addition. Let's name two general elements from W_1, say $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$. Then we know from the definition of W_1 that

$$\begin{aligned} x_1 - 2y_1 + z_1 &= 0 \\ x_2 - 2y_2 + z_2 &= 0. \end{aligned}$$

We want to show that $\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W_1$. So add the two equations above and group terms to obtain

$$(x_1 + x_2) - 2(y_1 + y_2) + (z_1 + z_2) = 0.$$

This equation shows that the coordinates of $\mathbf{u} + \mathbf{v}$ fit the requirement for being an element of W_1, i.e., $\mathbf{u} + \mathbf{v} \in W_1$. Similarly, if c is a scalar then we can multiply the equation that says $\mathbf{u} \in W_1$, i.e., $x_1 - 2y_1 + z_1 = 0$, by c to obtain

$$(cx_1) - 2(cy_1) + (cz_1) = c0 = 0.$$

This shows that the coordinates of $c\mathbf{v}$ fit the requirement for being in W_1, i.e., $c\mathbf{u} \in W_1$. It follows that W_1 is closed under both addition and scalar multiplication, so it is a subspace of $\mathbb{R}^3$.

(b) This one is easy. Any subspace must contain the zero vector $(0, 0, 0)$. Clearly W_2 does not. Hence it cannot be a subspace. Another way to see it is to notice that closure under scalar multiplication fails (try multiplying $(1, 1, 1)$ by -1).

(c) The only possible choice for arbitrary elements $\mathbf{u}, \mathbf{v}$, in this space is $\mathbf{u} = \mathbf{v} = (0, 0, 0)$. But then we see that W_3 obviously contains the zero vector and for any scalar c,

$$\begin{aligned}(0, 0, 0) + (0, 0, 0) &= (0, 0, 0),\\ c(0, 0, 0) &= (0, 0, 0).\end{aligned}$$

Therefore W_3 is a subspace of V by the subspace test.

(d) First of all, $0^2 - 0 = 0$, which means that $(0, 0, 0) \in W_4$. Likewise we see that $(1, 1, 0) \in W_4$ as well. But $(1, 1, 0) + (1, 1, 0) = (2, 2, 0)$, which is not an element of W_4 since $2^2 - 2 \neq 0$. Therefore, closure of addition fails and W_4 is not a subspace of V by the subspace test. □

Part (c) of this example highlights part of a simple fact about vector spaces. Every vector space V must have at least two subspaces, namely, $\{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in V, and V itself. These are not terribly surprising subspaces, so they are commonly called the *trivial* subspaces.

Trivial Subspaces

Example 3.14. Show that the subset $\mathcal{P}$ of $C[0, 1]$ consisting of all polynomial functions is a subspace of $C[0, 1]$ and that the subset $\mathcal{P}_n$ consisting of all polynomials of degree at most n is a subspace of $\mathcal{P}$.

Solution. Certainly $\mathcal{P}$ is a subset of $C[0, 1]$, since every polynomial is continuous on the interval $[0, 1]$ and $\mathcal{P}$ contains the zero constant function, which is a polynomial function. Let f and g be two polynomial functions on the interval $[0, 1]$, say

$$\begin{aligned}f(x) &= a_0 + a_1 x + \cdots + a_n x^n,\\ g(x) &= b_0 + b_1 x + \cdots + b_n x^n,\end{aligned}$$

where n is an integer equal to the maximum of the degrees of $f(x)$ and $g(x)$. Let c be any real number, and we see that

$$\begin{aligned}(f + g)(x) &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n,\\ (cf)(x) &= ca_0 + ca_1 x + \cdots + ca_n x^n,\end{aligned}$$

which shows that $\mathcal{P}$ is closed under function addition and scalar multiplication. By the subspace test, $\mathcal{P}$ is a subspace of $C[0, 1]$. The equations above also show that the subset $\mathcal{P}_n$ passes the subspace test, so it is a subspace of $C[0, 1]$. □

Example 3.15. Show that the set of all upper triangular matrices (see page 89) in the vector space $V = \mathbb{R}^{n,n}$ of $n \times n$ real matrices is a subspace of V.

Solution. Since the zero matrix is upper triangular, the subset W of all upper triangular matrices contains the zero element of V. Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be any two matrices in W and let c be any scalar. By the definition of upper triangular, we must have $a_{i,j} = 0$ and $b_{i,j} = 0$ if $i > j$. However,

$$\begin{aligned} A + B &= [a_{i,j} + b_{i,j}]\,, \\ cA &= [ca_{i,j}]\,, \end{aligned}$$

and for $i > j$ we have $a_{i,j} + b_{i,j} = 0 + 0 = 0$ and $ca_{i,j} = c0 = 0$, so that $A + B$ and cA are also upper triangular. It follows that W is a subspace of V by the subspace test. □

There is an extremely useful type of subspace that requires the notion of a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in the vector space V: an expression of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1,\ c_2, \ldots, c_n$ are scalars. We can consider the set of all possible linear combinations of a given list of vectors, which is what our next definition is about.

Linear Combinations and Span

Definition 3.6. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in the vector space V. The *span* of these vectors, denoted by $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, is the subset of V consisting of all possible linear combinations of these vectors, i.e.,

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid c_1, c_2, \ldots, c_n \text{ are scalars}\}$$

.

Caution: The scalars we are using really make a difference. For example, if $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$ are viewed as elements of the real vector space $\mathbb{R}^2$, then

$$\begin{aligned} \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} &= \{c_1(1, 0) + c_2(0, 1) \mid c_1, c_2 \in \mathbb{R}\} \\ &= \{(c_1, c_2) \mid c_1, c_2 \in \mathbb{R}\} \\ &= \mathbb{R}^2. \end{aligned}$$

Similarly, if we view $\mathbf{v}_1$ and $\mathbf{v}_2$ as elements of the complex vector space $\mathbb{C}^2$, then we see that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{C}^2$. Now $\mathbb{R}^2$ consists of those elements of $\mathbb{C}^2$ whose coordinates have zero imaginary parts, so $\mathbb{R}^2$ is a sub*set* of $\mathbb{C}^2$; but these are certainly not equal sets. By the way, $\mathbb{R}^2$ is definitely not a sub*space* of $\mathbb{C}^2$ either, since the subset $\mathbb{R}^2$ is not closed under multiplication by complex scalars.

We should take note here that the definition of span would work perfectly well with infinite sets, as long as we understand that linear combinations in the definition would be finite and therefore not involve all the vectors in the span. A situation in which this extension is needed is as follows: consider the space $\mathcal{P}$ of all polynomials with the standard addition and scalar multiplication. It makes perfectly good sense to write

$$\mathcal{P} = \operatorname{span}\left\{1, x, x^2, x^3, \ldots, x^n, \ldots\right\},$$

since every polynomial is a *finite* linear combination of various monomials x^k.

Example 3.16. Interpret the following linear spans in $\mathbb{R}^3$ geometrically:

$$W_1 = \operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}, \qquad W_2 = \operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}\right\}.$$

Solution. Elements of W_1 are simply scalar multiples of the single vector $(1,2,1)$. The set of all such multiples gives us a line through the origin $(0,0,0)$. On the other hand, elements of W_2 give all possible linear combinations of two vectors $(1,2,1)$ and $(2,0,0)$. The locus of points generated by these combinations is a plane in $\mathbb{R}^3$ containing the origin, so it is determined by the points with coordinates $(0,0,0)$, $(1,2,1)$, and $(2,0,0)$. See Figure 3.3 for a picture of a portion of these spans. □

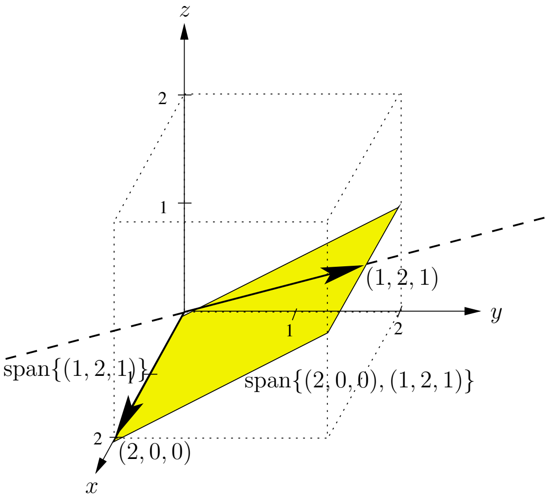


Fig. 3.3. Shaded portion of span $\{(2,0,0), (1,2,1)\}$ and dashed span $\{(1,2,1)\}$.

Spans are the premier examples of subspaces. In a certain sense, it can be said that *every* subspace is the span of some of its vectors. The following important fact is a very nice application of the subspace test.

Theorem 3.3. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in the vector space V. Then $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of V.

Proof. First, we observe that the zero vector can be expressed as the linear combination $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$, which is an element of W. Next, let c be any scalar and form general elements $\mathbf{u}, \mathbf{v} \in W$, say

$$\begin{aligned}\mathbf{u} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,\\ \mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n.\end{aligned}$$

Add these vectors and collect like terms to obtain

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_n + d_n)\mathbf{v}_n.$$

Thus $\mathbf{u}+\mathbf{v}$ is also a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, so W is closed under vector addition. Finally, form the product $c\mathbf{u}$ to obtain

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_n)\mathbf{v}_n,$$

which is again a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, so W is closed under scalar multiplication. By the subspace test, W is a subspace of V. □

Spanning Set

If $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, we say that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a *spanning set* for the vector space W, and that W *is spanned by the vectors* $\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_n$. There are a number of simple properties of spans that we will need from time to time. One of the most useful is this basic fact.

Theorem 3.4. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in the vector space V and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ be vectors in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then

$$\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\} \subseteq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

Proof. Suppose that for each index $j = 1, 2, \ldots k$,

$$\mathbf{w}_j = c_{1j}\mathbf{v}_1 + c_{2j}\mathbf{v}_2 + \cdots + c_{nj}\mathbf{v}_n.$$

Write a linear combination of the $\mathbf{w}_j$'s by regrouping the coefficients of each $\mathbf{v}_k$ as

$$\begin{aligned}d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + \cdots + d_k\mathbf{w}_k &= d_1(c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n)\\ &+ d_2(c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n) + \cdots + d_k(c_{1k}\mathbf{v}_1 + c_{2k}\mathbf{v}_2 + \cdots + c_{nk}\mathbf{v}_n)\\ &= \left(\sum_{i=1}^{k} d_i c_{1i}\right)\mathbf{v_1} + \left(\sum_{i=1}^{k} d_i c_{2i}\right)\mathbf{v_2} + \cdots + \left(\sum_{i=1}^{k} d_i c_{ni}\right)\mathbf{v_n}.\end{aligned}$$

It follows that each element of $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ belongs to the vector space $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, as desired. □

Here is a simple application of this theorem: if $\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}$ is a subset of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then

$$\operatorname{span}\{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}\} \subseteq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

The reason is that for $j = 1, 2, \ldots, k$,

$$\mathbf{w}_j = \mathbf{v}_{i_j} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_{i_j} + \cdots + 0\mathbf{v}_n,$$

so that the theorem applies to these vectors. Put another way, if we enlarge the list of spanning vectors, we enlarge the spanning set. However, we may not obtain a strictly larger spanning set, as the following example shows.

Example 3.17. Show that

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}.$$

Why might one prefer the first spanning set?

Solution. Label vectors $\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix}1\\2\end{bmatrix}$. Every element of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, since we can write $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$. So we certainly have that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. However, a little fiddling with numbers reveals this fact:

$$\begin{bmatrix}1\\2\end{bmatrix} = (-1)\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}.$$

In other words $\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2$. Therefore any linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ can be written as

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(-\mathbf{v}_1 + 2\mathbf{v}_2) \\ &= (c_1 - c_3)\mathbf{v}_1 + (c_2 + 2c_3)\mathbf{v}_2. \end{aligned}$$

Thus any element of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, so the two spans are equal. This is an algebraic representation of the geometric fact that the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ belong to the same plane in $\mathbb{R}^2$ that is spanned by the two vectors $\mathbf{v}_1, \mathbf{v}_2$. It seems reasonable that we should prefer the spanning set $\mathbf{v}_1, \mathbf{v}_2$ to the set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, since the former is smaller yet carries just as much information as the latter. As a matter of fact, we would get the same span if we used $\mathbf{v}_1, \mathbf{v}_3$ or $\mathbf{v}_2, \mathbf{v}_3$. The spanning set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ has "redundant" vectors in it. □

Subspaces of Geometrical Spaces

As another application of vector spans, let's consider the problem of determining all subspaces of the vector space $\mathbb{R}^2$, the plane, from a geometrical perspective. First, we have the trivial subspaces $\{(0,0)\}$ and $\mathbb{R}^2$. Next, consider the subspace $V = \operatorname{span}\{\mathbf{v}\}$, where $\mathbf{v} \neq \mathbf{0}$. It's easy to see that the set of all multiples of $\mathbf{v}$ constitutes a straight line through the origin. Finally, consider the subspace $V = \operatorname{span}\{\mathbf{v}, \mathbf{w}\}$, where $w \notin \operatorname{span}\{\mathbf{v}\}$. We can see that any point in the plane can be a corner of a parallelogram with edges that are multiples of $\mathbf{v}$ and $\mathbf{w}$. Hence $V = \mathbb{R}^2$. Consequently, the only subspaces of $\mathbb{R}^2$ are $\{(0,0)\}$, $\mathbb{R}^2$, and lines through the origin. In a similar fashion, you can convince yourself that the only subspaces of $\mathbb{R}^3$ are $\{(0, 0.0)\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$.

3.2 Exercises and Problems

In Exercises 1–10, determine whether the subset W is a subspace of the vector space V.

Exercise 1. $V = \mathbb{R}^3$ and $W = \{(a, b, a - b + 1) \mid a, b \in \mathbb{R}\}$.

Exercise 2. $V = \mathbb{R}^3$ and $W = \{(a, 0, a - b) \mid a, b \in \mathbb{R}\}$.

Exercise 3. $V = \mathbb{R}^3$ and $W = \{(a, b, c) \mid 2a - b + c = 0\}$.

Exercise 4. $V = \mathbb{R}^{2,3}$ and $W = \left\{ \begin{bmatrix} a & b & 0 \\ b & a & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$.

Exercise 5. $V = C[0, 1]$ and $W = \{f(x) \in C[0, 1] \mid f(1) + f(1/2) = 0\}$.

Exercise 6. $V = C[0, 1]$ and $W = \{f(x) \in C[0, 1] \mid f(1) \leq 0\}$.

Exercise 7. $V = \mathbb{R}^{n,n}$ and W is the set of all invertible matrices in V.

Exercise 8. $V = \mathbb{R}^{2,2}$ and W is the set of all matrices $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$, for some scalars a, b, c. (Such matrices are called *skew-symmetric* since $A^T = -A$.)

Exercise 9. V is the subset of geometrical vectors $(x_1, x_2, x_3, 0)$ in homogeneous space $W = \mathbb{R}^4$ with the standard vector addition and scalar multiplication.

Exercise 10. V is the subset of geometrical points $(x_1, x_2, x_3, 1)$ in homogeneous space $W = \mathbb{R}^4$ with vector addition and scalar multiplication given by $(x_1, x_2, x_3, 1) + (y_1, y_2, y_3, 1) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, 1)$ and
$c\,(x_1, x_2, x_3, 1) = (cx_1, cx_2, cx_3, 1)$.

Exercise 11. Show that $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Exercise 12. Show that $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Exercise 13. Which of the following spans equal the space $\mathcal{P}_2$ of polynomials of degree at most 2? Justify your answers.
(a) $\operatorname{span}\left\{1, 1+x, x^2\right\}$ (b) $\operatorname{span}\left\{x, 4x-2x^2, x^2\right\}$
(c) $\operatorname{span}\left\{1+x+x^2, 1+x, 3\right\}$ (d) $\operatorname{span}\left\{1-x^2, 1\right\}$

Exercise 14. Which of the following spans equal the space $\mathbb{R}^2$? Justify your answers.
(a) $\operatorname{span}\{(1,0),(-1,-1)\}$ (b) $\operatorname{span}\{(1,2),(2,4)\}$
(c) $\operatorname{span}\{(1,0),(0,0),(0,-1)\}$ (d) $\operatorname{span}\{(-1,-2),(-1,-1)\}$

Exercise 15. Let $\mathbf{u} = (2,-1,1)$, $\mathbf{v} = (0,1,1)$, and $\mathbf{w} = (2,1,3)$. Show that $\operatorname{span}\{\mathbf{u}+\mathbf{w}, \mathbf{v}-\mathbf{w}\} \subseteq \operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ and determine whether or not these spans are actually equal.

Exercise 16. Find two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that if $\mathbf{u} = (1,-1,1)$, then $\mathbb{R}^3 = \operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

*Problem 17. Let U and V be subspaces of W. Use the subspace test to prove the following.

(a) The set intersection $U \cap V$ is a subspace of W.

(b) The sum of the spaces, $U + V = \{u + v \mid u \in U \text{ and } v \in V\}$, is a subspace of W.

(c) The set union $U \cup V$ is not a subspace of W unless one of U or V is contained in the other.

Problem 18. Let V and W be subspaces of $\mathbb{R}^3$ given by

$$V = \{(x,y,z) \mid x = y = z \in \mathbb{R}\} \text{ and } W = \{(x,y,0) \mid x,y \in \mathbb{R}\}.$$

Show that $V + W = \mathbb{R}^3$ and $V \cap W = \{\mathbf{0}\}$.

*Problem 19. Prove that if $V = \mathbb{R}^{n,n}$, then the set of all diagonal matrices is a subspace of V.

*Problem 20. Let V be the space of 2×2 matrices and associate with each $A \in V$ the vector $\operatorname{vec}(A) \in \mathbb{R}^4$ obtained from A by stacking the columns of A underneath each other. (For example, $\operatorname{vec}\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right) = (1,-1,2,1)$.) Show the following.

(a) The vec operation establishes a one-to-one correspondence between matrices in V and vectors in $\mathbb{R}^4$.

(b) The vec operation, $\operatorname{vec} : \mathbb{R}^{2,2} \to \mathbb{R}^4$, is a linear operator.

Problem 21. You will need a computer algebra system (CAS) such as Mathematica or Maple for this exercise. Use the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and the vec method of the preceding exercise to turn powers of A into vectors. Then use your CAS to find a spanning set (or basis, which is a special spanning set) for subspaces $V_k = \operatorname{span}\left\{A^0, A^1, \ldots, A^k\right\}$, $k = 1, 2, 3, 4, 5, 6$. Based on this evidence, how many matrices will be required for a span of V_k? (Remember that $A^0 = I$.)

Problem 22. Show that the set $C^1[0,1]$ of continuous functions that have a continuous derivative on the interval $[0,1]$ is a subspace of the vector space $C[0,1]$.

3.3 Linear Combinations

We have seen in Section 3.2 that linear combinations give us a rich source of subspaces for a vector space. In this section we will take a closer look at linear combinations.

Linear Dependence

Lists and Sets

First we need to make precise the idea of redundant vectors that we encountered in Example 3.17. About lists and sets: *Lists* involve an ordering of elements (they can just as well be called finite sequences), while *sets* don't really imply any ordering of elements. Thus, every list of vectors, e.g., $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, gives rise to a unique set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. A different list $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2$ may define the same set $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Lists can have repeats in them, while sets don't. For instance, the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1$ defines the set $\{\mathbf{v}_1, \mathbf{v}_2\}$. The default meaning of the terminology "the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$" is "the list of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$," although occasionally it means a set or even both. For example, the definitions below work perfectly well for either sets or lists.

Redundant Vectors

Definition 3.7. The vector $\mathbf{v}_i$ is *redundant* in the vectors $\mathbf{v}_1, \mathbf{v}_{2,} \ldots, \mathbf{v}_n$ if the linear span $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ does not change when $\mathbf{v}_i$ is removed.

Example 3.18. Which vectors are redundant in the set consisting of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

Solution. As in Example 3.17, we notice that

$$\mathbf{v}_3 = (-1)\mathbf{v}_1 + 2\mathbf{v}_2.$$

Thus any linear combination involving $\mathbf{v}_3$ can be expressed in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. Therefore, $\mathbf{v}_3$ is redundant in the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. But there is more going on here. Let's write the equation above in a form that doesn't single out any one vector:

$$0 = (-1)\mathbf{v}_1 + 2\mathbf{v}_2 + (-1)\mathbf{v}_3.$$

Now we see that we could solve for *any* of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in terms of the remaining two vectors. Therefore, each of these vectors is redundant in the set. However, this doesn't mean that we can discard all three and get the same linear span. This is obviously false. What we can do is discard any *one* of them, then start over and examine the remaining set for redundant vectors. □

This example shows that what really counts for redundancy is that the vector in question occurs with a nonzero coefficient in a linear combination that equals 0. This situation warrants a name:

Linearly Dependent or Independent Vectors

Definition 3.8. The vectors v_1,v_2,…,v_n are said to be *linearly dependent* if there exist scalars $c_1, c_{2,} \ldots, c_n$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0. \tag{3.1}$$

Otherwise, the vectors are called *linearly independent.*

Trivial and Nontrivial Linear Combination

Zero Value Linear Combination

This fact inspires some notation: we will call a linear combination *trivial* if every coefficient is zero; otherwise it is *nontrivial.* We say that a linear combination has *value zero* if it sums to zero. Thus linear dependence is equivalent to the existence of a nontrivial linear combination with value zero. Just as with redundancy, linear dependence or independence is a property of the list or set in question, *not* of the individual vectors. Here is the key connection between linear dependence and redundancy.

Redundancy Test

Theorem 3.5. The list of vectors $\mathbf{v}_1, \mathbf{v}_{2,} \ldots, \mathbf{v}_n$ of a vector space has redundant vectors if and only if it is linearly dependent, in which case the redundant vectors are those that occur with nonzero coefficient in some linear combination with value zero.

Proof. Observe that if (3.1) holds and some scalar, say c_1, is nonzero, then we can use the equation to solve for $\mathbf{v}_1$ as a linear combination of the remaining vectors to obtain

$$\mathbf{v}_1 = \frac{-1}{c_1}\left(c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n\right).$$

Thus we see that any linear combination involving $\mathbf{v}_1, \mathbf{v}_{2,} \ldots, \mathbf{v}_n$ can be expressed using only $\mathbf{v}_2, \mathbf{v}_{3,} \ldots, \mathbf{v}_n$. It follows that

$$\operatorname{span}\{\mathbf{v}_2, \mathbf{v}_{3,} \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_{2,} \ldots, \mathbf{v}_n\}.$$

Conversely, if these spans are equal then $\mathbf{v}_1$ belongs to the left-hand side, so there are scalars $d_2, d_{3,} \ldots, d_n$ such that

$$\mathbf{v}_1 = d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + \cdots + d_n\mathbf{v}_n.$$

Now bring all terms to the right-hand side and obtain the nontrivial linear combination

$$-\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + \cdots + d_n\mathbf{v}_n = \mathbf{0}.$$

All of this works equally well for any index other than 1, so the theorem is proved. □

It is instructive to examine the simple case of two vectors $\mathbf{v}_1, \mathbf{v}_2$. What does it mean to say that these vectors are linearly dependent? Simply that one of the vectors can be expressed in terms of the other, in other words, that each vector is a scalar multiple of the other. However, matters are more complex when we proceed to three or more vectors, a point that is often overlooked. So we issue a warning here.

Caution: If we know that $\mathbf{v}_1, \mathbf{v}_{2,} \ldots, \mathbf{v}_n$ is linearly dependent, it does *not* necessarily imply that one of these vectors is a multiple of one of the others unless $n = 2$. In general, all we can say is that one of these vectors is a linear combination of the others.

Example 3.19. Which of the following lists of vectors have redundant vectors, i.e., are linearly dependent?

$$\text{(a)}\ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-2\end{bmatrix}\quad \text{(b)}\ \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\quad \text{(c)}\ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Solution. Let's try to see the big picture. Consider the vectors in each list to be designated as $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Define matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and vector $\mathbf{c} = (c_1, c_2, c_3)$. Then the general linear combination can be written as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = A\,\mathbf{c}.$$

This is the key idea of "linear combination as matrix–vector multiplication" that we saw in Theorem 2.1. Now we see that a nontrivial linear combination with value zero amounts to a nontrivial solution to the homogeneous equation $A\mathbf{c} = \mathbf{0}$. We know how to find these! In case (a) we have that

$$\begin{bmatrix}1&0&1\\1&1&-1\\0&1&-2\end{bmatrix}\xrightarrow[E_{21}(-1)]{}\begin{bmatrix}1&0&1\\0&1&-2\\0&1&-2\end{bmatrix}\xrightarrow[E_{32}(-1)]{}\begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\end{bmatrix},$$

so that the solutions to the homogeneous system are $c = (-c_3, 2c_3, c_3) = c_3(-1, 2, 1)$. Take $c_3 = 1$ and we have that

$$-1\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 0,$$

which shows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a linearly dependent list of vectors.

We'll solve (b) by a different method. Notice that

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1.$$

It follows that A is nonsingular, so the only solution to the system $A\mathbf{c} = 0$ is $\mathbf{c} = 0$. Since every linear combination of the columns of A takes the form $A\mathbf{c}$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must be linearly independent.

Finally, we see by inspection in (c) that since $\mathbf{v}_3$ is a repeat of $\mathbf{v}_1$, we have that

$$\mathbf{v}_1 + 0\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

Thus, this list of vectors is linearly dependent. Notice, by the way, that not every coefficient c_i has to be nonzero. □

Example 3.20. Show that any list of vectors that contains the zero vector is linearly dependent.

Solution. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be such a list and suppose that for some index j, $\mathbf{v}_j = 0$. Examine the following linear combination:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_j + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

This linear combination of value zero is nontrivial because the coefficient of the vector $\mathbf{v}_j$ is 1. Therefore this list is linearly dependent by the definition of dependence. □

The Basis Idea

We are now ready for one of the big ideas of vector space theory, the notion of a basis. We already know what a spanning set for a vector space V is. This is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. However, we saw back in Example 3.17 that some spanning sets are better than others because they are more economical. We know that a set of vectors has no redundant vectors in it if and only if it is linearly independent. This observation is the inspiration for the following definition.

Definition 3.9. A *basis* for the vector space V is a spanning set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ that is a linearly independent set.

Basis of Vector Space

Basis Is Minimal Spanning Set

We should take note here that we could have just as well defined a basis as a minimal spanning set, by which we mean a spanning set such that any proper subset is not a spanning set. The proof that this is equivalent to our definition of basis is left as an exercise.

Ordered Basis

Usually we think of a basis as a set of vectors and the order in which we list them is convenient but not important. Occasionally, ordering is important. In such a situation we speak of an *ordered basis* of $\mathbf{v}$, by which we mean a spanning list of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ that is a linearly independent list.

Example 3.21. Which subsets of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ yield bases of the vector space $\mathbb{R}^2$?

Solution. These are just the vectors of Example 3.17 and Example 3.18. Referring back to that example, we saw that

$$-\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = 0,$$

which told us that we could remove any one of these vectors and get the same span. Moreover, we saw that these three vectors span $\mathbb{R}^2$, so the same is true of any two of them. Clearly, a single vector cannot span $\mathbb{R}^2$, since the span of a single vector is a line through the origin. Therefore, the subsets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_3\}$ are all bases of $\mathbb{R}^2$. □

Example 3.22. Which subsets of $\left\{1 + x,\ x + x^2,\ 1,\ x\right\}$ yield bases of the vector space $\mathcal{P}_2$ of all polynomials of degree at most two?

Solution. Any linear combination of $1+x$, 1, and x yields a linear polynomial, so cannot equal x^2. Hence $x + x^2$ must be in the basis. On the other hand, any element of the set $\{x, 1 + x, 1\}$ can be expressed as a combination of the other two, so is redundant in the set. Discard redundant vectors from this set and we obtain three candidates for bases of $\mathcal{P}_2$: $\left\{x + x^2, 1 + x, 1\right\}$, $\left\{x + x^2, x, 1\right\}$, and $\left\{x + x^2, x, 1 + x\right\}$. It's easy to see that the span of any one of these sets contains 1, x, and x^2, so is a spanning set for $\mathcal{P}_2$. We leave it to the reader to check that each set contains no redundant vectors, hence is linearly independent. Therefore, each of these sets forms a basis of $\mathcal{P}_2$. □

Standard Basis

An extremely important generic type of basis is provided by the columns of the identity matrix. For future reference, we establish this notation. The *standard basis* of $\mathbb{R}^n$ or $\mathbb{C}^n$ is the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_j$ is the column vector of size n whose jth entry is 1 and all other entries 0.

Example 3.23. Let V be the standard vector space $\mathbb{R}^n$ or $\mathbb{C}^n$. Verify that the standard basis really is a basis of this vector space.

Solution. Let $\mathbf{v} = (c_1, c_2, \ldots, c_n)$ be a vector from V so that $c_1, c_2, \ldots, c_n$ are scalars of the appropriate type. Now we have

$$\begin{aligned}\mathbf{v} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ &= c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n.\end{aligned}$$

This equation tells us two things: first, every vector in V is a linear combination of the $\mathbf{e}_j$'s, so $V = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Second, if some linear combination of vectors has value zero, then each scalar coefficient of the combination is 0. Therefore, these vectors are linearly independent. Therefore the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis of V. □

Coordinates

In the case of the standard basis $\mathbf{e}_1, \mathbf{e}_2, , \mathbf{e}_3$ of $\mathbb{R}^3$ we know that it is very easy to write out any other vector $\mathbf{v} = (c_1, c_2, c_3)$ in terms of the standard basis:

$$\mathbf{v} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3.$$

We call the scalars c_1, c_2, c_3 the *coordinates* of the vector $\mathbf{v}$. Up to this point, this is the only sense in which we have used the term "coordinates." We can see that these coordinates are strongly tied to the standard basis. Yet $\mathbb{R}^3$ has many bases. Is there a corresponding notion of "coordinates" relative to other bases? The answer is a definite yes, thanks to the following fact.

Theorem 3.6. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis of the vector space V. Then every $\mathbf{v} \in V$ can be expressed uniquely as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, up to order of terms.

Uniqueness of Coordinates

Proof. To see this, note first that since

$$V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\},$$

there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Suppose that we could also write

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n.$$

Subtract these two equations and obtain

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_n - d_n)\mathbf{v}_n.$$

However, a basis is a linearly independent set, so it follows that each coefficient of this equation is zero, whence $c_j = d_j$, for $j = 1, 2, \ldots, n$. □

In view of this fact, we may speak of *coordinates of a vector relative to a basis.* Here is the notation that we employ:

Vector Coordinates and Coordinate Vector

Definition 3.10. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis B of the vector space V and $\mathbf{v} \in V$ with $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, then the scalars $c_1, c_2, \ldots, c_n$ are called the *coordinates of* $\mathbf{v}$ *with respect to the basis* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The coordinate vector of $\mathbf{v}$ with respect to B is $[\mathbf{v}]_B = (c_1, c_2, \ldots, c_n)$.

Standard Coordinates

As we have noted, coordinates of a vector with respect to the standard basis are what we have referred to as "coordinates" so far in this text. Perhaps we should call these the *standard coordinates* of a vector, but we will usually stick to the convention that an unqualified reference to a vector's coordinates assumes that we mean standard coordinates unless otherwise stated. Normally, vectors in $\mathbb{R}^n$ are given explicitly in terms of their standard coordinates, so these are trivial to identify. Coordinates with respect to other bases are fairly easy to calculate if we have enough information about the structure of the vector space.

Example 3.24. The following vectors form a basis of $\mathcal{P}_2$: $B = \left\{x + x^2, 1 + x, 1\right\}$ (see Example 3.22). Find the coordinate vector of $p(x) = 2 - 2x - x^2$ with respect to this basis.

Solution. The coordinates are c_1, c_2, c_3, where

$$2 - 2x - x^2 = c_1\left(x + x^2\right) + c_2(1 + x) + c_3 \cdot 1 = (c_2 + c_3) + (c_1 + c_2)x + c_1x^2.$$

We note here that the order in which we list the basis elements matters for the coordinates. Now we simply equate coefficients of like powers of x to obtain that $c_2 + c_3 = 2$, $c_1 + c_2 = -2$, and $c_1 = -1$. It follows that $c_2 = -2 - c_1 = -1$ and that $c_3 = 2 - c_2 = 3$. Thus, $[p(x)]_B = (-1, -1, 3)$. Incidentally, we note here that the order in which we list the basis elements matters for the coordinates. □

Example 3.25. The following vectors form a basis B of $\mathbb{R}^3$: $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (0, 2, 2)$, and $\mathbf{v}_3 = (1, 0, 1)$. Find the coordinate vector of $\mathbf{v} = (2, 1, 5)$ with respect to this basis.

Solution. Notice that the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ was given in terms of standard coordinates. Begin by writing

$$\begin{aligned}\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},\end{aligned}$$

where the coordinates c_1, c_2, c_3 of $\mathbf{v}$ relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are to be determined. This is a straightforward system of equations with coefficient matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and right-hand side $\mathbf{v}$. It follows that the solution we want is given by

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

This shows us that

$$\mathbf{v} = -1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

It does *not* prove that $\mathbf{v} = (-1, 1, 3)$, which is plainly false. Only in the case of the standard basis can we expect that a vector actually equals its vector of coordinates with respect to the basis. What we have is that the coordinate vector of $\mathbf{v}$ with respect to basis B is $[\mathbf{v}]_B = (-1, 1, 3)$. □

In general, vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent if and only if the system $A\mathbf{c} = 0$ has only the trivial solution, where $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$. This in turn is equivalent to the matrix A being of full column rank n (see Theorem 2.7, where we see that these are equivalent conditions for a matrix to be invertible). We can see how this idea can be extended, and doing so tells us something remarkable. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a basis of $V = \mathbb{R}^n$ and form the $n \times k$ matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$. By the same reasoning as in the example, for any $\mathbf{b} \in V$ there is a unique solution to the system $A\mathbf{x} = \mathbf{b}$. In view of Theorem 1.5 we see that A has full column rank k. Therefore, $k \leq n$. On the other hand, we can take $\mathbf{b}$ to be any one of the standard basis vectors e_j, $j = 1, 2, \ldots, n$, solve the resulting systems, and stack the solution vectors together to obtain a solution to the system $AX = I_n$. From our rank inequalities, we see that

Dimension Theorem for $\mathbb{R}^n$

$$n = \operatorname{rank} I_n = \operatorname{rank} AX \leq \operatorname{rank} A = k.$$

What this shows is that $k = n$, that is, every basis of $\mathbb{R}^n$ has exactly n elements in it, which would justify calling n the *dimension* of the space $\mathbb{R}^n$. Amazing! Does this idea extend to abstract vector spaces? Indeed it does, and we shall return to this issue in Section 3.5. Among other things, we have shown the following handy fact, which gives us yet one more characterization of invertible matrices to add to Theorem 2.7.

Theorem 3.7. An $n \times n$ real matrix A is invertible if and only if its columns are linearly independent, in which case they form a basis of $\mathbb{R}^n$.

Here is a problem that comes to us straight from analytical geometry (classification of conics) and shows how the matrix and coordinate tools we have developed can shed light on geometrical problems.

Example 3.26. Suppose we want to understand the character of the graph of the curve $x^2 - xy + y^2 - 6 = 0$. It is suggested to us that if we execute a change of variables by rotating the xy-axis by $\pi/4$ to get a new $x'y'$-axis, the graph will become more intelligible. OK, we do it. The algebraic connection between the coordinate pairs x, y and x', y' representing the same point in the plane and resulting from a rotation of θ can be worked out using a bit of trigonometry (which we omit) to yield

$$\begin{aligned} x' &= \quad x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta. \end{aligned}$$

Use matrix methods to formulate these equations and execute the change of variables.

Solution. First, we write the change of variable equations in matrix form as

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = G(\theta)\,\mathbf{x}.$$

Givens and Rotation Matrix

(Such a matrix $G(\theta)$ is often referred to as a *Givens* matrix.) This matrix isn't exactly what we need for substitution into our curve equation. Rather, we need x, y explicitly. That's easy enough. Simply invert $G(\theta)$ to obtain the rotationmatrix $R(\theta)$ as

$$G(\theta)^{-1} = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Therefore $\mathbf{x} = G(\theta)^{-1}\mathbf{x}' = R(\theta)\mathbf{x}'$. Now observe that the original equation can be put in the form (as in Example 2.33)

$$\begin{aligned} x^2 - xy + y^2 - 6 &= \mathbf{x}^T \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \mathbf{x} - 6 \\ &= (\mathbf{x}')^T\, \mathbf{R}(\theta)^T \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \mathbf{R}(\theta)\mathbf{x}' - 6. \end{aligned}$$

We leave it as an exercise to check that with $\theta = \pi/4$, so that $\cos\theta = 1/\sqrt{2} = \sin\theta$, the equation reduces to $\frac{1}{2}(x'^2 + 3y'^2) - 6 = 0$ or equivalently

$$\frac{x'^2}{12} + \frac{y'^2}{4} = 1.$$

This curve is an ellipse with semimajor axis of length $2\sqrt{3}$ and semiminor axis of length 2. With respect to the $x'y'$-axes, this ellipse is in so-called standard form. For a graph of the ellipse, see Figure 3.4. □

The change of variables we have just seen can be interpreted as a *change of coordinates* in the following sense: the variables x and y are just the standard coordinates (with respect to the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$) of a general vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2.$$

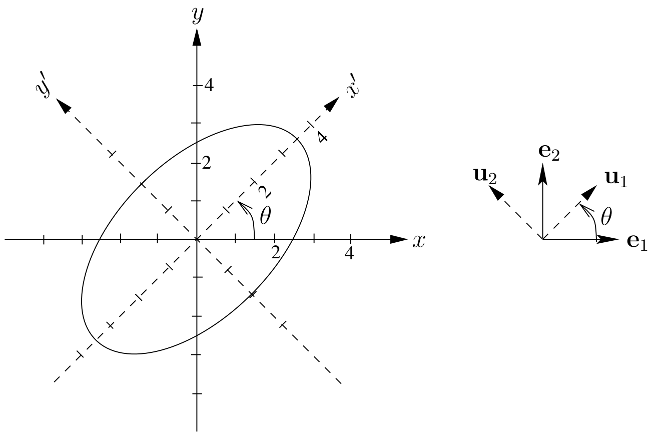


Fig. 3.4. Change of variables and the curve $x^2 - xy + y^2 - 6 = 0$.

The meaning of the variables x' and y' becomes clear when we set $\mathbf{x}' = (x', y')$ and write the matrix equation $\mathbf{x} = R(\theta)\mathbf{x}'$ out in detail as a linear combination of the columns of $R(\theta)$:

$$\mathbf{x} = R(\theta)\mathbf{x}' = x' \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y' \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = x'\mathbf{v}_1 + y'\mathbf{v}_2.$$

Thus the numbers x' and y' are just the coordinates of the vector $\mathbf{x}$ with respect to a new basis $C = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$. This basis consists of unit vectors in the direction of the x' and y' axes. See Figure 3.4 for a picture of the two bases. For these reasons, the matrix $R(\theta)$ is sometimes called a *change of coordinates* matrix.

The matrix $R(\theta)$ is also called a *change of basis* matrix, due to the fact that the coordinate equation above is equivalent to $[\mathbf{x}]_B = R(\theta)[\mathbf{x}]_C$. Thus $R(\theta)$ shows us how to change from the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ to another basis $C = \{\mathbf{v}_1, \mathbf{v}_2\}$. What makes a change of basis desirable is that sometimes a problem looks a lot easier if we look at it using a basis other than the standard one, such as in our example.

From a change of coordinates perspective, the vectors $\mathbf{x}$ and $\mathbf{x}'$ simply represent different coordinates for the same point and are connected by way of the formula $\mathbf{x} = R(\theta)\mathbf{x}'$. This is to be contrasted with the use of the rotation matrix $R(\theta)$ in Example 2.17. In that example we have only one coordinate system — the standard one — and we move a vector $\mathbf{x}$ by way of a rotation of θ in the counterclockwise direction to a new vector $\mathbf{y}$. This defined a linear operator, and the connection between the two vectors is that $\mathbf{y} = R(\theta)\mathbf{x} = T_{R(\theta)}(\mathbf{x})$.

Change of Basis Matrix

In general, a change of basis matrix from basis B to basis C of vector space V is a matrix P such that for any vector $\mathbf{v} \in V$, $[\mathbf{v}]_B = P[\mathbf{v}]_C$. These matrices are treated in more detail in Section 4.4. However, we will record this simple fact about change of basis matrices.

Change of Basis Formula

Theorem 3.8. If $V = \mathbb{R}^n$, B is the standard basis, and $C = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ any other basis, then the change of basis matrix from basis B to C is $P = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$.

Proof. To see this, note first that for any $\mathbf{v} \in V$, we have $\mathbf{v} = [\mathbf{v}]_B$ since B is the standard basis. Let

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

so that $c_1, c_2, \ldots, c_n$ are the coordinates of $\mathbf{v}$ relative to C. Then

$$\mathbf{v} = [v]_B = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]\,[c_1, c_2, \ldots, c_n]^T = P\,[\mathbf{u}_i]_C\,,$$

which shows that P is the change of basis matrix from B to C. □

3.3 Exercises and Problems

Exercise 1. Find the redundant vectors, if any, in the following lists.
(a) $(1,0,1)$, $(1,-1,1)$ (b) $(1,2,1)$, $(2,1,1)$, $(3,3,2)$, $(2,0,1)$
(c) $(1,0,-1)$, $(1,1,0)$, $(1,-1,-2)$ (d) $(0,1,-1)$, $(1,0,0)$, $(-1,1,3)$

Exercise 2. Find the redundant vectors, if any, in the following lists.
(a) x, $5x$ (b) 2, $2-x$, x^2, $1+x^2$
(c) $1+x$, $1+x^2$, $1+x+x^2$ (d) $x-1$, x^2-1, $x+1$

Exercise 3. Which of the following sets are linearly independent in $V = \mathcal{P}_3$? If not linearly independent, which vectors are redundant in the lists?
(a) $1, x, x^2, x^3$ (b) $1+x, 1+x^2, 1+x^3$
(c) $1-x^2, 1+x, 1-x-2x^2$ (d) $x^2-x^3, x, -x+x^2+3x^3$

Exercise 4. Which of the following sets are linearly independent in $V = \mathbb{R}^3$? If not linearly independent, which vectors are redundant in the lists?
(a) $(1,-1,0,1)\,,(-2,2,1,1)$ (b) $(1,1,0,0)\,,(1,0,1,0)\,,(1,0,0,1)\,,(-1,1,-2,0)$
(c) $(0,1,-1,2)\,,\ (0,1,3,4)\,,\ (0,2,2,6)$ (d) $(1,1,1,1)\,,\ (0,2,0,0)\,,\ (0,2,1,1)$

Exercise 5. Find the coordinates of $\mathbf{v}$ with respect to the following bases:
(a) $\mathbf{v} = (-1,1)$, basis $(2,1)$, $(2,-1)$ of $\mathbb{R}^2$.
(b) $\mathbf{v}$ $=2+x^2$, basis $1+x$, $x+x^2$, $1-x$ of $\mathcal{P}_2$.
(c) $\mathbf{v} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, basis $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ of the space of real symmetric 2×2 matrices.
(d) $\mathbf{v} = (1,2)$, basis $(2+\mathrm{i},1)$, $(-1,\mathrm{i})$ of $\mathbb{C}^2$.

Exercise 6. Find the coordinate vector of $\mathbf{v}$ with respect to the following bases:
(a) $\mathbf{v} = (0, 1, 2)$, basis $(2, 0, 1)$, $(-1, 1, 0)$, $(0, 1, 1)$ of $\mathbb{R}^3$.
(b)$\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, basis $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ of the space of upper triangular 2×2 matrices.
(c) $\mathbf{v} = (1, \mathrm{i}, \mathrm{i})$, basis $(1, 1, 0)$, $(0, 1, 1)$, $(0, 0, \mathrm{i})$ of $\mathbb{C}^3$.
(d) $\mathbf{v}$ =4, basis $1 + 2x$, $1 - x$ of $\mathcal{P}_1$.

Exercise 7. Let $\mathbf{u}_1 = (1, 0, 1)$ and $\mathbf{u}_2 = (1, -1, 1)$.
(a) Determine whether $\mathbf{v} = (2, 1, 2)$ belongs to the space span $\{\mathbf{u}_1, \mathbf{u}_2\}$.
(b) Find a basis of $\mathbb{R}^3$ that contains $\mathbf{u}_1$ and $\mathbf{u}_2$.

Exercise 8. Let $\mathbf{u}_1 = 1 - x + x^2$ and $\mathbf{u}_2 = x + 2x^2$.
(a) Determine whether $\mathbf{v} = 4 - 7x - x^2$ belongs to the space span $\{\mathbf{u}_1, \mathbf{u}_2\}$.
(b) Find a basis of $\mathcal{P}_2$ that contains $\mathbf{u}_1$ and $\mathbf{u}_2$.

Exercise 9. Given the information $\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}$, find all subsets of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that could form a minimal spanning set of span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Exercise 10. Given the information $2\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$ and $\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, find all subsets of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ that could form a minimal spanning set of span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Exercise 11. For what values of the parameter c is the set of vectors $(1, 1, c)$, $(2, c, 4)$, $(3c + 1, 3, -4)$ in $\mathbb{R}^3$ linearly independent?

Exercise 12. For what values of the parameter λ is the set of vectors $\big(1, \lambda^2, 1, 2\big)$, $(2, \lambda, 4, 8)$, $(0, 0, 1, 2)$ in $\mathbb{R}^4$ linearly dependent?

Exercise 13. Let e_{ij} be a matrix with a one in the (i, j)th entry and zeros elsewhere. Which 2×2 matrices e_{ij} can be added to the set below to form a basis of $\mathbb{R}^{2,2}$?

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Exercise 14. Which 2×2 matrices $e_{i,j}$ can be added to the set below to form a basis of $\mathbb{R}^{2,2}$?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Exercise 15. The Wronskian of smooth functions $f(x), g(x), h(x)$ is defined as

$$W(f, g, h)(x) = \det \begin{bmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{bmatrix}.$$

(A similar definition can be made for any number of functions.) Calculate the Wronskians of the polynomial functions of Exercise 2 (c) and (d). What does Problem 24 tell you about these calculations?

Exercise 16. Show that the functions e^x, x^3, and $\sin(x)$ are linearly independent in $C[0,1]$ in two ways:
(a) Use Problem 24.
(b) Assume that a linear combination with value zero exists and evaluate it at various points to obtain conditions on the coefficients.

Exercise 17. Let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $A = \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 1 \end{bmatrix}$. Calculate $R(\theta)^T A R(\theta)$ in the case that $\theta = \pi/4$.

Exercise 18. Use matrix methods as in Example 3.26 to express the equation of the curve $11x^2 + 10\sqrt{3}xy + y^2 - 16 = 0$ in new variables x', y' obtained by rotating the xy-axis by an angle of $\pi/4$.

Problem 19. Let $V = \mathbb{R}^{n,n}$ be the vector space of real $n \times n$ matrices and let $A, B \in \mathbb{R}^{n,n}$ be such that both are nonzero matrices, A is nilpotent (some power of A is zero), and B is idempotent ($B^2 = B$). Show that the subspace $W = \operatorname{span}\{A, B\}$ cannot be spanned by a single element of W.

Problem 20. Show that a basis is a minimal spanning set and conversely.

Problem 21. Let V be a vector space whose only subspaces are $\{\mathbf{0}\}$ and V. Show that V is the span of a single vector.

*Problem 22. Prove that a list of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with repeated vectors in it is linearly dependent.

Problem 23. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent elements of $\mathbb{R}^n$ and $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$. Show that $\operatorname{rank} A = k$.

*Problem 24. Show that smooth functions $f(x), g(x), h(x)$ are linearly dependent if and only if for all x, $W(f,g,h)(x) = 0$.

Problem 25. Show that a linear operator $T : V \to W$ maps a linearly dependent set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to linearly dependent set $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$, but if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ need not be linearly independent (give a specific counterexample).

*Problem 26. Suppose that a linear change of variables from old coordinates x_1, x_2 to new coordinates x_1', x_2' is defined by the equations

$$\begin{aligned} x_1 &= p_{11}x_1' + p_{12}x_2', \\ x_2 &= p_{21}x_1' + p_{22}x_2', \end{aligned}$$

where the 2×2 change of basis matrix $P = [p_{ij}]$ is invertible. Show that if a linear matrix multiplication function $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is given in old coordinates by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T_A(\mathbf{x}) = A\mathbf{x},$$

where $A = [a_{ij}]$ is any 2×2 matrix, then it is given by $\mathbf{y}' = P^{-1}AP\mathbf{x}' = T_{P^{-1}AP}(\mathbf{x}')$ in new coordinates.

3.4 Subspaces Associated with Matrices and Operators

Certain subspaces are a rich source of information about the behavior of a matrix or a linear operator. We define and explore the properties of these subspaces in this section.

Subspaces Defined by Matrices

There are three very useful subspaces that can be associated with a given matrix A. Understanding these subspaces is a great aid in vector space calculations that might have nothing to do with matrices per se, such as the determination of a minimal spanning set for a vector space. Each definition below is followed by an illustration using the following example matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}. \tag{3.2}$$

We make the default assumption that the scalars are the real numbers, but the definitions we will give can be stated just as easily for the complex numbers.

Caution: Do not confuse any of the spaces defined below with the matrix A itself. They are objects that are derived from the matrix, but do not even uniquely determine the matrix A.

Definition 3.11. The *column space* of the $m \times n$ matrix A is the subspace $\mathcal{C}(A)$ of $\mathbb{R}^m$ spanned by the columns of A. Column Space

Example 3.27. Describe the column space of the matrix A in equation (3.2).

Solution. Here we have that $\mathcal{C}(A) \subseteq \mathbb{R}^2$. Also

$$\mathcal{C}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Technically, this describes the column space in question, but we can do better. We saw in Example 3.17 that the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ was really redundant since it is a linear combination of the first two vectors. We also see that

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so that Theorem 3.4 shows us that

$$\mathcal{C}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

This description is much better, in that it exhibits a *basis* of $\mathcal{C}(A)$. It also shows that not all the columns of the matrix A are really needed to span the entire subspace $\mathcal{C}(A)$. □

Row Space **Definition 3.12.** The *row space* of the $m \times n$ matrix A is the subspace $\mathcal{R}(A)$ of $\mathbb{R}^n$ spanned by the transposes of the rows of A.

The "transpose" part of the preceding definition seems a bit odd. Why would we want rows to look like columns? It's a technicality, but later it will be convenient for us to have the row spaces live inside a $\mathbb{R}^n$ instead of an $(\mathbb{R}^n)^T$. Remember, we had to make a choice about $\mathbb{R}^n$ consisting of rows or columns. Just to make the elements of a row space look like rows, we can always adhere to the tuple notation instead of matrix notation. We gain one convenience: $\mathcal{R}(A) = \mathcal{C}(A^T)$, so that whatever we understand about column spaces can be applied to row spaces.

Example 3.28. Describe the row space of A in equation (3.2).

Solution. We have from the definition that

$$\mathcal{R}(A) = \operatorname{span}\{(1,1,1,-1),(0,1,2,1)\} \subseteq \mathbb{R}^4.$$

Now it's easy to see that neither one of these vectors can be expressed as a multiple of the other (if we had $c(1,1,1,-1) = (0,1,2,1)$, then read the first coordinates and obtain $c = 0$), so that span is given as economically as we can do, that is, the two vectors listed constitute a *basis* of $\mathcal{R}(A)$. □

Null Space **Definition 3.13.** The *null space* of the $m \times n$ matrix A is the subset $\mathcal{N}(A)$ of $\mathbb{R}^n$ defined by

$$\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

Observe that $\mathcal{N}(A)$ is the solution set to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. This means that null spaces are really very familiar. We were computing these solution sets way back in Chapter 1. We didn't call them subspaces at the time. Here is an application of this concept. Let A be a square matrix. We know that A is invertible exactly when the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (see Theorem 2.7). Now we can add one more equivalent condition to the long list of equivalences for invertibility: A is invertible exactly if $\mathcal{N}(A) = \{\mathbf{0}\}$. We next justify the subspace property implied by the term "null space."

Example 3.29. Use the subspace test to verify that $\mathcal{N}(A)$ really is a subspace of $\mathbb{R}^n$.

Solution. Since $A\mathbf{0} = \mathbf{0}$, the zero vector is in $\mathcal{N}(A)$. Now let c be a scalar and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ arbitrary elements of $\mathcal{N}(A)$. By definition, $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Add these two equations to obtain that

$$\mathbf{0} = \mathbf{0} + \mathbf{0} = A\mathbf{u} + A\mathbf{v} = A(\mathbf{u} + \mathbf{v}).$$

Therefore $\mathbf{u} + \mathbf{v} \in \mathcal{N}(A)$. Next multiply the equation $A\mathbf{u} = \mathbf{0}$ by the scalar c to obtain

$$\mathbf{0} = c\mathbf{0} = c(A\mathbf{u}) = A(c\mathbf{u}).$$

Thus we see from that definition that $c\mathbf{u} \in \mathcal{N}(A)$. The subspace test implies that $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$. □

Example 3.30. Describe the null space of the matrix A of equation (3.2).

Solution. Proceed as in Section 1.4. We find the reduced row echelon form of A, identify the free variables, and solve for the bound variables using the implied zero right-hand side and solution vector $x = [x_1, x_2, x_3, x_4]^T$:

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Pivots are in the first and second columns, so it follows that x_3 and x_4 are free, x_1 and x_2 are bound, and

$$\begin{aligned} x_1 &= x_3 + 2x_4 \\ x_2 &= -2x_3 - x_4. \end{aligned}$$

Let's write out the form of a general solution in terms of the free variables as a combination of x_3 times some vector plus x_4 times another vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

We have seen this clever trick before in Example 2.6. Remember that free variables can take on arbitrary values, so we see that the general solution to the homogeneous system has the form of an arbitrary linear combination of the two vectors on the right. In other words,

$$\mathcal{N}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4.$$

Neither of these vectors is a multiple of the other, so this is as economical an expression for $\mathcal{N}(A)$ as we can hope for. In other words, we have exhibited a minimal spanning set, that is, a *basis* of $\mathcal{N}(A)$. □

The following example relates null spaces to the idea of a limiting state for a Markov chain as discussed in Example 2.19. Recall that in that example we observed that the sequence of state vectors $\mathbf{x}^{(k)}$, $k = 0, 1, 2, \ldots$, appeared to converge to a steady-state vector $\mathbf{x}$, no matter what the initial (probability distribution) state vector $\mathbf{x}^{(0)}$. A stochastic matrix (Markov chain transition matrix) A that has this property is called *ergodic.* Null spaces can tell us something about such matrices.

Ergodic Matrix

Example 3.31. Suppose that a Markov chain has an ergodic transition matrix $A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$. Determine the steady-state vector for the Markov chain

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}.$$

Solution. We reason as follows: since the limit of the state vectors $\mathbf{x}^{(k)}$ is $\mathbf{x}$, and the state vectors are related by the formula

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)},$$

we can take the limits of both sides of this matrix equation and obtain that

$$\mathbf{x} = A\mathbf{x}.$$

Therefore

$$\mathbf{0} = \mathbf{x} - A\mathbf{x} = I\mathbf{x} - A\mathbf{x} = (I - A)\mathbf{x}.$$

It follows that $\mathbf{x} \in \mathcal{N}(I - A)$. Now

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.3 & -0.4 \\ -0.3 & 0.4 \end{bmatrix}.$$

Calculate the null space by Gauss–Jordan elimination:

$$\begin{bmatrix} 0.3 & -0.4 \\ -0.3 & 0.4 \end{bmatrix} \xrightarrow[E_1(1/0.3)]{E_{21}(1)} \begin{bmatrix} 1 & -4/3 \\ 0 & 0 \end{bmatrix}.$$

Therefore the null space of $I - A$ is spanned by the single vector $(4/3, 1)$. In particular, any multiple of this vector qualifies as a possible limiting vector. If we want a limiting vector whose entries are nonnegative and sum to 1 (which is required for states in a Markov chain), then the only choice is the vector resulting from dividing $(4/3, 1)$ by the sum of its coordinates to obtain

$$(3/7)(4/3, 1) = (4/7, 3/7) \approx (0.57143, 0.42857).$$

Interestingly enough, this is the vector that was calculated on page 78. □

Caution: We have no guarantee that the transition matrix A of the preceding example is actually ergodic. We have only experimental evidence so far. We will *prove* ergodicity using eigenvalue ideas in Chapter 5.

Here is a way of thinking about $\mathcal{C}(A)$. The key is the "linear combination as matrix–vector multiplication" idea that was first introduced in Example 2.9 and formalized in Theorem 2.1. Recall that it asserts that if matrix A has columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, i.e., $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$, and if $\mathbf{x} = [x_1, x_2, ..., x_n]^T$, then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

This equation shows that the column space of the matrix A can be thought of as the set of all possible matrix products $A\mathbf{x}$, i.e.,

$$\mathcal{C}(A) = \{A\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n\}\,.$$

An insight that follows from these observations: the linear combination of columns of A with coefficients from the vector $\mathbf{x}$ is zero exactly when $\mathbf{x} \in \mathcal{N}(A)$. Thus we can use null space calculations to identify redundant vectors in a set of column vectors, as in the next example.

Example 3.32. Find all possible linear combinations with value zero of the columns of matrix A of equation (3.2) and use this information to find a basis of $\mathcal{C}(A)$.

Solution. As in Example 3.30 we find the reduced row echelon form of A, identify the free variables, and solve for the bound variables using the implied zero right-hand side. The result is a solution vector $\mathbf{x} = (x_1, x_2, x_3, x_4) = (x_3 + 2x_4, -2x_3 - x_4, x_3, x_4)$. Write $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]$, and we see that the linear combinations of A are just

$$\mathbf{0} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = (x_3 + 2x_4)\,\mathbf{a}_1 - (2x_3 + x_4)\,\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4.$$

Here we think of x_3 and x_4 as free variables. Take $x_3 = 1$ and $x_4 = 0$, and we obtain $\mathbf{0} = \mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_3$, so that $\mathbf{a}_3$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. Similarly, take $x_3 = 0$ and $x_4 = 1$, and we obtain $\mathbf{0} = 2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_4$, so that $\mathbf{a}_4$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. Hence, $\mathcal{C}(A) = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\} = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$, the same conclusion we reached by trial and error in Example 3.27. □

Subspaces Defined by a Linear Operator

Suppose we are given a linear operator $T : V \to W$. We immediately have three spaces we can associate with the operator, namely, the domain V, target W, and range $\{\mathbf{y} \,|\, \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in V\}$ of the operator. The domain and range are vector spaces by definition of linear operator. That the range is a vector space is a nice example of using the subspace test.

Example 3.33. Show that if $T : V \to W$ is a linear operator, then range(T) is a subspace of W.

Solution. Apply the subspace test. First, we observe that range(T) contains $T(0)$. We leave it as an exercise for the reader to check that $T(0)$ is the zero element of W. Next let $\mathbf{y}$ and $\mathbf{z}$ be in range(T), say $\mathbf{y} = T(\mathbf{u})$ and $\mathbf{z} = T(\mathbf{v})$. We show closure of range(T) under addition: by the linearity property of T,

$$\mathbf{y} + \mathbf{z} = T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v}) \in \operatorname{range}(T),$$

where the latter term belongs to range(T) by the definition of image. Finally, we show closure under scalar multiplication: let c be a scalar, and we obtain from the linearity property of T that

$$c\mathbf{y} = cT(\mathbf{u}) = T(c\mathbf{u}) \in \mathrm{range}(T),$$

where the latter term belongs to range(T) by the definition of range. Thus, the subspace test shows that range(T) is a subspace of W. □

Here is another space that has proven to be very useful in understanding the nature of a linear operator.

Kernel of Operator

Definition 3.14. The *kernel* of the linear operator $T : V \to W$ is the subspace of V given by

$$\ker(T) = \{\mathbf{x} \in V \,|\, T(\mathbf{x}) = \mathbf{0}\}.$$

The definition claims that the kernel is a subspace and not merely a subset of the domain. This is true, and a proof of this fact is left to the exercises. In fact, we have been computing kernels since the beginning of the text. To see this, suppose that the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by matrix multiplication, that is, $T(\mathbf{x}) = T_A(\mathbf{x}) = A\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^n$. Then

$$\begin{aligned}\ker(T) &= \{\mathbf{x} \in \mathbb{R}^n \,|\, T_A(\mathbf{x}) = \mathbf{0}\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \,|\, A\mathbf{x} = \mathbf{0}\} \\ &= \mathcal{N}(A).\end{aligned}$$

In other words, for matrix operators kernels are the same thing as null spaces.

Here is one very nice application of kernels. Suppose we are interested in knowing whether a given operator $T : V \to W$ is one-to-one, i.e., whether the equation $T(\mathbf{u}) = T(\mathbf{v})$ always implies that $\mathbf{u} = \mathbf{v}$. For general functions this is a nontrivial question. If, for example, $V = W = \mathbb{R}$, then we could graph the function T and try to determine whether a horizontal line cut the graph twice. But for *linear* operators, the answer is very simple:

Theorem 3.9. The linear operator $T : V \to W$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Proof. If T is one-to-one, then only one element can map to $\mathbf{0}$ under T. Thus, $\ker(T)$ can consist of only one element. However, we know that $\ker(T)$ contains the zero vector since it is a subspace of the domain of T. Therefore, $\ker(T) = \{\mathbf{0}\}$.

Conversely, suppose that $\ker(T) = \{\mathbf{0}\}$. If $\mathbf{u}$ and $\mathbf{v}$ are such that $T(\mathbf{u}) = T(\mathbf{v})$, then subtract terms and use the linearity of T to obtain that

$$0 = T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u}) + (-1)T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}).$$

It follows that $\mathbf{u} - \mathbf{v} \in \ker(T) = \{\mathbf{0}\}$. Therefore, $\mathbf{u} - \mathbf{v} = \mathbf{0}$ and so $\mathbf{u} = \mathbf{v}$. □

Polynomials as Standard Vectors

Before we leave the topic of one-to-one linear mappings, let's digest its significance in a very concrete case. The space $\mathcal{P}_2 = \mathrm{span}\{1, x, x^2\}$ of polynomials of degree at most 2 has a basis of three elements, like $\mathbb{R}^3$, and it seems very reasonable to think that $\mathcal{P}_2$ is "just like" $\mathbb{R}^3$ in that a polynomial $p(x) = a + bx + cx^2$ is uniquely described by its vector of coefficients

$(a, b, c) \in \mathbb{R}^3$, and corresponding polynomials and vectors add and scalar multiply in a corresponding way. Here is the precise version of these musings: Define an operator $T : \mathcal{P}_2 \to \mathbb{R}^3$ by the formula $T(a + bx + cx^2) = (a, b, c)$. One can check that T is linear, the range of T is its target, $\mathbb{R}^3$, and $\ker(T) = 0$. By Theorem 3.9 the function T is one-to-one. Hence, it describes a one-to-one correspondence between elements of $\mathcal{P}_2$ and elements of $\mathbb{R}^3$ such that sums and scalar products in one space correspond to the corresponding sums and scalar products in the other. In plain words, this means we can get one of the vector spaces from the other simply by relabeling elements of one of the spaces. So, in a very real sense, they are "the same thing." More generally, whenever there is a one-to-one linear mapping of one vector space onto another, we say that the two vector spaces are *isomorphic*, which is a fancy way of saying that they are the same, up to a relabeling of elements. The mapping T itself is called an *isomorphism*. Actually, we have already encountered isomorphisms in the form of invertible linear operators. The following theorem, whose proof we leave as an exercise, explains the connection between these ideas.

Isomorphic Vector Spaces

Isomorphism

Theorem 3.10. The linear operator $T : V \to W$ is an isomorphism if and only if T is an invertible linear operator.

In summary, there are four important subspaces associated with a linear operator $T : V \to W$, the domain, target, kernel, and range. In symbols:

$$\begin{aligned}
\operatorname{domain}(T) &= V \\
\operatorname{target}(T) &= W \\
\ker(T) &= \{\mathbf{v} \in V \mid T(\mathbf{v}) = 0\} \\
\operatorname{range}(T) &= \{T(\mathbf{v}) \mid \mathbf{v} \in V\}.
\end{aligned}$$

There are important connections between these subspaces and those associated with a matrix. Let A be an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ the corresponding matrix operator defined by multiplication by A. We have

$$\begin{aligned}
\operatorname{domain}(T_A) &= \mathbb{R}^n \\
\operatorname{target}(T_A) &= \mathbb{R}^m \\
\ker(T_A) &= \mathcal{N}(A) \\
\operatorname{range}(T_A) &= \mathcal{C}(A).
\end{aligned}$$

The proofs of these are left to the exercises. One last example of subspaces associated with a linear operator $T : V \to W$ is really a whole family of subspaces. Suppose that U is a subspace of the domain V. Then we define the *image of U under T* to be the set

$$T(U) = \{T(u) \mid u \in U\}.$$

One can show that $T(U)$ is always a subspace of $\operatorname{range}(T)$. We leave the proof of this fact as an exercise. What this says is that a linear operator maps subspaces of its domain into subspaces of its range.

3.4 Exercises and Problems

Exercise 1. Find bases for null spaces of the following matrices.

(a) $\begin{bmatrix} 2 & -1 & 0 & 3 \\ 4 & -2 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 \\ -1 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -5 \\ 1 & 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Exercise 2. Find bases for null spaces of the following matrices.

(a) $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 6 & 2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 & \mathrm{i} \\ 2 & -2 & 2-\mathrm{i} \end{bmatrix}$

Exercise 3. Find bases for the column spaces of the matrices in Exercise 1.

Exercise 4. Find bases for the column spaces of the matrices in Exercise 2.

Exercise 5. Find bases for the row spaces of the matrices in Exercise 1.

Exercise 6. Find bases for the row spaces of the matrices in Exercise 2.

Exercise 7. For the following matrices find the null space of $I - A$ and find state vectors with nonnegative entries that sum to 1 in the null space, if any. Are these matrices ergodic (yes/no)?

(a) $A = \begin{bmatrix} 0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 8. Find the null space of $I - A$ and find state vectors (nonnegative entries that sum to 1) in the null space, if any, for the matrix $A = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$. Is this matrix ergodic? Explain your answer.

Exercise 9. For each of the following linear operators, find the kernel and range of the operator. Is the operator one-to-one? onto?

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $T((x_1, x_2, x_3)) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 - x_2 + 2x_3 \end{bmatrix}$

(b) $T : \mathcal{P}_2 \to \mathbb{R}$ and $T(p(x)) = p(1)$

Exercise 10. For each of the following linear operators, find the kernel and range of the operator. Is the operator one-to-one? onto?

(a) $T : \mathcal{P}_2 \to \mathcal{P}_3$ and $T(a + bx + cx^2) = ax + bx^2/2 + cx^3/3$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $T((x_1, x_2, x_3)) = \begin{bmatrix} 2x_2 \\ 3x_3 \end{bmatrix}$

Exercise 11. The linear operator $T : V \to \mathbb{R}^2$ is such that $T(\mathbf{v}_1) = (-1, 1)$, $T(\mathbf{v}_2) = (1, 1)$, and $T(\mathbf{v}_3) = (2, 0)$, where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis of V. Compute $\ker T$ and $\operatorname{range} T$. Is T one-to-one? onto? an isomorphism? *(Hint:* For the kernel calculation use Theorem 3.6 and find conditions on coefficients such that $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = \mathbf{0}$.)

Exercise 12. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a basis of the vector space V and let the linear operator $T : V \to \mathbb{R}^3$ be such that $T(\mathbf{v}_1) = (0, 1, 1)$, $T(\mathbf{v}_2) = (1, 1, 0)$, and $T(\mathbf{v}_3) = (-1, 0, 1)$. Compute $\ker T$ and $\operatorname{range} T$. Is T one-to-one? onto? an isomorphism?

Problem 13. Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the matrix multiplication operator defined by the $m \times n$ matrix A. Show that $\ker T_A = \mathcal{N}(A)$ and $\operatorname{range} T = \mathcal{C}(A)$.

Problem 14. Prove that if T is a linear operator, then for all $\mathbf{u}, \mathbf{v}$ in the domain of T and scalars c and d, we have $T(c\mathbf{u} - d\mathbf{v}) = cT(\mathbf{u}) - dT(\mathbf{v})$.

*Problem 15. Show that if $T : V \to W$ is a linear operator, then $T(\mathbf{0}) = \mathbf{0}$.

Problem 16. Show that if $T : V \to W$ is a linear operator, then the kernel of T is a subspace of V.

*Problem 17. Let the function $T : \mathbb{R}^3 \to \mathcal{P}_2$ be defined by

$$T([c_1, c_2, c_3]^T) = c_1 x + c_2(x - 1) + c_3 x^2.$$

Show that T is an isomorphism of vector spaces.

Problem 18. Let $T : V \to W$ be a linear operator and U a subspace of V. Show that the image of U, $T(U) = \{T(\mathbf{v}) \mid \mathbf{v} \in U\}$, is a subspace of W.

*Problem 19. Prove that if A is a nilpotent matrix then $\mathcal{N}(A) \neq \{\mathbf{0}\}$ and $\mathcal{N}(I - A) = \{\mathbf{0}\}$.

Problem 20. Let V be a vector space over the reals with basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Show that the linear operator $T : \mathbb{R}^n \to V$ given by

$$T((c_1, c_2, \ldots, c_n)) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

is an isomorphism of vector spaces.

3.5 Bases and Dimension

We have used the word “dimension” many times already, without really making the word precise. Intuitively, it makes sense when we say that $\mathbb{R}^2$ is “two-dimensional” or that $\mathbb{R}^3$ is “three-dimensional,” for we reason that it takes two coordinate numbers to determine a vector in $\mathbb{R}^2$ and three for a vector in $\mathbb{R}^3$. What can we say about general vector spaces? Is there some number that is a measure of the size of the vector space? We answer these questions in this section. In the familiar cases of geometrical vector spaces, the answers will confirm our intuition.

The Basis Theorem

We know that the standard vector spaces always have a basis: the standard basis. What about subspaces of a standard space? Or, for that matter, abstract vector spaces? It turns out that the answer in all cases is yes, but we will be satisfied to answer the question for a special class of abstract vector spaces. The following concept turns out to be helpful.

Finite-Dimensional Vector Space

Definition 3.15. The vector space V is called *finite-dimensional* if V has a finite spanning set.

Examples of finite-dimensional vector spaces are the standard spaces $\mathbb{R}^n$ and $\mathbb{C}^n$. As a matter of fact, we will see shortly that every subspace of a finite-dimensional vector space is finite-dimensional, and this includes most of the vector spaces we have studied so far. However, some very important vector spaces are *not* finite-dimensional, and accordingly, we call them *infinite-dimensional* spaces. Here is an example.

Example 3.34. Show that the space of all polynomial functions $\mathcal{P}$ is not a finite-dimensional space, while the subspaces $\mathcal{P}_n$ are finite-dimensional.

Solution. If $\mathcal{P}$ were a finite-dimensional space, then there would be a finite spanning set of polynomials $p_1(x), p_2(x), \ldots, p_m(x)$ for $\mathcal{P}$. This means that any other polynomial could be expressed as a linear combination of these polynomials. Let m be the maximum of all the degrees of the polynomials $p_j(x)$. Notice that any linear combination of polynomials of degree at most m must itself be a polynomial of degree at most m. (Remember that polynomial multiplication plays no part here, only addition and scalar multiplication.) Therefore, it is not possible to express the polynomial $q(x) = x^{m+1}$ as a linear combination of these polynomials, which means that they cannot be a basis. Hence, the space $\mathcal{P}$ has no finite spanning set.

On the other hand, it is obvious that the polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

is a linear combination of the monomials $1, x, \ldots, x^n$ from which it follows that $\mathcal{P}_n$ is a finite-dimensional space. □

Here is the first basic result about these spaces. It is simply a formalization of what we have already done with preceding examples.

Basis Theorem

Theorem 3.11. Every finite-dimensional vector space has a basis.

Proof. To see this, suppose that V is a finite-dimensional vector space with

$$V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

Now if the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ has a redundant vector in it, discard it and obtain a smaller spanning set of V. Continue discarding vectors until you reach a spanning set for V that has no redundant vectors in it. (Since you start with a finite set, this can't go on indefinitely.) By the redundancy test, this spanning set must be linearly independent. Hence it is a basis of V. □

The Dimension Theorem

No doubt you have already noticed that every basis of the vector space $\mathbb{R}^2$ must have exactly two elements in it. Similarly, one can reason geometrically that any basis of $\mathbb{R}^3$ must consist of exactly three elements. These numbers somehow measure the "size" of the space in terms of the degrees of freedom (number of coordinates) one needs to describe a general vector in the space. The dimension theorem asserts that this number can be unambiguously defined. As a matter of fact, the discussion on page 177 shows that every basis of $\mathbb{R}^n$ has exactly n elements. Our next stop: arbitrary finite-dimensional vector spaces. Along the way, we need a very handy theorem that is sometimes called the *Steinitz substitution principle.* This principle is a mouthful to swallow, so we will precede its statement with an example that illustrates its basic idea.

Example 3.35. Let $\mathbf{w}_1 = (1,-1,0)$, $\mathbf{w}_2 = (0,-1,1)$, $\mathbf{v}_1 = (0,1,0)$, $\mathbf{v}_2 = (1,1,0)$, and $\mathbf{v}_3 = (0,1,1)$. Then $\mathbf{w}_1, \mathbf{w}_2$ form a linearly independent set and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis of $V = \mathbb{R}^3$ (assume this). Show how to substitute both $\mathbf{w}_1$ and $\mathbf{w}_2$ into the set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ while substituting out some of the $\mathbf{v}_j$'s and at the same time retaining the basis property of the set.

Solution. Since $\mathbb{R}^3 = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we can express $\mathbf{w}_1$ as a linear combination of these vectors. We have a formal procedure for finding such combinations, but in this case we don't have to work too hard. A little trial and error shows that

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -2\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3,$$

so that $1\mathbf{w}_1 + 2\mathbf{v}_1 - \mathbf{v}_2 - 0\mathbf{v}_3 = 0$. It follows that $\mathbf{v}_1$ or $\mathbf{v}_2$ is redundant in the set $\mathbf{w}_1, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. So discard, say, $\mathbf{v}_2$, and obtain a spanning set $\mathbf{w}_1, \mathbf{v}_1, \mathbf{v}_3$. In fact, it is actually a basis of V since two vectors can span only a plane. Now start over: express $\mathbf{w}_2$ as a linear combination of this new basis. Again, a little trial and error shows that

$$\mathbf{w}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\mathbf{w}_1 - 2\mathbf{v}_1 + 1\mathbf{v}_3.$$

Therefore $\mathbf{v}_1$ or $\mathbf{v}_3$ is redundant in the set $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1, \mathbf{v}_3$. So discard, say, $\mathbf{v}_3$, and obtain a spanning set $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1$. Again, this set is actually a basis of V since two vectors can span only a plane; and this is the kind of set we were looking for. □

Theorem 3.12. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ be a linearly independent set in the space V and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis of V. Then $r \leq n$ and we may substitute all of the $\mathbf{w}_i$'s for some of the $\mathbf{v}_j$'s in such a way that the resulting set of vectors is still a basis of V.

Steinitz Substitution Principle

Proof. Let's do the substituting one step at a time. Suppose that $k < r$ and that we have relabeled the remaining $\mathbf{v}_i$'s so that

$$V = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$$

with $k + s \leq n$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ is a basis of V. (Notice that $k = 0$ and $s = n$ when we start, so $k + s = n$.)

We show how to substitute the next vector $\mathbf{w}_{k+1}$ into the basis. Certainly

$$V = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$$

as well, but this spanning set is linearly dependent since $\mathbf{w}_{k+1}$ is linearly dependent on $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$. Also, there have to be some $\mathbf{v}_i$'s left if $k < r$, for otherwise a proper subset of the $\mathbf{w}_j$'s would be a basis of V. Now use the redundancy test to discard, one at a time, as many of the $\mathbf{v}_j$'s from this spanning set as possible, all the while preserving the span. Again relabel the $\mathbf{v}_j$'s that are left so as to obtain for some $t \leq s$ a spanning set

$$\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k+1}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$$

of V from which no $\mathbf{v}_j$ can be discarded without shrinking the span. Could this set be linearly dependent? If so, there must be some equation of linear dependence among the vectors such that none of the vectors $\mathbf{v}_j$ occurs with a nonzero coefficient; otherwise, according to the redundancy test, such a $\mathbf{v}_j$ could be discarded and the span preserved. Therefore, there is an equation of dependency involving only the $\mathbf{w}_j$'s. This means that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ form a linearly dependent set, contrary to hypothesis. Hence, there is no such linear combination and the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k+1}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$ are linearly independent, as well as a spanning set of V. Now we must have discarded at least one of the $\mathbf{v}_i$'s since $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ is a linearly dependent set. Therefore, $t \leq s - 1$. It follows that

$$(k + 1) + t \leq k + 1 + s - 1 \leq k + s \leq n.$$

Now continue this process until $k = r$. □

Here is a nice application of the Steinitz substitution principle.

Corollary 3.2. Every linearly independent set in a finite-dimensional vector space can be expanded to a basis of the space.

Proof. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ be a linearly independent set in V and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ a basis of V. Apply the Steinitz substitution principle to the linearly independent set $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ and the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to obtain the desired basis of V that includes $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$. □

Next the dimension theorem is an easy consequence of Steinitz substitution, which has done the hard work for us.

Theorem 3.13. Let V be a finite-dimensional vector space. Then any two bases of V have the same number of elements, which is called the dimension of the vector space and denoted by $\dim V$.

Dimension Theorem

Proof. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be two given bases of V. Apply the Steinitz substitution principle to the linearly independent set $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ and the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to obtain that $r \leq n$. Now reverse the roles of these two sets in the substitution principle to obtain the reverse inequality $n \leq r$. We conclude that $r = n$, as desired. □

Remember that a vector space always carries a field of scalars with it. If we are concerned about that field we could specify it explicitly as part of the dimension notation. For instance, we could write

$$\dim \mathbb{R}^n = \dim_{\mathbb{R}} \mathbb{R}^n \text{ or } \dim \mathbb{C}^n = \dim_{\mathbb{C}} \mathbb{C}^n.$$

Usually, the field of scalars is clear from context and we don't need the subscript notation.

As a first application of the dimension theorem, let's dispose of the standard spaces. We already know from Example 3.23 that these vector spaces have a basis consisting of n elements, namely the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. According to the dimension theorem, this is all we need to specify the dimension of these spaces.

Corollary 3.3. For the standard spaces we have

$$\begin{aligned}\dim \mathbb{R}^n &= n\\ \dim \mathbb{C}^n &= n.\end{aligned}$$

There is one more question we want to answer. How do dimensions of a finite-dimensional vector space V and a subspace W of V relate to each other? At the outset, we don't even know whether W is finite-dimensional. Our intuition tells us that subspaces should have smaller dimension. Sure enough, our intuition is right this time!

Corollary 3.4. If W is a subspace of the finite-dimensional vector space V, then W is also finite-dimensional and

$$\dim W \leq \dim V,$$

with equality if and only if $V = W$.

Proof. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ be a linearly independent set in W and suppose that $\dim V = n$. According to the Steinitz substitution principle, $r \leq n$. So there is an upper bound on the number of elements of a linearly independent set in W. Now if the span of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ were smaller than W, then we could find a vector $\mathbf{w}_{r+1}$ in W but not in $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$. The new set $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{w}_{r+1}$ would also be linearly independent (we leave this fact as an exercise). Since we cannot continue adding vectors indefinitely, we have to conclude that at some point we obtain a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s$ for W. So W is finite-dimensional and furthermore, $s \leq n$, so we conclude that $\dim W \leq \dim V$. Finally, if we had equality, then a basis of W would be the same size as a basis of V. However, Steinitz substitution ensures that any linearly independent set can be expanded to a basis of V. It follows that a basis for W is also a basis for V, whence $W = V$. □

If U and V are subspaces of the vector space W, then $U + V = \{u + v \mid u \in U \text{ and } v \in V\}$ is also a subspace of W. These corollaries can be used to show how to calculate the dimension of $U + V$.

Corollary 3.5. If U and V are subspaces of the finite-dimensional vector space W, then $\dim(U + V) = \dim U + \dim V - \dim U \cap V$.

Proof. Corollary 3.4 shows that U, V, and $U \cap V$ are all finite-dimensional, say $\dim U = m$ and $\dim V = n$. Since $U \cap V$ is also a subspace of both U and V, $U \cap V$ has a basis, say $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$, with $r \leq m$ and $r \leq n$. Apply Corollary 3.2 to this basis to expand this basis to bases $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s$ of U and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$ of V. Then $r+s = m$ and $r+t = n$. We leave it as a exercise to verify that $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$ is a basis of $U + V$. Thus

$$\dim(U + V) = r + s + t = m + n - r = \dim U + \dim V - \dim U \cap V. \quad \square$$

3.5 Exercises and Problems

Exercise 1. Find all possible subsets of the following sets of vectors that form a basis of $\mathbb{R}^3$.

(a) $(1, 0, 1), (1, -1, 1)$ (b) $(1, 2, 1), (2, 1, 1), (3, 4, 1), (2, 0, 1)$

(c) $(2, -3, 1), (4, -2, -3), (0, -4, 5), (1, 0, 0), (0, 0, 0)$

Exercise 2. Find all possible subsets of the following sets of vectors that form a basis of $\mathbb{R}^{2,2}$.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

Exercise 3. Let $V = \mathbb{R}^3$ and $\mathbf{w}_1 = (2, 1, 0)$, $\mathbf{v}_1 = (1, 3, 1)$, $\mathbf{v}_2 = (4, 2, 0)$. The set $\mathbf{v}_1, \mathbf{v}_2$ is linearly independent in V. Determine which $\mathbf{v}_j$'s could be replaced by $\mathbf{w}_1$ while retaining the linear independence of the resulting set.

Exercise 4. Let $V = \mathbb{R}^3$ and $\mathbf{w}_1 = (0, 1, 0)$, $\mathbf{w}_2 = (1, 1, 1)$, $\mathbf{v}_1 = (1, 3, 1)$, $\mathbf{v}_2 = (2, -1, 1)$, $\mathbf{v}_3 = (1, 0, 1)$. The set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis of V. Determine which $\mathbf{v}_j$'s could be replaced by $\mathbf{w}_1$, and which $\mathbf{v}_j$'s could be replaced by both $\mathbf{w}_1$ and $\mathbf{w}_2$, while retaining the basis property.

Exercise 5. Let $V = C[0,1]$ and $\mathbf{w}_1 = \sin^2 x$, $\mathbf{w}_2 = \cos x$, $\mathbf{v}_1 = \sin x$, $\mathbf{v}_2 = \cos^2 x$, $\mathbf{v}_3 = 1$. The set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent in V. Determine which $\mathbf{v}_j$'s could be replaced by $\mathbf{w}_1$, and which $\mathbf{v}_j$'s could be replaced by both $\mathbf{w}_1$ and $\mathbf{w}_2$, while retaining the linear independence of the resulting set.

Exercise 6. Let $V = \mathcal{P}_2$ and $\mathbf{w}_1 = x$, $\mathbf{w}_2 = x^2$, $\mathbf{v}_1 = 1 - x$, $\mathbf{v}_2 = 2 + x$, $\mathbf{v}_3 = 1 + x^2$. The set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis of V. Determine which $\mathbf{v}_j$'s could be replaced by $\mathbf{w}_1$, and which $\mathbf{v}_j$'s could be replaced by both $\mathbf{w}_1$ and $\mathbf{w}_2$, while retaining the basis property.

Exercise 7. Let $\mathbf{w}_1 = (0, 1, 1)$. Expand $\{\mathbf{w}_1\}$ to a basis of $\mathbb{R}^3$.

Exercise 8. Let $\mathbf{w}_1 = x + 1$. Expand $\{\mathbf{w}_1\}$ to a basis of $\mathcal{P}_2$.

Exercise 9. Assume that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$, where V is a vector space of dimension n. Answer True/False to the following:
(a) If S is a basis of V then $k = n$.
(b) If S spans V then $k \leq n$.
(c) If S is linearly independent then $k \leq n$.
(d) If S is linearly independent and $k = n$ then S spans V.
(e) If S spans V and $k = n$ then S is a basis for V.
(f) If A is a 5 by 5 matrix and $\det A = 2$, then the first 4 columns of A span a 4 dimensional subspace of $\mathbb{R}^5$.

Exercise 10. Assume that V is a vector space of dimension n and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$. Answer True/False to the following:
(a) S is either a basis or contains redundant vectors.
(b) A linearly independent set contains no redundant vectors.
(c) If $V = \operatorname{span}\{\mathbf{v}_2, \mathbf{v}_3\}$ and $\dim V = 2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set.
(d) A set of vectors containing the zero vector is a linearly independent set.
(e) Every vector space is finite-dimensional.
(f) The set of vectors $[\mathrm{i}, 0]^T, [0, \mathrm{i}]^T, [1, \mathrm{i}]^T$ in $\mathbb{C}^2$ contains redundant vectors.

Problem 11. Let $V = \{\mathbf{0}\}$, a vector space with a single element. Explain why the element $\mathbf{0}$ is *not* a basis of V and the dimension of V must be 0.

*Problem 12. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ be linearly independent vectors in the vector space W. Show that if $\mathbf{w} \in W$ and $\mathbf{w} \notin \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$, then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{w}$ is a linearly independent set.

*Problem 13. Let $e_{i,j}$ be the $m \times n$ matrix with a unit in the (i, j)th entry and zeros elsewhere. Show that $\{e_{i,j} \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a basis of the vector space $\mathbb{R}^{m,n}$.

*Problem 14. Complete the proof of Corollary 3.5.

Problem 15. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be bases of U and V, respectively, where U and V are subspaces of the vector space W. Show by example that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are bases of U and V, respectively, and if $U \cap V \neq \{\mathbf{0}\}$, then their union need not be a basis if $U + V$.

*Problem 16. Determine the dimension of the subspace of $\mathbb{R}^{3,3}$ consisting of all symmetric matrices by exhibiting a basis.

Problem 17. Let U be the subspace of $W = \mathbb{R}^{3,3}$ consisting of all symmetric matrices and V the subspace of all skew-symmetric matrices.
(a) Show that $U \cap V = \{\mathbf{0}\}$ and $U + V = W$.
(b) Use Exercises 13, 16 and Corollary 3.5 to calculate $\dim V$.

Problem 18. Show that the functions $1, x, x^2, \ldots, x^n$ form a basis for the space $\mathcal{P}_n$ of polynomials of degree at most n.

Problem 19. Show that $C[0, 1]$ is an infinite-dimensional space.

Problem 20. Let $T : V \to W$ be a linear operator such that $\operatorname{range} T = W$ and $\ker T = \{0\}$. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis of V. Show that $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ is a basis of W.

*Problem 21. Let $p(x) = c_0 + c_1 x + \cdots + c_m x^m$ be a polynomial and A an $n \times n$ matrix. Use the result of Problem 13 to show that there exists a polynomial $p(x)$ of degree at most n^2 for which $p(A) = 0$. (Aside: this estimate is actually much too pessimistic. The Cayley–Hamilton theorem shows that n works in place of n^2.)

Problem 22. Show that a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in the vector space V is a basis if and only if it is a minimal spanning set, that is, it is a spanning set and no proper subset is a spanning set.

Problem 23. Let $T : V \to W$ be a linear operator where V is a finite-dimensional space and U is a subspace of V. Show that $\dim T(U) \leq \dim U$. (Show that the image of a spanning set for U under T is a spanning set for $T(U)$.)

Problem 24. Prove Corollary 3.2.

3.6 Linear Systems Revisited

We now have some very powerful tools for understanding the nature of solution sets of the standard linear system $A\mathbf{x} = \mathbf{b}$. This understanding will help us design practical computational methods for finding dimension and bases for vector spaces and other problems as well.

The first business at hand is to describe solution sets of inhomogeneous systems. Recall that every homogeneous system is consistent since it has the trivial solution. Inhomogeneous systems are another matter. We already have one criterion, namely that the rank of augmented matrix and coefficient matrix of the system must agree. Here is one more way to view the consistency of such a system in the language of vector spaces.

Consistency in Terms of Column Space

Theorem 3.14. The linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns is consistent if and only if $\mathbf{b} \in \mathcal{C}(A)$.

Proof. The key to this fact is Theorem 2.1, which says that the vector $A\mathbf{x}$ is a linear combination of the columns of A with the entries of $\mathbf{x}$ as scalar coefficients. Therefore, to say that $A\mathbf{x} = \mathbf{b}$ has a solution is simply to say that some linear combination of columns of A adds up to $\mathbf{b}$, i.e., $\mathbf{b} \in \mathcal{C}(A)$. □

Inclusion in a Span

The next example shows how to to determine whether a given vector belongs to a subspace specified by a spanning set of standard vectors.

Example 3.36. One of the following vectors belongs to the space V spanned by $\mathbf{v}_1 = (1, 1, 3, 3)$, $\mathbf{v}_2 = (0, 2, 2, 4)$, and $\mathbf{v}_3 = (1, 0, 2, 1)$. The vectors in question are $\mathbf{u} = (2, 1, 5, 4)$ and $\mathbf{w} = (1, 0, 0, 0)$. Which and why?

Solution. Theorem 3.14 tells us that if $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, then we need only determine whether the systems $A\mathbf{x} = \mathbf{u}$ and $A\mathbf{x} = \mathbf{w}$ are consistent. In the interests of efficiency, we may as well do both at once by forming the augmented matrix for both right-hand sides at once as

$$[A \,|\, \mathbf{u} \,|\, \mathbf{w}] = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & 0 \\ 3 & 2 & 2 & 5 & 0 \\ 3 & 4 & 1 & 4 & 0 \end{bmatrix}.$$

The reduced row echelon form of this matrix (whose calculation we leave as an exercise) is

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that there is a pivot in the fifth column but not in the fourth column. This tells us that the system with augmented matrix $[A \,|\, \mathbf{u}]$ is consistent, but the system with augmented matrix $[A \,|\, \mathbf{w}]$ is not consistent. Therefore

$\mathbf{u} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, but $\mathbf{w} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. As a matter of fact, the reduced row echelon form of $[A \,|\, \mathbf{u}]$ tells us what linear combinations will work, namely

$$\mathbf{u} = (2 - c_3)\mathbf{v}_1 - \frac{1}{2}(1 - c_3)\mathbf{v}_2 + c_3\mathbf{v}_3,$$

where c_3 is an arbitrary scalar. The reason for nonuniqueness of the coordinates of $\mathbf{u}$ is that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are not linearly independent. □

The next item of business is a description of the solution space itself, given that it is not empty. We already have a pretty good conceptual model for the solution of a homogeneous system $A\mathbf{x} = 0$. Remember that this is just the null space, $\mathcal{N}(A)$, of the matrix A. In fact, the definition of $\mathcal{N}(A)$ is the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. The important point here is that we proved that $\mathcal{N}(A)$ really is a subspace of the appropriate n-dimensional standard space $\mathbb{R}^n$ or $\mathbb{C}^n$. As such we can really picture it when n is 2 or 3: $\mathcal{N}(A)$ is either the origin, a line through the origin, a plane through the origin, or in the case $A = 0$, all of $\mathbb{R}^3$. What can we say about an inhomogeneous system? Here is a handy way of understanding these solution sets.

Form of General Solution

Theorem 3.15. Suppose the system $A\mathbf{x} = \mathbf{b}$ is consistent with a particular solution $\mathbf{x}_*$. Then the general solution $\mathbf{x}$ to this system can be described by the equation

$$\mathbf{x} = \mathbf{x}_* + \mathbf{z},$$

where $\mathbf{z}$ runs over all elements of $\mathcal{N}(A)$.

Proof. On the one hand, suppose we are given a vector of the form $\mathbf{x} = \mathbf{x}_* + \mathbf{z}$, where $A\mathbf{x}_* = \mathbf{b}$ and $\mathbf{z} \in \mathcal{N}(A)$. Then

$$A\mathbf{x} = A(\mathbf{x}_* + \mathbf{z}) = A\mathbf{x}_* + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Thus $\mathbf{x}$ is a solution to the system. Conversely, suppose we are given any solution $\mathbf{x}$ to the system and that $\mathbf{x}_*$ is a particular solution to the system. Then

$$A(\mathbf{x} - \mathbf{x}_*) = A\mathbf{x} - A\mathbf{x}_* = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Thus $\mathbf{x} - \mathbf{x}_* = \mathbf{z} \in \mathcal{N}(A)$, so that $\mathbf{x}$ has the required form $\mathbf{x}_* + \mathbf{z}$. □

This is really a pretty fact, so let's be clear about what it is telling us. It says that the solution space to a consistent system, as a set, can be described as the set of all translates of elements in the null space of A by some fixed vector. Such a set is sometimes called an *affine set* or a *flat.* When n is 2 or 3 this says that the solution set is either a single point, a line or a plane—*not* necessarily through the origin!

Example 3.37. Describe geometrically the solution sets to the system

$$\begin{aligned} x + 2y &= 3 \\ x + y + z &= 3. \end{aligned}$$

Solution. First solve the system, which has augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow[E_{21}(-1)]{} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{E_{12}(2) \\ E_2(-1)}]{} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

The general solution to the system is given in terms of the free variable z, which we will relabel as $z = t$ to obtain

$$\begin{aligned} x &= 3 - 2t \\ y &= t \\ z &= t. \end{aligned}$$

We may recognize this from calculus as a parametric representation of a line in three-dimensional space $\mathbb{R}^3$. Notice that this line does not pass through the origin since $z = 0$ forces $x = 3$. So the solution set is definitely not a subspace of $\mathbb{R}^3$. □

Now we turn to another computational matter. How do we find bases of vector spaces that are prescribed by a spanning set? How do we find the linear dependencies in a spanning set or implement the Steinitz substitution principle in a practical way? We have all the tools we need now to solve these problems. Let's begin with the question of finding a basis. We are going to solve this problem in two ways. Each has its own merits. First we examine the row space approach. We require two simple facts.

Theorem 3.16. Let A be any matrix and E an elementary matrix. Then

$$\mathcal{R}(A) = \mathcal{R}(EA).$$

Proof. Suppose the rows of A are the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, so that we have $\mathcal{R}(A) = \operatorname{span}\left\{\mathbf{r}_1^T, \mathbf{r}_2^T, \ldots, \mathbf{r}_n^T\right\}$. If $E = E_{ij}$, then the effect of multiplication by E is to switch the ith and jth rows, so the rows of EA are simply the rows of A in a different order. Hence, $\mathcal{R}(A) = \mathcal{R}(EA)$ in this case. If $E = E_i(a)$, with a a nonzero scalar, then the effect of multiplication by E is to replace the ith row by a nonzero multiple of itself. Clearly, this doesn't change the span of the rows either. To simplify notation, consider the case $E = E_{12}(a)$. Then the first row $\mathbf{r}_1$ is replaced by $\mathbf{r}_1 + a\mathbf{r}_2$, so that any combination of the rows of EA is expressible as a linear combination of the rows of A. Conversely, since $\mathbf{r}_1 = \mathbf{r}_1 + a\mathbf{r}_2 - a\mathbf{r}_2$, we see that any combination of $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ can be expressed in terms of the rows of EA. This proves the theorem. □

Theorem 3.17. If the matrix R is in a reduced row form, then the transposes of the nonzero rows of R form a basis of $\mathcal{R}(R)$.

Proof. Suppose the rows of R are given as $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, so that we have $\mathcal{R}(R)^T = \operatorname{span}\left\{\mathbf{r}_1^T, \mathbf{r}_2^T, \ldots, \mathbf{r}_k^T\right\}$, where the first k rows of R are nonzero and the remaining rows are zero rows. So certainly the nonzero rows span $\mathcal{R}(R)$. In order for these vectors to form a basis, they must also be a linearly independent set. If some linear combination of these vectors has value zero, say

$$\mathbf{0} = c_1\mathbf{r}_1 + \cdots + c_k\mathbf{r}_k,$$

we examine the first coordinate of this linear combination, corresponding to the column in which the first pivot appears. In that column $\mathbf{r}_1$ has a nonzero coordinate value and all other $\mathbf{r}_j$ have a value of zero. Therefore, the linear combination above yields that $c_1 = 0$. Repeat this argument for each index and we obtain that all $c_i = 0$. Hence the nonzero rows must be linearly independent. It follows that transposes of these vectors form a basis of $\mathcal{R}(R)$. □

These theorems are the foundations for the following algorithm for finding a basis for a vector space.

Row Space Algorithm

Given $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$.
(1) Form the $m \times n$ matrix A whose rows are $\mathbf{v}_1^T, \mathbf{v}_2^T, \ldots, \mathbf{v}_m^T$.
(2) Find a reduced row form R of A.
(3) List the nonzero rows of R. Their transposes form a basis of V.

Note 3.1. If *unique* answers are desired, we must use the reduced row echelon form of A, which is itself unique, rather than a reduced row form.

Example 3.38. Given that the vector space V is spanned by vectors $\mathbf{v}_1 = (1,1,3,3)$, $\mathbf{v}_2 = (0,2,2,4)$, $\mathbf{v}_3 = (1,0,2,1)$, and $\mathbf{v}_4 = (2,1,5,4)$, find a basis of V by the row space algorithm.

Solution. Form the matrix whose rows are these vectors:

$$A = \begin{bmatrix} 1&1&3&3 \\ 0&2&2&4 \\ 1&0&2&1 \\ 2&1&5&4 \end{bmatrix}.$$

Now find the reduced row echelon form of A:

$$\begin{bmatrix} 1&1&3&3 \\ 0&2&2&4 \\ 1&0&2&1 \\ 2&1&5&4 \end{bmatrix} \xrightarrow[\substack{E_{31}(-1) \\ E_{41}(-2) \\ E_2(1/2)}]{} \begin{bmatrix} 1&1&3&3 \\ 0&1&1&2 \\ 0&-1&-1&-2 \\ 0&-1&-1&-2 \end{bmatrix} \xrightarrow[\substack{E_{32}(1) \\ E_{42}(1) \\ E_{12}(-1)}]{} \begin{bmatrix} 1&0&2&1 \\ 0&1&1&2 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}.$$

From this we see that the vectors $(1,0,2,1)$ and $(0,1,1,2)$ form a basis for the row space of A. □

The second algorithm for computing a basis does more than find a basis: it formalizes an idea we encountered in Section 3.4 that determines when linear combinations have value zero.

Theorem 3.18. Let A be a matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Suppose the indices of the nonpivot columns in the reduced row echelon form of A are $i_1, i_2, \ldots, i_k$. Then every linear combination of value zero,

$$\mathbf{0} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n,$$

of the columns of A is uniquely determined by the values of $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$. In particular, if these coefficients are 0, then all the other coefficients must be 0.

Proof. Express the linear combination in the form

$$\mathbf{0} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]\mathbf{c} = A\mathbf{c},$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ and $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$. In other words, the column $\mathbf{c}$ of coefficients is in the null space of A. Every solution $\mathbf{c}$ to this system is uniquely specified as follows: assign arbitrary values to the free variables, then use the rows of the reduced row echelon form of A to solve for each bound variable. This is exactly what we wanted to show. □

In view of this theorem, we see that the columns of A corresponding to pivot columns in the reduced row echelon form of A must be themselves a linearly independent set. We also see from the proof that we can express any column corresponding to a nonpivot column in terms of columns corresponding to pivot columns by setting the free variable corresponding to the nonpivot column to 1, and all other free variables to 0. Therefore, the columns of A corresponding to pivot columns form a basis of $\mathcal{C}(A)$. This justifies the following algorithm for finding a basis for a vector space.

Column Space Algorithm

Given $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$ or $\mathbb{C}^m$.

(1) Form the $m \times n$ matrix A whose columns are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
(2) Find a reduced row form R of A.
(3) List the columns of A that correspond to pivot columns of R. These form a basis of V.

Caution: It is not the columns (or the rows) of the reduced row echelon form matrix R that yield the basis vectors for V. In fact, if E is an elementary matrix, in general we have $\mathcal{C}(A) \neq \mathcal{C}(EA)$.

Example 3.39. Given that the vector space V is spanned by vectors $\mathbf{v}_1 = (1, 1, 3, 3)$, $\mathbf{v}_2 = (0, 2, 2, 4)$, $\mathbf{v}_3 = (1, 0, 2, 1)$, and $\mathbf{v}_4 = (2, 1, 5, 4)$, find a basis of V by the column space algorithm.

Solution. Form the matrix A whose columns are these vectors and reduce the matrix to its reduced row echelon form:

$$\begin{bmatrix} 1&0&1&2\\ 1&2&0&1\\ 3&2&2&5\\ 3&4&1&4 \end{bmatrix} \xrightarrow[\substack{E_{21}(-1)\\ E_{31}(-3)\\ E_{41}(-3)}]{} \begin{bmatrix} 1&0&1&2\\ 0&2&-1&-1\\ 0&2&-1&-1\\ 0&4&-2&-2 \end{bmatrix} \xrightarrow[\substack{E_{32}(-1)\\ E_{42}(-2)\\ E_{2}(1/2)}]{} \begin{bmatrix} 1&0&1&2\\ 0&1&-1/2&-1/2\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}.$$

We can see from this calculation that the first and second columns will be pivot columns, while the third and fourth will not be. According to the column space algorithm, $\mathcal{C}(A)$ is a two-dimensional space with the first two columns of A for a basis. □

Note 3.2. While any reduced row form suffices, what is gained by the reduced row echelon form is the ability to use Theorem 3.18 to determine linear combinations of value zero easily.

Consider Example 3.39. From the first two rows of the reduced row echelon form of A we see that if $\mathbf{c} = (c_1, c_2, c_3, c_4)$ and $A\mathbf{c} = 0$, then

$$\begin{aligned} c_1 &= -\left(c_3 + 2c_4\right), \\ c_2 &= \frac{1}{2}\left(c_3 + c_4\right), \end{aligned}$$

and c_3, c_4 are free. Hence, the general linear combination with value zero is

$$-\left(c_3 + 2c_4\right)\mathbf{v}_1 + \frac{1}{2}\left(c_3 + c_4\right)\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

For example, take $c_3 = 0$ and $c_4 = 1$ to obtain

$$-2\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + 0\mathbf{v}_3 + 1\mathbf{v}_4 = \mathbf{0},$$

so that $\mathbf{v}_4 = 2\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$. A similar calculation with $c_3 = 1$ and $c_4 = 0$ shows that $\mathbf{v}_3 = \mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$.

Finally, we consider the problem of finding a basis for a null space. Actually, we have already dealt with this problem in an earlier example (Example 3.30), but now we will justify what we did there.

Theorem 3.19. Let A be an $m \times n$ matrix such that $\operatorname{rank} A = r$. Suppose the general solution to the homogeneous equation $A\mathbf{x} = 0$ with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is written in the form

$$\mathbf{x} = x_{i_1}\mathbf{w}_1 + x_{i_2}\mathbf{w}_2 + \cdots + x_{i_{n-r}}\mathbf{w}_{n-r},$$

where $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-r}}$ are the free variables. Then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-r}$ form a basis of $\mathcal{N}(A)$.

Proof. The vector $\mathbf{x} = 0$ occurs precisely when all the free variables are set equal to 0, for the bound variables are linear combinations of the free variables. This means that the only linear combinations with value zero of the vectors

$\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-r}$ are those for which all the coefficients $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-r}}$ are 0. Hence these vectors are linearly independent. They span $\mathcal{N}(A)$ since every element $\mathbf{x} \in \mathcal{N}(A)$ is a linear combination of them. Therefore, $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-r}$ form a basis of $\mathcal{N}(A)$. □

The formula in Theorem 3.19 shows that each of the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-r}$ is recovered from the general solution by setting one free variable to one and the others to zero. It also shows that the following algorithm is valid.

Null Space Algorithm

Given an $m \times n$ matrix A.

(1) Compute the reduced row echelon form R of A.
(2) Use R to find the general solution to the homogeneous system $A\mathbf{x} = 0$.
(3) Write the general solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to the homogeneous system in the form

$$\mathbf{x} = x_{i_1}\mathbf{w}_1 + x_{i_2}\mathbf{w}_2 + \cdots + x_{i_{n-r}}\mathbf{w}_{n-r},$$

where $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-r}}$ are the free variables.
(4) List the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-r}$. These form a basis of $\mathcal{N}(A)$.

Example 3.40. Find a basis for the null space of the matrix A in Example 3.39 by the null space algorithm.

Solution. We already found the reduced row echelon form of A as

$$R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The variables x_3 and x_4 are free, while x_1 and x_2 are bound. Hence the general solution of $A\mathbf{x} = 0$ can be written as

$$\begin{aligned} x_1 &= -x_3 - 2x_4, \\ x_2 &= \frac{1}{2}x_3 + \frac{1}{2}x_4, \\ x_3 &= x_3, \\ x_4 &= x_4, \end{aligned}$$

which becomes, in vector notation,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $\mathbf{w}_1 = (-1, 1/2, 1, 0)$ and $\mathbf{w}_1 = (-2, 1/2, 0, 1)$ form a basis of $\mathcal{N}(A)$. □

A summary of key dimensional facts that we have learned in this section:

Rank Theorem

Theorem 3.20. Let A be an $m \times n$ matrix such that $\operatorname{rank} A = r$. Then
(1) $\dim \mathcal{C}(A) = r$
(2) $\dim \mathcal{R}(A) = r$
(3) $\dim \mathcal{N}(A) = n - r$

The following example is an application of vector space tools to matrix computation issues.

Rank-One Matrix as Outer Product

Example 3.41. Show that every rank-one matrix can be expressed as an outer product of vectors.

Solution. Let A be an $m \times n$ rank-one matrix. Let the rows of A be $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Since $\dim \mathcal{R}(A) = 1$, the row space of A is spanned by a single row of A, say the kth one. Hence there are constants $c_1, c_2, \ldots, c_m$ such that $\mathbf{r}_j = c_j \, \mathbf{r}_k$, $k = 1, \ldots, m$. Let $\mathbf{c} = [c_1, c_2, \ldots, c_m]^T$ and $\mathbf{d} = \mathbf{r}_k^T$, and it follows that A and the outer product of $\mathbf{c}$ and $\mathbf{d}$, $\mathbf{c}\mathbf{d}^T$, have the same rows, hence are equal. □

3.6 Exercises and Problems

Exercise 1. Use the fact that B is a reduced row form of A to find bases for the row and column spaces of A with no calculations, and null space with minimum calculations, where $A = \begin{bmatrix} 3 & 5 & -1 & 5 & 1 \\ 1 & 2 & -1 & 2 & 0 \\ 2 & 3 & 0 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Exercise 2. Let $A = \begin{bmatrix} 3 & 1 & -2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & -1 & 1 & 2 & 2 \\ 3 & 2 & -1 & 1 & 1 & 8 & 9 \\ 0 & 2 & 2 & -1 & 1 & 6 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & -4 & -6 \\ 0 & 2 & 2 & 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & -2 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 & 6 & 7 \end{bmatrix}$, and repeat Exercise 1.

Exercise 3. Find two bases for the space spanned by each of the following sets of vectors by using the row space algorithm and column space algorithm with the reduced row echelon form.
(a) $(0, -1, 1)$, $(2, 1, 1)$ in $\mathbb{R}^3$.
(b) $(2, -1, 1)$, $(2, 0, 1)$, $(-4, 2, -2)$ in $\mathbb{R}^3$.
(c) $(1, -1)$, $(2, 2)$, $(-1, 2)$, $(2, 0)$ in $\mathbb{R}^2$.
(d) $1+x^2$, $-2-x+3x^2$, $5+x$, $4+4x^2$ in $\mathcal{P}_2$. (*Hint:* See the discussion following Theorem 3.9 of Section 3.4 for a way of thinking of polynomials as vectors.)

Exercise 4. Find two bases for each of the following sets of vectors by using the row space algorithm and the column space algorithm.
(a) $(1, -1)$, $(1, 1)$, $(2, 0)$ in $\mathbb{R}^2$.
(b) $(2, 2, -4)$, $(-4, -4, 8)$ in $\mathbb{R}^3$.
(c) $(1, 0, 0)$, $(1 + \mathrm{i}, 2, 2 - \mathrm{i})$, $(-1, 0, \mathrm{i})$ in $\mathbb{C}^3$.
(d) $1 + 2x + 2x^3$, $-2 - 5x + 5x^2 + 6x^3$, $-x + 5x^2 + 6x^3$, $x - 5x^2 + 4x^3$ in $\mathcal{P}_3$.

Exercise 5. Find bases for the row, column, and null space of each of the following matrices.

(a) $[2, 0, -1]$ (b) $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 3 & 6 & 2 & 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 1 & 3 & 5 & 2 & 1 \\ 2 & 3 & -5 & 10 & 0 \\ 2 & 4 & 0 & 8 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -3 & -1 \\ 0 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

Exercise 6. Find bases for the row, column, and null space of the following.

(a) $\begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 & 0 & -4 & 0 & -2 \\ 2 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 0 & -2 & 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & i & 0 \\ 1 & 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 2 & 2 \end{bmatrix}$

Exercise 7. Find all possible linear combinations with value zero of the following sets of vectors and the dimension of the space spanned by them.
(a) $(0,1,1)$, $(2,0,1)$, $(2,2,3)$, $(0,2,2)$ in $\mathbb{R}^3$.
(b) x, x^2+x, x^2-x in $\mathcal{P}_2$.
(c) $(1,1,2,2)$, $(0,2,0,2)$, $(1,0,2,1)$, $(2,1,4,4)$ in $\mathbb{R}^4$.

Exercise 8. Repeat Exercise 7 for the following sets of vectors.
(a) $(1,1,3,3)$, $(0,2,2,4)$, $(1,0,2,1)$, $(2,1,5,4)$ in $\mathbb{R}^4$.
(b) $1+x$, $1+x-x^2$, $1+x+x^2$, $x-x^2$, $1+2x$ in $\mathcal{P}_2$.
(c) $\cos(2x)$, $\sin^2 x$, $\cos^2 x$, 2 in $C[0,1]$.

Exercise 9. Let $A = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}$, $U = \mathcal{C}(A)$, and $V = \mathcal{C}(B)$.
(a) Compute $\dim U$ and $\dim V$.
(b) Use the column algorithm on the matrix $[A, B]$ to compute $\dim(U+V)$.
(c) Use Corollary 3.5 of Section 3.5 to determine $\dim U \cap V$.

Exercise 10. Repeat Exercise 9 with $A = \begin{bmatrix} 4 & 3 & 5 \\ 5 & 4 & 3 \\ 2 & 1 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 3 \\ -2 & -1 & -4 \\ 7 & 5 & 17 \end{bmatrix}$.

Exercise 11. Find a basis of $U \cap V$ in Exercise 9. (*Hint:* solve the system $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{0}$ and use the fact that any nonzero solution will give an element in the intersection, namely $A\mathbf{x}$ or $B\mathbf{y}$. Now just look for the right number of linearly independent elements in the intersection.)

Exercise 12. Find a basis of $U \cap V$ in Exercise 10.

*Problem 13. Suppose that the linear system $A\mathbf{x} = \mathbf{b}$ is a consistent system of equations, where A is an $m \times n$ matrix and $\mathbf{x} = [x_1, \ldots, x_n]^T$. Prove that if the set of columns of A has redundant vectors in it, then the system has more than one solution.

Problem 14. Use Theorem 3.16 and properties of invertible matrices to show that if P and Q are invertible and PAQ is defined, then $\operatorname{rank} PAQ = \operatorname{rank} A$.

*Problem 15. Let A be an $m \times n$ matrix of rank r. Suppose that there exists a vector $\mathbf{b} \in R^m$ such that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent. Use the consistency and rank theorems of this section to deduce that the system $A^T\mathbf{y} = \mathbf{0}$ must have nontrivial solutions.

Problem 16. Use the rank theorem and Problem 14 to prove that if P and Q are invertible and PAQ is defined, then $\dim \mathcal{N}(PAQ) = \dim \mathcal{N}(A)$.

3.7 *Computational Notes and Projects

Spaces Associated with a Directed Graph

An example will illustrate some of the basic ideas. You should also review the material of page 79.

Example 3.42. Suppose we have a communications network that connects five nodes, which we label $1, 2, 3, 4, 5$. Communications between points are not necessarily two-way. We specify the network by listing ordered pairs (i, j), the meaning of which is that information can be sent from point i to point j. For our problem the connection data is the set

$$E = \{(1,2), (3,1), (1,4), (2,3), (3,4), (3,5), (4,2), (5,3)\}.$$

A *loop* means a walk that starts and ends at the same node, i.e., a sequence that connects a node to itself. For example, the sequence $(3,5), (5,3)$ is a loop in our example. It is important to be able to account for loops in such a network. They give us subsets of nodes for which two-way communication between any two points is possible (start at one point and follow the arrows until you reach the other). Find all the loops of this example and formulate an algorithm that one could program to compute all the loops of the network.

Solution. These are data that can be modeled by a directed graph as in Example 2.18. Thus, nodes are vertices in the graph and connections are edges. See Figure 3.5 for a picture of this graph.

It isn't so simple to eyeball this picture and count all loops. In fact, if you count going around and around in the same loop as different from the original loop, there are infinitely many loops. Perhaps we should be a bit more systematic. Let's count the smallest loops only, that is, the loops that are not themselves a sum of other loops. It appears that there are only three of these, namely,

$$L_1 : (3,5), (5,3), \quad L_2 : (2,3), (3,4), (4,2), \quad L_3 : (1,2), (2,3), (3,1).$$

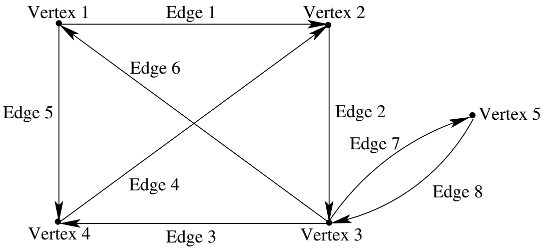


Fig. 3.5. Data from Example 3.42.

There are other loops, e.g., $L_4 : (2,3), (3,5), (5,3), (3,4), (4,2)$. But this loop is built up out of L_1 and L_2. In a certain sense, $L_4 = L_1 + L_2$. This suggests a "calculus of loops." Lurking in the background is another matrix, different from the adjacency matrix that we encountered in Section 2.3, that describes all the data necessary to construct the graph. It is called the *incidence matrix* of the graph and is given as follows: the incidence matrix has its rows indexed by the vertices of the graph and its columns by the edges. If the edge (i, j) is in the graph, then the column corresponding to this edge has a -1 in its ith row and a $+1$ in its jth row. All other entries are 0. In our example we see that the vertex set is $V = \{1, 2, 3, 4, 5\}$, the edge set is $E = \{(1,2), (2,3), (3,4), (4,2), (1,4), (1,3), (3,5), (5,3)\}$, and so the incidence matrix is $A = \begin{bmatrix} \mathbf{v}_1 \, \mathbf{v}_2 \, \mathbf{v}_3 \, \mathbf{v}_4 \, \mathbf{v}_5 \, \mathbf{v}_6 \, \mathbf{v}_7 \, \mathbf{v}_8 \end{bmatrix}$ with reduced row echelon form E, where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We leave the calculation of E to the reader.

We can now describe all loops. Each column of A defines an edge. Thus, linear combinations of these columns with integer coefficients, say $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_8\mathbf{v}_8 = A\mathbf{c}$, represent a listing of edges, possibly with repeats. Consider such a linear combination with defining vector of coefficients $\mathbf{c} = (c_1, \ldots, c_8)$. When will such a combination represent a loop? For one thing, the coefficients should all be nonnegative integers. But this isn't enough. Here's the key idea: examine this combination locally, that is, at each vertex. There we expect the total number of "in-arrows" (-1's) to be exactly canceled by the total number of "out-arrows" ($+1$'s). In other words, each coordinate of $\mathbf{v}$ should be 0 and so $\mathbf{c} \in \mathcal{N}(A)$. Now find a basis of $\mathcal{N}(A)$ using the reduced row echelon form E of A. We see that the free variables are c_4, c_5, c_6, c_8. The general form of an element of the null space takes the form $\mathbf{c} = c_4\mathbf{w}_4 + c_5\mathbf{w}_5 + c_6\mathbf{w}_6 + c_8\mathbf{w}_8$, where the columns are by the null space algorithm: $\mathbf{w}_4 = (0,1,1,1,0,0,0,0)$, $\mathbf{w}_5 = (-1,-1,-1,0,1,0,0,0)$, $\mathbf{w}_6 = (1,1,0,0,0,1,0,0)$, and $\mathbf{w}_8 = (0,0,0,0,0,0,1,1)$. Now we see that the "loop vectors" $\mathbf{w}_8, \mathbf{w}_4, \mathbf{w}_6$ represent the loops L_1, L_2, L_3, respectively. The vector $\mathbf{v}_5$ doesn't represent a loop in the sense that we originally defined them, since we allowed loops to move only in the direction of the edge arrow. However, we need $\mathbf{w}_5$ to describe all loops. For example, $\mathbf{w}_4 + \mathbf{w}_5 + \mathbf{w}_6 = (0,1,0,1,1,1,0,0)$ represents a loop that cannot be represented in terms of $\mathbf{w}_4, \mathbf{w}_6, \mathbf{w}_8$ alone. □

This null space calculation is trying to tell us something, namely that if we allowed for paths that moved against the direction of the edge arrows when the coefficient of the edge is negative, we would have *four* independent loops. These "algebraic" loops include our original "graphical" loops. They are much easier to calculate since we don't have to worry about all the coefficients c_i being positive. They may not be directly useful in the context of communications networks, since they don't specify a flow of information unless they are graphical; but in the context of electrical circuits they are very important. In fact, the correct definition of a "loop" in electrical circuit theory is an element $\mathcal{N}(A)$ with integer coefficients.

Project: Modeling with Directed Graphs II

Project Description: This assignment introduces you to another application of (directed) graph as mathematical modeling tool. You are given that the (directed) graph G has vertex set $V = \{1,2,3,4,5,6\}$ and edge set $E = \{(2,1),(1,5),(2,5),(5,4),(3,6),(4,2),(4,3),(3,2),(6,4),(6,1)\}$. We can draw a picture as in Figure 3.5. Answer the following questions about G.

1. What does the picture of this graph look like? You may leave space in your report and draw this by hand, or if you prefer, you may use the computer drawing applications available to you on your system.

2. Find $\mathcal{N}(A)$ and $\mathcal{N}(A^T)$ using a computer algebra system available to you. What does the former tell you about the loop structure of the circuit? Distinguish between graphical and "algebraic" loops.

3. Think of the digraph as representing an electrical circuit where an edge represents some electrical object like a resistor or capacitor. Each node represents the circuit space between these objects. and we can attach a potential value to each node, say the potentials are $x_1, \ldots, x_6$. The potential difference across an edge is the potential value of head minus tail. Kirchhoff's second law of electrical circuits says that the sum of potential differences around a circuit loop must be zero. Use the fact that $A\mathbf{x} = \mathbf{b}$ implies that for all $\mathbf{y} \in \mathcal{N}(A^T)$, $\mathbf{y}^T\mathbf{b} = 0$ to find conditions that a vector $\mathbf{b}$ must satisfy in order for it to be a vector of potential differences for some potential distribution on the vertices.

4. Across each edge of a circuit a current flows. Thus we can assign to each edge a "weight," namely the current flow along the edge. This is an example of a *weighted* digraph. However, not just any set of current weights will do, since Kirchhoff's first law of circuits says that the total flow of current in and out of any node should be 0. Use this law to find a matrix condition that must be satisfied by the currents.

4

GEOMETRICAL ASPECTS OF STANDARD SPACES

The standard vector spaces have many important extra features that we have ignored up to this point. These extra features made it possible to do sophisticated calculations in the spaces and enhanced our insight into vector spaces by appealing to geometry. For example, in the geometrical spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ that were studied in calculus, it was possible to compute the length of a vector and angles between vectors. These are visual concepts that feel very comfortable to us. In this chapter we are going to generalize these ideas to the standard spaces and their subspaces. We will abstract these ideas to general vector spaces in Chapter 6.

4.1 Standard Norm and Inner Product

Throughout this chapter vector spaces will be assumed to be subspaces of the standard vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$.

The Norm Idea

We dealt with sequences of vectors in Chapters 2 and 3 in an informal way. Consider this problem. How do we formulate precisely the idea of a sequence of vectors $\mathbf{u}_i$ converging to a limit vector $\mathbf{u}$, i.e.,

$$\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u},$$

in standard spaces? A reasonable answer is to mean that the distance between the vectors should tend to 0 as $n \to \infty$. By *distance* we mean the length of the difference. So what we need is some idea about the *length,* i.e., *norm*, of a vector. We have seen such an idea in the geometrical spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. There are different ways to measure length. We shall begin with the most standard method. It is one of the outcomes of geometry and the Pythagorean theorem. As with standard spaces, there is no compelling reason to stop at geometrical dimensions of two or three, so here is the general definition.

Standard Real Vector Norm

Definition 4.1. Let $\mathbf{u} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The *(standard) norm* of $\mathbf{u}$ is the nonnegative real number

$$\|\mathbf{u}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example 4.1. Compute the norms of the vectors $\mathbf{u} = (1, -1, 3)$ and $\mathbf{v} = [2, -1, 0, 4, 2]^T$.

Solution. From the definition,

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11} \approx 3.3166,$$

and

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 0^2 + 4^2 + 2^2} = \sqrt{25} = 5. \qquad \square$$

Even though we can't really "see" the five-dimensional vector y of this example, it is interesting to note that calculating its length is just as routine as calculating the length of the three-dimensional vector x. What about complex vectors? Shouldn't there be an analogous definition for such objects? The answer is yes, but we have to be a little careful. We can't use the same definition that we did for real vectors. Consider the vector $x = (1, 1 + \mathrm{i})$. The sum of the squares of the coordinates is just

$$1^2 + (1 + \mathrm{i})^2 = 1 + 1 + 2\mathrm{i} - 1 = 1 + 2\mathrm{i}.$$

This isn't good. We don't want "length" to be measured in complex numbers. The fix is very simple. We already have a way of measuring the length of a complex number z, namely the absolute value $|z|$, so length squared is $|z|^2$. That is the inspiration for the following definition, which is entirely consistent with our first definition when applied to real vectors:

Standard Complex Vector Norm

Definition 4.2. Let $\mathbf{u} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. The *(standard) norm* of $\mathbf{u}$ is the nonnegative real number

$$\|\mathbf{u}\| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}.$$

Notice that $|z|^2 = \overline{z}z$. (Remember that if $z = a + b\mathrm{i}$, then $\overline{z} = a - b\mathrm{i}$ and $\overline{z}z = a^2 + b^2 = |z|^2$.) Therefore,

$$\|\mathbf{u}\| = \sqrt{\overline{z}_1 z_1 + \overline{z}_2 z_2 + \cdots + \overline{z}_n z_n}.$$

Example 4.2. Compute the norms of the vectors $\mathbf{u} = (1, 1 + \mathrm{i})$ and $\mathbf{v} = (2, -1, \mathrm{i}, 3 - 2\mathrm{i})$.

Solution. From the definition,

$$\|\mathbf{u}\| = \sqrt{1^2 + (1 - \mathrm{i})(1 + \mathrm{i})} = \sqrt{1 + 1 + 1} \approx 1.7321$$

and

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{2^2 + (-1)^2 + (-\mathrm{i})\mathrm{i} + (3+2\mathrm{i})(3-2\mathrm{i})} \\ &= \sqrt{4+1+1+9+4} = \sqrt{19} \approx 4.3589.\end{aligned}$$

□

Here are the essential properties of the norm concept:

Basic Norm Laws

Let c be a scalar and $\mathbf{u}, \mathbf{v} \in V$ where the vector space V has the standard norm $\|\ \ \|$. Then the following hold.

(1) $\|\,\mathbf{u}\,\| \geq 0$ with $\|\,\mathbf{u}\,\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
(2) $\|\,c\mathbf{u}\,\| = |\,c\,|\ \|\,\mathbf{u}\,\|$.
(3) (Triangle Inequality) $\|\,\mathbf{u}+\mathbf{v}\,\| \leq \|\,\mathbf{u}\,\| + \|\,\mathbf{v}\,\|$.

That (1) is true is immediate from the definition of $\|\mathbf{u}\|$ as a sum of the lengths squared of the coordinates of $\mathbf{u}$. This sum is zero exactly when each term is zero. Condition (2) is fairly straightforward too. Suppose $\mathbf{u} = (z_1, z_2, \ldots, z_n)$, so that

$$\begin{aligned}\|c\mathbf{u}\| &= \sqrt{(\overline{cz}_1)cz_1 + (\overline{cz}_2)cz_2 + \cdots + (\overline{cz}_n)cz_n} \\ &= \sqrt{(\overline{c}c)(\overline{z}_1 z_1 + \overline{z}_2 z_2 + \cdots + \overline{z}_n z_n)} \\ &= \sqrt{|c|^2}\sqrt{\overline{z}_1 z_1 + \overline{z}_2 z_2 + \cdots + \overline{z}_n z_n} \\ &= |c| \cdot \|\mathbf{u}\|.\end{aligned}$$

The triangle inequality (which gets its name from the triangle with representatives of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ as its sides) can be proved easily in two- or three-dimensional geometrical space by appealing to the fact that the sum of lengths of any two sides of a triangle is greater that the length of the third side. A justification for higher dimensions is a nontrivial bit of algebra that we postpone until after the introduction of inner products below.

First we consider a few applications of the norm concept. We say that two vectors *determine the same direction* if one is a positive multiple of the other and *determine opposite directions* if one is a negative multiple of the other. The first application is the idea of "normalizing" a vector. This means finding a *unit vector*, which means a *vector of length* 1, that has the same direction as a given vector. This process is sometimes called "normalization." The following simple fact shows us how to do it.

Unit Vectors

Theorem 4.1. Let $\mathbf{u}$ be a nonzero vector. Then the vector

$$\mathbf{w} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$$

is a unit vector in the same direction as $\mathbf{u}$.

Proof. Since $\|\mathbf{u}\|$ is positive, we see immediately that $\mathbf{w}$ and $\mathbf{u}$ determine the same direction. Now check the length of $\mathbf{w}$ by using the basic norm law 2 to obtain that

$$\|\mathbf{w}\| = \left\| \frac{1}{\|\mathbf{u}\|}\mathbf{u} \right\| = \left| \frac{1}{\|\mathbf{u}\|} \right| \|\mathbf{u}\| = \frac{\|\mathbf{u}\|}{\|\mathbf{u}\|} = 1.$$

Hence $\mathbf{w}$ is a unit vector, as desired. □

Example 4.3. Use the normalization procedure to find unit vectors in the directions of vectors $\mathbf{u} = (2, -1, 0, 4)$ and $\mathbf{v} = (-4, 2, 0, -8)$. Conclude that these vectors determine opposite directions.

Solution. Let us find a unit vector in the same direction of each vector. We have norms

$$\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 0^2 + 4^2} = \sqrt{21}$$

and

$$\|\mathbf{v}\| = \sqrt{-4^2 + (2)^2 + +0^2 + (-8)^2} = \sqrt{84} = 2\sqrt{21}.$$

It follows that unit vectors in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively, are

$$\begin{aligned} \mathbf{w_1} &= (2, -1, 0, 4)/\sqrt{21}, \\ \mathbf{w_2} &= (-4, 2, 0, -8)/(2\sqrt{21}) = -(2, -1, 0, 4)/\sqrt{21} = -\mathbf{w_1}. \end{aligned}$$

Therefore $\mathbf{u}$ and $\mathbf{v}$ determine opposite directions. □

Example 4.4. Find a unit vector in the direction of the vector $\mathbf{v} = (2 + \mathrm{i}, 3)$.

Solution. We have

$$\|\mathbf{u}\| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}.$$

It follows that a unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{w} = \frac{1}{\sqrt{14}}(2 + \mathrm{i}, 3).$$ □

In order to work the next example we must express the idea of vector convergence of a sequence $\mathbf{u}_1, \mathbf{u}_2, \ldots$ to the vector $\mathbf{u}$ in a sensible way. The norm idea makes this straightforward: to say that the $\mathbf{u}_n$'s approach the vector $\mathbf{u}$ should mean that the distance between $\mathbf{u}$ and $\mathbf{u}_n$ goes to 0 as $n \to \infty$. But norm measures distance. Therefore the correct definition is as follows:

Convergence of Vectors

Definition 4.3. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots$ be a sequence of vectors in the vector space V and $\mathbf{u}$ also a vector in V. We say that the sequence converges to $\mathbf{u}$ and write

$$\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}$$

if the sequence of real numbers $\|\mathbf{u}_n - \mathbf{u}\|$ converges to 0, i.e.,

$$\lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\| = 0.$$

Example 4.5. Use the norm concept to justify the statement that

$$\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u},$$

where $\mathbf{u}_n = \big(1 + 1/n^2, 1/(n^2+1), \sin n/n\big)$ and $\mathbf{u} = (1, 0, 0)$.

Solution. In our case we have

$$\mathbf{u}_n - \mathbf{u} = \begin{bmatrix} 1 + 1/n^2 \\ 1/(n^2+1) \\ \sin n/n \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/n^2 \\ 1/(n^2+1) \\ \sin n/n \end{bmatrix},$$

so

$$\|\mathbf{u}_n - \mathbf{u}\| = \sqrt{\left(\frac{1}{n^2}\right)^2 + \left(\frac{1}{(n^2+1)}\right)^2 + \left(\frac{\sin n}{n}\right)^2} \underset{n\to\infty}{\to} \sqrt{0+0+0} = 0,$$

which is what we wanted to show. □

The Inner Product Idea

In addition to the norm concept we have another fundamental tool in our arsenal when we tackle two- and three-dimensional geometrical vectors. This tool is the so-called dot product of two vectors. It has many handy applications, but the most powerful of these is the ability to determine the angle between two vectors. In fact, some authors use this idea to define dot products as follows: let θ be the angle between representatives of the vectors $\mathbf{u}$ and $\mathbf{v}$. (See Figure 4.1.) The dot product of $\mathbf{u}$ and $\mathbf{v}$ is defined to be the quantity $\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$. It turns out that with some trigonometry (the law of cosines) and algebra, one can use this definition to derive a very convenient form for inner products; for example, if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{4.1}$$

This makes the calculation of dot products vastly easier since we don't have to use any trigonometry to compute it. A particularly nice application is that we can determine $\cos\theta$ quite easily from the dot product, namely

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}. \tag{4.2}$$

It is useful to try to extend these geometrical ideas to higher dimensions even if we can't literally use trigonometry and the like. So what we do is reverse the sequence of ideas we've discussed and take equation (4.1) as the prototype for our next definition. As with norms, we are going to have to distinguish carefully between the cases of real and complex scalars. First we focus on the more common case of real coefficients.

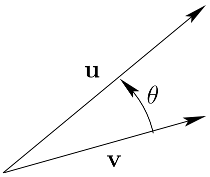


Fig. 4.1. Angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$.

Dot Product

Definition 4.4. Let $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{v} = (y_1, y_2, \ldots, y_n)$ be vectors in $\mathbb{R}^n$. The *(standard) inner product*, also called the *dot product* of $\mathbf{u}$ and $\mathbf{v}$, is the real number

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We can see from the first form of this definition where the term "inner product" came from. Recall from Section 2.4 that the matrix product $\mathbf{u}^T \mathbf{v}$ is called the inner product of these two vectors.

Example 4.6. Compute the dot product of the vectors $\mathbf{u} = (1, -1, 3, 2)$ and $\mathbf{v} = (2, -1, 0, 4)$ in $\mathbb{R}^4$.

Solution. From the definition,

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 2 + (-1) \cdot (-1) + 3 \cdot 0 + 2 \cdot 4 = 11. \qquad \square$$

There is a wonderful connection between the standard inner product and the standard norm for vectors that is immediately evident from the definitions. Here it is:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}. \tag{4.3}$$

Thus computing norms amounts to an inner product calculation followed by a square root. Actually, we can even avoid the square root and put the equation in the form

$$\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}.$$

We say that the standard norm is *induced* by the standard inner product. We would like this property to carry over to complex vectors. Now we have to be a bit careful. In general, the quantity $\mathbf{u}^T \mathbf{u}$ may not even be a real number, or may be negative. This means that $\sqrt{\mathbf{u}^T \mathbf{u}}$ could be complex, which doesn't seem like a good idea for measuring "length." So how can we avoid this problem? Recall that when we introduced transposes, we also introduced conjugate transposes and remarked that for complex vectors, this is a more natural tool than the transpose. Now we can back up that remark! Recall the definition for complex norm: for $\mathbf{u} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, the *norm* of $\mathbf{u}$ is the nonnegative real number

$$\|\mathbf{u}\| = \sqrt{\overline{z}_1 z_1 + \overline{z}_2 z_2 + \cdots + \overline{z}_n z_n} = \sqrt{\mathbf{u}^* \mathbf{u}}.$$

Therefore, in our definition of complex "dot products" we had better replace transposes by conjugate transposes. This inspires the following definition

Complex Dot Product

Definition 4.5. Let $\mathbf{u} = (w_1, w_2, \ldots, w_n)$ and $\mathbf{v} = (z_1, z_2, \ldots, z_n)$ be vectors in $\mathbb{C}^n$. The *(standard) inner product*, also called the *dot product* of $\mathbf{u}$ and $\mathbf{v}$, is the complex number

$$\mathbf{u} \cdot \mathbf{v} = \overline{w}_1 z_1 + \overline{w}_2 z_2 + \cdots + \overline{w}_n z_n = \mathbf{u}^* \mathbf{v}.$$

With this definition we still have the close connection given above in equation (4.3) between norm and standard inner product of complex vectors.

Example 4.7. Compute the dot product of the vectors $\mathbf{u} = (1 + 2\mathrm{i}, \mathrm{i}, 1)$ and $\mathbf{v} = (\mathrm{i}, -1 - \mathrm{i}, 0)$ in $\mathbb{C}^3$.

Solution. Simply apply the definition:

$$\mathbf{u} \cdot \mathbf{v} = \overline{(1 + 2\mathrm{i})}\mathrm{i} + \overline{\mathrm{i}}(-1 - \mathrm{i}) + 1 \cdot 0 = (1 - 2\mathrm{i})\mathrm{i} - \mathrm{i}(-1 - \mathrm{i}) = 1 + 2\mathrm{i}. \qquad \square$$

What are the essential defining properties of these standard inner products? It turns out that we can answer the question for both real and complex inner products at once. However, we should bear in mind that in most cases we will be dealing with real dot products, and in such cases all the dot products in question are real numbers, so that any reference to a complex conjugate can be omitted.

Basic Inner Product Laws

Let c be a scalar and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, where V is a vector space with the standard inner product. Then the following hold:

(1) $\mathbf{u} \cdot \mathbf{u} \geq 0$ with $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
(2) $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$.
(3) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
(4) $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.

That (1) is true is immediate from the fact that $\mathbf{u} \cdot \mathbf{u} = \mathbf{u}^* \mathbf{u}$ is a sum of the lengths squared of the coordinates of $\mathbf{u}$. This sum is zero exactly when each term is zero. Condition (2) follows from this line of calculation:

$$\overline{\mathbf{v} \cdot \mathbf{u}} = \overline{\mathbf{v}^* \mathbf{u}} = \left(\overline{\mathbf{v}^* \mathbf{u}}\right)^T = (\mathbf{v}^* \mathbf{u})^* = \mathbf{u}^* \mathbf{v} = \mathbf{u} \cdot \mathbf{v}.$$

One point that stands out in this calculation is the following:

Caution: A key difference between real and complex inner products is in the commutative law $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, which holds for real vectors but *not* for complex vectors, where instead $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$.

Conditions (3) and (4) are similarly verified and left to the exercises. We can also use (4) to prove this fact for real vectors:

$$(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{v} \cdot (c\mathbf{u}) = c(\mathbf{v} \cdot \mathbf{u}) = c(\mathbf{u} \cdot \mathbf{v}).$$

If we are dealing with complex dot products, matters are a bit trickier. One can show then that

$$(c\mathbf{u}) \cdot \mathbf{v} = \overline{c}(\mathbf{u} \cdot \mathbf{v}),$$

so we don't quite have the symmetry that we have for real products.

The Cross Product Idea

We complete this discussion of vector arithmetic with a tool that can be used only in three dimensions, the cross product of vectors. It has more sophisticated relatives, called *wedge products*, that operate in higher-dimensional spaces; this is an advanced topic in multilinear algebra that we shall not pursue. Unlike the dot product, cross products transform vectors into vectors.

In the traditional style of three-dimensional vector analysis, we use the symbols $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to represent the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$. Here is the definition of cross product along with a handy determinant mnemonic.

Cross Product of Vectors

Definition 4.6. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in $\mathbb{R}^3$. The cross product $\mathbf{u} \times \mathbf{v}$ of these vectors is defined to be the vector in $\mathbb{R}^3$ given by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\,\mathbf{i} + (u_3v_1 - u_1v_3)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Strictly speaking, the "determinant" of this definition is not a determinant in the usual sense. However, formal calculations with it are perfectly valid and provide us with useful insights. For example:

(1) Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, since a determinant with one row a multiple of another is zero. In particular, $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

(2) $\mathbf{w} \cdot \mathbf{u} \times \mathbf{v} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$, since the result of dotting $\mathbf{w} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ with $\mathbf{u} \times \mathbf{v}$ using the first form of the definition of cross product is equal to this determinant. (Note: parentheses are not needed since the only interpretation of $\mathbf{w} \cdot \mathbf{u} \times \mathbf{v}$ that makes sense is $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$.)

(3) $\mathbf{u} \cdot \mathbf{u} \times \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{u} \times \mathbf{v} = 0$, since a determinant with repeated rows is zero.

(4) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$, since interchanging two rows of a determinant changes its sign.

(5) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, as a direct calculation with the definition shows. Thus the products follow a circular pattern, with the product of any successive two yielding the next vector in the loop $\mathbf{i} \to \mathbf{j} \to \mathbf{k} \to \mathbf{i}$.

Example 4.8. Confirm by direct calculation that $\mathbf{u}\cdot\mathbf{u}\times\mathbf{v} = 0$ and $\mathbf{v}\cdot\mathbf{u}\times\mathbf{v} = 0$ if $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (1, 1, 0)$.

Solution. We calculate that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 0 \end{vmatrix} = (-1 \cdot 0 - 1 \cdot 3)\,\mathbf{i} - (2 \cdot 0 - 3 \cdot 1)\,\mathbf{j} + (2 \cdot 1 - (-1)\,1)\,\mathbf{k} \\ &= -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}. \end{aligned}$$

Thus

$$\mathbf{u}\cdot\mathbf{u}\times\mathbf{v} = (2\mathbf{i} - 1\mathbf{j} + 3\mathbf{k})\cdot(-3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) = -6 - 3 + 3 = 0$$
$$\mathbf{v}\cdot\mathbf{u}\times\mathbf{v} = (\mathbf{i} + \mathbf{j})\cdot(-3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) = -3 + 3 = 0.$$

□

Here is a summary of some of the basic laws of cross products:

Laws of Cross Products

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then

(1) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$.
(2) $(c\mathbf{u})\times\mathbf{v} = c\,(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(c\mathbf{v})$.
(3) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$.
(4) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}$.
(5) (Scalar triple product) $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$.
(6) (Vector triple product) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$.
(7) $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Items (1)–(7) can be verified directly from the definition of cross product and, in the case of item (7), trigonometric identities.

4.1 Exercises and Problems

Exercise 1. For the following pairs of vectors, calculate $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$.
(a) $(3,-5)$, $(2,4)$ (b) $(1,1,2)$, $(2,-1,3)$ (c) $(2,1,-2,-1)$, $(3,0,1,-4)$
(d) $(1+2\mathrm{i}, 2+\mathrm{i})$, $(4+3\mathrm{i}, 1)$ (e) $(3,1,2,-4)$, $(2,0,1,1)$ (f) $(2,2,-2)$, $(2,1,5)$

Exercise 2. For the following pairs of vectors, calculate $\mathbf{u}\cdot\mathbf{v}$ and unit vectors in the direction of $\mathbf{u}$ and $\mathbf{v}$.
(a) $(4,-2,2)$, $(1,3,2)$ (b) $(1,1)$, $(2,-2)$ (c) $(4,0,1,2-3\mathrm{i})$, $(1,1-2\mathrm{i},1,\mathrm{i})$
(d) $(\mathrm{i},-\mathrm{i})$, $(3\mathrm{i},1)$ (e) $(1,-1,1,-1)$, $(2,2,1,1)$ (f) $(4,1,2)$, $(1,0,0)$

Exercise 3. Let θ be the angle between the following pairs of real vectors and compute $\cos\theta$ using dot products.
(a) $(2,-5)$, $(4,2)$ (b) $(3,4)$, $(4,-3)$ (c) $(1,1,2)$, $(2,-1,3)$ (d) $\mathbf{j}+\mathbf{k}$, $2\mathbf{i}+\mathbf{k}$

Exercise 4. Compute an angle θ between the following pairs of real vectors.
(a) $(4,5)$, $(-4,4)$ (b) $\mathbf{i}-5\mathbf{j}$, $\mathbf{i}+\mathbf{k}$ (c) $(4,0,2)$, $(1,1,1)$

Exercise 5. Compute the cross product of the vector pairs in Exercise 4. (Express two-dimensional vectors in terms of $\mathbf{i}$ and $\mathbf{j}$ first.)

Exercise 6. Compute $\sin\theta$, where θ is the angle between the following pairs of real vectors, using cross products.
(a) $3\mathbf{i}-5\mathbf{j}$, $2\mathbf{i}+4\mathbf{j}$ (b)$3\mathbf{i}-5\mathbf{j}+2\mathbf{k}$, $2\mathbf{i}-4\mathbf{k}$ (c) $(-4,2,4)$, $(4,1,-5)$

Exercise 7. Let $c = 3$, $\mathbf{u} = (4, -1, 2, 3)$, and $\mathbf{v} = (-2, 2, -2, 2)$. Verify that the three basic norm laws hold for these vectors and scalars.

Exercise 8. Let $c = 2$, $\mathbf{u} = (-3, 2, 1)$, $\mathbf{v} = (4, 2, -3)$, and $\mathbf{w} = (1, -2, 1)$. Verify the four basic inner product laws for these vectors and scalars.

Exercise 9. Let $c = -2$, $\mathbf{u} = (0, 2, 1)$, $\mathbf{v} = (4, 0, -3)$, and $\mathbf{w} = (1, -2, 1)$. Verify cross product laws (1)–(4) for these vectors and scalars.

Exercise 10. Let $\mathbf{u} = (1, 2, 2)$, $\mathbf{v} = (0, 2, -3)$, and $\mathbf{w} = (1, 0, 1)$. Verify cross product laws (5)–(7) for these vectors and scalars.

Exercise 11. Verify that $\mathbf{u}_n = [2/n, (1 + n^2)/(2n^2 + 3n + 5)]^T$, $n = 1, 2, \ldots$, converges to a limit vector $\mathbf{u}$ by using the norm definition of vector limit.

Exercise 12. Let $\mathbf{u}_n = [\mathrm{i}, \left(n^2\mathrm{i} + 1\right) / \left((n\mathrm{i})^2 + n\right)]$, $n = 1, 2, \ldots$, and verify that $\mathbf{u}_n$ converges to a limit vector $\mathbf{u}$.

*Problem 13. Show that for real vectors $\mathbf{u}$, $\mathbf{v}$ and real number c one has

$$(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{v} \cdot (c\mathbf{u}) = c(\mathbf{v} \cdot \mathbf{u}) = c(\mathbf{u} \cdot \mathbf{v}).$$

Problem 14. Prove this basic norm law: $\|\mathbf{u}\| \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Problem 15. Show that if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ (or $\mathbb{C}^n$) and c is a scalar, then
(a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (b) $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

Problem 16. Show from the definition that if $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}$, where $\mathbf{u}_n = (x_n, y_n) \in \mathbb{R}^2$ and $\mathbf{u} = (x, y)$, then $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

*Problem 17. Prove that if $\mathbf{v}$ is a vector and c is a positive real, then normalizing $\mathbf{v}$ and normalizing $c\mathbf{v}$ yield the same unit vector. How are the normalized vectors related if c is negative?

Problem 18. Show that if A is a real $n \times n$ matrix and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^n$, then $\left(A^T\mathbf{u}\right) \cdot \mathbf{v} = \mathbf{u} \cdot (A\mathbf{v})$.

*Problem 19. Show that $|\|\mathbf{u}\| - \|\mathbf{v}\|| \leq \|\mathbf{u} - \mathbf{w}\|$ for any two vectors $\mathbf{u}, \mathbf{v}$ in the same space.

4.2 Applications of Norms and Inner Products

Projections and Angles

Now that we have dot products under our belts we can tackle geometrical issues such as angles between vectors in higher dimensions. For the matter of angles, we will stick to real vector spaces, though we could do it for complex vector spaces with a little extra work. What we would like to do is take equation (4.2) as the *definition* of the angle between two vectors. There's one slight problem: how do we know that it will give a quantity that could be a cosine? After all, cosines take on only values between -1 and 1. We could use some help and the Cauchy–Bunyakovsky–Schwarz inequality (CBS for short) is just what we need:

CBS Inequality

Theorem 4.2. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Proof. Let c be an arbitrary real number and compute the nonnegative quantity

$$\begin{aligned} f(c) &= \|\mathbf{u} + c\mathbf{v}\|^2 \\ &= (\mathbf{u} + c\mathbf{v}) \cdot (\mathbf{u} + c\mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot (c\mathbf{v}) + (c\mathbf{v}) \cdot \mathbf{u} + (c\mathbf{v}) \cdot (c\mathbf{v}) \\ &= \|\mathbf{u}\| + 2c(\mathbf{u} \cdot \mathbf{v}) + c^2 \|\mathbf{v}\| \, . \end{aligned}$$

The function $f(c)$ is therefore a quadratic in the variable c with nonnegative values. The low point of this quadratic occurs where $f'(c) = 0$, that is, where

$$0 = 2(\mathbf{u} \cdot \mathbf{v}) + 2c \|\mathbf{v}\| \, ,$$

that is,

$$c = \frac{-(\mathbf{u} \cdot \mathbf{v})}{\|\mathbf{v}\|^2}.$$

Evaluate f at this point to get that

$$0 \leq \|\mathbf{u}\|^2 - 2\frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2} + \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^4} \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2}.$$

Now add $(\mathbf{u} \cdot \mathbf{v})^2 / \|\mathbf{v}\|^2$ to both sides and multiply by $\|\mathbf{v}\|^2$ to obtain that

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \, .$$

Take square roots and we have the desired inequality. □

This inequality has a number of useful applications. Because of it we can articulate a definition of angle between vectors. Note that there is a certain ambiguity in discussing the angle between vectors, since more than one angle works. It is the cosine of these angles that is actually unique.

Angle Between Vectors

Definition 4.7. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we define the *angle* between $\mathbf{u}$ and $\mathbf{v}$ to be any angle θ satisfying

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Thanks to the CBS inequality, we know that $|\mathbf{u}\cdot\mathbf{v}|/(\|\mathbf{u}\|\,\|\mathbf{v}\|) \le 1$, so that this formula for $\cos\theta$ makes sense.

Example 4.9. Find the angle between the vectors $\mathbf{u} = (1, 1, 0, 1)$ and $\mathbf{v} = (1, 1, 1, 1)$ in $\mathbb{R}^4$.

Solution. We have that

$$\cos\theta = \frac{(1,1,0,1)\cdot(1,1,1,1)}{\|(1,1,0,1)\|\,\|(1,1,1,1)\|} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

Hence we can take $\theta = \pi/6$. □

Example 4.10. Use the laws of inner products and the CBS inequality to verify the triangle inequality for vectors $\mathbf{u}$ and $\mathbf{v}$. What happens to this inequality if we also know that $\mathbf{u}\cdot\mathbf{v} = 0$?

Solution. Here the trick is to avoid square roots. Square both sides of equation (4.3) to obtain that

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\
&= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2,
\end{aligned}$$

where the last inequality follows from the CBS inequality. If $\mathbf{u}\cdot\mathbf{v} = 0$, then the single inequality can be replaced by an equality. □

We have just seen a very important case of angles between vectors that warrants its own name. Recall from geometry that two vectors are *perpendicular* or *orthogonal* if the angle between them is $\pi/2$. Since $\cos\pi/2 = 0$, we see that this amounts to the equation $\mathbf{u}\cdot\mathbf{v} = 0$. Now we can extend the perpendicularity idea to arbitrary vectors, including complex vectors.

Orthogonal Vectors

Definition 4.8. Two vectors $\mathbf{u}$ and $\mathbf{v}$ in the same vector space are *orthogonal* if $\mathbf{u}\cdot\mathbf{v} = 0$. In this case we write $\mathbf{u} \perp \mathbf{v}$.

In the case that one of the vectors is the zero vector, we have the little oddity that the zero vector is orthogonal to every other vector, since the dot product is always 0 in this case. Some authors require that $\mathbf{u}$ and $\mathbf{v}$ be nonzero as part of the definition. It's a minor point and we won't worry about it. When $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, i.e., $\mathbf{u} \cdot \mathbf{v} = 0$, we see from the third equality in the derivation of CBS above that

Pythagorean Theorem

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 ,$$

which is really the Pythagorean theorem for vectors in $\mathbb{R}^n$.

Example 4.11. Determine whether the following pairs of vectors are orthogonal.

(a) $\mathbf{u} = (2, -1, 3, 1)$ and $\mathbf{v} = (1, 2, 1, -2)$
(b) $\mathbf{u} = (1 + \mathrm{i}, 2)$ and $\mathbf{v} = (-2\mathrm{i}, 1 + \mathrm{i})$.

Solution. For (a) we calculate

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + (-1)2 + 3 \cdot 1 + 1(-2) = 1,$$

so that $\mathbf{u}$ is not orthogonal to $\mathbf{v}$. For (b) we calculate

$$\mathbf{u} \cdot \mathbf{v} = (1 - \mathrm{i})(-2\mathrm{i}) + 2(1 + \mathrm{i}) = -2\mathrm{i} - 2 + 2 + 2\mathrm{i} = 0,$$

so that $\mathbf{u}$ is orthogonal to $\mathbf{v}$ in this case. □

The next example illustrates a handy little trick well worth remembering.

Example 4.12. Given a vector (a, b) in $\mathbb{R}^2$ or $\mathbb{C}^2$, find a vector orthogonal to (a, b).

Solution. Simply interchange coordinates, conjugate them (this does nothing if the entries are real), and insert a minus sign in front of one of the coordinates, say the first. We obtain $(-\overline{b}, \overline{a})$. Now check that

$$(a, b) \cdot (-\overline{b}, \overline{a}) = \overline{a}(-\overline{b}) + \overline{b}\overline{a} = 0. \qquad \square$$

By *parallel vectors* we mean two vectors that are nonzero scalar multiples of each other. Notice that parallel vectors may determine the same or opposite directions. Our next application of the dot product relates back to a fact that we learned in geometry: given two nonzero vectors in the plane, it is always possible to resolve one of them into a sum of a vector parallel to the other and a vector orthogonal to the other (see Figure 4.2). The parallel component is called the projection of one vector along the other. This idea is useful, for example, in physics problems where we want to resolve a force into orthogonal components. As a matter of fact, we can develop this same idea in arbitrary standard vector spaces. That is the content of the following useful fact.

Parallel Vectors

Theorem 4.3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a vector space with $\mathbf{v} \neq \mathbf{0}$. Let

Projection Formula for Vectors

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} \quad \text{and} \quad \mathbf{q} = \mathbf{u} - \mathbf{p}.$$

Then $\mathbf{p}$ is parallel to $\mathbf{v}$, $\mathbf{q}$ is orthogonal to $\mathbf{v}$, and $\mathbf{u} = \mathbf{p} + \mathbf{q}$.

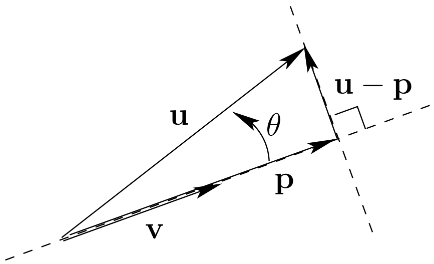


Fig. 4.2. Angle between vectors **u** and **v**, projection **p** of **u** along **v** and $(\mathbf{u}-\mathbf{p})\perp\mathbf{v}$.

Proof. Let $\mathbf{p} = c\mathbf{v}$, an arbitrary multiple of **v**. Then **p** is automatically parallel to **v**. Impose the constraint that $\mathbf{q} = \mathbf{u} - \mathbf{p}$ be orthogonal to **v**. This means, by definition, that

$$0 = \mathbf{v}\cdot\mathbf{q} = \mathbf{v}\cdot(\mathbf{u}-\mathbf{p}) = \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot(c\mathbf{v}).$$

Add $\mathbf{v}\cdot(c\mathbf{v})$ to both sides and pull the scalar c outside the dot product to obtain that

$$c(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\cdot\mathbf{u}$$

and therefore

$$c = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}.$$

So for this choice of c, **q** is orthogonal to **p**. Clearly, $\mathbf{u} = \mathbf{p} + \mathbf{u} - \mathbf{p}$, so the proof is complete. □

Projection Vector

It is customary to call the vector **p** of this theorem the *(parallel) projection of* **u** *along* **v**. As above, we write

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

Component of Vector

The projection of one vector along another is itself a vector quantity. A scalar quantity that is frequently associated with these calculations is the so-called *component* of **u** along **v**. It is defined as

$$\operatorname{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|}.$$

The connection between these two quantities is that

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \operatorname{comp}_{\mathbf{v}}\mathbf{u}\frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Notice that $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector in the same direction as **v**. Therefore, $\operatorname{comp}_{\mathbf{v}}\mathbf{u}$ is the signed magnitude of the projection of **u** along **v** and will be negative if the angle between **u** and **v** exceeds $\pi/2$.

The vector $\mathbf{q}$ of Theorem 4.3 that is orthogonal to $\mathbf{v}$ also has a name: the *orthogonal projection* of $\mathbf{u}$ to $\mathbf{v}$. We write

Orthogonal Projection

$$\operatorname{orth}_{\mathbf{v}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}.$$

Note, however, that the default meaning of "projection" is "parallel projection."

Example 4.13. Calculate the projection and component of $\mathbf{u} = (1, -1, 1, 1)$ along $\mathbf{v} = (0, 1, -2, -1)$ and verify that $\mathbf{u} - \mathbf{p} \perp \mathbf{v}$.

Solution. We have that

$$\begin{aligned}\mathbf{v} \cdot \mathbf{u} &= 0 \cdot 1 + 1(-1) + (-2)1 + (-1)1 = -4,\\ \mathbf{v} \cdot \mathbf{v} &= 0^2 + 1^2 + (-2)^2 + (-1)^2 = 6,\end{aligned}$$

so that

$$\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{-4}{6}(0, 1, -2, -1) = \frac{1}{3}(0, -2, 4, 2).$$

It follows that

$$\mathbf{u} - \mathbf{p} = \frac{1}{3}(3, -1, -1, 1)$$

and

$$(\mathbf{u} - \mathbf{p}) \cdot \mathbf{v} = \frac{1}{3}(3 \cdot 0 + 1(-1) + (-1)(-2) + 1(-1)) = 0.$$

Also, the component of $\mathbf{u}$ along $\mathbf{v}$ is

$$\operatorname{comp}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|} = \frac{-4}{\sqrt{6}}.$$ □

A hyperplane is a basic geometrical object on which inner product tools can shed light. Here is the definition.

Definition 4.9. A *hyperplane* in $\mathbb{R}^n$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{x} = b$, where the nonzero vector $\mathbf{a} \in \mathbb{R}^n$ and scalar b are given.

Hyperplane in $\mathbb{R}^n$

These are familiar objects. For example, a hyperplane in $\mathbb{R}^3$ is the set of points (x, y, z) that satisfy an equation $ax + by + cz = d$, which is simply a plane in three dimensions. A hyperplane in $\mathbb{R}^2$ is the set of points (x, y) that satisfy an equation $ax + by = c$, which is just a line in two dimensions. (Notice that in the absence of homogeneous space, a tuple like (x, y, z) has a dual interpretation as point or vector.) Here is a general geometrical interpretation of hyperplanes.

Theorem 4.4. Let H be the hyperplane in $\mathbb{R}^n$ defined by the equation $\mathbf{a} \cdot \mathbf{x} = b$ and let $\mathbf{x}_* \in H$. Then

Geometry of Hyperplanes

(1) $\mathbf{a}^\perp = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{y} = 0\}$ is a subspace of $\mathbb{R}^n$ of dimension $n - 1$.

(2) $H = \mathbf{x}_* + \mathbf{a}^\perp = \left\{\mathbf{x}_* + \mathbf{y} \,|\, \mathbf{y} \in \mathbf{a}^\perp\right\}$.

Proof. For (1), observe that $\mathbf{a}^\perp = \mathcal{N}\left(\mathbf{a}^T\right)$, which is a subspace of $\mathbb{R}^n$. According to the projection formula for vectors, any element of $\mathbb{R}^n$ can be expressed as a sum of a multiple of $\mathbf{a}$ and a vector orthogonal to $\mathbf{a}$. Therefore, $\mathbb{R}^n$ is spanned by a basis of $\mathbf{a}^\perp$ and $\mathbf{a}$. Since $\dim \mathbb{R}^n = n$, a basis of $\mathbf{a}^\perp$ must have at least $n-1$ elements. If it had n elements, then we would have $\mathbf{a}^\perp = \mathbb{R}^n$, which would imply that $\mathbf{a} \cdot \mathbf{a} = 0$ and therefore $\mathbf{a} = \mathbf{0}$, which is false. Therefore, $\dim \mathbf{a}^\perp = n$. Part (2) follows from Theorem 3.15 since $\mathbf{x}_*$ is a particular solution to the linear system $\mathbf{a}^T\mathbf{x} = 0$. □

Notice that the vector $\mathbf{a}$ can be read off immediately from the defining equation. For example, we see by inspection that a vector orthogonal to the plane given by $2x - 3y + z = 4$ is $\mathbf{a} = (2, -3, 1)$. Finding the defining equation is a bit more work.

Example 4.14. Find an equation that defines the plane containing the three (noncollinear) points P, Q, and R with coordinates $(1, 0, 2)$, $(2, 1, 0)$, and $(3, 1, 1)$, respectively.

Solution. First calculate displacement vectors

$$\begin{aligned}\overrightarrow{PQ} &= (2,1,0) - (1,0,2) = (1,1,-2)\\ \overrightarrow{PR} &= (3,1,1) - (1,0,2) = (2,1,-1)\,.\end{aligned}$$

These vectors are parallel to the plane. Therefore, their cross product, which is orthogonal to each vector, will be orthogonal to the plane. We calculate

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} - \mathbf{k}.$$

Hence the equation of the plane is $x - 3y - z = b$. To determine b, plug in the coordinates of P and obtain that $1 \cdot 1 - 3 \cdot 0 - 2 \cdot 1 = -1 = b$. Hence an equation of the plane is $x - 3y - z = -1$. □

Least Squares

Example 4.15. You are using a pound scale to measure weights for produce sales when you notice that your scale is broken. The vendor at the next stall is leaving and lends you another scale as she departs. You then realize that the new scale is in units you don't recognize. You happen to have some known weights that are approximately 2, 5, and 7 pounds respectively. When you weigh these items on the new scale you get the numbers 0.7, 2.4, and 3.2. You get your calculator out and hypothesize that the unit of weight should be some constant multiple of pounds. Model this information as a system of equations. Is it clear from this system what the units of the scale are?

Solution. Express the relationship between the weight p in pounds and the weight w in unknown units as $w \cdot c = p$, where c is an unknown constant of proportionality. Your data show that we have

$$\begin{aligned} 0.7c &= 2 \\ 2.4c &= 5 \\ 3.4c &= 7. \end{aligned}$$

As a system of three equations in one unknown you see immediately that this overdetermined system (too many equations) is inconsistent. After all, the pound weights were only approximate and there is always some error in measurement. What to do? You could just average the three inconsistent values of c, thereby obtaining

$$c = (2/0.7 + 5/2.4 + 7/3.4)/3 = 2.3331.$$

It isn't at all clear that this should be a good strategy. □

There really is a better way, and it will lead to a slightly different estimate of the number c. This method, called the *method of least squares*, was invented by C. F. Gauss to handle uncertainties in orbital calculations in astronomy.

Method of Least Squares

Here is the basic problem: suppose we have data that leads to a system of equations for unknowns that we want to solve for, but the data has errors in it and consequently leads to an inconsistent linear system

$$A\mathbf{x} = \mathbf{b}.$$

How do we find the "best" approximate solution? One could answer this in many ways. One of the most commonly accepted ideas is one that Gauss proposed: the so-called *residual* $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ should be $\mathbf{0}$, so its departure from $\mathbf{0}$ is a measure of our error. Thus we should try to find a value of the unknown $\mathbf{x}$ that minimizes the norm of the residual squared, i.e., a "solution" $\mathbf{x}$ such that

Residual Vector

$$\|\mathbf{b} - A\mathbf{x}\|^2$$

is minimized. Such a solution is called a "least squares" solution to the system. This technique is termed "linear regression" by statisticians, who use it in situations in which one has many estimates for unknown parameters that taken together are not perfectly consistent. It can be shown that if errors are normally distributed and the least squares solution unique, then it is an unbiased estimator of the true value in the statistical sense.

Let's try to get a fix on this problem. Even the one-variable case is instructive, so let's use the preceding example. In this case the coefficient matrix A is the column vector $\mathbf{a} = [0.7, 2.4, 3.4]^T$, and the right-hand-side vector is $\mathbf{b} = [2, 5, 7]^T$. What we are really trying to find is a value of the scalar $x = c$ such that $\mathbf{b} - Ax = \mathbf{b} - x\mathbf{a}$ is a minimum. Here is a geometrical interpretation: we want to find the multiple of the vector $\mathbf{a}$ that is closest to $\mathbf{b}$. Geometry suggests that this minimum occurs when $\mathbf{b} - x\mathbf{a}$ is orthogonal to $\mathbf{a}$, in other words, when $x\mathbf{a}$ is the projection of $\mathbf{b}$ along $\mathbf{a}$. Inspection of the projection formula shows us that we must have

$$x = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} = \frac{0.7 \cdot 2 + 2.4 \cdot 5 + 3.4 \cdot 7}{0.7 \cdot 0.7 + 2.4 \cdot 2.4 + 3.4 \cdot 3.4} \approx 2.0887.$$

Notice that this value doesn't solve any of the original equations exactly, but it is, in a certain sense, the best approximate solution to all three equations taken together. Also, this solution is *not* the same as the average of the solutions to the three equations, which we computed to be approximately 2.3331.

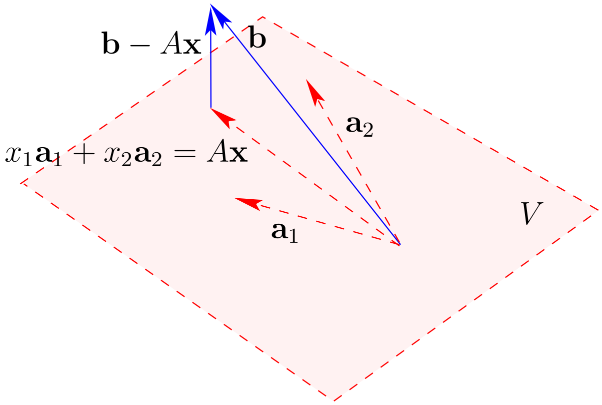

$\mathbf{b} - A\mathbf{x}$
$\mathbf{b}$
$\mathbf{a}_2$
$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = A\mathbf{x}$
V
$\mathbf{a}_1$

Fig. 4.3. The vector in the subspace $\mathcal{C}(A)$ nearest to $\mathbf{b}$.

Now how do we tackle the more general system $A\mathbf{x} = \mathbf{b}$? Since $A\mathbf{x}$ is just a linear combination of the columns, what we should find is the vector of this form that is closest to the vector $\mathbf{b}$. See Figure 4.3 for a picture of the situation with $n = 2$. Our experience with the 1-dimensional case suggests that we should require that the residual be orthogonal to each column of A, that is, $\mathbf{a}_i \cdot (\mathbf{b} - A\mathbf{x}) = \mathbf{a}_i^T(\mathbf{b} - A\mathbf{x}) = 0$, for all columns $\mathbf{a}_i$ of A. Each column gives rise to one equation. We can write all these equations at once in the form of the so-called *normal equations*:

Normal Equations

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

In fact, this is the same set of equations we get if we apply calculus to the scalar function of variables $x_1, x_2, \ldots, x_n$ given as $f(x) = \|\mathbf{b} - A\mathbf{x}\|^2$ and search for a local minimum by setting all partials equal to 0. Any solution to this system will minimize the norm of $\mathbf{b} - A\mathbf{x}$ as $\mathbf{x}$ ranges over all elements of $\mathbb{R}^n$. The coefficient matrix $B = A^T A$ of the normal system has some pleasant properties. For one, it is a symmetric matrix. For another, it is a *positive semidefinite matrix*, by which we mean that B is a square $n \times n$ matrix such

Positive Semidefinite Matrix

that $\mathbf{x}^T B\mathbf{x} \geq 0$ for all vectors $\mathbf{x} \in \mathbb{R}^n$. In fact, in some cases B is even better behaved because it is a *positive definite matrix*, by which we mean that B is a square $n \times n$ matrix such that $\mathbf{x}^T B\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x} \in \mathbb{R}^n$. (For complex matrices, the condition is $\mathbf{x}^* B\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x} \in \mathbb{C}^n$.)

Positive Definite Matrix

Does there exist a solution to the normal equations? The answer is yes. In general, any solution to the normal equations minimizes the residual norm and is called a *least squares solution* to the problem $A\mathbf{x} = \mathbf{b}$. Since we now have two versions of "solution" for the system $A\mathbf{x} = \mathbf{b}$, we should distinguish between them in situations that may refer to either. If the vector $\mathbf{x}$ actually satisfies the equation $A\mathbf{x} = \mathbf{b}$, we call $\mathbf{x}$ a *genuine solution* to the system to contrast it with a least squares solution. Certainly, every genuine solution is a least squares solution, but the converse will not be true if the original system is inconsistent. We leave the verifications as exercises.

Least Squares Solution

Genuine Solution

The normal equations are guaranteed to be consistent—a nontrivial fact—and will have infinitely many solutions if $A^T A$ is a singular matrix. Consider the most common case, namely that in which A is a rank-n matrix. Recall that in this case we say that A has *full column rank.* We can show that the $n \times n$ matrix $A^T A$ is also of rank n. This means that it is an invertible matrix and therefore the solution to the normal equations is *unique.* Here is the necessary fact.

Theorem 4.5. Suppose that the real $m \times n$ matrix A has full column rank n. Then the $n \times n$ matrix $A^T A$ also has rank n and is invertible.

Proof. Assume that A has rank n. Now suppose that for some vector $\mathbf{x}$ we have

$$\mathbf{0} = A^T A\mathbf{x}.$$

Multiply on the left by $\mathbf{x}^T$ to obtain that

$$0 = \mathbf{x}^T \mathbf{0} = \mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 ,$$

so that $A\mathbf{x} = \mathbf{0}$. However, we know by Theorem 1.5 that the homogeneous system with A as its coefficient matrix must have a unique solution. Of course, this solution is the zero vector. Therefore, $\mathbf{x} = \mathbf{0}$. It follows that the square matrix $A^T A$ has rank n and is also invertible by Theorem 2.7. □

Example 4.16. Two parameters, x_1 and x_2, are linearly related. Three samples are taken that lead to the system of equations

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 2. \end{aligned}$$

Show that this system is inconsistent, and find the least squares solution for $\mathbf{x} = (x_1, x_2)$. What is the minimum norm of the residual $\mathbf{b} - A\mathbf{x}$ in this case?

Solution. In this case it is obvious that the system is inconsistent: the first and third equations have the same quantity, $2x_1 + x_2$, equal to different values 0 and 2. Of course, we could have set up the augmented matrix of the system and found a pivot in the right-hand-side column as well. We see that the (rank 2) coefficient matrix A and right-hand side $\mathbf{b}$ are

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Thus

$$A^T A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

As predicted by the preceding theorem, $A^T A$ is invertible, and we use the 2×2 formula for the inverse:

$$(A^T A)^{-1} = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix},$$

so that the unique least squares solution is

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \frac{1}{2} \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The minimum value for the residual $\mathbf{b} - A\mathbf{x}$ occurs when $\mathbf{x}$ is a least squares solution, so we get

$$\mathbf{b} - A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and therefore

$$\|\mathbf{b} - A\mathbf{x}\| = \sqrt{2} \approx 1.414.$$

This isn't terribly small, but it's the best we can do with this system. This number tells us that the system is badly inconsistent. □

4.2 Exercises and Problems

In the following exercises, all vectors are real unless otherwise indicated.

Exercise 1. Find the angle θ in radians between the following pairs of vectors. (a) $(2, -5)$, $(3, 4)$ (b) $(4, 5, -3, 4)$, $(2, -4, 1, 3)$ (c) $(1, -2, 3, 4, 1)$, $(2, 3, 1, 5, 5)$

Exercise 2. Find the angle θ between the following pairs of vectors.
(a) $(1,0,1,0,2)$, $(2,1,-3,2,4)$ (b) $(7,-3,1,1,2,-2)$, $(2,3,-4,-3,2,2)$

Exercise 3. Find the projection and component of $\mathbf{u}$ along $\mathbf{v}$, where the pair $\mathbf{u},\mathbf{v}$ are
(a) $(-4,3)$, $(2,1)$ (b) $(3,0,4)$, $(2,2,-1)$ (c) $(1,0,-5,2)\,,(1,1,1,1)$

Exercise 4. Find the orthogonal projection of $\mathbf{u}$ to $\mathbf{v}$, where the pair $\mathbf{u},\mathbf{v}$ are
(a) $(1,-\sqrt{3})$, $(2,1)$ (b) $(2,1,3)$, $(8,2,-4)$ (c) $(3,2,1,1,1)\,,(1,1,1,0,1)$

Exercise 5. Verify the CBS inequality for the vectors $\mathbf{u}$ and $\mathbf{v}$, where the pair $\mathbf{u},\mathbf{v}$ are
(a) $\mathbf{i}-2\mathbf{j}$, $\mathbf{i}+\mathbf{j}-\mathbf{k}$ (b) $(3,-2,3)$, $(1,-5,2)$ (c) $(3,-2)$, $(-6,4)$

Exercise 6. Determine whether the following pairs of vectors $\mathbf{u}$, $\mathbf{v}$ are orthogonal, and if so, verify that the Pythagorean theorem holds for the pair.
(a) $(-2,1,3)$, $(1,2,0)$ (b) $(1,1,0,-1)$, $(1,-1,3,0)$ (c) $(\mathrm{i},2)$, $(2,\mathrm{i})$

Exercise 7. For the following orthogonal pairs $\mathbf{u},\mathbf{v}$ and matrix $M=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$, determine whether $M\mathbf{u}$ and $M\mathbf{v}$ are orthogonal.
(a) $(2,1,1)$, $(1,0,-2)$ (b) $(1,1,1)$, $(1,-1,1)$ (c) $(3,1,-2)$, $(1,3,3)$

Exercise 8. For each of the pairs of Exercise 7, determine whether $M\mathbf{u}$ and $\left(M^{-1}\right)^T\mathbf{v}$ are orthogonal.

Exercise 9. Find equations for the following planes in $\mathbb{R}^3$.
(a) The plane containing the points $(1,1,2)$, $(-1,3,2)$, $(2,4,3)$.
(b) The plane containing the points $(-2,1,1)$ and $(0,1,2)$ and orthogonal to the plane $2x-y+z=3$.

Exercise 10. Find equations for the following hyperplanes in $\mathbb{R}^4$.
(a) The plane parallel to the plane $2x_1+x_2-3x_3+x_4=2$ and containing the point $(2,1,1,3)$.
(b) The plane through the origin and orthogonal to the vector $(1,0,2,1)$.

Exercise 11. For each pair A, $\mathbf{b}$, solve the normal equations for the system $A\mathbf{x}=\mathbf{b}$ and find the residual vector and its norm. Are there any genuine solutions to the system?

(a) $\begin{bmatrix}1&3\\1&0\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}$ (b) $\begin{bmatrix}2&-2\\1&1\\3&1\end{bmatrix},\begin{bmatrix}2\\-1\\1\end{bmatrix}$ (c) $\begin{bmatrix}0&2&2\\1&1&0\\-1&1&2\\1&-2&-3\end{bmatrix},\begin{bmatrix}3\\1\\0\\0\end{bmatrix}$

Exercise 12. For each pair A, $\mathbf{b}$, solve the normal equations for the system $A\mathbf{x} = \mathbf{b}$ and find the residual vector and its norm. (Note: normal equations may not have unique solutions.)

(a) $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Exercise 13. (Linear regression) You have collected data points (x_k, y_k) that are theoretically linearly related by a line of the form $y = ax + b$. Each data point gives an equation for a and b. The collected data points are $(0, .3), (1, 1.1), (2, 2), (3, 3.5)$, and $(3.5, 3.6)$. Write out the resulting system of 5 equations, solve the normal equations to find the line that best fits this data, and calculate the residual norm. A calculator or computer might be helpful.

Exercise 14. (Text retrieval) You are given the following *term-by-document* matrix, that is, a matrix whose (i, j)th entry is the number of times term i occurs in document j. Columns of this matrix are document vectors, as is a query. We measure the quality of a match between query and document by the cosine of the angle θ between the two vectors, larger cosine being better. Which of the following nine documents D_i matches the query $(0, 1, 0, 1, 1)$ above the threshhold value $\cos\theta \geq 0.5$? Which is the best match to the query?

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
t_1	1	1	2	0	1	0	1	0	1
t_2	0	1	0	1	0	1	1	0	0
t_3	0	2	0	2	0	1	0	1	1
t_4	1	0	1	0	1	0	2	1	0
t_5	1	2	1	0	0	1	0	0	1

*Problem 15. Show that if two vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy the equation $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal.

Problem 16. Show that the CBS inequality is valid for complex vectors $\mathbf{u}$ and $\mathbf{v}$ by evaluating the nonnegative expression $\|\mathbf{u} + c\mathbf{v}\|^2$ with the complex dot product and evaluating it at $c = \|\mathbf{u}\|^2 / (\mathbf{u} \cdot \mathbf{v})$ in the case $\mathbf{u} \cdot \mathbf{v} \neq 0$.

Problem 17. Let A be an $m \times n$ real matrix and $B = A^T A$. Show the following:
(a) The matrix B is symmetric and positive semidefinite.
(b) If A has full column rank, then B is positive definite.

Problem 18. Show that if A is a real matrix and $A^T A$ is positive definite then A has full column rank.

Problem 19. In Example 4.15 two values of c are calculated: The average value and the least squares value. Calculate each resulting residual and its norm.

Problem 20. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors of the same length. Show that $\mathbf{u} - \mathbf{v}$ is orthogonal to $\mathbf{u} + \mathbf{v}$. Sketch a picture in the plane and interpret it geometrically.

*Problem 21. Show that if A is a rank-one real matrix, then the normal equations with coefficient matrix A are consistent.

Problem 22. Show that if A is a complex matrix, then A^*A is Hermitian and positive semidefinite.

*Problem 23. Show that Theorem 4.3 is valid for complex vectors.

Problem 24. It is hypothesized that sales of a certain product are linearly related to three factors. The sales output is quantified as z and the three factors as x_1, x_2, and x_3. Six samples are taken of the sales and the factor data. Results are contained in the following table. Does the hypothesis of a linear relationship seem reasonable? Explain your answer.

z	x_1	x_2	x_3
527	13	5	6
711	6	17	7
1291	12	16	23
625	11	13	4
1301	12	27	14
1350	5	14	31

4.3 Orthogonal and Unitary Matrices

Orthogonal Sets of Vectors

In our discussion of bases in Section 3.3, we saw that linear independence of a set of vectors was a key idea for understanding the nature of vector spaces. One of our examples of a linearly independent set was the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$. Here $\mathbf{e}_i$ is the vector with a 1 in the ith coordinate and zeros elsewhere. In the case of geometrical vectors and $n = 3$, these are just the familiar vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. These vectors have some particularly nice properties that go beyond linear independence. For one, each is a unit vector with respect to the standard norm. Furthermore, these vectors are mutually orthogonal to each other. These properties are so desirable that we elevate them to the status of a definition.

Definition 4.10. The set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a standard vector space are said to be an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. If, in addition, each vector has unit length, i.e., $\mathbf{v}_i \cdot \mathbf{v}_i = 1$, then the set of vectors is said to be an *orthonormal set* of vectors.

Orthogonal and Orthonormal Set of Vectors

Example 4.17. Which of the following sets of vectors are orthogonal? Orthonormal? Use the standard inner product in each case.

(a)$\{(3/5,4/5),(-4/5,3/5)\}$ (b) $\{(1,-1,0),(1,1,0),(0,0,1)\}$ (c) $\{(1,\mathrm{i}),(\mathrm{i},1)\}$

Solution. For (a) let $\mathbf{v}_1 = (3/5, 4/5)$, $\mathbf{v}_2 = (-4/5, 3/5)$ to obtain that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{-12}{25} + \frac{12}{25} = 0 \text{ and } \mathbf{v}_1 \cdot \mathbf{v}_1 = \frac{9}{25} + \frac{16}{25} = 1 = \mathbf{v}_2 \cdot \mathbf{v}_2.$$

It follows that the first set of vectors is an orthonormal set.

For (b) let $\mathbf{v}_1 = (1,-1,0), \mathbf{v}_2 = (1,1,0), \mathbf{v}_3 = (0,0,1)$ and check that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \cdot 1 - 1 \cdot 1 + 0 \cdot 0 = 0 \text{ and } \mathbf{v}_1 \cdot \mathbf{v}_3 = 1 \cdot 0 - 1 \cdot 0 + 0 \cdot 1 = 0 = \mathbf{v}_2 \cdot \mathbf{v}_3.$$

Hence this set of vectors is orthogonal, but $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1 \cdot 1 + (-1) \cdot (-1) + 0 = 2$, which is sufficient to show that the vectors do not form an orthonormal set.

For (c) let $\mathbf{v}_1 = (1, \mathrm{i})$, $\mathbf{v}_2 = (\mathrm{i}, 1)$ to obtain that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \bar{1}\mathrm{i} + \bar{\mathrm{i}}1 = \mathrm{i} - \mathrm{i} = 0 \text{ and } \mathbf{v}_1 \cdot \mathbf{v}_1 = 1 + 1 = 2 = \mathbf{v}_2 \cdot \mathbf{v}_2.$$

It follows that this set is orthogonal, but not orthonormal. □

One of the principal reasons that orthogonal sets are so desirable is the following key fact, which we call the *orthogonal coordinates theorem.*

Orthogonal Coordinates Theorem

Theorem 4.6. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthogonal set of nonzero vectors and suppose that $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then $\mathbf{v}$ can be expressed uniquely (up to order) as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, namely

$$\mathbf{v} = \frac{\mathbf{v}_1 \cdot \mathbf{v}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{v}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \cdots + \frac{\mathbf{v}_n \cdot \mathbf{v}}{\mathbf{v}_n \cdot \mathbf{v}_n}\mathbf{v}_n.$$

Proof. Since $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, we know that $\mathbf{v}$ is expressible as *some* linear combination of the $\mathbf{v}_i$'s, say

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Now we carry out a simple but wonderful trick that is used frequently with orthogonal sets, namely, take the inner product of both sides with the vector $\mathbf{v}_k$. Since $\mathbf{v}_k \cdot \mathbf{v}_j = 0$ if $j \neq k$, we obtain

$$\begin{aligned}\mathbf{v}_k \cdot \mathbf{v} &= \mathbf{v}_k \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots \cdots + c_n\mathbf{v}_n) \\ &= c_1\mathbf{v}_k \cdot \mathbf{v}_1 + c_2\mathbf{v}_k \cdot \mathbf{v}_2 + \cdots + c_n\mathbf{v}_k \cdot \mathbf{v}_n = c_k\mathbf{v}_k \cdot \mathbf{v}_k.\end{aligned}$$

Since $\mathbf{v}_k \neq 0$, we have $\|\mathbf{v}_k\|^2 = \mathbf{v}_k \cdot \mathbf{v}_k \neq 0$, so solve for c_k to obtain that

$$c_k = \frac{\mathbf{v}_k \cdot \mathbf{v}}{\mathbf{v}_k \cdot \mathbf{v}_k}.$$

This proves that the coefficients c_k are unique and establishes the formula of the theorem. □

The vector $\dfrac{\mathbf{v}_k \cdot \mathbf{v}}{\mathbf{v}_k \cdot \mathbf{v}_k}\mathbf{v}_k$ should look familiar. In fact, it is the projection of the vector $\mathbf{v}$ along the vector $\mathbf{v}_k$. Thus, Theorem 4.6 says that any linear combination of an orthogonal set of nonzero vectors is the sum of its projections in the direction of each vector in the set.

The coefficients c_k of Theorem 4.6 are also familiar: they are the *coordinates* of $\mathbf{v}$ relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, so that $[\mathbf{v}]_B = (c_1, c_2, \ldots, c_n)$. This terminology was introduced in Section 3.3. Theorem 4.6 shows us that coordinates are rather easy to calculate with respect to an orthogonal basis. Contrast this with Example 3.25.

Corollary 4.1. Every orthogonal set of nonzero vectors is linearly independent.

Proof. Consider a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. If some linear combination were to have value zero, say

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

it would follow from the preceding theorem that

$$c_k = \frac{\mathbf{v}_k \cdot 0}{\mathbf{v}_k \cdot \mathbf{v}_k} = 0.$$

It follows from the definition of linear independence that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. □

Caution: The converse of the corollary is false, that is, not every linearly independent set of vectors is orthogonal.

For an example, consider the linearly independent vectors $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (1, 1)$ in $V = \mathbb{R}^2$.

Given an orthogonal set of nonzero vectors, it is easy to manufacture an orthonormal set of vectors from them. Simply replace every vector in the original set by the vector divided by its length. The formula of Theorem 4.6 simplifies very nicely if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form an orthonormal set (which automatically consists of nonzero vectors!), namely

$$\mathbf{v} = (\mathbf{v}_1 \cdot \mathbf{v})\ \mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{v})\ \mathbf{v}_2 + \cdots + (\mathbf{v}_n \cdot \mathbf{v})\ \mathbf{v}_n.$$

The following theorem gives us a nice analogue to the fact that every linearly independent set of vectors can be expanded to a basis.

Theorem 4.7. Every orthogonal set of nonzero vectors in a standard vector space can be expanded to an orthogonal basis of the space.

Proof. Suppose that we have expanded our original orthogonal set in $\mathbb{R}^n$ to the orthogonal set of nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, where $k < n$. We show how to add one more element. This is sufficient, because by repeating this step we eventually fill up $\mathbb{R}^n$. Let $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]^T$ and let $\mathbf{v}_{k+1}$ be any nonzero solution to $A\mathbf{x} = \mathbf{0}$, which exists since $k < n$. This vector is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. □

Orthogonal and Unitary Matrices

In general, if we want to determine the coordinates of a vector $\mathbf{b}$ with respect to a certain basis of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, we stack the basis vectors together to form a matrix A, then solve the system $A\mathbf{x} = \mathbf{b}$ for the vector of coordinates $\mathbf{x}$ of $\mathbf{b}$ with respect to this basis. In fact, $\mathbf{x} = A^{-1}\mathbf{b}$. Now we have seen that if the basis vectors happen to form an orthonormal set, the situation is much simpler and we certainly don't have to find A^{-1}. Is this simplicity reflected in properties of the matrix A? The answer is yes and we can see this as follows: suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is an orthonormal basis of $\mathbb{R}^n$ and let $A = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$. Orthonormality says that $\mathbf{u}_i^T \mathbf{u}_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. This means that the matrix $A^T A$, whose (i, j)th entry is $\mathbf{u}_n^T \mathbf{u}_n$, is simply $[\delta_{ij}] = I$, that is, $A^T A = I$. Now recall that Theorem 2.7 shows that a square one-sided inverse of a square matrix is really the two-sided inverse. Hence, $A^{-1} = A^T$. A similar argument works if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is an orthonormal basis of $\mathbb{C}^n$ except that we use conjugate transpose instead of transpose. Matrices with these properties are important enough to be named.

Orthogonal and Unitary Matrix

Definition 4.11. A square real matrix Q is called *orthogonal if* $Q^T = Q^{-1}$. A square matrix U is called *unitary* if $U^* = U^{-1}$.

One could allow orthogonal matrices to be complex as well, but these are not particularly useful for us, so in this text we will always assume that orthogonal matrices have real entries. For real matrices Q, we have $Q^* = Q^T$. Hence we see from the definition that orthogonal matrices are exactly the real unitary matrices. The naming of orthogonal matrices is traditional in matrix theory, but a bit unfortunate because it can be a source of confusion.

Caution: Do not confuse "orthogonal vectors" and "orthogonal matrix." The objects and meaning are different.

By orthogonal *vectors* we mean a set of vectors with a certain relationship to each other, while an orthogonal *matrix* is a real matrix whose inverse is its transpose. To make matters more confusing, there actually is a close connection between the two terms, because a square matrix is orthogonal exactly when its columns form an orthonormal set.

Example 4.18. Show that the matrix $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}$ is unitary and that for any angle θ, the matrix $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution. It is sufficient to check that $U^*U = I$ and $R(\theta)^T R(\theta) = I$. So we calculate

$$\begin{aligned} U^*U &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \right)^* \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 - \mathrm{i}^2 & \mathrm{i} - \mathrm{i} \\ -\mathrm{i} + \mathrm{i} & 1 - \mathrm{i}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

which shows that U is unitary. For the real matrix $R(\theta)$ we have

$$\begin{aligned}
R(\theta)^T R(\theta) &= \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right)^T \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ -\cos\theta\sin\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

which shows that $R(\theta)$ is orthogonal. □

Orthogonal and unitary matrices have a certain "rigidity" quality about them that is nicely illustrated by the rotation matrix $R(\theta)$. We first saw this matrix in Example 2.17 of Chapter 2. The effect of multiplying a vector $\mathbf{x} \in \mathbb{R}^2$ by $R(\theta)$ is to rotate the vector counterclockwise through an angle of θ. This is illustrated in Figure 2.3 of Chapter 2. In particular, angles between vectors and lengths of vectors are preserved by such a multiplication. This is no accident of $R(\theta)$, but rather a property of orthogonal and unitary matrices in general. Here is a statement of these properties for orthogonal matrices. An analogous fact holds for complex unitary matrices with vectors in $\mathbb{C}^n$.

Theorem 4.8. Let Q be an orthogonal $n \times n$ matrix and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with the standard inner (dot) product. Then

$$\|Q\mathbf{x}\| = \|\mathbf{x}\| \qquad \text{and} \qquad Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Proof. We calculate the norm of $Q\mathbf{x}$:

$$\|Q\mathbf{x}\|^2 = Q\mathbf{x} \cdot Q\mathbf{x} = (Q\mathbf{x})^T Q\mathbf{x} = \mathbf{x}^T Q^T Q\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2,$$

which proves the first assertion, while similarly

$$Q\mathbf{x} \cdot Q\mathbf{y} = (Q\mathbf{x})^T Q\mathbf{y} = \mathbf{x}^T Q^T Q\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$ □

Here is another kind of orthogonal matrix that has turned out to be very useful in numerical calculations and has a very nice geometrical interpretation as well. As with rotation matrices, it gives us a simple way of forming orthogonal matrices directly without explicitly constructing an orthonormal basis.

Definition 4.12. A matrix of the form $H_{\mathbf{v}} = I - 2(\mathbf{v}\mathbf{v}^T)/(\mathbf{v}^T\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^n$, is called a *Householder* matrix.

Householder Matrix

Example 4.19. Let $\mathbf{v} = (3, 0, 4)$ and compute the Householder matrix $H_{\mathbf{v}}$. What is the effect of multiplying it by the vector $\mathbf{v}$?

Solution. We calculate $H_{\mathbf{v}}$ to be

$$\begin{aligned} I - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3^2+4^2}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{25}\begin{bmatrix} 9 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 7 & 0 & -24 \\ 0 & 25 & 0 \\ -24 & 0 & -7 \end{bmatrix}. \end{aligned}$$

Therefore multiplying $H_{\mathbf{v}}$ by $\mathbf{v}$ gives

$$H_{\mathbf{v}}\mathbf{v} = \frac{1}{25}\begin{bmatrix} 7 & 0 & -24 \\ 0 & 25 & 0 \\ -24 & 0 & -7 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -75 \\ 0 \\ -100 \end{bmatrix} = -\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}. \qquad \square$$

Multiplication by a Householder matrix can be thought of as a geometrical reflection that reflects the vector $\mathbf{v}$ to $-\mathbf{v}$ and leaves any vector orthogonal to $\mathbf{v}$ unchanged. This is implied by the following theorem. For a picture of this geometrical interpretation, see Figure 4.4. Notice that in this figure V is the plane perpendicular to $\mathbf{v}$ and the reflections are across this plane.

Theorem 4.9. Let $H_{\mathbf{v}}$ be the Householder matrix defined by $\mathbf{v} \in \mathbb{R}^n$ and let $\mathbf{w} \in \mathbb{R}^n$ be written as $\mathbf{w} = \mathbf{p} + \mathbf{u}$, where $\mathbf{p}$ is the projection of $\mathbf{w}$ along $\mathbf{v}$ and $\mathbf{u} = \mathbf{w} - \mathbf{p}$. Then

$$H_{\mathbf{v}}\mathbf{w} = -\mathbf{p} + \mathbf{u}.$$

Proof. With notation as in the statement of the theorem, we have $\mathbf{p} = \dfrac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v}$ and $\mathbf{w} = \mathbf{p} + \mathbf{u}$. We calculate that

$$\begin{aligned} H_{\mathbf{v}}\mathbf{w} &= \left(I - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T\right)(\mathbf{p}+\mathbf{u}) = \mathbf{p} + \mathbf{u} - 2\frac{\mathbf{v}^T\mathbf{w}}{(\mathbf{v}^T\mathbf{v})^2}\mathbf{v}\mathbf{v}^T\mathbf{v} - 2\frac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T\mathbf{u} \\ &= \mathbf{p} + \mathbf{u} - 2\frac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} - \mathbf{0} = \mathbf{p} + \mathbf{u} - 2\mathbf{p} = \mathbf{u} - \mathbf{p}. \qquad \square \end{aligned}$$

Example 4.20. Let $\mathbf{v} = (3, 0, 4)$ and $H_{\mathbf{v}}$ the corresponding Householder matrix (as in Example 4.19). The columns of this matrix form an orthonormal basis for the space $\mathbb{R}^3$. Find the coordinates of the vector $\mathbf{w} = (2, 1, -4)$ relative to this basis.

Solution. We have already calculated $H_{\mathbf{v}} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ in Example 4.19. The vector $\mathbf{c} = (c_1, c_2, c_3)$ of coordinates of $\mathbf{w}$ must satisfy the equations

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = H_{\mathbf{v}}\mathbf{c}.$$

Since $H_{\mathbf{v}}$ is orthogonal, it follows that

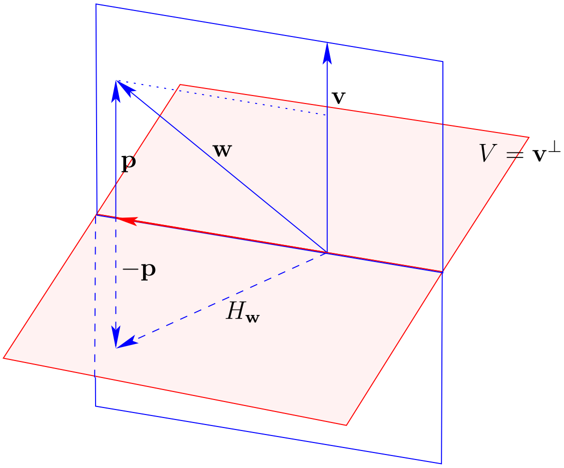


Fig. 4.4. Action of $\mathbf{H}_\mathbf{v}$ on $\mathbf{w}$ as a reflection across the plane V perpendicular to $\mathbf{v}$.

$$\mathbf{c} = H_\mathbf{v}^{-1}\mathbf{w} = H_\mathbf{v}^T\mathbf{w} = \frac{1}{25}\begin{bmatrix} 7 & 0 & -24 \\ 0 & 25 & 0 \\ -24 & 0 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 1.0 \\ -0.8 \end{bmatrix}. \qquad \square$$

Usually we work with real Householder matrices. Occasionally complex numbers are a necessary part of the scenery. In such situations we can define the *complex* Householder matrix by the formula $H_\mathbf{v} = I - 2(\mathbf{v}\mathbf{v}^*)/(\mathbf{v}^*\mathbf{v})$. The projection formula (Theorem 4.3) remains valid for complex vectors, so that the proof of Theorem 4.9 carries over to complex vectors provided that we replace all transposes by conjugate transposes.

Our last example is to generate orthogonal matrices with specified columns.

Example 4.21. Find orthogonal matrices with these orthonormal vectors as columns: (a) $\frac{1}{\sqrt{3}}(1,1,1)$ (b) $\frac{1}{3}(1,2,2,0)$, $\frac{1}{3}(-2,1,0,2)$

Solution. For (a), set $\mathbf{u}_1 = \frac{1}{\sqrt{3}}(1,1,1)$, and we see by inspection that a second orthonormal vector is $\mathbf{u}_2 = \frac{1}{\sqrt{2}}(1,-1,0)$. To obtain a third, take the cross product $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \frac{1}{\sqrt{6}}(1,1,-2)$. This vector is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ and has unit length. Hence the desired matrix is

$$P = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 1 \\ \sqrt{2} & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -2 \end{bmatrix}.$$

To keep the arithmetic simple in (b), form the system $A\mathbf{x} = \mathbf{0}$ where the rows of A are $(1,2,2,0)$ and $(-2,1,0,2)$. These are nonzero orthogonal vectors. Solve the system to get a general solution (the reader should check this) $\mathbf{x} = \left(-\frac{2}{5}x_3 + \frac{4}{5}x_4, -\frac{4}{5}x_3 - \frac{2}{5}x_4, x_3, x_4\right)$. So take $x_3 = 5$, $x_4 = 0$ and get a particular solution $(-2, -4, 5, 0)$. Take $x_3 = 0$, $x_4 = 5$ and get a particular solution $(4, -2, 0, 5)$. Normalize all four vectors to obtain the desired orthogonal matrix

$$P = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4] = \frac{1}{3\sqrt{5}} \begin{bmatrix} \sqrt{5} & -2\sqrt{5} & -2 & 4 \\ 2\sqrt{5} & \sqrt{5} & -4 & -2 \\ 2\sqrt{5} & 0 & 5 & 0 \\ 0 & 2\sqrt{5} & 0 & 5 \end{bmatrix}. \qquad \square$$

We'll see an efficient way to perform this calculation when we study the Gram–Schmidt algorithm in Chapter 6.

4.3 Exercises and Problems

Exercise 1. Determine whether the following sets of vectors are orthogonal, orthonormal, or linearly independent.
(a) $(1, -1, 2)$,$(2, 2, 0)$ (b) $(3, -1, 1)$,$(1, 2, -1)$, $(2, -1, 0)$ (c) $\frac{1}{5}(3, 4)$, $\frac{1}{5}(4, -3)$

Exercise 2. Determine whether these sets are orthogonal or orthonormal. If orthogonal but not orthonormal, normalize the set to form an orthonormal set.
(a) $(2, -3, 2, 1)$, $(2, 1, -1, 1)$ (b)$\frac{1}{3}(2, 2, 1)$, $\frac{1}{\sqrt{5}}(1, 0, -2)(c)(1 + \mathrm{i}, -1)$, $(1, 1 - \mathrm{i})$

Exercise 3. Let $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (-1, 1, 1)$, and $\mathbf{v}_3 = \frac{1}{2}(1, -1, 2)$. Show that this set is an orthogonal basis of $\mathbb{R}^3$ and find the coordinates of the following vectors $\mathbf{v}$ with respect to this basis by using the orthogonal coordinates theorem.
(a) $(1, 2, -2)$ (b) $(1, 0, 0)$ (c) $(4, -3, 2)$

Exercise 4. Let $\mathbf{v}_1 = (-1, 1, 1)$ and $\mathbf{v}_2 = (1, -1, 2)$. Determine whether each of the following vectors $\mathbf{v}$ is in span$\{\mathbf{v}_1, \mathbf{v}_2\}$ by testing the orthogonal coordinates theorem (if $\mathbf{v} \in$ span$\{\mathbf{v}_1, \mathbf{v}_2\}$ then Theorem 4.6 should yield an equality).
(a) $(1, -1, 8)$ (b) $(-2, 1, 3)$ (c) $(-4, 4, 1)$

Exercise 5. Determine whether the following matrices are orthogonal or unitary and if so, find their inverse.

(a) $\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

(d) $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$ (e) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2}\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \end{bmatrix}$ (f) $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 + \mathrm{i} & \mathrm{i} \\ \mathrm{i} & 1 - \mathrm{i} \end{bmatrix}$

Exercise 6. Find the coordinates of the following vectors with respect to the basis of column vectors of the corresponding matrices of Exercise 5.

(a) $(2,4)$ (b) $(3,1,1)$ (c) $(4,-3,1)$
(d) $(3,-2,4,1)$ (e) $(\mathrm{i},-2,1)$ (f) $(1,2)$

Exercise 7. Let $\mathbf{u}=(1,2,-2)$, $\mathbf{w}=(3,0,0)$, and $\mathbf{v}=\mathbf{u}-\mathbf{w}$. Construct the Householder matrix $H_{\mathbf{v}}$ and calculate $H_{\mathbf{v}}\mathbf{u}$ and $H_{\mathbf{v}}\mathbf{w}$.

Exercise 8. Find a matrix reflecting vectors in $\mathbb{R}^3$ across the plane $x+y+z=0$.

Exercise 9. Find orthogonal or unitary matrices that include the following orthonormal vectors in their columns.

(a) $\frac{1}{\sqrt{6}}(1,2,-1)$, $\frac{1}{\sqrt{3}}(-1,1,1)$ (b) $\frac{1}{5}(-4,3)$ (c) $(0,\mathrm{i})$

Exercise 10. Repeat Exercise 9 for these vectors.

(a) $\frac{1}{3}(1,2,-2)$ (b) $\frac{1}{2}(1,1,-1,-1)$, $\frac{1}{2}(1,-1,1,-1)$ (c)$\frac{1}{2}(1+\mathrm{i},1-\mathrm{i})$

Exercise 11. Let $P=\frac{1}{2}\begin{bmatrix}1&0&-1\\0&0&0\\-1&0&1\end{bmatrix}$. Verify that P is a *projection matrix*, that is, $P^T=P$ and $P^2=P$. Also verify that that $R=I-2P$ is a *reflection matrix*, that is, R is a symmetric orthogonal matrix.

Exercise 12. Let $R=\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}$ and $P=\frac{1}{2}(I-R)$. Verify that R is a reflection matrix and P is a projection matrix.

Problem 13. Show that if the real $n\times n$ matrix M is invertible and $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ are orthogonal, then so are $M\mathbf{u}$ and $\left(M^T\right)^{-1}\mathbf{v}$. What does this imply for orthogonal matrices?

*Problem 14. Show that if P is an orthogonal matrix, then $e^{i\theta}P$ is a unitary matrix for any real θ.

Problem 15. Let P be a real projection matrix and $R=I-2P$. Prove that R is a reflection matrix. (See Exercise 11 for definitions.)

Problem 16. Let R be a reflection matrix. Prove that $P=\frac{1}{2}(I-R)$ is a projection matrix.

Problem 17. Prove that every Householder matrix is a reflection matrix.

Problem 18. Show that the product of orthogonal matrices is orthogonal, and by example that the sum need not be orthogonal.

Problem 19. Let the quadratic function $f:\mathbb{R}^n\to\mathbb{R}$ be defined by the formula $y=f(\mathbf{x})=\mathbf{x}^TA\mathbf{x}$, where A is a real matrix. Suppose that an orthogonal change of variables is made in the domain, say $\mathbf{x}=Q\mathbf{x}'$, where Q is orthogonal. Show that in the new coordinates $y=\mathbf{x}'^T(Q^TAQ)\mathbf{x}'$.

4.4 *Change of Basis and Linear Operators

How much information do we need to uniquely identify an operator? For a general operator the answer is a lot! Specifically, we don't really know everything about it until we know how to find its value at every possible argument. This is an infinite amount of information. Yet we know that in some circumstances we can do better. For example, to know a polynomial function completely, we need only a finite amount of data, namely the coefficients of the polynomial. We have already seen that linear operators are special. Are they described by a finite amount of data? The answer is a resounding yes in the situation in which the domain and target are finite-dimensional.

Let's begin with some notation. We will indicate that $T : V \to W$ is a linear operator, $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V, and $C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a basis of W with the notation

$$T : V_B \to W_C \text{ or } V_B \xrightarrow{T} W_C.$$

Now let $\mathbf{v} \in V$ be given. We know that there exists a unique set of scalars, the coordinates $c_1, c_2, \ldots, c_n$ of $\mathbf{v}$ with respect to this basis, such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Thus by linearity of T we see that

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n).$$

It follows that we know everything about the linear operator T if we know the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$.

Now go a step further. Each vector $T(\mathbf{v}_j)$ can be expressed uniquely as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$, namely

$$T(\mathbf{v}_j) = a_{1,j}\mathbf{w}_1 + a_{2,j}\mathbf{w}_2 + \cdots + a_{m,j}\mathbf{w}_m. \tag{4.4}$$

In other words, the scalars $a_{1,j}, a_{2,j}, \ldots, a_{m,j}$ are the coordinates of $T(\mathbf{v}_j)$ with respect to the basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$. Stack these in columns and we now have the $m \times n$ matrix $A = [a_{i,j}]$, which contains everything we need to know in order to compute $T(\mathbf{v})$. In fact, with the above terminology, we have

$$\begin{aligned} T(\mathbf{v}) &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) \\ &= c_1\left(a_{1,1}\mathbf{w}_1 + a_{2,1}\mathbf{w}_2 + \cdots + a_{m,1}\mathbf{w}_m\right) + \\ &\qquad \cdots + c_n(a_{1,n}\mathbf{w}_1 + a_{2,n}\mathbf{w}_2 + \cdots + a_{m,n}\mathbf{w}_m) \\ &= (a_{1,1}c_1 + a_{1,2}c_2 + \cdots + a_{1,n}c_n)\mathbf{w}_1 + \\ &\qquad \cdots + \left(a_{m,1}c_1 + a_{m,2}c_2 + \cdots + a_{m,n}c_n\right)\mathbf{w}_m. \end{aligned}$$

Look closely and we see that the coefficients of these vectors are themselves coordinates of a matrix product, namely the matrix A times the column vector of coordinates of $\mathbf{v}$ with respect to the chosen basis of V. The result of this matrix multiplication is a column vector whose entries are the coordinates of $T(\mathbf{v})$ relative to the chosen basis of W. So in a certain sense, computing the value of a linear operator amounts to no more than multiplying a (coordinate) vector by the matrix A. Now we make the following definition.

Matrix of Linear Operator

Definition 4.13. The *matrix of the linear operator* $T : V_B \to W_C$ *relative to the bases* B and C is the matrix $[T]_{B,C} = [a_{i,j}]$ whose entries are specified by equation (4.4). In the case that $B = C$, we simply write $[T]_B$.

Recall that we denote the coordinate vector of a vector $\mathbf{v}$ with respect to a basis B by $[\mathbf{v}]_B$. Then the above calculation of $T(\mathbf{v})$ can be stated succinctly in matrix/vector terms as

$$[T(\mathbf{v})]_C = [T]_{B,C} [\mathbf{v}]_B . \tag{4.5}$$

Standard Matrix of Linear Operator

This equation has a very interesting application to the standard spaces. Recall that a matrix operator is a linear operator $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by the formula $T_A(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix. It turns out that *every* linear operator on the standard vector spaces is a matrix operator. The matrix A for which $T = T_A$ is called the *standard matrix* of T.

Linear Operator on Standard Spaces Is Matrix Operator

Theorem 4.10. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, B and C the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $A = [T]_{B,C}$, then $T = T_A$.

Proof. The proof is straightforward: for vectors $\mathbf{x}$, $\mathbf{y} = \mathbf{T}(\mathbf{x})$ in standard spaces with standard bases B, C, we have $\mathbf{x} = [\mathbf{x}]_B$ and $\mathbf{y} = [\mathbf{y}]_C$. Therefore,

$$T(\mathbf{x}) = \mathbf{y} = [\mathbf{y}]_C = [T(\mathbf{x})]_C = [T]_{B,C} [\mathbf{x}]_B = [T]_{B,C} \mathbf{x} = A\mathbf{x},$$

which proves the theorem. □

Even in the case of an operator as simple as the identity function $\mathrm{id}_V(\mathbf{v}) = \mathbf{v}$, the matrix of a linear operator can be useful and interesting.

Change of Basis Matrix

Definition 4.14. Let $\mathrm{id}_V : V_C \to V_B$ be the identity function of V. Then the matrix $[\mathrm{id}_V]_{C,B}$ is called the *change of basis* matrix from the basis B to the basis C.

Observe that this definition is consistent with the discussion in Section 3.3, since equation (4.5) shows us that for any vector $\mathbf{v} \in V$,

$$[\mathbf{v}]_B = [\mathrm{id}_V(\mathbf{v})]_B = [\mathrm{id}_V]_{C,B} [\mathbf{v}]_C .$$

Also note that change of basis matrix from basis B to basis C is quite easy if B is a standard basis: simply form the matrix that has the vectors of C listed as its columns.

Example 4.22. Let $V = \mathbb{R}^2$. What is the change of basis matrix from standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ to the basis $C = \left\{\mathbf{v}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\right\}$?

Solution. We see that

$$\begin{aligned} \mathbf{v}_1 &= \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \\ \mathbf{v}_2 &= -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2 . \end{aligned}$$

Compare these equations to (4.4) and we see that the change of basis matrix is

$$[\mathrm{id}_V]_{C,B} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R(\theta) .$$

As predicted, we have to form only the matrix that has the vectors of C listed as its columns. Now compare this to the discussion following Example 3.26. □

Next, suppose that $T : V \to W$ and $S : U \to V$ are linear operators. Can we relate the matrices of T, S and the function composition of these operators, $T \circ S$? The answer to this question is a very fundamental fact.

Matrix of Operator Composition

Theorem 4.11. If $U_D \overset{S}{\to} V_C \overset{T}{\to} W_D$, then $[T \circ S]_{D,C} = [T]_{B,C}[S]_{D,B}$.

Proof. Given a vector $\mathbf{u} \in U$, set $\mathbf{v} = S(\mathbf{u})$. With the notation of equation (4.5) we have that $[T \circ S]_{D,C} [\mathbf{u}]_D = [(T \circ S)(\mathbf{u})]_C$ and by definition of function composition that $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})) = T(\mathbf{v})$. Therefore

$$[T \circ S]_{D,C} [\mathbf{u}]_D = [(T \circ S)(\mathbf{u})]_C = [T(S(\mathbf{u}))]_C = [T(\mathbf{v})]_C .$$

On the other hand, equation (4.5) also implies that $[T(\mathbf{v})]_C = [T]_{B,C} [\mathbf{v}]_B$ and $[S(\mathbf{u})]_B = [S]\,_{D,C} [\mathbf{u}]_D$. Hence, we deduce that

$$[T \circ S]_{D,C} [\mathbf{u}]_D = [T]_{B,C} [\mathbf{v}]_B = [T]_{B,C} [S]\,_{D,C} [\mathbf{u}]_D .$$

If we choose $\mathbf{u}$ such that $\mathbf{e}_j = [\mathbf{u}]_D$, where $\mathbf{e}_j$ is the jth standard vector, then we obtain that the jth columns of left- and right-hand side agree for all j. Hence the matrices themselves agree, which is what we wanted to show. □

Operator Matrix Under Change of Bases

We can now also see exactly what happens when we make a change of basis in the domain and target of a linear operator and recalculate the matrix of the operator. Specifically, suppose that $T : V \to W$ and that B, B' are bases of V and C, C' are bases of W. Let P and Q be the change of basis matrices from B' to B and C' to C, respectively. From Problem 7 we obtain that Q^{-1} is the change of basis matrix from C to C'. Identify a matrix with its operator action by multiplication, and we have a chain of operators

$$V_{B'} \overset{\mathrm{id}_V}{\to} V_B \overset{T}{\to} W_C \overset{\mathrm{id}_W}{\to} W_{C'} .$$

Application of the theorem shows that

$$[T]_{B',C'} = [\mathrm{id}_W]_{C,C'}\,[T]_{B,C}\,[\mathrm{id}_V]_{B',B} = Q^{-1}[T]_{B,C}P.$$

We have just obtained a very important insight into the matrix of a linear transformation. Here is the form it takes for the standard spaces.

Change of Basis for Matrix Operator

Corollary 4.2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator, B a basis of $\mathbb{R}^n$, and C a basis of $\mathbb{R}^m$. Let P and Q be the change of basis matrices from the standard bases to the bases B and C, respectively. If A is the matrix of T with respect to the standard bases and M the matrix of T with respect to the bases B and C, then

$$M = Q^{-1}AP.$$

Example 4.23. Given the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^2$ by the rule

$$T(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 + 3x_2 - x_3 \\ 2x_1 + x_2 - x_4 \end{bmatrix},$$

find the standard matrix of T.

Solution. We see that

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad T(\mathbf{e}_4) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Since the standard coordinate vector of a standard vector is itself, we have

$$[T] = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}.$$

□

Example 4.24. With T as above, find the matrix of T with respect to the domain basis $B = \{(1,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,1)\}$ and range basis $C = \{(1,1), (1,-1)\}$

Solution. Let A be the matrix of the previous example, so it represents the standard matrix of T. Let $B' = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ and $C' = \{(1,0), (0,1)\}$ be the standard bases for the domain and target of T. Then we have

$$A = [T] = [T]_{B',C'}.$$

Further, we have only to stack columns of B and C to obtain change of basis matrices

$$P = [\mathrm{id}_{\mathbb{R}^4}]_{B,B'} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } Q = [\mathrm{id}_{\mathbb{R}^2}]_{C,C'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Now apply Corollary 4.2 to obtain that

$$\begin{aligned}[T]_{B,C} &= Q^{-1}AP \\ &= -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} & \frac{7}{2} & 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -1 & 0 \end{bmatrix}.\end{aligned}$$

□

4.4 Exercises and Problems

Exercise 1. Find the standard matrix, kernel, and range of the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left((x, y, z)\right) = (x + 2y, x - y, y + z)$.

Exercise 2. Find the standard matrix, kernel, and range of the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by $T\left((x_1, x_2, x_3, x_4)\right) = (x_2 - x_4 + 3x_3, 3x_2 - x_4 + x_3)$.

Exercise 3. Bases $B = \{(1, 1), (1, -1)\} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{(2, 0), (3, 1)\} = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ of $\mathbb{R}^2$ are given.
(a) Find the change of basis from the standard basis to each of these bases.
(b) Use (a) to compute the change of basis matrix from B to B' by applying Corollary 4.2 to $T = \mathrm{id}_{\mathbb{R}^2}$.
(c) Given that $\mathbf{w} = 3\mathbf{u}_1 + 4\mathbf{u}_2$, use (b) to express $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$ and $\mathbf{u}'_2$.

Exercise 4. Given bases $B = \{(0, 1, 1), (1, 0, 1), (1, 0, -1)\} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{(0, 0, -1), (0, 3, 1), (2, 0, 0)\} = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ of $\mathbb{R}^3$, find the change of basis matrix from B to B' and use it to express $\mathbf{w}' = 2\mathbf{u}'_1 + \mathbf{u}'_2 - 2\mathbf{u}'_3$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Exercise 5. Find the matrix of the operator $T_A : \mathbb{R}^3 \to \mathbb{R}^2$, where $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$, with respect to the bases $B = \{(1, 0, 1), (1, -1, 0)), (0, 0, 2)\}$ and $C = \{(3, 4), (4, -3)\}$.

Exercise 6. Find the matrix of the operator $T : \mathcal{P}_3 \to \mathcal{P}_2$, where T is given by $T\left(a + bx + cx^2 + dx^3\right) = b + 2cx + 3dx^2$, with respect to the bases $B = \left\{1, x, x^2, x^3\right\}$ and $C = \left\{1, x, 2x^2 - 1\right\}$.

Problem 7. Suppose the finite-dimensional vector space V has bases B and C. Let $T : V_B \to V_C$ be an invertible linear operator. Use Theorem 4.11 to show that $[T]_{B,C}^{-1} = \left[T^{-1}\right]_{C,B}$.

Problem 8. Two $n \times n$ matrices A and B are called *similar* if there exists an invertible matrix P such that $B = P^{-1}AP$. Use Corollary 4.2 to show that similar matrices A and B are both matrices of the same linear operator, namely T_A, with respect to different bases.

Problem 9. Show that a change of basis matrix from one orthonormal basis to another is an orthogonal matrix. Use this to simplify the change of basis formula of Corollary 4.2 in the case that C is an orthonormal basis.

*Problem 10. Define the *determinant* of a linear operator $T : V \to V$ to be the determinant of $[T]_B$, where B is any basis of the finite-dimensional vector space V. Show that this definition is independent of the basis B.

Problem 11. Let λ be a scalar and A, B similar $n \times n$ matrices, i.e., for some invertible matrix P, $B = P^{-1}AP$. Show that

$$\dim \mathcal{N}(\lambda I - A) = \dim \mathcal{N}(\lambda I - B).$$

4.5 *Computational Notes and Projects

Project: Least Squares

The Big Eight needs your help! Below is a table of scores from the games played thus far: The (i, j)th entry is team i's score in the game with team j. Your assignment is twofold. First, write a notebook (or script) in a CAS or MAS available to you that obtains team ratings and predicted point spreads based on the least squares and graph theory ideas you have seen. Include instructions for the illiterate on how to plug in data. Second, you are to write a brief report (one to three pages) on your project that describes the problem, your solution to it, its limitations, and the ideas behind it.

	CU	IS	KS	KU	MU	NU	OS	OU
CU		24		21	45		21	14
IS	12			42	21	16		7
KS				12	21	3	27	24
KU	9	14	30			10		14
MU	8	3	52			18	21	
NU		51	48	63	26		63	
OS	41		45		49	42		28
OU	17	35	70	63			31	

Implementation Notes: You will need to set up a suitable system of equations, form the normal equations, and have a computer algebra system solve the problem. For purposes of illustration, we assume in this project that the tool in use is Mathematica. If not, you will need to replace these commands with the appropriate ones that your computational tools provide. The equations in question are formed by letting the variables be a vector $\mathbf{x}$ of "potentials" $x(i)$, one for each team i, so that the "potential differences" best approximate the actual score differences (i.e., point spreads) of the games. To find the vector $\mathbf{x}$ of potentials you solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is the vector of observed potential differences. N.B: the matrix A is *not* the table given above. You will get one equation for each game played. For example, by checking the $(1,2)$th and $(2,1)$th entries, we see that CU beat IS by a score of 24 to 12. So the resulting equation for this game is $x(1) - x(2) = 24 - 12 = 12$. Ideally, the resulting potentials would give numbers that would enable you to predict the point spread of an as yet unplayed game: all you would have to do to determine the spread for team i versus team j is calculate the difference $x(j) - x(i)$. Of course, it doesn't really work out this way, but this is a reasonable use of the known data. When you set up this system, you obtain an inconsistent system. This is where least squares enter the picture. You will need to set up and solve the normal equations, one way or another. You might notice that the null space of the resulting coefficient matrix is nontrivial, so this matrix does not have full column rank. This makes sense: potentials are unique only up to a constant. To fix this, you could arbitrarily fix the value of one team's potential, that is, set the weakest team's potential value to zero by adding one additional equation to the system of the form $x(i) = 0$.

Note to the Instructor: the data above came from the now defunct Big Eight Conference. This project works better when adapted to your local environment. Pick a sport in season at your institution or locale. Have students collect the data themselves, make out a data table as above, and predict the spread for some (as yet) unplayed games of local interest. It can be very interesting to make it an ongoing project, where for a number of weeks the students collect the previous week's data and make predictions for the following week based on all data collected to date.

Project: Rotations in Computer Graphics I

Problem Description: the objective of this project is to implement a counterclockwise rotation of θ radians about an axis specified by the nonzero three-dimensional vector $\mathbf{v}$ using matrix multiplication. Assume that you are given this vector. Show how to calculate the appropriate matrix $R_{\mathbf{v}}$ and offer some justification (proofs aren't required). Illustrate the method with examples.

Implementation Notes: in principle, the desired matrix can be constructed in three steps: (1) Construct an orthonormal set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}/\|\mathbf{v}\|$. (2) Construct the orthogonal matrix P that maps $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. (3) To construct $R_{\mathbf{v}}$, apply P, do a rotation θ of the xy-plane about the z-axis via $R(\theta)$, then apply P^{-1}. Your job is to elaborate on the details of these steps and illustrate the result with examples.

Project: Rotations in Computer Graphics II

Problem Description: the objective of this project is to implement a counterclockwise rotation of θ radians about an axis specified by the nonzero three-dimensional vector $\mathbf{v}$ using quaternions. Assume that you are given this vector. Show how to calculate the appropriate quaternion $\mathbf{q}_\mathbf{v}$. Illustrate the method (and your mastery of quaternion arithmetic) with examples.

Quaternions

Background: Quaternions have a long and storied history in mathematics dating back to 1843, when they were discovered by Sir William Rowan Hamilton as a generalization of complex numbers. Three-dimensional vector dot and cross products originated as aids to quaternion arithmetic. In 1985 Ken Shoemake showed that quaternions were well suited for certain transforms in computer graphics, namely rotations about an axis in three-dimensional space. A quaternion that does the job requires only four numbers, in contrast to the nine needed for an orthogonal transform.

Implementation Notes: A quick google of "quaternions" will give you more than enough information. A brief précis: quaternion objects are simply elements of $\mathbb{H} = \mathbb{R}^4$, homogeneous space (see Section 3.1.) As such, $\mathbb{H}$ immediately has a vector space structure, standard inner product, and norm. Standard basis elements are denoted by $\mathbf{i} = \mathbf{e}_1$, $\mathbf{j} = \mathbf{e}_2$, $\mathbf{k} = \mathbf{e}_3$, and $\mathbf{h} = \mathbf{e}_4$. Hence quaternions can be written as $\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k} + q_w\mathbf{h} = \mathbf{q}_v + q_w\mathbf{h}$. The vector $\mathbf{q}_v$ is called the "imaginary" part of $\mathbf{q}$, and $q_w\mathbf{h}$ the "real" part. Inspired by complex numbers, we define the *conjugate* quaternion $q^* = q_w - \mathbf{q}_v$. Unlike homogeneous space, $\mathbb{H}$ carries a multiplicative structure. Multiplication is indicated by juxtaposition. We only need to know how to multiply basis elements, since the rest follows from using distributive and associative laws, which we assume to hold for quaternions (of course, everything can be proved formally). Here are the fundamental rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{h} = -\mathbf{h}^2.$$

It is a customary abuse of language to identify $\mathbf{h}$ with 1 and write $\mathbf{q} = \mathbf{q}_\mathbf{v} + q_w$. From these laws we can deduce that $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$, $\mathbf{ik} = -\mathbf{j}$, $\mathbf{kj} = -\mathbf{i}$ and $\mathbf{ji} = -\mathbf{k}$, which is everything we need to know to do arithmetic. A remarkable property of quaternions is that every nonzero element has a multiplicative inverse, namely

$$\mathbf{q}^{-1} = \frac{1}{\|\mathbf{q}\|^2}\mathbf{q}^*.$$

Finally, the connection to rotations can be spelled out as follows: let $\mathbf{p}, \mathbf{q} \in \mathbb{H}$, with $\mathbf{q}$ a unit quaternion, i.e., $\|\mathbf{q}\| = 1$, and $\mathbf{p}$ a quaternion that represents a geometrical point or vector in homogeneous space. Then (1) we can write $\mathbf{q} = \cos\phi + \sin\phi\mathbf{q}_v$ for some angle ϕ and unit vector $\mathbf{q}_v$ and (2) $\mathbf{qpq}^{-1}$ is the result of rotating $\mathbf{p}$ counterclockwise about the axis $\mathbf{q}_v$ through an angle of 2ϕ. Your job is elaborate on the details of this calculation and illustrate the result with examples. As an exercise in manipulation, prove item (1) (it isn't hard), but assume everything else.

5

THE EIGENVALUE PROBLEM

The first major problem of linear algebra is to understand how to solve the basis linear system $A\mathbf{x} = \mathbf{b}$ and what the solution means. We have explored this system from three points of view: In Chapter 1 we approached the problem from an operational point of view and learned the mechanics of computing solutions. In Chapter 2, we took a more sophisticated look at the system from the perspective of matrix theory. Finally, in Chapter 3, we viewed the problem from the vantage of vector space theory.

Now we begin a study of the second major problem of linear algebra, namely the eigenvalue problem. We had to tackle linear systems first because the eigenvalue problem is more sophisticated and will require most of the tools that we have thus far developed. This subject has many important applications, such as the analysis of discrete dynamical systems that we have seen in earlier chapters.

5.1 Definitions and Basic Properties

What Are They?

Good question. Let's get right to the point.

Eigenvector, Eigenvalue, and Eigenpair

Definition 5.1. Let A be a square $n \times n$ matrix. An *eigenvector* of A is a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$, if we are working over complex numbers) such that for some scalar λ, we have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The scalar λ is called an *eigenvalue* of the matrix A, and we say that the vector $\mathbf{x}$ is an *eigenvector belonging to the eigenvalue* λ. The pair $\{\lambda, \mathbf{x}\}$ is called an *eigenpair* for the matrix A.

Eigenvalues and eigenvectors are also known as characteristic values and characteristic vectors. In fact, the word "eigen" means (among other things) "characteristic" in German.

Right and Left Eigenvectors

Eigenvectors of A, as defined above, are also called *right eigenvectors* of A. Notice that if $A^T\mathbf{x} = \lambda\mathbf{x}$, then

$$\lambda\mathbf{x}^T = (\lambda\mathbf{x})^T = \left(A^T\mathbf{x}\right)^T = \mathbf{x}^T A.$$

For this reason, eigenvectors of A^T are called *left eigenvectors* of A.

The only kinds of matrices for which these objects are defined are square matrices, so we'll assume throughout this chapter that we are dealing with such matrices.

Zero Not Eigenvector

Caution: Be aware that the eigenvalue λ is allowed to be the 0 scalar, but an eigenvector $\mathbf{x}$ is, by definition, *never the* $\mathbf{0}$ *vector.*

As a matter of fact, it is quite informative to have an eigenvalue 0. This says that the system $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$. Therefore A is not invertible by Theorem 2.7. There are other reasons for the usefulness of the eigenvector/value concept that we will develop later, but we already see that knowledge of eigenvalues tells us about invertibility of a matrix.

Here are a few simple examples of eigenvalues and eigenvectors. Let $A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$, $\mathbf{x} = (-1, 1)$, and $\mathbf{y} = (4, 3)$. One checks that $A\mathbf{x} = (-3, 3) = 3\mathbf{x}$ and $A\mathbf{y} = (40, 30) = 10\mathbf{y}$. It follows that $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to eigenvalues 3 and 10, respectively.

Why should we have any interest in these quantities? A general answer goes something like this: knowledge of eigenvectors and eigenvalues gives us deep insights into the structure of the matrix A. Here is just one example: suppose that we would like to have a better understanding of the effect of multiplication of a vector $\mathbf{x}$ by powers of the matrix A, that is, of $A^k\mathbf{x}$. Let's start with the first power, $A\mathbf{x}$. If we knew that $\mathbf{x}$ were an eigenvector of A, then we would have that for some scalar λ,

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^2\mathbf{x} &= A(A\mathbf{x}) = A\lambda\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x} \\ &\vdots \\ A^k\mathbf{x} &= A(A^{k-1}\mathbf{x}) = \cdots = \lambda^k\mathbf{x}. \end{aligned}$$

This is very nice, because it reduces something complicated, namely matrix–vector multiplication, to something simple, namely scalar–vector multiplication.

We need some handles on these quantities. Let's ask how we could figure out what they are for specific matrices. Here are some of the basic points about eigenvalues and eigenvectors.

Theorem 5.1. Let A be a square $n \times n$ matrix. Then

(1) The eigenvalues of A are all the scalars λ that are solutions to the nth-degree polynomial equation

$$\det(\lambda I - A) = 0.$$

(2) For a given eigenvalue λ, the eigenvectors of the matrix A belonging to that eigenvalue are all the nonzero elements of $\mathcal{N}(\lambda I - A)$.

Proof. Note that $\lambda \mathbf{x} = \lambda I \mathbf{x}$. Thus we have the following chain of thought: A has eigenvalue λ if and only if $A\mathbf{x} = \lambda \mathbf{x}$, for some nonzero vector $\mathbf{x}$, which is true if and only if

$$\mathbf{0} = \lambda \mathbf{x} - A\mathbf{x} = \lambda I \mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x}$$

for some nonzero vector $\mathbf{x}$. This last statement is equivalent to the assertion that $\mathbf{0} \neq \mathbf{x} \in \mathcal{N}(\lambda I - A)$. The matrix $\lambda I - A$ is square, so it has a nontrivial null space precisely when it is singular (recall the characterizations of nonsingular matrices in Theorem 2.7). This occurs only when $\det(\lambda I - A) = 0$. If we expand this determinant down the first column, we see that the highest-order term involving λ that occurs is the product of the diagonal terms $(\lambda - a_{ii})$, so that the degree of the expression $\det(\lambda I - A)$ as a polynomial in λ is n. This proves (1).

We saw from this chain of thought that if λ is an eigenvalue of A, then the eigenvectors belonging to that eigenvalue are precisely the nonzero vectors $\mathbf{x}$ such that $(\lambda I - A)\mathbf{x} = \mathbf{0}$, that is, the nonzero elements of $\mathcal{N}(\lambda I - A)$, which is what (2) asserts. □

Here is some terminology that we will use throughout this chapter. We call a polynomial *monic* if the leading coefficient is 1.For example, $\lambda^2 + 2\lambda + 3$ is a monic polynomial in λ while $2\lambda^2 + \lambda + 1$ is not.

Monic Polynomial

Definition 5.2. Given a square $n \times n$ matrix A, the equation $\det(\lambda I - A) = 0$ is called the *characteristic equation* of A, and the nth-degree monic polynomial $p(\lambda) = \det(\lambda I - A)$ is called the *characteristic polynomial* of A.

Characteristic Equation and Polynomial

Suppose we already know the eigenvalues of A and want to find the eigenvalues of something like $3A + 4I$. Do we have to start over to find them? The next calculation is really a useful tool for answering such questions.

Theorem 5.2. If $B = cA + dI$ for scalars d and $c \neq 0$, then the eigenvalues of B are of the form $\mu = c\lambda + d$, where λ runs over the eigenvalues of A, and the eigenvectors of A and B are identical.

Proof. Let $\mathbf{x}$ be an eigenvector of A corresponding to the eigenvalue λ. Then by definition, $\mathbf{x} \neq \mathbf{0}$ and

$$A\mathbf{x} = \lambda \mathbf{x}.$$

Also, we have that

$$dI\mathbf{x} = d\mathbf{x}.$$

Now multiply the first equation by the scalar c and add these two equations to obtain

$$(cA + dI)\mathbf{x} = B\mathbf{x} = (c\lambda + d)\mathbf{x}.$$

It follows that every eigenvector of A belonging to λ is also an eigenvector of B belonging to the eigenvalue $c\lambda + d$. Conversely, if $\mathbf{y}$ is an eigenvector of B belonging to μ, then

$$B\mathbf{y} = \mu\mathbf{y} = (cA + dI)\mathbf{y}.$$

Now solve for $A\mathbf{y}$ to obtain that

$$A\mathbf{y} = \frac{1}{c}(\mu - d)\mathbf{y},$$

so that $\lambda = (\mu - d)/c$ is an eigenvalue of A with corresponding eigenvector $\mathbf{y}$. It follows that A and B have the same eigenvectors, and their eigenvalues are related by the formula $\mu = c\lambda + d$. □

Example 5.1. Let $A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$, $\mathbf{x} = (-1, 1)$, and $\mathbf{y} = (4, 3)$, so that $A\mathbf{x} = (-3, 3) = 3\mathbf{x}$ and $A\mathbf{y} = (40, 30) = 10\mathbf{y}$. Find the eigenvalues and corresponding eigenvectors for the matrix $B = 3A + 4I$.

Solution. From the calculations given to us, we observe that $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to the eigenvalues 3 and 10, respectively, for A. These are all the eigenvalues of A, since the characteristic polynomial of A is of degree 2, so has only two roots. According to Theorem 5.2, the eigenvalues of $3A+4I$ must be $\mu_1 = 3\cdot 3+4 = 13$ with corresponding eigenvector $\mathbf{x} = (-1, 1)$, and $\mu_2 = 3 \cdot 10 + 4 = 34$ with corresponding eigenvector $\mathbf{y} = (4, 3)$. □

Eigenspace

Definition 5.3. Given an eigenvalue λ of the matrix A, the *eigenspace* corresponding to λ is the subspace $\mathcal{N}(\lambda I - A)$ of $\mathbb{R}^n$ (or $\mathbb{C}^n$). We write $\mathcal{E}_\lambda(A) = \mathcal{N}(\lambda I - A)$.

Eigensystem

Definition 5.4. By an *eigensystem* of the matrix A, we mean a list of all the eigenvalues of A and, for each eigenvalue λ, a complete description of the eigenspace corresponding to λ.

The usual way to give a complete description of an eigenspace is to list a basis for the space. Remember that there is one vector in the eigenspace $\mathcal{N}(\lambda I - A)$ that is *not* an eigenvector, namely $\mathbf{0}$. In any case, the computational route is now clear. To call it an algorithm is really an abuse of language, since we don't have a complete computational description of the root-finding phase, but here it is:

Eigensystem Algorithm

Let A be an $n \times n$ matrix. To find an eigensystem of A:

(1) Find the scalars that are roots to the characteristic equation $\det(\lambda I - A) = 0$.

(2) For each scalar λ in (1), use the null space algorithm to find a basis of the eigenspace $\mathcal{N}(\lambda I - A)$.

As a matter of convenience, it is sometimes a little easier to work with $A - \lambda I$ when calculating eigenspaces (because there are fewer extra minus signs to worry about). This is perfectly OK, since $\mathcal{N}(A - \lambda I) = \mathcal{N}(\lambda I - A)$. It doesn't affect the eigenvalues either, since $\det(\lambda I - A) = \pm \det(A - \lambda I)$. Here is our first eigensystem calculation.

Example 5.2. Find an eigensystem for the matrix $A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$.

Solution. First solve the characteristic equation

$$\begin{aligned} 0 = \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 7 & -4 \\ -3 & \lambda - 6 \end{bmatrix} \\ &= (\lambda - 7)(\lambda - 6) - (-3)(-4) \\ &= \lambda^2 - 13\lambda + 42 - 12 \\ &= \lambda^2 - 13\lambda + 30 \\ &= (\lambda - 3)(\lambda - 10). \end{aligned}$$

Hence the eigenvalues are $\lambda = 3, 10$. Next, for each eigenvector calculate the corresponding eigenspace.

$\lambda = 3$: Then $A - 3I = \begin{bmatrix} 7 - 3 & 4 \\ 3 & 6 - 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$ and row reduction gives

$$\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \xrightarrow[\substack{E_{21}(-3/4) \\ E_1(1/4)}]{} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

so the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore a basis of $\mathcal{E}_3(A)$ is $\{(-1, 1)\}$.

$\lambda = 10$: Then $A - 10I = \begin{bmatrix} 7 - 10 & 4 \\ 3 & 6 - 10 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}$ and row reduction gives

$$\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \xrightarrow[\substack{E_{21}(1) \\ E_1(-1/3)}]{} \begin{bmatrix} 1 & -4/3 \\ 0 & 0 \end{bmatrix},$$

so the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (4/3)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}.$$

Therefore a basis of $\mathcal{E}_{10}(A)$ is $\{(4/3, 1)\}$. □

Concerning this example, there are several observations worth noting:

- Since the 2×2 matrix $A - \lambda I$ is singular for the eigenvalue λ, one row should always be a multiple of the other. Knowing this, we didn't have to do even the little row reduction we did above. However, its a good idea to check; it helps you avoid mistakes. Remember: any time that row reduction of $A - \lambda I$ leads to full rank (only trivial solutions), you have either made an arithmetic error or you do not have an eigenvalue.
- This matrix is familiar. In fact, $B = (0.1)A$ is the Markov chain transition matrix from Example 2.18. Therefore the eigenvalues of B are 0.3 and 1, by Example 5.2 with $c = 0.1$ and $d = 0$. The eigenvector belonging to $\lambda = 1$ is just a solution to the equation $B\mathbf{x} = \mathbf{x}$, which was discussed in Example 3.31.
- The vector
$$\mathbf{x} = \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix} = \frac{3}{7} \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$$
is an eigenvector of A belonging to the eigenvalue $\lambda = 10$ of A, so that from $A\mathbf{x} = 10\mathbf{x}$ we see that $B\mathbf{x} = 1\mathbf{x}$. Hence, $\mathbf{x} \in \mathcal{E}_1(B)$.

Example 5.3. How do we find eigenvalues of a triangular matrix? Illustrate the method with $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution. Eigenvalues are just the roots of the characteristic equation $\det(\lambda I - A) = 0$. Notice that $-A$ is triangular if A is. Also, the only entries in $\lambda I - A$ that are any different from the entries of $-A$ are the diagonal entries, which change from $-a_{ii}$ to $\lambda - a_{ii}$. Therefore, $\lambda I - A$ is triangular if A is. We already know that the determinant of a triangular matrix is easy to compute: just form the product of the diagonal entries. Therefore, the roots of the characteristic equation are the solutions to

$$0 = \det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}),$$

that is, $\lambda = a_{11}, a_{22}, \ldots, a_{nn}$. In other words, for a triangular matrix the eigenvalues are simply the diagonal elements! In particular, for the example A given above, we see with no calculations that the eigenvalues are $\lambda = 2, 1, -1$. □

Notice, by the way, that we don't quite get off the hook in the preceding example if we are required to find the eigenvectors. It will still be some work to compute each of the relevant null spaces, but much less than for a general matrix.

Example 5.3 can be used to illustrate another very important point. The reduced row echelon form of the matrix of that example is clearly the identity matrix I_3. This matrix has eigenvalues $1, 1, 1$, which are *not* the same as the eigenvalues of A (would that eigenvalue calculations were so easy!). In fact, a single elementary row operation on a matrix can change the eigenvalues. For example, simply multiply the first row of A above by $\frac{1}{2}$. This point warrants a warning, since it is the source of a fairly common mistake.

Caution: The eigenvalues of a matrix A and the matrix EA, where E is an elementary matrix, need not be the same.

Example 5.4. Find an eigensystem for the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Solution. For eigenvalues, compute the roots of the equation

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)^2 - (-1) = \lambda^2 - 2\lambda + 2. \end{aligned}$$

Now we have a little problem. Do we allow complex numbers? If not, we are stuck because the roots of this equation are

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2}}{2} = 1 \pm \mathrm{i}.$$

In other words, if we did not enlarge our field of scalars to the complex numbers, we would have to conclude that there are *no* eigenvalues or eigenvectors! Somehow, this doesn't seem like a good idea. It is throwing information away. Perhaps it comes as no surprise that complex numbers would eventually figure into the eigenvalue story. After all, finding eigenvalues is all about solving polynomial equations, and complex numbers were invented to overcome the inability of real numbers to provide solutions to all polynomial equations. Let's allow complex numbers as the scalars. Now our eigenspace calculations are really going on in the complex space $\mathbb{C}^2$ instead of $\mathbb{R}^2$.

$\lambda = 1 + \mathrm{i}$: Then $A - (1+\mathrm{i})I = \begin{bmatrix} 1-(1+\mathrm{i}) & -1 \\ 1 & 1-(1+\mathrm{i}) \end{bmatrix} = \begin{bmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{bmatrix}$ and row reduction gives (recall that $1/\mathrm{i} = -\mathrm{i}$)

$$\begin{bmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{bmatrix} \xrightarrow[E_1(1/(-\mathrm{i}))]{E_{21}(-\mathrm{i})} \begin{bmatrix} 1 & -\mathrm{i} \\ 0 & 0 \end{bmatrix},$$

so the general solution is

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \mathrm{i}z_2 \\ z_2 \end{bmatrix} = z_2 \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}.$$

Therefore a basis of $\mathcal{E}_{1+\mathrm{i}}(A)$ is $\{(\mathrm{i},1)\}$.

$\lambda = 1 - \mathrm{i}$: Then $A - (1-\mathrm{i})I = \begin{bmatrix} 1-(1-\mathrm{i}) & -1 \\ 1 & 1-(1-\mathrm{i}) \end{bmatrix} = \begin{bmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{bmatrix}$ and row reduction gives

$$\begin{bmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{bmatrix} \xrightarrow[\substack{E_{21}(\mathrm{i}) \\ E_1(1/\mathrm{i})}]{} \begin{bmatrix} 1 & \mathrm{i} \\ 0 & 0 \end{bmatrix},$$

so the general solution is

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\mathrm{i}z_2 \\ z_2 \end{bmatrix} = z_2 \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix}.$$

Therefore a basis of $\mathcal{E}_{1+\mathrm{i}}(A)$ is $\{(-\mathrm{i},1)\}$. □

In view of the previous example, we are going to adopt the following practice: if the eigenvalue calculation leads us to complex numbers, we take the point of view that the field of scalars should be enlarged to include the complex numbers and the eigenvalues in question. One small consolation for having to deal with complex eigenvalues is that in some cases our work may be cut in half.

Example 5.5. Show that if $\{\lambda, \mathbf{x}\}$ is an eigenpair for real matrix A, then so is $\{\overline{\lambda}, \overline{\mathbf{x}}\}$.

Solution. By hypothesis, $A\mathbf{x} = \lambda\mathbf{x}$. Apply complex conjugation to both sides and use the fact that A is real to obtain

$$\overline{A\mathbf{x}} = \overline{A}\,\overline{\mathbf{x}} = A\overline{\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}.$$

Thus $\{\overline{\lambda}, \overline{\mathbf{x}}\}$ is also an eigenpair for A. □

In view of this fact, we could have stopped with the calculation of eigenpair $\{1+\mathrm{i}, (\mathrm{i},1)\}$ in Example 5.4, since we automatically have that $\{1-\mathrm{i}, (-\mathrm{i},1)\}$ is also an eigenpair.

Multiplicity of Eigenvalues

The following example presents yet another curiosity about eigenvalues and eigenvectors.

Example 5.6. Find an eigensystem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

Solution. Here the eigenvalues are easy. This matrix is triangular, so they are $\lambda = 2, 2$. Next we calculate eigenvectors.

$\lambda = 2$: Then $A - 2I = \begin{bmatrix} 2-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and row reduction is not necessary here. Notice that the variable x_1 is free here, while x_2 is bound. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore a basis of $\mathcal{E}_2(A)$ is $\{(1,0)\}$. □

The manner in which we list the eigenvalues in this example is intentional. The number 2 occurs twice on the diagonal, suggesting that it should be counted twice. As a matter of fact, $\lambda = 2$ is a root of the characteristic equation $(\lambda - 2)^2 = 0$ of multiplicity 2. Yet there is a curious mismatch here. In all of our examples to this point, we have been able to come up with as many eigenvectors as eigenvalues, namely the size of the matrix if we allow complex numbers. In this case there is a deficiency in the number of eigenvectors, since there is only one eigenspace and it is one-dimensional. Does this failing always occur with multiple eigenvalues? The answer is no. The situation is a bit more complicated, as the following example shows.

Example 5.7. Discuss the eigenspace corresponding to the eigenvalue $\lambda = 2$ for these two matrices:

$$\text{(a)} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution. Notice that each of these matrices has eigenvalues $\lambda = 1, 2, 2$. Now for the eigenspace $\mathcal{E}_2(A)$.

(a) For this eigenspace calculation we have

$$A - 2I = \begin{bmatrix} 2-2 & 1 & 2 \\ 0 & 1-2 & -2 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

and row reduction gives

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{21}(1)]{} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that free variables are x_1, x_3 and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

Thus a basis for $\mathcal{E}_2(A)$ is $\{(1,0,0), (0,-2,1)\}$. Notice that in this case we get as many independent eigenvectors as the number of times that the eigenvalue $\lambda = 2$ occurs.

(b) For this eigenspace calculation we have

$$A - 2I = \begin{bmatrix} 2-2 & 1 & 1 \\ 0 & 1-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and row reduction gives

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{21}(1)]{} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{E_2(1/2) \\ E_{12}(-1)}]{} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the only free variable is x_1 and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus a basis for $\mathcal{E}_2(A)$ is $\{(1,0,0)\}$. Notice that in this case we don't get as many independent eigenvectors as the number of times that the eigenvalue $\lambda = 2$ occurs. □

This example shows that there are two kinds of "multiplicities" of an eigenvector. On the one hand there is the number of times that the eigenvalue occurs as a root of the characteristic equation. On the other hand there is the dimension of the corresponding eigenspace. One of these is algebraic in nature, the other is geometric. Here are the appropriate definitions.

Algebraic and Geometric Multiplicity

Definition 5.5. Let λ be a root of the characteristic equation $\det(\lambda I - A) = 0$. The *algebraic* multiplicity of λ is the multiplicity of λ as a root of the characteristic equation. The *geometric* multiplicity of λ is the dimension of the space $\mathcal{E}_\lambda(A) = \mathcal{N}(\lambda I - A)$.

We categorize eigenvalues as simple or repeated, according to the following definition.

Simple Eigenvalue

Definition 5.6. The eigenvalue λ of A is said to be *simple* if its algebraic multiplicity is 1, that is, the number of times it occurs as a root of the characteristic equation is 1. Otherwise, the eigenvalue is said to be *repeated*.

In Example 5.7 we saw that the repeated eigenvalue $\lambda = 2$ has algebraic multiplicity 2 in both (a) and (b), but geometric multiplicity 2 in (a) and 1 in (b). What can be said in general? The following theorem summarizes the facts. In particular, (2) says that *algebraic multiplicity is always greater than or equal to geometric multiplicity.* Item (1) is immediate since a polynomial of degree n has n roots, counting complex roots and multiplicities. We defer the proof of (2) to Section 5.3.

Theorem 5.3. Let A be an $n \times n$ matrix with characteristic polynomial $p(\lambda) = \det(\lambda I - A)$. Then:

(1) The number of eigenvalues of A, counting algebraic multiplicities and complex numbers, is n.

(2) For each eigenvalue λ of A, if $m(\lambda)$ is the algebraic multiplicity of λ, then

$$1 \leq \dim \mathcal{E}_\lambda(A) \leq m(\lambda).$$

Now when we wrote that each of the matrices of Example 5.7 has eigenvalues $\lambda = 1, 2, 2$, what we intended to indicate was a complete listing of the eigenvalues of the matrix, counting algebraic multiplicities. In particular, $\lambda = 1$ is a simple eigenvalue of the matrices, while $\lambda = 2$ is not. The geometric multiplicities of (a) are identical to the algebraic multiplicities in (a) but not those in (b). The latter kind of matrix is harder to deal with than the former. Following a time-honored custom of mathematicians, we call the more difficult matrix by a less than flattering name, namely, "defective."

Defective Matrix

Definition 5.7. A matrix is *defective* if one of its eigenvalues has geometric multiplicity less than its algebraic multiplicity.

Notice that the sum of the algebraic multiplicities of an $n \times n$ matrix is the size n of the matrix. This is due to the fact that the characteristic polynomial of the matrix has degree n, hence exactly n roots, counting multiplicities. Therefore, the sum of the geometric multiplicities of a defective matrix will be less than n.

5.1 Exercises and Problems

Exercise 1. Exhibit all eigenvalues of these matrices.

(a) $\begin{bmatrix} 7 & -10 \\ 5 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

Exercise 2. Compute the eigenvalues of these matrices.

(a) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1+\mathrm{i} & 3 \\ 0 & \mathrm{i} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Exercise 3. Find eigensystems for the matrices of Exercise 1. Specify the algebraic and geometric multiplicity of each eigenvalue.

Exercise 4. Find eigensystems for the matrices of Exercise 2 and identify any defective matrices.

Exercise 5. You are given that the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $1, -1$ and respective eigenvectors $(1, 1)$, $(1, -1)$. Use Theorem 5.2 to determine an eigensystem for $B = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$ without further eigensystem calculations.

Exercise 6. You are given that $A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and that $\{2, (-1, 0, 1)\}$ and $\{1 + \mathrm{i}, (2, 1 - \mathrm{i}, 0)\}$ are eigenpairs of A. Determine an eigensystem of A without further eigensystem calculations.

Exercise 7. The *trace* of a matrix A is the sum of all the diagonal entries of the matrix and is denoted by $\operatorname{tr} A$. Find the trace of each matrix in Exercise 1 and verify that it is the sum of the eigenvalues of the matrix.

Exercise 8. For each of the matrices in Exercise 2 show that the product of all eigenvalues is the determinant of the matrix.

Exercise 9. Show that for each matrix A of Exercise 1, A and A^T have the same eigenvalues.

Exercise 10. Find all left eigenvectors of each matrix in Exercise 1. Are right and left eigenspaces for each eigenvalue the same?

Exercise 11. For each matrix A of Exercise 1 determine whether $A^T A$ and A^2 have the same eigenvalues. (*Hint:* test eigenvalues of one matrix on the other.)

Exercise 12. For each matrix A of Exercise 2 show that the matrix $B = A^* A$ has nonnegative eigenvalues.

Exercise 13. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and let α be an eigenvalue of A, β an eigenvalue of B. Confirm or deny the hypotheses that (a) $\alpha + \beta$ is an eigenvalue of $A + B$, and (b) $\alpha\beta$ is an eigenvalue of AB.

Exercise 14. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Confirm or deny the hypothesis that eigenvalues of AB and BA are the same.

Problem 15. Show that if A is Hermitian, then right and left eigenvalues ane eigenvectors coincide.

Problem 16. Show from the definition of eigenvector that if $\mathbf{x}$ is an eigenvector for the matrix A belonging to the eigenvalue λ, then so is $c\mathbf{x}$ for any scalar $c \neq 0$.

*Problem 17. Prove that if A is invertible and λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

Problem 18. Show that if λ is an eigenvalue of an orthogonal matrix P, then $|\lambda| = 1$.

*Problem 19. Let A be a matrix whose eigenvalues are all less than 1 in absolute value. Show that every eigenvalue of $I - A$ is nonzero and deduce that $I - A$ is invertible.

*Problem 20. Show that A and A^T have the same eigenvalues.

Problem 21. Let A be a real matrix and $\{\lambda, \mathbf{x}\}$ an eigenpair for A. Show that $\{\overline{\lambda}, \overline{\mathbf{x}}\}$ is also an eigenpair for A.

*Problem 22. Show that if A and B are the same size, then AB and BA have the same eigenvalues.

Problem 23. Let T_k be the $k \times k$ tridiagonal matrix whose diagonal entries are 2 and off-diagonal nonzero entries are -1. Use a MAS or CAS to build an array y of length 30 whose kth entry is the minimum of the absolute value of the eigenvalues of T_{k+1}. Plot this array. Use the graph as a guide and try to approximate $y(k)$ as a simple function of k.

5.2 Similarity and Diagonalization

Diagonalization and Matrix Powers

Discrete Linear Dynamical System

Eigenvalues: why are they important? This is a good question and has many answers. We will try to demonstrate their importance by focusing on one special class of problems, namely, *discrete linear dynamical systems,* which were defined in Section 2.3. We have seen examples of this kind of system before, namely in Markov chains and difference equations. Here is the sort of question that we would like to answer: when is it the case that there is a limiting vector $\mathbf{x}$ for this sequence of vectors, and if so, how does one compute this vector? The answer to this question will explain the behavior of the Markov chain that was introduced in Example 2.18.

If there is such a limiting vector $\mathbf{x}$ for a Markov chain, we saw in Example 3.31 how to proceed: find the null space of the matrix $I - A$, that is, the set of all solutions to the system $(I - A)\mathbf{x} = 0$. However, the question whether all initial states $\mathbf{x}^{(0)}$ lead to this limiting vector is a more subtle issue, which requires the insights of the next section. We've already done some work on this problem. We saw in Section 2.3 that the entire sequence of vectors is uniquely determined by the initial vector and the transition matrix A in the explicit formula

$$\mathbf{x}^{(k)} = A^k \mathbf{x}^{(0)}.$$

Before proceeding further, let's consider another example that will indicate why we would be interested in limiting vectors.

Example 5.8. By some unfortunate accident a new species of frog has been introduced into an area where it has too few natural predators. In an attempt to restore the ecological balance, a team of scientists is considering introducing a species of bird that feeds on this frog. Experimental data suggests that the population of frogs and birds from one year to the next can be modeled by linear relationships. Specifically, it has been found that if the quantities F_k and B_k represent the populations of the frogs and birds in the kth year, then

$$\begin{aligned} B_{k+1} &= 0.6B_k + 0.4F_k, \\ F_{k+1} &= -rB_k + 1.4F_k, \end{aligned}$$

is a system that models their joint behavior reasonably well. Here the positive number r is a kill rate that measures the consumption of frogs by birds. It varies with the environment, depending on factors such as the availability of other food for the birds. Experimental data suggests that in the environment where the birds are to be introduced, $r = 0.35$. The question is this: in the long run, will the introduction of the birds reduce or eliminate growth of the frog population?

Solution. The discrete dynamical system concept introduced in the preceding discussion fits this situation very nicely. Let the population vector in the kth year be $\mathbf{x}^{(k)} = (B_k, F_k)$. Then the linear relationship above becomes

$$\begin{bmatrix} B_{k+1} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ -0.35 & 1.4 \end{bmatrix} \begin{bmatrix} B_k \\ F_k \end{bmatrix},$$

which is a discrete linear dynamical system. Notice that this is different from the Markov chains we studied earlier, since one of the entries of the coefficient matrix is negative. Before we can finish solving this example we need to have a better understanding of discrete dynamical systems and the relevance of eigenvalues. □

Let's try to understand how state vectors change in the general discrete dynamical system. We have $\mathbf{x}^{(k)} = A^k \mathbf{x}^{(0)}$. So what we really need to know is how the powers of the transition matrix A behave. In general, this is very hard!

Here is an easy case we can handle: what if $A = [a_{ij}]$ is diagonal? Since we'll make extensive use of diagonal matrices, let's recall a notation that was introduced in Chapter 2. The matrix $\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the $n \times n$ diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ down the diagonal. For example,

$$\operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

By matching up the ith row and jth column of A we see that the only time we could have a nonzero entry in A^2 is when $i = j$, and in that case the entry is a_{ii}^2. A similar argument applies to any power of A. In summary, we have this handy fact.

Theorem 5.4. If $D = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $D^k = \operatorname{diag}\{\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k\}$, for all positive integers k.

Just as an aside, this theorem has a very interesting consequence. We have seen in some exercises that if $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial, we can evaluate $f(x)$ at the square matrix A as long as we understand that the constant term a_0 is evaluated as $a_0 I$. In the case of a diagonal A, the following fact reduces evaluation of $f(A)$ to scalar calculations.

Corollary 5.1. If $D = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $f(x)$ is a polynomial, then

$$f(D) = \operatorname{diag}\{f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)\}.$$

Proof. Observe that if $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $f(D) = a_0 I + a_1 D + \cdots + a_n D^n$. Now apply the preceding theorem to each monomial D^k and add up the resulting terms in $f(D)$. □

Now for the powers of a more general A. For ease of notation, let's consider a 3×3 matrix A. What if we could find three linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$? We would have $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$, and $A\mathbf{v}_3 = \lambda_3 \mathbf{v}_3$. In matrix form,

$$A[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3] = [\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = [\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3] \operatorname{diag}\{\lambda_1,\lambda_2,\lambda_3\}.$$

Now set $P = [\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]$ and $D = \operatorname{diag}\{\lambda_1,\lambda_2,\lambda_3\}$. Then P is invertible since the columns of P are linearly independent. (Remember that any nonzero solution to $A\mathbf{x} = \mathbf{0}$ would give rise to a nontrivial linear combination of the column of A that sums to $\mathbf{0}$.) So the equation $AP = PD$, if multiplied on the left by P^{-1}, gives the equation

$$P^{-1}AP = D.$$

This is a beautiful equation, because it makes the powers of A simple to understand. The procedure we just went through is reversible as well. In other words, if P is an invertible matrix such that $P^{-1}AP = D$, then we deduce that $AP = PD$, identify the columns of P by the equation $P = [\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]$, and conclude that the columns of P are linearly independent eigenvectors of A. We make the following definition and follow it with a simple but key theorem relating similar matrices.

Definition 5.8. A matrix A is said to be *similar* to matrix B if there exists an invertible matrix P such that

Similar Matrices

$$P^{-1}AP = B.$$

The matrix P is called a *similarity transformation* matrix.

A simple size check shows that similar matrices have to be square and of the same size. Furthermore, if A is similar to B, then B is similar to A. To see this, suppose that $P^{-1}AP = B$ and multiply by P on the left and P^{-1} on the right to obtain that

$$A = PP^{-1}APP^{-1} = PBP^{-1} = (P^{-1})^{-1}BP^{-1}.$$

Similar matrices have much in common. For example, suppose that $B = P^{-1}AP$ and λ is an eigenvalue of A, say $A\mathbf{x} = \lambda\mathbf{x}$. One calculates

$$A\mathbf{x} = \lambda P^{-1}\mathbf{x} = P^{-1}A\mathbf{x} = P^{-1}AP\left(P^{-1}\mathbf{x}\right),$$

from which it follows that λ is an eigenvalue of B. Here is a slightly stronger statement.

Theorem 5.5. Suppose that A is similar to B, say $P^{-1}AP = B$. Then:

(1) For every polynomial $q(x)$,

$$q(B) = P^{-1}q(A)P.$$

(2) The matrices A and B have the same characteristic polynomial, hence the same eigenvalues.

Proof. We see that successive terms $P^{-1}P$ cancel out in the k-fold product

$$B^k = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$$

to give that

$$B^k = P^{-1}A^kP.$$

It follows easily that

$$a_0I + a_1B + \cdots + a_mB^m = P^{-1}\left(a_0I + a_1A + \cdots + a_mA^m\right)P,$$

which proves (1). For (2), remember that the determinant distributes over products, so that we can pull this clever little trick:

$$\begin{aligned}\det(\lambda I - B) &= \det(\lambda P^{-1}IP - P^{-1}AP)\\ &= \det(P^{-1}(\lambda I - A)P)\\ &= \det(P^{-1})\det(\lambda I - A)\det(P)\\ &= \det(\lambda I - A)\det(P^{-1}P)\\ &= \det(\lambda I - A).\end{aligned}$$

This proves (2). □

Now we can see the significance of the equation $P^{-1}AP = D$, where D is diagonal. It follows from this equation that for any positive integer k, we have $P^{-1}A^kP = D^k$, so multiplying on the left by P and on the right by P^{-1} yields

$$A^k = PD^kP^{-1}. \tag{5.1}$$

As we have seen, the term PD^kP^{-1} is easily computed. This gives us a way of constructing a formula for A^k.

Functions of Matrices

We can also use this identity to extend part (1) to transcendental functions like $\sin x$, $\cos x$, and e^x, which can be defined in terms of an infinite series (a limit of polynomials functions). One can show that for such functions $f(x)$, if $f(D)$ is defined, then $f(A) = Pf(D)P^{-1}$ uniquely defines $f(A)$. In particular, if $D = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, then we have $f(D) = \operatorname{diag}\{f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\}$. Thus we can define $f(A)$ for any matrix A similar to a diagonal matrix provided that $f(x)$ is defined for all scalars x.

Example 5.9. Illustrate the preceding discussion with the matrix in part (a) of Example 5.7 and $f(x) = \sin\left(\frac{\pi}{2}x\right)$.

Solution. The eigenvalues of this problem are $\lambda = 1, 2, 2$. We already found the eigenspace for $\lambda = 2$. Denote the two basis vectors by $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,-2,1)$. For $\lambda = 1$, apply Gauss–Jordan elimination to the matrix

$$A - 1I = \begin{bmatrix} 2-1 & 1 & 2\\ 0 & 1-1 & -2\\ 0 & 0 & 2-1\end{bmatrix} = \begin{bmatrix} 1 & 1 & 2\\ 0 & 0 & -2\\ 0 & 0 & 1\end{bmatrix} \xrightarrow[\substack{E_{23}(2)\\ E_{13}(-2)\\ E_{23}}]{} \begin{bmatrix} 1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix},$$

which gives a general eigenvector of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Hence the eigenspace $\mathcal{E}_1(A)$ has basis $\{(-1,1,0)\}$. Now set $\mathbf{v}_3 = (-1,1,0)$. Form the matrix

$$P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This matrix is nonsingular since $\det P = -1$, and a calculation, which we leave to the reader, shows that

$$P^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The discussion of the first part of this section shows us that P is a similarity transformation matrix that diagonalizes A, that is,

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D.$$

As we have seen, this means that for any positive integer k, we have

$$\begin{aligned} A^k &= PD^kP^{-1} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^k & 2^k - 1 & 2^{k+1} - 2 \\ 0 & 1 & -2^{k+1} + 2 \\ 0 & 0 & 2^k \end{bmatrix}. \end{aligned}$$

This is the formula we were looking for. It's *much* easier than calculating A^k directly!

To compute $\sin\left(\frac{\pi}{2}A\right)$, use the identity $f(A) = Pf(D)P^{-1}$. Thus

$$\sin(\pi A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sin(\pi) & 0 & 0 \\ 0 & \sin(\pi) & 0 \\ 0 & 0 & \sin\left(\frac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, we could evaluate this matrix A at any transcendental function. □

This example showcases some very nice calculations. Given a general matrix A, when can we pull off the same sort of calculation? First, let's give the favorable case a name.

Diagonalizable Matrix

Definition 5.9. The matrix A is *diagonalizable* if it is similar to a diagonal matrix, that is, there is an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. In this case we say that P is a *diagonalizing matrix* for A or that P *diagonalizes* A.

Can we be more specific about when a matrix is diagonalizable? We can. As a first step, notice that the calculations that we began the section with can easily be written in terms of an $n \times n$ matrix instead of a 3×3 matrix. What these calculations prove is the following basic fact.

Diagonalization Theorem

Theorem 5.6. The $n \times n$ matrix A is diagonalizable if and only if there exists a linearly independent set of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of A, in which case $P = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ is a diagonalizing matrix for A.

Can we be more specific about when a linearly independent set of eigenvectors exists? Actually, we can. Clues about what is really going on can be gleaned from a reexamination of Example 5.7.

Example 5.10. Apply the results of the preceding discussion to the matrix in part (b) of Example 5.7 or explain why they fail to apply.

Solution. The eigenvalues of this problem are $\lambda = 1, 2, 2$. We already found the eigenspace for $\lambda = 2$. Denote the single basis vector of $\mathcal{E}_2(A)$ by $\mathbf{v}_1 = (1, 0, 0)$. For $\lambda = 1$, apply Gauss–Jordan elimination to the matrix

$$A - 1I = \begin{bmatrix} 2-1 & 1 & 1 \\ 0 & 1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{E_{32}(-1) \\ E_{21}(-1)}]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives a general eigenvector of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Hence the eigenspace $\mathcal{E}_1(A)$ has basis $\{(-1, 1, 0)\}$. All we could come up with here is two eigenvectors. As a matter of fact, they are linearly independent since one is not a multiple of the other. But they aren't enough and there is no way to find a third eigenvector, since we have found them all! Therefore we have no hope of diagonalizing this matrix according to the diagonalization theorem. The problem is that A is defective, since the algebraic multiplicity of $\lambda = 2$ exceeds the geometric multiplicity of this eigenvalue. □

It would be very handy to have some working criterion for when we can manufacture linearly independent sets of eigenvectors. The next theorem gives us such a criterion.

Theorem 5.7. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a set of eigenvectors of the matrix A such that corresponding eigenvalues are all distinct. Then the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly independent.

Proof. Suppose the set is linearly dependent. Discard redundant vectors until we have a smallest linearly dependent subset such as $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ with $\mathbf{v}_i$ belonging to λ_i. All the vectors have nonzero coefficients in a linear combination that sums to zero, for we could discard the ones that have zero coefficient in the linear combination and still have a linearly dependent set. So there is some linear combination of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0} \tag{5.2}$$

with each $c_j \neq 0$ and $\mathbf{v}_j$ belonging to the eigenvalue λ_j. Multiply (5.2) by λ_1 to obtain the equation

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_1\mathbf{v}_2 + \cdots + c_m\lambda_1\mathbf{v}_m = \mathbf{0}. \tag{5.3}$$

Next multiply (5.2) on the left by A to obtain

$$\mathbf{0} = A(c_1\lambda_1\mathbf{v}_1 + c_2\lambda_1\mathbf{v}_2 + \cdots + c_m\lambda_1\mathbf{v}_m) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_mA\mathbf{v}_m,$$

that is,

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_k\lambda_m\mathbf{v}_m = \mathbf{0}. \tag{5.4}$$

Now subtract (5.4) from (5.3) to obtain

$$0\mathbf{v}_1 + c_2(\lambda_1 - \lambda_2)\mathbf{v}_2 + \cdots + c_k(\lambda_1 - \lambda_m)\mathbf{v}_m = \mathbf{0}.$$

This is a new nontrivial linear combination (since $c_2(\lambda_1 - \lambda_2) \neq 0$) of fewer terms, that contradicts our choice of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. It follows that the original set of vectors must be linearly independent. □

Actually, a little bit more is true: if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is such that for any eigenvalue λ of A, the subset of all these vectors belonging to λ is linearly independent, then the conclusion of the theorem is valid. We leave this as an exercise. Here's an application of the theorem that is useful for many problems.

Corollary 5.2. If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Proof. We can always find one nonzero eigenvector $\mathbf{v}_i$ for each eigenvalue λ_i of A. By the preceding theorem, the set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is linearly independent. Thus A is diagonalizable by the diagonalization theorem. □

Caution: Just because the $n \times n$ matrix A has fewer than n distinct eigenvalues, you may not conclude that it is not diagonalizable.

A simple example is the identity matrix, which is certainly diagonalizable (it's already diagonal!) but has only 1 as an eigenvalue.

5.2 Exercises and Problems

Exercise 1. Are the following matrices diagonalizable?

(a) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

Exercise 2. Use eigensystems to determine whether the following matrices are diagonalizable.

(a) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Exercise 3. Find a matrix P such that $P^{-1}AP$ is diagonal.

(a) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 4. For each matrix A in Exercise 3 use the matrix P to find a formula for A^k, k a positive integer.

Exercise 5. Given a matrix A, let $q(x)$ be the product of linear factors $x - \lambda$, where λ runs over each eigenvalue of A exactly once. For each of the following matrices, confirm or deny the hypothesis that if $p(A) = 0$, then A is diagonalizable.

(a) $\begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 6. Given a matrix A, let $p(x)$ be the characteristic polynomial of A. For each of the matrices of Exercise 5, confirm or deny the hypothesis that if $p(A) = 0$, then A is diagonalizable.

Exercise 7. Show that the matrix $J_\lambda(2) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is not diagonalizable for any scalar λ and calculate the second, third, and fourth powers of the matrix. What is a formula for $J_\lambda(2)^k$, k a positive integer, based on these calculations?

Exercise 8. Show that the matrix $J_\lambda(3) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ is not diagonalizable and calculate the third, fourth, and fifth powers of the matrix. What is a formula for $J_\lambda(3)^k$, $k > 2$, based on these calculations?

Exercise 9. Show that the matrices $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 \\ 0 & -2 \end{bmatrix}$ are similar as follows: find diagonalizing matrices P, Q for A, B, respectively, that yield identical diagonal matrices, set $S = PQ^{-1}$, and confirm that $S^{-1}AS = B$.

Exercise 10. Repeat Exercise 9 for the pair $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & -1 & 2 \end{bmatrix}$.

Exercise 11. Compute $\sin\left(\frac{\pi}{6}A\right)$ and $\cos\left(\frac{\pi}{6}A\right)$, where $A = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$.

Exercise 12. Compute $\exp(A)$ and $\arctan(A)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

*Problem 13. Show that any upper triangular matrix with identical diagonal entries is diagonalizable if and only if it is already diagonal.

Problem 14. Suppose that A is an invertible matrix that is diagonalized by the matrix P, that is, $P^{-1}AP = D$ is a diagonal matrix. Use this information to find a diagonalization for A^{-1}.

*Problem 15. Show that if A has no repeated eigenvalues, then the only matrices B that commute with A (i.e., $AB = BA$) are scalar matrices $B = cI$.

Problem 16. Show that if A is diagonalizable, then so is A^*.

*Problem 17. Prove the Cayley–Hamilton theorem for diagonalizable matrices: show that if $p(x)$ is the characteristic polynomial of the diagonalizable matrix A, then A satisfies its characteristic equation, that is, $p(A) = 0$.

Problem 18. Adapt the proof of Theorem 5.7 to prove that if eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are such that for any eigenvalue λ of A, the subset of all these vectors belonging to λ is linearly independent, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

*Problem 19. The thirteenth-century mathematician Leonardo Fibonacci discovered the sequence of integers $1, 1, 2, 3, 5, 8, \ldots$ called the *Fibonacci sequence.* These numbers have a way of turning up in many applications. They can be specified by the formulas

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_{k+2} &= f_{k+1} + f_k, \qquad k = 0, 1, \ldots. \end{aligned}$$

(a) Let $\mathbf{x}^{(k)} = (f_{k+1}, f_k)$ and show that these equations are equivalent to the matrix equations $\mathbf{x}^{(0)} = (1, 1)$ and $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$, $n = 0, 1, \ldots$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) Use part (a) and the diagonalization theorem to find an explicit formula for the kth Fibonacci number.

Problem 20. Suppose that the kill rate r of Example 5.8 is viewed as a variable positive parameter. There is a value of the number r for which the eigenvalues of the corresponding matrix are equal.

(a) Find this value of r and the corresponding eigenvalues by examining the characteristic polynomial of the matrix.

(b) Use the available MAS (or CAS) to determine experimentally the long-term behavior of populations for the value of r found in (a). Your choices of initial states should include $[100, 1000]$.

*Problem 21. Let A and B be matrices of the same size and suppose that A has no repeated eigenvalues. Show that $AB = BA$ if and only if A and B are simultaneously diagonalizable, that is, a single matrix P diagonalizes both A and B.

5.3 Applications to Discrete Dynamical Systems

Now we have enough machinery to come to a fairly complete understanding of the discrete dynamical system

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}.$$

Diagonalizable Transition Matrix

Let us first examine the case that A is diagonalizable. So we assume that the $n \times n$ matrix A is diagonalizable and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a complete linearly independent set of eigenvectors of A belonging to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A. Let us further suppose that these eigenvalues are ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

The eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$ or $\mathbb{C}^n$, whichever is appropriate. In particular, we may write $\mathbf{x}^{(0)}$ as a linear combination of these vectors by solving the system $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]\,\mathbf{c} = \mathbf{x}^{(0)}$ to obtain the coefficients $c_1, c_2, \ldots, c_n$ of the equation

$$\mathbf{x}^{(0)} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n. \tag{5.5}$$

Now we can see what the effect of multiplication by A is:

$$\begin{aligned} A\mathbf{x}^{(0)} &= A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \\ &= c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_n(A\mathbf{v}_n) \\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_n\lambda_n\mathbf{v}_n. \end{aligned}$$

Now apply A on the left repeatedly. Since $\mathbf{x}^{(k)} = A^k\mathbf{x}^{(0)}$, we see that

$$\mathbf{x}^{(k)} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n. \tag{5.6}$$

Equation (5.6) is the key to understanding how the state vector changes in a discrete dynamical system. Now we can see clearly that it is the size of the eigenvalues that governs the growth of successive states. Because of this fact, a handy quantity that can be associated with a matrix A (whether it is diagonalizable or not) is the following.

Spectral Radius

Definition 5.10. The *spectral radius* $\rho(A)$ of a matrix A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is defined to be the number

$$\rho(A) = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}.$$

If $\lambda_k = \rho(A)$ and λ_k is the only eigenvalue with this property, then λ_k is the *dominant eigenvalue* of A.

Thus $\rho(A)$ is the largest absolute value of the eigenvalues of A. We summarize a few of the conclusions about a matrix that can be drawn from the spectral radius.

Theorem 5.8. Let the transition matrix for a discrete dynamical system be the $n \times n$ diagonalizable matrix A as described above. Let $\mathbf{x}^{(0)}$ be an initial state vector given as in equation (5.5). Then the following are true:

(1) If $\rho(A) < 1$, then $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{0}$.
(2) If $\rho(A) = 1$, then the sequence of norms $\left\{\left\|\mathbf{x}^{(k)}\right\|\right\}_{k=0}^{\infty}$ is bounded.
(3) If $\rho(A) = 1$ and the only eigenvalues λ of A with $|\lambda| = 1$ are $\lambda = 1$, then $\lim_{k\to\infty} \mathbf{x}^{(k)}$ is an element of $\mathcal{E}_1(A)$, hence either an eigenvector or $\mathbf{0}$.
(4) If $\rho(A) > 1$, then for some choices of $\mathbf{x}^{(0)}$ we have $\lim_{k\to\infty} \|\mathbf{x}\| = \infty$.

Proof. Suppose that $\rho(A) < 1$. Then for all i, $\lambda_i^k \to 0$ as $k \to \infty$, so we see from equation (5.6) that $\mathbf{x}^{(k)} \to 0$ as $k \to \infty$, which is what (1) says. Next suppose that $\rho(A) = 1$. Then take norms of equation (5.6) to obtain that, since each $|\lambda_i| \leq 1$,

$$\begin{aligned}\left\|\mathbf{x}^{(k)}\right\| &= \left\|c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n\right\| \\ &\leq |\lambda_1|^k \|c_1\mathbf{v}_1\| + |\lambda_2|^k \|c_2\mathbf{v}_2\| + \cdots + |\lambda_n|^k \|c_n\mathbf{v}_n\| \\ &\leq \|c_1\mathbf{v}_1\| + \|c_2\mathbf{v}_2\| + \cdots + \|c_n\mathbf{v}_n\|.\end{aligned}$$

Therefore the sequence of norms $\left\|\mathbf{x}^{(k)}\right\|$ is bounded by a constant that depends only on $\left\|\mathbf{x}^{(0)}\right\|$, which proves (2). The proof of (3) follows from inspection of (5.6): observe that the eigenvalue powers λ_j^k are equal to 1 if $\lambda = 1$, and otherwise the powers tend to zero, since all other eigenvalues are less than 1 in absolute value. Hence if any coefficient c_j of an eigenvector $\mathbf{v}_j$ corresponding to 1 is not zero, the limiting vector is an eigenvector corresponding to $\lambda = 1$. Otherwise, the coefficients all tend to 0 and the limiting vector is $\mathbf{0}$. Finally, if $\rho(A) > 1$, then for $\mathbf{x}^{(0)} = c_1\mathbf{v}_1$, we have that $\mathbf{x}^{(k)} = c_1\lambda_1^k\mathbf{v}_1$. However, $|\lambda_1| > 1$, so that $\left|\lambda_1^k\right| \to \infty$, as $k \to \infty$, from which the desired conclusion for (4) follows. □

We should note that the cases of the preceding theorem are not quite exhaustive. One possibility that is not covered is the case that $\rho(A) = 1$ and A has other eigenvalues of absolute value 1. In this case the sequence of vectors $\mathbf{x}^{(k)}$ is bounded in norm, i.e., $\left\|\mathbf{x}^{(k)}\right\| \le K$ for some constant K and indices $k = 0, 1, \ldots$, but need not converge to anything. An example of this phenomenon is given in Example 5.13.

Example 5.11. Apply the preceding theory to the population of Example 5.8.

Solution. We saw in this example that the transition matrix is

$$A = \begin{bmatrix} 0.6 & 0.4 \\ -0.35 & 1.4 \end{bmatrix}.$$

The characteristic equation of this matrix is

$$\begin{aligned} \det \begin{bmatrix} 0.6-\lambda & 0.4 \\ -0.35 & 1.4-\lambda \end{bmatrix} &= (0.6-\lambda)(1.4-\lambda) + 0.35 \cdot 0.4 \\ &= \lambda^2 - 2\lambda + 0.84 + 0.14 \\ &= \lambda^2 - 2\lambda + 0.98, \end{aligned}$$

whence we see that the eigenvalues of A are

$$\lambda = 1.0 \pm \sqrt{4 - 3.92}/2 \approx 1.1414,\ 0.85858.$$

A calculation that we leave to the reader also shows that the eigenvectors of A corresponding to these eigenvalues are approximately $\mathbf{v}_1 = (1.684, 2.2794)$ and $\mathbf{v}_2 = (.8398, .54289)$, respectively. Since $\rho(A) \approx 1.1414 > 1$, it follows from (1) of Theorem 5.8 that for every initial state except a multiple of $\mathbf{v}_2$, the limiting state will grow without bound. Now if we imagine an initial state to be a random choice of values for the coefficients c_1 and c_2, we see that the probability of selecting $c_1 = 0$ is for all practical purposes 0. Therefore, with probability 1, we will make a selection with $c_1 \ne 0$, from which it follows that the subsequent states will tend to arbitrarily large multiples of the vector $\mathbf{v}_1 = (1.684, 2.2794)$.

Finally, we can offer some advice to the scientists who are thinking of introducing a predator bird to control the frog population of this example: don't do it! Almost any initial distribution of birds and frogs will result in a population of birds and frogs that grows without bound. Therefore, we will be stuck with both nonindigenous frogs *and* birds. To drive the point home, start with a population of 10,000 frogs and 100 birds. In 20 years we will have a population state of

$$\begin{bmatrix} 0.6 & 0.4 \\ -0.35 & 1.4 \end{bmatrix}^{20} \begin{bmatrix} 100 \\ 10{,}000 \end{bmatrix} \approx \begin{bmatrix} 197{,}320 \\ 267{,}550 \end{bmatrix}.$$

In view of our eigensystem analysis, we know that these numbers are no fluke. Almost any initial population will grow similarly. The conclusion is that we should try another strategy or leave well enough alone in this ecology. □

Example 5.12. Apply the preceding theory to the Markov chain Example 2.18.

Solution. Recall that this example led to a Markov chain whose transition matrix is given by

$$A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$$

Conveniently, we have already computed the eigenvalues and vectors of $10A$ in Example 5.2. There we found eigenvalues $\lambda = 10,\ 3$, with corresponding eigenvectors $\mathbf{v}_2 = (1, -1)$ and $\mathbf{v}_1 = (4/3, 1)$, respectively. It follows from Example 5.2 that the eigenvalues of A are $\lambda = 1,\ 0.3$, with the same eigenvectors. Therefore 1 is the dominant eigenvalue. Any initial state will necessarily involve $\mathbf{v}_1$ nontrivially, since multiples of $\mathbf{v}_2$ are not probability distribution vectors (the entries are of opposite signs). Thus we may apply part 3 of Theorem 5.8 to conclude that for any initial state, the only possible nonzero limiting state vector is some multiple of $\mathbf{v}_1$. Which multiple? Since the sum of the entries of each state vector $\mathbf{x}^{(k)}$ sum to 1, the same must be true of the initial vector. Since

$$\mathbf{x}^{(0)} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1\,(4/3) + c_2 1 \\ c_1 1 + c_2\,(-1) \end{bmatrix},$$

we see that

$$1 = c_1\,(4/3) + c_2 1 + c_1 1 + c_2\,(-1) = c_1(7/3),$$

so that $c_1 = 3/7$. Now use the facts that $\lambda_1 = 1$, $\lambda_2 = 0.3$, and equation (5.6) with $n = 2$ to see that the limiting state vector is

$$\lim_{k\to\infty} c_1 1^k \mathbf{v}_1 + c_2\,(4/3)^k\,\mathbf{v}_2 = c_1\mathbf{v}_1 = \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix} \approx \begin{bmatrix} .57143 \\ .42857 \end{bmatrix}.$$

Compare this vector with the result obtained by direct calculation in Example 2.19. □

When do complex eigenvalues occur and what do they mean? In general, all we can say is that the characteristic polynomial of a matrix, even if it is real, may have complex roots. This is an unavoidable fact, but it can be instructive. To see how this is so, consider the following example.

Example 5.13. Suppose that a discrete dynamical system has transition matrix $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, where a is a positive real number. What can be said about the states $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots$, if the initial state $\mathbf{x}^{(0)}$ is an arbitrary nonzero vector?

Solution. The eigenvalues of A are $\pm a\mathrm{i}$. Now if $a < 1$, then according to part 1 of Theorem 5.8 the limiting state is $\mathbf{0}$. Part 3 of that theorem cannot occur for our matrix A since 1 cannot be an eigenvalue. So suppose $a \geq 1$. Since the eigenvalues of A are distinct, there is an invertible matrix P such that

$$P^{-1}AP = D = \begin{bmatrix} a\mathrm{i} & 0 \\ 0 & -a\mathrm{i} \end{bmatrix}.$$

So we see from equation (5.1) that

$$A^k = PD^kP^{-1} = P \begin{bmatrix} (a\mathrm{i})^k & 0 \\ 0 & (-a\mathrm{i})^k \end{bmatrix} P^{-1}.$$

The columns of P are eigenvectors of A, hence complex. We may take real parts of the matrix D^k to get a better idea of what the powers of A do. Now $\mathrm{i} = e^{i\frac{\pi}{2}}$, so we may use de Moivre's formula to get

$$\Re((a\mathrm{i})^k) = a^k \cos(k\frac{\pi}{2}) = (-1)^{k/2}a^k \quad \text{if } k \text{ is even.}$$

We know that $\mathbf{x}^{(k)} = A^k\mathbf{x}^{(0)}$. In view of the above equation, we see that the states $\mathbf{x}^{(k)}$ will oscillate around the origin. In the case that $a = 1$ we expect the states to remain bounded, but if $a > 1$, we expect the values to become unbounded and oscillate in sign. This oscillation is fairly typical of what happens when complex eigenvalues are present, though it need not be as rapid as in this example. □

Nondiagonalizable Transition Matrix

How can a matrix be nondiagonalizable? All the examples we have considered so far suggest that nondiagonalizability is the same as being defective. Put another way, diagonalizable equals nondefective. This is exactly right, as the following shows.

Theorem 5.9. The matrix A is diagonalizable if and only if the geometric multiplicity of every eigenvalue equals its algebraic multiplicity.

Proof. Suppose that the $n \times n$ matrix A is diagonalizable. According to the diagonalization theorem, there exists a complete linearly independent set of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of the matrix A. The number of these vectors belonging to a given eigenvalue λ of A is a number $d(\lambda)$ at most the geometric multiplicity of λ, since they form a basis of the eigenspace $\mathcal{E}_\lambda(A)$. Hence their number is at most the algebraic multiplicity $m(\lambda)$ of λ by Theorem 5.3. Since the sum of all the numbers $d\,(\lambda)$ is n, as is the sum of all the algebraic multiplicities $m(\lambda)$, it follows that the sum of the geometric multiplicities must also be n. The only way for this to happen is that for each eigenvalue λ, we have that geometric multiplicity equals algebraic multiplicity. Thus, A is nondefective.

Conversely, if geometric multiplicity equals algebraic multiplicity, we can produce $m(\lambda)$ linearly independent eigenvectors belonging to each eigenvalue λ. Assemble all of these vectors and we have n eigenvectors such that for any eigenvalue λ of A, the subset of all these vectors belonging to λ is linearly independent. Therefore, the entire set of eigenvectors is linearly independent by the remark following Theorem 5.7. Now apply the diagonalization theorem to obtain that A is diagonalizable. □

The last item of business in our examination of diagonalization is to prove part 2 of Theorem 5.3, which asserts*: for each eigenvalue μ of A, if $m(\mu)$ is the algebraic multiplicity of μ, then*

$$1 \leq \dim \mathcal{E}_\mu(A) \leq m(\mu).$$

To see why this is true, suppose the eigenvalue μ has geometric multiplicity k and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a basis for the eigenspace $\mathcal{E}_\mu(A)$. We know from the Steinitz substitution theorem that this set can be expanded to a basis of the vector space $\mathbb{R}^n$ (or $\mathbb{C}^n$), say

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n.$$

Form the nonsingular matrix

$$S = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n].$$

Let

$$B = [S^{-1}A\mathbf{v}_{k+1}, S^{-1}A\mathbf{v}_{k+2}, \ldots, S^{-1}A\mathbf{v}_n] = \begin{bmatrix} F \\ G \end{bmatrix},$$

where F consists of the first k rows of B and G the remaining rows. Thus we obtain that

$$\begin{aligned} AS &= [A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n] \\ &= [\mu\mathbf{v}_1, \mu\mathbf{v}_2, \ldots, \mu\mathbf{v}_k, A\mathbf{v}_{k+1}, \ldots, A\mathbf{v}_n] \\ &= S \begin{bmatrix} \mu I_k & F \\ 0 & G \end{bmatrix}. \end{aligned}$$

Now multiply both sides on the left by S^{-1}, and we have

$$C = S^{-1}AS = \begin{bmatrix} \mu I_k & F \\ 0 & G \end{bmatrix}.$$

We see that the block upper triangular matrix C is similar to A. By part 2 of Theorem 5.5 we see that A and C have the same characteristic polynomial. However, the characteristic polynomial of C is

$$\begin{aligned} p(\lambda) &= \det\left(\lambda I_n - \begin{bmatrix} \mu I_k & F \\ 0 & G \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} (\lambda-\mu)I_k & F \\ 0 & G-\lambda I_{n-k} \end{bmatrix}\right) \\ &= \det(\lambda-\mu)I_k \cdot \det\left(G-\lambda I_{n-k}\right) \\ &= (\lambda-\mu)^k \det\left(G-\lambda I_{n-k}\right). \end{aligned}$$

The product term above results from Exercise 23 of Section 2.6. It follows that the algebraic multiplicity of μ as a root of $p(\lambda)$ is at least as large as k, which is what we wanted to prove.

Our newfound insight into nondiagonalizable matrices is somewhat of a negative nature: they are defective. Unfortunately, this isn't much help in determining the behavior of discrete dynamical systems with a nondiagonalizable transition matrix. If matrices are not diagonalizable, what simple kind of matrix *are* they reducible to? There is a very nice answer to this question; this answer requires the notion of a Jordan block, which can be defined as a $d \times d$ matrix of the form

$$J_d(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix},$$

where the entries off the main diagonal and first superdiagonal are understood to be zeros. This matrix is very close to being a diagonal matrix. Its true value comes from the following classical theorem, the proof of which is somewhat beyond the scope of this text. We refer the reader to the textbooks [12] and [11] of the bibliography for a proof. These texts are excellent references for higher-level linear algebra and matrix theory.

Jordan Canonical Form Theorem

Theorem 5.10. Every matrix A is similar to a block diagonal matrix that consists of Jordan blocks down the diagonal. Moreover, these blocks are uniquely determined by A up to order.

In particular, if $J = S^{-1}AS$, where J consists of Jordan blocks down the diagonal, we call J "the" Jordan canonical form of the matrix A, which suggests there is only one. This is a slight abuse of language, since the order of occurrence of the Jordan blocks of J could vary. To fix ideas, let's consider an example.

Example 5.14. Find all possible Jordan canonical forms for a 3×3 matrix A whose eigenvalues are $-2, 3, 3$.

Solution. Notice that each Jordan block $J_d(\lambda)$ contributes d eigenvalues λ to the matrix. Therefore, there can be only one 1×1 Jordan block for the eigenvalue -2 and either two 1×1 Jordan blocks for the eigenvalue 3 or one 2×2 block for the eigenvalue 3. Thus, the possible Jordan canonical forms for A (up to order of blocks) are

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}. \qquad \square$$

Notice that if all Jordan blocks are 1×1, then the Jordan canonical form of a matrix is simply a diagonal matrix. Thus, another way to say that a matrix is diagonalizable is to say that its Jordan blocks are 1×1. In reference to the previous example, we see that if the matrix has the first Jordan canonical form, then it is diagonalizable, while if it has the second, it is nondiagonalizable.

Now suppose that the matrix A is a transition matrix for a discrete dynamical system and A is not diagonalizable. What can one say? For one thing, the Jordan canonical form can be used to recover part 1 of Theorem 5.8. Part 4 remains valid as well; the proof we gave does not depend on A being diagonalizable. Unfortunately, things are a bit more complicated as regards parts (2) and (3). In fact, they fail to be true, as the following example shows.

Example 5.15. Let $A = J_2(1)$. Show how parts (2) and (3) of Theorem 5.8 fail to be true for this matrix.

Solution. We check that

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},$$

and in general,

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Now take $\mathbf{x}^{(0)} = (0, 1)$, and we see that

$$\mathbf{x}^{(k)} = A^k \mathbf{x}^{(0)} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}.$$

It follows that the norms $\left\| \mathbf{x}^{(k)} \right\| = \sqrt{k^2 + 1}$ are not a bounded sequence, so that part 2 of the theorem fails to be true. Also, the sequence of vectors $\mathbf{x}^{(k)}$ does not converge to any vector in spite of the fact that 1 is the largest eigenvalue of A. Thus (3) fails as well. □

In spite of this example, the news is not all negative. It can be shown by way of the Jordan canonical form that a restricted version of (3) holds. This kind of result is known as an *ergodic theorem.* Recall that stochastic matrices for which this theorem holds are called *ergodic.* Ergodic Matrix

Ergodic Theorem

Theorem 5.11. If A is the transition matrix for a discrete dynamical system and 1 is the dominant eigenvalue of A, then there is a vector $\mathbf{x}_*$ such that $\lim_{k\to\infty} \mathbf{x}^{(k)} = c\mathbf{x}_*$, where the scalar c is uniquely determined by $\mathbf{x}^{(0)}$.

5.3 Exercises and Problems

Exercise 1. Find the spectral radius of each of the following matrices and determine whether there is a dominant eigenvalue.

(a) $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 4 & -1 \\ -2 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ (d) $\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & -2 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$

Exercise 2. Find the spectral radius and dominant eigenvalue, if any.

(a) $\begin{bmatrix} -7 & -6 \\ 9 & 8 \end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Exercise 3. For initial state $\mathbf{x}^{(0)}$and transition matrix A below find an eigensystem of A and use this to produce a formula for the kth state $\mathbf{x}^{(k)}$ in the form of equation (5.6).

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix}$

Exercise 4. Repeat Exercise 3 for these pairs $\mathbf{x}^{(0)}$, A.

(a) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Exercise 5. If the matrices of Exercise 1 are transition matrices, for which do all $\mathbf{x}^{(k)}$ approach $\mathbf{0}$ as $k \to \infty$? Does the ergodic theorem apply to any of these?

Exercise 6. If the matrices of Exercise 2 are transition matrices, for which do all $\mathbf{x}^{(k)}$ remain bounded as $k \to \infty$? Are any of these matrices ergodic?

Exercise 7. You are given that a 5×5 matrix has eigenvalues $2, 2, 3, 3, 3$. What are the possible Jordan canonical forms for this matrix?

Exercise 8. What are the possible Jordan canonical forms for a 6×6 matrix with eigenvalues $-1, -1, -1, 4, 4, 4$?

Exercise 9. Let $A = J_3(2)$, a Jordan block. Show that the *Cayley–Hamilton theorem* is valid for A, that is, $p(A) = 0$, where $p(x)$ is the characteristic polynomial of A.

Exercise 10. Let $A = \begin{bmatrix} J_2(1) & 0 \\ 0 & J_2(1) \end{bmatrix}$. Verify that $p(A) = 0$, where $p(x)$ is the characteristic polynomial of A, and find a polynomial $q(x)$ of degree less than 4 such that $q(A) = 0$.

Exercise 11. The three-state insect model of Example 2.20 yields a transition matrix

$$A = \begin{bmatrix} 0.2 & 0 & 0.25 \\ 0.6 & 0.3 & 0 \\ 0 & 0.6 & 0.8 \end{bmatrix}.$$

Use a CAS or MAS to calculate the eigenvalues of this matrix. Deduce that A is diagonalizable and determine the approximate growth rate from one state to the next, given a random initial vector.

Exercise 12. The financial model of Example 2.24 gives rise to a discrete dynamical system $x^{(k+1)} = Ax^{(k)}$, where the transition matrix is

$$A = \begin{bmatrix} 1 & 0.06 & 0.12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Use a CAS or MAS to calculate the eigenvalues of this matrix. Deduce that A is diagonalizable and determine the approximate growth rate from one state to the next, given a random initial vector. Compare the growth rate with a flat interest rate.

Exercise 13. A (two) age structured population model results in a transition matrix $A = \begin{bmatrix} 0 & f_2 \\ s_1 & 0 \end{bmatrix}$ with positive per-capita reproductive rate f_2 and survival rate s_1. There exists a positive eigenpair $(\lambda, \mathbf{p})$ for A. Assume this and use the equation $A\mathbf{p} = \lambda\mathbf{p}$ to express $\mathbf{p} = (p_1, p_2)$ in terms of p_1, and to find a polynomial equation in terms of birth and survival rates that λ satisfies.

Exercise 14. Repeat Exercise 13 for the (three) age structured model with transition matrix

$$A = \begin{bmatrix} 0 & f_2 & f_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}$$

where f_2, f_3, s_1, s_2 are all positive.

*Problem 15. Let A be a 2×2 transition matrix of a Markov chain where A is not the identity matrix.

(a) Show that A can be written in the form $A = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$ for suitable real numbers $0 \leq a, b \leq 1$.

(b) Show that (b, a) and $(1, -1)$ are eigenvectors for A.

(c) Find a formula for the kth state $\mathbf{x}^{(k)}$ in the form of equation (5.6).

Problem 16. Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ be a transition matrix for a discrete dynamical system. Show that A is not ergodic for any choice of $a, b \in \mathbb{R}$.

Problem 17. Part (3) of Theorem 5.8 suggests that two possible limiting values are possible. Use your CAS or MAS to carry out this experiment: Compute a random 2×1 vector and normalize it by dividing by its length. Let the resulting initial vector be $\mathbf{x}^{(0)} = (x_1, x_2)$ and compute the state vector $\mathbf{x}^{(20)}$ using the transition matrix A of Example 5.12. Do this for a large number of times (say 500) and keep count of the number of times $\mathbf{x}^{(20)}$ is close to $\mathbf{0}$, say $\|\mathbf{x}^{(20)}\| < 0.1$. Conclusions?

Problem 18. Use a CAS or MAS to construct a 3×10 table whose jth column is $A^j\mathbf{x}$, where $\mathbf{x} = (1, 1, 1)$ and

$$A = \begin{bmatrix} 10 & 17 & 8 \\ -8 & -13 & -6 \\ 4 & 7 & 4 \end{bmatrix}.$$

What can you deduce about the eigenvalues of A based on inspection of this table? Give reasons. Check your claims by finding the eigenvalues of A.

Problem 19. A species of bird can be divided into three age groups: age less than 2 years for group 1, age between 2 and 4 years for group 2, and age between 4 and 6 years for the third group. Assume that these birds have at most a 6-year life span. It is estimated that the survival rates for birds in groups 1 and 2 are 50% and 75%, respectively. Also, birds in groups 1, 2, and 3 produce 0, 1, and 3 offspring on average in any biennium (period of 2 years). Model this bird population as a discrete dynamical system and analyze the long-term change in the population. If the survival rates are unknown, but the population is known to be stable, assume that survival rates for groups 2 and 3 are equal and estimate this number.

5.4 Orthogonal Diagonalization

Unitarily and Orthogonally Diagonalizable Matrices

We are going to explore some very remarkable facts about Hermitian and real symmetric matrices. These matrices are diagonalizable, and moreover, diagonalization can be accomplished by a unitary (orthogonal if A is real) matrix. This means that $P^{-1}AP = P^*AP$ is diagonal. In this situation we say that the matrix A is *unitarily (orthogonally) diagonalizable.* Orthogonal and unitary matrices are particularly attractive since the calculation is essentially free and error-free as well: $P^{-1} = P^*$.

Eigenvalue of Hermitian Matrices

As a first step, we need to observe a curious property of Hermitian matrices. It turns out that their eigenvalues are guaranteed to be real, even if the matrix itself is complex. This is one reason that one might prefer to work with these matrices.

Theorem 5.12. If A is a Hermitian matrix, then the eigenvalues of A are real.

Proof. Let λ be an eigenvalue of A with corresponding nonzero eigenvector $\mathbf{x}$, so that $A\mathbf{x} = \lambda\mathbf{x}$. Form the scalar $c = \mathbf{x}^* A\mathbf{x}$. We have that

$$\overline{c} = c^* = (\mathbf{x}^* A\mathbf{x})^* = \mathbf{x}^* A^* (\mathbf{x}^*)^* = \mathbf{x}^* A\mathbf{x} = c.$$

It follows that c is a real number. However, we also have that

$$c = \mathbf{x}^* \lambda\mathbf{x} = \lambda\mathbf{x}^*\mathbf{x} = \lambda \left\|\mathbf{x}\right\|^2$$

so that $\lambda = c/\left\|\mathbf{x}\right\|^2$ is also real. □

Example 5.16. Show that Theorem 5.12 is applicable if $A = \begin{bmatrix} 1 & 1-\mathrm{i} \\ 1+\mathrm{i} & 0 \end{bmatrix}$ and verify the conclusion of the theorem.

Solution. First notice that

$$A^* = \begin{bmatrix} 1 & 1-\mathrm{i} \\ 1+\mathrm{i} & 0 \end{bmatrix}^* = \begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 1-\mathrm{i} \\ 1+\mathrm{i} & 0 \end{bmatrix} = A.$$

It follows that A is Hermitian and the preceding theorem is applicable. Now we compute the eigenvalues of A by solving the characteristic equation

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1-\mathrm{i} \\ 1+\mathrm{i} & -\lambda \end{bmatrix} \\ &= (1-\lambda)(-\lambda) - (1+\mathrm{i})(1-\mathrm{i}) \\ &= \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2). \end{aligned}$$

Hence the eigenvalues of A are $\lambda = -1, 2$, which are real. □

Caution: Although the *eigenvalues* of a Hermitian matrix are guaranteed to be real, the *eigenvectors* may not be real unless the matrix in question is real.

The Principal Axes Theorem

A key fact about Hermitian matrices is the so-called *principal axes theorem*; its proof is a simple consequence of the Schur triangularization theorem which is proved in Section 5.5. We will content ourselves here with stating the theorem and supplying a proof for the case that the eigenvalues of A are distinct. This proof also shows us one way to carry out the diagonalization process.

Principal Axes Theorem

Theorem 5.13. Every Hermitian matrix is unitarily diagonalizable, and every real symmetric matrix is orthogonally diagonalizable.

Proof. Let us assume that the eigenvalues of the $n \times n$ matrix A are distinct. We saw in Theorem 5.12 that the eigenvalues of A are real. Let these eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Now find an eigenvector $\mathbf{v}_k$ for each eigenvalue λ_k. We can assume that each $\mathbf{v}_k$ is of unit length by replacing it by the vector divided by its length if necessary. We now have a diagonalizing matrix, as prescribed by Theorem 5.6 (the diagonalization theorem), namely the matrix $P = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$.

Recalling that $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$, $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, and that $A^* = A$, we see that

$$\lambda_k \mathbf{v}_j^* \mathbf{v}_k = \mathbf{v}_j^* \lambda_k \mathbf{v}_k = \mathbf{v}_j^* A \mathbf{v}_k = (A\mathbf{v}_j)^* \mathbf{v}_k = (\lambda_j \mathbf{v}_j)^* \mathbf{v}_k = \lambda_j \mathbf{v}_j^* \mathbf{v}_k.$$

Now bring both terms to one side of the equation and factor out the term $\mathbf{v}_j^* \mathbf{v}_k$ to obtain

$$(\lambda_k - \lambda_j)\mathbf{v}_j^* \mathbf{v}_k = 0.$$

Thus if $\lambda_k \neq \lambda_j$, it follows that $\mathbf{v}_j \cdot \mathbf{v}_k = \mathbf{v}_j^* \mathbf{v}_k = 0$. In other words the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form an orthonormal set. Therefore, the matrix P is unitary. If A is real, then so are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and P is orthogonal in this case. □

The proof we have just given suggests a practical procedure for diagonalizing a Hermitian or real symmetric matrix. The only additional information that we need for the complete procedure is advice on what to do if the eigenvalue λ is repeated. This is a sticky point. What we need to do in this case is find an orthogonal basis of the eigenspace $\mathcal{E}_\lambda(A) = \mathcal{N}(A - \lambda I)$. It is always possible to find such a basis using the so-called Gram–Schmidt algorithm, which is discussed in Chapter 6. For the hand calculations that we do in this chapter, the worst situation that we will encounter is that the eigenspace $\mathcal{E}_\lambda$ is two-dimensional, say with a basis $\mathbf{v}_1, \mathbf{v}_2$. In this case replace $\mathbf{v}_2$ by $\mathbf{v}_2^* = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{v}_2$. We know that $\mathbf{v}_2^*$ is orthogonal to $\mathbf{v}_1$ (see Theorem 6.4), so that $\mathbf{v}_1, \mathbf{v}_2^*$ is an orthogonal basis of $\mathcal{E}_\lambda(A)$. We illustrate the procedure with a few examples.

Example 5.17. Find an eigensystem for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and use this to orthogonally diagonalize A.

Solution. Notice that A is real symmetric, so diagonalizable by the principal axes theorem. First calculate the characteristic polynomial of A as

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} \\ &= \left((1-\lambda)(4-\lambda) - 2 \cdot 2\right)(5-\lambda) \\ &= -\lambda(\lambda - 5)^2, \end{aligned}$$

so that the eigenvalues of A are $\lambda = 0, 5, 5$.

Next find eigenspaces for each eigenvalue. For $\lambda = 0$, we find the null space by row reduction,

$$A - 0I = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[E_{21}(-2)]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[\substack{E_{23} \\ E_2\left(\frac{1}{5}\right)}]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the null space is spanned by the vector $(-2, 1, 0)$. Normalize this vector to obtain $\mathbf{v}_1 = (-2, 1, 0)/\sqrt{5}$. Next compute the eigenspace for $\lambda = 5$ via row reductions,

$$A - 5I = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{21}(1/2)]{} \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_1(-1/4)]{} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives two eigenvectors, $(1/2, 1, 0)$ and $(0, 0, 1)$. Normalize these to get $\mathbf{v}_2 = (1, 2, 0)/\sqrt{5}$ and $\mathbf{v}_3 = (0, 0, 1)$. In this case $\mathbf{v}_2$ and $\mathbf{v}_3$ are already orthogonal, so the diagonalizing matrix can be written as

$$P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}.$$

We leave it to the reader to check that $P^T AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. □

Example 5.18. Let $A = \begin{bmatrix} 1 & 1-\mathrm{i} \\ 1+\mathrm{i} & 0 \end{bmatrix}$ as in Example 5.16. Unitarily diagonalize this matrix.

Solution. In Example 5.16 we computed the eigenvalues to be $\lambda = -1, 2$. Next find eigenspaces for each eigenvalue. For $\lambda = -1$, we find the null space by row reduction,

$$A + I = \begin{bmatrix} 2 & 1-\mathrm{i} \\ 1+\mathrm{i} & 1 \end{bmatrix} \xrightarrow[E_{21}(-(1+\mathrm{i})/2)]{} \begin{bmatrix} 2 & 1-\mathrm{i} \\ 0 & 0 \end{bmatrix} \xrightarrow[E_1(1/2)]{} \begin{bmatrix} 1 & (1-\mathrm{i})/2 \\ 0 & 0 \end{bmatrix},$$

so that the null space is spanned by the vector $((-1+\mathrm{i})/2, 1)$. A similar calculation shows that a basis of eigenvectors for $\lambda = 2$ consists of the vector $((-1-\mathrm{i})/2, 1)$. Normalize these vectors to obtain $\mathbf{u}_1 = ((-1+\mathrm{i})/2, 1)/\sqrt{3/2}$ and $\mathbf{u}_2 = (-1, (-1-\mathrm{i})/2)/\sqrt{3/2}$. So set

$$U = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1+\mathrm{i}}{2} & -1 \\ 1 & \frac{-1-\mathrm{i}}{2} \end{bmatrix}$$

and obtain that (the reader should check this)

$$U^{-1}AU = U^*AU = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}. \qquad \square$$

5.4 Exercises and Problems

Exercise 1. Show that the following matrices are real symmetric and find orthogonal matrices that diagonalize these matrices.

(a) $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 36 \\ 36 & 23 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Exercise 2. Show that the following matrices are Hermitian and find unitary matrices that diagonalize these matrices.

(a) $\begin{bmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1+\mathrm{i} & 0 \\ 1-\mathrm{i} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Exercise 3. Show that these matrices are orthogonal and compute their eigenvalues. Determine whether it is possible to orthogonally or unitarily diagonalize these matrices. (*Hint:* look for orthogonal sets of eigenvectors.)

(a) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 4. Show that these matrices are unitary and compute their eigenvalues. Unitarily diagonalize these matrices.

(a) $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & \mathrm{i} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \end{bmatrix}$ (c) $\frac{1}{5\sqrt{2}}\begin{bmatrix} 5 & -3+4\mathrm{i} \\ 3+4\mathrm{i} & 5 \end{bmatrix}$

Exercise 5. A square matrix A is called *normal* if $AA^* = A^*A$. Which of the matrices in Exercises 3 and 1 are normal?

Exercise 6. Which of the matrices in Exercise 4 are normal or Hermitian?

Exercise 7. Use orthogonal diagonalization to find a formula for the kth power of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Exercise 8. Use unitary diagonalization to find a formula for the kth power of $A = \begin{bmatrix} 3 & \mathrm{i} \\ -\mathrm{i} & 3 \end{bmatrix}$.

Exercise 9. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. The eigenvalues of A are 1, 2, and 4. Find an orthogonal matrix P that diagonalizes A to $D = \operatorname{diag}\{1, 2, 4\}$, calculate $B = P \operatorname{diag}\left\{1, \sqrt{2}, 4\right\} P^T$, and show that B is a symmetric positive definite square root of A, that is, $B^2 = A$ and B is symmetric positive definite.

Exercise 10. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -\mathrm{i} \\ 0 & \mathrm{i} & 1 \end{bmatrix}$. The eigenvalues of A are 0, 1, and 3. Find a unitary matrix P that diagonalizes A to $D = \operatorname{diag}\{0, 1, 3\}$ and confirm that $B = P \operatorname{diag}\left\{0, 1, \sqrt{3}\right\} P^*$ is a Hermitian square root of A.

Problem 11. Show that if A is orthogonally diagonalizable, then so is A^T.

*Problem 12. Let B be a Hermitian matrix. Show that the eigenvalues of B are positive if and only if B is a positive definite matrix.

Problem 13. Show that if the real matrix A is orthogonally diagonalizable, then A is symmetric.

Problem 14. Show that if the real matrix A is skew-symmetric ($A^T = -A$), then iA is Hermitian.

Problem 15. Suppose that A is symmetric and orthogonal. Prove that the only possible eigenvalues of A are ± 1.

*Problem 16. Let A be real symmetric positive definite matrix. Show that A has a real symmetric positive definite square root, that is, there is a symmetric positive definite matrix S such that $S^2 = A$.

*Problem 17. Let A be any square real matrix and show that the eigenvalues of $A^T A$ are all nonnegative.

5.5 *Schur Form and Applications

Recall that matrices A and B are similar if there is an invertible matrix S such that $B = S^{-1}AS$; if the transformation matrix S is unitary, then $S^{-1} = S^*$. The main object of this section is to prove a famous theorem in linear algebra that provides a nice answer to the following question: if we wish to use only orthogonal (or unitary) matrices as similarity transformation matrices, what is the simplest form to which a given matrix A can be transformed? It would be nice if we could say something like “diagonal” or “Jordan canonical form.” Unfortunately, neither is possible. However, upper triangular matrices are very nice special forms of matrices. In particular, we can see the eigenvalues of an upper triangular matrix at a glance. That makes the following theorem extremely attractive. Its proof is also very interesting, in that it actually suggests an algorithm for computing the so-called Schur triangular form.

Schur Triangularization Theorem

Theorem 5.14. Let A be an arbitrary square matrix. Then there exists a unitary matrix U such that U^*AU is an upper triangular matrix. If A and its eigenvalues are real, then U can be chosen to be orthogonal.

Proof. To get started, take $k = 0$ and $V_0 = I$. Suppose we have reached the kth stage where we have a unitary matrix V_k such that

$$V_k^* A V_k = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ \vdots & \ddots & * & \vdots \\ 0 & \cdots & \lambda_k & * \\ 0 & \cdots & 0 & B \end{bmatrix} = \begin{bmatrix} R_k & C \\ 0 & B \end{bmatrix}$$

with the submatrix R_k upper triangular. Compute an eigenvalue λ_{k+1} of the submatrix B and a corresponding eigenvector $\mathbf{w}$ of unit length in the standard norm. Compute an eigenvalue λ_{k+1} of B and a corresponding eigenvector $\mathbf{w}$ of unit length in the standard norm. We may assume that the first coordinate of $\mathbf{w}$ is real. If not, replace $\mathbf{w}$ by $e^{-i\theta}\mathbf{w}$ where θ is a polar argument of the first coordinate of $\mathbf{w}$. This does not affect the length of $\mathbf{w}$, and any multiple of $\mathbf{w}$ is still an eigenvector of A. Now let $\mathbf{v} = \mathbf{w} - \mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, \ldots, 0)$. Form the (possibly complex) Householder matrix $H_\mathbf{v}$. Since $\mathbf{w} \cdot \mathbf{e}_1$ is real, it follows from Exercise 5 that $H_\mathbf{v}\mathbf{w} = \mathbf{e}_1$. Now recall that Householder matrices are unitary and symmetric, so that $H_\mathbf{v}^* = H_\mathbf{v} = H_\mathbf{v}^{-1}$. Hence

$$H_\mathbf{v}^* B H_\mathbf{v} \mathbf{e}_1 = H_\mathbf{v} B H_\mathbf{v}^{-1} \mathbf{e}_1 = H_\mathbf{v} B \mathbf{w} = H_\mathbf{v} \lambda_1 \mathbf{w} = \lambda_1 \mathbf{e}_1.$$

Therefore, the entries under the first row and in the first column of $H_\mathbf{v}^* B H_\mathbf{v}$ are zero. Form the unitary matrix

$$V_{k+1} = \begin{bmatrix} I_k & 0 \\ 0 & H_\mathbf{v} \end{bmatrix} V_k$$

and obtain that

$$\begin{aligned} V_{k+1}^* A V_{k+1} &= \begin{bmatrix} I_k & 0 \\ 0 & H_\mathbf{v} \end{bmatrix} V_k^* A V_k \begin{bmatrix} I_k & 0 \\ 0 & H_\mathbf{v} \end{bmatrix} \\ &= \begin{bmatrix} I_k & 0 \\ 0 & H_\mathbf{v} \end{bmatrix} \begin{bmatrix} R_k & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & H_\mathbf{v} \end{bmatrix} = \begin{bmatrix} R_k & CH_\mathbf{v} \\ 0 & H_\mathbf{v}^* B H_\mathbf{v} \end{bmatrix}. \end{aligned}$$

This new matrix is upper triangular in the first $k + 1$ columns, so we can continue in this fashion until we reach the last column, at which point we set $U = V_n$ to obtain that $U^H A U$ is upper triangular.

Finally, notice that if the eigenvalues and eigenvectors that we calculate are real, which would certainly be the case if A and the eigenvalues of A were real, then the Householder matrices used in the proof are all real, so that the matrix U is orthogonal. □

Of course, the upper triangular matrix T and triangularizing matrix U are not unique. Nonetheless, this is a very powerful theorem. Consider what it says in the case that A is Hermitian: the principal axes theorem is a simple special case of it.

Corollary 5.3. Every Hermitian matrix is unitarily (orthogonally, if the matrix is real) diagonalizable.

Principal Axes Theorem

Proof. Let A be Hermitian. According to the Schur triangularization theorem there is a unitary matrix U such that $U^*AU = R$ is upper triangular. We check that

$$R^* = (U^*AU)^* = U^*A^*(U^*)^* = U^*AU = R.$$

Therefore R is both upper and lower triangular. This makes R a diagonal matrix. If A is real symmetric, then A and its eigenvalues are real. By the triangularization theorem U can be chosen orthogonal. □

Another application of the Schur triangularization theorem is that we can show the real significance of *normal* matrices. This term has appeared in several exercises. Recall that a matrix A is normal if $A^*A = AA^*$. Clearly, every Hermitian matrix is normal.

Normal Matrix

Corollary 5.4. A matrix is unitarily diagonalizable if and only if it is normal.

Proof. We leave it as an exercise to show that a unitarily diagonalizable matrix is normal. Conversely, let A be normal. According to the Schur triangularization theorem there is a unitary matrix U such that $U^*AU = R$ is upper triangular. But then we have that $R^* = U^*A^*U$, so that

$$R^*R = U^*A^*UU^*AU = U^*A^*AU = U^*AA^*U = U^*AUU^*A^*U = RR^*.$$

Therefore R commutes with R^*, which means that R is diagonal by Problem 11 at the end of this section. This completes the proof. □

Our last application extends Theorem 5.2 to rational functions.

Corollary 5.5. Let $f(x)$ and $g(x)$ be polynomials and A a square matrix such that $g(A)$ is invertible. Then the eigenvalues of the matrix $f(A)g(A)^{-1}$ are of the form $f(\lambda)/g(\lambda)$, where λ runs over the eigenvalues of A.

Proof. We sketch the proof. As a first step, we make two observations about upper triangular matrices S and T with diagonal terms $\lambda_1, \lambda_2, \ldots, \lambda_n$, and $\mu_1, \mu_2, \ldots, \mu_n$, respectively. First, ST is upper triangular with diagonal terms $\lambda_1\mu_1, \lambda_2\mu_2, \ldots, \lambda_n\mu_n$. Next, if S is invertible, then S^{-1} is also an upper triangular matrix, whose diagonal terms are $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$.

Now, we have seen in Theorem 5.5 that for any invertible P of the right size, $P^{-1}f(A)P = f\left(P^{-1}AP\right)$. Similarly, if we multiply the identity $g(A)g(A)^{-1} = I$ by P^{-1} and P, we see that $P^{-1}g(A)^{-1}P = g(P^{-1}AP)^{-1}$. Thus, if P is a matrix that unitarily diagonalizes A, then

$$P^{-1}f(A)\,g(A)^{-1}P = f\left(P^{-1}AP\right)g\left(P^{-1}AP\right)^{-1},$$

so that by our first observations, this matrix is upper triangular with diagonal entries of the required form. Since similar matrices have the same eigenvalues, it follows that the eigenvalues of $f(A)g(A)^{-1}$ are of the required form. □

5.5 Exercises and Problems

Use a calculator or software for the following exercises.

Exercise 1. Apply one step of Schur triangularization to the following specified eigenvalues.

(a) $\lambda = -3$, $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (b) $\lambda = \sqrt{2}$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{bmatrix}$

Exercise 2. Apply Schur triangularization to the following matrices.

(a) $\begin{bmatrix} 4 & 4 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} \mathrm{i} & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ (c) $= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

Exercise 3. Use Schur triangularization to find eigenvalues of the following matrices.

(a) $\begin{bmatrix} 5 & 6 & 18 \\ 11 & 6 & 24 \\ -4 & -2 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 8 & 20 \\ 3 & 14 & 32 \\ -1 & -5 & -11 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 20 & 42 & 12 \\ 8 & 32 & 72 & 15 \\ -3 & -14 & -31 & -6 \\ -1 & -6 & -12 & -4 \end{bmatrix}$

Exercise 4. Find a unitary matrix that upper triangularizes the following matrices.

(a) $\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 8 & 10 \\ 3 & 14 & 0 \\ -1 & 5 & 1 \end{bmatrix}$ (c) $= \begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$

Exercise 5. Verify Corollary 5.5 in the case that $A = \begin{bmatrix} 22 & 10 \\ -50 & -23 \end{bmatrix}$, $f(x) = x^2 - 1$, and $g(x) = x^2 + 1$ by calculating the eigenvalues $f(A)/g(A)$ directly and comparing them to $f(\lambda)/g(\lambda)$, where λ runs over the eigenvalues of A.

Exercise 6. Verify that Corollary 5.5 fails in the case that $A = \begin{bmatrix} 22 & 10 \\ -50 & -23 \end{bmatrix}$, $f(x) = x - 1$, and $g(x) = x^2 + 4x + 3$ and explain why.

Problem 7. Show that every unitary matrix is normal. Give an example of a unitary matrix that is not Hermitian.

*Problem 8. Let A be an invertible matrix. Use Schur triangularization to reduce the problem $A\mathbf{x} = \mathbf{b}$ to a problem with triangular coefficient matrix.

Problem 9. Show that every unitarily diagonalizable matrix is normal.

Problem 10. Use Corollary 5.3 to show that the eigenvalues of a Hermitian matrix must be real.

*Problem 11. Prove that if an upper triangular matrix commutes with its Hermitian transpose, then the matrix must be diagonal.

5.6 *The Singular Value Decomposition

The object of this section is to develop yet one more factorization of a matrix that provides valuable information about the matrix. For simplicity, we stick with the case of a real matrix A and orthogonal matrices. However, the factorization we are going to discuss can be done with complex A and unitary matrices. This factorization is called the *singular value decomposition (SVD for short).* It has a long history in matrix theory, but was popularized in the 1960s as a powerful computational tool. We will see in Section 6.4 that multiplication on one side by an orthogonal matrix can produce an upper triangular matrix. This is called the *QR factorization.* Here is the basic question that the SVD answers: if multiplication on one side by an orthogonal matrix can produce an upper triangular matrix, how simple a matrix can be produced by multiplying on each side by a (possibly different) orthogonal matrix? The answer, as you might guess, is a matrix that is both upper and lower triangular, that is, diagonal. However, verification of this fact is much more subtle than that of the one-sided QR factorization of Section 6.4. Here is the key result:

Singular Value Decomposition Theorem

Theorem 5.15. Let A be an $m \times n$ real matrix. Then there exist an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ diagonal matrix Σ with diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, with $p = \min\{m, n\}$, such that $U^T AV = \Sigma$. Moreover, the numbers $\sigma_1, \sigma_2, \ldots, \sigma_p$ are uniquely determined by A.

Proof. There is no loss of generality in assuming that $n = \min\{m, n\}$. For if this is not the case, we can prove the theorem for A^T, and by transposing the resulting SVD for A^T, obtain a factorization for A. Form the $n \times n$ matrix $B = A^T A$. This matrix is symmetric and its eigenvalues are nonnegative (we leave these facts as exercises). Because they are nonnegative, we can write the eigenvalues of B in decreasing order of magnitude as the squares of nonnegative real numbers, say as $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$. Now we know from the principal axes theorem that we can find an orthonormal set of eigenvectors corresponding to these eigenvalues, say $B\mathbf{v}_k = \sigma_k^2 \mathbf{v}_k$, $k = 1, 2, \ldots, n$. Let $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$. Then V is an orthogonal $n \times n$ matrix. We may assume for some index r that $\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n$ are zero, while $\sigma_r \neq 0$.

Next set $\mathbf{u}_j = \frac{1}{\sigma_j} A\mathbf{v}_j$, $j = 1, 2, \ldots, r$. These are orthonormal vectors in $\mathbb{R}^m$ since

$$\mathbf{u}_j^T \mathbf{u}_k = \frac{1}{\sigma_j \sigma_k} \mathbf{v}_j^T A^T A \mathbf{v}_k = \frac{1}{\sigma_j \sigma_k} \mathbf{v}_j^T B \mathbf{v}_k = \frac{\sigma_k^2}{\sigma_j \sigma_k} \mathbf{v}_j^T \mathbf{v}_k = \begin{cases} 0, & \text{if } j \neq k, \\ 1, & \text{if } j = k. \end{cases}$$

Now expand this set to an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of $\mathbb{R}^m$. This is possible by Theorem 4.7 in Section 4.3. Set $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m]$. This matrix is orthogonal. We calculate that if $k > r$, then $\mathbf{u}_j^T A\mathbf{v}_k = 0$ since $A\mathbf{v}_k = \mathbf{0}$, and if $k < r$, then

$$\mathbf{u}_j^T A\mathbf{v}_k = \sigma_k \mathbf{u}_j^T \mathbf{u}_k = \begin{cases} 0, & \text{if } j \neq k, \\ \sigma_k, & \text{if } j = k. \end{cases}$$

It follows that $U^T AV = [\mathbf{u}_j^T A\mathbf{v}_k] = \Sigma$, which is the desired SVD.

Finally, if U, V are orthogonal matrices such that $U^T AV = \Sigma$, then $A = U\Sigma V^T$ and therefore

$$B = A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T,$$

so that the squares of the diagonal entries of Σ are the eigenvalues of B. It follows that the numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ are uniquely determined by A. □

The numbers $\sigma_1, \sigma_2, \ldots, \sigma_p$ are called the *singular values* of the matrix A, the columns of U are the *left singular vectors* of A, and the columns of V are the *right singular values* of A.

There is an interesting geometrical interpretation of this theorem from the perspective of linear transformations and change of basis as developed in Section 4.4. It can be stated as follows.

Corollary 5.6. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix A with respect to the standard bases. Then there exist orthonormal bases $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, such that the matrix of T with these bases is diagonal with nonnegative entries down the diagonal.

Proof. First observe that if $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m]$ and $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, then U and V are the change of basis matrices from the standard bases to the bases $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Also, $U^{-1} = U^T$. Now apply Corollary 4.2 of Section 4.4, and the result follows. □

Corollary 5.7. Let $U^T AV = \Sigma$ be the SVD of A and suppose that $\sigma_r \neq 0$ and $\sigma_{r+1} = 0$. Then

(1) $\operatorname{rank} A = r$.
(2) $A = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r] \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\} [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r]^T$.
(3) $\mathcal{N}(A) = \operatorname{span}\{\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_n\}$.
(4) $\mathcal{C}(A) = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.
(5) If $A^\dagger$ is given by

$$A^\dagger = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r] \operatorname{diag}\{1/\sigma_1, 1/\sigma_2, \ldots, 1/\sigma_r\} [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r]^T,$$

then $\mathbf{x} = A^\dagger \mathbf{b}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$.
(6) $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$.

Proof. Multiplication by invertible matrices does not change rank, and the rank of Σ is clearly r, so (1) follows. For (2), multiply the SVD equation by U on the left and V^T on the right to obtain

$$\begin{aligned} A = U\Sigma V^T &= [\sigma_1 \mathbf{u}_1, \sigma_2 \mathbf{u}_2, \ldots, \sigma_r \mathbf{u}_r, \mathbf{0}, \ldots, \mathbf{0}] [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]^T \\ &= \sum_{k=1}^{r} \sigma_k \mathbf{u}_k \mathbf{v}_k^T = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r] \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\} [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r]^T. \end{aligned}$$

This also proves item (6). The remaining items are left as exercises. □

Compact SVD and Pseudoinverse

Item (2) is called the *compact* SVD form for A. The matrix $A^\dagger$ of (5) is called the *pseudoinverse* of A and behaves in many ways like an inverse for matrices that need not be invertible or even square. Item (5) presents an important application of the pseudoinverse. We have only scratched the surface of the many facets of the SVD. Like most good ideas, it is rich in applications. We mention one more. It is based on item (6), which says that a matrix A of rank r can be written as a sum of r rank-one matrices. In fact, it can be shown that this representation is the most economical in the sense that the partial sums

$$\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T, \qquad k = 1, 2, \ldots, r,$$

give the rank-k approximation to A that is closest among all rank-k approximations to A. This suggests an intriguing way to compress data in a lossy way (i.e., with some loss of data). For example, suppose A is a matrix of floating-point numbers representing a picture. We might get a reasonably good approximation to the picture using only the σ_k larger than a certain threshold. Thus, with a $1{,}000 \times 1{,}000$ matrix A that has a very small σ_{21}, we could get by with the data $\sigma_k, \mathbf{u}_k, \mathbf{v}_k$, $k = 1, 2, \ldots, 20$. Consequently, we would store only these quantities, which add up to $1{,}000 \times 40 + 20 = 40{,}020$ numbers. Contrast this with storing the full matrix of $1{,}000 \times 1{,}000 = 1{,}000{,}000$ entries, and you can see the gain in economy.

5.6 Exercises and Problems

Exercise 1. Exhibit a singular value decomposition for the following matrices.

(a) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Exercise 2. Calculate a singular value decomposition for the following matrices.

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Exercise 3. Use a CAS or MAS to compute an orthonormal basis for the null space and column space of the following matrices with the SVD and Corollary 5.7. You will have to decide which nearly-zero terms are really zero.

(a) $\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 0 \\ 1 & -2 & 2 \\ 3 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 2 \\ 4 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 & 0 & -3 \\ 1 & 2 & 1 & -5 & 2 \\ 0 & 1 & 0 & -3 & 1 \\ 0 & 2 & -3 & 1 & 4 \end{bmatrix}$

Exercise 4. Use the pseudoinverse to find a least squares solution $A\mathbf{x} = \mathbf{b}$, where A is a matrix from Exercise 3 with corresponding right-hand side below.
(a) $(2, 2, 6, 5)$ (b) $(2, 3, 1)$ (c) $(4, 1, 2, 3)$

*Problem 5. Prove (3) and (4) of Corollary 5.7.

Problem 6. Show that if A is invertible, then A^{-1} is the pseudoinverse of A.

*Problem 7. Prove (5) of Corollary 5.7.

Problem 8. Digitize a picture into a 640×400 (standard VGA) matrix of grayscale pixels, where the value of each pixel is a number x, $0 \leq x \leq 1$, with black corresponding to $x = 0$ and white to $x = 1$. Compute the SVD of this image matrix and display various approximations using 10, 20, and 40 of the singular values and vector pairs. Do any of these give a good visual approximation to the picture? If not, find a minimal number that works. You will need computer support for this exercise.

5.7 *Computational Notes and Projects

Computation of Eigensystems

Nowadays, one can use an MAS such as Matlab or Octave on a home PC to find a complete eigensystem for, say a 100×100 matrix, in a fraction of a second. That's pretty remarkable and, to some extent, a tribute to the fast cheap hardware commonly available to the public. But hardware is only part of the story. Bad computational algorithms can bring the fastest computer to its knees. The rest of the story concerns the remarkable developments in numerical linear algebra over the past fifty years that have given us fast reliable algorithms for eigensystem calculation. We can only scratch the surface of these developments in this brief discussion. At the outset, we rule out the methods developed in this chapter as embodied in the eigensystem algorithm (page 254). These are for simple hand calculations and theoretical purposes. In a few special cases we can derive general formulas for eigenvectors and eigenvalues. One such example is a *Toeplitz* matrix (a matrix with constant entries down each diagonal) that is also tridiagonal. We outline the approach in a problem at the end of this section, but these complete solution formulas are the exception, not the rule.

We are going to examine some iterative methods for selectively finding eigenpairs of a real matrix whose eigenvalues are real and distinct. Hence the matrix A is diagonalizable. The hypothesis of diagonalizability may seem too constraining, but there is this curious aphorism that "numerically every matrix is diagonalizable." The reason is as follows: once you store and perform numerical calculations on the entries of A, you perturb them a small essentially random amount. This has the effect of perturbing the eigenvalues of the calculated A a small random amount. Thus, the probability that any two eigenvalues of A are numerically equal is quite small. To focus matters, consider the test matrix

$$A = \begin{bmatrix} -8 & -5 & 8 \\ 6 & 3 & -8 \\ -3 & 1 & 9 \end{bmatrix}.$$

Just for the record, the actual eigenvalues of A are -2, -1, and 5. Now we ask three questions about A:

1. How can we get a ballpark estimate of the location of the eigenvalues of A?
2. How can we estimate the *dominant* eigenpair $(\lambda, \mathbf{x})$ of A? (Recall that "dominant" means that λ is larger in absolute value than any other eigenvalue of A.)
3. Given a good estimate of any eigenvalue λ of A, how can we improve the estimate and compute a corresponding eigenvector?

An answer to question (1) is the following theorem, which predates modern numerical analysis, but has proved to be quite useful. Because it helps locate eigenvalues, it is called a "localization theorem."

Gershgorin Circle Theorem

Theorem 5.16. Let $A = [a_{ij}]$ be an $n \times n$ matrix and define disks D_j in the complex plane by

$$r_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{jk}|,$$
$$D_j = \{z \,|\, |z - a_{jj}| \leq r_j\}.$$

(1) Every eigenvalue of A is contained in some disk D_j.
(2) If k of the disks are disjoint from the others, then exactly k eigenvalues are contained in the union of these disks.

Proof. To prove (1), let λ be an eigenvalue of A and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ an eigenvector corresponding to λ. Suppose that x_j is the largest coordinate of $\mathbf{x}$ in absolute value. Divide $\mathbf{x}$ by this entry to obtain an eigenvector whose largest coordinate is $x_j = 1$. Without loss of generality, this vector is $\mathbf{x}$. Consider the jth entry of the zero vector $\lambda\mathbf{x} - A\mathbf{x}$, which is

$$(\lambda - a_{jj})1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{jk} x_k = 0.$$

Bring the sum to the right-hand side and take absolute values to obtain

$$|\lambda - a_{jj}| = |\sum_{\substack{k=1\\k\neq j}}^{n} a_{jk}x_k| \leq \sum_{\substack{k=1\\k\neq j}}^{n} |a_{jk}||x_k| \leq r_j,$$

since $|x_k| \leq 1$ for each x_k. This shows that $\lambda \in D_j$, which proves (1). We will not prove (2), since it requires some complex analysis (see the Horn and Johnson text [11], page 344, for a proof.)

Example 5.19. Apply the Gershgorin circle theorem to the test matrix A and sketch the resulting Gershgorin disks.

Solution. The disks are easily seen to be

$$\begin{aligned} D_1 &= \{ z \mid |z+8| \leq 13\}, \\ D_2 &= \{ z \mid |z-3| \leq 14\}, \\ D_3 &= \{ z \mid |z-9| \leq 4\}. \end{aligned}$$

A sketch of them is provided in Figure 5.1. □

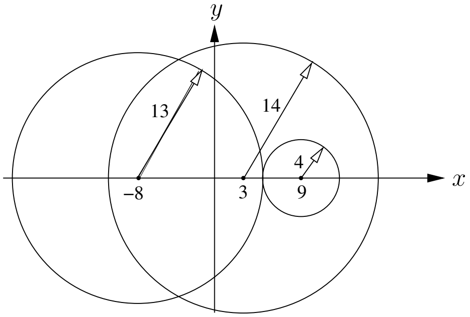


Fig. 5.1. Gershgorin disks for A.

Now we turn to question (2). One answer to it is contained in the following algorithm, known as the *power method*.

Power Method

Power Method: To compute an approximate eigenpair $(\lambda, \mathbf{x})$ of A with $\|\mathbf{x}\| = 1$ and λ the dominant eigenvalue.

(1) Input an initial guess $\mathbf{x}_0$ for $\mathbf{x}$
(2) For $k = 0, 1, \ldots$ until convergence of $\lambda^{(k)}$'s:
  (a) $\mathbf{y} = A\mathbf{x}_k$,
  (b) $\mathbf{x}_{k+1} = \dfrac{\mathbf{y}}{\|\mathbf{y}\|}$,
  (c) $\lambda^{(k+1)} = \mathbf{x}_{k+1}^T A\mathbf{x}_{k+1}$.

That's all there is to it! Why should this algorithm converge? The secret to this algorithm lies in a formula we saw earlier in our study of discrete dynamical systems, namely equation (5.6) which we reproduce here:

$$\mathbf{x}^{(k)} = A^k\mathbf{x}^{(0)} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n.$$

Here it is understood that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of eigenvectors corresponding to eigenvalues $\lambda_1\lambda_2, \ldots, \lambda_n$, which, with no loss of generality, we can assume to be unit-length vectors. Notice that at each stage of the power method we divided the computed iterate $\mathbf{y}$ by its length to get the next $\mathbf{x_{k+1}}$, and this division causes no directional change. Thus we would get exactly the same vector if we simply set $\mathbf{x}_{k+1} = \mathbf{x}^{(k+1)}/\left\|\mathbf{x}^{(k+1)}\right\|$. Now for large k the ratios $(\lambda_j/\lambda_1)^k$ can be made as small as we please, so we can rewrite the above equation as

$$\mathbf{x}^{(k)} = A^k\mathbf{x}^{(0)} = \lambda_1^k\left\{c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n\right\} \approx \lambda_1^k c_1\mathbf{v}_1.$$

Assuming that $c_1 \neq 0$, which is likely if $\mathbf{x}_0$ is randomly chosen, we see that

$$\mathbf{x}_{k+1} = \frac{A\mathbf{x}^{(k)}}{\left\|A\mathbf{x}^{(k)}\right\|} \approx \frac{\lambda_1^k c_1\lambda_1\mathbf{v}_1}{\left|\lambda_1^k c_1\lambda_1\right|} = \pm\mathbf{v}_1,$$
$$\lambda^{(k+1)} = \mathbf{x}_{k+1}^T A\mathbf{x}_{k+1} \approx (\pm\mathbf{v}_1)^T A(\pm\mathbf{v}_1) = \lambda_1.$$

Thus we see that the sequence of $\lambda^{(k)}$'s converges to λ_1 and the sequence of $\mathbf{x}_k$'s converges to $\pm\mathbf{v}_1$. The argument (it isn't rigorous enough to be called a proof) we have just given shows that the oscillation in sign in the entries of $\mathbf{x}_k$ occurs in the case $\lambda < 0$. You might notice also that the argument doesn't require the initial guess to be a real vector. Complex vectors are permissible.

If we apply the power method to our test problem with an initial guess of $\mathbf{x}_0 = (1, 1, 1)$, we get every third value as follows:

k	$\lambda^{(k)}$	$\mathbf{x}_k$
0		$(1, 1, 1)$
3	5.7311	$(0.54707, -0.57451, 0.60881)$
6	4.9625	$(0.57890, -0.57733, 0.57581)$
9	5.0025	$(0.57725, -0.57735, 0.57745)$
12	4.9998	$(0.57736, -0.57735, 0.57734)$

Notice that the eigenvector looks a lot like a multiple of $(1, -1, 1)$, and the eigenvalue looks a lot like 5. This is an exact eigenpair, as one can check.

Finally, we turn to question (3). One answer to it is contained in the following algorithm, known as the *inverse iteration method.*

Inverse Iteration Method: To compute an approximate eigenpair $(\lambda, \mathbf{x})$ of A with $\|\mathbf{x}\| = 1$.

Inverse Iteration Method

(1) Input an initial guess $\mathbf{x}_0$ for $\mathbf{x}$ and a *close* approximation $\mu = \lambda_0$ to λ.

(2) For $k = 0, 1, \ldots$ until convergence of the $\lambda^{(k)}$'s:

  (a) $\mathbf{y} = (A - \mu I)^{-1}\mathbf{x}_k$,

(b) $\mathbf{x}_{k+1} = \dfrac{\mathbf{y}}{\|\mathbf{y}\|}$,

(c) $\lambda^{(k+1)} = \mathbf{x}_{k+1}^T A \mathbf{x}_{k+1}$.

Notice that the inverse iteration method is simply the power method applied to the matrix $(A - \mu I)^{-1}$. In fact, it is sometimes called the inverse power method. The scalar μ is called a *shift.* Here is the secret of success for this method: we assume that μ is closer to a definite eigenvalue λ of A than to any other eigenvalue. But we don't want too much accuracy! We need $\mu \neq \lambda$. Theorem 5.2 in Section 1 of this chapter shows that the eigenvalues of the matrix $A - \mu I$ are of the form $\sigma - \mu$, where σ runs over the eigenvalues of A. Thus the matrix $A - \mu I$ is nonsingular since no eigenvalue is zero, and Exercise 17 of Section 5.1 shows us that the eigenvalues of $(A - \mu I)^{-1}$ are of the form $1/(\sigma - \mu)$, where σ runs over the eigenvalues of A. Since μ is closer to λ than to any other eigenvalue of A, the eigenvalue $1/(\lambda - \mu)$ is the dominant eigenvalue of $(A - \mu I)^{-1}$, which is exactly what we need to make the power method work on $(A - \mu I)^{-1}$. Indeed, if μ is *very* close (but not equal!) to λ, convergence should be very rapid.

In a general situation, we could now have the Gershgorin circle theorem team up with inverse iteration. Gershgorin would put us in the right ballpark for values of μ, and inverse iteration would finish the job. Let's try this with our test matrix and choices of μ in the interval suggested by Gershgorin. Let's try $\mu = 0$. Here are the results in tabular form:

k	$\lambda^{(k)}$	$\mathbf{x}_k$ with $\mu = 0.0$
0	0.0	$(1, 1, 1)$
3	0.77344	$(-0.67759, 0.65817, -0.32815)$
6	1.0288	$(-0.66521, 0.66784, -0.33391)$
9	0.99642	$(-0.66685, 0.66652, -0.33326)$
12	1.0004	$(-0.66664, 0.66668, -0.33334)$

It appears that inverse iteration is converging to $\lambda = 1$ and the eigenvector looks suspiciously like a multiple of $(-2, 2, -1)$. This is in fact an exact eigenpair.

There is much more to modern eigenvalue algorithms than we have indicated here. Central topics include deflation, the QR algorithm, numerical stability analysis, and many other issues. The interested reader might consult more advanced texts such as references [9], [7], [13], and [6], to name a few.

Project Topics

Project: Solving Polynomial Equations

In homework problems we solve for the roots of the characteristic polynomial in order to get eigenvalues. To this end we can use algebra methods or even Newton's method for numerical approximations to the roots. This is the conventional wisdom usually proposed in introductory linear algebra. But for larger problems than the simple 2×2 or 3×3 matrices we encounter, this method can be too slow and inaccurate. In fact, numerical methods hiding under the hood in a MAS (and some CASs) for finding eigenvalues are so efficient that it is better to turn this whole procedure on its head. Rather than find roots to solve linear algebra (eigenvalue) problems, we can use (numerical) linear algebra to find roots of polynomials. In this project we discuss this methodology and document it in a fairly nontrivial example.

Given a polynomial $f(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + x^n$, form the *companion matrix* of $f(x)$,

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

It is a key fact that the eigenvalues of $C(f)$ are precisely the roots of the equation $f(x) = 0$. Experiment with $n = 2, 3, 4$ and try to find a proof by expansion across the bottom row of $\det(A - \lambda I)$ that this result is true for all n.

Then use a CAS (or MAS) to illustrate this method by finding approximate roots of three polynomials: a cubic and quartic of your choice and then the polynomial

$$f(x) = 5 + 11x + 4x^2 + 6x^3 + x^4 - 15x^5 + 5x^6 - 3x^7 - 2x^8 + 8x^9 - 5x^{10} + x^{11}.$$

In each case use Newton's method to improve the values of some of the roots (it works with complex numbers as well as reals, provided one starts close enough to a root). Check your answers to this problem by evaluating the polynomial. Use your results to write the polynomial as a product of the linear factors $x - \lambda$, where λ is a root and check the correctness of this factorization.

Project: Finding a Jordan Canonical Form

A challenge: Find the Jordan canonical form of the 10×10 matrix A, which is given *exactly* as follows. The solution will require some careful work with a CAS or (preferably) MAS.

$$A = \begin{bmatrix} 1 & 1 & 1 & -2 & 1 & -1 & 2 & -2 & 4 & -3 \\ -1 & 2 & 3 & -4 & 2 & -2 & 4 & -4 & 8 & -6 \\ -1 & 0 & 5 & -5 & 3 & -3 & 6 & -6 & 12 & -9 \\ -1 & 0 & 3 & -4 & 4 & -4 & 8 & -8 & 16 & -12 \\ -1 & 0 & 3 & -6 & 5 & -4 & 10 & -10 & 20 & -15 \\ -1 & 0 & 3 & -6 & 2 & -2 & 12 & -12 & 24 & -18 \\ -1 & 0 & 3 & -6 & 2 & -5 & 15 & -13 & 28 & -21 \\ -1 & 0 & 3 & -6 & 2 & -5 & 15 & -11 & 32 & -24 \\ -1 & 0 & 3 & -6 & 2 & -5 & 15 & -14 & 37 & -26 \\ -1 & 0 & 3 & -6 & 2 & -5 & 15 & -14 & 36 & -25 \end{bmatrix}.$$

Your main task is to devise a strategy for identifying the Jordan canonical form matrix J. Do *not* expect to find the invertible matrix S for which $J = S^{-1}AS$. However, a key fact to keep in mind is that if A and B are similar matrices, i.e., $A = S^{-1}BS$ for some invertible S, then $\operatorname{rank} A = \operatorname{rank} B$. In particular, if S is a matrix that puts A into Jordan canonical form, then $J = S^{-1}AS$.

First prove this rank fact for A and B. Show that it applies to $A - cI$ and $B - cI$ as well, for any scalar c. Then extend it to powers of A and B.

Now you have the necessary machinery for determining numerically the Jordan canonical form. As a first step, one can use a CAS or MAS to find the eigenvalues of A. Of course, these will only be approximate, so one has to decide how many eigenvalues are really repeated.

Next, one has to determine the number of Jordan blocks of a given type. Suppose λ is an eigenvalue and find the rank of various powers of $A - \lambda I$. It would help greatly in understanding how all this counts blocks if you first experiment with a matrix already in Jordan canonical form, say, for example,

$$J = \begin{bmatrix} J_1(2) & 0 & 0 \\ 0 & J_2(2) & 0 \\ 0 & 0 & J_1(3) \end{bmatrix}.$$

Project: Classification of Quadratic Forms

You should review the change of coordinates material from Example 3.26 of Chapter 3. Recall from calculus that in order to classify all quadratic equations in x and y one went through roughly three steps. First, perform a rotation transformation of coordinates to get rid of mixed terms such as $2xy$ in the quadratic equation $x^2 + 2xy - y^2 + x - 3y = 4$. Second, do a translation of coordinates to put the equation in a "standard form." Third, identify the curve by your knowledge of the shape of a curve in the given standard form. Standard forms are equations like $x^2/4 + y^2/2 = 1$, an ellipse with its axes along the x- and y-axes. Also recall that it is the second-degree terms alone that determine the nature of a quadratic. For example, the second-degree terms of the equation above are x^2, $2xy$, and y^2. The discriminant of the equation is determined by these terms. In this case the discriminant is 8, which tells us that the curve represented by this equation is a hyperbola. Finally, recall that when it comes to *quadric* equations, i.e., quadratic equations in 3 unknowns, your text simply provides some examples in "standard form" (six of them to be exact) and perhaps suggested something about this list being essentially all surfaces represented by quadric equations.

Now you're ready for the rest of the story. Just as with curves in x and y, the basic shape of the surface of a quadric equation in x, y, and z is determined by the second-degree terms. Since this is so, we will focus on an example with no first-degree terms, namely,

$$Q\,(x, y, z) = 2x^2 + 4y^2 + 6z^2 - 4xy - 2xz + 2yz = 1.$$

The problem is this: find a change of coordinates that will make it clear which of the six standard forms is represented by this surface. Here is how to proceed: first you must express the so-called quadratic form $Q\,(x, y, z)$ in matrix form as $Q\,(x, y, z) = [x, y, z]A[x, y, z]^T$. It is easy to find such matrices A. But not any such A will do. Next, you must replace A by the equivalent matrix $(A+A^T)/2$. (Check that if A specifies the quadratic form Q, then so will $(A + A^T)/2$.) The advantage of this latter matrix is that it is symmetric. Now our theory of symmetric matrices can be brought to bear. In particular, we know that there is an orthogonal matrix P such that $P^T AP$ is diagonal, provided A is symmetric. So make the linear change of variables $[x, y, z]^T = P[x', y', z']^T$ and deduce that $Q\,(x, y, z) = [x', y', z']P^T AP[x', y', z']^T$. But when the matrix in the middle is diagonal, we end up with squares of x', y' and z', and no mixed terms.

Find a symmetric A for this problem and use the CAS or MAS available to you to calculate the eigenvalues of this A. From this data alone you will be able to classify the surface represented by the above equation. Also find unit-length eigenvectors for each eigenvalue. Put these together to form the desired orthogonal matrix P that eliminates mixed terms.

An outstanding reference on this topic and many others relating to matrix analysis is the recently republished textbook [3] by Richard Bellman, which is widely considered to be a classic in the field.

Report: Management of Sheep Populations

Description of the problem: You are working for the New Zealand Department of Agriculture on a project for sheep farmers. The species of sheep that these shepherds raise have a life span of 12 years. Of course, some live longer, but they are sufficiently few in number and their reproductive rate is so low that they may be ignored in your population study. Accordingly, you divide sheep into 12 age classes, namely those in the first year of life, etc. You have conducted an extensive survey of the demographics of this species of sheep and obtained the following information about the demographic parameters f_i and s_i, where f_i is the per-capita reproductive rate for sheep in the ith age class and s_i is the survival rate for sheep in that age class, i.e., the fraction of sheep in that age class that survive to the $(i + 1)$th class. (As a matter of fact, this table is related to real data. The interested reader might consult the article [5] in the bibliography.)

i	1	2	3	4	5	6	7	8	9	10	11	12
f_i	.000	.023	.145	.236	.242	.273	.271	.251	.234	.229	.216	.210
s_i	.845	.975	.965	.950	.926	.895	.850	.786	.691	.561	.370	-

The problem is as follows: in order to maintain a constant population of sheep, shepherds will harvest a certain number of sheep each year. Harvesting need not mean slaughter; it can be accomplished by selling animals to other shepherds, for example. It simply means removing sheep from the population. Denote the fraction of sheep that are removed from the ith age group at the end of each growth period (a year in our case) by h_i. If these numbers are constant from year to year, they constitute a *harvesting policy.* If, moreover, the yield of each harvest, i.e., total number of animals harvested each year, is a constant and the age distribution of the remaining populace is essentially constant after each harvest, then the harvesting policy is called *sustainable.* If all the h_i's are the same, say h, then the harvesting policy is called *uniform.* An advantage of uniform policies is that they are simple to implement: One selects the sheep to be harvested at random.

Your problem: Find a uniform sustainable harvesting policy to recommend to shepherds, and find the resulting distribution of sheep that they can expect with this policy. Shepherds who raise sheep for sale to markets are also interested in a sustainable policy that gives a maximum yield. If you can find such a policy that has a larger annual yield than the uniform policy, then recommend it. On the other hand, shepherds who raise sheep for their wool may prefer to minimize the annual yield. If you can find a sustainable policy whose yield is smaller than that of the uniform policy, make a recommendation accordingly. In each case find the expected distribution of your harvesting policies. Do you think there are optimum harvesting policies of this type? Do you think that there might be other economic factors that should be taken into account in this model? Organize your results for a report to be read by your supervisor and an informed public.

Procedure: Express this problem as a discrete linear dynamical system $\mathbf{x}^{(k+1)} = L\mathbf{x}^{(k)}$, where L is a so-called *Leslie matrix* of the form

$$L = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}.$$

It is understood that $0 < s_i \leq 1$, $0 \leq f_i$, and at least one f_i is nonzero. The facts you need to know (and may assume as standard facts about Leslie matrices) are as follows: such a matrix will have exactly one positive eigenvalue, which turns out to be a simple eigenvalue (not repeated). Moreover, if at least two adjacent entries of the first row are positive, this eigenvalue will be a *dominant* eigenvalue, i.e., it is strictly larger than any other eigenvalue in absolute value. In particular, if the positive eigenvalue is 1, then we know from Theorem 5.11 (ergodic theorem) that starting from any nonzero initial state with nonnegative entries, successive states converge to an eigenvector belonging to the eigenvalue 1. This eigenvector has all nonnegative entries since L and $\mathbf{x}^{(0)}$ are nonnegative. Scale this vector by dividing it by the sum of its components and one obtains an eigenvector that is a probability distribution vector, i.e., its entries are nonnegative and sum to 1. The entries of this vector give the long-term distribution of the population in the various age classes.

In regard to harvesting, let H be a diagonal matrix with the harvest fractions h_i down the diagonal. (Here $0 \leq h_i \leq 1$.) Then the population that results from this harvesting at the end of each period is given by $\mathbf{x}^{k+1} = L\mathbf{x}^k - HL\mathbf{x}^k = (I-H)L\mathbf{x}^k$. But the matrix $(I-H)L$ is itself a Leslie matrix, so the theory applies to it as well. There are other theoretical tools, but all you need to do is to find a matrix $H = hI$ such that 1 is the positive eigenvalue of $(I-H)L$. You can do this by trial and error, a method that is applicable to any harvesting policy, uniform or not. However, in the case of uniform policies it's simpler to note that $(I-H)L = (1-h)L$, where h is the diagonal entry of H.

Implementation Notes: your instructor may add local notes here and discuss available aids. For example, when I give this assignment under Maple or Mathematica, I create a notebook that has the correct vector of f_i's and s_i's in it to avoid a very common problem: data entry error.

5.7 Exercises and Problems

Exercise 1. The matrix of (c) below may have complex eigenvalues. Use the Gershgorin circle theorem to locate eigenvalues and the iteration methods of this section to compute an approximate eigensystem.

$$\text{(a)} \begin{bmatrix} 4 & -1 & 0 & 2 \\ 0 & 5 & 0 & -1 \\ -1 & -2 & 2 & 0 \\ 0 & 0 & 2 & 10 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 2 & 4 & -2 & 0 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Exercise 2. Use the Gershgorin circle theorem to locate eigenvalues and the iteration methods of this section to compute an approximate eigensystem.

$$\text{(a)} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 9 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & -2 & -2 & 0 \\ 6 & -7 & 21 & -18 \\ 4 & -8 & 22 & -18 \\ 2 & -4 & 13 & -13 \end{bmatrix}$$

*Problem 3. A square matrix is *strictly diagonally dominant* if in each row the sum of the absolute values of the off-diagonal entries is strictly less than the absolute value of the diagonal entry. Show that a strictly diagonally dominant matrix is invertible.

Problem 4. Let A be an $n \times n$ tridiagonal matrix with possibly complex entries a, b, c down the first subdiagonal, main diagonal, and first superdiagonal, respectively, where $a, c \neq 0$. Let $\mathbf{v} = (v_1, \ldots, v_n)$ satisfy $A\mathbf{v} = \lambda \mathbf{v}$.

(a) Show that $\mathbf{v}$ satisfies the difference equation $av_{j-1} + (b - \lambda)\, v_j + cv_{j+1} = 0$, $j = 1, \ldots, n$, with $v_0 = 0 = v_{n+1}$.

(b) Show that $v_j = Ar_1^j + Br_2^j$, where r_1, r_2 are (distinct) solutions to the auxiliary equation $a + (b - \lambda)\, r + cr^2 = 0$, is a solution to the difference equation in (a).

(c) Determine that $r_1 r_2 = a/c$, $r_1 + r_2 = (\lambda - b)\,/c$, and $(r_1/r_2)^{n+1} = e^{2i\pi s}$, $s = 1, \ldots, n$. Use these to find all r_1, r_2, and λ. (It helps to use the conditions $v_0 = 0 = v_{n+1}$ and examine the cases $j = 0$ and $j = n + 1$.)

(d) Conclude that a complete set of eigenpairs of A is given by

$$\lambda_j = b + 2c \left(\frac{a}{c}\right)^{1/2} \cos \frac{j\pi}{n+1} \text{ and } \mathbf{v}_j = \left(\left(\frac{a}{c}\right)^{j/2} \sin \frac{sj\pi}{n+1}\right)_{s=1}^{n}, \; j = 1, \ldots, n.$$

6

GEOMETRICAL ASPECTS OF ABSTRACT SPACES

Two basic ideas that we learn in geometry are those of length of a line segment and angle between lines. We have already seen how to extend these ideas to the standard vector spaces. The objective of this chapter is to extend these powerful ideas to general linear spaces. A surprising number of concepts and techniques that we learned in a standard setting can be carried over, almost word for word, to more general vector spaces. Once this is accomplished, we will be able to use our geometrical intuition in entirely new ways. For example, we will be able to have notions of size (length) and perpendicularity for nonstandard vectors such as functions in a function space. We will be able to give a sensible meaning to the size of the error incurred in solving a linear system with finite-precision arithmetic. We shall see that there are many more applications of this abstraction.

6.1 Normed Spaces

Definitions and Examples

The basic function of a norm is to measure length and distance, independent of any other considerations, such as angles or orthogonality. There are different ways to accomplish such a measurement. One method of measuring length might be more natural for a given problem, or easier to calculate than another. For these reasons, we would like to have the option of using different methods of length measurement. You may recognize the properties listed below from earlier in the text; they are the basic norm laws given in Section 4.1 for the standard norm. We are going to abstract the norm idea to arbitrary vector spaces.

Definition 6.1. A *norm* on the vector space V is a function $\|\cdot\|$ that assigns to each vector $\mathbf{v} \in V$ a real number $\|\mathbf{v}\|$ such that for c a scalar and $\mathbf{u}, \mathbf{v} \in V$ the following hold:

Abstract Norm

(1) $\|\mathbf{u}\| \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.
(2) $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$.
(3) (Triangle Inequality) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Normed Space

Distance Function

A vector space V, together with a norm $\|\cdot\|$ on the space V, is called a *normed space*. If $\mathbf{u}, \mathbf{v} \in V$, the distance between $\mathbf{u}$ and $\mathbf{v}$ is defined to be $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

Notice that if V is a normed space and W is any subspace of V, then W automatically becomes a normed space if we simply use the norm of V on elements of W. Obviously all the norm laws still hold, since they hold for elements of the bigger space V.

Standard Norms

Of course, we have already studied some very important examples of normed spaces, namely the standard vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, or any subspace thereof, together with the standard norms given by

$$\begin{aligned} \|(z_1, z_2, \ldots, z_n)\| &= \sqrt{z_1 \overline{z_1} + z_2 \overline{z_2} + \cdots + z_n \overline{z_n}} \\ &= \left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \right)^{1/2}. \end{aligned}$$

If the vectors are real then we can drop the conjugate bars. This norm is actually one of a family of norms that are commonly used.

p-norm

Definition 6.2. Let V be one of the standard spaces $\mathbb{R}^n$ or $\mathbb{C}^n$ and $p \geq 1$ a real number. The p-norm of a vector in V is defined by the formula

$$\|(z_1, z_2, \ldots, z_n)\|_p = \left(|z_1|^p + |z_2|^p + \cdots + |z_n|^p \right)^{1/p}.$$

Notice that when $p = 2$ we have the familiar example of the standard norm. Another important case is that in which $p = 1$. The last important instance of a p-norm is one that isn't so obvious: $p = \infty$. It turns out that the value of this norm is the limit of p-norms as $p \to \infty$. To keep matters simple, we'll supply a separate definition for this norm.

∞-norm

Definition 6.3. Let V be one of the standard spaces $\mathbb{R}^n$ or $\mathbb{C}^n$. The ∞-norm of a vector in V is defined by the formula

$$\|(z_1, z_2, \ldots, z_n)\|_\infty = \max \{ |z_1|, |z_2|, \ldots, |z_n| \}.$$

Example 6.1. Calculate $\|\mathbf{v}\|_p$, where $p = 1$, 2, or ∞ and $\mathbf{v} = (1, -3, 2, -1) \in \mathbb{R}^4$.

Solution. We calculate:

$$\begin{aligned} \|(1, -3, 2, -1)\|_1 &= |1| + |-3| + |2| + |-1| = 7 \\ \|(1, -3, 2, -1)\|_2 &= \sqrt{|1|^2 + |-3|^2 + |2|^2 + |-1|^2} = \sqrt{15} \\ \|(1, -3, 2, -1))\|_\infty &= \max \{ |1|, |-3|, |2|, |-1| \} = 3. \end{aligned}$$ □

It may seem a bit odd at first to speak of the same vector as having different lengths. You should take the point of view that choosing a norm is a bit like choosing a measuring stick. If you choose a yard stick, you won't measure the same number as you would by using a meter stick on an object.

Example 6.2. Calculate $\|\mathbf{v}\|_p$, where $p = 1$, 2, or ∞ and $\mathbf{v} = (2 - 3\mathrm{i}, 1 + \mathrm{i}) \in \mathbb{C}^2$.

Solution. We calculate:

$$\begin{aligned}
\|(2-3\mathrm{i},1+\mathrm{i})\|_1 &= |2-3\mathrm{i}| + |1+\mathrm{i}| = \sqrt{13}+\sqrt{2} \\
\|(2-3\mathrm{i},1+\mathrm{i})\|_2 &= \sqrt{|2-3\mathrm{i}|^2 + |1+\mathrm{i}|^2} = \sqrt{(2)^2+(-3)^2+1^2+1^2} = \sqrt{15} \\
\|(2-3\mathrm{i},1+\mathrm{i})\|_\infty &= \max\{|2-3\mathrm{i}|, |1+\mathrm{i}|\} = \max\left\{\sqrt{13},\sqrt{2}\right\} = \sqrt{13}. \qquad \square
\end{aligned}$$

Example 6.3. Verify that the norm properties are satisfied for the p-norm in the case that $p = \infty$.

Solution. Let c be a scalar, and let $\mathbf{u} = (z_1, z_2, \ldots, z_n)$, and $\mathbf{v} = (w_1, w_2, \ldots, w_n)$ be two vectors. Any absolute value is nonnegative, and any vector whose largest component in absolute value is zero must have all components equal to zero. Property (1) follows. Next, we have that

$$\begin{aligned}
\|c\mathbf{u}\|_\infty &= \|(cz_1, cz_2, \ldots, cz_n)\|_\infty \\
&= \max\{|cz_1|, |cz_2|, \ldots, |cz_n|\} \\
&= |c| \max\{|z_1|, |z_2|, \ldots, |z_n|\} = |c| \; \|\mathbf{u}\|_\infty,
\end{aligned}$$

which proves (2). For (3) we observe that

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|_\infty &= \max\{|z_1|+|w_1|, |z_2|+|w_2|, \ldots, |z_n|+|w_n|\} \\
&\le \max\{|z_1|, |z_2|, \ldots, |z_n|\} + \max\{|w_1|, |w_2|, \ldots, |w_n|\} \\
&\le \|\mathbf{u}\|_\infty + \|\mathbf{v}\|_\infty. \qquad \square
\end{aligned}$$

Unit Vectors

Unit Vector

Sometimes it is convenient to deal with vectors whose length is one. Such a vector is called a *unit vector*. We saw in Chapter 3 that it is easy to concoct a unit vector $\mathbf{u}$ in the same direction as a given nonzero vector $\mathbf{v}$ when using the standard norm, namely take

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}. \tag{6.1}$$

The same formula holds for any norm whatsoever because of norm property (2).

Example 6.4. Construct a unit vector in the direction of $\mathbf{v} = (1, -3, 2, -1)$, where the 1-norm, 2-norm, and ∞-norms are used to measure length.

Solution. We already calculated each of the norms of $\mathbf{v}$ in Example 6.1. Use these numbers in equation (6.1) to obtain unit-length vectors

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{7}(1, -3, 2, -1) \\ \mathbf{u}_2 &= \frac{1}{\sqrt{15}}(1, -3, 2, -1) \\ \mathbf{u}_\infty &= \frac{1}{3}(1, -3, 2, -1). \end{aligned}$$ □

From a geometric point of view there are certain sets of vectors in the vector space V that tell us a lot about distances. These are the so-called *balls about a vector (or point)* $\mathbf{v}_0$ *of radius* r, whose definition is as follows:

$$B_r(\mathbf{v}_0) = \{\mathbf{v} \in V \mid \|\mathbf{v} - \mathbf{v}_0\| \leq r\}.$$

Ball in Normed Space

Sometimes these are called *closed balls*, as opposed to *open balls,* which are defined using strict inequality. Here is a situation in which these balls are very helpful: imagine trying to find the distance from a given vector $\mathbf{v}_0$ to a closed (this means it contains all points on its boundary) set S of vectors that need not be a subspace. One way to accomplish this is to start with a ball centered at $\mathbf{v}_0$ such that the ball avoids S. Then expand this ball by increasing its radius until you have found a least radius r such that the ball $B_r(\mathbf{v}_0)$ intersects S nontrivially. Then the distance from $\mathbf{v}_0$ to this set is this number r. Actually, this is a reasonable *definition* of the distance from $\mathbf{v}_0$ to the set S. One expects these balls, for a given norm, to have the same shape, so it is sufficient to look at the unit balls, that is, the case $r = 1$.

Example 6.5. Sketch the unit balls centered at the origin for the 1-norm, 2-norm, and ∞-norms in the space $V = \mathbb{R}^2$.

Solution. In each case it's easiest to determine the boundary of the ball $B_1(0)$, i.e., the set of vectors $\mathbf{v} = (x, y)$ such that $\|\mathbf{v}\| = 1$. These boundaries are sketched in Figure 6.1, and the ball consists of the boundaries plus the interior of each boundary. Let's start with the familiar 2-norm. Here the boundary consists of points (x, y) such that

$$1 = \|(x, y)\|_2 = \sqrt{x^2 + y^2},$$

which is the familiar circle of radius 1 centered at the origin. Next, consider the 1-norm, in which case

$$1 = \|(x, y)\|_1 = |x| + |y|.$$

It's easier to examine this formula in each quadrant, where it becomes one of the four possibilities

$$\pm x \pm y = 1.$$

For example, in the first quadrant we get $x+y=1$. These equations give lines that connect to form a square whose sides are diagonal lines. Finally, for the ∞-norm we have

$$1 = |(x,y)|_\infty = \max\{|x|,|y|\},$$

which gives four horizontal and vertical lines $x = \pm 1$ and $y = \pm 1$. These intersect to form another square. Thus we see that the unit “balls” for the 1- and ∞-norms have corners, unlike the 2-norm. See Figure 6.1 for a picture of these balls. □

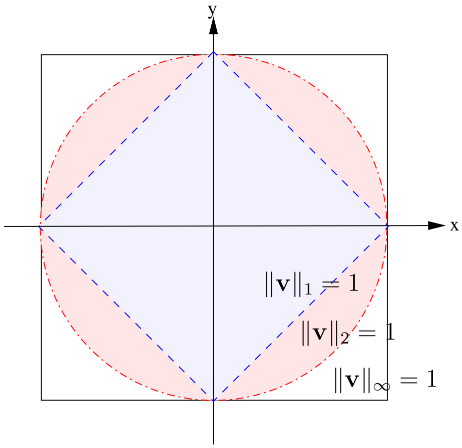


Fig. 6.1. Boundaries of unit balls in various norms.

Limit and Convergence of Vectors

Recall from Section 4.1 that one of the important applications of the norm concept is that it enables us to make sense out of the idea of limits and convergence of vectors. In a nutshell, $\lim_{n\to\infty} \mathbf{v}_n = \mathbf{v}$ was taken to mean that $\lim_{n\to\infty} \|\mathbf{v}_n - \mathbf{v}\| = 0$. In this case we said that the sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots$ *converges* to $\mathbf{v}$. Will we have to have a different notion of limits for different norms? For *finite-dimensional* spaces, the somewhat surprising answer is no. The reason is that given any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a finite-dimensional vector space, it is always possible to find positive real constants c and d such that for any vector $\mathbf{v}$,

$$\|\mathbf{v}\|_a \le c \cdot \|\mathbf{v}\|_b \text{ and } \|\mathbf{v}\|_b \le d\,\|\mathbf{v}\|_a\,.$$

Equivalent Norms

Hence, if $\|\mathbf{v}_n - \mathbf{v}\|$ tends to 0 in one norm, it will tend to 0 in the other norm. For this reason, any two norms satisfying these inequalities are called *equivalent*. It can be shown that all norms on a finite-dimensional vector space are equivalent (see Section 6.5). Indeed, it can be shown that the condition that $\|\mathbf{v}_n - \mathbf{v}\|$ tends to 0 in any one norm is equivalent to the condition that each coordinate of $\mathbf{v}_n$ converges to the corresponding coordinate of $\mathbf{v}$. We will verify the limit fact in the following example.

Example 6.6. Verify that $\lim_{n\to\infty} \mathbf{v}_n$ exists and is the same with respect to both the 1-norm and 2-norm, where

$$\mathbf{v}_n = \begin{bmatrix} (1-n)/n \\ e^{-n}+1 \end{bmatrix}.$$

Which norm is easier to work with?

Solution. First we have to know what the limit will be. Let's examine the limit in each coordinate. We have

$$\lim_{n\to\infty} \frac{1-n}{n} = \lim_{n\to\infty} \frac{1}{n} - 1 = 0 - 1 = -1 \text{ and } \lim_{n\to\infty} e^{-n} + 1 = 0 + 1 = 1.$$

So we try to use $\mathbf{v} = (-1, 1)$ as the limiting vector. Now calculate

$$\mathbf{v} - \mathbf{v}_n = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1-n}{n} \\ e^{-n}+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \\ e^{-n} \end{bmatrix},$$

so that

$$\|\mathbf{v} - \mathbf{v}_n\|_1 = \left|\frac{1}{n}\right| + \left|e^{-n}\right| \underset{n\to\infty}{\longrightarrow} 0$$

and

$$\|\mathbf{v} - \mathbf{v}_n\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(e^{-n}\right)^2} \underset{n\to\infty}{\longrightarrow} 0,$$

which shows that the limits are the same in either norm. In this case the 1-norm appears to be easier to work with, since no squaring and square roots are involved. □

Here are two examples of norms defined on nonstandard vector spaces:

Frobenius Norm

Definition 6.4. Let $V = \mathbb{R}^{m,n}$ (or $\mathbb{C}^{m,n}$). The Frobenius norm of an $m \times n$ matrix $A = [a_{ij}]$ is defined by the formula

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

We leave verification of the norm laws as an exercise.

Uniform (Infinity) Norm on Function Space

Definition 6.5. The uniform (or infinity) *norm* on $C[0,1]$ is defined by $\|f\|_\infty = \max_{0\le x\le 1} |f(x)|$.

This norm is well defined by the extreme value theorem, which guarantees that the maximum value of a continuous function on a closed interval exists. We leave verification of the norm laws as an exercise.

6.1 Exercises and Problems

Exercise 1. Find the 1-, 2-, and ∞-norms of each of the following real vectors and the distance between these pairs in each norm.

(a) $(2,1,3)$, $(-3,1,-1)$ (b) $(1,-2,0,1,3)$, $(2,2,-1,-1,-2)$

Exercise 2. Find the 1-, 2-, and ∞-norms of each of the following complex vectors and the distance between these pairs in each norm.

(a) $(1+\mathrm{i},-1,0,1)$,$(1,1,2,-4)$ (b) $(\mathrm{i},0,3-2\mathrm{i})$, $(\mathrm{i},1+\mathrm{i},0)$

Exercise 3. Find unit vectors in the direction of each of the following vectors with respect to the 1-, 2-, and ∞-norms.

(a) $(1,-3,-1)$ (b) $(3,1,-1,2)$ (c) $(2,1,3+\mathrm{i})$

Exercise 4. Find a unit vector in the direction of $f(x) \in C[0,1]$ with respect to the uniform norm, where $f(x)$ is one of the following.

(a) $\sin(\pi x)$ (b) $x(x-1)$ (c) e^x

Exercise 5. Verify the norm laws for the 1-norm in the case that $c=-2$, $\mathbf{u}=(0,2,3,1)$, and $\mathbf{v}=(1,-3,2,-1)$ in $V=\mathbb{R}^4$.

Exercise 6. Verify the norm laws for the Frobenius norm in the case that $c=-4$, $\mathbf{u}=\begin{bmatrix}1&0&-1\\1&2&0\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}-2&0&2\\1&0&-3\end{bmatrix}$ in $V=\mathbb{R}^{2,3}$.

Exercise 7. Find the distance from the point $\left(-1,-\frac{1}{2}\right)$ to the line $x+y=2$ using the ∞-norm by sketching a picture of the ball centered at that point that touches the line.

Exercise 8. Find the constant function that is nearest the function $f(x)=4x(1-x)\in V=C[0,1]$ with the infinity norm. (*Hint:* examine a graph of $f(x)$ and a constant function.)

Exercise 9. Describe in words the unit ball $B_1\left([1,1,1]^T\right)$ in the normed space $V=\mathbb{R}^3$ with the infinity norm.

Exercise 10. Describe in words the unit ball $B_1(g(x))$ in the normed space $V=C[0,1]$ with the uniform norm and $g(x)=2$.

Exercise 11. Verify that $\lim_{n\to\infty}\mathbf{v}_n$ exists and is the same with respect to both the 1- and 2-norms in $V=\mathbb{R}^2$, where $\mathbf{v}_n=((1-n)/n, e^{-n}+1)$.

Exercise 12. Calculate $\lim_{n\to\infty} f_n$ using the uniform norm on $V=C[0,1]$, where $f_n(x)=(x/2)^n+1$.

*Problem 13. Given the matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$, find the largest possible value of $\|A\mathbf{x}\|_\infty$, where $\mathbf{x}$ ranges over the vectors whose ∞-norm is 1.

*Problem 14. Verify that the 1-norm satisfies the definition of a norm.

*Problem 15. Show that the Frobenius norm satisfies the norm properties.

Problem 16. Show that the infinity norm on $C[0,1]$ satisfies the norm properties.

6.2 Inner Product Spaces

Definitions and Examples

We saw in Section 4.2 that the notion of a dot product of two vectors had many handy applications, including the determination of the angle between two vectors. This dot product amounted to the "standard" inner product of the two standard vectors. We now extend this idea to a setting that allows for abstract vector spaces.

Abstract Inner Product

Definition 6.6. An (abstract) *inner product* on the vector space V is a function $\langle \cdot, \cdot \rangle$ that assigns to each pair of vectors $\mathbf{u}, \mathbf{v} \in V$ a scalar $\langle \mathbf{u}, \mathbf{v} \rangle$ such that for c a scalar and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ the following hold:

(1) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
(2) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$
(3) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
(4) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

Inner Product Space

A vector space V, together with an inner product $\langle \cdot, \cdot \rangle$ on the space V, is called an *inner product space.*

Notice that in the case of the more common vector spaces over *real scalars*, property 2 becomes a commutative law: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. Also observe that if V is an inner product space and W is any subspace of V, then W automatically becomes an inner product space if we simply use the inner product of V on elements of W. For all the inner product laws still hold, since they hold for elements of the larger space V.

Standard Inner Products

Of course, we have the standard examples of inner products, namely the dot products on $\mathbb{R}^n$ and $\mathbb{C}^n$. Here is an example of a nonstandard inner product on a standard space that is useful in certain engineering problems.

Example 6.7. For vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $V = \mathbb{R}^2$, define an inner product by the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2.$$

Show that this formula satisfies the inner product laws.

Solution. First we see that

$$\langle \mathbf{u}, \mathbf{u} \rangle = 2u_1^2 + 3u_2^2,$$

so the only way for this sum to be 0 is for $u_1 = u_2 = 0$. Hence (1) holds. For (2) calculate

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 = 2v_1u_1 + 3v_2u_2 = \langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle},$$

since all scalars in question are real. For (3) let $\mathbf{w} = (w_1, w_2)$ and calculate

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle &= 2u_1\left(v_1+w_1\right)+3u_2\left(v_2+w_2\right)\\ &= 2u_1v_1+3u_2v_2+2u_1w_1+3u_2 = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle .\end{aligned}$$

For the last property, check that for a scalar c,

$$\langle \mathbf{u}, c\mathbf{v}\rangle = 2u_1cv_1+3u_2cv_2 = c\left(2u_1v_1+3u_2v_2\right) = c\left\langle \mathbf{u}, \mathbf{v}\right\rangle . \qquad \square$$

It follows that this "weighted" inner product is indeed an inner product according to our definition. In fact, we can do a whole lot more with even less effort. Consider this example, of which the preceding is a special case.

Example 6.8. Let A be an $n \times n$ Hermitian matrix ($A = A^*$) and define the product $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^* A\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$, where V is $\mathbb{R}^n$ or $\mathbb{C}^n$. Show that this product satisfies inner product laws (2), (3), and (4) and that if, in addition, A is positive definite, then the product satisfies (1) and is an inner product.

Solution. As usual, let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and let c be a scalar. For (2), remember that for a 1×1 scalar quantity q, $q^* = \overline{q}$, so we calculate

$$\langle \mathbf{v}, \mathbf{u}\rangle = \mathbf{v}^* A\mathbf{u} = \left(\mathbf{u}^* A\mathbf{v}\right)^* = \langle \mathbf{u}, \mathbf{v}\rangle^* = \overline{\langle \mathbf{u}, \mathbf{v}\rangle}.$$

For (3), we calculate

$$\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \mathbf{u}^* A(\mathbf{v}+\mathbf{w}) = \mathbf{u}^* A\mathbf{v} + \mathbf{u}^* A\mathbf{w} = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle .$$

For (4), we have that

$$\langle \mathbf{u}, c\mathbf{v}\rangle = \mathbf{u}^* Ac\mathbf{v} = c\mathbf{u}^* A\mathbf{v} = c\left\langle \mathbf{u}, \mathbf{v}\right\rangle .$$

Finally, if we suppose that A is also positive definite, then by definition,

$$\langle \mathbf{u}, \mathbf{u}\rangle = \mathbf{u}^* A\mathbf{u} > 0, \text{ for } \mathbf{u} \neq \mathbf{0},$$

which shows that inner product property (1) holds. Hence, this product defines an inner product. □

We leave it to the reader to check that if we take

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

then the inner product defined by this matrix is exactly the inner product of Example 6.7.

There is an important point to be gleaned from the previous example, namely, that a given vector space may have more than one inner product on it. In particular, $V = \mathbb{R}^2$ could have the standard inner product, i.e., dot product or something else like the previous example. The space V, together with each one of these inner products, provides us with two separate inner product spaces.

Here is a rather more exotic example of an inner product involving a nonstandard vector space.

Example 6.9. Let $V = C[a,b]$, the space of continuous functions on the interval $[a,b]$ with the usual function addition and scalar multiplication. Show that the formula

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

defines an inner product on the space V.

Solution. Certainly $\langle f,g\rangle$ is a real number. Now if $f(x)$ is a continuous function then $f(x)^2$ is nonnegative on $[a,b]$ and therefore $\int_0^1 f(x)^2dx = \langle f,f\rangle \geq 0$. Furthermore, if $f(x)$ is nonzero, then the area under the curve $y = f(x)^2$ must also be positive since $f(x)$ will be positive and bounded away from 0 on some subinterval of $[a,b]$. This establishes property (1) of inner products.

Now let $f(x), g(x), h(x) \in V$. For property (2), notice that

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = \langle g,f\rangle\,.$$

Also,

$$\begin{aligned}\langle f,g+h\rangle &= \int_a^b f(x)(g(x)+h(x))dx \\ &= \int_a^b f(x)g(x)dx + \int_a^b f(x)h(x)dx = \langle f,g\rangle + \langle f,h\rangle\,,\end{aligned}$$

which establishes property (3). Finally, we see that for a scalar c,

$$\langle f,cg\rangle = \int_a^b f(x)cg(x)\,dx = c\int_a^b f(x)g(x)\,dx = c\,\langle f,g\rangle\,,$$

which shows that property (4) holds. □

Function Space Standard Inner Product

We shall refer to this inner product on a function space as the *standard inner product* on the function space $C[a,b]$. (Most of our examples and exercises involving function spaces will deal with polynomials, so we remind the reader of the integration formula $\int_a^b x^m\,dx = \frac{1}{m+1}\left(b^{m+1} - a^{m+1}\right)$ and special case $\int_0^1 x^m\,dx = \frac{1}{m+1}$ for $m \geq 0$.)

Following are a few simple facts about inner products that we will use frequently. The proofs are left to the exercises.

Theorem 6.1. Let V be an inner product space with inner product $\langle\cdot,\cdot\rangle$. Then we have that for all $\mathbf{u},\mathbf{v},\mathbf{w} \in V$ and scalars a,

(1) $\langle\mathbf{u},\mathbf{0}\rangle = 0 = \langle\mathbf{0},\mathbf{u}\rangle$,
(2) $\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{w}\rangle + \langle\mathbf{v},\mathbf{w}\rangle$,
(3) $\langle a\mathbf{u},\mathbf{v}\rangle = \overline{a}\langle\mathbf{u},\mathbf{v}\rangle$.

Induced Norms and the CBS Inequality

It is a striking fact that we can accomplish all the goals we set for the standard inner product using general inner products: we can introduce the ideas of angles, orthogonality, projections, and so forth. We have already seen much of the work that has to be done, though it was stated in the context of the standard inner products. As a first step, we want to point out that every inner product has a "natural" norm associated with it.

Definition 6.7. Let V be an inner product space. For vectors $\mathbf{u} \in V$, the norm defined by the equation

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

is called the *norm induced by the inner product* $\langle \cdot, \cdot \rangle$ *on* V.

Induced Norm

As a matter of fact, this idea is not really new. Recall that we introduced the standard inner product on $V = \mathbb{R}^n$ or $\mathbb{C}^n$ with an eye toward the standard norm. At the time it seemed like a nice convenience that the norm could be expressed in terms of the inner product. It is, and so much so that we have turned this cozy relationship into a definition. Just calling the induced norm a norm doesn't make it so. Is the induced norm really a norm? We have some work to do. The first norm property is easy to verify for the induced norm: from property (1) of inner products we see that $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, with equality if and only if $\mathbf{u} = 0$. This confirms norm property (1). Norm property (2) isn't too hard either: let c be a scalar and check that

$$\|c\mathbf{u}\| = \sqrt{\langle c\mathbf{u}, c\mathbf{u} \rangle} = \sqrt{c\bar{c} \langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{|c|^2} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = |c| \, \|\mathbf{u}\| \, .$$

Norm property (3), the triangle inequality, remains. This one isn't easy to verify from first principles. We need a tool that we have seen before, the Cauchy–Bunyakovsky–Schwarz (CBS) inequality. We restate it below as the next theorem. Indeed, the very same proof that is given in Theorem 4.2 carries over word for word to general inner products over real vector spaces. We need only replace dot products $\mathbf{u} \cdot \mathbf{v}$ by abstract inner products $\langle \mathbf{u}, \mathbf{v} \rangle$. We can also replace dot products by inner products in Problem 16 of Chapter 4, which establishes CBS for complex inner products. Similarly, the proof of the triangle inequality as given in Example 4.10, carries over to establish the triangle inequality for abstract inner products. Hence property (3) of norms holds for any induced norm.

Theorem 6.2. Let V be an inner product space. For $\mathbf{u}, \mathbf{v} \in V$, if we use the inner product of V and its induced norm, then

CBS Inequality

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Henceforth, when the norm sign $\|\cdot\|$ is used in connection with a given inner product, it is understood that this norm is the induced norm of this inner product, unless otherwise stated.

Just as with the standard dot products, we can formulate the following definition thanks to the CBS inequality.

Angle Between Vectors

Definition 6.8. For vectors $\mathbf{u}, \mathbf{v} \in V$, a real inner product space, we define the *angle* between $\mathbf{u}$ and $\mathbf{v}$ to be any angle θ satisfying

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

We know that $|\langle \mathbf{u}, \mathbf{v} \rangle| \,/\, (\|\mathbf{u}\| \, \|\mathbf{v}\|) \leq 1$, so that this formula for $\cos\theta$ makes sense.

Example 6.10. Let $\mathbf{u} = (1, -1)$ and $\mathbf{v} = (1, 1)$ be vectors in $\mathbb{R}^2$. Compute an angle between these two vectors using the inner product of Example 6.7. Compare this to the angle found when one uses the standard inner product in $\mathbb{R}^2$.

Solution. According to 6.7 and the definition of angle, we have

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{2 \cdot 1 \cdot 1 + 3 \cdot (-1) \cdot 1}{\sqrt{2 \cdot 1^2 + 3 \cdot (-1)^2}\sqrt{2 \cdot 1^2 + 3 \cdot 1^2}} = \frac{-1}{5}.$$

Hence the angle in radians is

$$\theta = \arccos\left(\frac{-1}{5}\right) \approx 1.7722.$$

On the other hand, if we use the standard norm, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1 \cdot 1 + (-1) \cdot 1 = 0,$$

from which it follows that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and $\theta = \pi/2 \approx 1.5708$. □

In the previous example, it shouldn't be too surprising that we can arrive at two different values for the "angle" between two vectors. Using different inner products to measure angle is somewhat like measuring length with different norms. Next, we extend the perpendicularity idea to arbitrary inner product spaces.

Orthogonal Vectors

Definition 6.9. Two vectors $\mathbf{u}$ and $\mathbf{v}$ in the same inner product space are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Note that if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then $\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle} = 0$. Also, this definition makes the zero vector orthogonal to every other vector. It also allows us to speak of things like "orthogonal functions." One has to be careful with new ideas like this. Orthogonality in a function space is not something that can be as easily visualized as orthogonality of geometrical vectors. Inspecting the graphs of two functions may not be quite enough. If, however, graphical data is tempered with a little understanding of the particular inner product in use, orthogonality can be detected.

Example 6.11. Show that $f(x) = x$ and $g(x) = x - \frac{2}{3}$ are orthogonal elements of $C[0,1]$ with the inner product of Example 6.9 and provide graphical evidence of this fact.

Solution. According to the definition of inner product in this space,

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx = \int_0^1 x\left(x - \frac{2}{3}\right)dx = \left(\frac{x^3}{3} - \frac{x^2}{3}\right)\Bigg|_0^1 = 0.$$

It follows that f and g are orthogonal to each other. For graphical evidence, sketch $f(x)$, $g(x)$, and $f(x)g(x)$ on the interval $[0,1]$ as in Figure 6.2. The graphs of f and g are not especially enlightening; but we can see in the graph that the area below $f \cdot g$ and above the x-axis to the right of $(2/3, 0)$ seems to be about equal to the area to the left of $(2/3, 0)$ above $f \cdot g$ and below the x-axis. Therefore the integral of the product on the interval $[0,1]$ might be expected to be zero, which is indeed the case. □

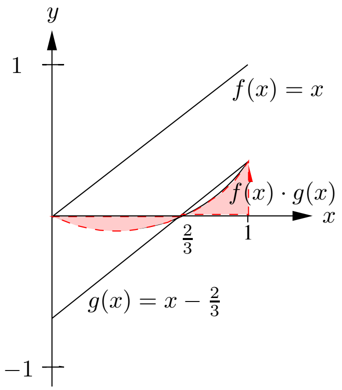


Fig. 6.2. Graphs of f, g, and $f \cdot g$ on the interval $[0,1]$.

Some of the basic ideas from geometry that fuel our visual intuition extend very elegantly to the inner product space setting. One such example is the famous Pythagorean theorem, which takes the following form in an inner product space.

Theorem 6.3. Let $\mathbf{u}, \mathbf{v}$ be orthogonal vectors in an inner product space V. Then $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$.

Pythagorean Theorem

Proof. Compute

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 . \end{aligned}$$ □

Here is an example of another standard geometrical fact that fits well in the abstract setting. This is equivalent to the law of parallelograms, which says that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of all four sides.

Law of Parallelograms

Example 6.12. Use properties of inner products to show that if we use the induced norm, then

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right).$$

Solution. The key to proving this fact is to relate induced norm to inner product. Specifically,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle,$$

while

$$\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle.$$

Now add these two equations and obtain by using the definition of induced norm again that

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{v}, \mathbf{v}\rangle = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right),$$

which is what was to be shown. □

It would be nice to think that every norm on a vector space is induced from some inner product. Unfortunately, this is not true, as the following example shows.

Example 6.13. Use the result of Example 6.12 to show that the infinity norm on $V = \mathbb{R}^2$ is not induced by any inner product on V.

Solution. Suppose the infinity norm were induced by some inner product on V. Let $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1/2)$. Then we have

$$\|\mathbf{u}+\mathbf{v}\|_\infty^2 + \|\mathbf{u}-\mathbf{v}\|_\infty^2 = \|(1, 1/2)\|_\infty^2 + \|1, -1/2\|_\infty^2 = 2,$$

while

$$2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) = 2\left(1 + 1/4\right) = 5/2.$$

This contradicts Example 6.12, so that the infinity norm cannot be induced from an inner product. □

One last example of a geometrical idea that generalizes to inner product spaces is the notion of projections of one vector along another. The projection formula for vectors in Section 4.2 works perfectly well for general inner products. Since the proof of this fact amounts to replacing dot products by inner products in the original formulation of the theorem (see Theorem 4.3), we omit it and simply state the result.

Projection Formula for Vectors

Theorem 6.4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space with $\mathbf{v} \neq \mathbf{0}$. Define the projection of $\mathbf{u}$ along $\mathbf{v}$ as

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

and let $\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}$, $\mathbf{q} = \mathbf{u} - \mathbf{p}$. Then $\mathbf{p}$ is parallel to $\mathbf{v}$, $\mathbf{q}$ is orthogonal to $\mathbf{v}$, and $\mathbf{u} = \mathbf{p} + \mathbf{q}$.

As with the standard inner product, it is customary to call the vector $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ of this theorem the *(parallel) projection of* $\mathbf{u}$ *along* $\mathbf{v}$. Likewise, components and orthogonal projections are defined as in the standard case. In summary, we have the two vector and one scalar quantities

Projection

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v},$$

Orthogonal Projection

$$\operatorname{orth}_{\mathbf{v}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u},$$

Component

$$\operatorname{comp}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|}.$$

Orthogonal Sets of Vectors

We have already seen the development of the ideas of orthogonal sets of vectors and bases in Chapter 4. Much of this development can be abstracted easily to general inner product spaces, simply by replacing dot products by inner products. Accordingly, we can make the following definition.

Orthogonal and Orthonormal Set of Vectors

Definition 6.10. The set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in an inner product space is said to be an *orthogonal set* if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ whenever $i \neq j$. If, in addition, each vector has unit length, i.e., $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$ for all i, then the set of vectors is said to be an *orthonormal set* of vectors.

The proof of the following key fact and its corollary are the same as those of Theorem 4.6 in Section 4.3. All we have to do is replace dot products by inner products. The observations that followed the proof of this theorem are valid for general inner products as well. We omit the proofs and refer the reader to Chapter 4.

Theorem 6.5. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthogonal set of nonzero vectors and suppose that $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then $\mathbf{v}$ can be expressed uniquely (up to order) as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, namely

Orthogonal Coordinates Formula

$$\mathbf{v} = \frac{\langle \mathbf{v}_1, \mathbf{v} \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{v}_2, \mathbf{v} \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{v}_n, \mathbf{v} \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle} \mathbf{v}_n.$$

Corollary 6.1. Every orthogonal set of nonzero vectors is linearly independent.

Another useful corollary is the following fact about the length of a vector, whose proof is left as an exercise.

Corollary 6.2. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal set of vectors and $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, then

$$\|\mathbf{v}\|^2 = c_1^2 \|\mathbf{v}_1\|^2 + c_2^2 \|\mathbf{v}_2\|^2 + \cdots + c_n^2 \|\mathbf{v}_n\|^2 .$$

Example 6.14. Find an orthogonal basis of $V = \mathbb{R}^2$ with respect to the inner product of Example 6.7 that includes $\mathbf{v}_1 = (1, -1)$. Calculate the coordinates of $\mathbf{v} = (1, 1)$ with respect to this basis and verify the formula of Corollary 6.2.

Solution. Recall that the inner product is given by $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$. Use the induced norm for $\|\cdot\|$. Let $\mathbf{w}$ be a nonzero solution to the equation

$$0 = \langle \mathbf{v}_1, \mathbf{w} \rangle = 2 \cdot 1 \cdot w_1 + 3 \cdot (-1)\, w_2,$$

say $\mathbf{w} = (3, 2)$. Then $\mathbf{v}_1$ and $\mathbf{v}_2 = \mathbf{w}$ are orthogonal, hence linearly independent and a basis of the two-dimensional space V. Now $\|\mathbf{v}_1\|^2 = 2 \cdot 1^2 + 3 \cdot (-1)^2 = 5$ and $\|\mathbf{v}_2\|^2 = 2 \cdot 3^2 + 3 \cdot 2^2 = 30$. The coordinates of $\mathbf{v}$ are easily calculated:

$$\begin{aligned} c_1 &= \frac{\langle \mathbf{v}_1, \mathbf{v} \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \frac{1}{5} (2 \cdot 1 \cdot 1 + 3(-1) \cdot 1) = \frac{-1}{5} \\ c_2 &= \frac{\langle \mathbf{v}_2, \mathbf{v} \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} = \frac{1}{30} (2 \cdot 3 \cdot 1 + 3 \cdot 2 \cdot 1) = \frac{2}{5}. \end{aligned}$$

From the definition we have that $\|\mathbf{v}\|^2 = 2 \cdot 1^2 + 3 \cdot 1^2 = 5$. Similarly, we calculate that

$$c_1^2 \|\mathbf{v}_1\|^2 + c_2^2 \|\mathbf{v}_2\|^2 = \left(\frac{-1}{5}\right)^2 5 + \left(\frac{2}{5}\right)^2 30 = 5 = \|\mathbf{v}\|^2 . \qquad \square$$

Note 6.1. In all exercises of this chapter use the standard inner products and induced norms for $\mathbb{R}^n$ and $C\,[a, b]$ unless otherwise specified.

6.2 Exercises and Problems

Exercise 1. Verify the Cauchy–Bunyakovsky–Schwarz inequality and calculate the angle between the vectors for the following pairs of vectors $\mathbf{u}$, $\mathbf{v}$ and specified inner product.
(a) $\mathbf{u} = (2, 3)$, $\mathbf{v} = (-1, 2)$, inner product $\langle (x, y), (w, z) \rangle = 4xw + 9yz$ on $\mathbb{R}^2$.
(b) $\mathbf{u} = x$, $\mathbf{v} = x^3$, inner product of Example 6.9 on $C\,[0, 1]$.

Exercise 2. Verify the CBS inequality and calculate the angle between the vectors for the following pairs of vectors and inner product.
(a) $(1,-1,1)$, $(-1,2,3)$, inner product $\langle(x,y,z),(u,v,w)\rangle = xu + 2yv + zw$.
(b) $(2,3)$, $(-1,2)$, inner product $\langle(x,y),(w,z)\rangle = 2xw + xz + yw + yz$.

Exercise 3. For each of the pairs $\mathbf{u},\mathbf{v}$ of vectors in Exercise 1, calculate the projection, component, and orthogonal projection of $\mathbf{u}$ to $\mathbf{v}$ using the specified inner product.

Exercise 4. For each of the pairs $\mathbf{u},\mathbf{v}$ of vectors in Exercise 2, calculate the projection, component, and orthogonal projection of $\mathbf{u}$ to $\mathbf{v}$ using the specified inner product.

Exercise 5. Find an equation for the hyperplane defined by $\langle\mathbf{a},\mathbf{x}\rangle = 2$ in $\mathbb{R}^3$ with inner product of Exercise 2(a) and $\mathbf{a} = (4,-1,2)$.

Exercise 6. Find an equation for the hyperplane defined by $\langle f,g\rangle = 2$ in $\mathcal{P}_3$ with the standard inner product of $C[0,1]$, $f(x) = x+3$, and $g(x) = c_0 + c_1x + c_2x^2 + c_3x^3$.

Exercise 7. The formula $\left\langle [x_1,x_2]^T,[y_1,y_2]^T\right\rangle = 3x_1y_1 - 2x_2y_2$ fails to define an inner product on $\mathbb{R}^2$. What laws fail?

Exercise 8. Do any inner product laws fail for the formula $\langle(x_1,x_2),(y_1,y_2)\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$ on $\mathbb{R}^2$. (*Hint:* $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.)

Exercise 9. Which of the following are orthogonal or orthonormal sets?
(a) $(2,-1,2)$, $(2,2,0)$ in $\mathbb{R}^3$ with the inner product of Exercise 2(a).
(b) 1, x, x^2 as vectors in $C[-1,1]$ with the standard inner product.
(c) $\frac{1}{5}(-2,1)$, $\frac{1}{30}(9,8)$ in $\mathbb{R}^2$ with the inner product of Exercise 1(a).

Exercise 10. Determine whether the following sets of vectors are linearly independent, orthogonal, or orthonormal.
(a) $\frac{1}{10}(3,4)$, $\frac{1}{10}(4,-3)$ in $\mathbb{R}^2$ with inner product $\langle(x,y),(w,z)\rangle = 4xw + 4yz$.
(b) 1, $\cos(x)$, $\sin(x)$ in $C[-\pi,\pi]$ with the standard inner product.
(c) $(2,4)$, $(1,0)$ in $\mathbb{R}^2$ with inner product $\langle\mathbf{x},\mathbf{y}\rangle = \mathbf{x}^T\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{y}$.

Exercise 11. Let $\mathbf{v}_1 = (1,3,2)$ and $\mathbf{v}_2 = (-4,1,-1)$. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal with respect to the inner product of Exercise 2(a) and use this to determine whether the following vectors $\mathbf{v}$ belong to $V = \operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}$ by checking whether Theorem 6.5 is satisfied.
(a) $(11,7,8)$ (b) $(5,1,3)$ (c) $(5,2,3)$

Exercise 12. Confirm that $p_1(x) = x$ and $p_2(x) = 3x^2 - 1$ are orthogonal elements of $C[-1,1]$ with the standard inner product and determine whether the following polynomials belong to span $\{p_1(x), p_2(x)\}$ using Theorem 6.5.
(a) x^2 (b) $1 + x - 3x^2$ (c) $1 + 3x - 3x^2$

Exercise 13. Let $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (-1,2,0)$, $\mathbf{v}_3 = (1,-2,3)$. Let $V = \mathbb{R}^3$ with inner product defined by the formula $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$, where $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Verify that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form an orthogonal basis of V and find the coordinates of the following vectors with respect to this basis.
(a) $(3,1,1)$ (b) $(0,0,1)$ (c) $(0,2,0)$

Exercise 14. Let $\mathbf{v}_1 = (1,3,2)$, $\mathbf{v}_2 = (-4,1,-1)$, and $\mathbf{v}_3 = (10,7,-26)$. Verify that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form an orthogonal basis of $\mathbb{R}^3$ with the inner product of Exercise 2(a). Convert this basis to an orthonormal basis and find the coordinates of the following vectors with respect to this basis.
(a) $(1,1,0)$ (b) $(2,1,1)$ (c) $(0,2,2)$ (c) $(0,0,-1)$

Exercise 15. Let $\mathbf{x} = (a,b)$ and $\mathbf{y} = (c,d)$. Let $V = \mathbb{R}^2$ with inner product defined by the formula $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$, where $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$. Calculate a formula for $\langle \mathbf{x}, \mathbf{y} \rangle$ in terms of coordinates a, b, c, d.

Exercise 16. Let $f(x) = a + bx$ and $g(x) = c + dx$. Let $V = \mathcal{P}_1$, the space of linear polynomials, with the standard function space inner product. Calculate a formula for $\langle f, g \rangle$ in terms of coordinates a, b, c, d. Compare with Exercise 15. Conclusions?

*Problem 17. Show that any inner product on $\mathbb{R}^2$ can be expressed as $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$ for some symmetric positive definite matrix A.

*Problem 18. Show that $\|\cdot\|_1$ is not an induced norm on $\mathbb{R}^2$.

*Problem 19. Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$ and let $\mathbf{u}, \mathbf{v} \in V$. Let A be a fixed $n \times n$ *nonsingular* matrix. Show that the matrix A defines an inner product by the formula $\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u})^* A\mathbf{v}$.

*Problem 20. Prove Theorem 6.1.

Problem 21. Prove Corollary 6.2.

*Problem 22. Let V be a real inner product space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Prove the *polarization identity*, which recovers the inner product from its induced norm:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4} \left\{ \|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 \right\}.$$

*Problem 23. Let $V = C^1\left[0,1\right]$, the space of continuous functions with a continuous derivative on the interval $[0,1]$ (see Exercise 22 of Section 3.2). Show that the formula

$$\langle f, g\rangle = \int_0^1 f'(x)g'(x)dx + \int_0^1 f(x)g(x)dx$$

defines an inner product on V (called the *Sobolev* inner product).

6.3 Gram–Schmidt Algorithm

We have seen that orthogonal bases have some very pleasant properties, such as easy coordinate calculations. Our goal in this section is the following: given a subspace V of some inner product space and a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ of V, to turn this basis into an orthogonal basis. The tool we need is the Gram–Schmidt algorithm.

Description of the Algorithm

Gram–Schmidt Algorithm

Theorem 6.6. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be a basis of the inner product space V. Define vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ recursively by the formula

$$\mathbf{v}_k = \mathbf{w}_k - \frac{\langle \mathbf{v}_1, \mathbf{w}_k\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{w}_k\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 - \cdots - \frac{\langle \mathbf{v}_{k-1}, \mathbf{w}_k\rangle}{\langle \mathbf{v}_{k-1}, \mathbf{v}_{k-1}\rangle}\mathbf{v}_{k-1}, \quad k = 1, \ldots, n.$$

Then

(1) The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form an orthogonal set.
(2) For each index $k = 1, \ldots, n$,

$$\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

Proof. In the case $k = 1$, we have that the single vector $\mathbf{v}_1 = \mathbf{w}_1$ is an orthogonal set and certainly $\operatorname{span}\{\mathbf{w}_1\} = \operatorname{span}\{\mathbf{v}_1\}$. Now suppose that for some index $k > 1$ we have shown that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$ is an orthogonal set such that $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k-1}\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$. Then it is true that $\langle \mathbf{v}_r, \mathbf{v}_s\rangle = 0$ for any indices r, s both less than k. Take the inner product of $\mathbf{v}_k$, as given by the formula above, with the vector $\mathbf{v}_j$, where $j < k$, and we obtain

$$\begin{aligned}\langle \mathbf{v}_j, \mathbf{v}_k\rangle &= \left\langle \mathbf{v}_j, \mathbf{w}_k - \frac{\langle \mathbf{v}_1, \mathbf{w}_k\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{w}_k\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 - \cdots - \frac{\langle \mathbf{v}_{k-1}, \mathbf{w}_k\rangle}{\langle \mathbf{v}_{k-1}, \mathbf{v}_{k-1}\rangle}\mathbf{v}_{k-1}\right\rangle \\ &= \langle \mathbf{v}_j, \mathbf{w}_k\rangle - \langle \mathbf{v}_1, \mathbf{w}_k\rangle \frac{\langle \mathbf{v}_j, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle} - \cdots - \langle \mathbf{v}_{k-1}, \mathbf{w}_k\rangle \frac{\langle \mathbf{v}_j, \mathbf{v}_{k-1}\rangle}{\langle \mathbf{v}_{k-1}, \mathbf{v}_{k-1}\rangle} \\ &= \langle \mathbf{v}_j, \mathbf{w}_k\rangle - \langle \mathbf{v}_j, \mathbf{w}_k\rangle \frac{\langle \mathbf{v}_j, \mathbf{v}_j\rangle}{\langle \mathbf{v}_j, \mathbf{v}_j\rangle} = 0.\end{aligned}$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is an orthogonal set. The Gram–Schmidt formula show us that one of $\mathbf{v}_k$ or $\mathbf{w}_k$ can be expressed as a linear combination of the other and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$. Therefore

$$\begin{aligned}\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k-1}, \mathbf{w}_k\} &= \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{w}_k\} \\ &= \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k\},\end{aligned}$$

which is the second part of the theorem. Repeat this argument for each index $k = 2, \ldots, n$ to complete the proof of the theorem. □

The Gram–Schmidt formula is easy to remember: subtract from the vector $\mathbf{w}_k$ all of the projections of $\mathbf{w}_k$ along the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$ to obtain the vector $\mathbf{v}_k$.

Example 6.15. Let $C[0,1]$ be the space of continuous functions on the interval $[0,1]$ with the usual function addition and scalar multiplication, and (standard) inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

as in Example 6.9. Let $V = \mathcal{P}_2 = \operatorname{span}\{1, x, x^2\}$ and apply the Gram–Schmidt algorithm to the basis $1, x, x^2$ to obtain an orthogonal basis for the space of quadratic polynomials.

Solution. Set $\mathbf{w}_1 = 1$, $\mathbf{w}_2 = x$, $\mathbf{w}_3 = x^2$ and calculate the Gram–Schmidt formulas:

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{w}_1 = 1, \\ \mathbf{v}_2 &= \mathbf{w}_2 - \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = x - \frac{1/2}{1}1 = x - \frac{1}{2}, \\ \mathbf{v}_3 &= \mathbf{w}_3 - \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 \\ &= x^2 - \frac{1/3}{1}1 - \frac{1/12}{1/12}(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}.\end{aligned}$$

□

Legendre Polynomials

Had we used $C[-1,1]$ and required that each polynomial have value 1 at $x = 1$, the same calculations would have given us the first three well-known functions called *Legendre polynomials.* These polynomials are used extensively in approximation theory and applied mathematics.

If we prefer to have an orthonormal basis rather than an orthogonal basis, then, as a final step in the orthogonalizing process, simply replace each vector $\mathbf{v}_k$ by the normalized vector $\mathbf{u}_k = \mathbf{v}_k / \|\mathbf{v}_k\|$. Here is an example to illustrate the whole scheme.

Example 6.16. Let $V = \mathcal{C}(A)$ with the standard inner product and compute an orthonormal basis of V, where

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & -1 & 3 & 2 \\ 1 & -1 & 3 & 2 \\ -1 & 1 & -3 & 1 \end{bmatrix}.$$

Solution. We know that V is spanned by the four columns of A. However, the Gram–Schmidt algorithm requests a basis of V and we don't know that the columns are linearly independent. We leave it to the reader to check that the reduced row echelon form of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows from the column space algorithm that columns 1, 2, and 4 of the matrix A yield a basis of V. So let $\mathbf{w}_1 = (1,1,1,-1)$, $\mathbf{w}_2 = (2,-1,-1,1)$, $\mathbf{w}_3 = (-1,2,2,1)$, and apply the Gram–Schmidt algorithm to obtain

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{w}_1 = (1,1,1,-1), \\
\mathbf{v}_2 &= \mathbf{w}_2 - \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 \\
&= (2,-1,-1,1) - \frac{-1}{4}(1,1,1,-1) = \frac{1}{4}(9,-3,-3,3), \\
\mathbf{v}_3 &= \mathbf{w}_3 - \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\
&= (-1,2,2,1) - \frac{2}{4}(1,1,1,-1) - \frac{-18}{108}(9,-3,-3,3) \\
&= \frac{1}{4}(-4,8,8,4) - \frac{1}{4}(2,2,2,-2) + \frac{1}{4}(6,-2,-2,2) = (0,1,1,2)\,.
\end{aligned}$$

Finally, normalize each vector to obtain the orthonormal basis

$$\begin{aligned}
\mathbf{u}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1,1,1,-1), \\
\mathbf{u}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{108}}(9,-3,-3,3) = \frac{1}{2\sqrt{3}}(3,-1,-1,1), \\
\mathbf{u}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{6}}(0,1,1,2).
\end{aligned}$$

□

There are several useful observations about the preceding example that are particularly helpful for hand calculations:

- If one encounters an inconvenient fraction, such as the $\frac{1}{4}$ in $\mathbf{v}_2$, replace the calculated $\mathbf{v}_2$ by $4\mathbf{v}_2$, thereby eliminating the fraction, and yet achieving the same results in subsequent calculations. The idea here is that for any nonzero scalar c,

$$\frac{\langle \mathbf{v}_2, \mathbf{w} \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 = \frac{\langle c\mathbf{v}_2, \mathbf{w} \rangle}{\langle c\mathbf{v}_2, c\mathbf{v}_2 \rangle} c\mathbf{v}_2.$$

So we could have replaced $\frac{1}{4}(9, -3, -3, 3)$ by $(3, -1, -1, 1)$ and achieved the same results.

- The same remark applies to the normalizing process, since in general,

$$\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{c\mathbf{v}_2}{\|c\mathbf{v}_2\|}.$$

The Gram–Schmidt algorithm is robust enough to handle linearly dependent spanning sets gracefully. We illustrate this fact with the following example:

Example 6.17. Suppose we had used all the columns of A in Example 6.16 instead of linearly independent ones, labeling them $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$. How would the Gram–Schmidt calculation work out?

Solution. Everything would have proceeded as above until we reached the calculation of $\mathbf{v}_3$, which would then yield

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{w}_3 - \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\
&= (0, 3, 3, -3) - \frac{9}{4}(1, 1, 1, -1) + \frac{1}{4}(9, -3, -3, 3) \\
&= \frac{1}{4}(0, 12, 12, -12) + \frac{9}{4}(-1, -1, -1, 1) - \frac{-27}{108}(9, -3, -3, 3) \\
&= (0, 0, 0, 0).
\end{aligned}$$

This tells us that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, which mirrors the fact that $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. Now discard $\mathbf{v}_3$ and continue the calculations to get that

$$\begin{aligned}
\mathbf{v}_4 &= \mathbf{w}_4 - \frac{\langle \mathbf{v}_1, \mathbf{w}_4 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{w}_4 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\
&= (-1, 2, 2, 1) - \frac{2}{4}(1, 1, 1, -1) - \frac{-18}{108}(9, -3, -3, 3) = (0, 1, 1, 2). \quad \square
\end{aligned}$$

Interestingly enough, this calculation yields the same third vector that we obtained in Example 6.16. The upshot of this calculation is that Gram–Schmidt can be applied to any spanning set, provided that one discards any zero vectors that result from the formula. The net result is still an orthogonal basis.

Application to Projections

We can use the machinery of orthogonal vectors to give a nice solution to a very practical and important question that can be phrased as follows (see Figure 6.3 for a graphical interpretation of it):

The Projection Problem: Given a finite-dimensional subspace V of a real inner product space W, together with a vector $\mathbf{b} \in W$, to find the vector $\mathbf{v} \in V$ which is closest to $\mathbf{b}$ in the sense that $\|\mathbf{b} - \mathbf{v}\|^2$ is minimized.

Observe that the quantity $\|\mathbf{b} - \mathbf{v}\|^2$ will be minimized exactly when $\|\mathbf{b} - \mathbf{v}\|$ is minimized, since the latter is always nonnegative. The squared term has the virtue of avoiding square roots that computing $\|\mathbf{b} - \mathbf{v}\|$ requires.

The projection problem looks vaguely familiar. It reminds us of the least squares problem of Chapter 4, which was to minimize the quantity $\|\mathbf{b} - A\mathbf{x}\|^2$, where A is an $m \times n$ real matrix and $\mathbf{b}, \mathbf{x}$ are standard vectors. Recall that $\mathbf{v} = A\mathbf{x}$ is a typical element in the column space of A. Therefore, the quantity to be minimized is

$$\|\mathbf{b} - A\mathbf{x}\|^2 = \|\mathbf{b} - \mathbf{v}\|^2,$$

where on the left-hand side $\mathbf{x}$ runs over all standard n-vectors and on the right-hand side $\mathbf{v}$ runs over all vectors in the space $V = \mathcal{C}(A)$. The difference between least squares and the projection problem is this: in the least squares problem we want to know the vector $\mathbf{x}$ of coefficients of $\mathbf{v}$ as a linear combination of columns of A, whereas in the projection problem we are interested only in $\mathbf{v}$. Knowing $\mathbf{v}$ doesn't tell us what $\mathbf{x}$ is, but knowing $\mathbf{x}$ easily gives $\mathbf{v}$ since $\mathbf{v} = A\mathbf{x}$.

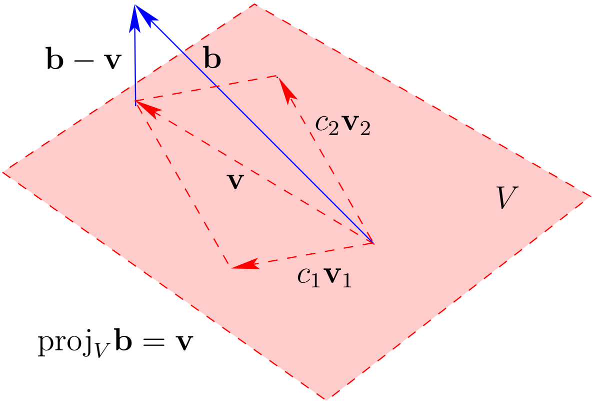

$\mathbf{b} - \mathbf{v}$
$\mathbf{b}$
$c_2\mathbf{v}_2$
$\mathbf{v}$
V
$c_1\mathbf{v}_1$
$\text{proj}_V\mathbf{b} = \mathbf{v}$

Fig. 6.3. Projection $\mathbf{v}$ of $\mathbf{b}$ into the subspace V spanned by the orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$.

To solve the projection problem we need the following key concept.

Definition 6.11. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthogonal basis for the subspace V of the inner product space W. For any $\mathbf{b} \in \mathbf{W}$, the *(parallel) projection of* $\mathbf{b}$ *into the subspace* V is the vector

Projection Formula for Subspaces

$$\text{proj}_V \mathbf{b} = \frac{\langle \mathbf{v}_1, \mathbf{b}\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 + \frac{\langle \mathbf{v}_2, \mathbf{b}\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{v}_n, \mathbf{b}\rangle}{\langle \mathbf{v}_n, \mathbf{v}_n\rangle}\mathbf{v}_n.$$

Notice that in the case of $n = 1$ the definition amounts to a familiar friend, the projection of $\mathbf{b}$ along the vector $\mathbf{v}_1$.

It appears that the definition of proj_V depends on the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, but we see from the next theorem that this is not the case.

Projection Theorem

Theorem 6.7. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthogonal basis for the subspace V of the inner product space W. For any $\mathbf{b} \in \mathbf{W}$, the vector $\mathbf{v} = \operatorname{proj}_V \mathbf{b}$ is the unique vector in V that minimizes $\|\mathbf{b} - \mathbf{v}\|^2$.

Proof. Let $\mathbf{v}$ be a solution to the projection problem and $\mathbf{p}$ the projection of $\mathbf{b} - \mathbf{v}$ to any vector in V. Use the Pythagorean theorem to obtain that

$$\|\mathbf{b} - \mathbf{v}\|^2 = \|\mathbf{b} - \mathbf{v} - \mathbf{p}\|^2 + \|\mathbf{p}\|^2 .$$

However, $\mathbf{v} + \mathbf{p} \in V$, so that $\|\mathbf{b} - \mathbf{v}\|$ cannot be the minimum distance $\mathbf{b}$ to a vector in V unless $\|\mathbf{p}\| = 0$. It follows that $\mathbf{b} - \mathbf{v}$ is orthogonal to any vector in V. Now let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an *orthogonal* basis of V and express the vector $\mathbf{v}$ in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Then for each $\mathbf{v}_k$ we must have

$$\begin{aligned} 0 &= \langle \mathbf{v}_k, \mathbf{b} - \mathbf{v} \rangle = \langle \mathbf{v}_k, \mathbf{b} - c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_n\mathbf{v}_n \rangle \\ &= \langle \mathbf{v}_k, \mathbf{b} \rangle - c_1 \langle \mathbf{v}_k, \mathbf{v}_1 \rangle - c_2 \langle \mathbf{v}_k, \mathbf{v}_2 \rangle - \cdots c_n \langle \mathbf{v}_k, \mathbf{v}_n \rangle \\ &= \langle \mathbf{v}_k, \mathbf{b} \rangle - c_k \langle \mathbf{v}_k, \mathbf{v}_k \rangle , \end{aligned}$$

from which we deduce that $c_k = \langle \mathbf{v}_k, \mathbf{b} \rangle / \langle \mathbf{v}_k, \mathbf{v}_k \rangle$. It follows that

$$\mathbf{v} = \frac{\langle \mathbf{v}_1, \mathbf{b} \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{v}_2, \mathbf{b} \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{v}_n, \mathbf{b} \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n = \operatorname{proj}_V \mathbf{b}.$$

This proves that there can be only one solution to the projection problem, namely the one given by the projection formula above.

To finish the proof one has to show that $\operatorname{proj}_V \mathbf{b}$ actually solves the projection problem. This is left to the exercises. □

The projection has the same nice properties that we observed in the case of standard inner products, namely, $\mathbf{p} = \operatorname{proj}_V \mathbf{b} \in V$ and $\mathbf{b} - \mathbf{p}$ is orthogonal to every $\mathbf{v} \in V$. For the latter assertion, notice that for any j,

$$\langle \mathbf{v}_j, \mathbf{b} - \mathbf{p} \rangle = \langle \mathbf{v}_j, \mathbf{b} \rangle - \sum_{k=1}^{n} \left\langle \mathbf{v}_j, \frac{\langle \mathbf{v}_k, \mathbf{b} \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle}\mathbf{v}_k \right\rangle = \langle \mathbf{v}_j, \mathbf{b} \rangle - \frac{\langle \mathbf{v}_j, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle} \langle \mathbf{v}_j, \mathbf{b} \rangle = 0.$$

One checks that the same is true if v_j is replaced by a $\mathbf{v} \in V$. In analogy with the standard inner products, we define the *orthogonal projection of* $\mathbf{b}$ to V by the formula

Orthogonal Projection

$$\operatorname{orth}_V \mathbf{b} = \mathbf{b} - \operatorname{proj}_V \mathbf{b}.$$

Let's specialize to standard real vectors and inner products and take a closer look at the formula for the projection operator in the case that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthonormal set. We then have $\langle \mathbf{v}_j, \mathbf{v}_j \rangle = 1$, so

$$\begin{aligned}\operatorname{proj}_V \mathbf{b} &= \langle \mathbf{v}_1, \mathbf{b} \rangle \mathbf{v}_1 + \langle \mathbf{v}_2, \mathbf{b} \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{v}_n, \mathbf{b} \rangle \mathbf{v}_n \\ &= \left(\mathbf{v}_1^T \mathbf{b}\right) \mathbf{v}_1 + \left(\mathbf{v}_2^T \mathbf{b}\right) \mathbf{v}_2 + \cdots + \left(\mathbf{v}_n^T \mathbf{b}\right) \mathbf{v}_n \\ &= \mathbf{v}_1 \mathbf{v}_1^T \mathbf{b} + \mathbf{v}_2 \mathbf{v}_2^T \mathbf{b} + \cdots + \mathbf{v}_n \mathbf{v}_n^T \mathbf{b} \\ &= \left(\mathbf{v}_1 \mathbf{v}_1^T + \mathbf{v}_2 \mathbf{v}_2^T + \cdots + \mathbf{v}_n \mathbf{v}_n^T\right) \mathbf{b} \\ &= P\mathbf{b}.\end{aligned}$$

Thus we have the following expression for the matrix P :

Projection Matrix Formula

$$P = \mathbf{v}_1 \mathbf{v}_1^T + \mathbf{v}_2 \mathbf{v}_2^T + \cdots + \mathbf{v}_n \mathbf{v}_n^T.$$

The significance of this expression for projections in standard spaces over the reals with the standard inner product is as follows: computing the projection of a vector into a subspace amounts to multiplying the vector by a matrix P that can be computed from V. Even in the one-dimensional case this gives us a new slant on projections:

$$\operatorname{proj}_V \mathbf{u} = (\mathbf{v}\mathbf{v}^T)\mathbf{u} = P\mathbf{u}.$$

Similarly, we see that the orthogonal projection has a matrix representation as

$$\operatorname{orth}_V \mathbf{u} = \mathbf{u} - P\mathbf{u} = (I - P)\,\mathbf{u}.$$

The general projection matrix P has some interesting properties. It is symmetric, i.e., $P^T = P$, and idempotent, i.e., $P^2 = P$. Therefore, this notation is compatible with the definition of projection matrix introduced in earlier exercises (see Exercise 11 of Section 4.3). Symmetry follows from the fact that $\left(\mathbf{v}_k \mathbf{v}_k^T\right)^T = \mathbf{v}_k \mathbf{v}_k^T$. For idempotence, notice that

$$\left(\mathbf{v}_j \mathbf{v}_j^T\right)\left(\mathbf{v}_k \mathbf{v}_k^T\right) = \left(\mathbf{v}_j^T \mathbf{v}_k\right)\left(\mathbf{v}_k \mathbf{v}_j^T\right) = \delta_{j,k} \mathbf{v}_k \mathbf{v}_j^T.$$

It follows that $P^2 = P$. One can show that the converse is true: if P is real symmetric and idempotent, then it is the projection matrix for the subspace $\mathcal{C}(P)$ (see Problem 16 at the end of this section.)

Example 6.18. Find the projection matrix for the subspace of $\mathbb{R}^3$ spanned by the orthonormal vectors $\mathbf{v}_1 = (1/\sqrt{2})[1, -1, 0]^T$ and $\mathbf{v}_2 = (1/\sqrt{3})[1, 1, 1]^T$ and use it to solve the projection problem with $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{b} = [2, 1, -3]^T$.

Solution. Use the formula developed above for the projection matrix

$$P = \mathbf{v}_1 \mathbf{v}_1^T + \mathbf{v}_2 \mathbf{v}_2^T = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}[1\ {-1}\ 0] + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}[1\ 1\ 1] = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Thus the solution to the projection problem for $\mathbf{b}$ is given by

$$\mathbf{v} = P\mathbf{b} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}. \qquad \square$$

The projection problem is closely related to another problem that we have seen before, namely the least squares problem of Section 4.2 in Chapter 4. Recall that the least squares problem amounted to minimizing the function $f(x) = \|\mathbf{b} - A\mathbf{x}\|^2$, which in turn led to the normal equations. Here A is an $m \times n$ real matrix. Now consider the projection problem for the subspace $V = \mathcal{C}(A)$ of $\mathbb{R}^m$, where $\mathbf{b} \in \mathbb{R}^m$. We know that elements of $\mathcal{C}(A)$ can be written in the form $\mathbf{v} = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$. Therefore, $\|\mathbf{b} - A\mathbf{x}\|^2 = \|\mathbf{b} - \mathbf{v}\|^2$, where $\mathbf{v}$ ranges over elements of V. It follows that when we solve a least squares problem, we are really solving a projection problem as well in the sense that the vector $A\mathbf{x}$ is the element of $\mathcal{C}(A)$ closest to the right-hand-side vector $\mathbf{b}$.

Least Squares as Projection Problem

The normal equations give us another way to generate projection matrices in the case of standard vectors and inner products. As above, let $V = \mathcal{C}(A) \subseteq \mathbb{R}^m$, and $\mathbf{b} \in \mathbb{R}^m$. Assume that the columns of A are linearly independent, i.e., that A has full column rank. Then, as we have seen in Theorem 4.5, the matrix A^TA is invertible and the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$ have the unique solution

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}.$$

Consequently, the solution to the projection problem is

$$\mathbf{v} = A\mathbf{x} = A(A^TA)^{-1}A^T\mathbf{b}.$$

Column Space Projection Formula

It is also true that $\mathbf{v} = P\mathbf{b}$; since this holds for all vectors $\mathbf{b}$, it follows that the projection matrix for this subspace is given by the formula

$$P = A(A^TA)^{-1}A^T.$$

Example 6.19. Find the projection matrix for the subspace $V = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ of $\mathbb{R}^3$ with $\mathbf{w}_1 = (1, -1, 0)$ and $\mathbf{w}_2 = (2, 0, 1)$.

Solution. Let $A = [\mathbf{w}_1, \mathbf{w}_2]$, so that

$$A^TA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}.$$

Thus

$$\begin{aligned} P &= A(A^TA)^{-1}A^T \\ &= \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}. \qquad \square \end{aligned}$$

Curiously, this is exactly the same matrix as the projection matrix found in the preceding example. What is the explanation? Notice that $\mathbf{w}_1 = \sqrt{2}\mathbf{v}_1$ and $\mathbf{w}_2 = \sqrt{2}\mathbf{v}_1 + \sqrt{3}\mathbf{v}_2$, so that $V = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Hence the subspaces of both examples, though specified by different bases, are the same subspace. Therefore we should expect the projection operators to be the same.

6.3 Exercises and Problems

Exercise 1. Apply the Gram–Schmidt algorithm to the columns of the following matrices in left-to-right order using the standard inner product.

$$\text{(a)} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ -1 & 2 & 1 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ -1 & 1 & 1 & 3 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 2. Apply the Gram–Schmidt algorithm to the following vectors using the specified inner product:
(a) $(1, -2, 0)$, $(0, 1, 1)$, $(1, 0, 2)$ in $\mathbb{R}^3$, inner product of Exercise 2, Section 6.2.
(b) $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$ in $\mathbb{R}^3$, inner product of Exercise 13, Section 6.2.
(c) 1, x, x^2 in $C^1[0, 1]$, inner product of Problem 23, Section 6.2.

Exercise 3. Find the projection matrix for the column space of each of the following matrices using the projection matrix formula (you will need an orthonormal basis).

$$\text{(a)} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ -1 & 2 & 0 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Exercise 4. Redo Exercise 3 using the column space projection formula (remember to use a matrix of full column rank for this formula, so you may have to discard columns).

Exercise 5. Let $V = \operatorname{span}\{(1, -1, 1), (1, 1, 0)\} \subseteq \mathbb{R}^3$. Compute $\operatorname{proj}_V \mathbf{w}$ and $\operatorname{orth}_V \mathbf{w}$ for the following $\mathbf{w}$.
(a) $(4, -1, 2)$ (b) $(1, 1, 1)$ (c) $(0, 0, 1)$

Exercise 6. Repeat Exercise 5 using the inner product $\langle (x, y, z), (u, v, w) \rangle = 2xu - xv - yu + 3yv + zw$.

Exercise 7. Find the projection of the polynomial $f(x) = x^3$ into the subspace $V = \operatorname{span}\{1, x\}$ of $C[0, 1]$ with the standard inner product and calculate $\|f - \operatorname{proj}_V f\|$.

Exercise 8. Repeat Exercise 7 using the Sobolev inner product of Problem 23, Section 6.2.

Exercise 9. Use the Gram–Schmidt algorithm to expand the orthogonal vectors $\mathbf{w}_1 = (-1, -1, 1, 1)$ and $\mathbf{w}_2 = (1, 1, 1, 1)$ to an orthogonal basis of $\mathbb{R}^4$ (you will need to supply additional vectors).

Exercise 10. Expand the unit vector $w_1 = \frac{1}{3}(1, 1, 1)$ of $\mathbb{R}^3$ to an orthonormal basis of $\mathbb{R}^3$ with the inner product $\langle (x, y, z), (u, v, w) \rangle = 2xu + 2yv + zw$.

Exercise 11. Show that the matrices $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 2 \\ 1 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 1 \\ 5 & 7 & 0 \\ 2 & -5 & 3 \end{bmatrix}$ have the same column space by computing the projection matrices into these column spaces.

Exercise 12. Use projection matrices to determine whether the row spaces of the matrices $A = \begin{bmatrix} 3 & -4 & 7 & 2 \\ 0 & 5 & -5 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & -3 & 4 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix}$ are equal; if not, exhibit vectors in one space but not the other, if possible.

Problem 13. Show that if P is a projection matrix, then so is $I - P$.

*Problem 14. Show that if $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is an orthonormal basis of $\mathbb{R}^3$, then $\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \mathbf{u}_3\mathbf{u}_3^T = I_3$.

Problem 15. Assume A has full column rank. Verify directly that if $P = A(A^TA)^{-1}A^T$, then P is symmetric and idempotent.

*Problem 16. Show that if P is an $n \times n$ projection matrix, then for every $\mathbf{v} \in \mathbb{R}^n$, $P\mathbf{v} \in \mathcal{C}(P)$ and $\mathbf{v} - P\mathbf{v}$ is orthogonal to every element of $\mathcal{C}(P)$.

Problem 17. Write out a proof of the Gram–Schmidt algorithm in the case that $n = 3$.

*Problem 18. Complete the proof of the Projection theorem (Theorem 6.7) by showing that $\operatorname{proj}_V \mathbf{b}$ solves the projection problem.

Problem 19. How does the orthogonal projection formula on page 329 have to be changed if the vectors in question are complex? Illustrate your answer with the orthonormal vectors $\mathbf{v}_1 = ((1+\mathrm{i})/2, 0, (1+\mathrm{i})/2)$, $\mathbf{v}_2 = (0, 1, 0)$ in $\mathbb{C}^2$.

Problem 20. Let $W = C[-1, 1]$ with the standard function space inner product. Suppose V is the subspace of linear polynomials and $\mathbf{b} = e^x$.
(a) Find an orthogonal basis for V.
(b) Find the projection $\mathbf{p}$ of $\mathbf{b}$ into V.
(c) Compute the "mean error of approximation" and compare it to the mean error of approximation $\|\mathbf{b} - \mathbf{q}\|$, where $\mathbf{q}$ is the first-degree Taylor series of $\mathbf{b}$ centered at 0.
(d) Use a CAS or MAS to plot $\mathbf{b} - \mathbf{p}$ and $\mathbf{b} - \mathbf{q}$.

6.4 Linear Systems Revisited

Once again we revisit our old friend, $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix. The notions of orthogonality can shed still more light on the nature of this system of equations, especially in the case of a homogeneous system $A\mathbf{x} = \mathbf{0}$. The kth entry of the column vector $A\mathbf{x}$ is simply the kth row of A multiplied by the column vector $\mathbf{x}$. Designate this row by $\mathbf{r}_k^T$, and we see that

$$\mathbf{r}_k \cdot \mathbf{x} = 0, \qquad k = 1, \dots, n.$$

In other words, $A\mathbf{x} = 0$, that is, $\mathbf{x} \in \mathcal{N}(A)$, precisely when $\mathbf{x}$ is orthogonal (with the standard inner product) to every row of A. We will see in Theorem 6.10 below that this means that $\mathbf{x}$ will be orthogonal to any linear combination of the rows of A. Thus, we could say

$$\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r} \cdot \mathbf{x} = 0 \text{ for every } \mathbf{r} \in \mathcal{R}(A)\}. \tag{6.2}$$

We are going to digress and put this equation in a more general context. Then we will return to linear systems with a new perspective on their meaning.

Orthogonal Complements and Homogeneous Systems

Definition 6.12. Let V be a subspace of an inner product space W. Then the *orthogonal complement* of V in W is the set

Orthogonal Complement

$$V^\perp = \{\mathbf{w} \in W \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{v} \in V\}.$$

We can see from the subspace test that $V^\perp$ is a subspace of W. Recall that if U and V are two subspaces of the vector space W, then two other subspaces that we can construct are the *intersection* and *sum* of these subspaces. The former is just the set intersection of the two subspaces, and the latter is the set of elements of the form $\mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in U$ and $\mathbf{v} \in V$. One can use the subspace test to verify that these are indeed subspaces of W (see Problem 17 of Section 3.2). In fact, it isn't too hard to see that $U + V$ is the smallest space containing all elements of both U and V. Basic facts about the orthogonal complement of V are summarized as follows.

Theorem 6.8. Let V be a subspace of the finite-dimensional inner product space W. Then the following are true:

(1) $V^\perp$ is a subspace of W.
(2) $V \cap V^\perp = \{\mathbf{0}\}$.
(3) $V + V^\perp = W$.
(4) $\dim V + \dim V^\perp = \dim W$.
(5) $\left(V^\perp\right)^\perp = V$.

Proof. We leave (1) and (2) as exercises. To prove (3), we notice that $V + V^{\perp} \subseteq W$ since W is closed under sums. Now suppose that $\mathbf{w} \in W$. Let $\mathbf{v} = \operatorname{proj}_V \mathbf{w}$. We know that $\mathbf{v} \in V$ and $\mathbf{w} - \mathbf{v}$ is orthogonal to every element of V. It follows that $\mathbf{w} - \mathbf{v} \in V^{\perp}$. Therefore every element of W can be expressed as a sum of an element in V and an element in $V^{\perp}$. This shows that $W \subseteq V + V^{\perp}$, from which it follows that $V + V^{\perp} = W$.

To prove (4), let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be a basis of V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s$ a basis of $V^{\perp}$. Certainly the union of the two sets spans V because of (3). Now if there were an equation of linear dependence, we could gather all terms involving $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ on one side of the equation, those involving $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s$ on the other side, and deduce that each is equal to zero separately, in view of (2). It follows that the union of these two bases must be an independent set. Therefore it forms a basis of W. It follows that $\dim W = r + s = \dim V + \dim V^{\perp}$.

Finally, apply (4) to $V^{\perp}$ in place of V and obtain that $\dim \left(V^{\perp}\right)^{\perp} = \dim W - \dim V^{\perp}$. But (4) implies directly that $\dim V = \dim W - \dim V^{\perp}$, so that $\dim \left(V^{\perp}\right)^{\perp} = \dim V$. Now if $\mathbf{v} \in V$, then certainly $\langle \mathbf{w}, \mathbf{v} \rangle = 0$ for all $\mathbf{w} \in V^{\perp}$. Hence $V \subseteq \left(V^{\perp}\right)^{\perp}$. Since these two spaces have the same dimension, they must be equal, which proves (5). □

Orthogonal complements of the sum and intersections of two subspaces have an interesting relationship to each other, whose proofs we leave as exercises.

Theorem 6.9. Let U and V be subspaces of the inner product space W. Then the following are true:

(1) $(U \cap V)^{\perp} = U^{\perp} + V^{\perp}$.
(2) $(U + V)^{\perp} = U^{\perp} \cap V^{\perp}$.

The following fact greatly simplifies the calculation of an orthogonal complement. It says that a vector is orthogonal to every element of a vector space if and only if it is orthogonal to every element of a spanning set of the space.

Theorem 6.10. Let $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a subspace of the inner product space W. Then

$$V^{\perp} = \{\mathbf{w} \in W \mid \langle \mathbf{w}, \mathbf{v}_j \rangle = 0,\ j = 1, 2, \ldots, n\}.$$

Proof. Let $\mathbf{v} \in V$, so that for some scalars $c_1, c_2, \ldots, c_n$,

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Take the inner product of both sides with a vector $\mathbf{w}$. We see by the linearity of inner products that

$$\langle \mathbf{w}, \mathbf{v} \rangle = c_1\langle \mathbf{w}, \mathbf{v}_1 \rangle + c_2\langle \mathbf{w}, \mathbf{v}_2 \rangle + \cdots + c_n\langle \mathbf{w}, \mathbf{v}_n \rangle,$$

so that if $\langle \mathbf{w}, \mathbf{v}_j \rangle = 0$ for each j then certainly $\langle \mathbf{w}, \mathbf{v} \rangle = 0$. Conversely, if $\langle \mathbf{w}, \mathbf{v}_j \rangle = 0$, for $j = 1, 2, \ldots, n$, then clearly $\langle \mathbf{w}, \mathbf{v}_j \rangle = 0$, which proves the theorem. □

Example 6.20. Compute $V^\perp$, where

$$V = \operatorname{span}\{(1,1,1,1), (1,2,1,0)\} \subseteq \mathbb{R}^4$$

with the standard inner product on $\mathbb{R}^4$.

Solution. Form the matrix A with the two spanning vectors of V as rows. According to Theorem 6.10, $V^\perp$ is the null space of this matrix. We have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[E_{21}(-1)]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

from which it follows that the null space of A consists of vectors of the form

$$\begin{bmatrix} -x_3 - 2x_4 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $V^\perp = \operatorname{span}\{(-1,0,1,0), (-2,1,0,1)\}$. □

Nothing prevents us from considering more exotic inner products as well. The arithmetic may be a bit more complicated, but the underlying principles are the same. Here is such an example.

Example 6.21. Let $V = \operatorname{span}\{1, x\} \subset W = \mathcal{P}_2$, where the space $\mathcal{P}_2$ of polynomials of degree at most 2 has the same standard inner product as $C[0,1]$. Compute $V^\perp$ and use this to verify that $\dim V + \dim V^\perp = \dim W$.

Solution. According to Theorem 6.10, $V^\perp$ consists of those polynomials $p(x) = c_0 + c_1 x + c_2 x^2$ for which

$$0 = \langle p, 1 \rangle = \int_0^1 \left(c_0 + c_1 x + c_2 x^2\right) 1\, dx = c_0 \int_0^1 1\, dx + c_1 \int_0^1 x\ dx + c_2 \int_0^1 x^2\, dx,$$

$$0 = \langle p, x \rangle = \int_0^1 \left(c_0 + c_1 x + c_2 x^2\right) x\, dx = c_0 \int_0^1 x\, dx + c_1 \int_0^1 x^2\ dx + c_2 \int_0^1 x^3\, dx.$$

Integrate, and we obtain the system of equations

$$\begin{aligned} c_0 + \frac{1}{2} c_1 + \frac{1}{3} c_2 &= 0, \\ \frac{1}{2} c_0 + \frac{1}{3} c_1 + \frac{1}{4} c_2 &= 0. \end{aligned}$$

Solve this system to obtain $c_0 = \frac{1}{6} c_2$, $c_1 = -c_2$, and c_2 is free. Therefore, $V^\perp$ consists of polynomials of the form

$$p(x) = \frac{1}{6}c_2 - c_2x + c_2x^2 = c_2\left(\frac{1}{6} - x + x^2\right).$$

It follows that $V^\perp = \operatorname{span}\left\{\frac{1}{6} - x + x^2\right\}$. In particular, $\dim V^\perp = 1$, and since $\{1, x\}$ is a linearly independent set, $\dim V = 2$. Therefore, $\dim V + \dim V^\perp = \dim \mathcal{P}_2 = \dim W$. □

Finally, we return to solutions to the homogeneous system $A\mathbf{x} = 0$. We have seen that the null space of A consists of elements that are orthogonal to the rows of A. One could turn things around and ask what we can say about a vector that is orthogonal to every element of the null space of A. This question has a surprisingly simple answer. In fact, there is a fascinating interplay between row spaces, column spaces, and null spaces that can be summarized in the following theorem:

Orthogonal Complements Theorem

Theorem 6.11. For a matrix A,

(1) $\mathcal{R}(A)^\perp = \mathcal{N}(A)$.
(2) $\mathcal{N}(A)^\perp = \mathcal{R}(A)$.
(3) $\mathcal{N}(A^T)^\perp = \mathcal{C}(A)$.

Proof. We have already seen item (1) in the discussion at the beginning of this section, where it was stated in equation (6.2). For item (2) we take orthogonal complements of both sides of (1) and use part (5) of Theorem 6.8 to obtain that

$$\mathcal{N}(A)^\perp = \left(\mathcal{R}(A)^\perp\right)^\perp = \mathcal{R}(A),$$

which proves (2). Finally, for (3) we observe that $\mathcal{R}(A^T) = \mathcal{C}(A)$. Apply (2) with A^T in place of A and the result follows. □

The connections spelled out by this theorem are powerful ideas. Here is one example of how they can be used. Consider the following problem: suppose we are given subspaces U and V of the standard space $\mathbb{R}^n$ with the standard inner product (the dot product) in some concrete form, and we want to compute a basis for the subspace $U \cap V$. How do we proceed? One answer is to use part (1) of Theorem 6.9 to see that $(U \cap V)^\perp = U^\perp + V^\perp$. Now use part (5) of Theorem 6.8 to obtain that

$$U \cap V = (U \cap V)^{\perp\perp} = (U^\perp + V^\perp)^\perp.$$

The strategy that this equation suggests is this: Express U and V as row spaces of matrices and compute bases for the null spaces of each. Put these bases together to obtain a spanning set for $U^\perp + V^\perp$. Use this spanning set as the rows of a matrix B. Then the complement of this space is, on the one hand, $U \cap V$, but by part (1) of the orthogonal complements theorem, it is also $\mathcal{N}(B)$. Therefore $U \cap V = \mathcal{N}(B)$, so all we have to do is calculate a basis for $\mathcal{N}(B)$, which we know how to do.

Example 6.22. Find a basis for $U \cap V$, where these subspaces of $\mathbb{R}^4$ are given as follows:

$$\begin{aligned} U &= \operatorname{span}\{(1,2,1,2),(0,1,0,1)\} \\ V &= \operatorname{span}\{(1,1,1,1),(1,2,1,0)\}. \end{aligned}$$

Solution. We have already determined in Example 6.20 that $V^\perp$ has a basis $(-1,0,1,0)$ and $(-2,1,0,1)$. Form the matrix A with the two spanning vectors of U as rows. By Theorem 6.10, $U^\perp = \mathcal{N}(A)$. We have

$$A = \begin{bmatrix} 1\,2\,1\,2 \\ 0\,1\,0\,1 \end{bmatrix} \xrightarrow[E_{12}(-2)]{} \begin{bmatrix} 1\,0\,1\,0 \\ 0\,1\,0\,1 \end{bmatrix},$$

from which it follows that the null space of A consists of vectors of the form

$$\begin{bmatrix} -x_3 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $U^\perp$ has basis $(-1,0,1,0)$ and $(0,-1,0,1)$. The vector $(-1,0,1,0)$ of this basis is repeated in the basis of $V^\perp$, so we only to need list it once. Form the matrix B whose rows are $(-1,0,1,0)$, $(-2,1,0,1)$, and $(0,-1,0,1)$, then calculate the reduced row echelon form of B:

$$B = \begin{bmatrix} -1 & 0\,1\,0 \\ -2 & 1\,0\,1 \\ 0 & -1\,0\,1 \end{bmatrix} \xrightarrow[\substack{E_{21}(-2) \\ E_1(-1)}]{} \begin{bmatrix} 1 & 0\,-1\,0 \\ 0 & 1\,-2\,1 \\ 0 & -1\;\;0\,1 \end{bmatrix}$$

$$\xrightarrow[E_{32}(1)]{} \begin{bmatrix} 1\,0\,-1\,0 \\ 0\,1\,-2\,1 \\ 0\,0\,-2\,2 \end{bmatrix} \xrightarrow[\substack{E_3(-1/2) \\ E_{23}(2) \\ E_{13}(1)}]{} \begin{bmatrix} 1\,0\,0\,-1 \\ 0\,1\,0\,-1 \\ 0\,0\,1\,-1 \end{bmatrix}.$$

It follows that $\mathcal{N}(B)$ consists of vectors of the form

$$\begin{bmatrix} x_4 \\ x_4 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, $U \cap V = \mathcal{N}(B)$ is a one-dimensional space spanned by the vector $(1,1,1,1)$. □

Our last application of the orthogonal complements theorem is another Fredholm alternative theorem (compare this to Corollary 2.3.)

Corollary 6.3. Given a square real linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq 0$, either the system is consistent or there is a solution $\mathbf{y}$ to the homogeneous system $A^T\mathbf{y} = \mathbf{0}$ such that $\mathbf{y}^T\mathbf{b} \neq 0$.

Fredholm Alternative

Proof. Let $V = \mathcal{C}(A)$. By (3) of Theorem 6.8, $\mathbb{R}^n = V + V^{\perp}$, where $\mathbb{R}^n$ has the standard inner product. From (3) of the orthogonal complements theorem, $\mathcal{C}(A) = \mathcal{N}(A^T)^{\perp}$. Take complements again and use (5) of Theorem 6.8 to get that $V^{\perp} = \mathcal{N}(A^T)$. Now the system either has a solution or does not. If the system has no solution, then by Theorem 3.14, $\mathbf{b}$ does not belong to $V = \mathcal{C}(A)$. Since $\mathbf{b} \notin V$, we can write $\mathbf{b} = \mathbf{v} + \mathbf{y}$, where $\mathbf{y} \neq 0$, $\mathbf{y} \in V^{\perp}$ and $\mathbf{v} \in V$. It follows that

$$\langle \mathbf{y}, \mathbf{b} \rangle = \mathbf{y} \cdot \mathbf{b} = \mathbf{y} \cdot (\mathbf{v} + \mathbf{y}) = 0 + \mathbf{y} \cdot \mathbf{y} \neq 0.$$

On the other hand, if the system has a solution $\mathbf{x}$, then for any vector $\mathbf{y} \in \mathcal{N}(A)$ we have $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b}$. It follows that if $\mathbf{y}^T A = \mathbf{0}$, then $\mathbf{y}^T \mathbf{b} = 0$. This completes the proof. □

The QR Factorization

We are going to use orthogonality ideas to develop one more way of solving the linear system $A\mathbf{x} = \mathbf{b}$, where the $m \times n$ real matrix A has full column rank. In fact, if the system is inconsistent, then this method will find the unique least squares solution to the system. Here is the basic idea: express the matrix A in the form $A = QR$, where the columns of the $m \times n$ matrix Q are orthonormal vectors and the $n \times n$ matrix R is upper triangular with nonzero diagonal entries. Such a factorization of A is called a *QR factorization* of A. It follows that the product $Q^T Q$ is equal to I_n. Now multiply both sides of the linear system on the left by Q^T to obtain that

$$Q^T A\mathbf{x} = Q^T QR\mathbf{x} = IR\mathbf{x} = R\mathbf{x} = Q^T b.$$

The net result is a simple square system with a triangular matrix, which we can solve by back solving. That is, we use the last equation to solve for x_n, then the next to the last to solve for x_{n-1}, and so forth. This is the back solving phase of Gaussian elimination as we first learned it in Chapter 1, before we were introduced to Gauss–Jordan elimination.

One has to wonder why we have any interest in such a factorization, since we already have Gauss–Jordan elimination for system solving. Furthermore, it can be shown that finding a QR factorization is harder by a factor of about 2, that is, requires about twice as many floating-point operations to accomplish. So why bother? There are many answers. For one, it can be shown that using the QR factorization has an advantage of higher accuracy than Gauss–Jordan elimination in certain situations. For another, QR factorization gives us a method for solving least squares problems. We'll see an example of this method at the end of this section.

Where can we find such a factorization? As a matter of fact, we already have the necessary tools, compliments of the Gram–Schmidt algorithm. To explain matters, let's suppose that we have a matrix $A = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$ with linearly independent columns. Application of the Gram–Schmidt algorithm leads to orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ by the following formulas:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{w}_1 \\
\mathbf{v}_2 &= \mathbf{w}_2 - \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{w}_3 - \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2.
\end{aligned}$$

Next, solve for $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ in the above equations to obtain

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 \\
\mathbf{w}_2 &= \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \mathbf{v}_2 \\
\mathbf{w}_3 &= \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 + \mathbf{v}_3.
\end{aligned}$$

In matrix form, these equations become

$$A = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \begin{bmatrix} 1 & \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} & \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \\ 0 & 1 & \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \\ 0 & 0 & 1 \end{bmatrix}.$$

Now normalize the $\mathbf{v}_j$'s by setting $\mathbf{q}_j = \mathbf{v}_j / \|\mathbf{v}_j\|$ and observe that

$$\begin{aligned}
A &= [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \begin{bmatrix} \|\mathbf{v}_1\| & 0 & 0 \\ 0 & \|\mathbf{v}_2\| & 0 \\ 0 & 0 & \|\mathbf{v}_3\| \end{bmatrix} \begin{bmatrix} 1 & \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} & \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \\ 0 & 1 & \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \\ 0 & 0 & 1 \end{bmatrix} \\
&= [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \begin{bmatrix} \|\mathbf{v}_1\| & \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\|\mathbf{v}_1\|} & \frac{\langle \mathbf{v}_1, \mathbf{w}_3 \rangle}{\|\mathbf{v}_1\|} \\ 0 & \|\mathbf{v}_2\| & \frac{\langle \mathbf{v}_2, \mathbf{w}_3 \rangle}{\|\mathbf{v}_2\|} \\ 0 & 0 & \|\mathbf{v}_3\| \end{bmatrix}.
\end{aligned}$$

This gives our QR factorization, which can be alternatively written as

$$A = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \begin{bmatrix} \|\mathbf{v}_1\| & \langle \mathbf{q}_1, \mathbf{w}_2 \rangle & \langle \mathbf{q}_1, \mathbf{w}_3 \rangle \\ 0 & \|\mathbf{v}_2\| & \langle \mathbf{q}_2, \mathbf{w}_3 \rangle \\ 0 & 0 & \|\mathbf{v}_3\| \end{bmatrix} = QR.$$

In general, the columns of A are linearly independent exactly when A has full column rank. It is easy to see that the argument we have given extends to any such matrix, so we have the following theorem.

Theorem 6.12. If A is an $m \times n$ full-column-rank matrix, then $A = QR$, where the columns of the $m \times n$ matrix Q are orthonormal vectors and the $n \times n$ matrix R is upper triangular with nonzero diagonal entries.

QR Factorization

Example 6.23. Let the full-column-rank matrix A be given as

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

Find a QR factorization of A and use this to find the least squares solution to the problem $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (1, 1, 1, 1)$. What is the norm of the residual $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ in this problem?

Solution. Notice that the columns of A are just the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of Example 6.16. Furthermore, the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ calculated in that example are just the $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ that we require. Thus we have from those calculations that

$$\|\mathbf{v}_1\| = \|(1,1,1,-1)\| = 2 \text{ and } \mathbf{q}_1 = \frac{1}{2}(1,1,1,-1),$$

$$\|\mathbf{v}_2\| = \left\| \frac{1}{4}(9,-3,-3,3) \right\| = \frac{3}{2}\sqrt{3} \text{ and } \mathbf{q}_2 = \frac{1}{2\sqrt{3}}(3,-1,-1,1),$$

$$\|\mathbf{v}_3\| = \|(0,1,1,2)\| = \sqrt{6} \text{ and } \mathbf{q}_3 = \frac{1}{\sqrt{6}}(0,1,1,2).$$

Now we calculate

$$\begin{aligned} \langle \mathbf{q}_1, \mathbf{w}_2 \rangle &= \frac{1}{2}(1,1,1,-1) \cdot (2,-1,-1,1) = -\frac{1}{2} \\ \langle \mathbf{q}_1, \mathbf{w}_3 \rangle &= \frac{1}{2}(1,1,1,-1) \cdot (-1,2,2,1) = 1 \\ \langle \mathbf{q}_2, \mathbf{w}_3 \rangle &= \frac{1}{2\sqrt{3}}(3,-1,-1,1) \cdot (-1,2,2,1) = -\sqrt{3}. \end{aligned}$$

It follows that

$$A = \begin{bmatrix} 1/2 & 3/(2\sqrt{3}) & 0 \\ 1/2 & -1/(2\sqrt{3}) & 1/\sqrt{6} \\ 1/2 & -1/(2\sqrt{3}) & 1/\sqrt{6} \\ -1/2 & 1/(2\sqrt{3}) & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1/2 & 1 \\ 0 & \frac{3}{2}\sqrt{3} & -\sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix} = QR.$$

Solving the system $R\mathbf{x} = Q^T\mathbf{b}$, where $\mathbf{b} = (1,1,1,1)$, by hand is rather tedious even though the system is a simple triangular one. We leave the detailed calculations to the reader. Better yet, use a CAS or MAS to obtain the solution $\mathbf{x} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$. Thus,

$$\mathbf{r} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that the system $A\mathbf{x} = \mathbf{b}$ is actually consistent, since the least squares solution turns out to be a genuine solution to the problem. □

QR as Least Squares Solver

Does this method really solve least squares problems? It does, and to see why, observe that with the above notation we have $A^T = (QR)^T = R^T Q^T$, so that the normal equations for the system $A\mathbf{x} = \mathbf{b}$ (which are given by $A^T A\mathbf{x} = A^T\mathbf{b}$) become

$$A^T A\mathbf{x} = R^T Q^T QR\mathbf{x} = R^T IR\mathbf{x} = R^T R\mathbf{x} = A^T\mathbf{b} = R^T Q^T\mathbf{b}.$$

But the triangular matrix R is invertible because its diagonal entries are nonzero; cancel it and obtain that the normal equations are equivalent to $R\mathbf{x} = Q^T\mathbf{b}$, which is exactly what the method we have described solves.

6.4 Exercises and Problems

Exercise 1. Let $V = \operatorname{span}\{(1, -1, 2, 0), (2, 0, -1, 1)\} \subset \mathbb{R}^4 = W$. Compute $V^\perp$ and use it to verify that $V + V^\perp = \mathbb{R}^4$.

Exercise 2. Let $V = \operatorname{span}\{(1, -1, 2)\} \subset \mathbb{R}^3 = W$. Compute $V^\perp$ and use it to verify that $V \cap V^\perp = \{\mathbf{0}\}$.

Exercise 3. Let $V = \operatorname{span}\left\{1 + x, x^2\right\} \subset W = \mathcal{P}_2$, where the space $\mathcal{P}_2$ of polynomials of degree at most 2 has the standard inner product of $C[0, 1]$. Compute $V^\perp$.

Exercise 4. Let $V = \operatorname{span}\left\{1 + x + x^3\right\} \subset W = \mathcal{P}_3$, where $\mathcal{P}_3$ has the standard inner product of $C[0, 1]$ and compute $V^\perp$.

Exercise 5. Let $V = \operatorname{span}\{(1, 0, 2), (0, 2, 1)\} \subset \mathbb{R}^3 = W$. Compute $V^\perp$and verify that $\left(V^\perp\right)^\perp = V$.

Exercise 6. Let $V = \operatorname{span}\{(4, 1, -2)\} \subset \mathbb{R}^3 = W$, where W has the weighted inner product $\langle(x, y, z), (u, v, w)\rangle = 2xu + 3yv + zw$. Compute $V^\perp$and verify that $\left(V^\perp\right)^\perp = V$.

Exercise 7. Use Gram–Schmidt to find QR factorizations for these matrices and use them to compute the least squares solutions of $A\mathbf{x} = \mathbf{b}$ with these $A, \mathbf{b}$.

(a) $\begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ -2 & 1 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 3 \\ 1 \end{bmatrix}$

Exercise 8. Carry out the method of computing $U \cap V$ discussed on page 335 using the subspaces $U = \operatorname{span}\{(1, 2, 1), (2, 1, 0)\}$ and $V = \operatorname{span}\{(1, 1, 1), (1, 1, 3)\}$ of $W = \mathbb{R}^3$.

Problem 9. Show that if V is a subspace of the inner product space W, then so is $V^{\perp}$.

Problem 10. Show that if V is a subspace of the inner product space W, then $V \cap V^{\perp} = \{\mathbf{0}\}$.

*Problem 11. Let U and V be subspaces of the inner product space W. Prove the following.
(a) $(U \cap V)^{\perp} = U^{\perp} + V^{\perp}$ (b) $(U + V)^{\perp} = U^{\perp} \cap V^{\perp}$

*Problem 12. Use the Fredholm alternative of this section to prove that the normal equations $A^T A\mathbf{x} = A^T \mathbf{b}$ are consistent for any matrix A.

6.5 *Operator Norms

The object of this section is to develop a useful notion of the norm of a matrix. For simplicity, we stick with real matrices, but all of the results in this section carry over to complex matrices. In Chapters 3 and 5 we studied the concept of a vector norm, which gave us a way of thinking about the "size" of a vector. We could easily extend this to matrices, just by thinking of a matrix as a vector that had been chopped into segments of equal length and re-stacked as a matrix. Thus, every vector norm on the space $\mathbb{R}^{mn}$ of vectors of length mn gives rise to a vector norm on the space $\mathbb{R}^{m,n}$ of $m \times n$ matrices. Experience has shown that with one exception—the standard norm—from which follows the Frobenius norm, this is not the best way to look for norms of matrices. After all, matrices are deeply intertwined with the operation of matrix multiplication. It would be too much to expect norms to distribute over products. The following definition takes a middle ground that has proved to be useful for many applications.

Matrix Norm

Definition 6.13. A vector norm $\|\cdot\|$ that is defined on the vector space $\mathbb{R}^{m,n}$ of $m \times n$ matrices, for any pair m, n, is said to be a *matrix norm* if for all pairs of matrices A, B that are conformable for multiplication,

$$\|AB\| \leq \|A\| \, \|B\| \, .$$

Our first example of such a norm is the Frobenius norm, which was introduced in Section 6.1; it is the one exception that we mentioned above.

Theorem 6.13. The Frobenius norm is a matrix norm.

Proof. Let A and B be matrices conformable for multiplication and suppose that the rows of A are $\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_m^T$, while the columns of B are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$. Then we have that $AB = \left[\mathbf{a}_i^T \mathbf{b}_j\right]$, so that by applying the definition and the CBS inequality, we obtain that

$$\begin{aligned}\|AB\|_F &= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\mathbf{a}_i^T\mathbf{b}_j\right|^2\right)^{1/2} \le \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\|\mathbf{a}_i\|^2\|\mathbf{b}_j\|^2\right)^{1/2}\\ &\le \left(\|A\|_F^2\|B\|_F^2\right)^{1/2} = \|A\|_F\|B\|_F\,. \qquad \square\end{aligned}$$

The most common multiplicative norms come from a rather general notion. Just as every inner product "induces" a norm in a natural way, every norm on the standard spaces induces a norm on matrices in a natural way. First recall that an *upper bound* for a set of real numbers is a number greater than or equal to any number in the set, and the *supremum* of a set of reals is the least (smallest) upper bound. We abbreviate this to "sup." For example, the sup of the open interval $(0,1)$ is 1.

Supremum

Definition 6.14. The *operator norm* induced on matrices by a norm on the standard spaces is defined by the formula

Operator Norm

$$\|A\| = \sup_{\mathbf{x}\neq 0}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}.$$

A useful fact about these norms is the following equivalence:

$$\|A\| = \sup_{\mathbf{x}\neq 0}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \sup_{\mathbf{x}\neq 0}\left\|A\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| = \sup_{\|\mathbf{v}\|=1}\|A\mathbf{v}\|\,.$$

Theorem 6.14. Every operator norm is a matrix norm.

Proof. For a given matrix A clearly $\|A\| \ge 0$ with equality if and only if $A\mathbf{x} = 0$ for all vectors $\mathbf{x}$, which is equivalent to $A = 0$. The remaining two norm properties are left as exercises. Finally, if A and B are conformable for multiplication, then

$$\|AB\| = \sup_{\mathbf{x}\neq 0}\frac{\|AB\mathbf{x}\|}{\|\mathbf{x}\|} \le \|A\|\sup_{\mathbf{x}\neq 0}\frac{\|B\mathbf{x}\|}{\|\mathbf{x}\|} = \|A\|\cdot\|B\|\,. \qquad \square$$

Incidentally, one difference between the Frobenius norm and operator norms is how the identity I_n is handled. Notice that $\|I_n\|_F = \sqrt{n}$, while with any operator norm $\|\cdot\|$ we have from the definition that $\|I_n\| = 1$.

How do we compute these norms? The next result covers the most common cases.

Theorem 6.15. If $A = [a_{ij}]_{m,n}$, then

(1) $\|A\|_1 = \max_{1\le i\le m}\{\sum_{j=1}^{n}|a_{ij}|\}$
(2) $\|A\|_\infty = \max_{1\le j\le n}\{\sum_{i=1}^{m}|a_{ij}|\}$
(3) $\|A\|_2 = \rho(A^TA)^{1/2}$

Proof. Items (1) and (3) are left as exercises. For the proof of (2), use the fact that $\|A\|_\infty = \sup_{\|\mathbf{v}\|_\infty = 1} \|A\mathbf{v}\|_\infty$. Now a vector has infinity norm 1 if each of its coordinates is 1 in absolute value. Notice that we can make the ith entry of $A\mathbf{v}$ as large as possible simply by choosing $\mathbf{v}$, so that the jth coordinate of $\mathbf{v}$ is ±1 and agrees with the sign of a_{ij}. Hence the infinity norm of $A\mathbf{v}$ is the maximum of the row sums of the absolute values of the entries of A, as stated in (2). □

One of the more important applications of the idea of a matrix norm is the famous Banach lemma. Essentially, it amounts to a matrix version of the familiar geometric series.

Banach Lemma

Theorem 6.16. If M is a square matrix such that $\|M\| < 1$ for some operator norm $\|\cdot\|$, then $I - M$ is invertible. Moreover, $\left\|(I-M)^{-1}\right\| \le 1/(1 - \|M\|)$ and

$$(I-M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots .$$

Proof. Form the telescoping series

$$(I-M)\left(I + M + M^2 + \cdots + M^k\right) = I - M^{k+1},$$

so that

$$I - (I-M)\left(I + M + M^2 + \cdots + M^k\right) = M^{k+1}.$$

Now by the multiplicative property of matrix norms and fact that $\|M\| < 1$,

$$\left\|M^{k+1}\right\| \le \|M\|^{k+1} \to 0, \text{ as } k \to \infty.$$

It follows that the matrix $\lim_{k\to\infty}\left(I + M + M^2 + \cdots + M^k\right) = B$ exists and that $I - (I-M)B = 0$, from which it follows that $B = (I-M)^{-1}$. Finally, note that

$$\begin{aligned}\left\|I + M + M^2 + \cdots + M^k\right\| &\le \|I\| + \|M\| + \|M\|^2 + \cdots + \|M\|^k \\ &\le 1 + \|M\| + \|M\|^2 + \cdots + \|M\|^k \\ &\le \frac{1}{1 - \|M\|}.\end{aligned}$$

Now take the limit as $k \to \infty$ to obtain the desired result. □

Condition Number

A fundamental idea in numerical linear algebra is the notion of the *condition number* of a matrix A. Roughly speaking, the condition number measures the degree to which changes in A lead to changes in solutions of systems $A\mathbf{x} = \mathbf{b}$. A large condition number means that small changes in A may lead to large changes in $\mathbf{x}$. In the case of an invertible matrix A, the condition number of A is defined to be

$$\operatorname{cond}(A) = \|A\| \left\|A^{-1}\right\|.$$

Of course this quantity is norm dependent. In the case of an operator norm, the Banach lemma has a nice application.

Corollary 6.4. If $A = I + N$, where $\|N\| < 1$, then

$$\operatorname{cond}(A) \le \frac{1 + \|N\|}{1 - \|N\|}.$$

We leave the proof as an exercise.

We conclude with a very fundamental result for numerical linear algebra. Here is the scenario: we desire to solve the linear system $A\mathbf{x} = \mathbf{b}$, where A is invertible. Due to arithmetic error or possibly input data error, we end up with a value $\mathbf{x}+\delta\mathbf{x}$ that solves exactly a "nearby" system $(A+\delta A)(\mathbf{x}+\delta\mathbf{x}) = \mathbf{b}+\delta\mathbf{b}$. (It can be shown using an idea called "backward error analysis" that this is really what happens when many algorithms are used to solve a linear system.) The question is, what is the size of the relative error $\|\delta\mathbf{x}\| / \|\mathbf{x}\|$? As long as the perturbation matrix $\|\delta A\|$ is reasonably small, there is a very elegant answer.

Theorem 6.17. Suppose that A is invertible, $A\mathbf{x} = \mathbf{b}$, $(A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$, and $\left\|A^{-1}\delta A\right\| = c < 1$ with respect to some operator norm. Then $A + \delta A$ is invertible and

Perturbation Theorem

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{\operatorname{cond}(A)}{1-c}\left\{\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}\right\}.$$

Proof. That the matrix $I + A^{-1}\delta A$ is invertible follows from hypothesis and the Banach lemma. Since A is also invertible by hypothesis, $A\left(I + A^{-1}\delta A\right) = A + \delta A$ is also invertible. Expand the perturbed equation to obtain

$$(A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = A\mathbf{x} + \delta A\mathbf{x} + A\delta\mathbf{x} + \delta A\,\delta\mathbf{x} = \mathbf{b} + \delta\mathbf{b}.$$

Now subtract the terms $A\mathbf{x} = \mathbf{b}$ from each side and solve for $\delta\mathbf{x}$ to obtain

$$(A + \delta A)\delta\mathbf{x} = A^{-1}(I + A^{-1}\delta A)\delta\mathbf{x} = -\delta A \cdot \mathbf{x} + \delta\mathbf{b},$$

so that

$$\delta\mathbf{x} = (I + A^{-1}\delta A)^{-1}A^{-1}\left\{-\delta A \cdot \mathbf{x} + \delta\mathbf{b}\right\}.$$

Take norms and use the additive and multiplicative properties and the Banach lemma to obtain

$$\|\delta\mathbf{x}\| \le \frac{\left\|A^{-1}\right\|}{1-c}\left\{\|\delta A\|\,\|\mathbf{x}\| + \|\delta\mathbf{b}\|\right\}.$$

Next divide both sides by $\|x\|$ to obtain

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{\left\|A^{-1}\right\|}{1-c}\left\{\|\delta A\| + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{x}\|}\right\}.$$

Finally, notice that $\|\mathbf{b}\| \le \|A\|\,\|\mathbf{x}\|$. Therefore, $1/\|\mathbf{x}\| \le \|A\| / \|\mathbf{b}\|$. Replace $1/\|\mathbf{x}\|$ in the right hand side by $\|A\| / \|\mathbf{b}\|$ and factor out $\|A\|$ to obtain

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{\|A\| \left\|A^{-1}\right\|}{1-c} \left\{ \frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} \right\},$$

which completes the proof, since by definition, $\operatorname{cond} A = \|A\| \left\|A^{-1}\right\|$. □

If we believe that the inequality in the perturbation theorem can be sharp (it can!), then it becomes clear how the condition number of the matrix A is a direct factor in how relative error in the solution vector is amplified by perturbations in the coefficient matrix.

Here is one more useful observation about operator norms that can be couched in very general terms.

Equivalent Norms **Definition 6.15.** Two norms $\|\cdot\|$ and $|||\cdot|||$ on the vector space V are said to be *equivalent* if there exist positive constants C, D such that for all $\mathbf{x} \in V$,

$$C \|\mathbf{x}\| \le |||\mathbf{x}||| \le D \|\mathbf{x}\| .$$

It is easily seen that this relation is symmetric, for we deduce from the definition that

$$\frac{1}{D} |||\mathbf{x}||| \le \|\mathbf{x}\| \le \frac{1}{C} |||\mathbf{x}||| .$$

Similarly, one checks that equivalence is a transitive relation, that is, if norm $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ and $\|\cdot\|_b$ is equivalent to $\|\cdot\|_c$, then $\|\cdot\|_a$ is equivalent to $\|\cdot\|_c$. Roughly speaking, the definition says that equivalent norms yield the same value up to fixed upper and lower scale factors. The significance of equivalence of norms is that convergence of a sequence of vectors in one norm implies convergence in the other equivalent norm. In general, a vector space can have inequivalent norms. However, in order to do so, the space must be infinite-dimensional. The following theorem applies to all finite-dimensional vector spaces, so it certainly applies to the space of $n \times n$ matrices $\mathbb{R}^{n,n}$ with a given operator norm. Thus, all operator norms are equivalent in the above sense.

Theorem 6.18. All norms on a finite-dimensional space are equivalent.

We sketch a proof. Let V be a finite-dimensional vector space. We know that there is an arithmetic preserving one-to-one correspondence between elements $\mathbf{x}$ of V and their coordinate vectors with respect to some basis of V, so that elements of V are identified with some $\mathbb{R}^n$. Without loss of generality $V = \mathbb{R}^n$. Now let $\|\cdot\|$ be any norm on V. First, one establishes that$\|\cdot\| : V \to \mathbb{R}$ is a continuous function by proving that for all $x, y \in V$, $|\|\mathbf{x}\| - \|\mathbf{y}\|| \le \|\mathbf{x} - \mathbf{y}\|$. The proof of Problem 19, Section 4.1, shows this inequality.

Next, one observes that the unit ball $B_1(\mathbf{0})$ in the infinity norm in V is a closed and bounded set that does not contain the origin. By the extreme value theorem of analysis, the function $\|\cdot\|$ assumes its maximum and minimum values on the ball, and these must be positive, say C, D. Thus for all nonzero vectors $\mathbf{x}$ we have

$$C \leq \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|_\infty} \right\| \leq D.$$

Multiply through by $\|x\|_\infty$, and we see that $C\|x\|_\infty \leq \|\mathbf{x}\| \leq D\|x\|_\infty$, which proves the equivalence of the given norm to the infinity norm. It follows from transitivity of the equivalence property that all norms are equivalent to each other. □

6.5 Exercises and Problems

Exercise 1. Compute the Frobenius, 1-, and ∞-norms of the following matrices.

(a) $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & -3 & 0 & -1 \\ 1 & 1 & -2 & 0 \\ -2 & 1 & 6 & 1 \end{bmatrix}$

Exercise 2. Compute the condition number of each matrix in Exercise 1 using the infinity norm.

Exercise 3. Verify that the perturbation theorem is valid for $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$, $\delta A = 0.05A$, and $\delta\mathbf{b} = 0.05\mathbf{b}$.

Exercise 4. Verify the inequality of Corollary 6.4 using the infinity norm and $N = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.

*Problem 5. Prove Corollary 6.4.

*Problem 6. Show that if A is invertible and $\|A^{-1}\delta A\| < 1$, then so is $A + \delta A$.

Problem 7. Prove that $\|A\|_1 = \max_{1\leq i\leq m}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}$.

Problem 8. Prove that $\|A\|_2 = \rho(A^TA)^{1/2}$.

Problem 9. Suppose we want to approximately solve a system of the form $A\mathbf{x} = \mathbf{b}$, where $A = I - M$ and $\|M\| < 1$ for some operator norm. Use the Banach lemma to devise such a scheme involving only a finite number of matrix additions and multiplications.

*Problem 10. Show that for any any operator norm $\|\cdot\|$, $\rho(A) \leq \|A\|$.

*Problem 11. Show that a square matrix A is *power bounded*, that is, $\|A^m\|_2 \leq C$ for all positive m and some constant C independent of m, if every eigenvalue of A is either strictly less than 1 in absolute value or of absolute value equal to 1 and simple.

Problem 12. Does it follow from Problem 11 and the equivalence of operator norms that power bounded in one operator norm implies power bounded in any other? Justify your answer.

*Problem 13. Let A be a real matrix and U, V orthogonal matrices.
(a) Show from the definition that $\left\|U^T AV\right\|_2 = \|A\|_2$.
(b) Determine $\|\Sigma\|_2$ if Σ is a diagonal matrix with nonnegative entries.
(c) Use (a) and (b) to express $\|A\|_2$ in terms of the singular values of A.

6.6 *Computational Notes and Projects

Error and Limit Measurements

We are going to consider a situation where infinity norms are both more natural to a problem and easier to use than the standard norm. This material is a simplified treatment of some of the concepts introduced in Section 6.5 and is independent of that section. The theorem below provides a solution to this question: how large an error in the solution to a linear system can there be, given that we have introduced an error in the right-hand side whose size we can estimate? (Such an error might be due to experimental error or input error.) The theorem, called a *perturbation theorem*, requires an extension of the idea of vector infinity norm to matrices for its statement.

Matrix Infinity Norm

Definition 6.16. Let A be an $n \times n$ matrix whose rows are $\mathbf{r}_1^T, \mathbf{r}_2^T, \ldots, \mathbf{r}_n^T$. The *infinity norm of the matrix* A is defined as

$$\|A\|_\infty = \max\left\{\|\mathbf{r}_1\|_1, \|\mathbf{r}_2\|_1, \ldots, \|\mathbf{r}_n\|_1\right\}.$$

If, moreover, A is invertible, then the condition number of A is defined to be

$$\operatorname{cond}(A) = \|A\|_\infty \left\|A^{-1}\right\|_\infty.$$

Example 6.24. Let $A = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$. Find $\|A\|_\infty$, $\left\|A^{-1}\right\|_\infty$, and $\operatorname{cond}(A)$.

Solution. We see that $A^{-1} = \begin{bmatrix} 101 & -10 \\ -10 & 1 \end{bmatrix}$. From the preceding definition we obtain that

$$\|A\|_\infty = \max\{|1| + |10|, |10| + |101|\} = 111$$

and

$$\left\|A^{-1}\right\|_\infty = \max\{|101| + |-10|, |-10| + |1|\} = 111,$$

so it follows that $\operatorname{cond}(A) = 111 \cdot 111 = 12321$. □

Perturbation Theorem

Theorem 6.19. Suppose that the $n \times n$ matrix A is nonsingular, $A\mathbf{x} = \mathbf{b}$, and $A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$. Then

$$\frac{\|\delta\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \le \operatorname{cond}(A)\frac{\|\delta\mathbf{b}\|_\infty}{\|\mathbf{b}\|_\infty}.$$

Proof. Subtract the first equation of the statement of the theorem from the second one to obtain that

$$A(\mathbf{x} + \delta\mathbf{x}) - A\mathbf{x} = A\delta\mathbf{x} = \mathbf{b} + \delta\mathbf{b} - \mathbf{b} = \delta\mathbf{b},$$

from which it follows that $\delta\mathbf{x} = A^{-1}\delta\mathbf{b}$. Now write $A^{-1} = [c_{ij}]$, $\delta\mathbf{b} = [d_i]$, and compute the ith coordinate of $\delta\mathbf{x}$:

$$(\delta\mathbf{x})_i = \sum_{j=1}^{n} c_{ij} d_j,$$

so that if $\mathbf{r}_i = (c_{i1}, c_{i2}, \ldots, c_{in})$ is the ith row of A^{-1}, then

$$|(\delta\mathbf{x})_i| \le \sum_{j=1}^{n} |c_{ij}|\,|d_j| \le \max\{|d_1|, \ldots, |d_n|\} \sum_{j=1}^{n} |c_{ij}| \le \|\delta\mathbf{b}\|_\infty \|\mathbf{r}_i\|_1 .$$

Therefore,

$$\|\delta\mathbf{x}\|_\infty \le \|\delta\mathbf{b}\|_\infty \left\|A^{-1}\right\|_\infty . \tag{6.3}$$

A similar calculation shows us that since $\mathbf{b} = A\mathbf{x}$, $\|\mathbf{b}\|_\infty \le \|\mathbf{x}\|_\infty \|A\|_\infty$. Divide both sides by $\|\mathbf{b}\|_\infty \|\mathbf{x}\|_\infty$ and obtain that

$$\frac{1}{\|\mathbf{x}\|_\infty} \le \|A\|_\infty \frac{1}{\|\mathbf{b}\|_\infty}. \tag{6.4}$$

Now multiply the inequalities 6.3 and 6.4 together to obtain the desired inequality. □

Allowing perturbations δA of the matrix A is a more complicated issue that is covered by Theorem 6.17.

Example 6.25. Suppose we wish to solve the nonsingular system $A\mathbf{x} = \mathbf{b}$ exactly, where the coefficient matrix A is as in Example 6.24 but the right-hand-side vector $\mathbf{b}$ is determined from measured data. Suppose also that the error of measurement is such that the ratio of the largest error in any coordinate of $\mathbf{b}$ to the largest coordinate of $\mathbf{b}$ (this ratio is called the *relative error*) is no more than 0.01 in absolute value. Estimate the size of the relative error in the solution.

Solution. In matrix notation, we can phrase the problem in this manner: let the correct value of the right-hand side be $\mathbf{b}$ and the measured value of the right-hand side be $\tilde{\mathbf{b}}$, so that the error of measurement is the vector

$\delta \mathbf{b} = \widetilde{\mathbf{b}} - \mathbf{b}$. Rather than solving the system $A\mathbf{x} = \mathbf{b}$, we end up solving the system $A\tilde{\mathbf{x}} = \widetilde{\mathbf{b}} = \mathbf{b} + \delta \mathbf{b}$, where $\tilde{\mathbf{x}} = \mathbf{x} + \delta \mathbf{x}$. The relative error in data is the quantity $\|\delta \mathbf{b}\|_\infty / \|\mathbf{b}\|_\infty$, while the relative error in the computed solution is $\|\delta \mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty$. This sets up very nicely for an application of Theorem 6.19. We calculated $\operatorname{cond}(A) = 12321$ in Example 6.24. It follows that the relative error in the solution satisfies the inequality

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq 12321 \cdot 0.01 = 123.21.$$

In other words, the relative error in our computed solution could be as large as 12321% which, of course, would make it quite worthless. Worthless answers do happen (see Exercise 2). □

A Practical QR Algorithm

In the preceding section we saw that the QR factorization can be used to solve systems including least squares. We also saw the factorization as a consequence of the Gram–Schmidt algorithm. As a matter of fact, the *classical* Gram–Schmidt algorithm that we have presented has certain numerical stability problems when used in practice. There is a so-called *modified* Gram–Schmidt algorithm that performs better. However, there is another approach to QR factorization that avoids Gram–Schmidt altogether. This approach uses the Householder matrices we introduced in Section 4.3. It is more efficient and stable than Gram–Schmidt. If you use an MAS to find the QR factorization of a matrix, it is likely that this is the method used by the system.

The basic idea behind this Householder QR is to use a succession of Householder matrices to zero out the lower triangle of a matrix, one column at a time. The key fact about Householder matrices is the following application of these matrices:

Theorem 6.20. Let $\mathbf{x}, \mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$ of the same length. Then there is a Householder matrix $H_\mathbf{v}$ such that $H_\mathbf{v}\mathbf{x} = \mathbf{y}$.

Proof. Let $\mathbf{v} = \mathbf{x} - \mathbf{y}$. Then we see that

$$(\mathbf{x}+\mathbf{y})^T(\mathbf{x}-\mathbf{y}) = \mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{x} - \mathbf{y}^T\mathbf{y} = \mathbf{x}^T\mathbf{x} - \mathbf{y}^T\mathbf{y} = 0,$$

since $\mathbf{x}$ and $\mathbf{y}$ have the same length. Now write

$$\mathbf{x} = \frac{1}{2}\{(\mathbf{x}-\mathbf{y}) + (\mathbf{x}+\mathbf{y})\} = \mathbf{p} + \mathbf{u}$$

and obtain from Theorem 4.9 that

$$H_\mathbf{v}\mathbf{x} = -\mathbf{p} + \mathbf{u} = \frac{1}{2}\{-(\mathbf{x}-\mathbf{y}) + (\mathbf{x}+\mathbf{y})\} = \frac{2\mathbf{y}}{2} = \mathbf{y},$$

which is what we wanted to show. □

Now we have a tool for massively zeroing out entries in a vector of the form $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Set $y = (\pm \|\mathbf{x}\|, 0, \ldots, 0)$ and apply the preceding theorem to construct Householder H such that $H_{\mathbf{v}}\mathbf{x} = \mathbf{y}$. It is standard to choose the ± to be the negative of the sign of x_1. In this way, the first term will not cause any loss of accuracy to subtractive cancellation. However, any choice of ± works fine in theory. We can picture this situation schematically very nicely by representing possibly nonzero entries by an × in the following simple version:

$$\mathbf{x} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix} \xrightarrow[H_{\mathbf{v}}]{} \begin{bmatrix} \pm \|\mathbf{x}\| \\ 0 \\ 0 \\ 0 \end{bmatrix} = H_{\mathbf{v}}\mathbf{x}.$$

We can extend this idea to zeroing out lower parts of $\mathbf{x}$ only, say

$$\mathbf{x} = \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ \times \\ \times \\ \times \end{bmatrix} \text{ by using } \mathbf{y} = \begin{bmatrix} \mathbf{z} \\ \pm \|\mathbf{w}\| \\ 0 \\ 0 \end{bmatrix} \text{ so } \mathbf{v} = \begin{bmatrix} 0 \\ \times \\ \times \\ \times \end{bmatrix} \text{ and } H_{\mathbf{v}}\mathbf{x} = \begin{bmatrix} 0 \\ \times \\ 0 \\ 0 \end{bmatrix}.$$

We can apply this idea to systematically zero out subdiagonal entries by successive multiplication by Householder (hence orthogonal) matrices; schematically we have this representation of a full-rank $m \times n$ matrix A:

$$A = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \xrightarrow[H_1]{} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \xrightarrow[H_2]{} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix} \xrightarrow[H_3]{} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{bmatrix} = R,$$

so that $H_3H_2H_1A = R$. Now we can check easily from the definition of a Householder matrix H that $H^T = H = H^{-1}$. Thus, if we set $Q = H_1^{-1}H_2^{-1}H_3^{-1} = H_1H_2H_3$, it follows that $A = QR$. Notice that we don't actually have to carry out the multiplications to compute Q unless they are needed, and the vectors needed to define these Householder matrices are themselves easily stored in a single matrix. What we have here is just a bit different from the QR factorization discussed in the last section. Here the matrix Q is a full $m \times m$ matrix and R is the same size as A. Even if A is not of full column rank, this procedure will work, provided we simply skip construction of H in the case that there are no nonzero elements to zero out in some column. Consequently, we have essentially proved the following theorem, which is sometimes called a *full* QR factorization, in contrast to the *reduced* QR factorization of Theorem 6.12.

Full QR Factorization

Theorem 6.21. Let A be a real $m \times n$ matrix. Then there exist an $m \times m$ orthogonal matrix Q and an $m \times n$ upper triangular matrix R such that $A = QR$.

All of the results we have discussed regarding QR factorization work for complex matrices, provided we use unitary matrices and conjugate transpose.

Project Topics

Project: Testing Least Squares Solvers

The object of this project is to test the quality of the solutions of three different methods for solving least squares problems $A\mathbf{x} = \mathbf{b}$:

1. Solution by solving the normal equations by Gaussian elimination.
2. Solution by reduced QR factorization obtained by Gram–Schmidt.
3. Solution by full QR factorization by Householder matrices.

Here is the test problem: suppose we want to approximate the curve $f(x) = e^{\sin(6x)}$, $0 \leq x \leq 1$, by a tenth-degree polynomial. The input data will be the sampled values of $f(x)$ at equally spaced nodes $x_k = kh$, $k = 0, 1, \ldots, 20$, $h = 0.05$. This gives 21 equations $f(x_k) = c_0 + c_1 x_k + \cdots + c_{10} x_k^{10}$ for the 11 unknown coefficients c_k, $k = 0, 1, \ldots, 20$. The coefficient matrix that results from this problem is called a Vandermonde matrix. Your MAS should have a built-in command for construction of such a matrix.

Procedure: First set up the system matrix A and right-hand-side vector $\mathbf{b}$. Method (a) is easily implemented on any CAS or MAS. The built-in procedure for computing a QR factorization will very likely be Householder matrices, which will take care of (c). You will need to check the documentation to verify this. The Gram–Schmidt method of finding QR factorization will have to be programmed by you.

Once you have solved the system by these three methods, make out a table that has the computed coefficients for each of the three methods. Then make plots of the difference between the function $f(x)$ and the computed polynomial for each method. Discuss your results.

There are a number of good texts that discuss numerical methods for least squares; see, e.g., references [6], [7], [9]. More advanced treatments can be found in references [1] and [10]. Or you can read from the master himself in [8] (Gauss's original text conveniently translated from the Latin with a very enlightening supplement by G. W. Stewart).

Project: Approximation Theory

Suppose you work for a manufacturer of calculators, and are involved in the design of a new calculator. The problem is this: as one of the "features" of this calculator, the designers decided that it would be nice to have a key that calculated a transcendental function, namely, $f(x) = \sin(\pi x)$, $-1 \leq x \leq 1$. Your job is to come up with an adequate way of calculating $f(x)$, say with an error no worse than .001

Polynomials are a natural idea for approximating functions. From a designer's point of view they are particularly attractive because they are so easy to implement. Given the coefficients of a polynomial, it is easy to design a very efficient and compact algorithm for calculating values of the polynomial. Such an algorithm, together with the coefficients of the polynomial, would fit nicely into a small ROM for the calculator, or could even be microcoded into the chip.

Your task is to find a low-degree polynomial that approximates $\sin(\pi x)$ to within the specified accuracy. For comparison, find a Taylor polynomial of lowest degree for $\sin x$ that gives sufficient accuracy. Next, use the projection problem idea to project the function $\sin x \in C[-1,1]$ with the standard inner product, into the subspace $\mathcal{P}_n$ of polynomials of degree at most n. You will need to find the smallest n that gives a projection whose difference from $\sin x$ is at most 0.001 on the interval $[-1,1]$. Is it of lower degree than the best Taylor polynomial approximation?

Use a CAS to do the computations and graphics. Then report on your findings. Include graphs that will be helpful in interpreting your conclusions. Also, give suggestions on how to compute this polynomial efficiently.

Report: Fourier Analysis

This project will introduce you to a very fascinating and important topic known as Fourier analysis. The setting is as follows: we are interested in finding approximations to functions in the vector space $\mathcal{C}_{2\pi}$ of continuous periodic functions on the closed interval $[-\pi,\pi]$. This vector space becomes an inner product space with the usual definition

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

In this space the sequence of trigonometric functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \ldots$$

forms an orthonormal set. Therefore, we can form the finite-dimensional subspaces V_n spanned by the first $2n+1$ of these elements and immediately obtain an orthonormal basis of V_n. We can also use the machinery of projections to approximate any function $f(x) \in \mathcal{C}_{2\pi}$ by its projection into the various subspaces V_n. The coefficients of the orthonormal basis functions in the projection formula of Definition 6.11 as applied to a function $f(x)$ are called the *Fourier coefficients* of $f(x)$. They are traditionally designated by the symbols

$$\frac{a_0}{2}, a_1, b_1, a_2, b_2, \ldots, a_k, b_k, \ldots.$$

In the first part of this project you will write a brief introduction to Fourier analysis in which you exhibit formulas for the Fourier coefficients of a function $f(x)$ and explain the form and meaning of the projection formula in this setting. Try to prove that the trigonometric functions given above are an orthonormal set. At minimum provide a proof for the first three functions.

In the second part you will explore the quality of these approximations for the following test functions. The functions are specified on the interval $[-\pi,\pi]$, and then each graph is replicated on adjacent intervals of length 2π, yielding periodic functions:

$$(1)\ f\left(x\right)=\sin\frac{x^2}{\pi},\quad (2)\ g(x)=x\left(x-\pi\right)\left(x+\pi\right),\quad (3)\ h\left(x\right)=x.$$

Notice that the last function violates the continuity condition.

For each test function you should prepare a graph that includes the test function and at least two projections of it into the V_n, $n=0,1,\ldots$. Discuss the quality of the approximations and report on any conclusions that you can draw from this data. You will need an MAS or CAS to carry out the calculations and graphs, since the calculations are very detailed. If you are allowed to do so, you could write up your report in the form of a notebook.

6.6 Exercises and Problems

Exercise 1. Let $A=\left[\begin{smallmatrix}2&7\\3&10\end{smallmatrix}\right]$, $\mathbf{b}=\left[5.7,8.2\right]^T$ and solve the system $A\mathbf{x}=\mathbf{b}$ for $\mathbf{x}$. Next, let $\delta\mathbf{b}=\left[0.096;-0.025\right]$ and $\mathbf{x}+\delta\mathbf{x}$ the solution to $A\left(\mathbf{x}+\delta\mathbf{x}\right)=\left(\mathbf{b}+\delta\mathbf{b}\right)$. Compute $\left\|\delta\mathbf{x}\right\|_\infty/\left\|\mathbf{x}\right\|_\infty$ and compare it to $\operatorname{cond}\left(A\right)\left\|\delta\mathbf{b}\right\|_\infty/\left\|\mathbf{b}\right\|_\infty$.

Exercise 2. Repeat Exercise 1 with A the matrix of Example 6.25, $\mathbf{b}=\left[0.985,9.95\right]^T$, and $\delta\mathbf{b}=\left[-0.0995;0.00985\right]$. How good is the solution?

Exercise 3. Let $A=\left[\begin{smallmatrix}3&2\\0&1\\4&1\end{smallmatrix}\right]$ and use Householder matrices to find a full QR factorization of A. Use this result to find the least squares solution to the system $A\mathbf{x}=\mathbf{b}$, where $\mathbf{b}=(1,2,3)$, and resulting residual.

Exercise 4. Calculate a full QR factorization of $A=\left[\begin{smallmatrix}1&10&20\\10&100&201\\1000&10001&20001\end{smallmatrix}\right]$ with an MAS. Inspect the R matrix and estimate the rank of A. Use QR to find the least squares solution to $A\mathbf{x}=\mathbf{b}$, where $\mathbf{b}=(1,2,3)$, and resulting residual.

Problem 5. The following is a simplified description of the *QR algorithm* (which is separate from the QR factorization, but involves it) for a real $n\times n$ matrix A :

```
$T_0 = A$, $Q_0 = I_n$
for $k = 0, 1, \ldots$
    $T_k = Q_{k+1}R_{k+1}$    (QR factorization of $T_k$)
    $T_{k+1} = R_{k+1}Q_{k+1}$
end
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Apply this algorithm to the following two matrices and, based on your results, speculate about what it is supposed to compute. You will need a CAS or MAS for this exercise and you will stop in a finite number of steps, but expect to take more than a few.

$$\text{(a) } A=\begin{bmatrix}1&2&0\\2&1&-2\\0&-2&1\end{bmatrix}\qquad\qquad \text{(b) } A=\begin{bmatrix}-8&-5&8\\6&3&-8\\-3&1&9\end{bmatrix}$$

Problem 6. Example 6.25 gives an upper bound on the error propagated to the solution of a system due to right-hand-side error. How pessimistic is it? Experiment with various random different erroneous right-hand-sides with your choice of error tolerance and compare actual error with estimated error.

Table of Symbols

Symbol	Meaning	Reference
$\emptyset$	Empty set	Page 10
$\in$	Member symbol	Page 10
$\subseteq$	Subset symbol	Page 10
$\subset$	Proper subset symbol	Page 10
$\cap$	Intersection symbol	Page 10
$\cup$	Union symbol	Page 10
$\otimes$	Tensor symbol	Page 136
$\approx$	Approximate equality sign	Page 79
$\overrightarrow{PQ}$	Displacement vector	Page 147
$\lvert z \rvert$	Absolute value of complex z	Page 13
$\lvert A \rvert$	determinant of matrix A	Page 115
$\lvert\lvert \mathbf{u} \rvert\rvert$	Norm of vector $\mathbf{u}$	Page 212
$\lvert\lvert \mathbf{u} \rvert\rvert_p$	p-norm of vector $\mathbf{u}$	Page 306
$\mathbf{u} \cdot \mathbf{v}$	Standard inner product	Page 216
$\langle \mathbf{u}, \mathbf{v} \rangle$	Inner product	Page 312
A_{cof}	Cofactor matrix of A	Page 122
$\operatorname{adj} A$	Adjoint of matrix A	Page 122
A^*	Conjugate (Hermitian) transpose of matrix A	Page 91
A^T	Transpose of matrix A	Page 91
$\mathcal{C}(A)$	Column space of matrix A	Page 183
$\operatorname{cond}(A)$	Condition number of matrix A	Page 344
$C[a, b]$	Function space	Page 151
$\mathbb{C}$	Complex numbers $a + bi$	Page 12
$\mathbb{C}^n$	Standard complex vector space	Page 149
$\operatorname{comp}_{\mathbf{v}} \mathbf{u}$	Component	Page 224
$\overline{z}$	Complex conjugate of z	Page 13
δ_{ij}	Kronecker delta	Page 65
$\dim V$	Dimension of space V	Page 195
$\det A$	Determinant of A	Page 115
$\operatorname{domain}(T)$	Domain of operator T	Page 189
$\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$	Diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ along diagonal	Page 103
E_{ij}	Elementary operation switch ith and jth rows	Page 25
$E_i(c)$	Elementary operation multiply ith row by c	Page 25
$E_{ij}(d)$	Elementary operation add d times jth row to ith row	Page 25

Symbol	Meaning	Reference
$\mathcal{E}_\lambda(A)$	Eigenspace	Page 254
$H_{\mathbf{v}}$	Householder matrix	Page 237
I, I_n	Identity matrix, $n \times n$ identity	Page 65
id_V	Identity function for V	Page 156
$\Im(z)$	Imaginary part of z	Page 12
$\ker(T)$	Kernel of operator T	Page 188
$M_{ij}(A)$	Minor of A	Page 117
$M(A)$	Matrix of minors of A	Page 122
$\max\{a_1, a_2, \ldots, a_m\}$	Maximum value	Page 40
$\min\{a_1, a_2, \ldots, a_m\}$	Minimum value	Page 40
$\mathcal{N}(A)$	Null space of matrix A	Page 184
$\mathbb{N}$	Natural numbers $1, 2, \ldots$	Page 10
$\mathrm{null}\, A$	Nullity of matrix A	Page 39
$\mathcal{P}$	Space of polynomials of any degree	Page 163
$\mathcal{P}_n$	Space of polynomials of degree $\leq n$	Page 163
$\mathrm{proj}_{\mathbf{v}}\, \mathbf{u}$	Projection vector along a vector	Page 224
$\mathrm{proj}_V\, \mathbf{u}$	Projection vector into subspace	Page 327
$\mathbb{Q}$	Rational numbers a/b	Page 11
$\Re(z)$	Real part of z	Page 12
$\mathcal{R}(A)$	Row space of matrix A	Page 184
$R(\theta)$	Rotation matrix	Page 178
$\mathbb{R}$	Real numbers	Page 11
$\mathbb{R}^n$	Standard real vector space	Page 147
$\mathbb{R}^{m,n}$	Space of $m \times n$ real matrices	Page 151
T_A	Matrix operator associated with A	Page 72
$\mathrm{range}(T)$	Range of operator T	Page 189
$\mathrm{rank}\, A$	Rank of matrix A	Page 39
$\rho(A)$	Spectral radius of A	Page 273
$\mathrm{span}\{S\}$	Span of vectors in S	Page 164
$\sup\{\mathrm{E}\}$	Supremum of set E of reals	Page 343
$\mathrm{target}(T)$	Target of operator T	Page 189
$[T]_{B,C}$	Matrix of operator T	Page 243
$V^\perp$	Orthogonal complement of V	Page 333
$\mathbb{Z}$	Integers $0, \pm 1, \pm 2, \ldots$	Page 11

Solutions to Selected Exercises

Section 1.1, Page 8

1 (a) $x=-1$, $y=1$ (b) $x=2$, $y=-2$, $z=1$ (c) $x=2$, $y=1$

3 (a) linear, $x-y-z=-2$, $3x-y=4$ (b) nonlinear (c) linear, $x+4y=0$, $2x-y=0$, $x+y=2$

5 (a) $m=3$, $n=3$, $a_{11}=1$, $a_{12}=-2$, $a_{13}=1$, $b_1=2$, $a_{21}=0$, $a_{22}=1$, $a_{23}=0$, $b_2=1$, $a_{31}=-1$, $a_{32}=0$, $a_{33}=1$, $b_3=1$ (b) $m=2$, $n=2$, $a_{11}=1$, $a_{12}=-3$, $b_1=1$, $a_{21}=0$, $a_{22}=1$, $b_2=5$

7 $\frac{47}{25}y_1-y_2=0$, $-y_1+\frac{47}{25}y_2-y_3=0$, $-y_2+\frac{47}{25}y_3-y_4=0$, $-y_3+\frac{47}{25}y_4=0$

9 $p_1=0.2p_1+0.1p_2+0.4p_3$, $p_2=0.3p_1+0.3p_2+0.2p_3$, $p_3=0.1p_1+0.2p_2+0.1p_3$

13 Counting inflow as positive, the equation for vertex v_1 is $x_1-x_4-x_5=0$.

Section 1.2, Page 19

1 (a) $\{0,1\}$ (b) $\{x \,|\, x\in\mathbb{Z} \text{ and } x>1\}$ (c) $\{x \,|\, x\in\mathbb{Z} \text{ and } x\le -1\}$ (d) $\{0,1,2,\ldots\}$ (e) A

3 (a) $\mathrm{e}^{3\pi\mathrm{i}/2}$ (b) $\sqrt{2}\mathrm{e}^{\pi\mathrm{i}/4}$ (c) $2\mathrm{e}^{2\pi\mathrm{i}/3}$ (d) $e^{0\mathrm{i}}$or 1 (e) $2\sqrt{2}\mathrm{e}^{7\pi\mathrm{i}/4}$ (f) $2\mathrm{e}^{\pi\mathrm{i}/2}$ (g) $\mathrm{e}^{\pi}\mathrm{e}^{0\mathrm{i}}$ or e^{π}

5 (a) $1+8\mathrm{i}$ (b) $10+10i$ (c) $\frac{3}{5}+\frac{4}{5}\mathrm{i}$ (d) $-\frac{3}{5}-\frac{4}{5}\mathrm{i}$ (e) $42+7\mathrm{i}$

7 (a) $\frac{6}{5}-\frac{8}{5}\mathrm{i}$, (b) $\pm\sqrt{2}\pm\mathrm{i}\sqrt{2}$, (c) $z=1$ (d) $z=-1,\ \pm\mathrm{i}$

9 (a) $\frac{1}{2}+\frac{1}{2}\mathrm{i}=\frac{1}{2}\sqrt{2}\mathrm{e}^{\pi\mathrm{i}/4}$ (b) $-1-\mathrm{i}\sqrt{3}=2\mathrm{e}^{4\pi\mathrm{i}/3}$ (c) $-1+\mathrm{i}\sqrt{3}=2\mathrm{e}^{2\pi\mathrm{i}/3}$ (d) $-\frac{1}{2}\mathrm{i}=\frac{1}{2}\mathrm{e}^{3\pi\mathrm{i}/2}$ (e) $\mathrm{i}\mathrm{e}^{\pi/4}=\mathrm{e}^{\pi/4}\mathrm{e}^{\pi\mathrm{i}/2}$

11 (a) $z=\frac{-1}{2}\pm\frac{\sqrt{11}}{2}i$, (b) $z=\pm\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$ (c) $z=1\pm(\frac{-\sqrt{2\sqrt{2}+2}}{2}-\frac{\sqrt{2\sqrt{2}-2}}{2}\mathrm{i})$ (d) $\pm 2\mathrm{i}$

13 (a) Circle of radius 2, center at origin (b) $\Re(z)=0$, the imaginary axis (c) Interior of circle of radius 1, center at $z=2$.

15 $\overline{2+4\mathrm{i}}+\overline{1-3\mathrm{i}}=2-4\mathrm{i}+1+3\mathrm{i}=3-\mathrm{i}$ and $\overline{(2+4\mathrm{i})+(1-3\mathrm{i})}=\overline{3+\mathrm{i}}=3-\mathrm{i}$

17 $z=1\pm\mathrm{i}$, $(z-(1+\mathrm{i}))(z-(1-\mathrm{i}))=z^2-2z+2$

21 Use $|z|^2=z\bar{z}$ and $\overline{z_1z_2}=\overline{z_1z_2}$.

24 Write $p(w)=a_0+a_1w+\cdots+a_nw^n=0$ and conjugate both sides.

Section 1.3, Page 30

1 (a) Size 2×4, $a_{11} = a_{14} = a_{23} = a_{24} = 1$, $a_{12} = -1$, $a_{21} = -2$, $a_{13} = a_{22} = 2$ (b) Size 3×2, $a_{11} = 0$, $a_{12} = 1$, $a_{21} = 2$, $a_{22} = -1$, $a_{31} = 0$, $a_{32} = 2$ (c) Size 2×1 , $a_{11} = -2$, $a_{21} = 3$ (d) Size 1×1, $a_{11} = 1 + \mathrm{i}$

3 (a) 2×3 augmented matrix $\begin{bmatrix} 2 & 3 & 7 \\ 1 & 2 & -2 \end{bmatrix}$, $x = 20$, $y = -11$ (b) 3×4 augmented matrix $\begin{bmatrix} 3 & 6 & -1 & -4 \\ -2 & -4 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $x_1 = -1 - 2x_2$, x_2 free, $x_3 = 1$, (c) 3×3 augmented matrix $\begin{bmatrix} 1 & 1 & -2 \\ 5 & 2 & 5 \\ 1 & 2 & -7 \end{bmatrix}$, $x_1 = 3$, $x_2 = -5$

5 (a) $x_1 = 1 - x_2$, $x_3 = -1$, x_2 free (b) $x_1 = -1 - 2x_2$, $x_3 = -2$, $x_4 = 3$, x_2 free (c) $x_1 = 3 - 2x_3$, $x_2 = -1 - x_3$, x_3 free (d) $x_1 = 1 + \frac{2}{3}\mathrm{i}$, $x_2 = 1 - \frac{1}{3}\mathrm{i}$ (e) $x_1 = \frac{7}{11}x_4$, $x_2 = \frac{-2}{11}x_4$, $x_3 = \frac{6}{11}x_4$, x_4 free

7 (a) $x_1 = 4$, $x_3 = -2$, x_2 free (b) $x_1 = 1$, $x_2 = 2$, $x_3 = 2$ (c) Inconsistent system (d) $x_1 = 1$, x_2 and x_3 free

9 (a) $x_1 = \frac{2}{3}b_1 + \frac{1}{3}b_2$, $x_2 = \frac{-1}{3}b_1 + \frac{1}{3}b_2$ (b) Inconsistent if $b_2 \neq 2b_1$, otherwise solution is $x_1 = b_1 + x_2$ and x_2 arbitrary. (c) $x_1 = \frac{1}{4}\left(2b_1 + b_2\right)\left(1 - \mathrm{i}\right)$, $x_2 = \frac{1}{4}\left(\mathrm{i}b_2 - 2b_1\right)\left(1 - \mathrm{i}\right)$

11 The only solution is the trivial solution $p_1 = 0$, $p_2 = 0$, and $p_3 = 0$, which has nonnegative entries.

13 Augmented matrix with three right-hand sides reduces to $\begin{bmatrix} 1 & 0 & 2 & -1 & -3 \\ 0 & 1 & 1 & -1 & -3 \end{bmatrix}$ given solutions (a) $x_1 = 2$, $x_2 = 1$ (b) $x_1 = -1$, $x_2 = -1$ (c) $x_1 = -3$, $x_2 = -3$.

15 (a) $x = 0$, $y = 0$ or divide by xy and get $y = -8/5$, $x = 4/7$ (b) Either two of x, y, z are zero and the other arbitrary or all are nonzero, divide by xyz and obtain $x = -2z$, $y = z$, and z is arbitrary nonzero.

17 Suppose not and consider such a solution (x, y, z, w). At least one variable is positive and largest. Now examine the equation corresponding to that variable.

19 (a) Equation for $x_2 = 1/2$ is $a + b \cdot 1/2 + c \cdot (1/2)^2 = e^{1/2}$.

Section 1.4, Page 42

1 (b) and (d) are in reduced row form, (a), (e), (f), and (h) are in reduced row echelon form. Leading entries (a) $(1,1)$, $(3,3)$ (b) $(1,1)$, $(2,2)$, $(3,4)$ (c) $(1,2)$, $(2,1)$ (d) $(1,1)$, $(2,2)$ (e) $(1,1)$ (f) $(1,1)$, $(2,2)$, $(3,3)$ (g) $(1,2)$ (h) $(1,1)$

3 (a) 3 (b) 0 (c) 3, (d) 1 (e) 1

5 (a) $E_{21}(-1)$, $E_{31}(-2)$, $E_{32}(-1)$, $E_2\left(\frac{1}{4}\right)$, $E_{12}(1)$, $\begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$, rank 2, nullity 1 (b) $E_{21}(1)$, $E_{23}(-15)$, $E_{13}(-9)$, $E_{12}(-1)$, $E_1\left(\frac{1}{3}\right)$, $\begin{bmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -33 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, rank 3, nullity 1 (c) E_{12}, $E_1\left(\frac{1}{2}\right)$, $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, rank 2, nullity 2 (d) $E_1\left(\frac{1}{2}\right)$, $E_{21}(-4)$, $E_{31}(-2)$, $E_{32}(1)$, $E_{12}(-2)$, $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, rank 2, nullity 1 (e) E_{12}, $E_{21}(-2)$, $E_2\left(\frac{1}{9}\right)$, $E_{12}(2)$ $\begin{bmatrix} 1 & 1 & 0 & \frac{22}{9} \\ 0 & 0 & 1 & \frac{2}{9} \end{bmatrix}$, rank 2, nullity 2 (f) E_{12}, $E_{31}(-1)$, E_{23}, $E_2(-1)$, $E_{32}(3)$, $E_3\left(\frac{-1}{4}\right)$, $E_{23}(1)$, $E_{13}(-1)$, $E_{12}(-2)$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, rank 3, nullity 0

7 Systems are not equivalent since (b) has trivial solution, (a) does not. (a) $\operatorname{rank}\left(\widetilde{A}\right) = 2$, $\operatorname{rank}(A) = 2$, $\{(-1 + x_3 + x_4, 3 - 2x_2, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\}$ (b) $\operatorname{rank}\left(\widetilde{A}\right) = 3$, $\operatorname{rank}(A) = 3$, $\{(-2x_2, x_2, 0, 0) \mid x_2 \in \mathbb{R}\}$

9 $0 < \operatorname{rank}(A) < 3$

11 (a) Infinitely many solutions for all c (b) Inconsistent for all c (c) Inconsistent if $c = -2$, unique solution otherwise

13 Rank of augmented matrix equals rank of coefficient matrix independently of right-hand side, so system is always consistent. Solution is $x_1 = -a + 2b - c + 4x_4$, $x_2 = -b + a + \frac{1}{2}c - 2x_4$, $x_3 = \frac{1}{2}c - x_4$, x_4 free.

15 (a) 3 (b) solution set (c) $E_{23}(-5)$ (d) 0 or 1

17 (a) false, (b) true, (c) false, (d) false, (e) false

20 Consider what you need to do to go from reduced row form to reduced row echelon form.

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3 (a) $\operatorname{rank} A = 3$, (b) $\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

4 Work of jth stage: $j + 2\left[(n-1) + (n-2) + \cdots + (n-j)\right]$.

Section 2.1, Page 60

1 (a) $\begin{bmatrix} -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 8 \\ 6 & 3 \end{bmatrix}$ (d) not possible (e) $\begin{bmatrix} 7 & 4 & -1 \\ 10 & 4 & 4 \\ 2 & 4 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} x - 2 + 4y \\ 3x - 2 + y \\ -1 \end{bmatrix}$

3 (a) not possible (b) $\begin{bmatrix} -1 & -3 & -2 \\ -4 & -1 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$ (d) not possible (e) $\begin{bmatrix} 5 & 8 & 3 \\ 13 & 5 & -6 \end{bmatrix}$

5 (a) $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ (b) $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (c) $x\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$ (d) $x\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + y\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

7 $a = \frac{-2}{3}$, $b = \frac{2}{3}$, $c = \frac{-4}{3}$

9 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

11 $A + (B + C) = \begin{bmatrix} -1 & 2 & -3 \\ 5 & 1 & 5 \end{bmatrix} = (A + B) + C$, $A + B = \begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & 5 \end{bmatrix} = B + A$

16 Solve for A in terms of B with the first equation and deduce $B = \frac{1}{4}I$.

Section 2.2, Page 68

1 (a) $[11+3i]$, (b) $\begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix}$, (c) impossible (d) impossible (e) $\begin{bmatrix} 15+3i & 20+4i \\ -3 & -4 \end{bmatrix}$ (f) impossible (g) $[10]$ (h) impossible

3 (a) $\begin{bmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & -3 \\ 2 & 2 & 4 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & 0 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

5 $\begin{bmatrix} 10 & -1 & 1 \\ 2 & -4 & -2 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

7 (a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2i \end{bmatrix}$

(c) $\begin{bmatrix} 10 & -1 & 1 \\ -4 & 2 & -2 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ -3x_2 \\ x_3 \end{bmatrix}$

9 $f(A) = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$, $g(A) = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, $h(A) = \begin{bmatrix} -1 & -6 \\ -3 & -4 \end{bmatrix}$

11 $A^2 = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}$, $BA = \begin{bmatrix} 6 & 16 \end{bmatrix}$, $AC = \begin{bmatrix} 11 \\ 16 \end{bmatrix}$, $AD = \begin{bmatrix} -1 & 7 & 2 \\ -2 & 9 & 3 \end{bmatrix}$, $BC = [22]$, $CB = \begin{bmatrix} 2 & 4 \\ 10 & 20 \end{bmatrix}$, $BD = \begin{bmatrix} -2 & 14 & 4 \end{bmatrix}$

13 (b) is not nilpotent, the others are.

15 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ are nilpotent, $A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not nilpotent.

17 $\mathbf{uv} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -2 & 2 & 2 \end{bmatrix} \xrightarrow[E_{31}(-2)]{E_1(-1)} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so rank $\mathbf{uv} = 1$

19 $A(BC) = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} = (AB)C$, $c(AB) = \begin{bmatrix} 0 & 16 \\ 0 & 4 \end{bmatrix} = (cA)B = A(cB)$

23 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and try simple B like $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

27 Let $A_{m\times n} = [a_{ij}]$ and $B_{m\times n} = [b_{ij}]$. If $\mathbf{b} = [1,0,\ldots,0]^T$, $\begin{bmatrix} a_{11} & 0\cdots 0 \\ \vdots & \vdots \\ a_{m1} & 0\cdots 0 \end{bmatrix} = \begin{bmatrix} b_{11} & 0\cdots 0 \\ \vdots & \vdots \\ b_{m1} & 0\cdots 0 \end{bmatrix}$ so $a_{11} = b_{11}$, etc. By similar computations, you can show that for each i, j, $a_{ij} = b_{ij}$.

Section 2.3, Page 84

1 $(\pm1,\pm1)$ map to (a) $(\pm1,\mp1)$ (b) $\pm\left(\frac{7}{5},\frac{1}{5}\right)$, $\pm\left(\frac{-1}{5},\frac{7}{5}\right)$ (c) $\pm(1,1)$, $\pm(1,-1)$ (d) $\pm(2,-1)$, $\pm(0,1)$

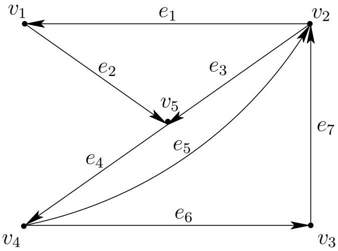
3 (a) $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 4 & -1 \end{bmatrix}$ (b) nonlinear (c) $\begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

5 Operator is T_A, $A = \begin{bmatrix} \sqrt{3} & -2 \\ 1 & 2\sqrt{3} \end{bmatrix}$, and in reverse order T_B, $B = \begin{bmatrix} \sqrt{3} & -1 \\ 2 & 2\sqrt{3} \end{bmatrix}$.

7 (d) is the only candidate and the only fixed point is $(0, 0, 0)$.

9 (a), (b) and (c) are Markov. First and second states are (a) $(0.2, 0.2, 0.6)$, $(0.08, 0.68, 0.24)$ (b) $\frac{1}{2}(0, 1, 1)$, $\frac{1}{2}(1, 1, 0)$ (c) $(0.4, 0.3, 0.4)$, $(0.26, 0.08, 0.66)$ (d) $(0, 0.25, 0.25)$, $(0.225, 0, 0.15)$

11 Powers of vertices 1–5 are 2, 4, 3, 5, 3, respectively. Graph is dominance directed, adjacency matrix is $\begin{bmatrix} 0&0&0&0&1 \\ 1&0&0&0&1 \\ 0&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&0 \end{bmatrix}$ and picture:



13 (a) $\begin{bmatrix} a_{k+3} \\ a_{k+2} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{5}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k+2} \\ a_{k+1} \\ a_k \end{bmatrix}$
(b) $\begin{bmatrix} a_{k+2} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

15 Points on a nonvertical line through the origin have the form (x, mx).

17 Use Exercise 27 of Section 2 and the definition of matrix operator.

Section 2.4, Page 97

1 (a) $\begin{bmatrix} 1&0&0 \\ 0&1&3 \\ 0&0&1 \end{bmatrix}$ (b) $\begin{bmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{bmatrix}$ (c) $\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&2 \end{bmatrix}$ (d) $\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&-1&1 \end{bmatrix}$ (e) $E_{12}(3)$ (f) $E_{31}(-a)$ (g) $E_2(3)$ (h) $E_{31}(2)$

3 (a) add 3 times third row to second (b) switch first and third rows (c) multiply third row by 2 (d) add -1 times second row to third (e) add 3 times second row to first (f) add $-a$ times first row to third (g) multiply second row by 3 (h) add 2 times first row to third

5 (a) $I_2 = E_{12}(-2)E_{21}(-1)\begin{bmatrix} 1&2 \\ 1&3 \end{bmatrix}$ (b) $\begin{bmatrix} 1&0&-1 \\ 0&1&1 \\ 0&0&0 \end{bmatrix} = E_{12}(-1)\, E_{32}(-2) \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 0&2&2 \end{bmatrix}$
(c) $\begin{bmatrix} 1&1&0 \\ 0&0&1 \end{bmatrix} = E_2\left(\frac{-1}{2}\right) E_{21}(-1) \begin{bmatrix} 1&1&0 \\ 1&1&-2 \end{bmatrix}$
(d) $\begin{bmatrix} 1&0&-2 \\ 0&1&\frac{1}{2}(1+\mathrm{i}) \end{bmatrix} = E\left(\frac{1}{1+\mathrm{i}}\right) E_{12} \begin{bmatrix} 0&1+\mathrm{i}&\mathrm{i} \\ 1&0&-2 \end{bmatrix}$

7 (a) strictly upper triangular, tridiagonal (b) upper triangular (c) upper and lower triangular, scalar (d) upper and lower triangular, diagonal (e) lower triangular, tridiagonal $\begin{bmatrix} 2&0 \\ 3&1 \end{bmatrix}$

9 $A = \begin{bmatrix} 0 & 2I_3 \\ C & D \end{bmatrix}$ with $C = [4, 1]$, $D = [2, 1, 3]$, $B = \begin{bmatrix} 0 & -I_2 \\ E & F \end{bmatrix}$ with $E = \begin{bmatrix} 0&0 \\ 2&2 \\ 1&1 \end{bmatrix}$ and $F = \begin{bmatrix} 1&2 \\ -1&1 \\ 3&2 \end{bmatrix}$,

$$AB = \begin{bmatrix} 0 + 2I_3E & 0\,(-I_2) + 2I_3F \\ C\,0 + DE & C\,(-I_2) + DF \end{bmatrix} = \begin{bmatrix} 2E & 2F \\ DE & -C + DF \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 4 & 4 & -2 & 2 \\ 2 & 2 & 6 & 4 \\ 5 & 5 & 6 & 10 \end{bmatrix}$$

11 $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

13 (a) $(1,-3,2)$, $(1,-3,2)$, not symmetric or Hermitian (b) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & -4 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & -4 \end{bmatrix}$, not symmetric or Hermitian (c) $\begin{bmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 2 \end{bmatrix}$, Hermitian, not symmetric (d) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & 2 \end{bmatrix}$, symmetric and Hermitian

15 (a) true (b) false (c) false (d) true (e) false

17 $Q(x, y, z) = \mathbf{x}^T A \mathbf{x}$ with $\mathbf{x} = [x, y, z]^T$ and $A = \begin{bmatrix} 2 & 2 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

19 $A^*A = \begin{bmatrix} 4 & -2 + 4\mathrm{i} \\ -2 - 4\mathrm{i} & 14 \end{bmatrix} = (A^*A)^*$ and $AA^* = \begin{bmatrix} 9 & 3 - 6\mathrm{i} \\ 3 + 6\mathrm{i} & 9 \end{bmatrix} = (AA^*)^*$

22 Since A and C are square, you can confirm that block multiplication applies and use it to square M.

29 Compare (i, j)th entries of each side.

32 Substitute the expressions for A into the right-hand sides and simplify them.

Section 2.5, Page 111

1 (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{-\mathrm{i}}{4} \\ 0 & \frac{1}{4} \end{bmatrix}$ (c) does not exist (DNE) (d) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

3 (a) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 20 \\ -11 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 6 & -1 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\frac{1}{15}\begin{bmatrix} 1 & -6 & 7 \\ 2 & 3 & -1 \\ 0 & 0 & 15 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix}$, $\frac{1}{3}\begin{bmatrix} -2 & 1 \\ 5 & -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

5 (a) $E_{21}(-3)$ (b) $E_2(-1/2)$ (c) $E_{21}(-1)\,E_{13}$ (d) $E_{12}(1)\,E_{23}(1)$ (e) $E_3\left(\frac{-1}{3}\right)E_1(-1)\,E_{21}(\mathrm{i})$

7 $\begin{bmatrix} 1 & -3 & -3 & 1 \\ 0 & 0 & -6 & -4 \\ 0 & -1 & -5 & -3 \end{bmatrix}$

9 Both sides give $\frac{1}{4}\begin{bmatrix} 2 & 1 & -2 \\ 2 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}$.

11 Both sides give $\frac{1}{12}\begin{bmatrix} 18 & 12 & 9 \\ 0 & 2 & -1 \\ -6 & 0 & 3 \end{bmatrix}$.

13 (a) any k, $\begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix}$ (b) $k \neq 1$, $\frac{1}{k-1}\begin{bmatrix} -1 & 0 & 1 \\ -k & k-1 & 1 \\ k & 0 & -1 \end{bmatrix}$ (c) $k \neq 0$, $\begin{bmatrix} 1 & 0 & 0 & \frac{-1}{k} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{-1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{k} \end{bmatrix}$

15 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, so both invertible, but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, not invertible.

17 (a) $N = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, $I + N + N^2 + N^3 = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $I + N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

19 $\mathbf{x} = (x, y)$, $\mathbf{x}^{(9)} \approx \begin{bmatrix} 1.00001 \\ -0.99999 \end{bmatrix}$, $\mathbf{F}\left(\mathbf{x}^{(9)}\right) \approx 10^{-6} \begin{bmatrix} -1.3422 \\ 2.0226 \end{bmatrix}$, $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x^2 + \sin(\pi x y) - 1 \\ x + y^2 + e^{x+y} - 3 \end{bmatrix}$, $J_{\mathbf{F}}(\mathbf{x}) = \begin{bmatrix} 2x + \cos(\pi x y), \pi y \cos(\pi x y)\, \pi x \\ 1 + e^{x+y}, \quad 2y + e^{x+y} \end{bmatrix}$

21 Move constant term to right-hand side and factor A on left.

24 Multiplication by elementary matrices does not change rank.

29 Assume M^{-1} has the same form as M and solve for the blocks in M using $MM^{-1} = I$.

Section 2.6, Page 125

1 (a) $A_{11} = -1$, $A_{12} = -2$, $A_{21} = -2$, $A_{22} = 1$ (b) $A_{11} = 1$, $A_{12} = 0$, $A_{21} = -3$, $A_{22} = 1$ (c) $A_{22} = 4$, all others are 0 (d)$A_{11} = 1$, $A_{12} = 0$, $A_{21} = -1 + \mathrm{i}$, $A_{22} = 1$

3 All except (c) are invertible. (a) 3, (b) $1 + i$, (c) 0, (d) -70, (e) 2i

5 Determinants of A and A^T are (a) -5 (b) 5 (c) 1 (d) 1

7 (a) $a \neq 0$ and $b \neq 1$(b) $c \neq 1$ (c) any θ

9 (a) $\begin{bmatrix} -2 & -2 & 2 \\ 4 & 4 & -4 \\ -3 & -3 & 3 \end{bmatrix}$, $0_{3,3}$ (b) $\begin{bmatrix} -1 & 0 & -3 \\ 0 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $-4I_3$ (c) $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$, $5I_2$ (d) $0_{4,4}$, $0_{4,4}$

11 (a) $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & -4 & -2 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & \mathrm{i} \\ -2\mathrm{i} & -1 \end{bmatrix}$

13 (a) $x = 5$, $y = 1$ (b) $x_1 = \frac{1}{4}(b_1 + b_2)$, $x_2 = \frac{1}{2}(b_1 - b_2)$ (c) $x_1 = \frac{-7}{6}$, $x_2 = \frac{5}{3}$, $x_3 = \frac{11}{2}$

17 Use elementary operations to clear the first column and factor out as many $(x_j - x_k)$ as possible in the resulting determinant.

19 Take determinants of both sides of the identity $AA^{-1} = I$.

21 Factor a term of -2 out of each row. What remains?

23 Use row operations to make the diagonal submatrices triangular.

24 Show that $J_n^2 = I_n$, which narrows down $\det J_n$.

Section 2.7, Page 143

1 $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4 & -3 \\ 0 & 0 & -1 \end{bmatrix}$, (a) $\mathbf{x} = (1, -2, 2)$ (b) $\mathbf{x} = \frac{1}{4}(3, -6, -4)$ (c) $\mathbf{x} = \frac{1}{4}(3, -2, -4)$ (d) $\mathbf{x} = \frac{1}{8}(3, 6, 8)$

3 $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & -2 & 4 & -2 & 2 & -1 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 4 & 4 & 2 & -2 & -2 & -1 \\ 2 & 0 & 2 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 & 0 & -1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -2 & -1 & 2 & 1 & -2 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ 2 & -4 & 0 & 0 & -2 & 4 \end{bmatrix}$ for (a), (b), (c), (d) same as (c)

5 $\begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & 2 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

7 If so, each factor must be nonsingular. Check the $(1,1)$th entry of an LU product.

13 For matrices M, N, block arithmetic gives $MN = [M\mathbf{n}_1, \ldots, M\mathbf{n}_n]$. Use this to show that $\operatorname{vec}(MN) = (I \otimes M)\operatorname{vec}(N)$. Also, $M\mathbf{n}_j = n_{1j}\mathbf{m}_1 + \cdots + n_{pj}\mathbf{m}_p$. Use this to show that $\operatorname{vec}(MN) = \left(N^T \otimes I\right)\operatorname{vec}(M)$. Then apply these to $AXB = A(XB)$.

Section 3.1, Page 158

1 (a)$(-2, 3, 1)$ (b) $(6, 4, 9)$

3 V is a vector space.

5 V is not a vector space because it is not closed under scalar multiplication.

7 V is not a vector space because it is not closed under vector addition or scalar multiplication.

9 V is a vector space.

11 (a) $T = T_A$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -4 \end{bmatrix}$, linear with range $\mathcal{C}(A) = \mathbb{R}^2$, equal to target (b) not linear (c) $T = T_A$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, linear with range $\mathcal{C}(A) = \operatorname{span}\{(1,1)\}$, not equal to target (d) not linear (e) not linear

13 (a) linear, range not V (b) not linear, (c) linear, range is V (d) linear, range not V

15 (a) identity operator is linear and invertible, $(\mathrm{id}_V)^{-1} = \mathrm{id}_V$

17 $M\mathbf{x} = (x_1 + 2, x_2 - 1, x_3 + 3, 1)$, so action of M is to translate the point in direction of vector $(2, -1, 3)$. $M^{-1} = \begin{bmatrix} I_3 & -\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ (think inverse action)

19 Write $c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$ by identity and distributive laws. Add $-(c\mathbf{0})$ to both sides.

27 Use the fact that $T_A \circ T_B = T_{AB}$ and $T_I = \mathrm{id}$

28 Use the fact that $T_A \circ T_B = T_{AB}$ and do matrix arithmetic.

Section 3.2, Page 168

1 W is not a subspace of V because W is not closed under addition and scalar multiplication.

3 W is a subspace.

5 W is a subspace.

7 Not a subspace, since W doesn't contain the zero element.

9 W is a subspace of V.

11 $\operatorname{span}\{(1,0),(0,1)\} = \mathbb{R}^2$ and $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{b}$ always has solution since coefficient matrix is invertible. So $\operatorname{span}\{(1,0),(-2,1)\} = \mathbb{R}^2$ and spans agree.

13 Write $ax^2 + bx + c = c_1 + c_2 x + c_3 x^2$ as matrix system $A\mathbf{c} = (a, b, c)$ by equating coefficients and see whether A is invertible, or use an ad hoc argument. (a) Spans $\mathcal{P}_2$. (b) Does not span $\mathcal{P}_2$ (can't get 1). (c) Spans $\mathcal{P}_2$. (d) Does not span $\mathcal{P}_2$ (can't get x).

15 $\mathbf{u} + \mathbf{w} = (4, 0, 4)$ and $\mathbf{v} - \mathbf{w} = (-2, 0, -2)$ so $\operatorname{span}\{\mathbf{u} + \mathbf{w}, \mathbf{v} - \mathbf{w}\} = \operatorname{span}\{(1,0,1)\} \subset \operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, since $\mathbf{u} + \mathbf{v}, \mathbf{v} - \mathbf{w} \in \operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. $\mathbf{u}$ is not a multiple of $(1, 0, 1)$, so spans are not equal.

17 (a) If $\mathbf{x}, \mathbf{y} \in U$ and $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x}, \mathbf{y} \in U \cap V$. Then $c\mathbf{x} \in U$ and $c\mathbf{x} \in V$ so $c\mathbf{x} \in U \cap V$, and $\mathbf{x} + \mathbf{y} \in U$ and $\mathbf{x} + \mathbf{y} \in V$ so $\mathbf{x} + \mathbf{y} \in U \cap V$. (b) Let $\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2 \in U + V$, where $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then $c\mathbf{u}_1 \in U$ and $c\mathbf{v}_1 \in V$ so $c(\mathbf{u}_1 + \mathbf{v}_1) = c\mathbf{u}_1 + c\mathbf{v}_1 \in U + V$, and similarly for sums.

19 Let A and B be $n \times n$ diagonal matrices. Then cA is diagonal matrix and $A + B$ is diagonal matrix so the set of diagonal matrices is closed under matrix addition and scalar multiplication.

20 (a) If $A = [a_{ij}]$, $\operatorname{vec}(A) = (a_{11}, a_{21}, a_{12}, a_{22})$ so for A there exists only one $\operatorname{vec}(A)$. If $\operatorname{vec}(A) = (a_{11}, a_{21}, a_{12}, a_{22})$, $A = [a_{ij}]$ so for $\operatorname{vec}(A)$ there exists only one A. Thus vec operation establishes a one-to-one correspondence between matrices in V and vectors in $\mathbb{R}^4$.

Section 3.3, Page 180

1 (a) none (b) $(1, 2, 1)$, $(2, 1, 1)$, $(3, 3, 2)$ (c) every vector redundant (d) none

3 (a) linearly independent (b) linearly independent, (c) every vector redundant (d) linearly independent

5 (a) $\left(\frac{1}{4}, \frac{-3}{4}\right)$ (b) $\left(\frac{1}{2}, 1, \frac{3}{2}\right)$ (c) (b, a, c) (d) $\left(\frac{1}{2} - \mathrm{i}, 1 - \frac{3}{2}\mathrm{i}\right)$

7 (a) $\mathbf{v} = 3\mathbf{u}_1 - \mathbf{u}_2 \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, (b) $\mathbf{u}_1$, $\mathbf{u}_2$, $(1, 0, -1)$

9 With the given information, $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ are possible minimal spanning sets.

11 All values except $c = 0$, 2 or $-\frac{7}{3}$

13 e_{11}

15 (c) $W = -2$, polynomials are linearly independent (d) $W = 4$, polynomials are linearly independent

17 $\begin{bmatrix} \frac{1}{1} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$

22 Assume $\mathbf{v}_i = \mathbf{v}_j$. Then there exists $c_i = -c_j \neq 0$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_i\mathbf{v}_i + \cdots + c_j\mathbf{v}_j + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

24 Start with a nontrivial linear combination of the functions that sums to 0 and differentiate it.

26 Domain and range elements $\mathbf{x}$ and $\mathbf{y}$ are given in terms of old coordinates. Express them in terms of new coordinates $\mathbf{x}'$, $\mathbf{y}'$ ($\mathbf{x} = P\mathbf{x}'$ and $\mathbf{y} = P\mathbf{y}'$.)

Section 3.4, Page 190

1 (a) $\left\{\left(-\frac{3}{2},0,3,1\right),\left(\frac{1}{2},1,0,0\right)\right\}$ (b) $\{(-4,1)\}$ (c) $\{(-3,1,1)\}$ (d) $\{\,\}$

3 (a) $\{(2,4),(0,1)\}$ (b) $\{(1,-1)\}$ (c) $\{(1,-2,1),(1,-1,2)\}$ (d) $\{(2,4,1),(-1,-2,1),(0,1,-1)\}$

5 (a) $\{(2,-1,0,3),(4,-2,1,3)\}$ (b) $\{(1,4)\}$ (c) $\{(1,1,2),(2,1,5)\}$ (d) $\{(2,-1,0),(4,-2,1),(1,1,-1)\}$

7 (a) $\operatorname{span}\{(2,2,1)\}$, $\left(\frac{2}{5},\frac{2}{5},\frac{1}{5}\right)$, yes (b) $\operatorname{span}\{(1,1)\}$, $\left(\frac{1}{2},\frac{1}{2}\right)$, no

9 (a) kernel $\operatorname{span}\{(1,0,-1)\}$, range $\operatorname{span}\{(1,1,2),(-2,1,-1)\}$, not onto or one-to-one (b) kernel $\left\{a+bx+cx^2 \mid a+b+c=0\right\}$, range $\mathbb{R}$ onto but not one-to-one

11 $\ker T = \operatorname{span}\{\mathbf{v}_1-\mathbf{v}_2+\mathbf{v}_3\}$, $\operatorname{range} T = \mathbb{R}^2$, T is onto but not one-to-one, hence not an isomorphism.

15 Calculate $T(\mathbf{0}+\mathbf{0})$.

17 Use definition of isomorphism, Theorem 3.9 and for onto, solve $c_1x+c_2(x-1)+c_3x^2=a+bx+cx^2$ for c_i's.

19 Since A is nilpotent, there exists m such that $A^m=\mathbf{0}$ so $\det(A^m)=(\det A)^m=0$ and $\det A=0$. Also since A is nilpotent, by Exercise 17 of Section 2.4, $(I-A)^{-1}=I+A+A^2+\ldots+A^{m-1}$.

Section 3.5, Page 196

1 (a) None (b) Any subset of three vectors (c) Any two of $\{(2,-3,1),(4,-2,-3),(0,-4,5)\}$ and $(1,0,0)$

3 $\mathbf{w}_1$ could replace $\mathbf{v}_2$.

5 $\mathbf{w}_1$ could replace $\mathbf{v}_2$ or $\mathbf{v}_3$, $\mathbf{w}_2$ could replace any of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ and $\mathbf{w}_1,\mathbf{w}_2$ could replace any two of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$.

7 $(0,1,1)$, $(1,0,0)$, $(0,1,0)$ is one choice among many.

9 (a) true (b) false (c) true (d) true (e) true (f) true

12 Suppose not and form a nontrivial linear combination of $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_r,\mathbf{w}$. Could the coefficient of $\mathbf{w}$ be nonzero?

13 If $c_{1,1}e_{1,1}+\cdots+c_{n,n}e_{n,n}=0$, $c_{a,b}=0$ for each a,b because $e_{a,b}$ is the only matrix with a nonzero entry in the (a,b)th position.

14 The union of bases for U and V will work. The fact that if $\mathbf{u}+\mathbf{v}=\mathbf{0}$, $\mathbf{u}\in U$, $\mathbf{v}\in V$, then $\mathbf{u}=\mathbf{v}=\mathbf{0}$, helps.

16 Dimension of the space is $n(n+1)/2$.

21 $\left\{I,A,A^2,\ldots,A^{n^2}\right\}$ must be linearly dependent since $\dim(\mathbb{R}^{n,n})=n^2$. Examine a nontrivial linear combination summing to zero.

Section 3.6, Page 206

1 Bases for row, column, and null spaces: $\{(1,0,3,0,2),(0,1,-2,1,-1)\}$, $\{(3,1,2),(5,2,3)\}$, $\{(-3,2,1,0,0),(0,-1,0,1,0),(-2,1,0,0,1)\}$

3 Bases by row and column algorithms: (a) $\{(1,0,1),(0,1,-1)\}$, $\{(0,-1,1),(2,1,1)\}$ (b) $\left\{\left(1,0,\frac{1}{2}\right),(0,1,0)\right\}$, $\{(2,-1,1),(2,0,1))\}$ (c) $\{(1,0),(0,1)\}$,

$\{(1,-1),(2,2)\}$ (d) $\left\{1+x^2, x-5x^2\right\}$, $\left\{1+x^2,-2-x+3x^2\right\}$

5 Bases for row, column, and null spaces: (a) $\{(2,0,-1)\}$, $\{1\}$, $\left\{\left(\frac{1}{2},0,1\right),(0,1,0)\right\}$ (b) $\{(1,2,0,0,1),(0,0,1,1,0)\}$, $\{(1,1,3),\ (0,1,2)\}$, $\{(-2,1,0,0,0),(0,0,-1,1,0),(-1,0,0,0,1)\}$ (c) $\{(1,0,-10,8,0),(0,1,5,-2,0),(0,0,0,0,1)\}$, $\{(1,1,2,2),(2,3,3,4),(0,1,0,1)\}$, $\{(10,-5,1,0,0),(-8,2,0,1,0)\}$ (d) $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, $\{\}$

7 (a) $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3+c_4\mathbf{v}_4=\mathbf{0}$, where $c_1=-2c_3-2c_4$, $c_2=-c_3$, and c_3,c_4 are free, $\dim\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}=2$ (b) $c_1x+c_2\left(x^2+x\right)+c_3\left(x^2-x\right)=0$ where $c_1=2c_3$, $c_2=-c_3$, and c_3 is free, $\dim\operatorname{span}\left\{x,x^2+x,x^2-x\right\}=2$ (c) $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3+c_4\mathbf{v}_4=\mathbf{0}$, where $c_1=-c_3$, $c_2=\frac{1}{2}c_3$, $c_4=0$ and c_3 is free, $\dim\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}=3$

9 (a) $\dim\mathcal{C}(A)=2$, $\dim\mathcal{C}(B)=2$ (b) $\dim\mathcal{C}\left(\left[\,A\ B\,\right]\right)=3$ (c) $\dim\mathcal{C}(A)\cap\mathcal{C}(B)=2+2-3=1$

11 $\mathcal{C}(A)\cap\mathcal{C}(B)=\operatorname{span}\{(1,1,-1)\}$

13 Since $A\mathbf{x}=\mathbf{b}$ is a consistent, $\mathbf{b}\in\mathcal{C}(A)$. If $\{\mathbf{a}_i\}$, the set of columns of A, has redundant vectors in it, $c_1\mathbf{a}_1+c_2\mathbf{a}_2+\cdots+c_n\mathbf{a}_n=\mathbf{0}$ for some nontrivial $\mathbf{c}$.

15 What does $\mathbf{b}\notin\mathcal{C}(A)$ tell you about r and m?

Section 4.1, Page 219

1 (a) -14, $\sqrt{34}$, $2\sqrt{5}$ (b) 7, $\sqrt{6}$, $\sqrt{14}$ (c) 8, $\sqrt{10}$, $\sqrt{26}$ (d) $12-6\mathrm{i}$, $\sqrt{10}$, $\sqrt{26}$ (e) 4, $\sqrt{30}$, $\sqrt{6}$ (f) -4, $2\sqrt{3}$, $\sqrt{30}$

3 (a) $\sqrt{145}/145$ (b) 0 (c) $\sqrt{21}/6$ (d) $\sqrt{10}/10$

5 (a) $36\mathbf{k}$ (b) $-5\mathbf{i}-\mathbf{j}+5\mathbf{k}$ (c) $(-2,-2,4)$

7 $\|\mathbf{u}\|=\sqrt{30}$, $\|c\mathbf{u}\|=3\sqrt{30}$,$\|\mathbf{v}\|=4$,$\|\mathbf{u}+\mathbf{v}\|=\sqrt{30}$, $\|\mathbf{u}+\mathbf{v}\|\leq\|\mathbf{u}\|+\|\mathbf{v}\|=4+\sqrt{30}$

9 $\mathbf{u}\times\mathbf{v}=(-6,4,-8)$, $\mathbf{v}\times\mathbf{u}=(6,-4,8)$, $(c\mathbf{u})\times\mathbf{v}=c(\mathbf{u}\times\mathbf{v})=\mathbf{u}\times(c\mathbf{v})=(12,-8,16)$, $\mathbf{u}\times\mathbf{w}=(4,1,-2)$, $\mathbf{v}\times\mathbf{w}=-(6,7,8)$ $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=(-2,5,-10)$, $(\mathbf{u}+\mathbf{v})\times\mathbf{w}=-(2,6,10)$

11 $\mathbf{u}_n=\left(\frac{2}{n},\frac{\frac{1}{n^2}+1}{2+\frac{3}{n}+\frac{5}{n^2}}\right)\to\left(0,\frac{1}{2}\right)$

13 Let $\mathbf{u}=(u_1,\ldots,u_n)\in\mathbb{R}^n$, $\mathbf{v}=(v_1,\ldots,v_n)\in\mathbb{R}^n$, and $c\in\mathbb{R}$. Then $(c\mathbf{u})\cdot\mathbf{v}=(cu_1)v_1+\cdots+(cu_n)v_n$ and $\mathbf{v}\cdot(c\mathbf{u})=v_1(cu_1)+\cdots+v_n(cu_n)$ so $(c\mathbf{u})\cdot\mathbf{v}=\mathbf{v}\cdot(c\mathbf{u})$. Similarly, show $(c\mathbf{u})\cdot\mathbf{v}=\mathbf{v}\cdot(c\mathbf{u})=c(\mathbf{v}\cdot\mathbf{u})=c(\mathbf{u}\cdot\mathbf{v})$.

17 $\|c\mathbf{v}\|=|c|\,\|\mathbf{v}\|$ by basic norm law (2). Since $c\in\mathbb{R}$ and $c>0$, $\|c\mathbf{v}\|=c\,\|\mathbf{v}\|$. So a unit vector in direction of $c\mathbf{v}$ is $c\mathbf{v}/c\,\|\mathbf{v}\|=\mathbf{v}/\,\|\mathbf{v}\|$.

19 Apply the triangle inequality to $\mathbf{u}+(\mathbf{v}-\mathbf{u})$ and $\mathbf{v}+(\mathbf{u}-\mathbf{v})$.

Section 4.2, Page 230

1 (a) 2.1176 (b) 1.6383 (c) 1.0018

3 (a) $(-2,-1)$, $-\sqrt{5}$ (b) $\frac{10}{9}(2,2,1)$, $\frac{10}{3}$ (c) $\frac{-1}{2}(1,1,1,1)$, -1

5 (a) $|\mathbf{v}_1\cdot\mathbf{v}_2|=1\leq\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|=\sqrt{15}$ (b) $|\mathbf{v}_1\cdot\mathbf{v}_2|=0\leq\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|=\sqrt{6}$ (c) $|\mathbf{v}_1\cdot\mathbf{v}_2|=19\leq\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|=2\sqrt{165}$

7 (a) $(M\mathbf{u})\cdot(M\mathbf{v})=1$, no (b) $(M\mathbf{u})\cdot(M\mathbf{v})=0$, yes (c) $(M\mathbf{u})\cdot(M\mathbf{v})=-13$, no

9 (a) $x+y-4z=-6$ (b) $x-2z=-4$

11 (a) $\mathbf{x}=\left(3,-\frac{2}{3}\right)$, $\mathbf{b}-A\mathbf{x}=\mathbf{0}$, $\|\mathbf{b}-A\mathbf{x}\|=0$, yes (b) $\mathbf{x}=\frac{1}{21}(9,-14)$, $\mathbf{b}-A\mathbf{x}=\frac{1}{21}(-4,-16,8)$,

$\|\mathbf{b}-A\mathbf{x}\| = \frac{\sqrt{336}}{21}$, no (c) $\mathbf{x} = \left(x_3+\frac{12}{13}, -x_3+\frac{23}{26}, x_3\right)$ where x_3 is free, $\mathbf{b}-A\mathbf{x} = \frac{1}{26}(32,-21,1,22)$, $\|\mathbf{b}-A\mathbf{x}\| = \frac{\sqrt{1950}}{26}$, no

13 $b = 0.3$, $a+b = 1.1$, $2a+b = 2$, $3a+b = 3.5$, $3.5a+b = 3.6$, least squares solution $a \approx 1.00610$, $b \approx 0.18841$, residual norm is $\|\mathbf{b}-A\mathbf{x}\| \approx 0.39962$

15 Express each norm in terms of dot products.

21 Use Example 3.41.

23 Examine the proof of Theorem 4.3 for points where real and complex dots might differ.

Section 4.3, Page 240

1 (a) orthogonal, linearly independent (b) linearly independent (c) orthonormal, orthogonal, linearly independent

3 $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$, and $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$ so $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is an orthogonal basis. $\mathbf{v}_1 \cdot \mathbf{v}_1 = 2$, $\mathbf{v}_2 \cdot \mathbf{v}_2 = 3$, and $\mathbf{v}_3 \cdot \mathbf{v}_3 = \frac{3}{2}$. Coordinates with respect to $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ are (a) $\left(\frac{3}{2}, \frac{-1}{3}, \frac{-5}{3}\right)$ (b) $\left(\frac{1}{2}, \frac{-1}{3}, \frac{1}{3}\right)$ (c) $\left(\frac{1}{2}, \frac{-5}{3}, \frac{11}{3}\right)$

5 (a) orthogonal, $\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ (b) not orthogonal (c) not orthogonal (d) not orthogonal (e) unitary, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -\mathrm{i} \\ 0 & -\sqrt{2}\mathrm{i} & 0 \\ 1 & 0 & \mathrm{i} \end{bmatrix}$ (f) unitary, $\frac{1}{\sqrt{3}}\begin{bmatrix} 1-\mathrm{i} & -\mathrm{i} \\ -\mathrm{i} & 1+\mathrm{i} \end{bmatrix}$

7 $H_{\mathbf{v}} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$, $H_{\mathbf{v}}\mathbf{u} = (3,0,0)$, $H_{\mathbf{v}}\mathbf{w} = (1,2,-2)$

9 (a) $\begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \end{bmatrix}$ (b) $\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} \mathrm{i} & 0 \\ 0 & 1 \end{bmatrix}$

11 Calculate both sides of each equation.

14 Let $\mathbf{u}, \mathbf{v}$ be columns of P, calculate $\|e^{\mathrm{i}\theta}\mathbf{u}\|$ and $\left(e^{\mathrm{i}\theta}\mathbf{u}\right) \cdot \left(e^{\mathrm{i}\theta}\mathbf{v}\right)$.

Section 4.4, Page 246

1 $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\operatorname{range}(T) = \operatorname{span}\{(1,1,0),(2,-1,1),(0,0,1)\}$, $\ker(T) = \{\mathbf{0}\}$

3 (a) $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ (b) $[\mathrm{id}]_{B,B'} = Q^{-1}I_2P = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ (c) $[\mathbf{w}]_{B'} = [\mathrm{id}]_{B,B'}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

5 $\frac{1}{25}\begin{bmatrix} 7 & 6 & -6 \\ 1 & 8 & -8 \end{bmatrix}$

10 Let B' be any other basis and use the chain of operators $V_{B'} \xrightarrow{\mathrm{id}_V} V_B \xrightarrow{T} V_B \xrightarrow{\mathrm{id}_V} V_{B'}$.

Section 5.1, Page 261

1 (a) -3, 2 (b) -1, -1, -1 (c) 2, 2, 3 (d) -2, 2 (e) $-2\mathrm{i}$, $2\mathrm{i}$

3 Eigenvalue, algebraic multiplicity, geometric multiplicity, bases: (a) $\lambda = -3$, 1, 1, $\{(2,1)\}$, $\lambda = 2$, 1, 1, $\{(1,1)\}$ (b) $\lambda = -1$, 3, 1, $\{(0,0,1)\}$, (c) $\lambda = 2$, 2, 2, $\{(1,0,0),(0,-1,1)\}$, $\lambda = 3$, 1, 1, $\{(1,1,0)\}$ (d) $\lambda = -2$, 1, 1, $\{(-1,1)\}$, $\lambda = 2$, 1, 1, $\{(1,1)\}$ (e) $\lambda = -2\mathrm{i}$, 1, 1, $\{(\mathrm{i},-1)\}$, $\lambda = 2\mathrm{i}$, 1, 1, $\{(\mathrm{i},1)\}$

5 $B = 3I - 5A$, so eigensystem for B consists of eigenpairs $\{-2,(1,1)\}$ and $\{8,(1,-1)\}$.

7 (a) $\operatorname{tr} A = 7 - 8 = -1 = -3 + 2$, (b) $\operatorname{tr} A = -1 - 1 - 1 = -3$, (c) $\operatorname{tr} A = 7 = 2{+}2{+}3$ (d) $\operatorname{tr} A = 0{+}0 = 0 = -1{+}1$ (e) $\operatorname{tr} A = 0 + 0 = 0 = -2\mathrm{i} + 2\mathrm{i}$

9 Eigenvalues of A and A^T are the same.

11 (a) No (b) No (c) No (d) Yes (e) No

13 Eigenvalues of A are $1, 2$. Eigenvalues of B are $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$. Eigenvalues of $A + B$ are $3 \pm \sqrt{3}$. Eigenvalues of AB are $3 \pm \sqrt{7}$. (a) Deny – $3 + \sqrt{3}$ not sum of 1 or 2 plus $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$. (b) Deny – $3 + \sqrt{7}$ not product of 1 or 2 times $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$.

17 If A is invertible, $\lambda \neq 0$, then $A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v}$.

19 For λ eigenvalue of A with eigenvector $\mathbf{v}$, $(I - A)\mathbf{v} = I\mathbf{v} - A\mathbf{v} = \mathbf{v} - \lambda\mathbf{v} = (1 - \lambda)\mathbf{v}$. Since $|\lambda| < 1$, $1 - \lambda > 0$.

20 Use part (1) of Theorem 5.1.

22 Deal with the 0 eigenvalue separately. If λ is an eigenvalue of AB, multiply the equation $AB\mathbf{x} = \lambda\mathbf{x}$ on the left by B.

Section 5.2, Page 270

1 All except (d) have distinct eigenvalues, so are diagonalizable. For $\lambda = 1$ (d) has eigenspace of dimension two, so is not diagonalizable.

3 (a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ -2 & 1 & 0 & -1 \\ 0 & -1 & 0 & 3 \\ 0 & 2 & 0 & 0 \end{bmatrix}$

5 True in every case. (a), (b), and (c) satisfy $p(A) = 0$ and are diagonalizable, (d) is not diagonalizable and $p(A) \neq 0$.

7 $\mathcal{E}_\lambda(J_\lambda(2)) = \operatorname{span}\{(1,0)\}$, so $J_\lambda(2)$ is not diagonalizable (not enough eigenvectors). $J_\lambda(2)^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$, $J_\lambda(2)^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$, $J_\lambda(2)^4 = \begin{bmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{bmatrix}$, which suggests $J_\lambda(2)^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$.

9 $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

11 $\sin\left(\frac{\pi}{6}A\right) = \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{4}{5} + \frac{2}{5}\sqrt{3} \\ 0 & -1 \end{bmatrix}$, $\cos\left(\frac{\pi}{6}A\right) = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ 0 & 0 \end{bmatrix}$

13 Similar matrices have the same eigenvalues.

15 Examine $DB = BD$, with D diagonal and no repeated diagonal entries.

17 You may find Exercise 16 and Corollary 5.1 helpful.

19 (b) $f_n = (\frac{1+\sqrt{5}}{2})^n(\frac{5+\sqrt{5}}{10}) + (\frac{1-\sqrt{5}}{2})^n(\frac{5-\sqrt{5}}{10})$.

21 In one direction use the fact that diagonal matrices commute. In the other direction, prove it for a diagonal A first, then use the diagonalization theorem.

Section 5.3, Page 280

1 (a) 2, dominant eigenvalue 2 (b) 0, no dominant eigenvalue (c) 0, no dominant eigenvalue (d) 1, dominant eigenvalue -1 (e) $\frac{1}{2}$, dominant eigenvalue $\frac{-1}{2}$

3 (a) $\mathbf{x}^{(k)} = \begin{bmatrix} 2\left(\frac{1}{2}\right)^k - \left(\frac{-1}{2}\right)^k \\ -2\left(\frac{1}{2}\right)^k + 2\left(\frac{-1}{2}\right)^k \end{bmatrix}$ (b) $\mathbf{x}^{(k)} = \begin{bmatrix} 2^k \\ 3^{k+1} - 2^k \\ 2^k \end{bmatrix}$ (c) $\mathbf{x}^{(k)} = \begin{bmatrix} 13 \cdot 2^k - 10 \cdot 3^k \\ -13 \cdot 2^k + 15 \cdot 3^k \end{bmatrix}$

5 (b), (c), and (e) give matrices for which all $\mathbf{x}^{(k)} \to \mathbf{0}$ as $k \to \infty$. Ergodic theorem applies to (d).

7 $\operatorname{diag}\{A, B\}$, where possibilities for A are $\operatorname{diag}\{J_2(1), J_2(1)\}$, $J_2(2)$ and possibilities for B are $\operatorname{diag}\{J_3(1), J_3(1), J_3(1)\}$, $\operatorname{diag}\{J_3(1), J_3(2)\}$, $\operatorname{diag}\{J_3(3)\}$

9 Characteristic polynomial for $J_3(2)$ is $(\lambda - 2)^3$ and $(J_3(2) - 2I_3)^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = 0.$

11 Eigenvalues are $\lambda \approx 0.1636 \pm 0.3393\mathrm{i}, 0.973$ with absolute values 0.3766 and 0.973. So population will decline at rate of approximately 2.7% per time period.

13 $\lambda^2 = s_1 f_2$, $\mathbf{p} = p_1\left(1, \sqrt{s_1/f_2}\right)$

15 (a) Sum of each column is 1. (c) Since a and b are nonnegative, (a, b) and $(1, -1)$ are linearly independent eigenvectors. Use diagonalization theorem.

Section 5.4, Page 286

1 A is real and $A = A^T$ in each case. (a) $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ (b) $\frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}$ (c) $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix}$

3 $P^T P = I$ in each case. (a) Unitarily diagonalizable by $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -\mathrm{i} & \mathrm{i} \\ 0 & 1 & 1 \\ \sqrt{2} & 0 & -1 \end{bmatrix}$ (b) Unitarily diagonalizable by $\frac{1}{\sqrt{2}}\begin{bmatrix} -\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{bmatrix}$ (c) Orthogonally diagonalizable by $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$

5 All of these matrices are normal.

7 Orthogonalize by $\begin{bmatrix} -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \end{bmatrix}$, let $a = (-1)^k + 2^{k+1}$, $b = (-1)^{k-1} + 2^k$, $c = (-1)^k + 2^{k-1}$ and $A^k = \frac{1}{3}\begin{bmatrix} a & b & b \\ b & c & c \\ b & c & c \end{bmatrix}$

9 $P = \begin{bmatrix} -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{-\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{-\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{bmatrix}$,
$B = P \operatorname{diag}\left\{1, \sqrt{2}, 4\right\} P^T$
$= \begin{bmatrix} \frac{2}{3} + \frac{\sqrt{2}}{2} & \frac{1}{3} & \frac{-2}{3} + \frac{\sqrt{2}}{2} \\ \frac{1}{3} & \frac{5}{3} & \frac{-1}{3} \\ \frac{-2}{3} + \frac{\sqrt{2}}{2} & \frac{-1}{3} & \frac{2}{3} + \frac{\sqrt{2}}{2} \end{bmatrix}$

12 Use orthogonal diagonalization and change of variable $\mathbf{x} = P\mathbf{y}$ for a general B to reduce the problem to one of a diagonal matrix.

16 First show it for a diagonal matrix with positive diagonal entries. Then use

Exercise 12 and the principal axes theorem.

17 $A^T A$ is symmetric. Now calculate $\|A\mathbf{x}\|^2$ for an eigenvector $\mathbf{x}$.

Section 5.5, Page 287

1 (a) $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2.5764 & -1.5370 \\ 0 & -1.5370 & 2.5764 \end{bmatrix}$
(b) $\begin{bmatrix} 1.41421 & 0 & 0 \\ 0 & -1.25708 & 0.44444\mathrm{i} \\ 0 & -0.44444\mathrm{i} & -0.15713 \end{bmatrix}$

3 (a) $-2, 3, 2$ (b) $3, 1, 2$ (c) $2, -1, \pm\sqrt{2}$

5 Eigenvalues of A are $2, -3$ and eigenvalues of $f(A)/g(A)$ are $0.6, 0.8$.

8 Do a change of variables $\mathbf{x} = P\mathbf{y}$, where P upper triangularizes A.

11 Equate $(1,1)$th coefficients of the equation $R^*R = RR^*$ and see what can be gained from it. Proceed to the $(2,2)$th coefficient, etc.

Section 5.6, Page 291

1 (a) $U = E_2(-1)$, $\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $V = I_3$ (b) $U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$, $V = I_2$ (c) $U = E_{12}E_{23}$, $\Sigma = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{5}}{5} & 0 & \frac{2\sqrt{5}}{5} \\ \frac{-2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{bmatrix}$ (d) $U = E_{12}E_2(-1)$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, $V = I_3$

3 Calculate U, Σ, V, null space, column space bases: (a) First three columns of $\underline{U}$, $\{\}$ (b) First two columns of U, third column of V

5 For (3), use a change of variables $\mathbf{x} = V\mathbf{y}$.

7 Use a change of variables $\mathbf{x} = V\mathbf{y}$ and check that $\|\mathbf{b} - A\mathbf{x}\| = \left\|U^T(\mathbf{b} - A\mathbf{x})\right\| = \left\|U^T\mathbf{b} - U^T AV\mathbf{y}\right\|$.

Section 5.7, Page 294

1 (a) -0.04283, 5.08996, 2.97644 ± 0.5603 (b) -0.48119, 3.17009, 1.3111 (c) $3.3123 \pm 2.8466\mathrm{i}$, $1.6877 \pm 0.8466\mathrm{i}$

3 Use Gershgorin to show that 0 is not an eigenvalue of the matrix.

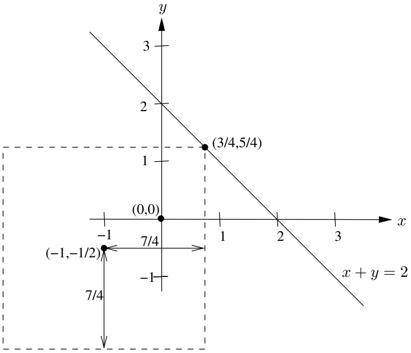
Section 6.1, Page 311

1 (a) 1-norms 6, 5, 2-norms $\sqrt{14}$, $\sqrt{11}$, ∞-norms 3, 3, distance ($\|(-5, 0, -4)\|$) in each norm 9, $\sqrt{41}$, 5 (b) 1-norms 7, 8, 2-norms $\sqrt{15}$, $\sqrt{14}$, ∞-norms 3, 2, distance ($\|(1, 4, -1, -2, -5)\|$) in each norm 13, $\sqrt{47}$, 5

3 (a) $\frac{1}{5}(1, -3, -1)$, $\frac{1}{\sqrt{11}}(1, -3, -1)$, $\frac{1}{3}(1, -3, 1)$ (b) $\frac{1}{7}(3, 1, -1, 2)$, $\frac{1}{\sqrt{15}}(3, 1, -1, 2)$, $\frac{1}{3}(3, 1, -1, 2)$ (c) $\frac{1}{3+\sqrt{10}}(2, 1, 3 + \mathrm{i})$, $\frac{1}{\sqrt{15}}(2, 1, 3 + \mathrm{i})$, $\frac{1}{\sqrt{10}}(2, 1, 3 + \mathrm{i})$

5 $\|\mathbf{u}\|_1 = 6$, $\|\mathbf{v}\|_1 = 7$, (1) $\|\mathbf{u}\|_1 > 0$, $\|\mathbf{v}\|_1 > 0$ (2) $\|-2\,(0,2,3,1)\|_1 = 12 = |-2|\,6$ (3) $\|(0,2,3,1)+(1,-3,2,-1)\|_1 = 7 \leq 6+7$

7 Ball of radius $7/4$ touches the line, so distance from point to line in ∞-norm is $7/4$.



9 Unit ball $B_1\left((1,1,1)\right)$ in $\mathbb{R}^3$ with infinity norm is set of points (x,y,z) which are between the pairs of planes (1) $x=0$, $x=2$, (2) $y=0$, $y=2$ and (3) $z=0$, $z=2$.

11 Set $\mathbf{v} = (-1,1)$, $\mathbf{v}-\mathbf{v}_n = \left(\frac{-1}{n}, -e^{-n}\right)$ so $\|\mathbf{v}-\mathbf{v}_n\|_1 = \left(\frac{1}{n}+e^{-n}\right)\overrightarrow{\scriptstyle n\longrightarrow\infty}0$ and $\|\mathbf{v}-\mathbf{v}_n\|_2 = \sqrt{(\frac{1}{n})^2+(e^{-n})^2} \to 0$, as $n \to \infty$. So $\lim_{n\to\infty}\mathbf{v}_n$ is the same in both norms.

13 Answer: $\max\{|\,\{|a|+|b|\,, |c|+|d|\}$. Note that a vector of length one has one coordinate equal to ± 1 and the other at most 1 in absolute value.

14 Let $\mathbf{u} = (u_1,\ldots,u_n)$, $\mathbf{v} = (v_1,\ldots,v_n)$, so $|u_1|+\cdots+|u_n| \geq 0$. Also $|cu_1|+\cdots+|cu_n| = |c|\,|u_1|+\cdots+|c|\,|u_n|$ and $|u_1+v_1|+\cdots+|u_n+v_n| \leq |u_1|+\cdots+|u_n|+|v_1|+\cdots+|v_n|$.

15 Observation that $\|A\|_F = \|\mathrm{vec}\,(A)\|_2$ enables you to use known properties of the 2-norm.

Section 6.2, Page 320

1 (a) $|\langle\mathbf{u},\mathbf{v}\rangle| = 46$, $\|\mathbf{u}\| = \sqrt{97}$, $\|\mathbf{v}\| = \sqrt{40}$ and $46 \leq \sqrt{94}\sqrt{40} \approx 61.32$ (b) $|\langle\mathbf{u},\mathbf{v}\rangle| = \frac{1}{5}$, $\|\mathbf{u}\| = \frac{1}{\sqrt{3}}$, $\|\mathbf{v}\| = \frac{1}{2}$ and $\frac{1}{5} = 0.2 \leq \frac{1}{2\sqrt{3}} \approx 0.288$

3 $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$, $\mathrm{comp}_{\mathbf{v}}\,\mathbf{u}$, $\mathrm{orth}_{\mathbf{v}}\,\mathbf{u}$: (a) $\left(\frac{-23}{20},\frac{23}{10}\right)$, $\frac{46}{\sqrt{40}}$, $\left(\frac{63}{20},\frac{7}{10}\right)$ (b) $\frac{4}{5}x^3$, $\frac{2}{5}$, $x-\frac{4}{5}x^3$

5 If $\mathbf{x} = (x,y,z)$, equation is $4x-2y+2z=2$.

7 Only (1), since if, e.g., $\mathbf{x} = (0,1)$, then $\langle\mathbf{x},\mathbf{x}\rangle = -2 < 0$.

9 (a) orthogonal (b) not orthogonal or orthonormal (c) orthonormal

11 $1(-4)+2\cdot 3\cdot 1+2\,(-1) = 0$. For each $\mathbf{v}$ calculate $\frac{\langle\mathbf{v}_1,\mathbf{v}\rangle}{\langle\mathbf{v}_1,\mathbf{v}_1\rangle}\mathbf{v}_1 + \frac{\langle\mathbf{v}_2,\mathbf{v}\rangle}{\langle\mathbf{v}_2,\mathbf{v}_2\rangle}\mathbf{v}_2$. (a) $(11,7,8)$, $(11,7,8) \in V$ (b) $\left(\frac{2255}{437},\frac{486}{437},\frac{1129}{437}\right)$, $(5,1,3) \notin V$ (c) $(5,2,3)$, $(5,2,3) \in V$

13 $\mathbf{v}_i^T A\mathbf{v}_j = 0$ for $i \neq j$. Coordinate vectors: (a) $\left(\frac{7}{2},\frac{5}{6},\frac{1}{3}\right)$ (b) $\left(0,\frac{1}{3},\frac{1}{3}\right)$ (c) $(1,1,0)$

15 $ac+\frac{1}{2}\,(ad+bc)+\frac{1}{3}bd$

17 Express $\mathbf{u}$ and $\mathbf{v}$ in terms of the standard basis $\mathbf{e}_1,\mathbf{e}_2$ and calculate $\langle\mathbf{u},\mathbf{v}\rangle$.

18 Use the same technique as in Example 6.13.

19 Follow Example 6.8 and use the fact that $\|A\mathbf{u}\|^2 = (A\mathbf{u})^*\,A\mathbf{u}$.

20 (1) Calculate $\langle\mathbf{u},\mathbf{0}+\mathbf{0}\rangle$. (2) Use norm law (2), (3) and (2) on $\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle$.

22 Express $\|\mathbf{u}+\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2$ in terms of inner products and add.

23 Imitate the steps of Example 6.9.

Section 6.3, Page 331

1 (a) $(1,1,-1)$, $\frac{1}{3}(-2,7,5)$, $\frac{1}{13}(8,-2,6)$ (b) $(1,0,1)$, $\frac{1}{2}(1,8,-1)$ (c) $(1,1)$, $\frac{1}{2}(-1,1)$ (d) $(1,1,-1,-1)$, $(0,1,1,0)$, $\frac{1}{4}(5,-1,1,3)$

3 (a) $\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\frac{1}{15}\begin{bmatrix} 14 & 1 & -2 & 3 \\ 1 & 14 & 2 & -3 \\ -2 & 2 & 11 & 6 \\ 3 & -3 & 6 & 6 \end{bmatrix}$ (d) $\frac{1}{9}\begin{bmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{bmatrix}$

5 $\operatorname{proj}_V \mathbf{w}$, $\operatorname{orth}_V \mathbf{w}$: (a) $\frac{1}{6}(23,-5,7)$, $\frac{1}{6}(1,-1,-2)$ (b) $\frac{1}{3}(4,2,1)$, $\frac{1}{3}(-1,1,2)$ (c) $\frac{1}{3}(1,-1,1)$, $\frac{1}{3}(-1,1,2)$

7 $\operatorname{proj}_V x^3 = \frac{1}{10}(9x-2)$, $\left\|x^3 - \frac{1}{10}(9x-2)\right\| = \frac{3}{10\sqrt{7}}$

9 Use Gram–Schmidt algorithm on $\mathbf{w}_1 = (-1,1,1,-1)$, $\mathbf{w}_2 = (1,1,1,1)$, $\mathbf{w}_3 = (1,0,0,0)$, $\mathbf{w}_4 = (0,0,1,0)$ to obtain orthogonal basis $\mathbf{v}_1 = (-1,1,1,-1)$, $\mathbf{v}_2 = (1,1,1,1)$, $\mathbf{v}_3 = \left(\frac{1}{2},0,0,\frac{-1}{2}\right)$, $\mathbf{v}_4 = \left(0,\frac{-1}{2},\frac{1}{2},0\right)$.

11 Use Gram–Schmidt on columns of A and normalize to obtain orthonormal $\frac{1}{\sqrt{3}}(1,1,1)$ and $\frac{1}{\sqrt{42}}(1,2,-5)$, then projection matrix $\frac{1}{14}\begin{bmatrix} 5 & 6 & 3 \\ 6 & 10 & -2 \\ 3 & -2 & 13 \end{bmatrix}$. Use Gram–Schmidt on columns of B and normalize to obtain orthonormal $\frac{1}{3\sqrt{5}}(4,5,2)$, $\frac{1}{3\sqrt{70}}(-1,10,-23)$, then obtain same projection matrix.

14 If a vector $\mathbf{x} \in \mathbb{R}^3$ is projected into $\mathbb{R}^3$, the result is $\mathbf{x}$.

16 Use matrix arithmetic to calculate $\langle P\mathbf{u}, \mathbf{v} - P\mathbf{v}\rangle$.

18 For any $\mathbf{v} \in V$, write $\mathbf{b} - \mathbf{v} = (b - \mathbf{p}) + (\mathbf{p} - \mathbf{v})$, note that $\mathbf{b} - \mathbf{p}$ is orthogonal to $\mathbf{p} - \mathbf{v}$, which belongs to V, and take norms.

Section 6.4, Page 341

1 $V^\perp = \operatorname{span}\left\{\left(\frac{1}{2},\frac{5}{2},1,0\right), \left(\frac{-1}{2},\frac{-1}{2},0,1\right)\right\}$ and if A consists of the columns $\left(\frac{1}{2},\frac{5}{2},1,0\right)$, $\left(\frac{-1}{2},\frac{-1}{2},0,1\right)$, $(1,-1,2,0)$, $(2,0,-1,1)$, then $\det A = 18$ which shows that the columns of A are linearly independent, hence a basis of $\mathbb{R}^4$.

3 $V^\perp = \operatorname{span}\left\{\frac{3}{14} - \frac{38}{35}x + x^2\right\}$

5 $V^\perp = \operatorname{span}\left\{\left(-2,\frac{-1}{2},1\right)\right\}$ and $\left(V^\perp\right)^\perp = \operatorname{span}\left\{\left(\frac{1}{2},0,1\right), \left(\frac{-1}{4},1,0\right)\right\}$ which is V since $(1,0,2) = 2\left(\frac{1}{2},0,1\right)$ and $(0,2,1) = \left(\frac{1}{2},0,1\right) + 2\left(\frac{-1}{4},1,0\right)$.

7 Q, R, $\mathbf{x}$: (a) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5} & \frac{-3}{5\sqrt{2}} \end{bmatrix}$, $\begin{bmatrix} 5 & 2 \\ 0 & \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} \frac{9}{5} \\ \frac{-5}{2} \end{bmatrix}$ (b) Caution: this matrix is rank deficient. $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \end{bmatrix}$, $\begin{bmatrix} \sqrt{5} & 0 & \frac{-10}{\sqrt{5}} \\ 0 & \sqrt{6} & \frac{12}{\sqrt{6}} \end{bmatrix}$, $\begin{bmatrix} 2x_3 - 3 \\ -2x_3 + 2 \\ x_3 \end{bmatrix}$, x_3 free (c) $\frac{1}{2}\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 1 & \sqrt{2} & \frac{-1}{3} \\ -1 & \sqrt{2} & \frac{1}{3} \\ -1 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & \frac{3}{2} \\ 0 & \sqrt{2} & \frac{3}{2}\sqrt{2} \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$, $\begin{bmatrix} \frac{-1}{2} \\ \frac{9}{2} \\ \frac{-5}{3} \end{bmatrix}$

11 (a) Inclusion $U^\perp + V^\perp \subset (U \cap V)^\perp$follows from the definition and inclusion $U \cap V \subset U + V$. For the converse, show that $(\mathbf{v} - \operatorname{proj}_U \mathbf{v})$ is orthogonal to all $\mathbf{u} \in U$. (b) Use (a) on $U^\perp$, $V^\perp$.

12 Show that if $A^T A\mathbf{y} = \mathbf{0}$, then $A\mathbf{y} = \mathbf{0}$.

Section 6.5, Page 347

1 Frobenius, 1-, and ∞-norms: (a) $\sqrt{14}$, 3, 5 (b) $3\sqrt{3}$, 5, 5 (c) $2\sqrt{17}$, 10, 9 $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & -3 & 0 & -1 \\ 1 & 1 & -2 & 0 \\ -2 & 1 & 6 & 0 \end{bmatrix}$

3 Verify that the perturbation theorem is valid for $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$, $\delta A = 0.05A$, and $\delta\mathbf{b} = -0.1\mathbf{b}$. Calculate $c = \left\|A^{-1}\delta A\right\| = 0.05\left\|I_3\right\| = 0.05 < 1$, $\frac{\|\delta A\|}{\|A\|} = 0.05$, $\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} = 0.05$, $\operatorname{cond}(A) \approx 6.7807$. Hence, $\frac{\operatorname{cond}(A)}{1-c}\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}\right] \approx 0.71376$. Now calculate $\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{1}{7}$ and $\frac{1}{7} \approx 0.142 < 0.71376$.

5 Use the triangle inequality on A and and Banach lemma on A^{-1}.

6 Factor out A and use Banach lemma.

10 Examine $\|A\mathbf{x}\|$ with $\mathbf{x}$ an eigenvector belonging to λ with $\rho(A) = |\lambda|$ and use definition of matrix norm.

11 If eigenvalue λ satisfies $|\lambda| > 1$, consider $\|A^m\mathbf{x}\|$ with $\mathbf{x}$ an eigenvector belonging to λ. For the rest, use the Jordan canonical form theorem.

13 (a) Make change of variables $\mathbf{x} = V\mathbf{y}$ and note $\left\|U^T AV\mathbf{x}\right\|_2 = \|A\mathbf{y}\|_2$, $\|\mathbf{x}\| = \|V\mathbf{y}\|$. (c) Use SVD of A.

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1 $\mathbf{x} = (0.4, 0.7)$, $\|\delta\mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty = 1.6214$, $\operatorname{cond}(A)\|\delta\mathbf{b}\|_\infty / \|\mathbf{b}\|_\infty = 1.8965$

3 $Q = \frac{\sqrt{2}}{10}\begin{bmatrix} 3\sqrt{2} & 4 & -4 \\ 0 & 5 & 5 \\ 4\sqrt{2} & -3 & 3 \end{bmatrix}$, $R = \begin{bmatrix} 5 & 2 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$, $\mathbf{x} = \frac{1}{10}(4, 5)$, $\|\mathbf{b} - A\mathbf{x}\|_2 = \sqrt{\frac{9}{2}}$

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