附录 A 部分习题答案

第 1 章

1. 67 600 000; 19 656 000 **2.** 1296 **4.** 24; 4 **5.** 144; 18 **6.** 2401 **7.** 720; 72; 144; 72 **8.** 120; 1260; 34 650 **9.** 27 720 **10.** 40 320; 10 080; 1152; 2880; 384 **11.** 720; 72; 144 **12.** 24 300 000; 17 100 720 **13.** 190 **14.** 2 598 960 **16.** 42; 94 **17.** 604 800 **18.** 600 **19.** 896; 1000; 910 **20.** 36; 26 **21.** 35 **22.** 18 **24.** 48 **25.** $52!/(13!)^4$ **27.** 27 720 **28.** 65 536; 2520 **29.** 12 600; 945 **30.** 564 480 **31.** 165; 35 **32.** 1287; 14 112 **33.** 220; 572

第 2 章

9. 74 **10.** 0.4; 0.1 **11.** 70; 2 **12.** 0.5; 0.32; 149/198 **13.** 20 000; 12 000; 11 000; 68 000; 10 000 **14.** 1.057 **15.** 0.0020; 0.4226; 0.0475; 0.0211; 0.000 24 **17.** $9.109\ 47 \times 10^{-6}$ **18.** 0.048; **19.** 5/18 **20.** 0.9052 **22.** $(n+1)/2^n$ **23.** 5/12 **25.** 0.4 **26.** 0.492 929 **27.** 0.0888; 0.2477; 0.1243; 0.2099 **30.** 1/18; 1/6; 1/2 **31.** 2/9; 1/9 **33.** 70/323 **36.** 0.0045; 0.0588 **37.** 0.0833; 0.5 **38.** 4 **39.** 0.48 **40.** 1/64; 21/64; 36/64; 6/64 **41.** 0.5177 **44.** 0.3; 0.2; 0.1 **46.** 5 **48.** 1.0604×10^{-3} **49.** 0.4329 **50.** 2.6084×10^{-6} **52.** 0.091 45; 0.4268 **53.** 12/35 **54.** 0.0511 **55.** 0.2198; 0.0343

第 3 章

1. 1/3 **2.** 1/6; 1/5; 1/4; 1/3; 1/2; 1 **3.** 0.339 **5.** 6/91 **6.** 1/2 **7.** 2/3 **8.** 1/2 **9.** 7/11 **10.** 0.22 **11.** 1/17; 1/33 **12.** 0.504; 0.3629 **14.** 35/768; 210/768 **15.** 0.4848 **16.** 0.9835 **17.** 0.0792; 0.264 **18.** 0.331; 0.383; 0.286; 48.62 **19.** 44.29; 41.18 **20.** 0.4; 1/26 **21.** 0.496; 3/14; 9/62 **22.** 5/9; 1/6; 5/54 **23.** 4/9; 1/2 **24.** 1/3; 1/2 **26.** 20/21; 40/41 **28.** 3/128; 29/1536 **29.** 0.0893 **30.** 7/12; 3/5 **33.** 0.76, 49/76 **34.** 27/31 **35.** 0.62, 10/19 **36.** 1/2 **37.** 1/3; 1/5; 1 **38.** 12/37 **39.** 46/185 **40.** 3/13; 5/13; 5/52; 15/52 **41.** 43/459 **42.** 34.48 **43.** 4/9 **45.** 1/11 **48.** 2/3 **50.** 17.5; 38/165; 17/33 **51.** 0.65; 56/65; 8/65; 1/65; 14/35; 12/35; 9/35 **52.** 0.11; 16/89; 12/27; 3/5; 9/25 **55.** 9 **57.** (c) 2/3 **60.** 2/3; 1/3; 3/4 **61.** 1/6; 3/20 **65.** 9/13; 1/2 **69.** 9; 9; 18; 110; 4; 4; 8; 120 所有被 128 除 **70.** 1/9; 1/18 **71.** 38/64; 13/64; 13/64 **73.** 1/16; 1/32; 5/16; 1/4; 31/32 **74.** 9/19 **75.** 3/4; 7/12 **78.** $p^2/(1-2p+2p^2)$ **79.**

0.5550 **81.** 0.9530 **83.** 0.5; 0.6; 0.8 **84.** 9/19; 6/19; 4/19; 7/15; 53/165; 7/33 **89.** 97/142; 15/26; 33/102

第 4 章

1. $p(4)=6/91$; $p(2)=8/91$; $p(1)=32/91$; $p(0)=1/91$; $p(-1)=16/91$; $p(-2)=28/91$ **4.** 1/2; 5/18; 5/36; 5/84; 5/252; 1/252; 0; 0; 0; 0 **5.** $n-2i$; $i=0,\cdots,n$ **6.** $p(3)=p(-3)=1/8$; $p(1)=p(-1)=3/8$ **12.** $p(4)=1/16$; $p(3)=1/8$; $p(2)=1/16$; $p(0)=1/2$; $p(-i)=p(i)$; $p(0)=1$ **13.** $p(0)=0.28$; $p(500)=0.27$; $p(1000)=0.315$; $p(1500)=0.09$; $p(2000)=0.045$ **14.** $p(0)=1/2$; $p(1)=1/6$; $p(2)=1/12$; $p(3)=1/20$; $p(4)=1/5$ **17.** 1/4; 1/6; 1/12; 1/2 **19.** 1/2; 1/10; 1/5; 1/10; 1/10 **20.** 0.5918; 否; -0.108 **21.** 39.28; 37 **24.** $p=11/18$; 最大值 $=23/72$ **25.** 0.46, 1.3 **26.** 11/2; 17/5 **27.** $A/(p+1/10)$ **28.** 3/5 **31.** p^* **32.** $11-10\times(0.9)^{10}$ **33.** 3 **35.** -0.067; 1.089 **37.** 82.2; 84.5 **39.** 3/8 **40.** 11/243 **42.** $p\geqslant 1/2$ **45.** 3 **50.** 1/10; 1/10 **51.** $\mathrm{e}^{-0.2}$; $1-1.2\mathrm{e}^{-0.2}$ **53.** $1-\mathrm{e}^{-0.6}$; $1-\mathrm{e}^{-219.18}$ **56.** 253 **57.** 0.5768; 0.6070 **59.** 0.3935; 0.3033; 0.0902 **60.** 0.8886 **61.** 0.4082 **63.** 0.0821; 0.2424 **65.** 0.3935; 0.2293; 0.3935 **66.** $2/(2n+1)$; $2/(2n-2)$; e^{-1} **67.** $2/n$; $(2n-3)/(n-1)^2$; e^{-2} **68.** $(1-\mathrm{e}^{-5})^{80}$ **70.** $p+(1-p)\mathrm{e}^{-\lambda t}$ **71.** 0.1500; 0.1012 **73.** 5.8125 **74.** 32/243; 4864/6561; 160/729; 160/729 **78.** $18\times(17)^{n-1}/(35)^n$ **81.** 3/10; 5/6; 75/138 **82.** 0.3439 **83.** 1.5

第 5 章

2. $3.5\mathrm{e}^{-5/2}$ **3.** 否; 否 **4.** 1/2 **5.** $1-(0.01)^{1/5}$ **6.** 4; 0; ∞ **7.** 3/5; 6/5 **8.** 2 **10.** 2/3; 2/3 **11.** 2/5 **13.** 2/3; 1/3 **15.** 0.7977; 0.6827; 0.3695; 0.9522; 0.1587 **16.** $(0.9938)^{10}$ **18.** 22.66 **19.** (c)1/2; (d)1/4 **20.** 0.9994; 0.75; 0.977 **22.** 9.5; 0.0019 **23.** 0.9258; 0.1762 **26.** 0.0606; 0.0525 **28.** 0.8363 **29.** 0.9993 **32.** e^{-1}; $\mathrm{e}^{-1/2}$ **34.** e^{-1}; 1/3 **38.** 3/5 **40.** $1/y$

第 6 章

2. (a)14/39; 10/39; 10/39; 5/39 (b)84/429; 70/429; 70/429; 70/429; 40/429; 40/429; 40/429; 15/429 **3.** 15/26; 5/26; 5/26; 1/26 **4.** 25/169; 40/169; 40/169; 64/169 **7.** $p(i,j)=p^2(1-p)^{i+j}$ **8.** $c=1/8$; $E[X]=0$ **9.** $(12x^2+6x)/7$; 15/56; 0.8625; 5/7; 8/7 **10.** 1/2; $1-\mathrm{e}^{-a}$ **11.** 0.1458 **12.** $39.3\mathrm{e}^{-5}$ **13.** 1/6; 1/2 **15.** $\pi/4$ **16.** $n(1/2)^{n-1}$ **17.** 1/3 **18.** 7/9 **19.** 1/2 **21.** 2/5; 2/5 **22.** 否; 1/3 **23.** 1/2; 2/3; 1/20; 1/18 **25.** $\mathrm{e}^{-1}/i!$ **28.** $1/2\mathrm{e}^{-t}$; $1-3\mathrm{e}^{-2}$ **29.** 0.0326 **30.** 0.3372; 0.2061 **31.** 0.0829; 0.3766 **32.** e^{-2}; $1-3\mathrm{e}^{-2}$ **35.** 5/13; 8/13 **36.** 1/6; 5/6; 1/4; 3/4 **41.** $(y+1)^2x\mathrm{e}^{-x(y+1)}$; $x\mathrm{e}^{-xy}$; e^{-x} **42.** $1/2+3y/(4x)-y^3/(4x^3)$ **46.** $(1-2d/L)^3$ **47.** 0.79297 **48.** $1-\mathrm{e}^{-5\lambda a}$; $(1-\mathrm{e}^{-\lambda a})^5$ **52.** r/π **53.** r **56.** (a) $u/(v+1)^2$

第 7 章

1. 52.5/12　**2.** 324; 199.6　**3.** 1/2; 1/4; 0　**4.** 1/6; 1/4; 1/2　**5.** 3/2　**6.** 35　**7.** 0.9; 4.9; 4.2　**8.** $(1-(1-p)^N)/p$　**10.** 0.6; 0　**11.** $2(n-1)p(1-p)$　**12.** $(3n^2-n)/(4n-2)$; $3n^2/(4n-2)$　**14.** $m/(1-p)$　**15.** 1/2　**18.** 4　**21.** 0.9301; 87.5755　**22.** 14.7　**23.** 147/110　**26.** $n/(n+1)$; $1/(n+1)$　**29.** 437/35; 12; 4; 123/35　**31.** 175/6　**33.** 14　**34.** 20/19; 360/361　**35.** 21.2; 18.929; 49.214　**36.** $-n/36$　**37.** 0　**38.** 1/8　**41.** 6; 112/33　**42.** 100/19; 16 200/6137; 10/19; 3240/6137　**45.** 1/2; 0　**47.** $1/(n-1)$　**48.** 6; 7; 5.8192　**49.** 6.06　**50.** $2y^2$　**51.** $y^3/4$　**53.** 12　**54.** 8　**56.** $N(1-\mathrm{e}^{-10/N})$　**57.** 12.5　**63.** $-96/145$　**65.** 5.16　**66.** 218　**67.** $x[1+(2p-1)^2]^n$　**69.** 1/2; 1/16; 2/81　**70.** 1/2; 1/3　**72.** $1/i$; $[i(i+1)]^{-1}$; ∞　**73.** μ; $1+\sigma^2$; 是; σ^2　**79.** 0.176; 0.141

第 8 章

1. $\geqslant 19/20$　**2.** 15/17; $\geqslant 3/4$; $\geqslant 10$　**3.** $\geqslant 3$　**4.** $\leqslant 4/3$; 0.8428　**5.** 0.1416　**6.** 0.9431　**7.** 0.3085　**8.** 0.6932　**9.** $(327)^2$　**10.** 117　**11.** $\geqslant 0.057$　**13.** 0.0162; 0.0003; 0.2514; 0.2514　**14.** $n \geqslant 23$　**16.** 0.013; 0.018; 0.691　**18.** $\leqslant 0.2$　**23.** 0.769; 0.357; 0.4267; 0.1093; 0.112 184

第 9 章

1. 1/9; 5/9　**3.** 0.9735; 0.9098; 0.7358; 0.5578　**10.** (b)1/6　**14.** 2.585; 0.5417; 3.1267　**15.** 5.5098

附录 B　自检习题答案

第 1 章

1. (a) C,D,E,F 这 4 个字母共有 4! 种不同的排序方法. 对于每一种这样的排列, 可将 A,B 放在 5 个位置. 即可把它们放在 C,D,E,F 字母的前面, 或放在第 2 个位置, 等等. 但 A,B 本身又可以以 AB 或 BA 的方式嵌入这 5 个位置, 因此, 一共有 $2\times5\times4!=240$ 种方法. 另一种方法是想象 B 是粘在 A 的后面, 这样一共有 5! 种方法, 但也有可以 B 粘在 A 的前面, 这样也有 5! 种方法, 故一共有 $2\times5!=240$ 种方法.

(b) 6 个字母一共有 $6!=720$ 种排列方式, 其中有一半 A 在 B 前, 一半 A 在 B 后, 因此, A 在 B 之前的排列共有 $720/2=360$ 种.

(c) 由于 A, B, C 三个字母的排列共有 6 种, 因此, 在 720 种全部排列中, 有 $720/6=120$ 种排列为 A B C 这种顺序.

(d) A 在 B 之前的排列共有 $6!/2=360$ 种, 其中一半是 C 在 D 之前. 因此, A 在 B 前, C 在 D 前的排列共有 180 种.

(e) 若将 B 粘于 A 的后面, C 粘于 D 的后面, 这样共有 $4!=24$ 种方法. 但由于 A,B 的位置可以颠倒过来, 即 A 可以粘于 B 的后面, 类似, D 可粘于 C 的后面. 一共有 4 种不同情况, 因此, 共有 $4\times24=96$ 种排列.

(f) E 在最后共有 5! 种排列, 因此, 它不在最后共有 $6!-5!=600$ 种排列.

2. 由于 3 个国家有 3! 种次序. 而每个国家的人也有一个排序问题, 因此, 一共有 3!4!3!3! 种排序法.

3. (a) $10\times9\times8=720$

(b) $8\times7\times6+2\times3\times8\times7=672$

若A, B 都不入选, 则共有 $8\times7\times6$ 种选法. 若只选 A , 没有 B , 则有 $3\times8\times7$ 种选法. 故, A,B 中只有一人入选, 一共有 $2\times3\times8\times7$ 种选法.

(c) $8\times7\times6+3\times2\times8=384$

(d) $3\times9\times8=216$

(e) $9\times8\times7+9\times8=576$

4. (a) $\binom{10}{7}$　(b) $\binom{5}{3}\binom{5}{4}+\binom{5}{4}\binom{5}{3}+\binom{5}{5}\binom{5}{2}$

5. $\binom{7}{3,2,2}=210$

6. 一共有 $\binom{7}{3}=35$ 种位置的选择, 每种选择可做成 $(26)^3(10)^4$ 种牌子. 因此, 总共可做成 $35\times(26)^3(10)^4$ 种不同的牌子.

7. n 个中选 r 个等价于 n 个中剔除 $(n-r)$ 个. 因此, 等式两边经过计算是相等的.

8. (a) $10\times9\times9\cdots9=10\times9^{n-1}$

(b) $\binom{n}{i}9^{n-i}$, 一共有 $\binom{n}{i}$ 种位置选择, 这种位置上放上 0, 而其余 $n-i$ 个位置上可以任意放 $1,2,\cdots,9$.

9. (a) $\binom{3n}{3}$　(b) $3\binom{n}{3}$　(c) $\binom{3}{1}\binom{2}{1}\binom{n}{2}\binom{n}{1}=3n^2(n-1)$

(d) n^3　(e) $\binom{3n}{3}=3\binom{n}{3}+3n^2(n-1)+n^3$

10. 一共有 $9\times 8\times 7\times 6\times 5$ 个数, 其中没有两个数字是相同的. 若容许某一指定数可重复一次, 则共有 $\binom{5}{2}\times 8\times 7\times 6$ 个数, 因此, 只容许有一个数可重复出现两次的一共有 $9\times\binom{5}{2}\times 8\times 7\times 6$ 个数. 若在 5 位数中有两个数可重复, 对于确定的两个数, 一共有$7\times\dfrac{5!}{2!2!}$ 个数. 这样, 一共有 $\binom{9}{2}7\times\dfrac{5!}{2!2!}$ 个数, 其中有两个数字重复一次. 这样, 总共有

$$9\times 8\times 7\times 6\times 5+9\times\binom{5}{2}\times 8\times 7\times 6+\binom{9}{2}7\times\frac{5!}{2!2!}$$

个 5 位数.

11. (a) 我们可以将这个问题看成一个 7 阶段的试验. 首先从 10 对夫妇中选择 6 对夫妇, 这种选择的方法共有 $\binom{10}{6}$ 种方法. 然后在选出的 6 对夫妇中的每一对选出 1 人, 这样一共选出 6 人. 根据推广的计数法则可知, 一共有 $\binom{10}{6}2^6$ 种不同的选择方法.

(b) 首先从 10 对夫妻中选出 6 对夫妻, 这种选法一共有 $\binom{10}{6}$ 种. 然后, 从中选择 3 对夫妻, 这 3 对夫妻中的男人就是选中的人员, 这种选法一共有 $\binom{6}{3}$ 种选法. 依据计数法则可知, 一共有 $\binom{10}{6}\binom{6}{3}=\dfrac{10!}{4!3!3!}$ 种选择方法. 另一种方法是, 先从 10 个男人中选出 3 个男人, 然后从与这些男人无关的女人中选出 3 个女人. 这样一共有 $\binom{10}{7}\binom{7}{3}=\dfrac{10!}{3!3!4!}$ 种选择方法.

12. $\binom{8}{3}\binom{7}{3}+\binom{8}{4}\binom{7}{2}=3430$

上式左边第一项给出由 3 个女的和 3 个男的组成一个委员会的可能组成方式. 第二项是由 4 个女的和 2 个男的组成委员会的可能组成方式.

13. $(x_1+\cdots+x_5=4$的解的组数$)(x_1+\cdots+x_5=5$的解的组数$)(x_1+\cdots+x_5=6$的解的组数$)=\binom{8}{4}\binom{9}{4}\binom{10}{4}$

14. 总和为 j 的正向量共有 $\binom{j-1}{n-1}$ 个, 因此, 一共有 $\sum_{j=n}^{k}\binom{j-1}{n-1}$ 个这样的向量.

15. 先假定有 k 个学生通过了考试, 这样可有 $\binom{n}{k}$ 组. 由于每个组内各人成绩还有顺序, 因此, 由 k 个学生通过考试时, 一共有$\binom{n}{k}k!$ 种可能性. 显然, 总起来有 $\sum_{k=0}^{n}\binom{n}{k}k!$ 种可能的结果.

16. 由 4 个数组成的集合个数为 $\binom{20}{4} = 4845$. 其中不含前 5 个数的子集为 $\binom{15}{4} = 1365$. 它的反面, 即至少含有 $\{1, 2, \cdots, 5\}$ 中一个数的组数有 $4845 - 1365 = 3480$ 个. 另一种计算方法是 $\sum_{i=1}^{4}\binom{5}{i}\binom{15}{4-i}$.

17. 两边乘以 2, 得

$$n(n-1) = k(k-1) + 2k(n-k) + (n-k)(n-k-1)$$

上式右边经过整理得

$$k^2(1-2+1) + k(-1+2n-n-n+1) + n(n-1)$$

作为组合解释, 可考虑 n 个学生中, 有 k 个女生. 从 n 个学生中找出 2 个代表, 一共有 $\binom{n}{2}$ 种方法. 若两个代表全由女生组成的话, 一共有 $\binom{k}{2}$ 种组成方式. 若由一男一女组成的话, 一共有 $k(n-k)$ 种组成方法. 若全由男生组成一共有 $\binom{n-k}{2}$ 种方法. 将这些组合方法加起来, 就是 $\binom{n}{2}$.

18. 有 3 种方法可从单亲且有一个孩子的家庭中选择; 有 $3 \times 1 \times 2 = 6$ 种方法可从单亲且有二个孩子的家庭中选择; 有 $5 \times 2 \times 1 = 10$ 种方法可从有独生子的双亲家庭中选择; 有 $7 \times 2 \times 2 = 28$ 种方法可从有二个孩子的双亲家庭中选择; 有 $6 \times 2 \times 3 = 36$ 种方法可从有三个孩子的双亲家庭中选择. 总起来, 一共有 80 种可能的选择方法.

19. 首先选定三个位置放置数字, 然后在相应的位置中放置数字或字母. 这样一共有 $\binom{8}{3} \times 26 \times 25 \times 24 \times 23 \times 22 \times 10 \times 9 \times 8$ 块牌子. 如果三个数字必须放在连续的位置上, 数字的位置只有 6 种可能, 这样一共可有 $6 \times 26 \times 25 \times 24 \times 23 \times 22 \times 10 \times 9 \times 8$ 块牌子.

第 2 章

1. (a) $2 \times 3 \times 4 = 24$ (b) $2 \times 3 = 6$ (c) $3 \times 4 = 12$

(d) $AB = \{(c, \text{面}, i), (c, \text{米饭}, i), (c, \text{土豆}, i)\}$ (e) 8 (f) $ABC = \{(c, \text{米饭}, i)\}$

2. 记 A 为“买一套西服”, B 为“买一件衬衫”, C 为“买一条领带”, 则

$$P(A \cup B \cup C) = 0.22 + 0.30 + 0.28 - 0.11 - 0.14 - 0.10 + 0.06 = 0.51$$

(a) $1 - 0.51 = 0.49$

(b) 买两样以上的概率为

$$P(AB \cup AC \cup BC) = 0.11 + 0.14 + 0.10 - 0.06 - 0.06 - 0.06 + 0.06 = 0.23$$

因此, 正好买一样东西的概率为 $0.51 - 0.23 = 0.28$.

3. 根据对称性, 第 14 张牌可以是 52 张牌中的任意一张, 因此相关概率为 4/52. 更形式的说法是：52! 个结果中, 第 14 张牌为“A”(指红桃“A”, 或方块“A”, $\cdots$) 的概率为

$$p = \frac{4 \times 51 \times 50 \cdots 2 \times 1}{(52)!} = \frac{4}{52}$$

记事件 A 为"第 1 张'A'出现在第 14 张牌", 我们有

$$P(A)=\frac{48\times 47\cdots 36\times 4}{52\times 51\times\cdots 40\times 39}=0.0312$$

4. 记 D 为事件"最小温度为 70°C". 则

$$P(A\cup B)=P(A)+P(B)-P(AB)=0.7-P(AB)$$
$$P(C\cup D)=P(C)+P(D)-P(CD)=0.2+P(D)-P(DC)$$

利用事实, $A\cup B=C\cup D, AB=CD$, 将上述两式相减得:

$$0=0.5-P(D)$$

或 $P(D)=0.5$.

5.

$$\text{(a)}\ \frac{52\times 48\times 44\times 40}{52\times 51\times 50\times 49}=0.6761\quad \text{(b)}\ \frac{52\times 39\times 26\times 13}{52\times 51\times 50\times 49}=0.1055$$

6. 记 R 为"两球均为红球"的事件, B 为"两球均为黑球"的事件, 则

$$P(R\cup B)=P(R)+P(B)=\frac{3\times 4}{6\times 10}+\frac{3\times 6}{6\times 10}=\frac{1}{2}$$

7. (a) $1\Big/\binom{40}{8}=1.3\times 10^{-8}$　(b) $\binom{8}{7}\binom{32}{1}\Big/\binom{40}{8}=3.3\times 10^{-6}$

(c) $\binom{8}{6}\binom{32}{2}\Big/\binom{40}{8}+1.3\times 10^{-8}+3.3\times 10^{-6}=1.8\times 10^{-4}$

8. (a) $\dfrac{3\times 4\times 4\times 3}{\binom{14}{4}}=0.1439$　(b) $\binom{4}{2}\binom{4}{2}\Big/\binom{14}{4}=0.0360$　(c) $\binom{8}{4}\Big/\binom{14}{4}=0.0699$

9. 令 $S=\bigcup_{i=1}^{n}A_i$, 考虑随机地从 S 种选一元素, 则 $P(A)=N(A)/N(S)$, 有关结果可从命题 4.3 和 4.4 得到.

10. 当 1 号马的名次确定的情况下, 一共有 $5!=120$ 种可能排名. 因此, $N(A)=360$, 类似地, $N(B)=120, N(AB)=2\times 4!=48$. 由自检习题 9, 我们得 $N(A\cup B)=432$.

11. 一种办法是先计算它的补事件: 至少有一种花色在这一副牌中不出现. 记 A_i 表示"牌中没有花色 i", $i=1,2,3,4$. 则

$$\begin{aligned}P\Big(\bigcup_{i=1}^{4}A_i\Big)&=\sum_i P(A_i)-\sum_j\sum_{i:i<j}P(A_iA_j)+\cdots-P(A_1A_2A_3A_4)\\&=4\binom{39}{5}\Big/\binom{52}{2}-\binom{4}{2}\binom{26}{5}\Big/\binom{52}{5}+\binom{4}{3}\binom{13}{5}\Big/\binom{52}{5}\\&=4\binom{39}{5}\Big/\binom{52}{2}-6\binom{26}{5}\Big/\binom{52}{5}+4\binom{13}{5}\Big/\binom{52}{5}\end{aligned}$$

用 1 减去上述概率值就是所求事件的概率. 也可以从另一角度来解这个问题. 记 A 为"一副牌中 4 种花色都出现"的事件, 利用等式

$$P(A)=P(\mathrm{n},\mathrm{n},\mathrm{n},\mathrm{n},o)+P(\mathrm{n},\mathrm{n},\mathrm{n},o,\mathrm{n})+P(\mathrm{n},\mathrm{n},o,\mathrm{n},\mathrm{n})+P(\mathrm{n},o,\mathrm{n},\mathrm{n},\mathrm{n})$$

式中 $P(\mathrm{n},\mathrm{n},\mathrm{n},o,\mathrm{n})$ 表示第 1 张为新花色, 第 2、3 张都为新花色, 第 4 张为老花色 (即第 4 张花色为之前出现过的花色), 第 5 张为新花色. 这样

$$\begin{aligned}P(A)&=\frac{52\times39\times26\times13\times48+52\times39\times26\times36\times13}{52\times51\times50\times49\times48}\\&\quad+\frac{52\times39\times24\times26\times13+52\times12\times39\times26\times13}{52\times51\times50\times49\times48}\\&=\frac{52\times39\times26\times13\times(48+36+24+12)}{52\times51\times50\times49\times48}=0.2637\end{aligned}$$

12. 一共有 $(10)!/2^5$ 种分配方法, 将 5 对分配到 5 间房. 例如某两个人分配在第一间房, 又两个人分配在第二间房, 等等. 如果不计房间次序, 那么应该有 $(10)!/(5!2^5)$ 种分配方案. 一共有 $\binom{6}{2}\binom{4}{2}$ 种方法从前后卫中各选出两个人来组成前后卫组合. 但是将这 4 个人分成对的时候又有两种方式, 至于剩下的 2 个后卫, 只能分成一个组了. 剩下的 4 个前卫有 3 种方式分成 2 对 $(3=4!/(2!2^2))$, 这样,

$$P\{\text{刚好有两个房间是一个前卫和一个后卫混住的}\}=\frac{\binom{6}{2}\binom{4}{2}\times2\times3}{(10)!/(5!2^5)}=0.5714$$

13. 用 R 表示“两次均选上字母 R ”的事件, 对于事件 E 和 V 之定义是类似的. 则

$$P\{\text{两次选上同一字母}\}=P(R)+P(E)+P(V)=\frac{2}{7}\times\frac{1}{8}+\frac{3}{7}\times\frac{1}{8}+\frac{1}{7}\times\frac{1}{8}=\frac{3}{28}$$

14. 记 $B_1=A_1, B_i=A_i\Big(\bigcup_{j=1}^{i-1}A_j\Big)^{\mathrm{c}}, i>1$, 则

$$P\left(\bigcup_{i=1}^{\infty}A_i\right)=P\left(\bigcup_{i=1}^{\infty}B_i\right)=\sum_{i=1}^{\infty}P(B_i)\leqslant\sum_{i=1}^{\infty}P(A_i)$$

此处最后一个等式利用 B_i 互不相容, 而不等式利用了 $B_i\subset A_i$.

15.

$$P\Big(\bigcap_{i=1}^{\infty}A_i\Big)=1-P\bigg(\Big(\bigcap_{i=1}^{\infty}A_i\Big)^{\mathrm{c}}\bigg)=1-P\Big(\bigcup_{i=1}^{\infty}A_i^{\mathrm{c}}\Big)\geqslant1-\sum_{i=1}^{\infty}P(A_i^{\mathrm{c}})=1$$

16. 分割中含 $\{1\}$ 作为子集的, 一共有 $T_{k-1}(n-1)$ 种分割, 其中 $T_{k-1}(n-1)$ 表示将剩余的 $n-1$ 个元素 $\{2,\cdots,n\}$ 分成 $k-1$ 非空个子集的方法数. 另外, 分割中不含 $\{1\}$, 此时, “1”必与其他元素在一起. 将 $\{2,\cdots,n\}$ 分成 k 个非空子集的分割, 一共有 $T_k(n-1)$ 不同的分割, 将每一种分割的某一集合加上 1, 就成为 $\{1,2,\cdots,n\}$ 的一个分割, 而加“1”的方式共有 k 种, 因此, 不含 $\{1\}$ 的分割共有 $kT_k(n-1)$ 个, 由此得到题中的等式.

17. 记 R 为“5 个取出来的球中没有红球”, W 代表“没有白球”, B 代表“没有蓝球”, 则

$$\begin{aligned}&P(R\cup W\cup B)=P(R)+P(W)+P(B)-P(RW)-P(RB)-P(WB)+P(RWB)\\&=\binom{13}{5}\Big/\binom{18}{5}+\binom{12}{5}\Big/\binom{18}{5}+\binom{11}{5}\Big/\binom{18}{5}-\binom{7}{5}\Big/\binom{18}{5}-\binom{6}{5}\Big/\binom{18}{5}-\binom{5}{5}\Big/\binom{18}{5}\\&\approx0.2933\end{aligned}$$

这样, 5 个球出现所有颜色的概率近似地等于 $1-0.2933=0.7067$.

18. (a) $\dfrac{8\times7\times6\times5\times4}{17\times16\times15\times14\times13}=\dfrac{2}{221}$

(b) 由于一共有 9 个球不是蓝色的球, 所求的概率为 $\dfrac{9\times8\times7\times6\times5}{17\times16\times15\times14\times13}=\dfrac{9}{442}$.

(c) 三个不同颜色球的组合一共有 $4\times8\times5$ 种方式, 而每一种组合又有 3! 中排列方式. 这样所求的概率为 $\dfrac{3!\times4\times8\times5}{17\times16\times15}=\dfrac{4}{17}$.

(d) 4 个红球作为一个集体, 它们的位置连在一起的占位方式一共有 14 种方式, 而这 4 个球在一个占位方式下又有 4! 种排列方式. 这样, 4 个红球排在一起的概率为 $\dfrac{14\times4!}{17\times16\times15\times14}=\dfrac{1}{170}$.

19. (a) 10 张牌中出现 4 张黑桃、3 张红桃、2 张方块和一张梅花的概率为 $\dfrac{\binom{13}{4}\binom{13}{3}\binom{13}{2}\binom{13}{1}}{\binom{52}{10}}$.

而这 4 种花色的角色又可以互相调换, 这样概率变成 $\dfrac{24\times\binom{13}{4}\binom{13}{3}\binom{13}{2}\binom{13}{1}}{\binom{52}{10}}$.

(b) 先选定两种花色, 从这两种花色中选取三张牌, 然后在选取一种花色, 从这种花色中选定 4 张牌. 单就花色的选定方式就有 $\dbinom{4}{2}\times2=12$ 种. 这样所求的概率为 $\dfrac{12\times\binom{13}{3}\binom{13}{3}\binom{13}{4}}{\binom{52}{10}}$.

20. 红球先拿光的充要条件是蓝球最后拿走. 而每个球被最后拿走的概率是相同的. 因此, 红球先拿光的概率为 10/30.

第 3 章

1. (a) $P(\text{没有“}A\text{”})=\dbinom{35}{13}\Big/\dbinom{39}{13}$

(b) $1-P(\text{没有“}A\text{”})-4\dbinom{35}{12}\Big/\dbinom{39}{13}$

(c) $P(i\text{个“}A\text{”})=\dbinom{3}{i}\dbinom{36}{13-i}\Big/\dbinom{39}{13}$

2. 令 L_i 表示事件“汽车电池寿命超过 $10000\times i$ 英里”.

(a) $P(L_2|L_1)=P(L_1L_2)/P(L_1)=P(L_2)/P(L_1)=1/2$

(b) $P(L_3|L_1)=P(L_1L_3)/P(L_1)=P(L_3)/P(L_1)=1/8$

3. 将 1 个白球和 0 个黑球放在坛子 1, 将 9 个白球和 10 个黑球放在坛子 2.

4. 记 T 为“转移的球为白球”这一事件, W 表示“从坛子 B 中随机抽取一个白球”, 则

$$\begin{aligned}P(T|W)&=\frac{P(W|T)P(T)}{P(W|T)P(T)+P(W|T^c)P(T^c)}\\&=\frac{2/7\times2/3}{2/7\times2/3+1/7\times1/3}=4/5\end{aligned}$$

5. (a) $\dfrac{r}{r+w}$, 因为 $r+w$ 个球以完全相同的概率在第 i 次被抽中.

(b) (c)

$$P(R_j|R_i) = \frac{P(R_iR_j)}{P(R_i)} = \frac{\binom{r}{2}/\binom{r+w}{2}}{r/(r+w)} = \frac{r-1}{r+w-1}$$

可用这样的说法解释：对于 $i \neq j$, 已知第 i 次抽出是红球的情况下, 其他 $(r+w-1)$ 个球

以完全相同的概率在第 j 次被抽出来, 而红球个数为 $(r-1)$.

6. 用 B_i 表示“第 i 次抽出的球是黑球”, 令 $R_i = B_i^c$, 则

$$\begin{aligned} P\{B_1|R_2\} &= \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)} \\ &= \frac{[r/(b+r+c)][b/(b+r)]}{[r/(b+r+c)][b/(b+r)] + [(r+c)/(b+r+c)][r/(b+r)]} \\ &= b/(b+r+c) \end{aligned}$$

7. 记 B 为“两张牌均为‘A’”的事件.

(a)

$$\begin{aligned} P\{B|\text{肯定其中一张为黑桃“A”}\} &= \frac{P\{B, \text{肯定其中一张为黑桃“A”}\}}{P\{\text{肯定其中一张为黑桃“A”}\}} \\ &= \frac{\binom{1}{1}\binom{3}{1}}{\binom{52}{2}} \Bigg/ \frac{\binom{1}{1}\binom{51}{1}}{\binom{52}{2}} = 3/51 \end{aligned}$$

(b) 由于第 2 张可以是余下 51 张牌中的任意一张, 而且各张牌都是以相同的概率出现, 因此, 其解仍然是 3/51.

(c) 由于我们可以交换次序, 因此, 答案与 (b) 是一样的. 但也可以由下面形式推导：

$$\begin{aligned} P\{B|\text{第二张为“A”}\} &= \frac{P\{B, \text{第二张为“A”}\}}{P\{\text{第二张为“A”}\}} \\ &= \frac{P\{B\}}{P\{B\} + P\{\text{第一张不是“A”}, \text{第二张是“A”}\}} \\ &= \frac{(4/52)(3/51)}{(4/52)(3/51) + (48/52)(4/51)} = 3/51 \end{aligned}$$

(d)

$$\begin{aligned} P\{B|\text{至少有一张为“A”}\} &= \frac{B}{P\{\text{至少有一张是“A”}\}} \\ &= \frac{(4/52)(3/51)}{1-(48/52)(47/51)} = 1/33 \end{aligned}$$

8.

$$\frac{P(H|E)}{P(G|E)} = \frac{P(HE)}{P(GE)} = \frac{P(H)P(E|H)}{P(G)P(E|G)}$$

新的证据出现后, H 出现的概率是 G 出现的概率的 1.5 倍.

9. 用 A 表示“植物存活”的事件, W 表示“给植物浇水”的事件.

(a)

$$P(A) = P(A|W)P(W) + P(A|W^c)P(W^c)$$
$$= (0.85)(0.9) + (0.2)(0.1) = 0.785$$

(b)

$$P(W^c|A^c) = \frac{P(A^c|W^c)P(W^c)}{P(A^c)} = \frac{(0.8)(0.1)}{0.215} = \frac{16}{43}$$

10. (a) $1 - P(\text{没有红球}) = 1 - \dfrac{\binom{22}{6}}{\binom{30}{6}}$

(b) 若已知红球没有被选中, 那么在剩余的 22 个非红球中的任意 6 个球具有相同的被选中概率. 因此

$$P(2\text{绿}|\text{没有红球}) = \frac{\binom{10}{2}\binom{12}{4}}{\binom{22}{6}}$$

11. 记 W 表示事件 “电池工作”, 用 C 和 D 分别表示所用的电池为 C 型和 D 型电池。

(a) $P(W) = P(W|C)P(C) + P(W|D)P(D) = 0.7 \times (8/14) + 0.4 \times (6/14) = 4/7$

(b)

$$P(C|W^c) = \frac{P(CW^c)}{P(W^c)} = \frac{P(W^c|C)P(C)}{3/7} = \frac{0.3 \times (8/14)}{3/7} = 0.4$$

12. 记 L_i 表示事件 “Maria 喜欢书 i”, $i = 1, 2$. 于是

$$P(L_2|L_1^c) = \frac{P(L_1^c L_2)}{P(L_1^c)} = \frac{P(L_1^c L_2)}{0.4}$$

再利用事件 L_2 是两个不相容的事件 $L_1 L_2$ 和 $L_1^c L_2$ 的和, 我们得到

$$0.5 = P(L_2) = P(L_1 L_2) + P(L_1^c L_2) = 0.4 + P(L_1^c L_2)$$

这样,

$$P(L_2|L_1^c) = 0.1/0.4 = 0.25$$

13. (a) 这个问题等价于：从坛子中最后拿走的球是一个蓝色球. 由于这 30 个球中的任意一个球以相同的概率在最后被取走, 因此所求的概率为 1/3.

(b) 这个问题等价于：从坛子中拿完红球或蓝球时, 最后拿着的球是一个蓝色球. 而这 30 个 (红球或蓝球) 完全处于相同的可能性. 它们中的哪一个最后被拿走的可能性是相同的 (尽管坛子里还可能剩下若干绿色球没有被拿走). 因此所求的概率仍然是 1/3.

(c) 记 B_1, R_2, G_3 分别表示, 第一次拿走的是蓝色球, 第二次拿走的是红色球, 第三次拿走的是绿色球. 我们有

$$P(B_1 R_2 G_3) = P(G_3)P(R_2|G_3)P(B_1|R_2 G_3) = \frac{8}{38} \times \frac{20}{30} = \frac{8}{57}$$

上式中 $P(G_3)$ 是这样计算的：事件 G_3 刚好是事件 “绿色球最后被拿走”, 因此它的概率为 8/38. $P(R_2|G_3)$ 的计算方法如下：已知绿色球最后被拿走的条件下, 这 20 个红球和 10 个蓝球中的每一个球都有相同的机会排在这个集合的最后一位被拿走. 因此其概率为 20/30. 至于 $P(B_1|R_2 G_3)$, 显然等于 1.

(d) $P(B_1)=P(B_1G_2R_3)+P(B_1R_2G_3)=\dfrac{20}{38}\times\dfrac{8}{18}+\dfrac{8}{58}=\dfrac{64}{171}$

14. 记 H 为硬币正面向上的事件, T_h 是事件 “B 被告知 ‘正面向上’ (实际上，不一定真实)”, F 表示 “A 忘记了他所投掷的结果”, C 表示 B 被告知的是真实的情况. 这样

(a)

$$\begin{aligned}P(T_h)&=P(T_h|F)P(F)+P(T_h|F^c)P(F^c)\\&=0.5\times0.4+P(H)\times0.6\\&=0.68\end{aligned}$$

(b)

$$\begin{aligned}P(C)&=P(C|F)P(F)+P(C|F^c)P(F^c)\\&=0.5\times0.4+1\times0.6\\&=0.80\end{aligned}$$

(c) 利用公式

$$P(H|T_h)=\frac{P(HT_h)}{P(T_h)}$$

和

$$\begin{aligned}P(HT_h)&=P(HT_h|F)P(F)+P(HT_h|F^c)P(F^c)\\&=P(H|F)P(T_h|HF)P(F)+P(H)P(F^c)\\&=0.8\times0.5\times0.4+0.8\times0.6\\&=0.64\end{aligned}$$

可得 $P(H|T_h)=0.64/0.68=16/17$.

15. 记 A 为“第一次试验结果大于第二次结果”, B 为“第二次结果大于第一次”, E 为“两次的结果相等”, 则 $1=P(A)+P(B)+P(E)$. 由对称性, $P(A)=P(B)$, 因此,

$$P(B)=\frac{1-P(E)}{2}=\frac{1-\sum\limits_{i=1}^{n}p_i^2}{2}$$

此题的另一种解法是

$$P(B)=\sum_i\sum_{j>i}P\{\text{第一次结果为}i,\text{第二次结果为}j\}=\sum_i\sum_{j>i}p_ip_j$$

由下面的恒等式看出两种方法得到的结果是相同的：

$$1=\sum_{i=1}^{n}p_i\sum_{j=1}^{n}p_j=\sum_i\sum_j p_ip_j=\sum_i p_i^2+\sum_i\sum_{j\neq i}p_ip_j=\sum_i p_i^2+2\sum_i\sum_{j>i}p_ip_j$$

16. 记 $E = \{A比B得到更多正面朝上\}$,
$A_w = \{n次掷硬币后, A的正面朝上次数比B的正面朝上次数多\}$,
$B_w = \{n次掷硬币后, B的正面朝上次数比A的正面朝上次数多\}$,
$A_e = \{n次掷硬币后, A的正面朝上次数与B的正面朝上次数相同\}$, 则

$$\begin{aligned} P(E) &= P(E|A_w)P(A_w) + P(E|B_w)P(B_w) + P(E|A_e)P(A_e) \\ &= 1 \times P(A_w) + 0 \times P(B_w) + \frac{1}{2}P(A_e) = P(A_w) + \frac{1}{2}P(A_e) \end{aligned}$$

由于 $P(A_w) = P(B_w)$. 由等式 $1 = P(A_w) + P(B_w) + P(A_e)$ 得 $P(A_w) = \frac{1}{2} - \frac{1}{2}P(A_e)$. 由此可知,

$$P(E) = P(A_w) + \frac{1}{2}P(A_e) = \frac{1}{2} - \frac{1}{2}P(A_e) + \frac{1}{2}P(A_e) = \frac{1}{2}$$

17. (a) 不真. 在掷两个骰子的游戏中, 记 $E = \{和为\ 7\ 点\}$, $F = \{第一次掷的结果不是\ 4\}$, $G = \{第二次不是\ 3\}$. 可以验证, E, F 相互独立, E, G 相互独立, 但

$$P(E|F \cup G) = \frac{P\{和为\ 7,\ 但没有\{4,3\}\}}{P\{没有\{4,3\}\}} = \frac{5/36}{35/36} = \frac{5}{35} \neq P(E)$$

(b)

$$\begin{aligned} P(E(F \cup G)) &= P(EF \cup FG) = P(EF) + P(EG) \qquad 因为EFG = \emptyset \\ &= P(E)[P(F) + P(G)] = P(E)P(F \cup G) \qquad FG = \emptyset \end{aligned}$$

(c)

$$\begin{aligned} P(G|EF) &= \frac{P(EFG)}{P(EF)} = \frac{P(E)P(FG)}{P(EF)} \qquad 由于E与FG相互独立 \\ &= \frac{P(E)P(F)P(G)}{P(E)P(F)} \qquad 由独立性假设 \\ &= P(G) \end{aligned}$$

18. (a) 一定不对. 若它们互不相容, 则 $0 = P(AB) \neq P(A)P(B)$.
(b) 一定不对. 若它们相互独立, 则 $P(AB) = P(A) \times P(B) > 0$.
(c) 一定不对. 若它们互不相容, 则 $P(A \cup B) = P(A) + P(B) = 1.2$.
(d) 可能正确.

19. (a), (b), (c) 三个概率分别为 $0.5, (0.8)^3 = 0.512, (0.9)^7 \approx 0.4783$.

20. 记 $D_i(i = 1, 2)$ 为第 i 个收音机是坏的. 又令 $A(B)$ 表示“这批收音机是由工厂 A(工厂 B) 生产的”.

$$\begin{aligned} P(D_2|D_1) &= \frac{P(D_1D_2)}{P(D_1)} = \frac{P(D_1D_2|A)P(A) + P(D_1D_2|B)P(B)}{P(D_1|A)P(A) + P(D_1|B)P(B)} \\ &= \frac{0.05^2 \times 1/2 + 0.01^2 \times 1/2}{0.05 \times 1/2 + 0.01 \times 1/2} = \frac{13}{300} \end{aligned}$$

21. $P(A|B)=1$ 即 $P(AB)=P(B)$, 而 $P(B^c|A^c)=1$ 即 $P(A^cB^c)=P(A^c)$. 因此, 为证明本题, 只需由 $P(AB)=P(B)\Rightarrow P(A^cB^c)=P(A^c)$. 这可由下式推得:

$$\begin{aligned}P(B^cA^c)&=P((A\cup B)^c)=1-P(A\cup B)\\&=1-P(A)-P(B)+P(AB)=1-P(A)=P(A^c)\end{aligned}$$

22. 当 $n=0$ 时, 结论显然成立. 记 A_i 表示"经过 n 步以后, 在坛子内有 i 个红球", 依归纳法假设

$$P(A_i)=\frac{1}{n+1}\qquad i=1,\cdots,n+1$$

用 B_j 表示"经过 $n+1$ 步以后坛子里有 j 个红球"这一事件, 则

$$\begin{aligned}P(B_j)&=\sum_{i=1}^{n+1}P(B_j|A_i)P(A_i)=\frac{1}{n+1}\sum_{i=1}^{n+1}P(B_j|A_i)\\&=\frac{1}{n+1}[P(B_j|A_{j-1})+P(B_j|A_j)]\end{aligned}$$

经过 n 步以后, 坛子内一共有 $n+2$ 个球. $P(B_j|A_{j-1})$ 表示坛子中有 $n+2$ 个球, 其中 $j-1$ 个红球, 从中随机地取出的是一个红球的概率, 这样, $n+1$ 步以后, 坛子内就有 i 个红球, 显然

$$P(B_j|A_{j-1})=\frac{j-1}{n+2}$$

而相应的 $P(B_j|A_j)$ 表示在抽球之前坛子内有 j 个红球, $n+2-j$ 个蓝球, 而取出的是一个蓝球的概率, 这样

$$P(B_j|A_j)=\frac{n+2-j}{n+2}$$

将这些概率代入 $P(B_j)$ 的公式, 得

$$P(B_j)=\frac{1}{n+1}[\frac{j-1}{n+2}+\frac{n+2-j}{n+2}]=\frac{1}{n+2}$$

按归纳法, 完成了证明.

23. 记 A_i 为"第 i 个人宣称拿到了 'A'", 则

$$P(A_i)=1-\binom{2n-2}{n}\Big/\binom{2n}{n}=1-\frac{1}{2}\times\frac{n-1}{2n-1}=\frac{3n-1}{4n-2}$$

A_1A_2 表示"第一个人只能从 2 张 'A' 中选一张, 从 $2n-2$ 张非 ' A' 中选 $n-1$ 张". 这样,

$$P(A_1A_2)=\left(\binom{1}{2}\binom{2n-2}{n-1}\right)\Big/\binom{2n}{n}=\frac{n}{2n-1}$$

因此,

$$P(A_2^c|A_1)=1-P(A_2|A_1)=1-\frac{P(A_1A_2)}{p(A_1)}=\frac{n-1}{3n-1}$$

可以将分牌的结果看成两次试验, 试验 i 成功表示第 i 张"A"给了第一个玩牌者, 当 n 充分大时, 这两个试验就相互独立, 成功的概率为 1/2, 这样, 问题变成了两次试验至少有一次成功的条件下, 求两次都成功的概率 (= 1/3), 因此, n 充分大时, 可用例 2b 那样的伯努利试验来逼近.

24. (a) 设 n 次收集到的优惠券的顺序为 $i_1, i_2, \cdots, i_n$, 其相应的概率为 $p_{i_1}\cdots p_{i_n}=\prod_{i=1}^n p_i$. 因此, n 次收集到 n 种不同的优惠券的概率为 $n!\prod_{i=1}^n p_i$.

(b) 设 $i_1, \cdots, i_k$ 各不相同, 则

$$P(E_{i_1}\cdots E_{i_k})=\left(\frac{n-k}{n}\right)^n$$

此处 $E_{i_1}, \cdots, E_{i_k}$ 表示没有 $i_1, \cdots, i_k$ 类型的优惠券, 在 n 次收集优惠券, 每次都没有收集到 $i_1, \cdots, i_k$. 而各次收集优惠券又相互独立, 因此 $P(E_{i_1}\cdots E_{i_n})$ 有上述表达式. 现在利用事件和的概率公式得到

$$P\left(\bigcup_{i=1}^n E_i\right)=\sum_{k=1}^n(-1)^{k+1}\binom{n}{k}\left(\frac{n-k}{n}\right)^n$$

由于 $1-P\left(\bigcup_{i=1}^n E_i\right)$ 表示 n 种优惠券都收集到的概率, 由 (a) 知这个数等于 n/n^n, 将这个值代入 $P\big(\bigcup_{i=1}^n E_i\big)$ 的展开式中, 得到

$$1-\frac{n!}{n^n}=\sum_{k=1}^n(-1)^{k+1}\binom{n}{k}\Big(\frac{n-k}{n}\Big)^n$$

或

$$n!=n^n-\sum_{k=1}^n(-1)^{k+1}\binom{n}{k}(n-k)^n$$

或

$$n!=\sum_{k=0}^n(-1)^k\binom{n}{k}(n-k)^n$$

25. 记 $A=EF^{\mathrm{c}}, B=FE^{\mathrm{c}}, C=E\cap F$, 则 A, B, C 互不相容, 且

$$E\cup F=A\cup B\cup C$$

$$E=A\cup C \qquad F=B\cup C$$

$$P(E|E\cup F)=\frac{P(E\cap(E\cup F))}{P(E\cup F)}=\frac{P(A)P(C)}{P(A)+P(B)+P(C)}$$

$$P(E|F)=\frac{P(EF)}{P(F)}=\frac{P(C)}{P(B)+P(C)}$$

由上式看, $P(E|E\cup F)\geqslant P(E|F)$ 是显然的.

第 4 章

1. 由于概率之和为 1, 我们利用这个条件得

$$4P\{X=3\}+0.5=1$$

从而 $P\{X=0\}=0.375, P\{X=3\}=0.125$, 故 $E[X]=1\times0.3+2\times0.2+3\times0.125=1.075$.

2. 利用题中的关系, 得 $p_i = c^i p_0, i = 1, 2$, 其中 $p_i = P\{X = i\}$. 由于这些概率之和为 1, 得

$$p_0(1 + c + c^2) = 1 \Rightarrow p_0 = \frac{1}{1 + c + c^2}$$

因此,

$$E[X] = p_1 + 2p_2 = \frac{c + 2c^2}{1 + c + c^2}$$

3. 令 X 为掷硬币的次数, X 的分布列为

$$p_2 = p^2 + (1-p)^2 \qquad p_3 = 1 - p_2 = 2p(1-p)$$

因此,

$$E[X] = 2p_2 + 3p_3 = 2p_2 + 3(1 - p_2) = 3 - p^2 - (1-p)^2$$

4. 随机地选定一个家庭, 而这个家庭有 i 个儿童的概率为 n_i/m , 因此

$$E[X] = \sum_{i=1}^{r} i n_i / m$$

由于有 i 个儿童的家庭总数为 n_i , 因此这些儿童的总数为 in_i, 抽到的儿童来自这样的家庭的概率为 $in_i / \sum_{j=1}^{r} in_i$. 因此,

$$E[Y] = \frac{\sum\limits_{i=1}^{r} i^2 n_i}{\sum\limits_{i=1}^{r} i n_i}$$

因此, 我们必须证明

$$\frac{\sum\limits_{i=1}^{r} i^2 n_i}{\sum\limits_{i=1}^{r} i n_i} \geqslant \frac{\sum\limits_{i=1}^{r} i n_i}{\sum\limits_{i=1}^{r} n_i}$$

上式等价于

$$\sum_{j=1}^{r} n_j \sum_{i=1}^{r} i^2 n_i \geqslant \sum_{i=1}^{r} i n_i \sum_{j=1}^{r} j n_j$$

$$\Leftrightarrow \sum_{i=1}^{r} \sum_{j=1}^{r} i^2 n_i n_j \geqslant \sum_{i=1}^{r} \sum_{j=1}^{r} ij n_i n_j$$

对于固定的 (i, j), 左边和式 $n_i n_j$ 的系数为 $i^2 + j^2$, 右边和式 $n_i n_j$ 系数为 $2ij$, 所以上式等价于

$$i^2 + j^2 \geqslant 2ij$$

而这个不等式是明显的.[1]

1. 也可这样解. 若令 $p_i = \dfrac{n_i}{\sum n_i}$, 则上式等价于 $(\sum p_i)(\sum i^2 p_i) \geqslant \left(\sum i p_i\right)^2$. 而这个不等式是显然的. ——译者注

5. 记 $p = P\{X = 1\}$, 则 $E[X] = p, \mathrm{Var}(X) = p(1-p)$, 由问题条件知

$$p = 3p(1-p)$$

解此方程得 $p = 2/3$. 因此 $P\{X = 0\} = 1/3$.

6. 假定你押上 x, 而赢 x 的概率为 p, 输 x 的概率为 $1-p$. 此时, 你赢钱的期望为

$$xp - x(1-p) = (2p-1)x$$

当 $p > 1/2$ 时, 这个值为正, 当 $p < 1/2$, 这个值为负. 若告诉你正面朝上的概率为 0.6, 则你应该押 10 元 (最大容许的押宝的值) 若告诉你是 0.3, 则你应该押 0 元, 这样你期望的利润为

$$\frac{1}{2} \times (1.2-1) \times 10 + \frac{1}{2} \times 0 - C = 1 - C$$

其中 C 为信息费. 若没有这个信息, 则你的期望的所得为

$$\frac{1}{2}(2 \times 0.6 - 1)x + \frac{1}{2}(2 \times 0.3 - 1)x = \left[\frac{1}{2} \times 0.2 + \frac{1}{2} \times (-0.4)\right]x = -0.1x$$

因此, 在没有信息的情况下, 你应该押 0 元, 使损失最小. 比较这两种赌博方式可看出, 只要 $C < 1$ 你就买这个信息.

7. (a) 若你翻开红纸, 观察得到 x, 若你转向蓝纸, 你的期望收入为

$$2x(1/2) + x/2(1/2) = 5x/4 > x$$

因此, 你应该转向蓝纸, 而期望得到更多.

(b) 设慈善家写的数为 x(写在红纸上), 则在蓝纸上写上 $2x$ 或 $x/2$, 注意若 $y < x/2$, 此时蓝纸上写的数字总是比 y 大, 因此按规定接受了蓝纸上提供的数字, 即 $2x$ 或 $x/2$.

$$E[R_y(x)] = 5x/4 \qquad x/2 \geqslant y$$

如果 $x/2 < y \leqslant 2x$, 此时, 如果蓝纸上, 写的是 $2x$, 你就接受 $2x$; 若蓝纸上写上 $x/2$, 你就转向红纸, 此时, 你得到的是

$$E[R_y(x)] = 2x(1/2) + x(1/2) = 3x/2 \qquad x/2 < y \leqslant 2x$$

最后, 若 $2x < y$, 此时蓝纸上的数被拒绝, 你的收入为

$$R_y(x) = x \qquad 2x < y$$

对于 y 值, 期望收入为

$$E[R_y(x)] = \begin{cases} x & x < y/2 \\ 3x/2 & y/2 \leqslant x < 2y \\ 5x/4 & x \geqslant 2y \end{cases}$$

8. 设 n 次独立重复试验成功的概率为 p, 成功数小于或等于 i 的充要条件为失败数大于或等于 $n-i$, 但是每次试验失败的概率为 $1-p$. 因此失败数的分布为二项分布, 故

$$\begin{aligned}P\{\mathrm{Bin}(n,p)\leqslant i\}&=P\{\mathrm{Bin}(n,1-p)\geqslant n-i\}\\&=1-P\{\mathrm{Bin}(n,1-p)\leqslant n-i-1\}\end{aligned}$$

上面最后一个等式是利用事件和它的对立事件的概率之间的关系.

9. 由 $E[X]=np,\mathrm{Var}(X)=np(1-p)$. 通过 $np=6,np(1-p)=2.4$, 解得 $p=0.6,n=10$, 故

$$P\{X=5\}=\binom{10}{5}(0.6)^5(0.4)^5$$

10. 令 X_i 为第 i 次取出的球的号码, 则

$$\begin{aligned}P\{X\leqslant k\}&=P\{X_1\leqslant k,X_2\leqslant k,\cdots,X_m\leqslant k\}\\&=P\{X_1\leqslant k\}P\{X_2\leqslant k\}\cdots P\{X_m\leqslant k\}=\left(\frac{k}{n}\right)^m\end{aligned}$$

因此,

$$P\{X=k\}=P\{X\leqslant k\}-P\{X\leqslant k-1\}=\left(\frac{k}{n}\right)^m-\left(\frac{k-1}{n}\right)^m$$

11. (a) 给定 A 赢第一局, A 赢全局的充要条件在 B 赢 3 局以前赢 2 局. 故

$$P\{\text{A 赢}|\text{A 赢第一局}\}=p^2+2p^2(1-p)+3p^2(1-p)^2$$

(b)

$$\begin{aligned}P\{\text{A 赢第一局}|\text{A 赢}\}&=\frac{P\{\text{A 赢}|\text{A 赢第一局}\}P\{\text{A 赢第一局}\}}{P\{\text{A 赢}\}}\\&=\frac{p^3+2p^3(1-p)+3p^3(1-p)^2}{p^3+3p^3(1-p)+\binom{4}{2}p^3(1-p)^2}\\&=\frac{1+2(1-p)+3(1-p)^2}{1+3(1-p)+6(1-p)^2}\end{aligned}$$

12. 为计算至少赢三场的概率, 必须计算本周赢或输的条件之下的事件的概率, 因此, 答案为

$$0.5\sum_{i=3}^{4}\binom{4}{i}0.4^i0.6^{4-i}+0.5\sum_{i=3}^{4}\binom{4}{i}0.7^i0.3^{4-i}$$

13. 记 C 为"陪审团作出正确决定"的事件, 记 F 为"其中有 4 个审判员的结论相同", 则

$$P(C)=\sum_{i=4}^{7}\binom{7}{i}0.7^i0.3^{7-i}$$

$$P(C|F)=\frac{P(CF)}{P(F)}=\frac{\binom{7}{4}0.7^40.3^3}{\binom{7}{4}0.7^40.3^3+\binom{7}{3}0.7^30.3^4}=0.7$$

14. 假定飓风次数为泊松分布. 这样, 我们的解为

$$\sum_{i=0}^{3} \mathrm{e}^{-5.2}(5.2)^i/i!$$

15.

$$E[Y] = \sum_{i=1}^{\infty} iP\{X=i\}/P\{X>0\} = E[X]/P\{X>0\} = \frac{\lambda}{1-\mathrm{e}^{-\lambda}}$$

16. (a) $1/n$

(b) 记 D 表示女生 i 和女生 j 选择不同的男生, 此时我们有

$$\begin{aligned} P(G_iG_j) &= P(G_iG_j|D)P(D) + P(G_iG_j|D^c)P(D^c) \\ &= P(G_iG_j|D)P(D) + 0 \qquad (P(G_iG_j|D^c) = 0) \end{aligned}$$

由于

$$\begin{aligned} P(D^c) &= P\{i,j\text{选择同一男生}\} \\ &= \sum_{k=1}^{n} P\{i,j\text{选择同一男生}k\} = nP\{i,j\text{同时选上男生 }1\} = n \times \frac{1}{n^2} = \frac{1}{n} \end{aligned}$$

这样,

$$P(G_iG_j) = P(G_iG_j|D)(1-\frac{1}{n}) = \left(\frac{1}{n}\right)^2\left(1-\frac{1}{n}\right) = \frac{n-1}{n^3}$$

因此 $P(G_i|G_j) = P(G_iG_j)/P(G_j) = (n-1)/n^2$.

(c) (d) 当 n 充分大时, $P(G_i|G_j)$ 很小, 并且与 $P(G_i)$ 很接近. 由此可知形成夫妇的对数近似于泊松分布, 均值为 $\sum_{i=1}^{n} P(G_i) = 1$. 从而, $P_0 \sim \mathrm{e}^{-1}, P_k \approx \mathrm{e}^{-1}/k!$.

(e) 为求给定 k 个女生都被配成夫妇的概率, 利用条件概率计算. 记 D 为"这 k 个女生找到不同的男生".

$$\begin{aligned} &P(G_{i_1}\cdots G_{i_k}) = P(G_{i_1}\cdots G_{i_k}|D)P(D) + P(G_{i_1}\cdots G_{i_k}|D^c)P(D^c) \\ &= P(G_{i_1}\cdots G_{i_k}|D)P(D) = \left(\frac{1}{n}\right)^k \frac{n(n-1)\cdots(n-k+1)}{n^k} = \frac{n!}{(n-k)!n^{2k}} \end{aligned}$$

因此,

$$\sum_{i_1<\cdots<i_k} P(G_{i_1}\cdots G_{i_k}) = \binom{n}{k} P(G_{i_1}\cdots G_{i_k}) = \frac{n!n!}{(n-k)!(n-k)!k!n^{2k}}$$

利用事件和的概率公式得

$$1 - P_0 = P\left(\bigcup_{i=1}^{n} G_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \frac{n!n!}{(n-k)!(n-k)!k!n^{2k}}$$

17. (a) 由于第 i 个妇女与其余每个人结成对的可能性相同, 因此 $P(W_i) = 1/(2n-1)$.

(b) 由于在 W_j 的条件下, 第 i 个妇女与其余 $2n-3$ 个人结成对的可能性相同, 因此 $P(W_i|W_j) = 1/(2n-3)$.

(c) 当 n 很大时, 妇女和她的丈夫结成对的数目近似服从泊松分布, 其期望近似为 $\sum_{i=1}^{n} P(W_i) = n/(2n-1) \approx 1/2$. 因此, 没有夫妻结成对的概率近似等于 $\mathrm{e}^{-1/2}$.

(d) 这个问题变成了配对问题 (见 16 题).

18. (a) $\binom{8}{3}(9/19)^3(10/19)^5(9/19) = \binom{8}{3}(9/19)^4(10/19)^5$

(b) 记 W 为她的最后所得, X 为赌的次数, 由于她要赢 4 次, 输 $X-4$ 次, 所以她的所得为

$$W = 20 - 5(X-4) = 40 - 5X$$

因此

$$E[W] = 40 - 5E[X] = 40 - 5 \times [4/(9/19)] = -20/9$$

19. 当三个人抛掷硬币的结果相同时, 就不会产生"奇人", 此概率为 $1/4$.

(a) $(1/4)^2(3/4) = 3/64$ (b) $(1/4)^4 = 1/256$.

20. 令 $q = 1-p$,

$$\begin{aligned}
E[1/X] &= \sum_{i=1}^{\infty} \frac{1}{i} q^{i-1} p = \frac{p}{q} \sum_{i=1}^{\infty} q^i / i \\
&= \frac{p}{q} \sum_{i=1}^{\infty} \int_0^q x^{i-1}\,\mathrm{d}x = \frac{p}{q} \int_0^q \sum_{i=1}^{\infty} x^{i-1}\,\mathrm{d}x \\
&= \frac{p}{q} \int_0^q \frac{1}{1-x}\,\mathrm{d}x = \frac{p}{q} \int_p^1 \frac{1}{y}\,\mathrm{d}y = -\frac{p}{q} \ln p
\end{aligned}$$

21. 由于 $(X-b)/(a-b)$ 以概率 p 为 1, 以概率 $(1-p)$ 为 0, 故它是一个伯努利随机变量, 其参数为 p, 方差为 $p(1-p)$. 即

$$p(1-p) = \mathrm{Var}\Big(\frac{X-b}{a-b}\Big) = \frac{1}{(a-b)^2}\mathrm{Var}(X-b) = \frac{1}{(a-b)^2}\mathrm{Var}(X)$$

故

$$\mathrm{Var}(X) = (a-b)^2 p(1-p)$$

22. 记 X 为你玩的点盘数, Y 为你失败的盘数.

(a) 玩了 4 盘以后, 你再继续玩, 直到你输为止. 因此 $X-4$ 是几何分布, 其参数为 $(1-p)$, 故

$$E[X] = E[4 + (X-4)] = 4 + E[X-4] = 4 + \frac{1}{1-p}$$

(b) 令 Z 为前 4 盘中输的盘数, 则 Z 为二项随机变量, 其参数为 $(4, 1-p)$. 由于 $Y = Z+1$, 我们有

$$E[Y] = E[Z+1] = E[Z] + 1 = 4(1-p) + 1$$

23. "在抽出 m 个黑球以前抽出 n 个白球"这一事件等价于"在前 $n+m-1$ 次至少抽出 n 个白球"(与第 3 章例 4j 的问题进行比较). 记 X 为前 $n+m-1$ 次抽出的球中的白球个数, X 是超几何随机变量.

$$P\{X \geqslant n\} = \sum_{i=n}^{n+m-1} P\{X = i\} = \sum_{i=n}^{n+m-1} \frac{\dbinom{N}{i}\dbinom{M}{n+m-1-i}}{\dbinom{N+M}{n+m-1}}$$

24. 由于每个球以相同的概率 p_i, 并且相互独立地进入坛子 i, X_i 的分布是参数 $n=10, p=p_i$ 的二项分布.

同样的理由可知, X_i+X_j 的分布是参数 $n=10, p=p_i+p_j$ 的二项分布.

同样的理由可知, $X_1+X_2+X_3$ 的分布是参数 $n=10, p=p_1+p_2+p_3$ 的二项分布, 故

$$P\{X_1+X_2+X_3=7\}=\binom{10}{7}(p_1+p_2+p_3)^7(p_4+p_5)^3$$

25. 如果第 i 个人拿到自己的帽子, 则记 $X_i=1$, 否则记 $X_i=0$. 这样

$$X=\sum_{i=1}^{n}X_i$$

是拿到自己帽子的人数. 等式两边求期望得到

$$E[X]=E[\sum_{i=1}^{n}X_i]=\sum_{i=1}^{n}E[X_i]=\sum_{i=1}^{n}P\{X_i=1\}=\sum_{i=1}^{n}1/n=1$$

倒成第二个等式是利用事实: 第 i 个人以相等的概率拿到任何一个帽子, 因此他拿到自己帽子的概率为 $1/n$

利用式 (9.1), 得到

$$E[X^2]=\sum_{i=1}^{n}E[X_i]+\sum_{i=1}^{n}\sum_{j\neq i}E[X_iX_j]$$

对于 $i\neq j$,

$$E[X_iX_j]=P\{X_i=1,X_j=1\}=P\{X_i=1\}P\{X_j=1|X_i=1\}=\frac{1}{n}\times\frac{1}{n-1}$$

因此,

$$\begin{aligned}E[X^2]&=1+\sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{n(n-1)}\\&=1+n(n-1)\frac{1}{n(n-1)}=2\end{aligned}$$

这样

$$\mathrm{Var}(X)=2-1^2=1$$

第 5 章

1. 设 X 是玩球的时间 (分)

(a) $P\{X>15\}=1-P\{X\leqslant 15\}=1-5\times 0.025=0.875$
(b) $P\{20<X<35\}=10\times 0.05+5\times 0.025=0.625$
(c) $P\{X<30\}=10\times 0.025+10\times 0.05=0.75$
(d) $P\{X>36\}=4\times 0.025=0.1$

2. (a) $1 = \int_0^1 cx^n\,\mathrm{d}x = c/(n+1) \Rightarrow c = n+1$
(b) $P\{X > x\} = (n+1)\int_x^1 x^n\,\mathrm{d}x = x^{n+1}|_x^1 = 1 - x^{n+1}$

3. 首先由下式确定 c 的值

$$1 = \int_0^2 cx^4\mathrm{d}x = 32c/5 \Rightarrow c = 5/32$$

(a) $E[X] = \dfrac{5}{32}\int_0^2 x^5\,\mathrm{d}x = \dfrac{5}{32}\dfrac{64}{6} = 5/3$
(b) $E[X^2] = \dfrac{5}{32}\int_0^2 x^6\,\mathrm{d}x = \dfrac{5}{32}\dfrac{128}{7} = 20/7 \Rightarrow \mathrm{Var}(X) = 20/7 - (5/3)^2 = 5/63$

4. 由

$$1 = \int_0^1 (ax + bx^2)\,\mathrm{d}x = a/2 + b/3$$
$$0.6 = \int_0^1 (ax^2 + bx^3)\,\mathrm{d}x = a/3 + b/4$$

我们得到 $a = 3.6, b = -2.4$. 因此,

(a) $P[X < 1/2] = \int_0^{1/2}(3.6x - 2.4x^2)\,\mathrm{d}x = (1.8x^2 - 0.8x^3)|_0^{1/2} = 0.35$
(b) $E[X^2] = \int_0^1 (3.6x^3 - 2.4x^4)\,\mathrm{d}x = 0.42 \Rightarrow \mathrm{Var}(X) = 0.06$

5. 对于 $i = 1, \cdots, n$

$$P\{X = i\} = P\{[\mathrm{Int}(nU)] = i - 1\} = P\{i - 1 \leqslant nU < i\}$$
$$= P\left\{\frac{i-1}{n} \leqslant U < \frac{i}{n}\right\} = 1/n$$

(式中 $[nU]$ 表示 nU 的整数部分.)

6. 如果你的竞价为 $x, 70 < x < 140$, 则你将以概率 $(140-x)/70$ 赢得该工程, 利润为 $x-100$, 或者失去工程, 利润为 0. 因此, 若你竞价 x, 期望利润为

$$\frac{1}{70}(x - 100)(140 - x) = \frac{1}{70}(240x - x^2 - 14000)$$

上式求导并使之为 0, 得方程

$$240 - 2x = 0$$

因此, 你应该竞价 120(千元), 期望利润为 40/7(千元).

7. (a) $P\{U > 0.1\} = 9/10$
(b) $P\{U > 0.2|U > 0.1\} = P\{U > 0.2\}/P\{U > 0.1\} = 8/9$
(c) $P\{U > 0.3|U > 0.2, U > 0.1\} = P\{U > 0.3\}/P\{U > 0.2\} = 7/8$
(d) $P\{U > 0.3\} = 7/10$

将 (a),(b),(c) 所得的概率相乘得到 (d) 的概率.

8. 记 X 为测试数据, 令 $Z = (X - 100)/15$, 注意 Z 是标准正态随机变量.
(a) $P\{X > 125\} = P\{Z > 25/15\} \approx 0.0478$
(b)

$$P\{90 < X < 110\} = P\{-10/15 < Z < 10/15\}$$
$$= P\{Z < 2/3\} - P\{Z < -2/3\} = P\{Z < 2/3\} - [1 - P\{Z < 2/3\}] \quad \approx 0.4950$$

9. 设 X 是路上花的时间, 我们需要确定 x, 使

$$P\{X > x\} = 0.05$$

它等价于

$$P\left\{\frac{X-40}{7} > \frac{x-40}{7}\right\} = 0.05$$

或

$$P\left\{Z > \frac{x-40}{7}\right\} = 0.05$$

其中 Z 为标准正态随机变量. 但是

$$P\{Z > 1.645\} = 0.05$$

因此

$$\frac{x-40}{7} = 1.645 \quad \text{或} \quad x = 51.515$$

这样, 你应该在 12 点过 8.485 分以前动身.

10. 令 X 为轮胎的寿命 (单位: 1000 英里), 令 $Z = (X-34)/4$, 则 Z 为标准正态随机变量.

(a) $P\{X > 40\} = P\{Z > 1.5\} \approx 0.0668$

(b) $P\{30 < X < 35\} = P\{-1 < Z < 0.25\} = P\{Z < 0.25\} - P\{Z > 1\} \approx 0.44$

(c)

$$\begin{aligned} P\{X > 40 | X > 30\} &= P\{X > 40\}/P\{X > 30\} \\ &= P\{Z > 1.5\}/P\{Z > -1\} \approx 0.079 \end{aligned}$$

11. 令 X 为下一年的雨量, 记 $Z = (X-40.2)/8.4$

(a) $P\{X > 44\} = P\{Z > 3.8/8.4\} \approx P\{Z > 0.4524\} \approx 0.3255$

(b) $\binom{7}{3}(0.3255)^3(0.6745)^4$

12. 记 M_i 为样本中每年至少有收入 i (单位: 千元) 的男人数. W_i 为相应的女人数, 令 Z 为标准正态随机变量.

(a)

$$\begin{aligned} P\{W_{25} \geqslant 70\} &= P\{W_{25} \geqslant 69.5\} \\ &= P\left\{\frac{W_{25} - 200 \times 0.34}{\sqrt{200 \times 0.34 \times 0.66}} \geqslant \frac{69.5 - 200 \times 0.34}{\sqrt{200 \times 0.34 \times 0.66}}\right\} \\ &\approx P\{Z \geqslant 0.2239\} \approx 0.4114 \end{aligned}$$

(b)

$$\begin{aligned} P\{M_{25} \leqslant 120\} &= P\{M_{25} \leqslant 120.5\} \\ &= P\left\{\frac{M_{25} - 200 \times 0.587}{\sqrt{200 \times 0.587 \times 0.413}} \leqslant \frac{120.5 - 200 \times 0.587}{\sqrt{200 \times 0.587 \times 0.413}}\right\} \\ &\approx P\{Z \leqslant 0.4452\} \approx 0.6719 \end{aligned}$$

(c)

$$\begin{aligned} P\{M_{20} \geqslant 150\} &= P\{M_{20} \geqslant 149.5\} \\ &= P\left\{\frac{M_{20} - 200 \times 0.745}{\sqrt{200 \times 0.745 \times 0.255}} \leqslant \frac{149.5 - 200 \times 0.745}{\sqrt{200 \times 0.745 \times 0.255}}\right\} \\ &\approx P\{Z \geqslant 0.0811\} \approx 0.4677 \end{aligned}$$

(d)

$$\begin{aligned}P\{W_{20}\geqslant 100\}&=P\{W_{20}\geqslant 99.5\}\\&=P\Big\{\frac{W_{20}-200\times 0.534}{\sqrt{200\times 0.534\times 0.466}}\geqslant\frac{99.5-200\times 0.534}{\sqrt{200\times 0.534\times 0.466}}\Big\}\\&\approx P\{Z\geqslant -1.0348\}\approx 0.8496\end{aligned}$$

因此, $P\{M_{20}\geqslant 150\}P\{W_{20}\geqslant 100\}\approx 0.3974$.

13. 由于指数分布是无记忆的, 其结果为 $\mathrm{e}^{-4/5}$.

14. (a) $\mathrm{e}^{-2^2}=\mathrm{e}^{-4}$

(b) $F(3)-F(1)=\mathrm{e}^{-1}-\mathrm{e}^{-9}$

(c) $\lambda(t)=2t\mathrm{e}^{-t^2}/\mathrm{e}^{-t^2}=2t$

(d) 令 Z 为标准正态随机变量, 利用恒等式 $E[X]=\int_0^\infty P\{X>x\}\,\mathrm{d}x$, 得到

$$E[X]=\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x=2^{-1/2}\int_0^\infty \mathrm{e}^{-y^2/2}\,\mathrm{d}y=2^{-1/2}\sqrt{2\pi}P\{Z>0\}=\sqrt{\pi}/2$$

(e) 利用理论习题 5, 得到

$$E[X^2]=\int_0^\infty 2x\mathrm{e}^{-x^2}\,\mathrm{d}x=-\mathrm{e}^{-x^2}|_0^\infty=1$$

因此, $\mathrm{Var}(X)=1-\pi/4$.

15. (a) $P\{X>6\}=\exp\{-\int_0^6\lambda(t)\mathrm{d}t\}=\mathrm{e}^{-3.45}$

(b)

$$\begin{aligned}P\{X<8|X>6\}&=1-P\{X>8|X>6\}=1-P\{X>8\}/P\{X>6\}\\&=1-\mathrm{e}^{-5.65}/\mathrm{e}^{-3.45}\approx 0.8892\end{aligned}$$

16. 对于 $x\geqslant 0$

$$F_{1/X}(x)=P\{1/X\leqslant x\}=P\{X\leqslant 0\}+P\{X\geqslant 1/x\}=1/2+1-F_X(1/x)$$

上式求导, 得

$$f_{1/X}(x)=x^{-2}f_X(1/x)=\frac{1}{x^2\pi(1+(1/x)^2)}=f_X(x)$$

对于 $x<0$ 的证明是相似的.

17. 令 X 表示 n 次赌博中你赢的次数, 你所赢的钱数为

$$35X-(n-X)=36X-n$$

你赢钱的概率为

$$P\{36X-n>0\}=P\{X>n/36\}$$

而 X 是二项随机变量, 参数为 $n,p=1/38$.

(a) 当 $n=34$,

$$\begin{aligned}
p&=P\{X\geqslant 34/36\}=P\{X>0.5\} \qquad \text{(连续性修正)}\\
&=P\Big\{\frac{X-34/38}{\sqrt{34\times 1/38\times 37/38}}>\frac{0.5-34/38}{\sqrt{34\times 1/38\times 37/38}}\Big\}\\
&=P\Big\{\frac{X-34/38}{\sqrt{34\times 1/38\times 37/38}}>-0.4229\Big\}\approx\Phi(0.4229)\approx 0.6638
\end{aligned}$$

通过 34 次赌博, 如果你能赢一次以上的话, 你就会赢. 准确概率为 $1-(37/38)^{34}=0.5961$.

(b) 当 $n=1000$,

$$\begin{aligned}
P\{X>27.5\}&=P\Big\{\frac{X-1000/38}{\sqrt{1000\times 1/38\times 37/38}}>\frac{27.5-1000/38}{\sqrt{1000\times 1/38\times 37/38}}\Big\}\\
&\approx 1-\Phi(0.2339)\approx 0.4075
\end{aligned}$$

准确概率为 0.3961.

(c) 当 $n=100000$,

$$\begin{aligned}
P\{X>2777.5\}&=P\Big\{\frac{X-100\,000/38}{\sqrt{100\,000\times 1/38\times 37/38}}>\frac{2777.5-100\,000/38}{\sqrt{100\,000\times 1/38\times 37/38}}\Big\}\\
&\approx 1-\Phi(2.883)\approx 0.0020
\end{aligned}$$

准确概率为 0.0021.

18. 设 X 表示电池的寿命. 所求的概率为 $P\{X>s+t|X>t\}$, 故

$$\begin{aligned}
P\{X>s+t|X>t\}&=\frac{P\{X>s+t,X>t\}}{P\{X>t\}}=\frac{P\{X>s+t\}}{P\{X>t\}}\\
&=\frac{P\{X>s+t|\text{类型 1 电池}\}p_1+P\{X>s+t|\text{类型 2 电池}\}p_2}{P\{X>t|\text{类型 1 电池}\}p_1+P\{X>t|\text{类型 2 电池}\}p_2}\\
&=\frac{\mathrm{e}^{-\lambda_1(s+t)}p_1+\mathrm{e}^{-\lambda_2(s+t)}p_2}{\mathrm{e}^{-\lambda_1 t}p_1+\mathrm{e}^{-\lambda_2 t}p_2}
\end{aligned}$$

另一个方法是以电池类型为条件, 然后利用指数分布无记忆性,

$$\begin{aligned}
P\{X>s+t|X>t\}=&P\{X>s+t|X>t,\text{类型 1}\}P\{\text{类型 1}|X>t\}\\
&+P\{X>s+t|X>t,\text{类型 2}\}P\{\text{类型 2}|X>t\}\\
=&\mathrm{e}^{-\lambda_1 s}P\{\text{类型 1}|X>t\}+\mathrm{e}^{-\lambda_2 s}P\{\text{类型 2}|X>t\}
\end{aligned}$$

对于类型 i,

$$\begin{aligned}
P\{\text{类型}i,X>t\}&=\frac{P\{\text{类型}i,X>t\}}{P\{X>t\}}\\
&=\frac{P\{X>t|\text{类型}i\}p_i}{P\{X>t|\text{类型}1\}p_1+P\{X>t|\text{类型}2\}p_2}=\frac{\mathrm{e}^{-\lambda_i t}p_i}{\mathrm{e}^{-\lambda_1 t}p_1+\mathrm{e}^{-\lambda_2 t}p_2}
\end{aligned}$$

计算结果与前一种方法是一致的.

19. 令 X_i 为指数分布随机变量, 具有期望 $i, i=1,2$.

(a) c 的值应满足 $P\{X_1 > c\} = 0.05$, 故

$$\mathrm{e}^{-c} = 0.05 = 1/20$$

或

$$c = \ln 20 = 2.996$$

(b) $P\{X_2 > c\} = \mathrm{e}^{-c/2} = \dfrac{1}{\sqrt{20}} = 0.2236$

20. (a)

$$\begin{aligned}
E[(Z-c)^+] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-c)^+\mathrm{e}^{-x^2/2}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{c}^{\infty}(x-c)\mathrm{e}^{-x^2/2}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{c}^{\infty}x\mathrm{e}^{-x^2/2}\mathrm{d}x - \frac{1}{\sqrt{2\pi}}\int_{c}^{\infty}c\mathrm{e}^{-x^2/2}\mathrm{d}x \\
&= -\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,|_c^{\infty} - c(1-\Phi(c)) \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-c^2/2} - c(1-\Phi(c))
\end{aligned}$$

(b) 利用 X 与 $\mu+\sigma Z$ 具有相同分布的事实, 其中 Z 为标准正态随机变量, 得到

$$\begin{aligned}
E[(X-c)^+] &= E[(\mu+\sigma Z-c)^+] \\
&= E\left[\left(\sigma\left(Z-\frac{c-\mu}{\sigma}\right)\right)^+\right] \\
&= E\left[\sigma\left(Z-\frac{c-\mu}{\sigma}\right)^+\right] \\
&= \sigma\left[\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-a^2/2} - a(1-\Phi(a))\right]
\end{aligned}$$

其中 $a = (c-\mu)/\sigma$.

第 6 章

1. (a) $3C + 6C = 1 \Rightarrow C = 1/9$

(b) 令 $p(i,j) = P\{X=i, Y=j\}$, 则

$$p(1,1) = 4/9 \qquad p(1,0) = 2/9 \qquad p(0,1) = 1/9 \qquad p(0,0) = 2/9$$

(c) $\dfrac{(12)!}{2^6}(1/9)^6(2/9)^6$　(d) $\dfrac{(12)!}{(4!)^3}(1/3)^{12}$　(e) $\sum\limits_{i=8}^{12}\dbinom{12}{i}(2/3)^i(1/3)^{12-i}$

2. (a) 记 $p_j = P\{XYZ = j\}$, 我们有

$$p_6 = p_2 = p_4 = p_{12} = 1/4$$

因此, $E[XYZ] = (6+2+4+12)/4 = 6$

(b) 记 $q_j = P\{XY + XZ + YZ = j\}$, 我们有

$$q_{11} = q_5 = q_8 = q_{16} = 1/4$$

因此, $E[XY + XZ + YZ] = (11 + 5 + 8 + 16)/4 = 10$

3. 此题中, 我们要用到恒等式

$$\int_0^\infty \mathrm{e}^{-x} x^n \mathrm{d}x = n!$$

实际上, $\mathrm{e}^{-x}x^n/n!$, $x > 0$, 是 Γ 随机变量的分布密度 (参数 $n+1, \lambda = 1$).

(a)

$$1 = C\int_0^\infty \mathrm{e}^{-y}\int_{-y}^{y}(y-x)\,\mathrm{d}x\,\mathrm{d}y = C\int_0^\infty \mathrm{e}^{-y}2y^2\,\mathrm{d}y = 4C$$

因此, $C = 1/4$.

(b) 联合密度只在 $-y < x < y, y > 0$ 上为非零.

$x > 0$ 时,

$$f_X(x) = \frac{1}{4}\int_x^\infty (y-x)\mathrm{e}^{-y}\,\mathrm{d}y = \frac{1}{4}\int_0^\infty u\mathrm{e}^{-(x+u)}\,\mathrm{d}u = \frac{1}{4}\mathrm{e}^{-x}$$

当 $x < 0$ 时,

$$f_X(x) = \frac{1}{4}\int_{-x}^\infty (y-x)\mathrm{e}^{-y}\,\mathrm{d}y = \frac{1}{4}[-y\mathrm{e}^{-y} - \mathrm{e}^{-y} + x\mathrm{e}^{-y}]|_{-x}^{\infty} = (-2x\mathrm{e}^{x} + \mathrm{e}^{x})/4$$

(c) $f_Y(y) = \frac{1}{4}\mathrm{e}^{-y}\int_{-y}^{y}(y-x)\,\mathrm{d}x = \frac{1}{2}y^2\mathrm{e}^{-y}, y > 0$

(d)

$$\begin{aligned}E[X] &= \frac{1}{4}\Big[\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x + \int_{-\infty}^{0}(-2x^2\mathrm{e}^{x} + x\mathrm{e}^{x})\,\mathrm{d}x\Big]\\ &= \frac{1}{4}\Big[1 - \int_0^\infty (2y^2\mathrm{e}^{-y} + y\mathrm{e}^{-y})\mathrm{d}y\Big] = \frac{1}{4}[1 - 4 - 1] = -1\end{aligned}$$

(e) $E[Y] = \frac{1}{2}\int_0^\infty y^3\mathrm{e}^{-y}\,\mathrm{d}y = 3$

4. 设 $X_i, i = 1, \ldots, r$ 是多项分布随机变量, 其中 X_i 表示 n 次独立重复试验中结果 i 发生的次数, 每次试验的可能结果为 $1, \ldots, r$, 每个结果发生的概率分别为 $p_i, i = 1\ldots, r$. 现在设想把所有可能结果分成 k 类, 试验结果 $1, \ldots, r_1$ 归成第一类, 试验结果 $r_1+1, \ldots, r_1+r_2$ 归成第二类, 如此等等. 这样定义以后, Y_i 就是在 n 次独立重复试验中第 i 类结果发生的次数, 其相应发生的概率为 $\sum_{j=r_{i-1}+1}^{r_{i-1}+r_i} p_j, i = 1\ldots, k$. 按定义, $Y_1, \ldots, Y_k$ 是多项随机变量.

5. (a)

$$1 = \int_0^1\int_1^5 (x/5 + cy)\,\mathrm{d}y\mathrm{d}x = \int_0^1 (4x/5 + 12c)\,\mathrm{d}x = 12c + 2/5$$

因此, $c = 1/20$.

(b) X, Y 不独立, 不能将密度函数分解.

(c)

$$\begin{aligned}P\{X+Y>3\}&=\int_0^1\int_{3-x}^5(x/5+y/20)\,\mathrm{d}y\,\mathrm{d}x\\&=\int_0^1[(2+x)x/5+25/40-(3-x)^2/40]\,\mathrm{d}x\\&=1/5+1/15+5/8-19/120=11/15\end{aligned}$$

6. (a) X,Y 相互独立, 密度函数可分解因子.

(b) $f_X(x)=x\int_0^2 y\,\mathrm{d}y=2x\qquad 0<x<1$

(c) $f_Y(y)=y\int_0^1 x\,\mathrm{d}x=y/2\qquad 0<y<2$

(d)

$$\begin{aligned}P\{X<x,Y<y\}&=P\{X<x\}P\{Y<y\}\\&=\min(1,x^2)\min(1,y^2/4)\qquad x>0,y>0\end{aligned}$$

(e) $E[Y]=\int_0^2 y^2/2\,\mathrm{d}y=4/3$

(f)

$$P\{X+Y<1\}=\int_0^1 x\int_0^{1-x}y\,\mathrm{d}y\,\mathrm{d}x=\frac{1}{2}\int_0^1 x(1-x)^2\mathrm{d}x=1/24$$

7. 记 T_i 表示第 i 种冲击的来临时刻, $i=1,2,3$. 对于 $s>0,t>0$,

$$\begin{aligned}&P\{X_1>s,X_2>t\}=P\{T_1>s,T_2>t,T_3>\max(s,t)\}\\&=P\{T_1>s\}P\{T_2>t\}P\{T_3>\max(s,t)\}=\exp\{-\lambda_1 s\}\exp\{-\lambda_2 t\}\exp\{-\lambda_3\max(s,t)\}\\&=\exp\{-(\lambda_1 s+\lambda_2 t+\lambda_3\max(s,t))\}\end{aligned}$$

8. (a) 不. 若在一页上有很多广告, 那么这些广告被选中的机会比具有较少广告那些页上的广告被选中的机会小.

(b) $\frac{1}{m}\frac{n(i)}{n}$

(c) $\sum_{i=1}^m n(i)/(mn)=\bar{n}/n$,其中$\bar{n}=\sum_{i=1}^m n(i)/m$

(d) $(1-\bar{n}/n)^{k-1}\frac{1}{m}\frac{n(i)}{n}\frac{1}{n(i)}=(1-\bar{n}/n)^{k-1}/(nm)$

(e) $\sum_{k=1}^\infty\frac{1}{nm}(1-\bar{n}/n)^{k-1}=\frac{1}{\bar{n}m}$

(f) 循环次数的分布为几何分布, 其期望值为 $n\sqrt{n}$

9. (a) $P\{X=i\}=1/m,i=1,\cdots,m$.

(b) **第 2 步** 产生一个随机数 $U[(0,1)$上均匀随机变量], 若 $U<n(X)/n$, 到第 3 步, 否则

回到第 1 步.

第 3 步 产生一随机数 U, 选择第 X 页上第 $[n(X)U]+1$ 个元素.

10. 是, 它们相互独立. 我们可以这样地看, 当我们知道这个序列在某时刻 N 超过 c 的时候, 并不影响这个超过 c 的随机变量的分布, 它仍然为 $(c,1)$ 上均匀分布.

11. 记 p_i 为掷一箭得到 i 点的概率, 则

$$\begin{aligned}p_{30}&=\pi/36\\p_{20}&=4\pi/36-p_{30}=\pi/12\\p_{10}&=9\pi/36-p_{20}-p_{30}=5\pi/36\\p_0&=1-p_{10}-p_{20}-p_{30}=1-\pi/4\end{aligned}$$

(a) π/12 (b) π/9 (c) 1 − π/4

(d) π(30/36+20/12+50/36) = 35π/9 (e) $(\pi/4)^2$ (f) 2(π/36)(1−π/4)+2(π/12)(5π/36)

12. 令 Z 为标准正态随机变量.

(a)

$$P\Big\{\sum_{i=1}^{4} X_i > 0\Big\} = P\left\{\frac{\sum\limits_{i=1}^{4} X_i - 6}{\sqrt{24}} > \frac{-6}{\sqrt{24}}\right\} \approx P\{Z > -1.2247\} \approx 0.8897$$

(b)

$$P\Big\{\sum_{i=1}^{4} X_i > 0 \Big| \sum_{i=1}^{2} X_i = -5\Big\} = P\{X_3 + X_4 > 5\}$$
$$= P\left\{\frac{X_3 + X_4 - 3}{\sqrt{12}} > 2\sqrt{12}\right\} \approx P\{Z > 0.5774\} \approx 0.2818$$

(c)

$$P\Big\{\sum_{i=1}^{4} X_i > 0 | X_1 = 5\Big\} = P\{X_2 + X_3 + X_4 > -5\}$$
$$= P\left\{\frac{X_2 + X_3 + X_4 - 4.5}{\sqrt{18}} > -9.5\sqrt{18}\right\} \approx P\{Z > -2.239\} \approx 0.9874$$

13. 在下面式中, 常数 C 不依赖于 n.

$$P\{N = n | X = x\} = f_{X|N}(x|n)P\{N = n\}/f_X(x)$$
$$= C\frac{1}{(n-1)!}(\lambda x)^{n-1}(1-p)^{n-1} = C(\lambda(1-p)x)^{n-1}/(n-1)!$$

它指出, 在 $X = x$ 之条件下, $N - 1$ 是泊松随机变量, 其均值为 $\lambda(1-p)x$, 也即

$$P\{N = n | X = x\} = P\{N - 1 = n - 1 | X = x\}$$
$$= \mathrm{e}^{-\lambda(1-p)x}\frac{(\lambda(1-p)x)^{n-1}}{(n-1)!} \qquad n \geqslant 1$$

14. (a) 这个变换的雅可比值为

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

由方程 $u = x, v = x + y$ 解得 $x = u, y = v - u$, 我们得到

$$f_{U,V}(u,v) = f_{X,Y}(u, v-u) = 1 \qquad 0 < u < 1, 0 < v - u < 1$$

或

$$f_{U,V}(u,v) = 1 \qquad \max(v-1, 0) < u < \min(v, 1)$$

(b) 对于 $0 < v < 1$,

$$f_V(v) = \int_0^v \mathrm{d}u = v$$

对于 $1 \leqslant v \leqslant 2$,

$$f_V(v) = \int_{v-1}^1 \mathrm{d}u = 2 - v$$

15. 记 U 为 $(7,11)$ 上均匀随机变量, 如果你出价 $x, 7 \leqslant x \leqslant 10$, 你会以概率

$$(P\{U < x\})^3 = \left(P\left\{\frac{U-7}{4} < \frac{x-7}{4}\right\}\right)^3 = \left(\frac{x-7}{4}\right)^3$$

赢得这个项目. 因此, 你赚的钱数期望值为

$$E[G(x)] = \frac{1}{4}(x-7)^3(10-x)$$

由计算知, 当 $x = 37/4$ 时, 你赚的钱数达到最大值.

16. 记 $i_1, \cdots, i_n$ 为 $1, 2, \cdots, n$ 的一个排列.

$$\begin{aligned} P\{X_1 = i_1, X_2 = i_2, \cdots, X_n = i_n\} &= P\{X_1 = i_1\}P\{X_2 = i_2\}\cdots P\{X_n = i_n\} \\ &= p_{i_1}p_{i_2}\cdots p_{i_n} = p_1p_2\cdots p_n \end{aligned}$$

因此, 所求概率为 $n_!p_1\cdots p_n$. 当所有 $p_i = 1/n$ 时, 该概率变成 $n!/(n^n)$.

17. (a) 由于 $\sum_{i=1}^n X_i = \sum_{i=1}^n Y_i$, 故 $N = 2M$.

(b) 我们先在 $(Y_1, \cdots, Y_n)$ 固定的条件下求出 M 的分布. 若这个分布与 $Y_1, \cdots, Y_n$ 的值无关, 则 M 的条件分布就是 M 的无条件分布. 现假定 $(Y_1, \cdots, Y_n)$ 的值为

$$(1, \cdots, 1, 0, \cdots, 0)$$

即 $Y_1 = \cdots = Y_k = 1, Y_{k+1} = \cdots = Y_n = 0$. 我们把 $0, \cdots, 0$ 看成红球, 一共有 $n-k$ 个红球. 现在再看 X 的值, 它是一个随机的序列, 这个序列中有 k 个 1, $n-k$ 个 0.

$$\begin{aligned} Y &= (1, \cdots, 1, 0, 0, \cdots, 0) \\ X &= (i_1, \cdots, i_k, i_{k+1}, \cdots, i_n) \end{aligned}$$

将 $i_l = 1$, 看成取到第 l 个球这样, M 刚好为随机取 k 个球, 其中 $\{X_i = 1, Y_i = 0\}$ 的个数. 由于把 $Y = 0$ 解释为红球, 这样 M 就是从 n 个球中随机地抓 k 个球以后, 其中红球的个数, 而红球的总个数是 $n-k$, 这个分布是超几何分布. 由于这个分布与向量 Y 中 0 或 1 的位置排列无关, 因此我们求得的 M 的分布也是无条件分布.

(c) $E[N] = E[2M] = 2E[M] = 2k(n-k)/n$

(d) 利用第 4 章例 8j 中关于超几何分布的方差公式

$$\mathrm{Var}(N) = 4\mathrm{Var}(M) = 4\frac{n-k}{n-1}k(1-\frac{k}{n})(k/n)$$

18. (a) 由于 $S_n - S_k = \sum_{i=k+1}^n Z_i$, 它具有均值 0 和方差 $n-k$, 并且与 S_k 相互独立. 因此, 给定 $S_k = y$, S_n 是一个具有期望 y , 方差为 $n-k$ 的正态随机变量.

(b) 在求 $S_n = x$ 之下, S_k 的密度 $f_{S_k|S_n}(y|x)$ 的过程中, 将 x 看成一个与 y 无关的常数. 下面推论中, $C_i, i = 1, 2, 3, 4$ 都是与 y 无关的常数.

$$
\begin{aligned}
f_{S_k|S_n}(y|x) &= \frac{f_{S_k,S_n}(y,x)}{f_{S_n(x)}} \\
&= C_1 f_{S_n|S_k}(x|y) f_{S_k}(y) \qquad C_1 = \frac{1}{f_{s_n}(x)} \\
&= C_1 \frac{1}{\sqrt{2\pi}\sqrt{n-k}} \mathrm{e}^{-(x-y)^2/2(n-k)} \frac{1}{\sqrt{2\pi}\sqrt{k}} \mathrm{e}^{-y^2/2k} \\
&= C_2 \exp\left\{-\frac{(x-y)^2}{2(n-k)} - \frac{y^2}{2k}\right\} \\
&= C_3 \exp\left\{\frac{2xy}{2(n-k)} - \frac{y^2}{2(n-k)} - \frac{y^2}{2k}\right\} \\
&= C_3 \exp\left\{-\frac{n}{2k(n-k)}(y^2 - 2\frac{k}{n}xy)\right\} \\
&= C_3 \exp\left\{-\frac{n}{2k(n-k)}[(y-\frac{k}{n}x)^2 - (\frac{k}{n}x)^2]\right\} \\
&= C_4 \exp\left\{-\frac{n}{2k(n-k)}(y-\frac{k}{n}x)^2\right\}
\end{aligned}
$$

由上式可知, 这个密度为正态分布密度, 期望为 kx/n, 方差为 $k(n-k)/n$.

19. (a)

$$
\begin{aligned}
&P\{X_6 > X_1 | X_1 = \max(X_1, \cdots, X_5)\} \\
&= \frac{P\{X_6 > X_1, X_1 = \max(X_1, \cdots, X_5)\}}{P\{X_1 = \max(X_1, \cdots, X_5)\}} \\
&= \frac{P\{X_6 = \max(X_1, \cdots, X_6), X_1 = \max(X_1, \cdots, X_5)\}}{1/5} \\
&= 5 \times \frac{1}{6} \times \frac{1}{5} = \frac{1}{6}
\end{aligned}
$$

因此 X_6 达到最大值与前面 5 个 X_i 中哪个最大是无关的.

(b) 注意

$$P\{X_6 > X_2 | X_1 = \max(X_1, \cdots, X_5), X_6 > X_1\} = 1$$

另一方面, 在 $X_1 = \max(X_1, \cdots, X_5), X_6 < X_1$ 的条件下, X_6 与 X_2 的地位是对称的, 故

$$P\{X_6 > X_2 | X_1 = \max(X_1, \cdots, X_5), X_6 < X_1\} = \frac{1}{2}$$

利用上面的公式, 得到

$$
\begin{aligned}
&P\{X_6 > X_2 | X_1 = \max(X_1, \cdots, X_5)\} \\
&= P\{X_6 > X_2 | X_1 = \max(X_1, \cdots, X_5), X_6 > X_1\} P\{X_6 > X_1 | X_1 = \max(X_1, \cdots, X_5)\} \\
&\quad + P\{X_6 > X_2 | X_1 = \max(X_1, \cdots, X_5), X_6 < X_1\} P\{X_6 < X_1 | X_1 = \max(X_1, \cdots, X_5)\} \\
&= 1 \times \frac{1}{6} + \frac{1}{2} \times \frac{5}{6} = \frac{7}{12}
\end{aligned}
$$

第 7 章

1. (a) $d = \sum_{i=1}^{m} 1/n(i)$

(b) $P\{X=i\}=P\{[mU]=i-1\}=P\{i-1\leqslant mU<i\}=1/m, \qquad i=1,\cdots,m$

(c) $E\left[\frac{m}{n(X)}\right]=\sum_{i=1}^{m}\frac{m}{n(i)}P\{X=i\}=\sum_{i=1}^{m}\frac{m}{n(i)}\frac{1}{m}=d$

2. 令

$$I_j=\begin{cases}1 & \text{若第}j\text{次抽出的是白球, 而第}j+1\text{次抽出的是黑球}\\ 0 & \text{其他}\end{cases}$$

设 X 是抽出一个白球紧接着抽出一个是黑球的次数, 则

$$X=\sum_{j=1}^{n+m-1}I_j$$

因此,

$$E[X]=\sum_{j=1}^{n+m-1}E[I_j]=\sum_{j=1}^{n+m-1}P\{\text{第}j\text{次抽出白球, 第}j+1\text{次抽出黑球}\}$$

$$=\sum_{j=1}^{n+m-1}P\{\text{第}j\text{次抽出白球}\}P\{\text{第}j+1\text{次抽出黑球}\}$$

$$=\sum_{j=1}^{n+m-1}\frac{n}{n+m}\times\frac{m}{n+m-1}=\frac{nm}{n+m}$$

前面的论证中用到这样的事实, $n+m$ 个球中的任意一个球都具有相同的机会在第 j 次被抽出, 因此, 白球被抽出的概率为 $n/(n+m)$. 当第 j 次抽出白球之后, 在剩下的 $n+m-1$ 个球中, 任意一个球都有相同的机会被抽出, 因此, 在 j 次抽出白球的条件下, 第 $j+1$ 次抽出黑球的条件概率为 $m/(n+m-1)$.

3. 将各对夫妇编上号, 令 $I_j=1$ 表示第 j 对夫妇坐在同一桌, 否则 $I_j=0$. 若 X 代表坐在同一桌的夫妇的对数, 我们有

$$X=\sum_{j=1}^{10}I_j$$

因此

$$EX=\sum_{j=1}^{10}E[I_j]$$

(a) 为计算 $E[I_j]$, 考虑妇女 j, 其余 19 人的任意 3 人组合都有相同的机会与她同桌, 这种三人组合共有 $\binom{19}{3}$ 个. 这样, 她与丈夫的可能性为

$$\binom{1}{1}\binom{18}{2}\Big/\binom{19}{3}=\frac{3}{19}$$

因此, $E[I_j]=3/19$, 且

$$E[X]=30/19$$

(b) 这种情况下, 10 个男人的任何组合都有相同的机会与她同桌, 她的丈夫在这两人组内的可能性是 2/10. 因此,

$$E[I_j]=2/10 \qquad E[X]=2$$

4. 在例 2i 中, 我们已经指出, 要使所有 1 点到 6 点均出现, 所需掷骰子数的平均值为 $6(1+1/2+1/3+1/4+1/5+1/6)=14.7$. 现在令 X_i 表示在这个过程中 i 点出现的次数, 由于 $\sum_{i=1}^{6} X_i$ 表示出现全部 6 个点数所需的掷骰子数, 这样

$$14.7=E\Big[\sum_{i=1}^{6} X_i\Big]=\sum_{i=1}^{6} E[X_i]$$

由对称性, 所有 $E[X_i]$ 都是相等的. 故 $E[X_1]=14.7/6=2.45$.

5. 令 $I_j=1$ 若当第 j 张红牌翻过来时, 我们赢 1 个单位, $I_j=0$ 其他情况. 若 X 是我们赢的单位数, 则

$$EX=E\Big[\sum_{j=1}^{n} I_j\Big]=\sum_{j=1}^{n} E[I_j]$$

此处 $I_j = 1$, 如果已经翻出黑牌的张数比 j 少 (此时已翻出 j 张红牌), 由对称性, $E[I_j]=1/2$. 故 $E[X]=n/2$.

6. 先证明: $N\leqslant n-1+I$. 若所有事件都出现, 则此不等式两边相等. 若不是所有事件都发生, 显然 $N\leqslant n-1$, 这样该不等式成立. 现在将此不等式两边求期望, 得

$$E[N]\leqslant n-1+E[I]$$

若令 I_i 为事件 A_i 的示性函数, 即当 A_i 发生 $I_i=1$, 否则 $I_i=0$. 则

$$E[N]=E\Big[\sum_{i=1}^{n} I_i\Big]=\sum_{i=1}^{n} E[I_i]=\sum_{i=1}^{n} P(A_i)$$

但 $E[I]=P(A_1\cdots A_n)$, 故结论成立.

7. 我们想象一共有 n 个球, 其中 k 个红球, $n-k$ 个白球. 随机地从这 n 个球中取出 1 个球, 并在球上标上 1 号, 然后无放回地抽出第 2 个球, 标上 2 号, 依次下去, 直到最后一个球, 将它标上 n 号, 现在看一看这 k 个红球, 这 k 个红球的号码是 $\{1,2,\cdots,n\}$ 的一个大小为 k 的子集 $(i_1,\cdots,i_k)$, 显然 $(i_1,\cdots,i_k)$ 是随机子集, 并且对所有 $\binom{n}{k}$ 个子集都有相同的机会被取到. 这样 $(i_1,\cdots,i_k)$ 就是 $\{1,2,\cdots,n\}$ 的一个简单随机抽样. 另一方面, 不妨设 $i_1<\cdots<i_k$, 其中 i_1 就是第一次抽到红球时取球的次数, 由例 3e 可知, 它是负超几何分布, 其平均值为 $1+\dfrac{n-k}{k+1}=\dfrac{n+1}{k+1}$.

也可用下列形式求得 X 的分布, 其中 X 表示从 $\{1,2,\cdots,n\}$ 中随机抽取 k 个数中的最小数. $\{X\geqslant j\}$ 表示抽取的 k 个数都比 $j-1$ 大. 故

$$P\{X\geqslant j\}=\binom{n-j+1}{k}\Big/\binom{n}{k}=\binom{n-k}{j-1}\Big/\binom{n}{j-1}$$

X 的分布就是超几何分布.

8. 记 X 表示在桑切斯家离开以后离开机场的户数, 将其余的人家任意编号, $i=1,2,\cdots,N-1$. 记 $I_i=1$, 若 i 家比桑切斯家晚离开机场. $I_i=0$, 若 i 家比桑切斯家早离开机场. 此时, X 与 I_i 之间有如下关系:

$$X=\sum_{i=1}^{N-1} I_i$$

两边求期望得

$$E[X]=\sum_{i=1}^{N-1}P\{i\text{家在桑切斯家后离开机场}\}$$

现在考虑 i 家, 设 i 家有 k 件行李, 而桑切斯家有 j 件行李. 两家一共有 $k+j$ 件行李, 这 $k+j$ 件在行李线上排成了一个队. (当然中间还会有其他家庭的行李, 但是我们关心的只是这两家的行李. 这两家的行李也形成了一个次序, 排成了一个队.) 这两家的行李的排序决定了哪一家先离开机场, 若这 $k+j$ 件行李中排在最后的一件行李是桑切斯家的, 那么, 桑切斯家比 i 家后离开机场. 否则, 桑切斯家比 i 家早离开机场. 由于这 $k+j$ 件行李中的每一件都以相同的机会排在最后, 因此 i 家比桑切斯家晚离开机场的概率为 $k/(k+j)$. 对于除了桑切斯以外的家庭, 具有 k 件行李的户数为 $n_k, k\neq j$, 当 $k=j$ 时, 共有 n_j-1. 这样, 我们得到

$$E[X]=\sum_k\frac{kn_k}{k+j}-\frac{1}{2}$$

9. 对于单位圆周上的一个点, 它的邻域是指从这个点出发逆时针方向距离为 1 的那样的一段弧 (这与几何上的邻域的概念有区别). 现在在这一圆周上随机地取一个点, 这个点在长度为 x 的弧上的概率为 $x/2\pi$. 记 X 表示圆周上的 19 个点在这个随机点的邻域上的点数. 令 $I_j=1$, 如果第 j 个点在这个随机点的邻域上, 其他情况, $I_j=0$. 则

$$X=\sum_{j=1}^{19}I_j$$

两边取期望得

$$E[X]=\sum_{j=1}^{19}P\{\text{第}j\text{个点在随机点的邻域内}\}$$

事实上, 任意一个点在这个随机点的概率等于 $1/2\pi$. 这样

$$E[X]=19/2\pi>3$$

由 $E[X]>3$ 可知, 至少有一个 X 的可能值使得 $X>3$, 即至少有一个随机点, 使得在这个点的邻域内有 4 个以上的点.

10. 令 $g(x)=x^{1/2}$, 则

$$g'(x)=\frac{1}{2}x^{-1/2}\qquad g''(x)=-\frac{1}{4}x^{-3/2}$$

因此, $\sqrt{X}$ 在 λ 处泰勒展开得:

$$\sqrt{X}\approx\sqrt{\lambda}+\frac{1}{2}\lambda^{-1/2}(X-\lambda)-\frac{1}{8}\lambda^{-3/2}(X-\lambda)^2$$

两边求期望得

$$\begin{aligned}E[\sqrt{X}]&\approx\sqrt{\lambda}+\frac{1}{2}\lambda^{-1/2}E[X-\lambda]-\frac{1}{8}\lambda^{-3/2}E[(X-\lambda)^2]\\&=\sqrt{\lambda}-\frac{1}{8}\lambda^{-3/2}\lambda=\sqrt{\lambda}-\frac{1}{8}\lambda^{-1/2}\end{aligned}$$

因此

$$\mathrm{Var}(X)=E[X]-(E[\sqrt{X}])^2\approx\lambda-(\sqrt{\lambda}-\frac{1}{8}\lambda^{-1/2})^2=1/4-\frac{1}{64\lambda}\approx1/4$$

11. 将桌子编号, $1,2,3$ 是 4 座位桌子, $4,5,6,7$ 是 2 座位桌子. 令 $X_{ij}=1$, 如果妻子 i 与她的丈夫坐在桌子 j, 否则 $X_{ij}=0$. 注意

$$E[X_{ij}]=\binom{2}{2}\binom{18}{2}\Big/\binom{20}{4}=\frac{3}{95}\qquad j=1,2,3$$

和

$$E[X_{ij}]=1\Big/\binom{20}{2}=\frac{1}{190}\qquad j=4,5,6,7$$

记 X 表示夫妻坐在同一桌的对数, 我们有

$$\begin{aligned}E[X]&=E\Big[\sum_{i=1}^{10}\sum_{j=1}^{7}X_{ij}\Big]=\sum_{i=1}^{10}\sum_{j=1}^{3}E[X_{ij}]+\sum_{i=1}^{10}\sum_{j=4}^{7}E[X_{ij}]\\&=30(3/95)+40(1/190)=22/19\end{aligned}$$

12. 记 $X_i=1$, 如果第 i 个人没有招聘到任何人, $X_i=0$, 其他情况.

$$\begin{aligned}E[X_i]&=P\{i\text{没有招聘到}i+1,\cdots,n\text{中的任意一人}\}\\&=\frac{i-1}{i}\cdot\frac{i}{i+1}\cdots\frac{n-2}{n-1}=\frac{i-1}{n-1}\end{aligned}$$

因此

$$E\Big[\sum_{i=1}^{n}X_i\Big]=\sum_{i=1}^{n}\frac{i-1}{n-1}=\frac{n}{2}$$

由于 X_i 为伯努利随机变量, 我们有

$$\mathrm{Var}(X_i)=\frac{i-1}{n-1}(1-\frac{i-1}{n-1})=\frac{(i-1)(n-i)}{(n-1)^2}$$

对于 $i<j$,

$$E[X_iX_j]=\frac{i-1}{i}\cdots\frac{j-2}{j-1}\times\frac{j-2}{j}\times\frac{j-1}{j+1}\cdots\frac{n-3}{n-1}=\frac{(i-1)(j-2)}{(n-2)(n-1)}$$

故

$$\mathrm{Cov}(X_i,X_j)=\frac{(i-1)(j-2)}{(n-2)(n-1)}-\frac{i-1}{n-1}\frac{j-1}{n-1}=\frac{(i-1)(j-n)}{(n-2)(n-1)^2}$$

从而

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{n}X_i\right)&=\sum_{i=1}^{n}\mathrm{Var}(X_i)+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mathrm{Cov}(X_i,X_j)\\&=\sum_{i=1}^{n}\frac{(i-1)(n-i)}{(n-1)^2}+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{(i-1)(j-n)}{(n-2)(n-1)^2}\\&=\frac{1}{(n-1)^2}\sum_{i=1}^{n}(i-1)(n-i)\\&\quad-\frac{1}{(n-2)(n-1)^2}\sum_{i=1}^{n-1}(i-1)(n-i)(n-i-1)\end{aligned}$$

13. 令 $X_i = 1$, 如果第 i 个三人组内包含每种类型的球员. 其他情形, $X_i = 0$. 此时

$$E[X_i] = \binom{2}{1}\binom{3}{1}\binom{4}{1}\Big/\binom{9}{3} = \frac{2}{7}$$

(a) 我们得到

$$E\Big[\sum_{i=1}^{3} X_i\Big] = 6/7$$

由于 X_i 为伯努利随机变量, 我们得到

$$\mathrm{Var}(X_i) = (2/7)(1-2/7) = 10/49$$

对于 $i \neq j$

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1\}P\{X_j = 1|X_i = 1\}$$

$$= \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}} \times \frac{\binom{1}{1}\binom{2}{1}\binom{3}{1}}{\binom{6}{3}} = 6/70$$

(b)

$$\mathrm{Var}\Big(\sum_{i=1}^{3} X_i\Big) = \sum_{i=1}^{3}\mathrm{Var}(X_i) + 2\sum_{i=1}^{3}\sum_{j>1}\mathrm{Cov}(X_i, X_j)$$

$$= 30/49 + 2 \times \binom{3}{2} \times (6/70 - 4/49) = \frac{312}{490}$$

14. 令 $X_i = 1$, 若第 i 张牌是“A”, 其他情形 $X_i = 0$. 又令 $Y_j = 1$, 若第 j 张牌是黑桃, 否则 $Y_j = 0$. $i, j = 1, 2 \cdots, 13$. 一张牌的花色不会改变另一张牌的花色, 这一点, 只要计算相应的条件概率. 令 $A_{i,s}, A_{i,h}, A_{i,d}, A_{i,c}$ 分别为第 i 张牌是黑桃, 红心, 方块, 梅花的事件, 则

$$P\{Y_j = 1\} = \frac{1}{4}(P\{Y_j = 1|A_{i,s}\} + P\{Y_j = 1|A_{i,h}\} + P\{Y_j = 1|A_{i,d}\} + P\{Y_j = 1|A_{i,c}\})$$

而等式右边四项相等, 所以

$$P\{Y_j = 1\} = P\{Y_j = 1|A_{i,s}\}$$

因此, X_i 与 Y_j 是相互独立的, $i \neq j$, 进一步可以证明, 即使 $i = j$, 也是相互独立的. 利用这个事实, 我们得到

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}\Big(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_i, Y_j) = 0$$

但是 X, Y 是不独立的. 事实上, $P\{Y = 13\}$ 表示得到一副 13 张全是黑桃的一副牌, 显然其概率不为 0. 但是 $P\{Y = 13|X = 4\} = 0$. (已知一副牌中有 4 张“A”, 显然 $P\{Y = 13|X = 4\} = 0$.) 这说明 X 与 Y 相互不独立.

15. (a) 当没有任何信息时, 你的期望收入为 0.

(b) 当知道 $p>1/2$ 时, 你应该猜正面朝上; 当 $p\leqslant 1/2$ 时, 应猜反面朝上.

(c) 当知道 V(硬币的 p) 的值, 则你赢得的期望值为

$$\begin{aligned}&\int_0^1 E[\text{赢得}|V=p]\mathrm{d}p\\&=\int_0^{1/2}[1\times(1-p)-1\times p]\mathrm{d}p+\int_{1/2}^1[1\times p-1\times(1-p)]\mathrm{d}p=1/2\end{aligned}$$

16. 首先指出列表有 m 个位置, 而构造的随机变量 X 是在 m 个位置 $\{1,2,\cdots,m\}$ 上均匀分布的随机变量. 当 $X=i$ 时, $n(X)=n(i)$ 而 $n(i)$ 是列表上与位置 i 上的名称相同的名称的个数. 首先我们指出

$$E[m/n(X)]=\sum_{i=1}^m\frac{m}{n(i)}P\{X=i\}=\sum_{i=1}^m\frac{m}{n(i)}\times\frac{1}{m}=\sum_{i=1}^m\frac{1}{n(i)}=d$$

我们需要对上面的最后一个等式作一说明, 将和号分成 d 个部分之和

$$\sum_{i=1}^m\frac{1}{n(i)}=\sum_{j=1}^d\sum_{i\in A_j}\frac{1}{n(i)}$$

其中 A_j 表示具有相同名称的位置 i 的集合. A_j 中有 $n(i)$ 个位置, 因此

$$\sum_{i\in A_j}\frac{1}{n(i)}=n(i)\times\frac{1}{n(i)}=1$$

这样便得到 $\sum_{i=1}^m\frac{1}{n(i)}=d$, 但是 $n(X)$ 不好计算. 我们用 I 代替 $1/n(X)$. 现在计算 $E[I|n(X)]$.

$$E[I|n(X)=n(i)]=P\{I=1|n(X)=n(i)\}=\frac{1}{n(i)}$$

上式最后一式是由于当 $n(X)=n(i)$ 时, X 可能存在 $n(i)$ 种情况, 哪一种情况都是等可能的, 只有 X 取其中最小值时, I 才等于 1. 两边再取期望, 得

$$E[I]=E[E[I|n(X)]]=E[1/n(X)]$$

这样我们得到

$$E[mI]=E[m/n(X)]=d^1$$

17. 令 $X_i=1$, 如果第 i 件物品放入某房间时发生碰撞; 否则 $X_i=0$. 这样碰撞总数

$$X=\sum_{i=1}^m X_i$$

因此,

$$E[X]=\sum_{i=1}^m E[X_i]$$

---

1. (a) 在求期望过程中, 我们假定不同的名称, 具有不同的 $n(i)$, 若不同的名称, 其相应的 $n(i)$ 可以相同, 其结论还成立, 不过论证需稍作修改.

(b) mI 是 d 的无偏估计. 但它绝不是一个好估计, 因为 mI 只取两个值, m 或 0, 这都是极端情况, 特别是 0, 根本是一个不合理的值. 因此, 实际中不会用这一估计. ——译者注

为求 $E[X]$, 可以利用条件期望,

$$\begin{aligned}
E[X_i] &= \sum_j E[X_j|\text{物品}i\text{放入房间}j]p_j \\
&= \sum_j P\{\text{物品 } i \text{ 形成碰撞}|\text{物品 } i \text{ 放入房间 } j\}p_j \\
&= \sum_j [1-(1-p_j)^{i-1}]p_j = 1 - \sum_j (1-p_j)^{i-1}p_j
\end{aligned}$$

上面最后第二个等式是这样解释的, 在物品 i 放入房间 j 的条件下, 形成碰撞的意思是前面 $i-1$ 个物品中至少有一个已经放入房间 j, 而它的概率刚好是 $1-(1-p_j)^{i-1}$. 有了 $E[X_i]$ 的等式以后, 我们得到

$$E[X] = m - \sum_{i=1}^{m}\sum_{j=1}^{n}(1-p_j)^{i-1}p_j$$

改变求和次序得

$$E[X] = m - n + \sum_{j=1}^{n}(1-p_j)^m$$

由上式可以看出, 这个等式可以以更加容易的方式导出, 只需对下面的恒等式求期望即可.

$$m - X = \text{非空的房间数}$$

其中 m 为物品总数, 当 m 大于非空房间数时, 必定有房间放入 2 个或 2 个以上的物品, 那些多余的物品数就是碰撞次数. 两边求期望时, 求非空房间数的期望, 还需要一个技巧, 将非空房间数分解成 n 个示性函数的和, 而每个示性函数的期望就是某房间是否为非空房间的概率.

18. 记 L 为第一个游程的长度, 以第一个值为条件求期望可得

$$E[L] = E[L|\text{第一个值为 } 1]\frac{n}{n+m} + E[L|\text{第一个值为 } 0]\frac{m}{n+m}$$

现在考虑 $E[L|\text{第一个值为 } 1]$. 此时, 这个序列具有形式

$$1\ 1\ 0\ 0\ 1\ \cdots\ 10$$

一共有 n 个 1, m 个 0, 但第一个值为 1, 若将这个序列的第一个 1 去掉, 这样, 这个子序列成为

$$1\ 0\ 0\ 1\ \cdots\ 10$$

其中有 $n-1$ 个 1, m 个 0. 原来序列的第一个游程的长度就是这个子序列中第一个 0 的位置. 例如, 在我们列出的第一个游程的长度为 2(两个 1), 它就是子序列中第一个 0 的位置. 现在的问题化成将 $n-1$ 个 1, m 个 0, 随机地排成一个序列, 求这个序列的第一个 0 的位置的期望值, 这个值刚好等于从 $n-1$ 个白球, m 个红球中, 随机地一个一个往外取球, 直到拿出第一个红球的所需的平均次数. 利用例 3e 的结果, 这个平均数等于 $(n+m)/(m+1)$. 对于 $E[L|\text{第一个值为 } 0]$ 的计算是类似的. 这样, 我们得到

$$E[L] = \frac{n+m}{m+1}\frac{n}{n+m} + \frac{n+m}{n+1}\frac{m}{n+m} = \frac{n}{m+1} + \frac{m}{n+1}$$

19. 设 X 是将两个盒子内物件拿光所需要的抛掷次数. 记 Y 为前 $n+m$ 次掷硬币所得正面朝上次数, 利用条件期望的性质

$$E[X]=\sum_{i=0}^{n+m}E[X|Y=i]P\{Y=i\}=\sum_{i=0}^{n+m}E[X|Y=i]\binom{n+m}{i}p^i(1-p)^{n+m-i}$$

现在假定在 $n+m$ 次抛掷硬币中, 得到正面朝上 i 次, $i<n$. 此时, 附加的次数只是出现 $n-i$ 个正面朝上所需的次数. 若 $i=n$, 两个盒子中的物件已经取光, 无需再做附加的抛掷硬币试验. 若 $i>n$, 此时盒子 H 内的物件已经取光, 在盒子 T 内还有 $i-n$ 个物件尚未取光. 附加的掷硬币次数只是出现 $i-n$ 次反面朝上所需的掷硬币次数. 这样

$$E[X]=\sum_{i=0}^{n}\frac{n-i}{p}\binom{n+m}{i}p^i(1-p)^{n+m-i}+\sum_{i=n+1}^{n+m}\frac{i-n}{1-p}\binom{n+m}{i}p^i(1-p)^{n+m-i}$$

20. 利用提示中的等式, 两边求期望得

$$\begin{aligned}E[X^n]&=E\Big[n\int_0^\infty x^{n-1}I_X(x)\mathrm{d}x\Big]=n\int_0^\infty E\Big[x^{n-1}I_X(x)\Big]\mathrm{d}x\\&=n\int_0^\infty x^{n-1}E[I_X(x)]\mathrm{d}x=n\int_0^\infty x^{n-1}\bar{F}(x)\mathrm{d}x\end{aligned}$$

上述论证中积分号与期望号的可交换性是由于所涉及的随机变量均为非负.

21. 考虑一个随机排列 $I_1,\cdots,I_n$, 它取任何一个具体的排列 $i_1,\cdots,i_n$ 的概率都是相同的 (等于 $1/n!$). 这样,

$$\begin{aligned}&E[a_{I_j}a_{I_{j+1}}]=\sum_k E[a_{I_j}a_{I_{j+1}}|I_j=k]P\{I_j=k\}\\&=\frac{1}{n}\sum_k a_kE[a_{I_{j+1}}|I_j=k]=\frac{1}{n}\sum_k a_k\sum_i a_iP\{I_{j+1}=i|I_j=k\}\\&=\frac{1}{n(n-1)}\sum_k a_k\sum_{i\neq k}a_i=\frac{1}{n(n-1)}\sum_k a_k(-a_k)<0\end{aligned}$$

其中最后第二个等式是由于 $\sum_{i=1}^n a_i=0$, 有了上述不等式, 我们可得

$$E\Big[\sum_{j=1}^{n-1}a_{I_j}a_{I_{j+1}}\Big]<0$$

这说明必有一个排列, 使得

$$\sum_{j=1}^{n-1}a_{i_j}a_{i_{j+1}}<0$$

22. (a) $E[X]=\lambda_1+\lambda_2, E[Y]=\lambda_2+\lambda_3$

(b)

$$\begin{aligned}\mathrm{Cov}(X,Y)&=\mathrm{Cov}(X_1+X_2,X_2+X_3)\\&=\mathrm{Cov}(X_1,X_2+X_3)+\mathrm{Cov}(X_2,X_2+X_3)\\&=\mathrm{Cov}(X_2,X_2)=\mathrm{Var}(X_2)=\lambda_2\end{aligned}$$

(c) 利用条件期望的性质

$$\begin{aligned}
P\{X=i,Y=j\} &= \sum_k P\{X=i,Y=j|X_2=k\}P\{X_2=k\} \\
&= \sum_k P\{X_1=i-k,X_3=j-k|X_2=k\}\mathrm{e}^{-\lambda_2}\lambda_2^k/k! \\
&= \sum_k P\{X_1=i-k,X_3=j-k\}\mathrm{e}^{-\lambda_2}\lambda_2^k/k! \\
&= \sum_k P\{X_1=i-k\}P\{X_3=j-k\}\mathrm{e}^{-\lambda_2}\lambda_2^k/k! \\
&= \sum_{k=0}^{\min(i,j)} \mathrm{e}^{-\lambda_1}\frac{\lambda_1^{i-k}}{(i-k)!}\mathrm{e}^{-\lambda_3}\frac{\lambda_3^{j-k}}{(j-k)!}\mathrm{e}^{-\lambda_2}\frac{\lambda_2^k}{k!}
\end{aligned}$$

23.

$$\begin{aligned}
\mathrm{Corr}\Big(\sum_i X_i,\sum_j Y_j\Big) &= \frac{\mathrm{Cov}\big(\sum_i X_i,\sum_j Y_j\big)}{\sqrt{\mathrm{Var}\big(\sum_i X_i\big)\mathrm{Var}\big(\sum_j Y_j\big)}} = \frac{\sum_i\sum_j \mathrm{Cov}\big(X_i,Y_j\big)}{\sqrt{n\sigma_x^2 n\sigma_y^2}} \\
&= \frac{\sum_i \mathrm{Cov}(X_i,Y_i)+\sum_i\sum_{j\neq i}\mathrm{Cov}(X_i,Y_j)}{n\sigma_x\sigma_y} \\
&= \frac{n\rho\sigma_x\sigma_y}{n\sigma_x\sigma_y}=\rho
\end{aligned}$$

其中最后第二个等式是利用了 $\mathrm{Cov}(X_i,Y_i)=\rho\sigma_x\sigma_y$.

24. 令

$$X_i=\begin{cases}1 & \text{如果第}i\text{张牌为“A”} \\ 0 & \text{其他}\end{cases}$$

这样,

$$X=\sum_{i=1}^3 X_i$$

且 $E[X_i]=P\{X_i=1\}=1/13$(牌“A”是 1 点!). 现在用 A 表示事件“黑桃已被选中”.

$$\begin{aligned}
E[X] &= E[X|A]P(A)+E[X|A^{\mathrm{c}}]P(A^{\mathrm{c}}) \\
&= E[X|A]\frac{3}{52}+E[X|A^{\mathrm{c}}]\frac{49}{52} = E[X|A]\frac{3}{52}+\frac{49}{52}E\Big[\sum_{i=1}^3 X_i|A^{\mathrm{c}}\Big] \\
&= E[X|A]\frac{3}{52}+\frac{49}{52}\sum_{i=1}^3 E\Big[X_i|A^{\mathrm{c}}\Big] = E[X|A]\frac{3}{52}+\frac{49}{52}\times 3\times\frac{3}{51}
\end{aligned}$$

利用 $E[X]=3/13$, 可求得

$$E[X|A]=\frac{52}{3}\Big(\frac{3}{13}-\frac{49}{52}\frac{3}{17}\Big)=\frac{19}{17}=1.1176$$

类似地, 令 L 表示“至少有一张‘A’被选中”, 此时

$$\begin{aligned}
E[X] &= E[X|L]P(L)+E[X|L^{\mathrm{c}}]P(L^{\mathrm{c}}) \\
&= E[X|L]P(L) = E[X|L]\Big(1-\frac{48\times 47\times 46}{52\times 51\times 50}\Big)
\end{aligned}$$

这样,

$$E[X|L] = \frac{3/13}{1-(48\times 47\times 46)/(52\times 51\times 50)} \approx 1.0616$$

另一种解法是将牌“A”进行编号, 将黑桃“A”编号为 1, 令

$$Y_i = \begin{cases} 1 & \text{如果第}i\text{ 张“A”被选中} \\ 0 & \text{其他} \end{cases}$$

此时,

$$E[X|A] = E\Big[\sum_{i=1}^{4} Y_i|Y_1=1] = 1 + \sum_{i=2}^{4} E[Y_i|Y_1=1] = 1 + 3\times\frac{2}{51} = 19/17$$

同样,

$$E[X|L] = E\Big[\sum_{i=1}^{4} Y_i|L\Big] = \sum_{i=1}^{4} E[Y_i|L] = 4P\{Y_1=1|L\}$$

但

$$P\{Y_1=1|L\} = P(A|L) = \frac{P(AL)}{P(L)} = \frac{P(A)}{P(L)} = \frac{3/52}{1-(48\times 47\times 46)/(52\times 51\times 50)}$$

与前面的结果完全相同.

25. (a) $E[I|X=x] = P\{Z<X|X=x\} = P\{Z<x|X=x\} = P\{Z<x\} = \Phi(x)$

(b) 由 (a) 知 $E[I|X] = \Phi(X)$, 因此

$$E[I] = E[E[I|X]] = E[\Phi(X)]$$

再由 $E[I] = P\{I=1\} = P\{Z<X\}$ 可得所需结论.

(c) 由于 $X-Z$ 为正态随机变量, 均值为 μ, 方差为 2, 我们有

$$\begin{aligned} P\{X>Z\} &= P\{X-Z>0\} = P\Big\{\frac{X-Z-\mu}{\sqrt{2}} > \frac{-\mu}{\sqrt{2}}\Big\} \\ &= 1-\Phi\Big(\frac{-\mu}{\sqrt{2}}\Big) = \Phi\Big(\frac{\mu}{\sqrt{2}}\Big) \end{aligned}$$

26. 设前 $n+m-2$ 次抛掷硬币时出现正面朝上的次数为 N. 令 $M=\max(X,Y)$ 表示抛掷硬币一直到出现 n 个正面朝上和 m 个反面朝上所需的抛掷硬币次数. (关于 X,Y 的定义, 可见本题的提示.) 利用条件期望的性质,

$$\begin{aligned} E[M] &= \sum_i E[M|N=i]P\{N=i\} \\ &= \sum_{i=0}^{n-1} E[M|N=i]P\{N=i\} + \sum_{i=n}^{n+m-1} E[M|N=i]P\{N=i\} \end{aligned}$$

现在假定在 $n+m-1$ 次试验中一共 i 个正面朝上. 若 $i<n$, 此时, 我们至少已经有 m 个反面朝上的硬币. 为了达到 M 次试验, 我们只需进行附加试验, 获取另外 $n-i$ 个正面朝上即可. 而 $E[M|N=i] = n+m-1+(n-i)/p$. 类似地, 如果 $i \geqslant n$, 此时正面朝上数已满足了要求, 为了达到既有 n 个正面朝上, 又有 m 个反面朝上, 我们只需继续做附加

试验, 达到 $m-(n+m-1-i)$ 次反面朝上即可. 此时, $E[M|N-i]=(i+1-n)/(1-p)$. 这样, 我们得到

$$\begin{aligned}E[M]=&\sum_{i=0}^{n-1}\left(n+m-1+\frac{n-i}{p}\right)P\{N=i\}\\&+\sum_{i=n}^{n+m-1}\left(n+m-1+\frac{i+1-n}{1-p}\right)P\{N=i\}\\=&n+m-1+\sum_{i=0}^{n-1}\frac{n-i}{p}\binom{n+m-1}{i}p^i(1-p)^{n+m-1-i}\\&+\sum_{i=n}^{n+m-1}\frac{i+1-n}{1-p}\binom{n+m-1}{i}p^i(1-p)^{n+m-1-i}\end{aligned}$$

这样, $E[\min(X,Y)]$ 可由下式给出

$$E[\min(X,Y)]=E[X+Y-M]=\frac{n}{p}+\frac{m}{1-p}-E[M]$$

27. 这个问题与例 2i 中的收集优惠券的问题是一样的, 即求收集到 $n-1$ 种优惠券所需要的平均次数 (优惠券的总数为 n). 这个例子的解为

$$1+\frac{n}{n-1}+\frac{n}{n-2}+\cdots+\frac{n}{2}$$

28.

$$E[X]=\sum_{i=1}^{\infty}P\{X\geqslant i\}=\sum_{i=1}^{n}P\{X\geqslant i\}=\sum_{i=1}^{n}q^{i-1}=\frac{1-q^n}{p}$$

式中 $q=1-p$.

29.

$$\mathrm{Cov}(X,Y)=E[XY]-E[X]E[Y]=P(X=1,Y=1)-P(X=1)P(Y=1)$$

因此

$$\mathrm{Cov}(X,Y)=0\Longleftrightarrow P(X=1,Y=1)=P(X=1)P(Y=1)$$

因为

$$\mathrm{Cov}(X,Y)=\mathrm{Cov}(1-X,1-Y)=-\mathrm{Cov}(1-X,Y)=-\mathrm{Cov}(X,1-Y)$$

从前面的结论可以看出，当 X 和 Y 的分布是伯努利分布时，下面的结论是相互等价的：

1. $\mathrm{Cov}(X,Y)=0$
2. $P(X=1,Y=1)=P(X=1)P(Y=1)$
3. $P(1-X=1,1-Y=1)=P(1-X=1)P(1-Y=1)$
4. $P(1-X=1,Y=1)=P(1-X=1)P(Y=1)$
5. $P(X=1,1-Y=1)=P(X=1)P(1-Y=1)$

30. 首先将人员编号, 记 (i,j) 为第 j 个戴 i 号帽子的人. 如果 (i,j) 拿到 i 号帽子, 则记 $X_{i,j}=1$, 否则 $X_{i,j}=0$. 这样, 拿到自己帽子型号的总人数为

$$X=\sum_{i=1}^{r}\sum_{j=1}^{n_i}X_{i,j}$$

因此

$$E[X]=\sum_{i=1}^{r}\sum_{j=1}^{n_i}E[X_{i,j}]=\sum_{i=1}^{r}\sum_{j=1}^{n_i}\frac{h_i}{n}=\frac{1}{n}\sum_{i=1}^{r}h_in_i$$

第 8 章

1. 设 X 为下一周的汽车销售量, 利用马尔可夫不等式

(a) $P\{X>18\}=P\{X\geqslant 19\}\leqslant\dfrac{E[X]}{19}=16/19$

(b) $P\{X>25\}=P\{X\geqslant 26\}\leqslant\dfrac{E[X]}{26}=16/26$

2. (a)

$$\begin{aligned}P\{10\leqslant X\leqslant 22\}&=P\{|X-16|\leqslant 6\}=P\{|X-\mu|\leqslant 6\}\\&=1-P\{|X-\mu|>6\}\geqslant 1-9/36=3/4\end{aligned}$$

(b) $P\{X\geqslant 19\}=P\{X-16\geqslant 3\}\leqslant\dfrac{9}{9+9}=1/2$

在 (a) 中利用了切比雪夫不等式, 而 (b) 利用了单边的切比雪夫不等式. (见命题 5.1)

3. 关于 $X-Y$, 有下列结论:

$$\begin{aligned}&E[X-Y]=0\\&\mathrm{Var}(X-Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)-2\mathrm{Cov}(X,Y)=28\end{aligned}$$

下面不等式中, (a) 利用了切比雪夫不等式, (b),(c) 利用了单边不等式.

(a) $P\{|X-Y|>15\}\leqslant 28/225$
(b) $P\{X-Y>15\}\leqslant 28/(28+225)=28/253$
(c) $P\{Y-X>15\}\leqslant 28/(28+225)=28/253$

4. 设工厂 A 的生产数为 X, 工厂 B 的生产数为 Y, 则

$$E[Y-X]=-2\qquad \mathrm{Var}(Y-X)=36+9=45$$

$$P\{Y-X>0\}=P\{Y-X\geqslant 1\}=P\{Y-X+2\geqslant 3\}\leqslant\frac{45}{45+9}=45/54$$

5. 注意到

$$E[X_i]=\int_0^1 2x^2\,\mathrm{d}x=2/3$$

利用强大数律可得

$$r=\lim_{n\to\infty}\frac{n}{S_n}=\lim_{n\to\infty}\frac{1}{S_n/n}=\frac{1}{\lim\limits_{n\to\infty}S_n/n}=1/(2/3)=3/2$$

6. 上题中得到 $E[X_i]=2/3$, 由

$$E[X_i^2]=\int_0^1 2x^3\,\mathrm{d}x=1/2$$

我们得到 $\mathrm{Var}(X_i)=1/2-(2/3)^2=1/18$. 因此, 若一共有 n 个元件,

$$\begin{aligned}P\{S_n\geqslant 35\}&=P\{S_n\geqslant 34.5\}\qquad\text{连续性修正}\\&=P\left\{\frac{S_n-2n/3}{\sqrt{n/18}}\geqslant\frac{34.5-2n/3}{\sqrt{n/18}}\right\}\approx P\left\{Z\geqslant\frac{34.5-2n/3}{\sqrt{n/18}}\right\}\end{aligned}$$

其中 Z 为标准正态随机变量. 由于

$$P\{Z>-1.284\}=P\{Z<1.284\}\approx 0.90$$

元件数 n 应满足

$$34.5-2n/3\approx -1.284\sqrt{n/18}$$

由计算给出 $n=55$.

7. 设 X 是修理一台机器所用的时间, 则

$$E[X]=0.2+0.3=0.5$$

利用指数分布的方差等于它的期望的平方, 得

$$\mathrm{Var}(X)=0.2^2+0.3^2=0.13$$

现设 $X_i, i=1,2,\cdots,20$ 为 20 台机器的修理时间, Z 为标准正态随机变量,

$$\begin{aligned}P\{X_1+\cdots+X_{20}<8\}&=P\left\{\frac{X_1+\cdots+X_{20}-10}{\sqrt{2.6}}<\frac{8-10}{\sqrt{2.6}}\right\}\\&\approx P\{Z<-1.24035\}\approx 0.1074\end{aligned}$$

8. 首先设 X 为赌徒一次所赢的钱数 (若负数为输). 则

$$E[X]=-0.7-0.4+1=-0.1\qquad E[X^2]=0.7+0.8+10=11.5$$

$$\Rightarrow \mathrm{Var}(X)=11.49$$

$$\begin{aligned}P\{X_1+\cdots+X_{100}\leqslant -0.5\}&=P\left\{\frac{X_1+\cdots+X_{100}+10}{\sqrt{1149}}\leqslant\frac{-0.5+10}{\sqrt{1149}}\right\}\\&\approx P\{Z\leqslant 0.2803\}\approx 0.6104\end{aligned}$$

9. 利用自检习题 7 中的记号,

$$P\{X_1+\cdots+X_{20}<t\}=P\left\{\frac{X_1+\cdots+X_{20}-10}{\sqrt{2.6}}<\frac{t-10}{\sqrt{2.6}}\right\}\approx P\left\{Z<\frac{t-10}{\sqrt{2.6}}\right\}$$

由于 $P\{Z<1.645\}\approx 0.95$, t 应满足

$$\frac{t-10}{\sqrt{2.6}}\approx 1.645$$

得 $t\approx 12.65$.

10. 如果烟草公司宣布的结论是正确的, 记尼古丁的含量为 $\bar{X}$, 平均含量超过 3.1 的概率为

$$P\{\bar{X} > 3.1\} = P\left\{\frac{\sqrt{100} \times (X - 2.2)}{0.3} > \frac{\sqrt{100} \times (3.1 - 2.2)}{0.3}\right\}$$
$$\approx P\{Z > 30\} \approx 0$$

其中 Z 为标准正态随机变量.

本问题是统计检验问题, 若真像公司宣称那样, 那么试验结果是不可能出现的. 因此, 推翻了公司的结论.

11. (a) 随机地将电池编号, 记 X_i 为第 i 号电池的寿命 $(i = 1, \ldots, 40)$, 则 X_i 为独立同分布的随机变量序列. 现在计算 X_1 的期望和方差. 记 $I = 1$ 表示电池的型号为 A, $I = 0$ 表示电池的型号为 B. 由

$$E[X_1|I = 1] = 50, \quad E[X_1|I = 0] = 30$$

可知

$$E[X_1] = 50 \times P\{I = 1\} + 30 \times P\{I = 0\} = 50 \times (1/2) + 30 \times (1/2) = 40$$

在利用公式 $E[W^2] = (E[W])^2 + \mathrm{Var}(W)$, 可得

$$E[X_1^2|I = 1] = (50)^2 + (15)^2 = 2725, \quad E[X_1^2|I = 0] = (30)^2 + (6)^2 = 936$$

这样

$$E[X_1^2] = (2725) \times (1/2) + (936) \times (1/2) = 1830.5$$

这样, $X_1, \ldots, X_{40}$ 是独立同分布的随机变量序列, 其公共期望为 40, 公共方差为 $1830.5 - 1600 = 230.5$. 对于 $S = \sum_{i=1}^{40} X_i$, 我们有

$$E[S] = 40(40) = 1600, \quad \mathrm{Var}(S) = 40 \times (230.5) = 9220$$

再利用中心极限定理,

$$P\{S > 1700\} = P\left\{\frac{S - 1600}{\sqrt{9220}} > \frac{1700 - 1600}{\sqrt{9220}}\right\}$$
$$\approx P\{Z > 1.041\}$$
$$= 1 - \Phi(1.041) = 0.149$$

(b) 记 S_A 为所有类型为 A 的电池的总寿命, S_B 为所有类型为 B 的电池的总寿命. 利用中心极限定理可得, S_A 为近似正态分布，其期望为 $20 \times 50 = 1000$, 方差为 $20 \times 225 = 4500$, S_B 为近似正态分布, 其期望为 $20 \times 30 = 600$, 方差为 $20 \times 36 = 720$. 由于独立正态随机变量的和也是正态随机变量, $S_A + S_B$ 为近似正态随机变量, 其期望为 1600, 方差为 5220. 记 $S = S_A + S_B$, 得

$$P\{S > 1700\} = P\left\{\frac{S - 1600}{\sqrt{5220}} > \frac{1700 - 1600}{\sqrt{5220}}\right\}$$
$$\approx P\{Z > 1.384\}$$
$$= 1 - \Phi(1.384) = 0.084$$

12. 求对数, 并应用强大数律, 得到

$$\ln\left[\left(\prod_{i=1}^{n} X_i\right)^{1/n}\right] = \frac{1}{n}\sum_{i=1}^{n}\ln(X_i) \to E[\ln(X)]$$

因此,

$$\left(\prod_{i=1}^{n} X_i\right)^{1/n} \to \mathrm{e}^{E[\ln(X)]}$$

第 9 章

1. 由泊松过程定义的条件 (iii) 知在 8 到 10 之间发生的事件数的分布与 0 到 2 之间发生的事件数的分布是相同的. 这个分布是泊松分布, 均值为 6. 对于问题 (a)(b), 其答案为

(a) $P\{N(10)-N(8)=0\}=\mathrm{e}^{-6}$

(b) $E[N(10)-N(8)]=6$

(c) 由泊松过程的条件 (ii) 和 (iii) 知, 对于以任何时间点作为起点开始, 关于这个时间轴上的过程都是具有相同参数的泊松过程, 因此, 从 2:00PM 以后, 第 5 个事件的平均发生时间为 $2+5/3$, 或者是 3:40PM.

2. (a)

$$\begin{aligned}
P\{N(1/3)=2|N(1)=2\} &= \frac{P\{N(1/3)=2, N(1)=2\}}{P\{N(1)=2\}}\\
&= \frac{P\{N(1/3)=2, N(1)-N(1/3)=0\}}{P\{N(1)=2\}}\\
&= \frac{P\{N(1/3)=2\}P\{N(1)-N(1/3)=0\}}{P\{N(1)=2\}} \qquad \text{根据泊松过程定义条件 (ii)}\\
&= \frac{P\{N(1/3)=2\}P\{N(2/3)=0\}}{P\{N(1)=2\}} \qquad \text{根据泊松过程定义条件 (iii)}\\
&= \frac{\mathrm{e}^{-\lambda/3}(\lambda/3)^2/2!\mathrm{e}^{-2\lambda/3}}{\mathrm{e}^{-\lambda}\lambda^2/2!} = 1/9
\end{aligned}$$

(b)

$$\begin{aligned}
&P\{N(1/2)\geqslant 1|N(1)=2\} = 1-P\{N(1/2)=0|N(1)=2\}\\
&= 1-\frac{P\{N(1/2)=0, N(1)=2\}}{P\{N(1)=2\}} = 1-\frac{P\{N(1/2)=0, N(1)-N(1/2)=2\}}{P\{N(1)=2\}}\\
&= 1-\frac{P\{N(1/2)=0\}P\{N(1)-N(1/2)=2\}}{P\{N(1)=2\}} = 1-\frac{P\{N(1/2)=0\}P\{N(1/2)=2\}}{P\{N(1)=2\}}\\
&= 1-\frac{\mathrm{e}^{-\lambda/2}\mathrm{e}^{-\lambda/2}(\lambda/2)^2/2!}{\mathrm{e}^{-\lambda}\lambda^2/2!} = 1-1/4 = 3/4
\end{aligned}$$

3. 在路上取一观察点, 记 $X_n=0$ 表示通过该点的第 n 辆车是小汽车, $X_n=1$, 若第 n 辆车是大卡车, $n\geqslant 1$. 现在将 X_n 看成一个马尔可夫链, 其转移概率为

$$P_{0,0}=5/6 \qquad P_{0,1}=1/6 \qquad P_{1,0}=4/5 \qquad P_{1,1}=1/5$$

记 π_0 表示路过某点为小汽车的概率, π_1 为大卡车的概率. 它们是下列方程组的解:

$$\pi_0=\pi_0(5/6)+\pi_1(4/5) \qquad \pi_1=\pi_0(1/6)+\pi_1(1/5) \qquad \pi_0+\pi_1=1$$

解此方程组, 得

$$\pi_0 = 24/29 \qquad \pi_1 = 5/29$$

这样, 在路上, $\dfrac{2400}{29}\% \approx 83\%$ 的车是小汽车.

4. 每天的气候分类形成一个马尔可夫链, 令状态 0 为雨天, 1 为晴天, 2 为多云, 此时转移矩阵为

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

各种气候的比例应该满足

$$\pi_0 = \pi_1(1/3) + \pi_2(1/3) \qquad \pi_1 = \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3)$$
$$\pi_2 = \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3) \qquad 1 = \pi_0 + \pi_1 + \pi_2$$

这组方程的解为：

$$\pi_0 = 1/4 \qquad \pi_1 = 3/8 \qquad \pi_2 = 3/8$$

因此, 3/8 为晴天, 1/4 为雨天.

5. (a) 直接计算之结果

$$H(X)/H(Y) \approx 1.06$$

(b) 首先指出, 在 X 的取值空间内有两个值, 使得 X 取值的概率分别为 0.35 和 0.05. 在 Y 的取值空间中也有两个值, 它们取值概率也是 0.35 和 0.05. 但是, 当 X 不取这两个值的时候, X 以相同的概率取值其余的三个值, 而 Y 却不是这样的. 根据理论习题 13 的结论, $H(X)$ 应该大于 $H(Y)$.

第 10 章

1. (a)

$$1 = C\int_0^1 \mathrm{e}^x \,\mathrm{d}x \Rightarrow C = 1/(e-1)$$

(b)

$$F(x) = C\int_0^x \mathrm{e}^y \,\mathrm{d}y = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1} \qquad 0 \leqslant x \leqslant 1$$

如果令 $X = F^{-1}(U)$, 可知

$$U = (\mathrm{e}^X - 1)/(\mathrm{e} - 1)$$

或

$$X = \ln(U(\mathrm{e} - 1) + 1)$$

我们可以通过随机数 $U, X = \ln(U(\mathrm{e}-1)+1)$ 模拟得到随机变量 X.

2. 利用舍取法. 取 $g(x) \equiv 1, 0 < x < 1$. 利用微积分知识可知, $f(x)/g(x)$ 在 $[0,1]$ 上的极大点必满足下列方程:

$$2x - 6x^2 + 4x^3 = 0$$

或等价地,

$$4x^2 - 6x + 2 = (4x-2)(x-1) = 0$$

$f(x)/g(x)$ 的最大值在 $x = 1/2$ 处取得, 并且

$$C = \max f(x)/g(x) = 30(1/4 - 2/8 + 1/16) = 15/8$$

因此, 可采用以下算法:

第 1 步 产生一随机数 U_1.
第 2 步 产生一随机数 U_2.
第 3 步 若 $U_2 < 16(U_1^2 - 2U_1^3 + U_1^4)$, 则令 $X = U_1$, 否则转向第 1 步.

3. 最有效的方法是首先检验具有最大概率的值. 本题中, 采用下面的算法:

第 1 步 产生一随机数 U.
第 2 步 若 $U \leqslant 0.35$, 则令 $X = 3$, 并且停止程序.
第 3 步 若 $U \leqslant 0.65$, 则令 $X = 4$, 并且停止程序.
第 4 步 若 $U \leqslant 0.85$, 则令 $X = 2$, 并且停止程序.
第 5 步 $X = 1$.

4. $2\mu - X$

5. (a) 产生 $2n$ 个独立同分布的随机变量 $X_i, Y_i, i = 1, \cdots, n$. 它们的公共分布为指数分布, 均值为 1. 然后利用

$$\sum_{i=1}^{n} \mathrm{e}^{X_i Y_i}/n$$

作为估计量.

(b) 可以利用 XY 作为控制变量, 然后利用下式作为估计:

$$\sum_{i=1}^{n} (\mathrm{e}^{X_i Y_i} + cX_i Y_i)/n$$

另一种方法是用 $XY + X^2Y^2/2$ 作为控制变量, 得到估计量

$$\sum_{i=1}^{n} \left(\mathrm{e}^{X_i Y_i} + c(X_i Y_i + X_i^2 Y_i^2/2 - 1/2)\right)/n$$

我们之所以用这样的控制变量, 是由于 e^{xy} 展开式的前三项为 $1 + xy + x^2y^2/2$.

Chapter 1

Problems

1. (a) By the generalized basic principle of counting there are

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 67{,}600{,}000$$

(b) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 19{,}656{,}000$

2. $6^4 = 1296$

3. An assignment is a sequence $i_1, \ldots, i_{20}$ where i_j is the job to which person j is assigned. Since only one person can be assigned to a job, it follows that the sequence is a permutation of the numbers 1, …, 20 and so there are 20! different possible assignments.

4. There are 4! possible arrangements. By assigning instruments to Jay, Jack, John and Jim, in that order, we see by the generalized basic principle that there are $2 \cdot 1 \cdot 2 \cdot 1 = 4$ possibilities.

5. There were $8 \cdot 2 \cdot 9 = 144$ possible codes. There were $1 \cdot 2 \cdot 9 = 18$ that started with a 4.

6. Each kitten can be identified by a code number i, j, k, l where each of i, j, k, l is any of the numbers from 1 to 7. The number i represents which wife is carrying the kitten, j then represents which of that wife's 7 sacks contain the kitten; k represents which of the 7 cats in sack j of wife i is the mother of the kitten; and l represents the number of the kitten of cat k in sack j of wife i. By the generalized principle there are thus $7 \cdot 7 \cdot 7 \cdot 7 = 2401$ kittens

7. (a) $6! = 720$
(b) $2 \cdot 3! \cdot 3! = 72$
(c) $4!3! = 144$
(d) $6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 72$

8. (a) $5! = 120$
(b) $\dfrac{7!}{2!2!} = 1260$
(c) $\dfrac{11!}{4!4!2!} = 34{,}650$
(d) $\dfrac{7!}{2!2!} = 1260$

9. $\dfrac{(12)!}{6!4!} = 27{,}720$

10. (a) $8! = 40{,}320$
(b) $2 \cdot 7! = 10{,}080$
(c) $5!4! = 2{,}880$
(d) $4!2^4 = 384$

11. (a) 6!
(b) 3!2!3!
(c) 3!4!

12. (a) 30^5
(b) $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$

13. $\binom{20}{2}$

14. $\binom{52}{5}$

15. There are $\binom{10}{5}\binom{12}{5}$ possible choices of the 5 men and 5 women. They can then be paired up in 5! ways, since if we arbitrarily order the men then the first man can be paired with any of the 5 women, the next with any of the remaining 4, and so on. Hence, there are $5!\binom{10}{5}\binom{12}{5}$ possible results.

16. (a) $\binom{6}{2}+\binom{7}{2}+\binom{4}{2} = 42$ possibilities.
(b) There are $6 \cdot 7$ choices of a math and a science book, $6 \cdot 4$ choices of a math and an economics book, and $7 \cdot 4$ choices of a science and an economics book. Hence, there are 94 possible choices.

17. The first gift can go to any of the 10 children, the second to any of the remaining 9 children, and so on. Hence, there are $10 \cdot 9 \cdot 8 \cdots 5 \cdot 4 = 604{,}800$ possibilities.

18. $\binom{5}{2}\binom{6}{2}\binom{4}{3} = 600$

19. (a) There are $\binom{8}{3}\binom{4}{3}+\binom{8}{3}\binom{2}{1}\binom{4}{2} = 896$ possible committees.

There are $\binom{8}{3}\binom{4}{3}$ that do not contain either of the 2 men, and there are $\binom{8}{3}\binom{2}{1}\binom{4}{2}$ that contain exactly 1 of them.

(b) There are $\binom{6}{3}\binom{6}{3}+\binom{2}{1}\binom{6}{2}\binom{6}{3} = 1000$ possible committees.

(c) There are $\binom{7}{3}\binom{5}{3}+\binom{7}{2}\binom{5}{3}+\binom{7}{3}\binom{5}{2}$ = 910 possible committees. There are $\binom{7}{3}\binom{5}{3}$ in which neither feuding party serves; $\binom{7}{2}\binom{5}{3}$ in which the feuding women serves; and $\binom{7}{3}\binom{5}{2}$ in which the feuding man serves.

20. $\binom{6}{5}+\binom{2}{1}\binom{6}{4}, \binom{6}{5}+\binom{6}{3}$

21. $\dfrac{7!}{3!4!}$ = 35. Each path is a linear arrangement of 4 *r*'s and 3 *u*'s (*r* for right and *u* for up). For instance the arrangement *r*, *r*, *u*, *u*, *r*, *r*, *u* specifies the path whose first 2 steps are to the right, next 2 steps are up, next 2 are to the right, and final step is up.

22. There are $\dfrac{4!}{2!2!}$ paths from A to the circled point; and $\dfrac{3!}{2!1!}$ paths from the circled point to B. Thus, by the basic principle, there are 18 different paths from A to B that go through the circled piont.

23. $3!2^3$

25. $\binom{52}{13, 13, 13, 13}$

27. $\binom{12}{3, 4, 5} = \dfrac{12!}{3!4!5!}$

28. Assuming teachers are distinct.
(a) 4^8
(b) $\binom{8}{2, 2, 2, 2} = \dfrac{8!}{(2)^4}$ = 2520.

29. (a) (10)!/3!4!2!

(b) $3\binom{3}{2}\dfrac{7!}{4!2!}$

30. $2 \cdot 9! - 2^2 8!$ since $2 \cdot 9!$ is the number in which the French and English are next to each other and $2^2 8!$ the number in which the French and English are next to each other and the U.S. and Russian are next to each other.

31. (a) number of nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 = 8$.

Hence, answer is $\binom{11}{3} = 165$

(b) here it is the number of positive solutions—hence answer is $\binom{7}{3} = 35$

32. (a) number of nonnegative solutions of $x_1 + \ldots + x_6 = 8$

answer = $\binom{13}{5}$

(b) (number of solutions of $x_1 + \ldots + x_6 = 5$) $\times$ (number of solutions of $x_1 + \ldots + x_6 = 3$) = $\binom{10}{5}\binom{8}{5}$

33. (a) $x_1 + x_2 + x_3 + x_4 = 20$, $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$

Let $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, $y_3 = x_3 - 2$, $y_4 = x_4 - 3$

$$y_1 + y_2 + y_3 + y_4 = 13,\ y_i > 0$$

Hence, there are $\binom{12}{3} = 220$ possible strategies.

(b) there are $\binom{15}{2}$ investments only in 1, 2, 3

there are $\binom{14}{2}$ investments only in 1, 2, 4

there are $\binom{13}{2}$ investments only in 1, 3, 4

there are $\binom{13}{2}$ investments only in 2, 3, 4

$$\binom{15}{2} + \binom{14}{2} + 2\binom{13}{2} + \binom{12}{3} = 552 \text{ possibilities}$$

Theoretical Exercises

2. $\sum_{i=1}^{m} n_i$

3. $n(n-1)\cdots(n-r+1) = n!/(n-r)!$

4. Each arrangement is determined by the choice of the r positions where the black balls are situated.

5. There are $\binom{n}{j}$ different $0-1$ vectors whose sum is j, since any such vector can be characterized by a selection of j of the n indices whose values are then set equal to 1. Hence there are $\sum_{j=k}^{n}\binom{n}{j}$ vectors that meet the criterion.

6. $\binom{n}{k}$

7. $$\binom{n-1}{r}+\binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!}+\frac{(n-1)!}{(n-r)!(r-1)!}$$
$$= \frac{n!}{r!(n-r)!}\left[\frac{n-r}{n}+\frac{r}{n}\right]=\binom{n}{r}$$

8. There are $\binom{n+m}{r}$ gropus of size r. As there are $\binom{n}{i}\binom{m}{r-i}$ groups of size r that consist of i men and $r-i$ women, we see that

$$\binom{n+m}{r}=\sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i}.$$

9. $$\binom{2n}{n}=\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n}\binom{n}{i}^2$$

10. Parts (a), (b), (c), and (d) are immediate. For part (e), we have the following:

$$k\binom{n}{k} = \frac{k!n!}{(n-k)!k!}=\frac{n!}{(n-k)!(k-1)!}$$
$$(n-k+1)\binom{n}{k-1}=\frac{(n-k+1)n!}{(n-k+1)!(k-1)!}=\frac{n!}{(n-k)!(k-1)!}$$
$$n\binom{n-1}{k-1} = \frac{n(n-1)!}{(n-k)!(k-1)!}=\frac{n!}{(n-k)!(k-1)!}$$

11. The number of subsets of size k that have i as their highest numbered member is equal to $\binom{i-1}{k-1}$, the number of ways of choosing $k-1$ of the numbers $1, \ldots, i-1$. Summing over i yields the number of subsets of size k.

12. Number of possible selections of a committee of size k and a chairperson is $k\binom{n}{k}$ and so $\sum_{k=1}^{n} k\binom{n}{k}$ represents the desired number. On the other hand, the chairperson can be anyone of the n persons and then each of the other $n-1$ can either be on or off the committee. Hence, $n2^{n-1}$ also represents the desired quantity.

(i) $\binom{n}{k}k^2$

(ii) $n2^{n-1}$ since there are n possible choices for the combined chairperson and secretary and then each of the other $n-1$ can either be on or off the committee.

(iii) $n(n-1)2^{n-2}$

(c) From a set of n we want to choose a committee, its chairperson its secretary and its treasurer (possibly the same). The result follows since

(a) there are $n2^{n-1}$ selections in which the chair, secretary and treasurer are the same person.

(b) there are $3n(n-1)2^{n-2}$ selection in which the chair, secretary and treasurer jobs are held by 2 people.

(c) there are $n(n-1)(n-2)2^{n-3}$ selections in which the chair, secretary and treasurer are all different.

(d) there are $\binom{n}{k}k^3$ selections in which the committee is of size k.

13. $(1-1)^n = \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-1}$

14. (a) $\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$

(b) From (a), $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1} = \binom{n}{i}2^{n-i}$

(c) $$\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}(-1)^{n-j} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1}(-1)^{n-j}$$

$$= \binom{n}{i}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{n-i-k} = 0$$

15. (a) The number of vectors that have $x_k = j$ is equal to the number of vectors $x_1 \le x_2 \le \ldots \le x_{k-1}$ satisfying $1 \le x_i \le j$. That is, the number of vectors is equal to $H_{k-1}(j)$, and the result follows.

(b)

$H_2(1) = H_1(1) = 1$
$H_2(2) = H_1(1) + H_1(2) = 3$
$H_2(3) = H_1(1) + H_1(2) + H_1(3) = 6$
$H_2(4) = H_1(1) + H_1(2) + H_1(3) + H_1(4) = 10$
$H_2(5) = H_1(1) + H_1(2) + H_1(3) + H_1(4) + H_1(5) = 15$
$H_3(5) = H_2(1) + H_2(2) + H_2(3) + H_2(4) + H_2(5) = 35$

16. (a) $1 < 2 < 3$, $1 < 3 < 2$, $2 < 1 < 3$, $2 < 3 < 1$, $3 < 1 < 2$, $3 < 2 < 1$,
$1 = 2 < 3$, $1 = 3 < 2$, $2 = 3 < 1$, $1 < 2 = 3$, $2 < 1 = 3$, $3 < 1 = 2$, $1 = 2 = 3$

(b) The number of outcomes in which i players tie for last place is equal to $\binom{n}{i}$, the number of ways to choose these i players, multiplied by the number of outcomes of the remaining $n - i$ players, which is clearly equal to $N(n - i)$.

(c) $$\sum_{i=1}^{n}\binom{n}{i}N(n-1) = \sum_{i=1}^{n}\binom{n}{n-i}N(n-i) = \sum_{j=0}^{n-1}\binom{n}{j}N(j)$$

where the final equality followed by letting $j = n - i$.

(d) $N(3) = 1 + 3N(1) + 3N(2) = 1 + 3 + 9 = 13$
$N(4) = 1 + 4N(1) + 6N(2) + 4N(3) = 75$

17. A choice of r elements from a set of n elements is equivalent to breaking these elements into two subsets, one of size r (equal to the elements selected) and the other of size $n - r$ (equal to the elements not selected).

18. Suppose that r labelled subsets of respective sizes $n_1, n_2, \ldots, n_r$ are to be made up from elements $1, 2, \ldots, n$ where $n = \sum_{i=1}^{r} n_i$. As $\binom{n-1}{n_1,\ldots,n_i-1,\ldots n_r}$ represents the number of possibilities when person n is put in subset i, the result follows.

19. By induction:

$$(x_1 + x_2 + \ldots + x_r)^n$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} (x_2 + \ldots + x_r)^{n-i_1} \text{ by the Binomial theorem}$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} \underset{\substack{i_2,\ldots,i_r \\ i_2+\ldots+i_r = n-i_1}}{\sum \cdots \sum} \binom{n-i_1}{i_2,\ldots,i_r} x_1^{i_2} \ldots x_r^{i_2}$$

$$= \underset{\substack{i_1,\ldots,i_r \\ i_1+i_2+\ldots+i_r = n}}{\sum \sum \cdots \sum} \binom{n}{i_1,\ldots,i_r} x_1^{i_1} \ldots x_r^{i_r}$$

where the second equality follows from the induction hypothesis and the last from the identity $\binom{n}{i_1}\binom{n-i_1}{i_2,\ldots,i_n} = \binom{n}{i_1,\ldots,i_r}$.

20. The number of integer solutions of

$$x_1 + \ldots + x_r = n,\ x_i \ge m_i$$

is the same as the number of nonnegative solutions of

$$y_1 + \ldots + y_r = n - \sum_1^r m_i\ ,\ y_i \ge 0.$$

Proposition 6.2 gives the result $\binom{n - \sum_1^r m_i + r - 1}{r-1}$.

21. There are $\binom{r}{k}$ choices of the k of the x's to equal 0. Given this choice the other $r - k$ of the x's must be positive and sum to n.
By Proposition 6.1, there are $\binom{n-1}{r-k-1} = \binom{n-1}{n-r+k}$ such solutions.
Hence the result follows.

22. $\binom{n+r-1}{n-1}$ by Proposition 6.2.

23. There are $\binom{j+n-1}{j}$ nonnegative integer solutions of

$$\sum_{i=1}^{n} x_i = j$$

Hence, there are $\sum_{j=0}^{k} \binom{j+n-1}{j}$ such vectors.

Chapter 2

Problems

1. (a) $S = \{(r, r), (r, g), (r, b), (g, r), (g, g), (g, b), (b, r), b, g), (b, b)\}$
   (b) $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$

2. $S = \{(n, x_1, \ldots, x_{n-1}), n \geq 1, x_i \neq 6, i = 1, \ldots, n - 1\}$, with the interpretation that the outcome is $(n, x_1, \ldots, x_{n-1})$ if the first 6 appears on roll n, and x_i appears on roll, i, $i = 1, \ldots, n - 1$. The event $(\cup_{n=1}^{\infty} E_n)^c$ is the event that 6 never appears.

3. $EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$.
   $E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1. $FG = \{(1, 4), (4, 1)\}$.
   EF^c is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.

4. $A = \{1{,}0001{,}0000001, \ldots\}$ $B = \{01, 00001, 00000001, \ldots\}$
   $(A \cup B)^c = \{00000 \ldots, 001, 000001, \ldots\}$

5. (a) $2^5 = 32$
   (b)
   $W = \{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 0, 1, 0)$
   $(1, 1, 0, 0, 1), (1, 1, 0, 0, 0), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)$
   $(0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$

   (c) 8
   (d) $AW = \{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0)\}$

6. (a) $S = \{(1, g), (0, g), (1, f), (0, f), (1, s), (0, s)\}$
   (b) $A = \{(1, s), (0, s)\}$
   (c) $B = \{(0, g), (0, f), (0, s)\}$
   (d) $\{(1, s), (0, s), (1, g), (1, f)\}$

7. (a) 6^{15}
   (b) $6^{15} - 3^{15}$
   (c) 4^{15}

8. (a) .8
   (b) .3
   (c) 0

9. Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

   $$P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74.$$

   Therefore, 74 percent of the establishment's customers carry at least one of the two types of credit cards that it accepts.

10. Let R and N denote the events, respectively, that the student wears a ring and wears a necklace.

(a) $P(R \cup N) = 1 - .6 = .4$

(b) $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$
Thus, $P(RN) = .1$

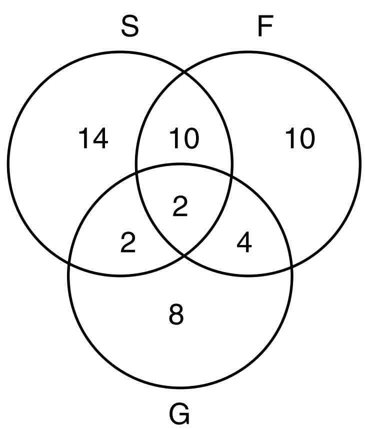
11. Let A be the event that a randomly chosen person is a cigarette smoker and let B be the event that she or he is a cigar smoker.

(a) $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$. Hence, 70 percent smoke neither.

(b) $P(A^cB) = P(B) - P(AB) = .07 - .05 = .02$. Hence, 2 percent smoke cigars but not cigarettes.

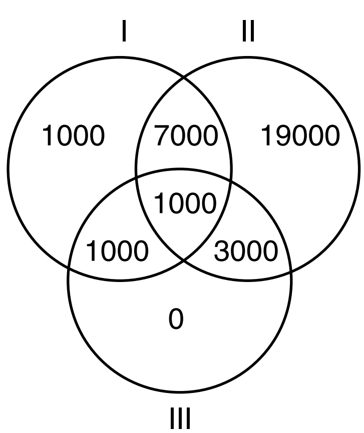
12. (a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$
The desired probability is $1 - 1/2 = 1/2$.

(b) Use the Venn diagram below to obtain the answer 32/100.



(c) since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\binom{50}{2}\Big/\binom{100}{2} = 49/198$ and so the probability that at least one is taking a course is 149/198.

13.



(a) 20,000
(b) 12,000
(c) 11,000
(d) 68,000
(e) 10,000

14. $P(M) + P(W) + P(G) - P(MW) - P(MG) - P(WG) + P(MWG) = .312 + .470 + .525 - .086 - .042 - .147 + .025 = 1.057$

15. (a) $4\binom{13}{5}\Big/\binom{52}{5}$

(b) $13\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(c) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}\Big/\binom{52}{5}$

(d) $13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(e) $13\binom{4}{4}\binom{48}{1}\Big/\binom{52}{5}$

16. (a) $\dfrac{6\cdot 5\cdot 4\cdot 3\cdot 2}{6^5}$ (b) $\dfrac{6\binom{5}{2}5\cdot 4\cdot 3}{6^5}$ (c) $\dfrac{\binom{6}{2}4\binom{5}{2}\binom{3}{2}}{6^5}$

(d) $\dfrac{6\cdot 5\cdot 4\binom{5}{3}}{21}$ (e) $\dfrac{6\cdot 5\binom{5}{3}}{6^5}$ (f) $\dfrac{6\cdot 5\binom{5}{4}}{6^5}$

(g) $\dfrac{6}{6^5}$

17. $\dfrac{\prod_{i=1}^{8} i^2}{64\cdot 63\cdots 58}$

18. $\dfrac{2\cdot 4\cdot 16}{52\cdot 51}$

19. $4/36 + 4/36 + 1/36 + 1/36 = 5/18$

20. Let A be the event that you are dealt blackjack and let B be the event that the dealer is dealt blackjack. Then,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= \frac{4\cdot 4\cdot 16}{52\cdot 51} + \frac{4\cdot 4\cdot 16\cdot 3\cdot 15}{52\cdot 51\cdot 50\cdot 49} \\ &= .0983 \end{aligned}$$

where the preceding used that $P(A) = P(B) = 2\times\dfrac{4\cdot 16}{52\cdot 51}$. Hence, the probability that neither is dealt blackjack is .9017.

21. (a) $p_1 = 4/20, p_2 = 8/20, p_3 = 5/20, p_4 = 2/20, p_5 = 1/20$

(b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,

$$q_1 = 4/48, q_2 = 16/48, q_3 = 15/48, q_4 = 8/48, q_5 = 5/48$$

22. The ordering will be unchanged if for some k, $0 \le k \le n$, the first k coin tosses land heads and the last $n - k$ land tails. Hence, the desired probability is $(n + 1/2^n$

23. The answer is 5/12, which can be seen as follows:

$$\begin{aligned}1 &= P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\}\\ &= 2P\{\text{second higher}\} + p\{\text{same}\}\\ &= 2P\{\text{second higher}\} + 1/6\end{aligned}$$

Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

25. $P(E_n) = \left(\dfrac{26}{36}\right)^{n-1}\dfrac{6}{36}, \quad \displaystyle\sum_{n=1}^{\infty} P(E_n) = \frac{2}{5}$

27. Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3\cdot 9! + 7\cdot 6\cdot 3\cdot 7! + 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 5! + 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 3\cdot 3!}{10!}$$

28. $P\{\text{same}\} = \dfrac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$

$$P\{\text{different}\} = \binom{5}{1}\binom{6}{1}\binom{8}{1}\Big/\binom{19}{3}$$

If sampling is with replacement

$$P\{\text{same}\} = \frac{5^3 + 6^3 + 8^3}{(19)^3}$$

$$\begin{aligned}P\{\text{different}\} &= P(RBG) + P\{BRG) + P(\text{RGB}) + \ldots + P(GBR)\\ &= \frac{6\cdot 5\cdot 6\cdot 8}{(19)^3}\end{aligned}$$

29. (a) $\dfrac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}$

(b) Putting all terms over the common denominator $(n + m)^2(n + m - 1)$ shows that we must prove that

$$n^2(n + m - 1) + m^2(n + m - 1) \geq n(n - 1)(n + m) + m(m - 1)(n + m)$$

which is immediate upon multiplying through and simplifying.

30. (a) $\dfrac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = 1/18$

(b) $\dfrac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} - 1/18 = 1/6$

(c) $\dfrac{\binom{7}{3}\binom{8}{4}+\binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} = 1/2$

31. $P(\{\text{complete}\} =$
$P\{\text{same}\} =$

32. $\dfrac{g(b+g-1)!}{(b+g)!} = \dfrac{g}{b+g}$

33. $\dfrac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = \dfrac{70}{323}$

34. $\binom{32}{13}\Big/\binom{52}{13}$

35. (a) $$\frac{\binom{12}{3}\binom{16}{2}\binom{18}{2}}{\binom{46}{7}}$$

(b) $$1-\frac{\binom{34}{7}}{\binom{46}{7}}-\frac{\binom{12}{1}\binom{34}{6}}{\binom{46}{7}}$$

(c) $$\frac{\binom{12}{7}+\binom{16}{7}+\binom{18}{7}}{\binom{46}{7}}$$

(d) $$P(R_3 \cup B_3) = P(R_3) + P(B_3) - P(R_3B_3) = \frac{\binom{12}{3}\binom{34}{4}}{\binom{46}{7}}+\frac{\binom{16}{3}\binom{30}{4}}{\binom{46}{7}}-\frac{\binom{12}{3}\binom{16}{3}\binom{18}{1}}{\binom{46}{7}}$$

36. (a) $\binom{4}{2}\Big/\binom{52}{2} \approx .0045$,

(b) $13\binom{4}{2}\Big/\binom{52}{2} = 1/17 \approx .0588$

37. (a) $\binom{7}{5}\Big/\binom{10}{5} = 1/12 \approx .0833$

(b) $\binom{7}{4}\binom{3}{1}\Big/\binom{10}{5} + 1/12 = 1/2$

38. $1/2 = \binom{3}{2}\Big/\binom{n}{2}$ or $n(n-1) = 12$ or $n = 4$.

39. $$\frac{5\cdot 4\cdot 3}{5\cdot 5\cdot 5}=\frac{12}{25}$$

40. $P\{1\} = \dfrac{4}{44} = \dfrac{1}{64}$

$P\{2\} = \dbinom{4}{2}\left[4 + \dbinom{4}{2} + 4\right] \Big/ 4^4 = \dfrac{84}{256}$

$P\{3\} = \dbinom{4}{3}\dbinom{3}{1}\dfrac{4!}{2!} \Big/ 4^4 = \dfrac{36}{64}$

$P\{4\} = \dfrac{4!}{4^4} = \dfrac{6}{64}$

41. $1 - \dfrac{5^4}{6^4}$

42. $1 - \left(\dfrac{35}{36}\right)^n$

43. $\dfrac{2(n-1)(n-2)}{n!} = \dfrac{2}{n}$ in a line

$\dfrac{2n(n-2)!}{n!} = \dfrac{2}{n-1}$ if in a circle, $n \geq 2$

44. (a) If A is first, then A can be in any one of 3 places and B's place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when B is first, we see that the probability in this case is $2 \cdot 3 \cdot 3!/5! = 3/10$

(b) $2 \cdot 2 \cdot 3!/5! = 1/5$

(c) $2 \cdot 3!/5! = 1/10$

45. $1/n$ if discard, $\dfrac{(n-1)^{k-1}}{n^k}$ if do not discard

46. If n in the room,

$$P\{\text{all different}\} = \frac{12 \cdot 11 \cdot \quad \cdot (13-n)}{12 \cdot 12 \cdot \quad \cdot 12}$$

When $n = 5$ this falls below 1/2. (Its value when $n = 5$ is .3819)

47. $12!/(12)^{12}$

48. $\dbinom{12}{4}\dbinom{8}{4}\dfrac{(20)!}{(3!)^4(2!)^4} \Big/ (12)^{20}$

49. $\dbinom{6}{3}\dbinom{6}{3} \Big/ \dbinom{12}{6}$

50. $\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}\Big/\binom{52}{13}\binom{39}{13}$

51. $\binom{n}{m}(n-1)^{n-m}/N^n$

52. (a) $\dfrac{20\cdot18\cdot16\cdot14\cdot12\cdot10\cdot8\cdot6}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13}$

(b) $\dfrac{\binom{10}{1}\binom{9}{6}\frac{8!}{2!}2^6}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13}$

53. Let A_i be the event that couple i sit next to each other. Then

$$P(\cup_{i=1}^4 A_i) = 4\frac{2\cdot7!}{8!} - 6\frac{2^2\cdot6!}{8!} + 4\frac{2^3\cdot5!}{8!} - \frac{2^4\cdot4!}{8!}$$

and the desired probability is 1 minus the preceding.

54. $P(S \cup H \cup D \cup C) = P(S) + P(H) + P(D) + P(\text{C}) - P(SH) - \ldots - P(SHDC)$

$$= \frac{4\binom{39}{13}}{\binom{52}{13}} - \frac{6\binom{26}{13}}{\binom{52}{13}} + \frac{4\binom{13}{13}}{\binom{52}{13}}$$

$$= \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}}$$

55. (a) $P(S \cup H \cup D \cup C) = P(S) + \ldots - P(SHDC)$

$$= \frac{4\binom{2}{2}}{\binom{52}{13}} - \frac{6\binom{2}{2}\binom{2}{2}\binom{48}{9}}{\binom{52}{13}} + \frac{4\binom{2}{2}^3\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{2}{2}^4\binom{44}{5}}{\binom{52}{13}}$$

$$= \frac{4\binom{50}{11} - 6\binom{48}{9} + 4\binom{46}{7} - \binom{44}{5}}{\binom{52}{13}}$$

(b) $P(1 \cup 2 \cup \ldots \cup 13) = \dfrac{13\binom{48}{9}}{\binom{52}{13}} - \dfrac{\binom{13}{2}\binom{44}{5}}{\binom{52}{13}} + \dfrac{\binom{13}{3}\binom{40}{1}}{\binom{52}{13}}$

56. Player B. If Player A chooses spinner (a) then B can choose spinner (c). If A chooses (b) then B chooses (a). If A chooses (c) then B chooses (b). In each case B wins probability 5/9.

Theoretical Exercises

5. $F_i = E_i \overset{i=1}{\underset{j=1}{\cap}} E_j^c$

6. (a) EF^cG^c

   (b) EF^cG

   (c) $E \cup F \cup G$

   (d) $EF \cup EG \cup FG$

   (e) EFG

   (f) $E^cF^cG^c$

   (g) $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$

   (h) $(EFG)^c$

   (i) $EFG^c \cup EF^cG \cup E^cFG$

   (j) S

7. (a) E

   (b) EF

   (c) $EG \cup F$

8. The number of partitions that has $n + 1$ and a fixed set of i of the elements 1, 2, …, n as a subset is T_{n-i}. Hence, (where $T_0 = 1$). Hence, as there are $\binom{n}{i}$ such subsets.

$$T_{n+1} = \sum_{i=0}^{n}\binom{n}{i}T_{n-i} = 1 + \sum_{i=0}^{n-1}\binom{n}{i}T_{n-i} = 1 + \sum_{k=1}^{n}\binom{n}{k}T_k \; .$$

11. $1 \ge P(E \cup F) = P(E) + P(F) - P(EF)$

12. $P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$
$= P(E) - P(EF) + P(F) - P(EF)$

13. $E = EF \cup EF^c$

15. $$\frac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}$$

16. $P(E_1 \ldots E_n) \geq P(E_1 \ldots E_{n-1}) + P(E_n) - 1$ by Bonferonni's Ineq.

$$\geq \sum_{1}^{n-1} P(E_i) - (n-2) + P(E_n) - 1 \text{ by induction hypothesis}$$

19. $$\frac{\binom{n}{r-1}\binom{m}{k-r}(n-r+1)}{\binom{n+m}{k-1}(n+m-k+1)}$$

21. Let $y_1, y_2, \ldots, y_k$ denote the successive runs of losses and $x_1, \ldots, x_k$ the successive runs of wins. There will be $2k$ runs if the outcome is either of the form $y_1, x_1, \ldots, y_k\, x_k$ or $x_1 y_1, \ldots\, x_k, y_k$ where all x_i, y_i are positive, with $x_1 + \ldots + x_k = n$, $y_1 + \ldots + y_k = m$. By Proposition 6.1 there are $2\binom{n-1}{k-1}\binom{m-1}{k-1}$ number of outcomes and so

$$P\{2k \text{ runs}\} = 2\binom{n-1}{k-1}\binom{m-1}{k-1}\Bigg/\binom{m+n}{n}.$$

There will be $2k + 1$ runs if the outcome is either of the form $x_1, y_1, \ldots, x_k, y_k, x_{k+1}$ or $y_1, x_1, \ldots, y_k, x_k\, y_{k+1}$ where all are positive and $\sum x_i = n$, $\sum y_i = m$. By Proposition 6.1 there are $\binom{n-1}{k}\binom{m-1}{k-1}$ outcomes of the first type and $\binom{n-1}{k-1}\binom{m-1}{k}$ of the second.

Chapter 3

Problems

1. $P\{6 \mid \text{different}\} = P\{6, \text{different}\}/P\{\text{different}\}$

$$= \frac{P\{1\text{st} = 6, 2\text{nd} \neq 6\} + P\{1\text{s}t \neq 6, 2\text{nd} = 6\}}{5/6}$$

$$= \frac{2\ 1/6\ 5/6}{5.6} = 1/3$$

could also have been solved by using reduced sample space—for given that outcomes differ it is the same as asking for the probability that 6 is chosen when 2 of the numbers 1, 2, 3, 4, 5, 6 are randomly chosen.

2. $P\{6 \mid \text{sum of } 7\} = P\{(6,1)\}/1/6 = 1/6$

$P\{6 \mid \text{sum of } 8\} = P\{(6,2)\}/5/36 = 1/5$

$P\{6 \mid \text{sum of } 9\} = P\{(6,3)\}/4/36 = 1/4$

$P\{6 \mid \text{sum of } 10\} = P\{(6,4)\}/3/36 = 1/3$

$P\{6 \mid \text{sum of } 11\} = P\{(6,5)\}/2/36 = 1/2$

$P\{6 \mid \text{sum of } 12\} = 1.$

3. $P\{E \text{ has } 3 \mid N-S \text{ has } 8\} = \dfrac{P\{E \text{ has } 3, N-S \text{ has } 8\}}{P\{N-S \text{ has } 8\}}$

$$= \frac{\binom{13}{8}\binom{39}{18}\binom{5}{3}\binom{21}{10}\Big/\binom{52}{26}\binom{26}{13}}{\binom{13}{8}\binom{39}{18}\Big/\binom{52}{26}} = .339$$

4. $P\{\text{at least one } 6 \mid \text{sum of } 12\} = 1$. Otherwise twice the probability given in Problem 2.

5. $\dfrac{6}{15}\dfrac{5}{14}\dfrac{9}{13}\dfrac{8}{12}$

6. In both cases the one black ball is equally likely to be in either of the 4 positions. Hence the answer is 1/2.

7. $P1\ g \text{ and } 1\ b \mid \text{at least one } b\} = \dfrac{1/2}{3/4} = 2/3$

8. 1/2

9. $$P\{A = w \mid 2w\} = \frac{P\{A = w, 2w\}}{P\{2w\}}$$

$$= \frac{P\{A = w, B = w, C \neq w\} + P\{A = w, B \neq w, C = w\}}{P\{2w\}}$$

$$= \frac{\frac{1}{3}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4}}{\frac{1}{2}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{2}{3}\frac{1}{4}} = \frac{7}{11}$$

10. 11/50

11. (a) $P(B \mid A_s) = \dfrac{P(BA_s)}{P(A_s)} = \dfrac{\frac{1}{52}\frac{3}{21} + \frac{3}{52}\frac{1}{51}}{\frac{2}{52}} = \dfrac{1}{17}$

Which could have been seen by noting that, given the ace of spades is chosen, the other card is equally likely to be any of the remaining 51 cards, of which 3 are aces.

(b) $P(B \mid A) = \dfrac{P(B)}{P(A)} = \dfrac{\frac{4}{52}\frac{3}{51}}{1 - \frac{48}{52}\frac{47}{51}} = \dfrac{1}{33}$

12. (a) (.9)(.8)(.7) = .504

(b) Let F_i denote the event that she failed the ith exam.

$$P(F_2 \mid F_1^c\, F_2^c F_3^c)^c) = \frac{P(F_1^c F_2)}{1 - .504} = \frac{(.9)(.2)}{.496} = .3629$$

13. $P(E_1) = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13}$, $\quad P(E_2 \mid E_1) = \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13}$

$P(E_3 \mid E_1E_2) = \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13}$, $\quad P(E_4 \mid E_1E_2E_3) = 1.$

Hence,

$$p = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13}\cdot\binom{3}{1}\binom{36}{12}\Big/\binom{39}{13}\cdot\binom{2}{1}\binom{24}{12}\Big/\binom{26}{13}$$

14. $\dfrac{5}{12}\dfrac{7}{14}\dfrac{7}{16}\dfrac{9}{18} - \dfrac{35}{768}$.

15. Let E be the event that a randomly chosen pregnant women has an ectopic pregnancy and S the event that the chosen person is a smoker. Then the problem states that

$$P(E|S) = 2P(E|S^c),\ P(S) = .32$$

Hence,

$$\begin{aligned} P(S|E) &= P(SE)/P(E) \\ &= \frac{P(E|S)P(S)}{P(E|S)P(S) + P(E|S^c)P(S^C)} \\ &= \frac{2P(S)}{2P(S) + P(S^c)} \\ &= 32/66 \approx .4548 \end{aligned}$$

16. With S being survival and C being C section of a randomly chosen delivery, we have that

$$\begin{aligned} .98 = P(S) &= P(S|C).15 + P(S|C^2)\ .85 \\ &= .96(.15) + P(S|C^2)\ .85 \end{aligned}$$

Hence

$$P(S|C^c) \approx .9835.$$

17. $P(D) = .36$, $P(\mathrm{C}) = .30$, $P(C|D) = .22$

(a) $P(DC) = P(D)\ P(C|D) = .0792$
(b) $P(D|C) = P(DC)/P(C) = .0792/.3 = .264$

18. (a) $P(\text{Ind}|\text{voted}) = \dfrac{P(\text{voted}|\text{Ind})P(\text{Ind})}{\sum P(\text{voted}|\textit{type})P(\text{type})}$

$$= \frac{.35(.46)}{.35(.46) + .62(.3) + .58(.24)} \approx 331$$

(b) $P\{\text{Lib}|\text{voted}\} = \dfrac{.62(.30)}{.35(.46) + .62(.3) + .58(.24)} \approx .383$

(c) $P\{\text{Con}|\text{voted}\} = \dfrac{.58(.24)}{.35(.46) + .62(.3) + .58(.24)} \approx .286$

(d) $P\{\text{voted}\} = .35(.46) + .62(.3) + .58(.24) = .4862$
That is, 48.62 percent of the voters voted.

19. Choose a random member of the class. Let A be the event that this person attends the party and let W be the event that this person is a woman.

(a) $P(W|A) = \dfrac{P(A|W)P(W)}{P(A|W)P(W)+P(A|M)P(M)}$ where $M = W^c$

$$= \frac{.48(.38)}{.48(.38)+.37(.62)} \approx .443$$

Therefore, 44.3 percent of the attendees were women.

(b) $P(A) = .48(.38) + .37(.62) = .4118$

Therefore, 41.18 percent of the class attended.

20. (a) $P(F|C) = \dfrac{P(FC)}{P(C)} = .02/.05 = .40$

(b) $P(C|F) = P(FC)/P(F) = .02/.52 = 1/26 \approx .038$

21. (a) $P\{\text{husband under 25}\} = (212 + 36)/500 = .496$

(b) $P\{\text{wife over} | \text{husband over}\} = P\{\text{both over}\}/P\{\text{husband over}\}$

$$= (54/500)/(252/500)$$
$$= 3/14 \approx .214$$

(c) $P\{\text{wife over} | \text{husband under}\} = 36/248 \approx .145$

22. a. $\dfrac{6\cdot5\cdot4}{6\cdot6\cdot6} = \dfrac{5}{9}$

b. $\dfrac{1}{3!} = \dfrac{1}{6}$

c. $\dfrac{5}{9}\dfrac{1}{6} = \dfrac{5}{54}$

23. $P(w|w \text{ transferred}\}P\{w \text{ tr.}\} + P(w|R \text{ tr.}\}P\{R \text{ tr.}\} = \dfrac{2}{3}\dfrac{1}{3} + \dfrac{1}{3}\dfrac{2}{3} = \dfrac{4}{9}$.

$$P\{w \text{ transferred} | w\} = \frac{P\{w|w \text{ tr.}\}P\{w \text{ tr.}\}}{P\{w\}} = \frac{\frac{2}{3}\frac{1}{3}}{\frac{4}{9}} = 1/2.$$

24. (a) $P\{g-g \mid \text{at least one } g\} = \dfrac{1/4}{3/4} = 1/3.$

(b) Since we have no information about the ball in the urn, the answer is 1/2.

26. Let M be the event that the person is male, and let C be the event that he or she is color blind. Also, let p denote the proportion of the population that is male.

$$P(M \mid C) = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} = \frac{(.05)p}{(.05)p + (.0025)(1-p)}$$

27. Method (b) is correct as it will enable one to estimate the average number of workers per car. Method (a) gives too much weight to cars carrying a lot of workers. For instance, suppose there are 10 cars, 9 transporting a single worker and the other carrying 9 workers. Then 9 of the 18 workers were in a car carrying 9 workers and so if you randomly choose a worker then with probability 1/2 the worker would have been in a car carrying 9 workers and with probability 1/2 the worker would have been in a car carrying 1 worker.

28. Let A denote the event that the next card is the ace of spades and let B be the event that it is the two of clubs.

(a) $P\{A\} = P\{\text{next card is an ace}\}P\{A \mid \text{next card is an ace}\}$

$$= \frac{3}{32}\frac{1}{4} = \frac{3}{128}$$

(b) Let C be the event that the two of clubs appeared among the first 20 cards.

$$P(B) = P(B \mid C)P(C) + P(B \mid C^c)P(C^c)$$
$$= 0\frac{19}{48} + \frac{1}{32}\frac{29}{48} = \frac{29}{1536}$$

29. Let A be the event that none of the final 3 balls were ever used and let B_i denote the event that i of the first 3 balls chosen had previously been used. Then,

$$P(A) = P(A \mid B_0)P(B_0) + P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$$
$$= \sum_{i=0}^{3} \frac{\binom{6+i}{3}}{\binom{15}{3}} \frac{\binom{6}{i}\binom{9}{3-i}}{\binom{15}{3}}$$
$$= .083$$

30. Let B and W be the events that the marble is black and white, respectively, and let B be the event that box i is chosen. Then,

$$P(B) = P(B \mid B_1)P(B_1) + P(B \mid B_2)P(B_2) = (1/2)(1/2) = (2/3)(1/2) = 7/12$$
$$P(B_1 \mid W) = \frac{P(W|B_1)P(B_1)}{P(W)} = \frac{(1/2)(1/2)}{5/12} = 3/5$$

31. Let C be the event that the tumor is cancerous, and let N be the event that the doctor does not call. Then

$$\beta = P(C \mid N) = \frac{P(NC)}{P(N)}$$
$$= \frac{P(N \mid C)P(C)}{P(N \mid C)P(C) + P(N \mid C^c)P(C^c)}$$
$$= \frac{\alpha}{\alpha + \frac{1}{2}(1-\alpha)}$$
$$= \frac{2\alpha}{1+\alpha} \geq \alpha$$

with strict inequality unless $\alpha = 1$.

32. Let E be the event the child selected is the eldest, and let F_j be the event that the family has j children. Then,

$$P(F_j \mid E) = \frac{P(EF_j)}{P(E)}$$
$$= \frac{P(F_j)P(E \mid F_j)}{\sum_j P(F_j)P(E \mid F_j)}$$
$$= \frac{p_j(1/j)}{.1 + .25(1/2) + .35(1/3) + .3(1/4)} = .24$$

Thus, $P(F_1 \mid E) = .24$, $P(F_4 \mid E) = .18$.

33. Let E and R be the events that Joe is early tomorrow and that it will rain tomorrow.

(a) $P(E) = P(E \mid R)P(R) + P(E \mid R^c)P(R^c) = .7(.7) + .9(.3) = .76$

(b) $P(R \mid E) = \dfrac{P(E \mid R)P(R)}{P(E)} = 49/76$

34. $P(G \mid C) = \dfrac{P(C \mid G)P(G)}{P(C \mid G)P(G) + P(C \mid G^c)P(G^c)} = 54/62$

35. Let U be the event that the present is upstairs, and let M be the event it was hidden by mom.

(a) $P(U) = P(U \mid M)P(M) + P(U \mid M^c)P(M^c) = .7(.6) + .5(.4) = .62$

(b) $P(M^c \mid U^c) = P(\text{dad} \mid \text{down}) = \dfrac{P(\text{down} \mid \text{dad})\text{P(dad)}}{1-.62} = \dfrac{.5(.4)}{.38} = 10/19$

36. $$P\{C \mid \text{woman}\} = \frac{P\{\text{women} \mid C\}P\{C\}}{P\{\text{women} \mid A\}P\{A\} + P\{\text{women} \mid B\}P\{B\} + P\{\text{women} \mid C\}P\{C\}}$$

$$= \frac{.7\frac{100}{225}}{.5\frac{50}{225} + .6\frac{75}{225} + .7\frac{100}{225}} = \frac{1}{2}$$

37. (a) $P\{\text{fair} \mid h\} = \dfrac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{3}$.

(b) $P\{\text{fair} \mid hh\} = \dfrac{\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{5}$.

(c) 1

38. $$P\{\text{tails} \mid w\} = \frac{\frac{3}{15}\frac{1}{2}}{\frac{3}{15}\frac{1}{2} + \frac{5}{12}\frac{1}{2}} = \frac{36}{36+75} = \frac{36}{111}.$$

39. $$P\{\text{acc.} \mid \text{no acc.}\} = \frac{P\{\text{no acc., acc.}}{P\{\text{no acc.}\}}$$

$$= \frac{\frac{3}{10}(.4)(.6) + \frac{7}{10}(.2)(.8)}{\frac{3}{10}(.6) + \frac{7}{10}(.8)} = \frac{46}{185}.$$

40. (a) $\dfrac{7}{12}\dfrac{8}{13}\dfrac{9}{14}$

(b) $3\dfrac{7 \cdot 8 \cdot 5}{12 \cdot 13 \cdot 14}$

(c) $\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

(d) $3\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

41. $P\{\text{ace}\} = P\{\text{ace} \mid \text{interchanged selected}\}\dfrac{1}{27}$

$$+P\{\text{ace} \mid \text{interchanged not selected}\}\ \frac{26}{27}$$

$$= 1\frac{1}{27} + \frac{3}{51}\frac{26}{27} = \frac{129}{51 \cdot 27}.$$

42. $P\{A \mid \text{failure}\} = \dfrac{(.02)(.5)}{(.02)(.5) + (.03)(.3) + (.05)(.2)} = \dfrac{10}{29}$

43. $P\{2 \text{ headed} \mid \text{heads}\} = \dfrac{\frac{1}{3}(1)}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{3}{4}} = \dfrac{4}{4+2+3} = \dfrac{4}{9}.$

45. $P\{5\text{th} \mid \text{heads}\} = \dfrac{P\{\text{heads} \mid 5^{\text{th}}\}P\{5^{\text{th}}\}}{\sum_i P\{h \mid i^{\text{th}}\}P\{i^{th}\}}$

$$= \frac{\frac{5}{10}\frac{1}{10}}{\sum_{i=1}^{10}\frac{i}{10}\frac{1}{10}} = \frac{1}{11}.$$

46. Let M and F denote, respectively, the events that the policyholder is male and that the policyholder is female. Conditioning on which is the case gives the following.

$$P(A_2 \mid A_1) = \frac{P(A_1A_2)}{P(A_1)}$$

$$= \frac{P(A_1A_2 \mid M)\alpha + P(A_1A_2 \mid F)(1-\alpha)}{P(A_1 \mid M)\alpha + P(A_1 \mid F)(1-\alpha)}$$

$$= \frac{p_m^2\alpha + p_f^2(1-\alpha)}{p_m\alpha + p_f(1-\alpha)}$$

Hence, we need to show that

$$p_m^2\alpha + p_f^2[1-\alpha) > (p_m\alpha + p_f(1-\alpha))^2$$

or equivalently, that

$$p_m^2(\alpha - \alpha^2) + p_f^2[1 - \alpha - (1-a)^2] > 2\alpha(1-\alpha)p_fp_m$$

Factoring out $\alpha(1 - \alpha)$ gives the equivalent condition

$$p_m^2 + p_f^2 > 2pf_m$$

or

$$(p_m - p_f)^2 > 0$$

which follows because $p_m \neq p_f$. Intuitively, the inequality follows because given the information that the policyholder had a claim in year 1 makes it more likely that it was a type policyholder having a larger claim probability. That is, the policyholder is more likely to me male if $p_m > p_f$ (or more likely to be female if the inequality is reversed) than without this information, thus raising the probability of a claim in the following year.

47. $$P\{\text{all white}\} = \frac{1}{6}\left[\frac{5}{15} + \frac{5}{15}\frac{4}{14} + \frac{5}{15}\frac{4}{14}\frac{3}{13} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}\right]$$

$$P\{3 \mid \text{all white}\} = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P\{\text{all white}\}}$$

48. (a) $P\{\text{silver in other} \mid \text{silver found}\}$

$$= \frac{P\{S \text{ in other, } S \text{ found}\}}{P\{S \text{ found}\}}.$$

To compute these probabilities, condition on the cabinet selected.

$$= \frac{1/2}{P\{S \text{ found} \mid A\}1/2 + P\{S \text{ found} \mid B\}1/2}$$

$$= \frac{1}{1 + 1/2} = \frac{2}{3}.$$

49. Let C be the event that the patient has cancer, and let E be the event that the test indicates an elevated PSA level. Then, with $p = P(C)$,

$$P(C \mid E) = \frac{P(E \mid C)P(C)}{P(E \mid C)P(C) + P(E \mid C^c)P(C^c)}$$

Similarly,

$$P(C \mid E^c) = \frac{P(E^c \mid C)P(C)}{P(E^c \mid C)P(C) + P(E^c \mid C^c)P(C^c)}$$

$$= \frac{.732p}{.732p + .865(1 - p)}$$

50. Choose a person at random

$$P\{\text{they have accident}\} = P\{\text{acc.} \mid \text{good}\}P\{g\} + P\{\text{acc.} \mid \text{ave.}\}P\{\text{ave.}\}$$
$$+ P\{\text{acc.} \mid \text{bad } P(\text{b})\}$$
$$= (.05)(.2) + (.15)(.5) + (.30)(.3) = .175$$

$$P\{A \text{ is good} \mid \text{ no accident}\} = \frac{.95(2)}{.825}$$
$$P\{A \text{ is average} \mid \text{no accident}\} = \frac{(.85)(.5)}{.825}$$

51. Let R be the event that she receives a job offer.

(a) $P(R) = P(R \mid \text{strong})P(\text{strong}) + P(R \mid \text{moderate})P(\text{moderate}) + P(R \mid \text{weak})P(\text{weak})$
$= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65$

(b) $$P(\text{strong} \mid R) = \frac{P(R \mid \text{ strong})\,P(\text{strong})}{P(R)}$$
$$= \frac{(.8)(.7)}{.65} = \frac{56}{65}$$

Similarly,

$$P(\text{moderate} \mid R) = \frac{8}{65},\ P(\text{weak} \mid R) = \frac{1}{65}$$

(c) $$P(\text{strong} \mid R^c) = \frac{P(R^c \mid \text{strong})P(\text{strong})}{P(R^c\ )}$$
$$= \frac{(.2)(.7)}{.35} = \frac{14}{35}$$

Similarly,

$$P(\text{moderate} \mid R^c) = \frac{12}{35},\ P(\text{weak} \mid R^c) = \frac{9}{35}$$

52. Let M, T, W, Th, F be the events that the mail is received on that day. Also, let A be the event that she is accepted and R that she is rejected.

(a) $P(M) = P(M \mid A)P(A) + P(M \mid R)P(R) = (.15)(.6) + (.05)(.4) = .11$

(b) $P(T \mid M^c) = \dfrac{P(T)}{P(M^c)}$

$$= \frac{P(T \mid A)\,P(A) + P(T \mid R)P(R)}{1 - P(M)}$$

$$= \frac{(.2)(.6) + (.1)(.4)}{.89}\frac{16}{89}$$

(c) $P(A \mid M^cT^cW^c) = \dfrac{P(M^cT^cW^c \mid A)\,P(A)}{P(M^cT^cW^c)}$

$$= \frac{(1 - .15 - .20 - .25)(.6)}{(.4)(.6) + (.75)(.4)} = \frac{12}{27}$$

(d) $P(A \mid Th) = \dfrac{P(Th \mid A)P(A)}{P(Th)}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.15)(.4)} = \frac{3}{5}$$

(e) $P(A \mid \text{no mail}) = \dfrac{P(\text{no mail} \mid A)P(A)}{P(\text{no mail})}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.4)(.4)} = \frac{9}{25}$$

53. Let W and F be the events that component 1 works and that the system functions.

$$P(W \mid F) = \frac{P(WF)}{P(F)} = \frac{P(W)}{1 - P(F^c)} = \frac{1/2}{1 - (1/2)^{n-1}}$$

55. $P\{\text{Boy}, F\} = \dfrac{4}{16 + x}$ $\quad P\{\text{Boy}) = \dfrac{10}{16 + x}$ $\quad P\{F\} = \dfrac{10}{16 + x}$

so independence $\Rightarrow 4 = \dfrac{10 \cdot 10}{16 + x} \Rightarrow 4x = 36$ or $x = 9$.

A direct check now shows that 9 sophomore girls (which the above shows is necessary) is also sufficient for independence of sex and class.

56. $P\{\text{new}\} = \sum_i P\{\text{new} \mid \text{type } i\}p_i = \sum_i (1 - p_i)^{n-1}\,p_i$

57. (a) $2p(1-p)$

(b) $\binom{3}{2}p^2(1-p)$

(c) $P\{\text{up on first} \mid \text{up 1 after 3}\}$
$= P\{\text{up first, up 1 after 3}\}/[3p^2(1-p)]$
$= p2p(1-p)/[3p^2(1-p)] = 2/3.$

58. (a) All we know when the procedure ends is that the two most flips were either H, T, or T, H. Thus,

$$P(\text{heads}) = P(H, T \mid H, T \text{ or } T, H)$$
$$= \frac{P(H,T)}{P(H,T)+P(T,H)} = \frac{p(1-p)}{p(1-p)+(1-p)p} = \frac{1}{2}$$

(b) No, with this new procedure the result will be heads (tails) whenever the first flip is tails (heads). Hence, it will be heads with probability $1-p$.

59. (a) 1/16

(b) 1/16

(c) The only way in which the pattern H, H, H, H can occur first is for the first 4 flips to all be heads, for once a tail appears it follows that a tail will precede the first run of 4 heads (and so T, H, H, H will appear first). Hence, the probability that T, H, H, H occurs first is 15/16.

60. From the information of the problem we can conclude that both of Smith's parents have one blue and one brown eyed gene. Note that at birth, Smith was equally likely to receive either a blue gene or a brown gene from each parent. Let X denote the number of blue genes that Smith received.

(a) $P\{\text{Smith blue gene}\} = P\{X = 1 \mid X \le 1\} = \dfrac{1/2}{1-1/4} = 2/3$

(b) Condition on whether Smith has a blue-eyed gene.
$P\{\text{child blue}\} = P\{\text{blue} \mid \text{blue gene}\}(2/3) + P\{\text{blue} \mid \text{no blue}\}(1/3)$
$= (1/2)(2/3) = 1/3$

(c) First compute

$$P\{\text{Smith blue} \mid \text{child brown}\} = \frac{P\{\text{child brown} \mid \text{Smith blue}\}2/3}{2/3}$$
$$= 1/2$$

Now condition on whether Smith has a blue gene given that first child has brown eyes.
$P\{\text{second child brown}\} = P\{\text{brown} \mid \text{Smith blue}\}1/2 + P\{\text{brown} \mid \text{Smith no blue}\}1/2$
$= 1/4 + 1/2 = 3/4$

61. Because the non-albino child has an albino sibling we know that both its parents are carriers. Hence, the probability that the non-albino child is not a carrier is

$$P(A, A \mid A, a \text{ or } a, A \text{ or } A, A) = \frac{1}{3}$$

Where the first gene member in each gene pair is from the mother and the second from the father. Hence, with probability 2/3 the non-albino child is a carrier.

(a) Condition on whether the non-albino child is a carrier. With C denoting this event, and O_i the event that the i^{th} offspring is albino, we have:

$$\begin{aligned} P(O_1) &= P(O_1 \mid C)P(C) + P(O_1 \mid C^c)P(C^c) \\ &= (1/4)(2/3) + 0(1/3) = 1/6 \end{aligned}$$

(b) $$\begin{aligned} P(O_2 \mid O_1^c) &= \frac{P(O_1^c O_2)}{P(O_1^c)} \\ &= \frac{P(O_1^c O_2 \mid C)\,P(C) + P(O_1^c O_2 \mid C^c)\,P(C^c)}{5/6} \\ &= \frac{(3/4)(1/4)(2/3) + 0(1/3)}{5/6} = \frac{3}{20} \end{aligned}$$

62. (a) $$\begin{aligned} P\{\text{both hit} \mid \text{at least one hit}\} &= \frac{P\{\text{both hit}\}}{P\{\text{at least one hit}\}} \\ &= p_1p_2/(1 - q_1q_2) \end{aligned}$$

(b) $P\{\text{Barb hit} \mid \text{at least one hit}\} = p_1/(1 - q_1q_2)$
$Q_i = 1 - p_i$, and we have assumed that the outcomes of the shots are independent.

63. Consider the final round of the duel. Let $q_x = 1 - p_x$

(a) $$\begin{aligned} P\{A \text{ not hit}\} &= P\{A \text{ not hit} \mid \text{at least one is hit}\} \\ &= P\{A \text{ not hit, } B \text{ hit}\}/P\{\text{at least one is hit}\} \\ &= q_Bp_A/(1 - q_Aq_B) \end{aligned}$$

(b) $$\begin{aligned} P\{\text{both hit}\} &= P\{\text{both hit} \mid \text{at least one is hit}\} \\ &= P\{\text{both hit}\}/P\{\text{at least one hit}\} \\ &= p_Ap_B/(1 - q_Aq_B) \end{aligned}$$

(c) $(q_Aq_B)^{n-1}(1 - q_Aq_B)$

(d) $$\begin{aligned} P\{n \text{ rounds} \mid A \text{ unhit}\} &= P\{n \text{ rounds, } A \text{ unhit}\}/P\{A \text{ unhit}\} \\ &= \frac{(q_Aq_B)^{n-1}\,p_Aq_B}{q_Bp_A/(1-q_Aq_B)} \\ &= (q_Aq_B)^{n-1}(1 - q_Aq_B) \end{aligned}$$

(e) $P(n \text{ rounds} \mid \text{both hit}) = P\{n \text{ rounds both hit}\}/P\{\text{both hit}\}$

$$= \frac{(q_A q_B)^{n-1} p_A p_B}{p_B p_A/(1-q_A q_B)}$$

$$= (q_A q_B)^{n-1}(1 - q_A q_B)$$

Note that (c), (d), and (e) all have the same answer.

64. If use (a) will win with probability p. If use strategy (b) then

$$\begin{aligned} P\{\text{win}\} &= P\{\text{win} \mid \text{both correct}\}p^2 + P\{\text{win} \mid \text{exactly 1 correct}\}2p(1-p) \\ &\quad + P\{\text{win} \mid \text{neither correct}\}(1-p)^2 \\ &= p^2 + p(1-p) + 0 = p \end{aligned}$$

Thus, both strategies give the same probability of winning.

65. (a) $P\{\text{correct} \mid \text{agree}\} = P\{\text{correct, agree}\}/P\{\text{agree}\}$

$$\begin{aligned} &= p^2/[p^2 + (1-p)^2] \\ &= 36/52 = 9/13 \qquad \text{when } p = .6 \end{aligned}$$

(b) 1/2

66. (a) $[I - (1 - P_1P_2)(1 - P_3P_4)]P_5 = (P_1P_2 + P_3P_4 - P_1P_2P_3P_4)P_5$

(b) Let $E_1 = \{1 \text{ and } 4 \text{ close}\}$, $E_2 = \{1, 3, 5 \text{ all close}\}$

$E_3 = \{2, 5 \text{ close}\}$, $E_4 = \{2, 3, 4 \text{ close}\}$. The desired probability is

67. $$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup E_4) &= P(E_1) + P(E_2) + P(E_3) + P(E_4) - P(E_1E_2) - P(E_1E_3) - P(E_1E_4) \\ &\quad - P(E_2E_3) - P(E_2E_4) + P(E_3E_4) + P(E_1E_2E_3) + P(E_1E_2E_4) \\ &\quad + P(E_1E_3E_4) + P(E_2E_3E_4) - P(E_1E_2E_3E_4) \end{aligned}$$

$$\begin{aligned} &= P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4 \\ &\quad - P_1P_2P_3P_5 - P_2P_3P_4P_5 - 2P_1P_2P_3P_4P_5 + 3P_1P_2P_3P_4P_5. \end{aligned}$$

(a) $$\begin{aligned} &P_1P_2(1 - P_3)(1 - P_4) + P_1(1 - P_2)P_3(1 - P_4) + P_1(1 - P_2(1 - P_3)P_4 \\ &+ P_2P_3(1 - P_1)(1 - P_4) + (1 - P_1)P_2(1 - P_3)P_4 + (1 - P_1)(1 - P_2)P_3P_4 \\ &+ P_1P_2P_3(1 - P_4) + P_1P_2(1 - P_3)P_4 + P_1(1 - P_2)P_3P_4 + (1 - P_1)P_2P_3P_4 + P_1P_2P_3P_4. \end{aligned}$$

(c) $\displaystyle\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$

68. Let C_i denote the event that relay i is closed, and let F be the event that current flows from A to B.

$$P(C_1C_2 \mid F) = \frac{P(C_1C_2F)}{P(F)}$$
$$= \frac{P(F \mid C_1C_2)P(C_1C_2)}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)}$$
$$= \frac{p_5p_1p_2}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)}$$

69. 1. (a) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(b) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(c) $\frac{18}{128}$

(d) $\frac{110}{128}$

2. (a) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(b) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(c) $\frac{1}{16}$

(d) $\frac{15}{16}$

70. (a) $P\{\text{carrier} \mid 3 \text{ without}\}$

$$= \frac{1/8\,1/2}{1/8\,1/2 + 1\,1/2} = 1/9.$$

(b) 1/18

71. $P\{\text{Braves win}\} = P\{B \mid B \text{ wins 3 of 3}\}\ 1/8 + P\{B \mid B \text{ wins 2 of 3}\}\ 3/8$
$+ P\{B \mid B \text{ wins 1 of 3}\}\ 3/8 + P\{B \mid B \text{ wins 0 of 3}\}\ 1/8$

$$= \frac{1}{8} + \frac{3}{8}\left[\frac{1}{4}\frac{1}{2} + \frac{3}{4}\right] + \frac{3}{8} \quad \frac{3}{4}\frac{1}{2} = \frac{38}{64}$$

where $P\{B \mid B \text{ wins } i \text{ of 3}\}$ is obtained by conditioning on the outcome of the other series. For instance

$$P\{B \mid B \text{ win 2 of 3}\} = P\{B \mid D \text{ or } G \text{ win 3 of 3}, B \text{ win 2 of 3}\}\ 1/4$$
$$= P\{B \mid D \text{ or } G \text{ win 2 of 3}, B \text{ win 2 of 3}\}\ 3/4$$
$$= \frac{1}{2}\frac{1}{4} + \frac{3}{4}.$$

By symmetry $P\{D \text{ win}\} = P\{G \text{ win}\}$ and as the probabilities must sum to 1 we have.

$$P\{D \text{ win}\} = P\{G \text{ win}\} = \frac{13}{64}.$$

72. Let f denote for and a against a certain place of legislature. The situations in which a given steering committees vote is decisive are as follows:

given member	other members of S.C.	other council members
for	both for	3 or 4 against
for	one for, one against	at least 2 for
against	one for, one against	at least 2 for
against	both for	3 of 4 against

$$
\begin{aligned}
P\{\text{decisive}\} = {} & p^3 4p(1-p)^3 + p^2 p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) \\
& + (1-p)2p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) \\
& + (1-p)p^2 4p(1-p)^3.
\end{aligned}
$$

73. (a) 1/16, (b) 1/32, (c) 10/32, (d) 1/4, (e) 31/32.

74. Let P_A be the probability that A wins when A rolls first, and let P_B be the probability that B wins when B rolls first. Using that the sum of the dice is 9 with probability 1/9, we obtain upon conditioning on whether A rolls a 9 that

$$P_A = \frac{1}{9} + \frac{8}{9}(1 - P_B)$$

Similarly,

$$P_B = \frac{5}{36} + \frac{31}{36}(1 - P_A)$$

Solving these equations gives that $P_A = 9/19$ (and that $P_B = 45/76$.)

75. (a) The probability that a family has 2 sons is 1/4; the probability that a family has exactly 1 son is 1/2. Therefore, on average, every four families will have one family with 2 sons and two families with 1 son. Therefore, three out of every four sons will be eldest sons.

Another argument is to choose a child at random. Letting E be the event that the child is an eldest son, letting S be the event that it is a son, and letting A be the event that the child's family has at least one son,

$$
\begin{aligned}
P(E \mid S) &= \frac{P(ES)}{P(S)} \\
&= 2P(E) \\
&= 2\left[P(E|A)\frac{3}{4} + P(E|A^c)\frac{1}{4}\right] \\
&= 2\left[\frac{1}{2}\frac{3}{4} + 0\frac{1}{4}\right] = 3/4
\end{aligned}
$$

(b) Using the preceding notation

$$\begin{aligned} P(E \mid S) &= \frac{P(ES)}{P(S)} \\ &= 2P(E) \\ &= 2\left[P(E|A)\frac{7}{8} + P(E|A^c)\frac{1}{8}\right] \\ &= 2\left[\frac{1}{3}\frac{7}{8}\right] = 7/12 \end{aligned}$$

76. Condition on outcome of initial trial

$$\begin{aligned} P(E \text{ before } F) &= P(E \text{ b } F \mid E)P(E) + P(E \text{ b } F \mid F)P(F) \\ &\quad + P(E \text{ b } F \mid \text{neither } E \text{ or } F)[1 - P(E) - P(F)] \\ &= P(E) + P(E \text{ b } F)(1 - P(E) - P(F)]. \end{aligned}$$

Hence,

$$P(E \text{ b } F) = \frac{P(E)}{P(E)+P(F)}.$$

77. (a) This is equal to the conditional probability that the first trial results in outcome 1 (F_1) given that it results in either 1 or 2, giving the result 1/2. More formally, with L_3 being the event that outcome 3 is the last to occur

$$P(F_1 \mid L_3) = \frac{P(L_3|F_1)P(F_1)}{P(L_3)} = \frac{(1/2)(1/3)}{1/3} = 1/2$$

(b) With S_1 being the event that the second trial results in outcome 1, we have

$$P(F_1S_1 \mid L_3) = \frac{P(L_3|F_1S_1)P(F_1S_1)}{P(L_3)} = \frac{(1/2)(1/9)}{1/3} = 1/6$$

78. (a) Because there will be 4 games if each player wins one of the first two games and then one of them wins the next two, $P(4 \text{ games}) = 2p(1-p)[p^2 + (1-p)^2]$.

(b) Let A be the event that A wins. Conditioning on the outcome of the first two games gives

$$\begin{aligned} P(A &= P(A \mid a, a)p^2 + P(A \mid a, b)p(1-p) + P(A \mid b, a)(1-p)p + P(A \mid b, b)(1-p)^2 \\ &= p^2 + P(A)2p(1-p) \end{aligned}$$

where the notation a, b means, for instance, that A wins the first and B wins the second game. The final equation used that $P(A \mid a, b) = P(A \mid b, a) = P(A)$. Solving, gives

$$P(A) = \frac{p^2}{1-2p(1-p)}$$

79. Each roll that is either a 7 or an even number will be a 7 with probability

$$p = \frac{P(7)}{P(7)+P(\text{even})} = \frac{1/6}{1/6+1/2} = 1/4$$

Hence, from Example 4*i* we see that the desired probability is

$$\sum_{i=2}^{7}\binom{7}{i}(1/4)^i(3/4)^{7-i} = 1 - (3/4)^7 - 7(3/4)^6(1/4)$$

80. (a) $P(A_i) = (1/2)^i$, if $i < n$
$= (1/2)^{n-1}$, if $i = n$

(b) $$\frac{\sum_{i=1}^{n} i(1/2)^i + n(1/2)^{n-1}}{2^n - 1} = \frac{1}{2^{n-1}}$$

(c) Condition on whether they initially play each other. This gives

$$P_n = \frac{1}{2^n-1} + \frac{2^n-2}{2^n-1}\left(\frac{1}{2}\right)^2 P_{n-1}$$

where $\left(\frac{1}{2}\right)^2$ is the probability they both win given they do not play each other.

(d) There will be $2^n - 1$ losers, and thus that number of games.

(e) Since the 2 players in game i are equally likely to be any of the $\binom{2^n}{2}$ pairs it follows that

$P(B_i) = 1\Big/\binom{2^n}{2}$.

(f) Since the events B_i are mutually exclusive

$$P(\cup B_i) = \sum P(B_i) = (2^n - 1)\Big/\binom{2^n}{2} = (1/2)^{n-1}$$

81. $$\frac{1-(9/11)^{15}}{1-(9/11)^{30}}$$

82. (a) $P(A) = P_1^2 + \left(1-P_1^2\right)\left[\left(1-P_2^2\right)P(A)\right]$ or $P(A) = \dfrac{P_1^2}{P_1^2 + P_2^2 - P_1^2P_2^2}$

(c) similar to (a) with P_i^3 replacing P_i^2.

(b) and (d) Let $P_{ij}\,(\overline{P}_{ij})$ denote the probability that *A* wins when *A* needs *i* more and *B* needs *j* more and *A*(*B*) is to flip. Then

$$P_{ij} = P_1 P_{i-1,j} + (1-P_1)\overline{P}_{ij}$$
$$\overline{P}_{ij} = P_2 \overline{P}_{i,j-1} + (1-P_2)P_{ij}.$$

These equations can be recursively solved starting with

$$P_{01} = 1,\ P_{1,0} = 0.$$

83. (a) Condition on the coin flip

$$P\{\text{throw } n \text{ is red}\} = \frac{1}{2}\frac{4}{6} + \frac{1}{2}\frac{2}{6} = \frac{1}{2}$$

(b) $$P\{r \mid rr\} = \frac{P\{rrr\}}{P\{rr\}} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{3}\right)^3}{\frac{1}{2}\left(\frac{2^2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)^2} = \frac{3}{5}$$

(c) $$P\{A \mid rr\} = \frac{P\{rr \mid A\}P(A)}{P\{rr\}} = \frac{\left(\frac{2^2}{3}\right)\frac{1}{2}}{\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)^2\frac{1}{2}} = 4/5$$

84. (b) $$P(A \text{ wins}) = \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}$$
$$P(B \text{ wins}) = \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5}$$
$$P(C \text{ wins}) = \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}$$

85. Part (a) remains the same. The possibilities for part (b) become more numerous.

86. Using the hint

$$P\{A \subset B\} = \sum_{i=0}^{n} (2^i/2^n)\binom{n}{i}\Big/2^n = \sum_{i=0}^{n}\binom{n}{i}2^i/4^n = (3/4)^n$$

where the final equality uses

$$\sum_{i=0}^{n}\binom{n}{i}2^i 1^{n-i} = (2+1)^n$$

(b) $P(AB = \phi) = P(A \subset B^c) = (3/4)^n$, by part (a), since B^c is also equally likely to be any of the subsets.

87. $$P\{i^{\text{th}} \mid \text{all heads}\} = \frac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}.$$

88. No—they are conditionally independent given the coin selected.

89. (a) $P(J_3 \text{ votes guilty} \mid J_1 \text{ and } J_2 \text{ vote guilty}\}$

$= P\{J_1, J_2, J_3 \text{ all vote guilty}\}/P\{J_1 \text{ and } J_2 \text{ vote guilty}\}$

$$= \frac{\frac{7}{10}(.7)^3 + \frac{3}{10}(.2)^3}{\frac{7}{10}(.7)^2 + \frac{3}{10}(.2)^2} = \frac{97}{142}.$$

(b) $P(J_3 \text{ guilty} \mid \text{one of } J_1, J_2 \text{ votes guilty}\}$

$$= \frac{\frac{7}{10}(.7)2(.7)(.3) + \frac{3}{10}(2.)2(.2)(.8)}{\frac{7}{10}2(.7)(.3) + \frac{3}{10}2(.2)(.8)} = \frac{15}{26}.$$

(c) $P\{J_3 \text{ guilty} \mid J_1, J_2 \text{ vote innocent}\}$

$$= \frac{\frac{7}{10}(.7)(.3)^2 + \frac{3}{10}(.2)(.8)^2}{\frac{7}{10}(.3)^2 + \frac{3}{10}(.8)^2} = \frac{33}{102}.$$

E_i are conditionally independent given the guilt or innocence of the defendant.

90. Let N_i denote the event that none of the trials result in outcome i, $i = 1, 2$. Then

$$\begin{aligned} P(N_1 \cup N_2) &= P(N_1) + P(N_2) - P(N_1N_2) \\ &= (1 - p_1)^n + (1 - p_2)^n - (1 - p_1 - p_2)^n \end{aligned}$$

Hence, the probability that both outcomes occur at least once is
$1 - (1 - p_1)^n - (1 - p_2)^n + (p_0)^n$.

Theoretical Exercises

1. $$P(AB|A) = \frac{P(AB)}{P(A)} \geq \frac{P(AB)}{P(A \cup B)} = P(AB|A \cup B)$$

2. If $A \subset B$

$$P(A|B) = \frac{P(A)}{P(B)},\ P(A|B^c) = 0, \qquad P(B|A) = 1, \qquad P(B|A^c) = \frac{P(BA^c)}{P(A^c)}$$

3. Let F be the event that a first born is chosen. Also, let S_i be the event that the family chosen in method a is of size i.

$$P_a(F) = \sum_i P(F|S_i)P(S_i) = \sum_i \frac{1}{i}\frac{n_i}{m}$$

$$P_b(F) = \frac{m}{\sum_i in_i}$$

Thus, we must show that

$$\sum_i in_i \sum_i n_i / i \geq m^2$$

or, equivalently,

$$\sum_i in_i \sum_j n_j / j \geq \sum_i n_i \sum_j n_j$$

or,

$$\sum\sum_{i \neq j} \frac{i}{j} n_i n_j \geq \sum\sum_{i \neq j} n_i n_j$$

Considering the coefficients of the term $n_i n_j$, shows that it is sufficient to establish that

$$\frac{i}{j} + \frac{j}{i} \geq 2$$

or equivalently

$$i^2 + j^2 \geq 2ij$$

which follows since $(i - j)^2 \geq 0$.

4. Let N_i denote the event that the ball is not found in a search of box i, and let B_j denote the event that it is in box j.

$$P(B_j | N_i) = \frac{P(N_i | B_j)P(B_j)}{P(N_i | B_i)P(B_i) + P(N_i | B_i^c)P(B_i^c)}$$

$$= \frac{P_j}{(1-\alpha_i)P_i + 1 - P_i} \text{ if } j \neq i$$

$$= \frac{(1-\alpha_i)P_i}{(1-\alpha_i)P_i + 1 - P_i} \text{ if } j = i$$

5. None are true.

6. $P\left(\bigcup_1^n E_i\right) = 1 - P\left(\bigcap_1^n E_i^c\right) = 1 - \prod_1^n [1 - P(E_i)]$

7. (a) They will all be white if the last ball withdrawn from the urn (when all balls are withdrawn) is white. As it is equally likely to by any of the $n + m$ balls the result follows.

(b) $P(RBG) = \dfrac{g}{r+b+g} P(RBG | G \text{ last}) = \dfrac{g}{r+b+g}\dfrac{b}{r+b}$.

Hence, the answer is $\dfrac{bg}{(r+b)(r+b+g)} + \dfrac{b}{r+b+g}\dfrac{g}{r+g}$.

8. (a) $P(A) = P(A | C)P(C) + P(A | C^c)P(C^c) > P(B | C)P(C) + P(B | C^c)P(C^c) = P(B)$

(b) For the events given in the hint

$$P(A | C) = \frac{P(C|A)P(A)}{3/36} = \frac{(1/6)(1/6)}{3/36} = 1/3$$

Because $1/6 = P(A$ is a weighted average of $P(A | C)$ and $P(A | C^c)$, it follows from the result $P(A | C) > P(A)$ that $P(A | C^c) < P(A)$. Similarly,

$$1/3 = P(B | C) > P(B) > P(B | C^c)$$

However, $P(AB | C) = 0 < P(AB | C^c)$.

9. $P(A) = P(B) = P(C) = 1/2$, $P(AB) = P(AC) = P(BC) = 1/4$. But, $P(ABC) = 1/4$.

10. $P(A_{i,j}) = 1/365$. For $i \neq j \neq k$, $P(A_{i,j}A_{j,k}) = 365/(365)^3 = 1/(365)^2$. Also, for $i \neq j \neq k \neq r$, $P(A_{i,j}A_{k,r}) = 1/(365)^2$.

11. $1 - (1 - p)^n \geq 1/2$, or, $n \geq -\dfrac{\log(2)}{\log(1-p)}$

12. $a_i \prod_{j=1}^{i-1}(1-a_j)$ is the probability that the first head appears on the i^{th} flip and $\prod_{i=1}^{\infty}(1-a_i)$ is the probability that all flips land on tails.

13. Condition on the initial flip. If it lands on heads then A will win with probability $P_{n-1,m}$ whereas if it lands tails then B will win with probability $P_{m,n}$ (and so A will win with probability $1 - P_{m,n}$).

14. Let N go to infinity in Example 4*j*.

15. $P\{r$ successes before m failures$\}$
$= P\{r^{\text{th}}$ success occurs before trial $m + r\}$
$$= \sum_{n=r}^{m+r-1} \binom{n-1}{r-1} p^r (1-p)^{n-r} .$$

16. If the first trial is a success, then the remaining $n - 1$ must result in an odd number of successes, whereas if it is a failure, then the remaining $n - 1$ must result in an even number of successes.

17. $P_1 = 1/3$
$P_2 = (1/3)(4/5) + (2/3)(1/5) = 2/5$
$P_3 = (1/3)(4/5)(6/7) + (2/3)(4/5)(1/7) + (1/3)(1/5)(1/7) = 3/7$
$P_4 = 4/9$

(b) $P_n = \dfrac{n}{2n+1}$

(c) Condition on the result of trial n to obtain

$$P_n = (1 - P_{n-1})\frac{1}{2n+1} + P_{n-1}\frac{2n}{2n+1}$$

(d) Must show that

$$\frac{n}{2n+1} = \left[1 - \frac{n-1}{2n-1}\right]\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

or equivalently, that

$$\frac{n}{2n+1} = \frac{n}{2n-1}\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

But the right hand side is equal to

$$\frac{n+2n(n-1)}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

18. Condition on when the first tail occurs.

19. $P_{n,i} = p^{P}_{n-1,i+1} + (1-p)P_{n-1,i-1}$

20. $\alpha_{n+1} = \alpha_n p + (1 - \alpha_n)(1 - p^1)$
$P_n = \alpha_n p + (1 - \alpha_n)p^1$

21. (b) $P_{n,1} = P\{A \text{ receives first 2 votes}\} = \dfrac{n(n-1)}{(n+1)n} = \dfrac{n-1}{n+1}$

$P_{n,2} = P\{A \text{ receives first 2 and at least 1 of the next 2}\}$

$$= \frac{n}{n+2}\frac{n-1}{n+1}\left\{1-\frac{2\cdot 1}{n(n-1)}\right\} = \frac{n-2}{n+2}$$

(c) $P_{n,m} = \dfrac{n-m}{n+m}$, $n \geq m$.

(d) $P_{n,m} = P\{A \text{ always ahead}\}$

$$= P\{A \text{ always} \mid A \text{ receives last vote}\}\ \frac{n}{n+m}$$

$$+ P\{A \text{ always} \mid B \text{ receives last vote}\}\ \frac{m}{n+m}$$

$$= \frac{n}{n+m}P_{n-1,m} + \frac{m}{n+m}P_{n,m-1}$$

(e) The conjecture of (c) is true when $n + m = 1$ ($n = 1$, $m = 0$).
Assume it when $n + m = k$. Now suppose that $n + m = k + 1$. By (d) and the induction hypothesis we have that

$$P_{n,m} = \frac{n}{n+m}\frac{n-1-m}{n-1+m} + \frac{m}{n+m}\frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$$

which completes the proof.

22. $P_n = P_{n-1}p + (1 - P_{n-1})(1 - p)$

$= (2p - 1)P_{n-1} + (1 - p)$

$= (2p-1)\left[\dfrac{1}{2}+\dfrac{1}{2}(2p-1)^{n-1}\right] + 1 - p$ by the induction hypothesis

$= \dfrac{2p-1}{2}+\dfrac{1}{2}(2p-1)^{n} + 1 - p$

$= \dfrac{1}{2}+\dfrac{1}{2}(2p-1)^{n}$.

23. $P_{1,1} = 1/2$. Assume that $P_{a,b} = 1/2$ when $k \geq a + b$ and now suppose $a + b = k + 1$. Now

$$P_{a,b} = P\{\text{last is white} \mid \text{first } a \text{ are white}\} \frac{1}{\binom{a+b}{a}}$$

$$+ P\{\text{last is white} \mid \text{first } b \text{ are black}\} \frac{1}{\binom{b+a}{b}}$$

$$+ P\{\text{last is white} \mid \text{neither first } a \text{ are white nor first } b \text{ are black}\}$$

$$\left[1 - \frac{1}{\binom{a+b}{a}} - \frac{1}{\binom{b+a}{b}}\right] = \frac{a!b!}{(a+b)!} + \frac{1}{2}\left[1 - \frac{a!b!}{(a+b)!} - \frac{a!b!}{(a+b)!}\right] = \frac{1}{2}$$

where the induction hypothesis was used to obtain the final conditional probability above.

24. The probability that a given contestant does not beat all the members of some given subset of k other contestants is, by independence, $1 - (1/2)^k$. Therefore $P(B_i)$, the probability that none of the other $n - k$ contestants beats all the members of a given subset of k contestants, is $[1 - (1/2)^k]^{n-k}$. Hence, Boole's inequality we have that

$$P(\cup B_i) \leq \binom{n}{k}[1 - (1/2)^k]^{n-k}$$

Hence, if $\binom{n}{k}[1 - (1/2)^k]^{n-k} < 1$ then there is a positive probability that none of the $\binom{n}{k}$ events B_i occur, which means that there is a positive probability that for every set of k contestants there is a contestant who beats each member of this set.

25. $P(E \mid F) = P(EF)/P(F)$

$$P(E \mid FG)P(G \mid F) = \frac{P(EFG)}{P(FG)}\frac{P(FG)}{P(F)} = \frac{P(EFG)}{P(F)}$$

$$P(E \mid FG^c)P(G^c \mid F) = \frac{P(EFG^c)}{P(F)}.$$

The result now follows since

$$P(EF) = P(EFG) + P(EFG^c)$$

27. $E_1, E_2, \ldots, E_n$ are conditionally independent given F if for all subsets $i_1, \ldots, i_r$ of $1, 2, \ldots, n$

$$P\left(E_{i_1} \ldots E_{i_r} \middle| F\right) = \prod_{j=1}^{r} P\left(E_{i_j|F}\right).$$

28. Not true. Let $F = E_1$.

29. $P\{\text{next } m \text{ heads} \mid \text{first } n \text{ heads}\}$
$= P\{\text{first } n + m \text{ are heads}\}/P(\text{first } n \text{ heads}\}$
$= \int_0^1 p^{n+m} dp \Big/ \int_0^1 p^n dp = \dfrac{n+1}{n+m+1}.$

Chapter 4

Problems

1. $P\{X=4\} = \dfrac{\binom{4}{2}}{\binom{14}{2}} = \dfrac{6}{91}$ $\qquad$ $\text{P}\{X=0\} = \dfrac{\binom{2}{2}}{\binom{14}{2}} = \dfrac{1}{91}$

$P\{X=2\} = \dfrac{\binom{4}{2}\binom{2}{1}}{\binom{14}{2}} = \dfrac{8}{91}$ $\qquad$ $P\{X=-1\} = \dfrac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \dfrac{16}{91}$

$P\{X=1\} = \dfrac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \dfrac{32}{91}$ $\qquad$ $P\{X=-2\} = \dfrac{\binom{8}{2}}{\binom{14}{2}} = \dfrac{28}{91}$

2.

$p(1) = 1/36$	$p(5) = 2/36$	$p(9) = 1/36$	$p(15) = 2/36$	$p(24) = 2/36$
$p(2) = 2/36$	$p(6) = 4/36$	$p(10) = 2/36$	$p(16) = 1/36$	$p(25) = 1/36$
$p(3) = 2/36$	$p(7) = 0$	$p(11) = 0$	$p(18) = 2/36$	$p(30) = 2/36$
$p(4) = 3/36$	$p(8) = 2/36$	$p(12) = 4/36$	$p(20) = 2/36$	$p(36) = 1/36$

4. $P\{X=1\} = 1/2$, $P\{X=2\} = \dfrac{5}{10}\dfrac{5}{9} = \dfrac{5}{18}$, $P\{X=3\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{5}{8} = \dfrac{5}{36}$,

$P\{X=4\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{3}{8}\dfrac{5}{7} = \dfrac{10}{168}$, $P\{X=5\} = \dfrac{5\cdot4\cdot3\cdot2}{10\cdot9\cdot8\cdot7}\dfrac{5}{6} = \dfrac{5}{252}$,

$P\{X=6\} = \dfrac{5\cdot4\cdot3\cdot2\cdot1}{10\cdot9\cdot8\cdot7\cdot6} = \dfrac{1}{252}$

5. $n - 2i$, $i = 0, 1, \ldots, n$

6. $P(X = 3\} = 1/8$, $P\{X = 1\} = 3/8$, $P\{X = -1\} = 3/8$, $P\{X = -3\} = 1/8$

8. (a) $p(6) = 1 - (5/6)^2 = 11/36$, $p(5) = 2\ 1/6\ \ 4/6 + (1/6)^2 = 9/36$
$p(4) = 2\ 1/6\ \ 3/6 + (1/6)^2 = 7/36$, $p(3) = 2\ 1/6\ \ 2/6 + (1/6)^2 = 5/36$
$p(2) = 2\ 1/6\ \ 1/6 + (1/6)^2 = 3/36$, $p(1) = 1/36$

(d) $p(5) = 1/36$, $p(4) = 2/36$, $p(3) = 3/36$, $p(2) = 4/36$, $p(1) = 5/36$
$p(0) = 6/36$, $p(-j) = p(j)$, $j > 0$

11. (a) $P\{\text{divisible by } 3\} = \dfrac{333}{1000}$ $P\{\text{divisible by } 105\} = \dfrac{9}{1000}$

$P\{\text{divisible by } 7\} = \dfrac{142}{1000}$

$P\{\text{divisible by } 15\} = \dfrac{66}{1000}$

In limiting cases, probabilities converge to 1/3, 1/7, 1/15, 1/10

(b) $P\{\mu(N) \neq 0\} = P\{N \text{ is not divisible by } p_i^2, i \geq 1\}$

$= \prod_i P\{N \text{ is not divisible by } p_i^2\}$

$= \prod_i (1 - 1/p_i^2) = 6/\pi^2$

13. $p(0) = P\{\text{no sale on first and no sale on second}\}$
$= (.7)(.4) = .28$
$p(500) = P\{1 \text{ sale and it is for standard}\}$
$= P\{1 \text{ sale}\}/2$
$= [P\{\text{sale, no sale}\} + P\{\text{no sale, sale}\}]/2$
$= [(.3)(.4) + (.7)(.6)]/2 = .27$

$p(1000) = P\{2 \text{ standard sales}\} + P\{1 \text{ sale for deluxe}\}$
$= (.3)(.6)(1/4) + P\{1 \text{ sale}\}/2$
$= .045 + .27 = .315$

$p(1500) = P\{2 \text{ sales, one deluxe and one standard}\}$
$= (.3)(.6)(1/2) = .09$

$p(2000) = P\{2 \text{ sales, both deluxe}\} = (.3)(.6)(1/4) = .045$

14. $P\{X = 0\} = P\{1 \text{ loses to } 2\} = 1/2$

$P\{X = 1\} = P\{\text{of } 1, 2, 3\text{: } 3 \text{ has largest, then } 1, \text{ then } 2\}$
$= (1/3)(1/2) = 1/6$

$P\{X = 2\} = P\{\text{of } 1, 2, 3, 4\text{: } 4 \text{ has largest and } 1 \text{ has next largest}\}$
$= (1/4)(1/3) = 1/12$

$P\{X = 3\} = P\{\text{of } 1, 2, 3, 4, 5\text{: } 5 \text{ has largest then } 1\}$
$= (1/5)(1/4) = 1/20$

$P\{X = 4\} = P\{1 \text{ has largest}\} = 1/5$

15. $P\{X = 1\} = 11/66$

$$P\{X = 2\} = \sum_{j=2}^{11}\left(\frac{12-j}{66}\right)\left(\frac{11}{54+j}\right)$$

$$P\{X = 3\} = \sum_{\substack{k\neq 1\\ k\neq j}}\sum_{j=2}\left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+j+k}\right)$$

$$P\{X = 4\} = 1 - \sum_{i=1}^{3} P\{X = 1\}$$

16. $P\{Y_1 = i\} = \dfrac{12-i}{66}$

$$P\{Y_2 = i\} = \sum_{j\neq i}\left(\frac{12-j}{66}\right)\left(\frac{12-i}{54+j}\right)$$

$$P\{Y_3 = i\} = \sum_{\substack{k\neq j\\ k\neq i}}\sum_{j\neq i}\left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+k+j}\right)$$

All sums go from 1 to 11, except for prohibited values.

20. (a) $P\{x > 0\} = P\{\text{win first bet}\} + P\{\text{lose, win, win}\}$
$= 18/38 + (20/38)(18/38)^2 \approx .5918$

(b) No, because if the gambler wins then he or she wins \$1.
However, a loss would either be \$1 or \$3.

(c) $E[X] = 1[18/38 + (20/38)(18/38)^2] - [(20/38)2(20/38)(18/38)] - 3(20/38)^3 \approx -.108$

21. (a) $E[X]$ since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

(b) $P\{X = i\} = i/148$, $i = 40, 33, 25, 50$

$E[X] = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/148 \approx 39.28$
$E[Y] = (40 + 33 + 25 + 50)/4 = 37$

22. Let N denote the number of games played.

(a) $E(N) = 2[p^2 + (1-p)^2] + 3[2p(1-p)] = 2 + 2p(1-p)$

The final equality could also have been obtained by using that $N = 2 +]$ where I is 0 if two games are played and 1 if three are played. Differentiation yields that

$$\frac{d}{dp}E[N] = 2 - 4p$$

and so the minimum occurs when $2 - 4p = 0$ or $p = 1/2$.

(b) $E[N] = 3[p^3 + (1-p)^3 + 4[3p^2(1-p)p + 3p(1-p)^2(1-p)]$
$+ 5[6p^2(1-p)^2 = 6p^4 - 12p^3 + 3p^2 + 3p + 3$

Differentiation yields

$$\frac{d}{dp}E[N] = 24p^3 - 36p^2 + 6p + 3$$

Its value at $p = 1/2$ is easily seen to be 0.

23. (a) Use all your money to buy 500 ounces of the commodity and then sell after one week. The expected amount of money you will get is

$$E[\text{money}] = \frac{1}{2}500 + \frac{1}{2}2000 = 1250$$

(b) Do not immediately buy but use your money to buy after one week. Then

$$E[\text{ounces of commodity}] = \frac{1}{2}1000 + \frac{1}{2}250 = 625$$

24. (a) $p - (1-p)\frac{3}{4} = \frac{7}{4}p - 3/4$, (b) $-\frac{3}{4}p + (1-p)2 = -\frac{11}{4}p + 2$

$\frac{7}{4}p - 3/4 = -\frac{11}{4}p + 2 \Rightarrow p = 11/18$, maximum value = 23.72

(c) $q - \frac{3}{4}(1-q)$ (d) $-\frac{3}{4}q + 2(1-q)$, minimax value = 23/72
attained when $q = 11/18$

25. (a) $P(X = 1) = .6(.3) + .4(.7) = .46$

(b) $E[X] = 1(.46) + 2(.42) = 1.3$

27. $C - Ap = \frac{A}{10} \Rightarrow C = A\left(p + \frac{1}{10}\right)$

28. $3 \cdot \frac{4}{20} = 3/5$

29. If check 1, then (if desired) 2: Expected Cost $= C_1 + (1-p)C_2 + pR_1 + (1-p)R_2$;
if check 2, then 1: Expected Cost $= C_2 + pC_1 + pR_1 + (1-p)R_2$ so 1, 2, best if

$C_1 + (1-p)C_2 \le C_2 + pC_1$, or $C_1 \le \frac{p}{1-p}C_2$

30. $E[X] = \sum_{n=1}^{\infty} 2^n (1/2)^n = \infty$

(a) probably not

(b) yes, if you could play an arbitrarily large number of games

31. $E[\text{score}] = p^*[1 - (1 - P)^2 + (1 - p^*)(1 - p^2)$

$$\frac{d}{dp} = 2(1 - p)p^* - 2p(1 - p^*)$$
$$= 0 \Rightarrow p = p^*$$

32. If T is the number of tests needed for a group of 10 people, then

$$E[T] = (.9)^{10} + 11[1 - (.9)^{10}] = 11 - 10(.9)^{10}$$

35. If X is the amount that you win, then

$$P\{X = 1.10\} = 4/9 = 1 - P\{X = -1\}$$
$$E[X] = (1.1)4/9 - 5/9 = -.6/9 \approx =-.067$$
$$\text{Var}(X) = (1.1)^2(4/9) + 5/9 - (.6/9)^2 \approx 1.089$$

36. Using the representation

$$N = 2 + I$$

where I is 0 if the first two games are won by the same team and 1 otherwise, we have that

$$\text{Var}(N) = \text{Var}(I) = E[I]^2 - E^2[I]$$

Now, $E[I]^2 = E[I\} = P\{I = 1\} = 2p\{1 - p\}$ and so
$\text{Var}(N) = 2p(1 - p)[1 - 2p(1 - p)] = 8p^3 - 4p^4 - 6p^2 + 2p$

Differentiation yields

$$\frac{d}{dp}\text{Var}(N) = 24p^2 - 16p^3 - 12p + 2$$

and it is easy to verify that this is equal to 0 when $p = 1/2$.

37. $E[X^2] = [(40)^3 + (33)^3 + (25)^3 + (50)^3]/148 \approx 1625.4$

$$\text{Var}(X = E[X^2] - (E[X])^2 \approx 82.2$$

$E[Y^2] = = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/4 = 1453.5$, Varr($Y$) = 84.5

38. (a) $E[(2 + X)^2] = \text{Var}(2 + X) + (E[2 + X])^2 = \text{Var}(X) + 9 = 14$

(b) $\text{Var}(4 + 3X) = 9\ \text{Var}(X) = 45$

39. $\binom{4}{2}(1/2)^4 = 3/8$

40. $\binom{5}{4}(1/3)^4(2/3)^1 + (1/3)^5 = 11/243$

41. $\displaystyle\sum_{i=7}^{10}\binom{10}{i}(1/2)^{10}$

42. $\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 \geq \binom{3}{2}p^2(1-p) + p^3$

$\Leftrightarrow 6p^3 - 15p^2 + 12p - 3 \geq 0$

$\Leftrightarrow 6(p - 1/2)(p - 1)^2 \geq 0$

$\Leftrightarrow p \geq 1/2$

43. $\binom{5}{3}(.2)^3(.8)^2 + \binom{5}{4}(.2)^4(.8) + (.2)^5$

44. $\displaystyle\alpha\sum_{i=k}^{n}\binom{n}{i}p_1^i(1-p_1)^{n-i} + (1-\alpha)\sum_{i=k}^{n}\binom{n}{i}p_2^i(1-p_2)^{n-i}$

45. with 3: $P\{\text{pass}\} = \dfrac{1}{3}\left[\binom{3}{2}(.8)^2(.2) + (.8)^3\right] + \dfrac{2}{3}\left[\binom{3}{2}(.4)^2(.6) + (.4)^3\right]$

$= .533$

with 5: $P\{\text{pass}\} = \dfrac{1}{3}\displaystyle\sum_{i=3}^{5}\binom{5}{i}(.8)^i(.2)^{5-i} + \dfrac{2}{3}\sum_{i=3}^{5}\binom{5}{i}(.4)^i(.6)^{5-i}$

$= .3038$

46. Let C be the event that the jury is correct, and let G be the event that the defendant is guilty. Then

$$\begin{aligned}P(C) &= P(C|G)P(G) + P(C|G^c)P(G^c)\\ &= \sum_{i=0}^{3}(.2)^i(.8)^{12-i}(.65) + \sum_{i=0}^{8}(.1)^i(.9)^{12-i}(.35)\end{aligned}$$

Let CV be the event that the defendant is convicted. Then

$$\begin{aligned}P(CV) &= P(CV|G)P(G) + P(CV|G^c)P(G^c)\\ &= \sum_{i=0}^{3}(.2)^i(.8)^{12-i}(.65) + \left(1-\sum_{i=0}^{8}(.1)^i(.9)^{12-i}(.35)\right)\\ &= \sum_{i=0}^{3}(.2)^i(.8)^{12-i}(.65) + \sum_{i=9}^{12}(.1)^i(.9)^{12-i}(.35)\end{aligned}$$

47. (a) and (b): (i) $\sum_{i=5}^{9}\binom{9}{i}p^i(1-p)^{9-i}$, (ii) $\sum_{i=5}^{8}\binom{8}{i}p^i(1-p)^{8-i}$,

(iii) $\sum_{i=4}^{7}\binom{7}{i}p^i(1-p)^{7-i}$ where $p = .7$ in (a) and $p = .3$ in (b).

48. The probability that a package will be returned is $p = 1 - (.99)^{10} - 10(.99)^9(.01)$. Hence, if someone buys 3 packages then the probability they will return exactly 1 is $3p(1-p)^2$.

49. (a) $\frac{1}{2}\binom{10}{7}.4^7.6^3 + \frac{1}{2}\binom{10}{7}.7^7.3^3$

(b) $\dfrac{\frac{1}{2}\binom{9}{6}.4^7.6^3 + \frac{1}{2}.7^7.3^3}{.55}$

50. (a) $P\{H, T, T | 6 \text{ heads}\}$

$$\begin{aligned}&= P(H, T, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\}\\ &= P\{H, T, T\}P\{6 \text{ heads} | H, T, T\}/P\{6 \text{ heads}\}\\ &= pq^2\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4\\ &=1/10\end{aligned}$$

(b) $P\{T, H, T | 6 \text{ heads}\}$

$$\begin{aligned}&= P(T, H, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\}\\ &= P\{T, H, T\}P\{6 \text{ heads} | T, H, T\}/P\{6 \text{ heads}\}\\ &= q^2p\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4\\ &=1/10\end{aligned}$$

51. (a) $e^{-.2}$ (b) $1 - e^{-.2} - .2e^{-.2} = 1 - 1.2e^{-.2}$
Since each letter has a small probability of being a typo, the number of errors should approximately have a Poisson distribution.

52. (a) $1 - e^{-3.5} - 3.5e^{-3.5} = 1 - 4.5e^{-3.5}$

(b) $4.5e^{-3.5}$

Since each flight has a small probability of crashing it seems reasonable to suppose that the number of crashes is approximately Poisson distributed.

53. (a) The probability that an arbitrary couple were both born on April 30 is, assuming independence and an equal chance of having being born on any given date, $(1/365)^2$. Hence, the number of such couples is approximately Poisson with mean $80{,}000/(365)^2 \approx .6$. Therefore, the probability that at least one pair were both born on this date is approximately $1 - e^{-.6}$.

(b) The probability that an arbitrary couple were born on the same day of the year is 1/365. Hence, the number of such couples is approximately Poisson with mean $80{,}000/365 \approx 219.18$. Hence, the probability of at least one such pair is $1 - e^{-219.18} \approx 1$.

54. (a) $e^{-2.2}$ (b) $1 - e^{-2.2} - 2.2e^{-2.2} = 1 - 3.2e^{-2.2}$

55. $\frac{1}{2}e^{-3} + \frac{1}{2}e^{-4.2}$

56. The number of people in a random collection of size n that have the same birthday as yourself is approximately Poisson distributed with mean $n/365$. Hence, the probability that at least one person has the same birthday as you is approximately $1 - e^{-n/365}$. Now, $e^{-x} = 1/2$ when $x = \log(2)$. Thus, $1 - e^{-n/365} \ge 1/2$ when $n/365 \ge \log(2)$. That is, there must be at least 365 log(2) people.

57. (a) $1 - e^{-3} - 3e^{-3} - e^{-3}\frac{3^2}{2} = 1 - \frac{17}{2}e^{-3}$

(b) $P\{X \ge 3 \mid X \ge 1\} = \dfrac{P\{X \ge 3\}}{P\{X \ge 1\}} = \dfrac{1 - \frac{17}{2}e^{-3}}{1 - e^{-3}}$

59. (a) $1 - e^{-1/2}$

(b) $\frac{1}{2}e^{-1/2}$

(c) $1 - e^{-1/2} - \frac{1}{2}e^{-1/2} = 1 - \frac{3}{2}e^{-1/2}$

60. $$P\{\text{beneficial} \mid 2\} = \frac{P\{2 \mid \text{beneficial}\}3/4}{P\{2 \mid \text{beneficial}\}3/4 + P\{2 \mid \text{not beneficial}\}1/4}$$
$$= \frac{e^{-3}\frac{3^2}{2}\frac{3}{4}}{e^{-3}\frac{3^2}{2}\frac{3}{4} + e^{-5}\frac{5^2}{2}\frac{1}{4}}$$

61. $1 - e^{-1.4} - 1.4e^{-1.4}$

62. For $i < j$, say that trial pair (i, j) is a success if the same outcome occurs on trials i and j. Then (i, j) is a success with probability $\sum_{k=1}^{n} p_k^2$. By the Poisson paradigm the number of trial pairs that result in successes will approximately have a Poisson distribution with mean
$$\sum_{i<j}\sum_{k=1}^{n} p_k^2 = n(n-1)\sum_{k=1}^{n} p_k^2/2$$
and so the probability that none of the trial pairs result in a success is approximately $\exp\left(-n(n-1)\sum_{k=1}^{n} p_k^2/2\right)$.

63. (a) $e^{-2.5}$

(b) $1 - e^{-2.5} - 2.5e^{-2.5} - \frac{(2.5)^2}{2}e^{-2.5} - \frac{(2.5)^3}{3!}e^{-2.5}$

64. (a) $1 - \sum_{i=0}^{7} e^{-4}4^i/i! \equiv p$

(b) $1 - (1 - p)^{12} - 12p(1 - p)^{11}$

(c) $(1 - p)^{i-1}p$

65. (a) $1 - e^{-1/2}$

(b) $P\{X \geq 2 \mid X \geq 1\} = \frac{1 - e^{-1/2} - \frac{1}{2}e^{-1/2}}{1 - e^{-1/2}}$

(c) $1 - e^{-1/2}$

(d) $1 - \exp\{-500 - i)/1000\}$

66. Assume $n > 1$.

(a) $\frac{2}{2n-1}$

(b) $\frac{2}{2n-2}$

(c) $\exp\{-2n/(2n - 1)\} \approx e^{-1}$

67. Assume $n > 1$.

(a) $\dfrac{2}{n}$

(b) Conditioning on whether the man of couple j sits next to the woman of couple i gives the result: $\dfrac{1}{n-1}\dfrac{1}{n-1}+\dfrac{n-2}{n-1}\dfrac{2}{n-1}=\dfrac{2n-3}{(n-1)^2}$

(c) e^{-2}

68. $\exp(-10e^{-5}\}$

69. With P_j equal to the probability that 4 consecutive heads occur within j flips of a fair coin, $P_1 = P_2 = P + 3 = 0$, and

$P_4 = 1/16$
$P_5 = (1/2)P_4 + 1/16 = 3/32$
$P_6 = (1/2)P_5 + (1/4)P_4 + 1/16 = 1/8$
$P_7 = (1/2)P_6 + (1/4)P_5 + (1/8)P_4 + 1/16 = 5/32$
$P_8 = (1/2)P_7 + (1/4)P_6 + (1/8)P_5 + (1/16)P_4 + 1/16 = 6/32$
$P_9 = (1/2)P_8 + (1/4)P_7 + (1/8)P_6 + (1/16)P_5 + 1/16 = 111/512$
$P_{10} = (1/2)P_9 + (1/4)P_8 + (1/8)P_7 + (1/16)P_6 + 1/16 = 251/1024 = .2451$

The Poisson approximation gives

$$P_{10} \approx 1 - \exp\{-6/32 - 1/16\} = 1 - e^{-.25} = .2212$$

70. $e^{-\lambda t} + (1 - e^{-\lambda t})p$

71. (a) $\left(\dfrac{26}{38}\right)^5$

(b) $\left(\dfrac{26}{38}\right)^3\dfrac{12}{38}$

72. $P\{\text{wins in } i \text{ games}\} = \dbinom{i-1}{3}(.6)^4(.4)^{i-4}$

73. Let N be the number of games played. Then

$P\{N = 4\} = 2(1/2)^4 = 1/8,$ $\quad P\{N = 5\} = 2\dbinom{4}{1}(1/2)(1/2)^4 = 1/4$

$P\{N = 6\} = 2\dbinom{5}{2}(1/2)^2(1/2)^4 = 5/16,$ $\quad P\{N = 7\} = 5/16$

$E[N] = 4/8 + 5/4 + 30/16 + 35.16 = 93/16 = 5.8125$

74. (a) $\left(\frac{2}{3}\right)^5$

(b) $\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3+\binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2+\binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)^8$

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5\frac{1}{3}$

(d) $\binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2$

76. $\binom{N_1+N_2-k}{N_1}(1/2)^{N_1+N_2-k}(1/2)+\binom{N_1+N_2-k}{N_2}(1/2)^{N_1+N_2-k}(1/2)$

77. $2\binom{2N-k}{N}(1/2)^{2N-k}$

$2\binom{2N-k-1}{N-1}(1/2)^{2N-k-1}(1/2)$

79. (a) $P\{X=0\}=\dfrac{\binom{94}{10}}{\binom{100}{10}}$

(b) $P\{X>2\}=1-\dfrac{\binom{94}{10}+\binom{94}{9}\binom{6}{1}+\binom{94}{8}\binom{6}{2}}{\binom{100}{10}}$

80. $P\{\text{rejected} \mid 1 \text{ defective}\}=3/10$

$P\{\text{rejected} \mid 4 \text{ defective}\}=1-\binom{6}{3}\Big/\binom{10}{3}=5/6$

$P\{4 \text{ defective} \mid \text{rejected}\}=\dfrac{\frac{5}{6}\frac{3}{10}}{\frac{5}{6}\frac{3}{10}+\frac{3}{10}\frac{7}{10}}=75/138$

81. $P\{\text{rejected}\}=1-(.9)^4$

83. Let X_i be the number of accidents that occur on highway i. Then

$$E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3] = 1.5$$

84. Let X_i equal 1 if box i does not have any balls, and let it equal 0 otherwise. Then

$$E\left[\sum_{i=1}^{5} X_i\right] = \sum_{i=1}^{5} E[X_i] = \sum_{i=1}^{5} P(X_i = 1) = \sum_{i=1}^{5} (1 - p_i)^{10}$$

Let Y_i equal 1 if box i has exactly one ball, and let it equal 0 otherwise. Then

$$E\left[\sum_{i=1}^{5} Y_i\right] = \sum_{i=1}^{5} E[Y_i] = \sum_{i=1}^{5} P(Y_i = 1) = \sum_{i=1}^{5} 10\, p_i (1 - p_i)^9$$

where we used that the number of balls that go into box i is binomial with parameters 10 and p_i.

85. Let X_i equal 1 if there is at least one type i coupon in the set of n coupons. Then

$$E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E[X_i] = \sum_{i=1}^{k} P(X_i = 1) = \sum_{i=1}^{k} \left(1 - (1 - p_i)^n\right) = k - \sum_{i=1}^{k} (1 - p_i)^n$$

Theoretical Exercises

1. Let E_i = {no type i in first n selections}

$$P\{T > n\} = P\left(\bigcup_{i=1}^{N} E_i\right)$$
$$= \sum_i (1 - P_i)^n - \sum\sum_{I<J} (1 - P_i - P_j)^n + \sum\sum\sum_{i<j<k} (1 - p_{_J} - p_k)^n$$
$$\ldots + (-1)^N \sum_i P_i^n$$

$$P\{T = n\} = P\{\mathrm{T} > n - 1\} - P\{T > n\}$$

2. Not true. Suppose $P\{X = b\} = \varepsilon > 0$ and $b_n = b + 1/n$. Then $\lim_{b_n \to b} P(X < b_n\} = P\{X \le b\} \ne P\{X < b\}$.

3. When $\alpha > 0$

$$P\{\alpha X + \beta \le x\} = P\left\{x \le \frac{x-\beta}{\alpha}\right\} = F\left(\frac{x-\beta}{\alpha}\right)$$

When $\alpha < 0$

$$P\{\alpha X + \beta \le x\} = P\left\{X \ge \frac{x-\beta}{\alpha}\right\} = 1 - \lim_{h \to 0^+} F\left(\frac{x-\beta}{\alpha} - 1\right).$$

4. $$\sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} P\{N = K\}$$
$$= \sum_{k=1}^{\infty} kP\{N = k\} = E[N].$$

5. $$\sum_{i=0}^{\infty} i\, P\{N > i\} = \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty} P\{N = k\} \sum_{i=0}^{k-1} i$$
$$= \sum_{k=1}^{\infty} P\{N = k\}(k-1)k/2$$
$$= \left(\sum_{k=1}^{\infty} k^2 P\{N = k\} - \sum_{k=1}^{\infty} kP\{N = k\}\right)\Big/2$$

6. $$E[c^X] = cp + c^{-1}(1-p)$$

Hence, $1 = E[c^X]$ if

$$cp + c^{-1}(1-p) = 1$$

or, equivalently

$$pc^2 - c + 1 - p = 0$$

or

$$(pc - 1 + p)(c - 1) = 0$$

Thus, $c = (1-p)/p$.

7. $$E[Y] = E[X/\sigma - \mu/\sigma] = \frac{1}{\sigma}E[X] - \mu/\sigma = \mu/\sigma - \mu/\sigma = 0$$

$$\text{Var}(Y) = (1/\sigma)^2\,\text{Var}(X) = \sigma^2/\sigma^2 = 1.$$

8. Let I equal 1 if $X = a$ and let it equal 0 if $X = b$, then

$$X = a + (b-a)I$$

yielding the result

$$\text{Var}(X) = (b-a)^2\text{Var}(I) = (b-a)^2p(1-p)$$

9. $$1 = \sum_{i=0}^{n} P(X=i) = \sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}$$

If x and y are positive numbers, then letting $p = \dfrac{x}{x+y}$, gives

$$1 = \sum_{i=0}^{n}\binom{n}{i}\left(\frac{x}{x+y}\right)^n\left(\frac{y}{x+y}\right)^{n-i}$$

or, upon multiplying both sides by $(x+y)^n$, that

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^i y^{n=i}$$

10. $$E[1/(X+1)] = \sum_{i=0}^{n} \frac{1}{i+1} \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i}$$
$$= \sum_{i=0}^{n} \frac{n!}{(n-i)!(i+1)!} p^i (1-p)^{n-i}$$
$$= \frac{1}{(n+1)p} \sum_{i=0}^{n} \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i}$$
$$= \frac{1}{(n+1)p} \sum_{j=1}^{n+1} \binom{n+1}{j} p^j (1-p)^{n+1-j}$$
$$= \frac{1}{(n+1)p} \left[1 - \binom{n+1}{0} p^0 (1-p)^{n+1-0}\right]$$
$$= \frac{1}{(n+1)p} [1 - (1-p)^{n+1}]$$

11. For any given arrangement of k successes and $n - k$ failures:

$$P\{\text{arrangement} \mid \text{total of } k \text{ successes}\}$$
$$= \frac{P\{\text{arrangement}\}}{P\{k \text{ successes}\}} = \frac{p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

12. Condition on the number of functioning components and then use the results of Example 4c of Chapter 1:

$$\text{Prob} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \left[\binom{i+1}{n-i} \Big/ \binom{n}{i}\right]$$

where $\binom{i+1}{n-i} = 0$ if $n - i > i + 1$. We are using the results of Exercise 11.

13. Easiest to first take log and then determine the p that maximizes $\log P\{X = k\}$.

$$\log P\{X = k\} = \log \binom{n}{k} + k \log p + (n-k) \log (1-p)$$

$$\frac{\partial}{\partial p} \log P\{x = k\} = \frac{k}{p} - \frac{n-k}{1-p}$$
$$= 0 \Rightarrow p = k/n \text{ maximizes}$$

14. (a) $1 - \sum_{n=1}^{\infty} \alpha p^n = 1 - \dfrac{\alpha p}{1-p}$

(b) Condition on the number of children: For $k > 0$

$$\begin{aligned} P\{k \text{ boys}\} &= \sum_{n=1}^{\infty} P\{k \mid n \text{ children}\}\alpha p^n \\ &= \sum_{n=k}^{\infty} \binom{n}{k} (1/2)^n \alpha p^n \end{aligned}$$

$$P\{0 \text{ boys}\} = 1 - \frac{\alpha p}{1-p} + \sum_{n=1}^{\infty} \alpha p^n (1/2)^n$$

17. (a) If X is binomial (n, p) then, from exercise 15,

$$\begin{aligned} P\{X \text{ is even}\} &= [1 + (1 - 2p)^n]/2 \\ &= [1 + (1 - 2\lambda/n)^n]/2 \quad \text{when } \lambda = np \\ &\to (1 + e^{-2\lambda})/2 \quad \text{as } n \text{ approaches infinity} \end{aligned}$$

(b) $P\{X \text{ is even}\} = e^{-\lambda} \sum_n \lambda^{2n}/(2n)! = e^{-\lambda}(e^{\lambda} + e^{-\lambda})/2$

18. $\log P\{X = k\} = -\lambda + k \log \lambda - \log (k!)$

$$\begin{aligned} \frac{\partial}{\partial \lambda} \log P\{X = k\} &= -1 + \frac{k}{\lambda} \\ &= 0 \Rightarrow \lambda = k \end{aligned}$$

19. $$E[X^n] = \sum_{i=0}^{\infty} i^n e^{-\lambda}\lambda^i / i!$$
$$= \sum_{i=1}^{\infty} i^n e^{-\lambda}\lambda^i / i!$$
$$= \sum_{i=1}^{\infty} i^{n-1} e^{-\lambda}\lambda^i / (i-1)!$$
$$= \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^{j+1} / j!$$
$$= \lambda\sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^{j} / j!$$
$$= \lambda E[(X+1)^{n-1}]$$

Hence $[X^3] = \lambda E(X + 1)^2]$
$$= \lambda\sum_{i=0}^{\infty}(i+1)^2 e^{-\lambda}\lambda^i / i!$$
$$= \lambda\left[\sum_{i=0}^{\infty} i^2 e^{-\lambda}\lambda^i / i! + 2\sum_{i=0}^{\infty} i e^{-\lambda}\lambda^i / i! + \sum_{i=0}^{\infty} e^{-\lambda}\lambda^i / i!\right]$$
$$= \lambda[E[X^2] + 2E[X] + 1)$$
$$= \lambda(\mathrm{Var}(X) = E^2[X] + 2E[X] + 1)$$
$$= \lambda(\lambda + \lambda^2 + 2\lambda + 1) = \lambda(\lambda^2 + 3\lambda + 1)$$

20. Let S denote the number of heads that occur when all n coins are tossed, and note that S has a distribution that is approximately that of a Poisson random variable with mean λ. Then, because X is distributed as the conditional distribution of S given that $S > 0$,

$$P\{X = 1\} = P\{S = 1 \mid S > 0\} = \frac{P\{S = 1\}}{P\{S > 0\}} \approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

21. (i) 1/365
(ii) 1/365
(iii) 1 The events, though independent in pairs, are not independent.

22. (i) Say that trial i is a success if the i^{th} pair selected have the same number. When n is large trials 1, …, k are roughly independent.
(ii) Since, $P\{\text{trial } i \text{ is a success}\} = 1/(2n - 1)$ it follows that, when n is large, M_k is approximately Poisson distributed with mean $k/(2n - 1)$. Hence,

$$P\{M_k = 0\} \approx \exp[-k/(2n - 1)]$$

(iii) and (iv) $P\{T > \alpha n\} = P\{M_{\alpha n} = 0\} \approx \exp[-\alpha n/(2n - 1)] \to e^{-\alpha/2}$

23. (a) $P(E_i) = 1 - \sum_{j=0}^{2} \binom{365}{j} (1/365)^j (364/365)^{365-j}$

(b) $\exp(-365P(E_1)\}$

24. (a) There will be a string of k consecutive heads within the first n trials either if there is one within the first $n - 1$ trials, or if the first such string occurs at trial n; the latter case is equivalent to the conditions of 2.

(b) Because cases 1 and 2 are mutually exclusive

$$P_n = P_{n-1} + (1 - P_{n-k-1})(1 - P)p^k$$

25.
$$\begin{aligned} P(m \text{ counted}) &= \sum_n P(m|n \text{ events}) e^{-\lambda} \lambda^n / n! \\ &= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} e^{-\lambda} \lambda^n / n! \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \sum_{n=m}^{\infty} \frac{[\lambda(1-p)]^{n-m}}{(n-m)!} e^{-\lambda(1-p)} \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \end{aligned}$$

Intuitively, the Poisson λ random variable arises as the approximate number of successes in n (large) independent trials each having a small success probability α (and λ $n\alpha$). Now if each successful trial is counted with probability p, than the number counted is Binomial with parameters n (large) and αp (small) which is approximately Poisson with parameter $\alpha pn = \lambda p$.

27.
$$\begin{aligned} P\{X = n + k \mid X > n\} &= \frac{P\{X = n+k\}}{P\{X > n\}} \\ &= \frac{p(1-p)^{n+k-1}}{(1-p)^n} \\ &= p(1-p)^{k-1} \end{aligned}$$

If the first n trials are fall failures, then it is as if we are beginning anew at that time.

28. The events $\{X > n\}$ and $\{Y < r\}$ are both equivalent to the event that there are fewer than r successes in the first n trials; hence, they are the same event.

29.
$$\begin{aligned} \frac{P\{X = k+1\}}{P\{X = k\}} &= \frac{\binom{Np}{k+1}\binom{N-np}{n-k-1}}{\binom{Np}{k}\binom{N-Np}{n-k}} \\ &= \frac{(Np-k)(n-k)}{(k+1)(N-Np-n+k+1)} \end{aligned}$$

30. $P\{Y=j\} = \binom{j-1}{n-1}\Big/\binom{N}{n},\ n \le j \le N$

$$E[Y] = \sum_{j=n}^{N}\binom{j-1}{n-1}\Big/\binom{N}{n}$$

$$= \frac{n}{\binom{N}{n}}\sum_{j=n}^{N}\binom{j}{n}$$

$$= \frac{n}{\binom{N}{n}}\sum_{i=n+1}^{N+1}\binom{i-1}{n+1-1}$$

$$= \frac{n}{\binom{N}{n}}\binom{N+1}{n+1}$$

$$= \frac{n(N+1)}{n+1}$$

31. Let Y denote the largest of the remaining m chips. By exercise 28

$$P\{Y=j\} = \binom{j-1}{m-1}\Big/\binom{m+n}{m},\ m \le j \le n+m$$

Now, $X = n + m - Y$ and so

$$P\{X=i\} = P\{Y=m+n-i\} = \binom{m+n-i-1}{m-1}\Big/\binom{m+n}{m},\ i \le n$$

32. $P\{X=k\} = \dfrac{k-1}{n}\prod_{i=0}^{k-2}\dfrac{n-i}{n},\ k > 1$

34. $$E[X] = \sum_{k=0}^{n} \frac{k\binom{n}{k}}{2^n - 1} = \frac{n2^{n-1}}{2^n - 1}$$

$$E[X^2] = \sum_{k=0}^{n} \frac{k^2\binom{n}{k}}{2^n - 1} = \frac{2^{n-2}n(n+1)}{2^n - 1}$$

$$\text{Var}(X) = E[X^2] - \{E[X])^2 = \frac{n2^{2n-2} - n(n+1)2^{n-2}}{(2^n - 1)^2}$$

$$\sim \frac{n2^{2n-2}}{2^{2n}} = \frac{n}{4}$$

$$E[Y] = \frac{n+1}{2},\ E[Y^2] = \sum_{i=1}^{n} i^2/n \sim \int_1^{n+1} \frac{\mathrm{x}^2\,dx}{n} \sim \frac{n^2}{3}$$

$$\text{Var}(Y) \sim \frac{n^2}{3} - \left(\frac{n+1}{2}\right)^2 \sim \frac{n^2}{12}$$

35. (a) $P\{X > i\} = \frac{1}{2}\frac{2}{3}\cdots\frac{i}{i+1} = \frac{1}{i+1}$

(b) $P(X < \infty\} = \lim_{i\to\infty} P\{X \le i\}$

$= \lim_i (1 - 1/(i+1)) = 1$

(c) $E[X] = \sum_i iP\{X = i\}$

$= \sum_i i(P\{X > i-1\} - P\{X > i\}$

$= \sum_i i\left(\frac{1}{i} - \frac{1}{i+1}\right)$

$= \sum_i \frac{1}{i+1}$

$= \infty$

36. (a) This follows because $\{X + Y = z_k\} = \cup_{(i,j)\in A_k} \{X = x_i, Y = y_j\}$, and the events $\{X = x_i, Y = y_j\}$, $(i, j) \in A_k$, are mutually exclusive.

(b)

$$\begin{aligned}E[X+Y] &= \sum_k z_k P\{X+Y=z_k\}\\ &= \sum_k z_k \sum_{(i,j)\in A_k} P\{X=x_i, Y=y_j\}\\ &= \sum_k \sum_{(i,j)\in A_k} z_k P\{X=x_i, Y=y_j\}\\ &= \sum_k \sum_{(i,j)\in A_k} (x_i+y_j) P\{X=x_i, Y=y_j\}\end{aligned}$$

(c) This follows from (b) because every pair i, j is in exactly one of the sets A_k.

(d) This follows because

$$\{X = x_i\} = \cup_j \{X = x_i, Y = y_j\}$$

(e) From the preceding

$$\begin{aligned}E[X+Y] &= \sum_k \sum_{(i,j)\in A_k} (x_i+y_j) P\{X=x_i, Y=y_j\}\\ &= \sum_{i,j} (x_i+y_j) P\{X=x_i, Y=y_j\}\\ &= \sum_{i,j} x_i P\{X=x_i, Y=y_j\} + \sum_{i,j} y_j P\{X=x_i, Y=y_j\}\\ &= \sum_i x_i \sum_j P\{X=x_i, y=y_j\} + \sum_j y_j \sum_i P\{X=x_i, Y=y_j\}\\ &= \sum_i x_i P\{X=x_i\} + \sum_j y_j P\{Y=y_j\}\\ &= E[X]+E[Y]\end{aligned}$$

Chapter 5

Problems

1. (a) $c\int_{-1}^{1}(1-x^2)dx = 1 \Rightarrow c = 3/4$

(b) $F(x) = \dfrac{3}{4}\int_{-1}^{x}(1-x^2)dx = \dfrac{3}{4}\left(x - \dfrac{x^3}{3} + \dfrac{2}{3}\right), -1 < x < 1$

2. $\int xe^{-x/2}dx = -2xe^{-x/2} - 4e^{-x/2}$. Hence,

$$c\int_{0}^{\infty} xe^{-x/2}dx = 1 \Rightarrow c = 1/4$$

$$P\{X > 5\} = \frac{1}{4}\int_{5}^{\infty} xe^{-x/2}dx = \frac{1}{4}[10e^{-5/2} + 4e^{-5/2}]$$
$$= \frac{14}{4}e^{-5/2}$$

3. No. $f(5/2) < 0$

4. (a) $\int_{20}^{\infty}\dfrac{10}{x^2}dx = \dfrac{-10}{x}\Big|_{20}^{\infty} = 1/2$.

(b) $F(y) = \int_{10}^{y}\dfrac{10}{x^2}dx = 1 - \dfrac{10}{y}$, $y > 10$. $F(y) = 0$ for $y < 10$.

(c) $\sum_{i=3}^{6}\binom{6}{i}\left(\dfrac{2}{3}\right)^{i}\left(\dfrac{1}{3}\right)^{6-i}$ since $\bar{F}(15) = \dfrac{10}{15}$. Assuming independence of the events that the devices exceed 15 hours.

5. Must choose c so that

$$.01 = \int_{c}^{1} 5(1-x)^4 dx = (1-c)^5$$

so $c = 1 - (.01)^{1/5}$.

6. (a) $E[X] = \dfrac{1}{4}\int_0^\infty x^2 e^{-x/2}dx = 2\int_0^\infty y^2 e^{-y}dx = 2\Gamma(3) = 4$

(b) By symmetry of $f(x)$ about $x= 0$, $E[X] = 0$

(c) $E[X] = \int_5^\infty \dfrac{5}{x}dx = \infty$

7. $\int_0^1 (a+bx^2)dx = 1$ or $a + \dfrac{b}{3} = 1$

$\int_0^1 x(a+bx^2)dx = \dfrac{3}{5}$ or $\dfrac{a}{2} + \dfrac{b}{4} = 3/5$. Hence,

$a = \dfrac{3}{5}$, $b = \dfrac{6}{5}$

8. $E[X] = \int_0^\infty x^2 e^{-x}dx = \Gamma(3) = 2$

9. If s units are stocked and the demand is X, then the profit, $P(s)$, is given by

$$\begin{aligned} P(s) &= bX - (s - X)P && \text{if } X \le s \\ &= sb && \text{if } X > s \end{aligned}$$

Hence

$$\begin{aligned} E[P(s)] &= \int_0^s (bx - (s-x)\ell)f(x)dx + \int_s^\infty sbf(x)dx \\ &= (b+\ell)\int_0^s xf(x)dx - s\ell\int_0^s f(x)dx + sb\left[1 - \int_0^s f(x)dx\right] \\ &= sb + (b+\ell)\int_0^s (x-s)f(x)dx \end{aligned}$$

Differentiation yields

$$\begin{aligned} \frac{d}{ds}E[P(s)] &= b + (b+\ell)\frac{d}{ds}\left[\int_0^s xf(x)dx - s\int_0^s f(x)dx\right] \\ &= b + (b+\ell)\left[sf(s) - sf(s) - \int_0^s f(s)dx\right] \\ &= b - (b+\ell)\int_0^s f(x)dx \end{aligned}$$

Equating to zero shows that the maximal expected profit is obtained when s is chosen so that

$$F(s) = \frac{b}{b+\ell}$$

where $F(s) = \int_0^s f(x)dx$ is the cumulative distribution of demand.

10. (a) $P\{\text{goes to } A\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60\}$.
$= 2/3$ since X is uniform (0, 60).

(b) same answer as in (a).

11. X is uniform on (0, L).

$$P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < 1/4\right\}$$
$$= 1 - P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) > 1/4\right\}$$
$$= 1 - P\left\{\frac{X}{L-X} > 1/4, \frac{L-X}{X} > 1/4\right\}$$
$$= 1 - P\{X > L/5, X < 4L/5\}$$
$$= 1 - P\left\{\frac{L}{5} < X < 4L/5\right\}$$
$$= 1 - \frac{3}{5} = \frac{2}{5}.$$

13. $P\{X > 10\} = \frac{2}{3}$, $P\{X > 25 \mid X > 15\} = \frac{P\{X > 25}{P\{X > 15\}} = \frac{5/30}{15/30} = 1/3$
where X is uniform (0, 30).

14. $E[X^n] = \int_0^1 x^n dx = \frac{1}{n+1}$

$P\{X^n \le x\} = P\{X \le x^{1/n}\} = x^{1/n}$

$E[X^n] = \int_0^1 x \frac{1}{n} x^{\left(\frac{1}{n}-1\right)} dx = \frac{1}{n}\int_0^1 x^{1/n} dx = \frac{1}{n+1}$

15. (a) $\Phi(.8333) = .7977$
(b) $2\Phi(1) - 1 = .6827$
(c) $1 - \Phi(.3333) = .3695$
(d) $\Phi(1.6667) = .9522$
(e) $1 - \Phi(1) = .1587$

16. $P\{X > 50\} = P\left\{\frac{X-40}{4} > \frac{10}{4}\right\} = 1 - \Phi(2.5) = 1 - .9938$

Hence, $(P\{X < 50\})^{10} = (.9938)^{10}$

17. $E[\text{Points}] = 10(1/10) + 5(2/10) + 3(2/10) = 2.6$

18. $.2 = P\left\{\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right\} = P\{Z > 4/\sigma\}$ where Z is a standard normal. But from the normal table $P\{Z < .84) \approx .80$ and so

$$.84 \approx 4/\sigma \text{ or } \sigma \approx 4.76$$

That is, the variance is approximately $(4.76)^2 = 22.66$.

19. Letting $Z = (X - 12)/2$ then Z is a standard normal. Now, $.10 = P\{Z > (c - 12)/2\}$. But from Table 5.1, $P\{Z < 1.28\} = .90$ and so

$$(c - 12)/2 = 1.28 \quad \text{or} \quad c = 14.56$$

20. Let X denote the number in favor. Then X is binomial with mean 65 and standard deviation $\sqrt{65(.35)} \approx 4.77$. Also let Z be a standard normal random variable.

(a) $P\{X \geq 50\} = P\{X \geq 49.5\} = P\{X - 65\}/4.77 \geq -15.5/4.77$
$\approx P\{Z \geq -3.25\} \approx .9994$

(b) $P\{59.5 \leq X \leq 70.5\} \approx P\{-5.5/4.77 \leq Z \leq 5.5/4.77\}$
$= 2P\{Z \leq 1.15\} - 1 \approx .75$

(c) $P\{X \leq 74.5\} \approx P\{Z \leq 9.5/4.77\} \approx .977$

22. (a) $P\{.9000 - .005 < X < .9000 + .005\}$

$$= P\left\{-\frac{.005}{.003} < Z < \frac{.005}{.003}\right\}$$
$$= P\{-1.67 < Z < 1.67\}$$
$$= 2\Phi(1.67) - 1 = .9050.$$

Hence 9.5 percent will be defective (that is each will be defective with probability $1 - .9050 = .0950$).

(b) $P\left\{-\frac{.005}{\sigma} < Z < \frac{.005}{\sigma}\right\} = 2\Phi\left(\frac{.005}{\sigma}\right) - 1 = .99$ when

$$\Phi\left(\frac{.005}{\sigma}\right) = .995 \Rightarrow \frac{.005}{\sigma} = 2.575 \Rightarrow \sigma = .0019\,.$$

23. (a) $$P\{149.5 < X < 200.5\} = P\left\{\frac{149.5-\frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}} < Z < \frac{200.5-\frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}}\right\}$$
$$= \Phi\left(\frac{200.5-166.7}{\sqrt{5000/36}}\right) - \Phi\left(\frac{149.5-166.7}{\sqrt{5000/36}}\right)$$
$$\approx \Phi(2.87) + \Phi(1.46) - 1 = .9258.$$

(b) $$P\{X < 149.5\} = P\left\{Z < \frac{149.5-800(1/5)}{\sqrt{800\frac{1}{5}\frac{4}{5}}}\right\}$$
$$= P\{Z < -.93\}$$
$$= 1 - \Phi(.93) = .1762.$$

24. With C denoting the life of a chip, and ϕ the standard normal distribution function we have

$$P\{C < 1.8 \times 10^6\} = \phi\left(\frac{1.8\times 10^6 - 1.4\times 10^6}{3\times 10^5}\right)$$
$$= \phi(1.33)$$
$$= .9082$$

Thus, if N is the number of the chips whose life is less than 1.8×10^6 then N is a binomial random variable with parameters (100, .9082). Hence,

$$P\{N > 19.5\} \approx 1 - \phi\left(\frac{19.5-90.82}{90.82(.0918)}\right) = 1 - \phi(-24.7) \approx 1$$

25. Let X denote the number of unacceptable items among the next 150 produced. Since X is a binomial random variable with mean 150(.05) = 7.5 and variance 150(.05)(.95) = 7.125, we obtain that, for a standard normal random variable Z.

$$P\{X \le 10\} = P\{X \le 10.5\}$$
$$= P\left\{\frac{X-7.5}{\sqrt{7.125}} \le \frac{10.5-7.5}{\sqrt{7.125}}\right\}$$
$$\approx P\{Z \le 1.1239\}$$
$$= .8695$$

The exact result can be obtained by using the text diskette, and (to four decimal places) is equal to .8678.

27. $$P\{X > 5{,}799.5\} = P\left\{Z > \frac{799.5}{\sqrt{2{,}500}}\right\}$$
$$= P\{Z > 15.99\} = \text{negligible}.$$

28. Let X equal the number of lefthanders. Assuming that X is approximately distributed as a binomial random variable with parameters $n = 200$, $p = .12$, then, with Z being a standard normal random variable,

$$\begin{aligned} P\{X > 19.5\} &= P\left\{\frac{X - 200(.12)}{\sqrt{200(.12)(.88)}} > \frac{19.5 - 200(.12)}{\sqrt{200(.12)(.88)}}\right\} \\ &\approx P\{Z > -.9792\} \\ &\approx .8363 \end{aligned}$$

29. Let s be the initial price of the stock. Then, if X is the number of the 1000 time periods in which the stock increases, then its price at the end is

$$su^X d^{1000\text{-}X} = sd^{1000}\left(\frac{u}{d}\right)^X$$

Hence, in order for the price to be at least 1.3s, we would need that

$$d^{1000}\left(\frac{u}{d}\right)^X > 1.3$$

or

$$X > \frac{\log(1.3) - 1000\log(d)}{\log(u/d)} = 469.2$$

That is, the stock would have to rise in at least 470 time periods. Because X is binomial with parameters 1000, .52, we have

$$\begin{aligned} P\{X > 469.5\} &= P\left\{\frac{X - 1000(.52)}{\sqrt{1000(.52)(.48)}} > \frac{469.5 - 1000(.52)}{\sqrt{1000(.52)(.48)}}\right\} \\ &\approx P\{Z > -3.196\} \\ &\approx .9993 \end{aligned}$$

30. $$\begin{aligned} P\{\text{in black}\} &= \frac{P\{5 \mid \text{black}\}\alpha}{P\{5 \mid \text{black}\}\alpha + P\{5 \mid \text{white}\}(1-\alpha)} \\ &= \frac{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8}\alpha}{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8}\alpha + (1-\alpha)\frac{1}{3\sqrt{2\pi}} e^{-(5-6)^2/18}} \\ &= \frac{\frac{\alpha}{2}e^{-1/8}}{\frac{\alpha}{2}e^{-1/8} + \frac{(1-\alpha)}{3}e^{-1/8}} \end{aligned}$$

α is the value that makes preceding equal 1/2

31. (a) $E\left[|X-a|\right] = \int_a^A (x-a)\frac{dx}{A} + \int_0^a (a-x)\frac{dx}{A} = \frac{A}{2} - \left(a - \frac{a^2}{A}\right)$

$$\frac{d}{da}(\quad) = \frac{2a}{A} - 1 = 0 \Rightarrow a = A/2$$

(b) $E\left[|X-a|\right] = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx$

$$= a(1-e^{-\lambda a}) + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - ae^{-\lambda a}$$

Differentiation yields that the minimum is attained at $\bar{a}$ where

$$e^{-\lambda\bar{a}} = 1/2 \text{ or } \bar{a} = \log 2/\lambda$$

(c) Minimizing a = median of F

32. (a) e^{-1}
(b) $e^{-1/2}$

33. e^{-1}

34. (a) $P\{X > 20\} = e^{-1}$

(b) $P\{X > 30 \mid X > 10\} = \frac{P\{X > 30\}}{P\{X > 10\}} = \frac{1/4}{3/4} = 1/3$

35. (a) $\exp\left[-\int_{40}^{50} \lambda(t)dt\right] = e^{-.35}$

(b) $e^{-1.21}$

36. (a) $1 - F(2) = \exp\left[-\int_0^2 t^3 dt\right] = e^{-4}$

(b) $\exp[-(.4)^4/4] - \exp[-(1.4)^4/4]$

(c) $\exp\left[-\int_1^2 t^3 dt\right] = e^{-15/4}$

37. (a) $P\{|X| > 1/2\} = P\{X > 1/2\} + P\{X < -1/2\} = 1/2$

(b) $P\{|X| \le a\} = P\{-a \le X \le a\} = a$, $0 < a < 1$. Therefore,
$f_{|X|}(a) = 1$, $0 < a < 1$

That is, $|X|$ is uniform on (0, 1).

38. For both roots to be real the discriminant $(4Y)^2 - 44(Y + 2)$ must be ≥ 0. That is, we need that $Y^2 \geq Y + 2$. Now in the interval $0 < Y < 5$.

$$Y^2 \geq Y + 2 \Leftrightarrow Y \geq 2 \text{ and so}$$
$$P\{Y^2 \geq Y + 2\} = P\{Y \geq 2\} = 3/5.$$

39. $F_Y(y) = P\{\log X \leq y\}$
$\quad = P\{X \leq e^y\} = F_X(e^y)$

$$f_Y(y) = f_X(e^y)e^y = e^y e^{-e^y}$$

40. $F_Y(y) = P\{e^X \leq y\}$
$\quad = F_X(\log y)$

$$f_Y(y) = f_X(\log y)\frac{1}{y} = \frac{1}{y},\ 1 < y < e$$

Theoretical Exercises

1. The integration by parts formula $\int u dv = uv - \int v du$ with $dv = -2bxe^{-bx^2}$, $u = -x/2b$ yields that

$$\int_0^\infty x^2 e^{-bx^2} dx = \frac{-xe^{-bx^2}}{2b}\Big|_0^\infty + \frac{1}{2b}\int_0^\infty e^{-bx^2} dx$$

$$= \frac{1}{(2b)^{3/2}}\int_0^\infty e^{-y^2/2} dy \text{ by } y = x\sqrt{2b}$$

$$= \frac{\sqrt{2\pi}}{2}\frac{1}{(2b)^{3/2}} = \frac{\sqrt{\pi}}{4b^{3/2}}$$

where the above uses that $\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-y^2/2} dy = 1/2$. Hence, $a = \frac{4b^{3/2}}{\sqrt{\pi}}$

2. $$\int_0^\infty P\{Y < -y\} dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x) dx\, dy$$

$$= \int_{-\infty}^{0} \int_0^{-x} f_Y(x) dy\, dx = -\int_{-\infty}^{0} x f_Y(x)\, dx$$

Similarly,

$$\int_0^\infty P\{Y > y\} dy = \int_0^\infty x f_Y(x)\, dx$$

Subtracting these equalities gives the result.

4. $E[aX + b] = \int (ax+b) f(x)\, dx = a\int x f(x) dx + b\int f(x) dx$
$= aE[X] + b$

5. $E[X^n] = \int_0^\infty P\{X^n > t\} dt$

$$= \int_0^\infty P\{X^n > x^n\} n x^{n-1} dx \text{ by } t = x^n,\ dt = nx^{n-1}dx$$

$$= \int_0^\infty P\{X > x\} n x^{n-1} dx$$

6. Let X be uniform on (0, 1) and define E_a to be the event that X is unequal to a. Since $\bigcap_a E_a$ is the empty set, it must have probability 0.

7. $SD(aX + b) = \sqrt{\text{Var}(aX+b)} = \sqrt{a^2\sigma^2} = |a|\sigma$

8. Since $0 \le X \le c$, it follows that $X^2 \le cX$. Hence,

$$\begin{aligned}\text{Var}(X) &= E[X^2] - (E[X])^2 \\ &\le E[cX - (E[X])^2 \\ &= cE[X] - (E[X])^2 \\ &= E[X](c - E[X]) \\ &= c^2[\alpha(1-\alpha)] \quad \text{where } \alpha = E[X]/c \\ &\le c^2/4\end{aligned}$$

where the last inequality first uses the hypothesis that $P\{0 \le X \le c\} = 1$ to calculate that $0 \le \alpha \le 1$ and then uses calculus to show that $\underset{0\le\alpha\le1}{\text{maximum}}\ \alpha(1-\alpha) = 1/4$.

9. The final step of parts (a) and (b) use that $-Z$ is also a standard normal random variable.

(a) $P\{Z > x\} = P\{-Z < -x\} = P\{Z < -x\}$

(b) $P\{|Z| > x\} = P\{Z > x\} + P\{Z < -x\} = P\{Z > x\} + P\{-Z > x\}$
$= 2P\{Z > x\}$

(c) $P\{|Z| < x\} = 1 - P\{|Z| > x\} = 1 - 2P\{Z > x\}$ by (b)
$= 1 - 2(1 - P\{Z < x\})$

10. With $c = 1/\left(\sqrt{2\pi}\sigma\right)$ we have

$$f(x) = ce^{-(x-\mu)^2/2\sigma^2}$$
$$f'(x) = -ce^{-(x-\mu)^2/2\sigma^2}(x-\mu)/\sigma^2$$
$$f''(x) = c\sigma^{-4}e^{-(x-\mu)^2/2\sigma^2}(x-\mu)^2 - c\sigma^{-2}e^{-(x-\mu)^2/2\sigma^2}$$

Therefore,

$$f''(\mu+\sigma) = f''(\mu-\sigma) = c\sigma^{-2}e^{-1/2} - c\sigma^{-2}e^{-1/2} = 0$$

11. (a) Integrate

$$E[g'(Z)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g'(x)e^{-x^2/2}dx$$

by parts $\left(dv = g'(x)dx, u = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)$ to obtain the result.

(b) Let $g(z) = z^n$ and apply part (a).

(c) Let $n = 3$ in part (b) to obtain $E[Z^4] = 3E[Z^2] = 3$

12. $E[X^2] = \int_0^\infty P\{X > x\}2x^{2-1}dx = 2\int_0^\infty xe^{-\lambda x}dx = \frac{2}{\lambda}E[X] = 2/\lambda^2$

13. (a) $\dfrac{b+a}{2}$

(b) μ

(c) $1 - e^{-\lambda m} = 1/2$ or $m = \dfrac{1}{\lambda}\log 2$

14. (a) all values in (a, b)

(b) μ

(c) 0

15. $P\{cX < x\} = P\{X < x/c\} = 1 - e^{-\lambda x/c}$

16. $\lambda(t) = \dfrac{f(t)}{\bar{F}(t)} = \dfrac{1/a}{(a-t)/a} = \dfrac{1}{a-t}$, $0 < t < a$

17. If X has distribution function F and density f, then for $a > 0$

$$F_{aX}(t) = P\{aX \le t\} = F(t/a)$$

and

$$f_{ax} = \frac{1}{a} f(t/a)$$

Thus,

$$\lambda_{aX}(t) = \frac{\frac{1}{a} f(t/a)}{1 - F(t/a)} = \frac{1}{a}\lambda_X(t/a).$$

19. $$E[X^k] = \int_0^\infty x^k \lambda e^{-\lambda x} dx = \lambda^{-k}\int_0^\infty \lambda e^{-\lambda x}(\lambda x)^k dx$$
$$= \lambda^{-k}\,\Gamma(k+1) = k!/\lambda^k$$

20. $$E[X^k] = \frac{1}{\Gamma(t)}\int_0^\infty x^k \lambda e^{-\lambda x}(\lambda x)^{t-1}dx$$
$$= \frac{\lambda^{-k}}{\Gamma(t)}\int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{t+k-1}dx$$
$$= \frac{\lambda^{-k}}{\Gamma(t)}\Gamma(t+k)$$

Therefore,

$E[X] = t/\lambda$,
$E[X^2] = = \lambda^{-2}\Gamma(t + 2)/\Gamma(t) = (t + 1)t/\lambda^2$

and thus

$\mathrm{Var}(X) = (t + 1)t/\lambda^2 - t^2/\lambda^2 = t/\lambda^2$

21. $$\Gamma(1/2) = \int_0^\infty e^{-x}x^{-1/2}dx$$
$$= \sqrt{2}\int_0^\infty e^{-y^2/2}dy \text{ by } x = y^2/2, dx = ydy = \sqrt{2x}\ dy$$
$$= 2\sqrt{\pi}\int_0^\infty (2\pi)^{-1/2}e^{-y^2/2}dy$$
$$= 2\sqrt{\pi}P\{Z > 0\} \text{ where } Z \text{ is a standard normal}$$
$$= \sqrt{\pi}$$

22. $$1/\lambda(s) = \int_{x\ge s} \lambda e^{-\lambda x}(\lambda x)^{t-1}dx / \lambda e^{-\lambda s}(\lambda s)^{t-1}$$
$$= \int_{x\ge s} e^{-\lambda(x-s)}(x/s)^{t-1}dx$$
$$= \int_{y\ge 0} e^{-\lambda y}(1+y/s)^{t-1}dy \text{ by letting } y = x - s$$

As the above, equal to the inverse of the hazard rate function, is clearly decreasing in s when $t \ge 1$ and increasing when $t \le 1$ the result follows.

23. $\lambda(s) = c(s - v)^{\beta-1}$, $s > v$ which is clearly increasing when $\beta \ge 1$ and decreasing otherwise.

24. $F(\alpha) = 1 - e^{-1}$

25. Suppose X is Weibull with parameters v, α, β. Then

$$P\left\{\left(\frac{X-v}{\alpha}\right)^{\beta} \le x\right\} = P\left\{\frac{X-v}{\alpha} \le x^{1/\beta}\right\}$$
$$= P\{X \le v + \alpha x^{1/\beta}\}$$
$$= 1 - \exp\{-x\}.$$

26. We use Equation (6.3).

$$E[X] = B(a+1, b)/B(A, b) = \frac{\Gamma(a+1)}{\Gamma(a+b+1)}\frac{\Gamma(a+b)}{\Gamma(a)} = \frac{a}{a+b}$$

$$E[X^2] = B(a+2, b)/B(a, b) = \frac{\Gamma(a+2)}{\Gamma(a+b+2)}\frac{\Gamma(a+b)}{\Gamma(a)} = \frac{(a+1)a}{(a+b+1)(a+b)}$$

Thus,

$$\mathrm{Var}(X) = \frac{(a+1)a}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2} = \frac{ab}{(a+b+1)(a+b)^2}$$

27. $(X - a)/(b - a)$

29. $P\{F(X \le x\} = P\{X \le F^{-1}(x)\}$
$= F(F^{-1}(x))$
$= x$

30. $F_Y(x) = P\{aX + b \le x\}$

$= P\left\{X \le \frac{x-b}{a}\right\}$ when $a > 0$

$= F_X((x-b)/a)$ when $a > 0$.

$f_Y(x) = \dfrac{1}{a} f_X((x-b)/a)$ if $a > 0$.

When $a< 0$, $F_Y(x) = P\left\{X \ge \dfrac{x-b}{a}\right\} = 1 - F_X\left(\dfrac{x-b}{a}\right)$ and so

$f_Y(x) = -\dfrac{1}{a} f_X\left(\dfrac{x-b}{a}\right)$.

31. $F_Y(x) = P\{e^X \le x\}$
$= P\{X \le \log x\}$
$F_X(\log x)$

$f_Y(x) = f_X(\log x)/x$

$$= \frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log x - \mu)^2/2\sigma^2}$$

Chapter 6

Problems

2. (a) $p(0, 0) = \dfrac{8 \cdot 7}{13 \cdot 12} = 14/39$,

$p(0, 1) = p(1, 0) = \dfrac{8 \cdot 5}{13 \cdot 12} = 10/39$

$p(1, 1) = \dfrac{5 \cdot 4}{13 \cdot 12} = 5/39$

(b) $p(0, 0, 0) = \dfrac{8 \cdot 7 \cdot 6}{13 \cdot 12 \cdot 11} = 28/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = \dfrac{8 \cdot 7 \cdot 5}{13 \cdot 12 \cdot 11} = 70/429$

$p(0, 1, 1) = p(1, 0, 1) = p(1, 1, 0) = \dfrac{8 \cdot 5 \cdot 4}{13 \cdot 12 \cdot 11} = 40/429$

$p(1, 1, 1) = \dfrac{5 \cdot 4 \cdot 3}{13 \cdot 12 \cdot 11} = 5/143$

3. (a) $p(0, 0) = (10/13)(9/12) = 15/26$

$p(0, 1) = p(1, 0) = (10/13)(3/12) = 5/26$

$p(1, 1) = (3/13)(2/12) = 1/26$

(b) $p(0, 0, 0) = (10/13)(9/12)(8/11) = 60/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = (10/13)(9/12)(3/11) = 45/286$

$p(i, j, k) = (3/13)(2/12)(10/11) = 5/143 \qquad \text{if } i + j + k = 2$

$p(1, 1, 1) = (3/13)(2/12)(1/11) = 1/286$

4. (a) $p(0, 0) = (8/13)^2$, $p(0, 1) = p(1, 0) = (5/13)(8/13)$, $p(1, 1) = (5/13)^2$

(b) $p(0, 0, 0) = (8/13)^3$

$p(i, j, k) = (8/13)^2(5/13)$ if $i + j + k = 1$

$p(i, j, k) = (8/13)(5/13)^2$ if $i + j + k = 2$

5. $p(0, 0) = (12/13)^3(11/12)^3$

$p(0, 1) = p(1, 0) = (12/13)^3[1 - (11/12)^3]$

$p(1, 1) = (2/13)[(1/13) + (12.13)(1/13)] + (11/13)(2/13)(1/13)$

8. $f_Y(y) = c\int_{-y}^{y} (y^2 - x^2)e^{-y}dx$

$= \frac{4}{3}cy^3e^{-y}$, $-0 < y < \infty$

$\int_0^\infty f_Y(y)dy = 1 \Rightarrow c = 1/8$ and so $f_Y(y) = \frac{y^3e^{-y}}{6}$, $0 < y < \infty$

$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty} (y^2 - x^2)e^{-y}dy$

$= \frac{1}{4}e^{-|x|}(1+|x|)$ upon using $-\int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$

9. (b) $f_X(x) = \frac{6}{7}\int_0^2 \left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}(2x^2 + x)$

(c) $P\{X > Y\} = \frac{6}{7}\int_0^1\int_0^x \left(x^2 + \frac{xy}{2}dydx\right) = \frac{15}{56}$

(d) $P\{Y > 1/2 \mid X < 1/2\} = P\{Y > 1/2, X < 1/2\}/P\{X < 1/2\}$

$$= \frac{\int_{1/2}^{2}\int_0^{1/2}\left(x^2 + \frac{xy}{2}dxdy\right)}{\int_0^{1/2}(2x^2 + x)dx}$$

10. (a) $f_X(x) = e^{-x}$, $f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

$P\{X < Y\} = 1/2$

(b) $P\{X < a\} = 1 - e^{-a}$

11. $\frac{5!}{2!1!2!}(.45)^2(.15)(.40)^2$

12. $e^{-5} + 5e^{-5} + \frac{5^2}{2!}e^{-5} + \frac{5^3}{3!}e^{-5}$

14. Let X and Y denoted respectively the locations of the ambulance and the accident of the moment the accident occurs.

$$P\{\,|Y-X|\,<a\} = P\{Y<X<Y+a\} + P\{X<Y<X+a\}$$

$$= \frac{2}{L^2}\int_0^L \int_y^{\min(y+a,L)} dxdy$$

$$= \frac{2}{L^2}\left[\int_0^{L-a}\int_y^{y+a} dxdy + \int_{L-a}^{L}\int_y^{L} dxdy\right]$$

$$= 1 - \frac{L-a}{L} + \frac{a}{L^2}(L-a) = \frac{a}{L}\left(2-\frac{a}{L}\right),\ \ 0<a<L$$

15. (a) $1 = \int\int f(x,y)dydx = \int\int_{(x,y)\in R} c\ dydx\ = cA(R)$

where $A(R)$ is the area of the region R.

(b) $f(x, y) = 1/4,\ -1 \le x, y \le 1$
$= f(x)f(y)$
where $f(v) = 1/2,\ -1 \le v \le 1$.

(c) $P\{X^2 + Y^2 \le 1\} = \dfrac{1}{4}\int\int_c dydx\ =$ (area of circle)/4 = $\pi/4$.

16. (a) $A = \cup A_i$,
(b) yes
(c) $P(A) = \sum P(A_i)\ = n(1/2)^{n-1}$

17. $\dfrac{1}{3}$ since each of the 3 points is equally likely to be the middle one.

18. $P\{Y-X>L/3\} = \int\int_{y-x>L/3} \dfrac{4}{L^2}dydx$

$$\frac{L}{2}<y<L$$
$$0<x<\frac{L}{2}$$

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dydx + \int_{L/6}^{L/2}\int_{x+L/3}^{L} dydx\right]$$

$$= \frac{4}{L^2}\left[\frac{L^2}{12}+\frac{5L^2}{24}-\frac{7L^2}{72}\right] = 7/9$$

19. $\int_0^1 \int_0^x \frac{1}{x}\, dydx = \int_0^1 dx = 1$

(a) $\int_y^1 \frac{1}{x} dx = -\ln(y)$, $0 < y < 1$

(b) $\int_0^x \frac{1}{x} dy = 1$, $0 < y < 1$

(c) $\frac{1}{2}$

(d) Integrating by parts gives that

$$\int_0^1 y \ \ln(y) dy = -1 - \int_0^1 (y \ln(y) - y) dy$$

yielding the result

$$E[Y] = -\int_0^1 y \ln(y) dy = 1/4$$

20. (a) yes: $f_X(x) = xe^{-x}$, $f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

(b) no: $f_X(x) = \int_x^1 f(x, y) dy = 2(1 - x)$, $0 < x < 1$

$$f_Y(y) = \int_0^y f(x, y) dx = 2y, \ 0 < y < 1$$

21. (a) We must show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$. Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy &= \int_0^1 \int_0^{1-y} 24xy \ dxdy \\
&= \int_0^1 12y(1-y)^2 dy \\
&= \int_0^1 12(y - 2y^2 + y^3) dy \\
&= 12(1/2 - 2/3 + 1/4) = 1
\end{aligned}$$

(b)
$$\begin{aligned}
E[X] &= \int_0^1 x f_X(x) dx \\
&= \int_0^1 x \int_0^{1-x} 24xy \ dydx \\
&= \int_0^1 12x^2(1-x)^2 dx = 2/5
\end{aligned}$$

(c) 2/5

22. (a) No, since the joint density does not factor.

(b) $f_X(x) = \int_0^1 (x+y)dy = x + 1/2,\ 0 < x < 1.$

(c) $P\{X + Y < 1\} = \int_0^1 \int_0^{1-x} (x+y)dydx$

$= \int_0^1 [x(1-x) + (1-x)^2/2]dx = 1/3$

23. (a) yes

$$f_X(x) = 12x(1-x)\int_0^1 ydy = 6x(1-x),\ 0 < x < 1$$

$$f_Y(y) = 12y\int_0^1 x(1-x)dx = 2y,\ 0 < y < 1$$

(b) $E[X] = \int_0^1 6x^2(1-x)dx = 1/2$

(c) $E[Y] = \int_0^1 2y^2 dy = 2/3$

(d) $\text{Var}(X) = \int_0^1 6x^3(1-x)dx - 1/4 = 1/20$

(e) $\text{Var}(Y) = \int_0^1 2y^3 dy - 4/9 = 1/18$

24. $P\{N = n\} = p_0^{n-1}(1-p_0)$

(b) $P\{X = j\} = p_j/(1-p_0)$

(c) $P\{N = n, X = j\} = p_0^{n-1}p_j$

25. $\dfrac{e^{-1}}{i!}$ by the Poisson approximation to the binomial.

26. (a) $F_{A,B,C}(a, b, c) = abc \quad 0 < a, b, c < 1$

(b) The roots will be real if $B^2 \geq 4AC$. Now

$$P\{AC \leq x\} = \iint\limits_{\substack{c \leq x/a \\ 0 \leq a \leq 1 \\ 0 \leq c \leq 1}} dadc = \int_0^x \int_0^1 dcda + \int_x^1 \int_0^{x/a} dcda$$

$= x - x \log x.$

Hence, $F_{AC}(x) = x - x \log x$ and so

$f_{AC}(x) = -\log x,\ 0 < x < 1$

$$P\{B^2/4 \geq AC\} = -\int_0^1 \int_0^{b^2/4} \log x dx db$$
$$= \int_0^1 \left[\frac{b^2}{4} - \frac{b^2}{4}\log(b^2/4)\right] db$$
$$= \frac{\log 2}{6} + \frac{5}{36}$$

where the above uses the identity

$$\int x^2 \log x dx = \frac{x^3 \log x}{3} - \frac{x^3}{9}.$$

27. $$P\{X_1/X_2 < a\} = \int_0^\infty \int_0^{ay} \lambda_1 e^{-\lambda} 1^x \lambda_2 e^{-\lambda_2 y} dx dy$$
$$= \int_0^\infty \left(1 - e^{-\lambda_1 ay}\right) \lambda_2 e^{-\lambda_2 y} dy$$
$$= 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 a} = \frac{\lambda_1 a}{a\lambda_1 + \lambda_2}$$

$$P\{X_1/X_2 < 1\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

28. (a) $\frac{1}{2}e^{-t}$, since e^{-t} is the probability that AJ is still in service when MJ arrives, and 1/2 is the conditional probability that MJ then finishes first.

(b) Using that the time at which MJ finishes is gamma with parameters 2, 1 yields the result: $1 - 3e^{-2}$

29. (a) If $W = X_1 + X_2$ is the sales over the next two weeks, then W is normal with mean 4,400 and standard deviation $\sqrt{2(230)^2}$ = 325.27. Hence, with Z being a standard normal, we have

$$P\{W > 5000\} = P\left\{Z > \frac{5000 - 4400}{325.27}\right\}$$
$$= P\{Z > 1.8446\} = .0326$$

(b) $P\{X > 2000\} = P\{Z > (2000 - 2200)/230\}$
$= P\{Z > -.87\} = P\{Z < .87\} = .8078$

Hence, the probability that weekly sales exceeds 2000 in at least 2 of the next 3 weeks $p^3 + 3p^2(1 - p)$ where $p = .8078$.

We have assumed that the weekly sales are independent.

30. Let X denote Jill's score and let Y be Jack's score. Also, let Z denote a standard normal random variable.

(a)
$$\begin{aligned} P\{Y > X\} &= P\{Y - X > 0\} \\ &\approx P\{Y - X > .5\} \\ &= P\left\{\frac{Y - X - (160 - 170)}{\sqrt{(20)^2 + (15)^2}} > \frac{.5 - (160 - 170)}{\sqrt{(20)^2 + (15)^2}}\right\} \\ &\approx P\{Z > .42\} \approx .3372 \end{aligned}$$

(b)
$$\begin{aligned} P\{X + Y > 350\} &= P\{X + Y > 350.5\} \\ &= P\left\{\frac{X + Y - 330}{\sqrt{(20)^2 + (15)^2}} > \frac{20.5}{\sqrt{(20)^2 + (15)^2}}\right\} \\ &\approx P\{Z > .82\} \approx .2061 \end{aligned}$$

31. Let X and Y denote, respectively, the number of males and females in the sample that never eat breakfast. Since

$$E[X] = 50.4,\ \mathrm{Var}(X) = 37.6992,\ \ E[Y] = 47.2,\ \mathrm{Var}(Y) = 36.0608$$

it follows from the normal approximation to the binomial that X is approximately distributed as a normal random variable with mean 50.4 and variance 37.6992, and that Y is approximately distributed as a normal random variable with mean 47.2 and variance 36.0608. Let Z be a standard normal random variable.

(a)
$$\begin{aligned} P\{X + Y \geq 110\} &= P\{X + Y \geq 109.5\} \\ &= P\left\{\frac{X + Y - 97.6}{\sqrt{73.76}} \geq \frac{109.5 - 97.6}{\sqrt{73.76}}\right\} \\ &\approx P\{Z > 1.3856\} \approx .0829 \end{aligned}$$

(b)
$$\begin{aligned} P\{Y \geq X\} &= P\{Y - X \geq -.5\} \\ &= P\left\{\frac{Y - X - (-3.2)}{\sqrt{73.76}} \geq \frac{-.5 - (-3.2)}{\sqrt{73.76}}\right\} \\ &\approx P\{Z \geq .3144\} \approx .3766 \end{aligned}$$

32. (a) e^{-2}

(b) $1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$

The number of typographical errors on each page should approximately be Poisson distributed and the sum of independent Poisson random variables is also a Poisson random variable.

33. (a) $1 - e^{-2.2} - 2.2e^{-2.2} - e^{-2.2}(2.2)^2/2!$

(b) $1 - \sum_{i=0}^{4} e^{-4.4}(4.4)^i / i!$, (c) $1 - \sum_{i=0}^{5} e^{-6.6}(6.6)^i / i!$

The reasoning is the same as in Problem 26.

34. Use the distribution of the sum of independent geometric random variables to obtain the result: $4(.7)^{12} - 3(.6)^{12}$

35. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 5/13 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) same as in (a)

36. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = 2/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = 3/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$

37. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = p(1, 1)/[1 - (12/13)^3] = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = p(1, 0)/(12/13)^3 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$
where $p(1, 1)$ and $p(1, 0)$ are given in the solution to Problem 5.

38. (a) $P\{X = j, Y = i\} = \frac{1}{5}\frac{1}{j}$, $j = 1, \ldots, j$, $i = 1, \ldots, j$

(b) $P\{X = j \mid Y = i\} = \frac{1}{5j} \Big/ \sum_{k=i}^{5} 1/5k = \frac{1}{j} \Big/ \sum_{k=i}^{5} 1/k,\ 5 \ge j \ge i.$

(c) No.

39. For $j = i$: $P\{Y = i \mid X = i\} = \dfrac{P\{Y = i, X = i\}}{P\{X = i\}} = \dfrac{1}{36P\{X = i\}}$

For $j < i$: $P\{Y = j \mid X = i\} = \dfrac{2}{36P\{X = i\}}$

Hence

$$1 = \sum_{j=1}^{i} P\{Y = j \mid X = i\} = \frac{2(i-1)}{36P\{X = i\}} + \frac{1}{36P\{X = i\}}$$

and so, $P\{X = i\} = \dfrac{2i-1}{36}$ and

$$P\{Y = j \mid X = i\} = \begin{cases} \dfrac{1}{2i-i} & j = i \\ \dfrac{2}{2i-1} & j < i \end{cases}$$

41. (a) $f_{X|Y}(x \mid y) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dx} = (y + 1)^2 xe^{-x(y+1)}$, $0 < x$

(b) $f_{Y|X}(y \mid x) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dy} = xe^{-xy}$, $0 < y$

$$\begin{aligned} P\{XY < a\} &= \int_0^\infty \int_0^{a/x} xe^{-x(y+1)}dydx \\ &= \int_0^\infty (1 - e^{-a})e^{-x}dx = 1 - e^{-a} \end{aligned}$$

$f_{XY}(a) = e^{-a}$, $0 < a$

42. $$\begin{aligned} f_{Y|X}(y \mid x) &= \frac{(x^2 - y^2)e^{-x}}{\int_{-x}^{x} (x^2 - y^2)e^{-x}dx} \\ &= \frac{3}{4x^3}(x^2 - y^2), \ -x < y < x \end{aligned}$$

$$\begin{aligned} F_{Y|X}(y \mid x) &= \frac{3}{4x^3}\int_{-x}^{y} (x^2 - y^2)dy \\ &= \frac{3}{4x^3}(x^2 y - y^3/3 + 2x^3/3), \ -x < y < x \end{aligned}$$

43. $$f(\lambda \mid n) = \frac{P\{N = n \mid \lambda\}g(\lambda)}{P\{N = n\}}$$
$$= C_1 e^{-\lambda}\lambda^n \alpha e^{-\alpha\lambda}(\alpha\lambda)^{s-1}$$
$$= C_2 e^{-(\alpha+1)\lambda}\lambda^{n+s-1}$$

where C_1 and C_2 do not depend on λ. But from the preceding we can conclude that the conditional density is the gamma density with parameters $\alpha + 1$ and $n + s$. The conditional expected number of accidents that the insured will have next year is just the expectation of this distribution, and is thus equal to $(n + s)/(\alpha + 1)$.

44. $P\{X_1 > X_2 + X_3\} + P\{X_2 > X_1 + X_3\} + P\{X_3 > X_1 + X_2\}$

$$= 3P\{X_1 > X_2 + X_3\}$$

$$= 3\underset{\substack{x_1 > x_2 > x_3 \\ 0 \le x_i \le 1 \\ i = 1,2,3}}{\iiint} dx_1 dx_2 dx_3 \qquad \text{(take } a = 0,\ b = 1)$$

$$= 3\int_0^1 \int_0^{1-x_3} \int_{x_2+x_3}^1 dx_1 dx_2 dx_3 = 3\int_0^1 \int_0^{1-x_3} (1 - x_2 - x_3)dx_2 dx_3$$

$$= 3\int_0^1 \frac{(1-x_3)^2}{2} dx_3 = 1/2\,.$$

45. $$f_{X_{(3)}}(x) = \frac{5!}{2!2!}\left[\int_0^x xe^{-x}dx\right]^2 xe^{-x}\left[\int_x^\infty xe^{-x}dx\right]^2$$
$$= 30(x + 1)^2 e^{-2x} xe^{-x}[1 - e^{-x}(x + 1)]^2$$

46. $$\left(\frac{L - 2d}{L}\right)^3$$

47. $$\int_{1/4}^{3/4} f_{X_{(3)}}(x)dx = \frac{5!}{2!2!}\int_{1/4}^{3/4} x^2(1-x)^2\,dx$$

48. (a) $P\{\min X_i \le a\} = 1 - P\{\min X_i > a\} = 1 - \prod P\{X_i > a\} = 1 - e^{-5\lambda a}$

(b) $P\{\max X_i \le a\} = \prod P\{X_i \le a\} = (1 - e^{-\lambda a})^5$

49. It is uniform on $(s_{n-1}, 1)$

50. Start with

$$f_{z_1,z_2}(z_1,z_2) = \frac{1}{2\pi}e^{-\left(z_1^2+z_2^2)/2\right)}$$

Making the transformation – using that its Jacobian is 1 – yields that

$$f_{X,Y}(x,y) = f_{Z_1Z_2}(x, y-x) = \frac{1}{2\pi}e^{-\left(x^2+(y-x)^2\right)/2}$$

51. $f_{X_{(1)},X_{(4)}}(x,y) = \frac{4!}{2!}2x\left(\int_X^Y 2zdz\right)^2 2y\,,\ x < y$

$= 48xy(y^2 - x^2).$

$$P(X_{(4)} - X_{(1)} \le a\} = \int_0^{1-a}\int_0^{a+x} 48xy(y^2 - x^2)dydx$$
$$+ \int_{1-a}^{1}\int_0^{1} 48xy(y^2 - x^2)dydx$$

52. $f_{R_1}(r,\theta) = \frac{r}{\pi} = 2r\frac{1}{2\pi},\ 0 \le r \le 1,\ 0 \le \theta < 2\pi.$

Hence, R and θ are independent with θ being uniformly distributed on $(0, 2\pi)$ and R having density $f_R(\mathrm{r}) = 2r$, $0 < r < 1$.

53. $f_{R,\theta}(r,\theta) = r,\quad 0 < r\sin\theta < 1,\quad 0 < r\cos\theta < 1,\quad 0 < \theta < \pi/2,\quad 0 < r < \sqrt{2}$

54. $J = \begin{vmatrix} \frac{1}{2}x^{-1/2}\cos u\sqrt{2} & \frac{1}{2}z^{-1/2}\sin u\sqrt{2} \\ -\sqrt{2z}\sin u & \sqrt{2z}\cos u \end{vmatrix} = \cos^2 u + \sin^2 u = 1$

$f_{u,z}(u, z)$ - $\frac{1}{2\pi}e^{-z}$. But $x^2 + y^2 = 2z$ so

$f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$

55. (a) If $u = xy$, $v = xy$, then $J = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -2\frac{x}{y}$ and

$y = \sqrt{u/v}$, $x = \sqrt{vu}$. Hence,

(b) $f_{u,v}(u, v) = \dfrac{1}{2v} f_{X,Y}\left(\sqrt{vy}, \sqrt{u/v}\right) = \dfrac{1}{2vu^2}$, $u \geq 1$, $\dfrac{1}{u} < v < u$

$$f_u(u) = \int_{1/u}^{u} \frac{1}{2vu^2}\,dv = \frac{1}{u^2}\log u\,,\ u \geq 1.$$

For $v > 1$

$$f_V(v) = \int_{v}^{\infty} \frac{1}{2vu^2}\,du = \frac{1}{2v^2}\,,\ v > 1$$

For $v < 1$

$$f_V(v) = \int_{1/2}^{\infty} \frac{1}{2vu^2}\,du = \frac{1}{2},\ 0 < v < 1.$$

56. (a) $u = x + y$, $v = x/y \Rightarrow y = \dfrac{u}{v+1}$, $x = \dfrac{uv}{v+1}$

$$J = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = -\left(\frac{x}{y^2} + \frac{1}{y}\right) = \frac{-1}{y^2}(x + y) = \frac{-(v+1)^2}{u}$$

$$f_{u,v}(u, v) = \frac{u}{(v+1)^2}\,,\ 0 < uv < 1 + v,\ 0 < u < 1 + v$$

58. $y_1 = x_1 + x_2$, $y_2 = e^{x_1}$. $J = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = -e^{x_1} = -y_2$

$x_1 = \log y_2$, $x_2 = y_1 - \log y_2$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{y_2}\lambda e^{-\lambda \log y_2}\lambda e^{-\lambda(y_1 - \log y_2)}$$
$$= \frac{1}{y_2}\lambda^2 e^{-\lambda y_1}\,,\ 1 \leq y_2,\ y_1 \geq \log y_2$$

59. $u = x + y,\ v = x + z,\ w = y + z \Rightarrow z = \dfrac{v+w-u}{2}, x = \dfrac{v-w+u}{2}, y = \dfrac{w-v+u}{2}$

$$J = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

$$f(u, v, w) = \frac{1}{2}\exp\left\{-\frac{1}{2}(u+v+w)\right\}, u + v > w,\ u + w > v,\ v + w + u$$

60. $P(Y_j = i_j, j = 1, \ldots, k + 1\} = P\{Y_j = i_j, j = 1, \ldots, k\}\ P(Y_{k+1} = i_{k+1} \mid Y_j = i_j, j = 1, \ldots, k\}$

$$= \frac{k!(n-k)!}{n!} P\{n+1-\sum_{i=1}^{k} Y_i = i_{k+1} \big| Y_j = i_j, j = 1,\ldots,k\}$$

$$= \begin{cases} k!(n-k)!/n!, & \text{if } \sum_{j=1}^{k+1} i_j = n+1 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the joint mass function is symmetric, which proves the result.

61. The joint mass function is

$$P\{X_i = x_i, i = 1, \ldots, n\} = 1/\binom{n}{k}, x_i \in \{0, 1\}, i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i = k$$

As this is symmetric in $x_1, \ldots, x_n$ the result follows.

Theoretical Exercises

1. $$\begin{aligned} P\{X \le a_2,\ Y \le b_2\} = &P\{a_1 < X \le a_2,\ b_1 < Y \le b_2\} \\ &+ P\{X \le a_1,\ b_1 < Y \le b_2\} \\ &+ P\{a_1 < X \le a_2,\ Y \le b_1\} \\ &+ P\{X \le a_1,\ Y \le b_1\}. \end{aligned}$$

The above following as the left hand event is the union of the 4 mutually exclusive right hand events. Also,

$$\begin{aligned} P\{X \le a_1, Y \le b_2\} = &P\{X \le a_1,\ b_1 < Y \le b_2\} \\ &+ P\{X \le a_1,\ Y \le b_1\} \end{aligned}$$

and similarly,

$$\begin{aligned} P\{X \le a_2, Y \le b_1\} = &P\{a_1 \le X \le a_2, < Y \le b_1\} \\ &+ P\{X \le a_1,\ Y \le b_1\}. \end{aligned}$$

Hence, from the above

$$\begin{aligned} F(a_2, b_2) = &P\{a_1 < X \le a_2, b_1 < Y \le b_2\} + F(a_1, b_2) - F(a_1, b_1) \\ &+ F(a_2, b_1) - F(a_1, b_1) + F(a_1, b_1). \end{aligned}$$

2. Let X_i denote the number of type i events, i= 1, …, n.

$$\begin{aligned} P\{X_1 = r_1, \ldots, X_n = r_n\} &= P\left\{X_1 = r_1, \ldots, X_n = r_n \,\middle|\, \sum_{`1}^{n} r_i \text{ events}\right\} \\ &\quad \times e^{-\lambda}\lambda^{\sum_1^n r_i} \Big/ \left(\sum_1^n r_i\right)! \\ &= \frac{\left(\sum_1^n r_i\right)!}{r_1!\ldots r_n!} P_1^{r_1}\ldots p_n^{r_n} \frac{e^{-\lambda}\lambda^{\sum_1^n r_i}}{\left(\sum_1^n r_i\right)!} \\ &= \prod_{i=1}^{n} e^{-\lambda P_i} (\lambda_{p_i})^{r_i} \big/ r_i! \end{aligned}$$

3. Throw a needle on a table, ruled with equidistant parallel lines a distance D apart, a large number of times. Let L, $L < D$, denote the length of the needle. Now estimate π by $\dfrac{2L}{fD}$ where f is the fraction of times the needle intersects one of the lines.

5. (a) For $a > 0$

$$F_Z(a) = P\{X \le aY\}$$
$$= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)dxdy$$
$$= \int_0^{\infty} F_X(ay) f_Y(y)dy$$
$$f_Z(a) = \int_0^{\infty} f_X(ay) y f_Y(y)dy$$

(b)
$$F_Z(a) = P\{XY < a\}$$
$$= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)dxdy$$
$$= \int_0^{\infty} F_X(a/y) f_Y(y)dy$$
$$f_Z(a) = \int_0^{\infty} f_X(a/y)\frac{1}{y} f_Y(y)dy$$

If X is exponential with rate λ and Y is exponential with rate μ then (a) and (b) reduce to

(a) $F_Z(a) = \int_0^{\lambda} \lambda e^{-\lambda ay} y \mu e^{-\mu y} dy$

(b) $F_Z(a) = \int_0^{\infty} \lambda e^{-\lambda a/y} \frac{1}{y} \mu e^{-\mu y} dy$

6.

$$F_{X+Y}(t) = \int_{x+y\le t} f_{X,Y}(x,y)dydx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{t-x} f_{X,Y}(x,y)dydx$$

Differentiation yields that

$$f_{X+Y}(t) = \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{t-x} f_{X,Y}(x,y)dydx$$
$$= \int_{-\infty}^{\infty}\frac{d}{dt}\int_{-\infty}^{t-x} f_{X,Y}(x,y)dydx$$
$$= \int_{-\infty}^{\infty} f_{X,Y}(x,t-x)dx$$

7. (a) $P\{cX \le a\} = P\{X \le a/c\}$ and differentiation yields

$$f_{cX}(a) = \frac{1}{c} f_X(a/c) = \frac{\lambda}{c} e^{-\lambda a/c} (\lambda a/c)^{t-1} \Gamma(t) .$$

Hence, cX is gamma with parameters $(t, \lambda/c)$.

(b) A chi-squared random variable with $2n$ degrees of freedom can be regarded as being the sum of n independent chi-square random variables each with 2 degrees of freedom (which by Example is equivalent to an exponential random variable with parameter λ). Hence by Proposition X_{2n}^2 is a gamma random variable with parameters $(n, 1/2)$ and the result now follows from part (a).

8. (a) $P\{W \le t\} = 1 - P\{W > t\} = 1 - P\{X > t, Y > t\} = 1 - [1 - F_X(t)]\,[1 - F_Y(t)]$

(b) $f_W(t) = f_X(t)[1 - F_Y(t)] + f_Y(t)\,[1 - F_X(t)]$

Dividing by $[1 - F_X(t)][1 - F_Y(t)]$ now yields

$$\lambda_W(t) = f_X(t)/[1 - F_X(t)] + f_Y(t)/[1 - F_Y(t)] = \lambda_X(t) + \lambda_Y(t)$$

9. $$\begin{aligned} P\{\min(X_1, \ldots, X_n) > t\} &= P\{X_1 > t, \ldots, X_n > t\} \\ &= e^{-\lambda t} \ldots e^{-\lambda t} = e^{-n\lambda t} \end{aligned}$$

thus showing that the minimum is exponential with rate $n\lambda$.

10. If we let X_i denote the time between the i^{th} and $(i + 1)^{\text{st}}$ failure, $i = 0, \ldots, n - 2$, then it follows from Exercise 9 that the X_i are independent exponentials with rate 2λ. Hence, $\sum_{i=0}^{n-2} X_i$ the amount of time the light can operate is gamma distributed with parameters $(n - 1, 2\lambda)$.

11. $$\begin{aligned} I &= \underset{x_1 < x_2 > x_3 < x_4 > x_5}{\int\!\!\int\!\!\int\!\!\int\!\!\int} f(x_1) \ldots f(x_5) dx_1 \ldots dx_5 \\ &= \underset{\substack{u_1 < u_2 > u_3 < u_4 > u_5 \\ 0 < u_i < 1}}{\int\!\!\int\!\!\int\!\!\int\!\!\int} du_1 \ldots du_5 \qquad \text{by } u_i = F(x_i),\ i = 1, \ldots, 5 \\ &= \int\!\!\int\!\!\int\!\!\int u_2 du_2 \ldots du_5 \\ &= \int\!\!\int\!\!\int (1 - u_3^2)/2 \quad du_3 \ldots \\ &= \int\!\!\int [u_4 - u_4^3/3]/2 du_4 du_5 \\ &= \int_0^1 [u^2 - u^4/3]/2 du = 2/15 \end{aligned}$$

12. Assume that the joint density factors as shown, and let

$$C_i = \int_{-\infty}^{\infty} g_i(x)dx,\ \ i = 1, \ldots, n$$

Since the n-fold integral of the joint density function is equal to 1, we obtain that

$$1 = \prod_{i=1}^{n} C_i$$

Integrating the joint density over all x_i except x_j gives that

$$f_{X_j}(x_j) = g_j(x_j)\prod_{i \neq j} C_i = g_j(x_j)/C_j$$

If follows from the preceding that

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n} f_{X_j}(x_j)$$

which shows that the random variables are independent.

13. No. Let $X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & -- \end{cases}$. Then

$$f_{X|X_1}, \ldots, X_{n+m}(x \mid x_1, \ldots, x_{n+m}) = \frac{P\{x_1, \ldots, x_{n+m} \mid X = x\}}{P\{x_1, \ldots, x_{n+m}\}} f_X(x)$$
$$= cx^{\sum x_i}(1-x)^{n+m-\sum x_i}$$

and so given $\sum_{1}^{n+m} X_i = n$ the conditional density is still beta with parameters $n + 1$, $m + 1$.

14. $P\{X = i \mid X + Y = n\} = P\{X = i, Y = n - i\}/P\{X + Y = n\}$

$$= \frac{p(1-p)^{i-1} p(1-p)^{n-i-1}}{\binom{n-1}{1} p^2(1-p)^{n-2}} = \frac{1}{n-1}$$

15. Let X denote the trial number of the kth success, and let $s, s, f, f, s, \ldots, f$ be an outcome of the first $n - 1$ trials that contains a total of $k - 1$ successes and $n - k$ failures. Using that X is a negative binomial random, we have

$$P\left(s, s, f, f, s, \ldots, f \middle| X = n\right) = \frac{P(s, s, f, f, s, \ldots, f, X = n)}{P\{X = n\}}$$
$$= \frac{P(s, s, f, f, s, \ldots, f, s)}{\binom{n-1}{k-1} p^k (1-p)^{n-k}}$$
$$= \frac{p^k (1-p)^{n-k}}{\binom{n-1}{k-1} p^k (1-p)^{n-k}}$$
$$= \frac{1}{\binom{n-1}{k-1}}$$

and the result if proven.

16. $P\{X = k \mid X + Y = m\} = \dfrac{P\{X = k, X + Y = m\}}{P\{X + Y = m\}}$

$$= \frac{P\{X = k, Y = m - k\}}{P\{X + Y = m\}}$$
$$= \frac{\binom{n}{k} p^k (1-p)^{n-k} \binom{n}{m-k} p^{m-k} (1-p)^{n-m+k}}{\binom{2n}{m} p^m (1-p)^{2n-m}}$$
$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

17. $P(X = n, Y = m) = \sum_i P(X = n, Y = m \mid X_2 = i) P(X_2 = i)$

$$= e^{-(\lambda_1 + \lambda_2 + \lambda_3)} \sum_{i=0}^{\min(n,m)} \frac{\lambda_1^{n-i}}{(n-1)!} \frac{\lambda_3^{m-i}}{(m-i)!} \frac{\lambda_2^i}{i!}$$

18. Starting with

$$p(i|j) = \frac{P(X = i, Y = j)}{P(Y = j)}$$

$$q(j|i) = \frac{P(X = i, Y = j)}{P(X = i)}$$

we see that

$$\frac{p(i|j)}{q(j|i)} = \frac{P(X = i)}{P(Y = j)}$$

which gives that

$$\sum_i \frac{p(i|j)}{q(j|i)} = \frac{1}{P(Y = j)}$$

and the result follows.

19. (a) $P\{X_1 > X_2 \mid X_1 > X_3\} = \dfrac{P\{X_1 = \max(X_1, X_2, X_3)\}}{P\{X_1 > X_3\}} = \dfrac{1/3}{1/2} = 2/3$

(b) $P\{X_1 > X_2 \mid X_1 < X_3\} = \dfrac{P\{X_3 > X_1 > X_2\}}{P\{X_1 < X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(c) $P\{X_1 > X_2 \mid X_2 > X_3\} = \dfrac{P\{X_1 > X_2 > X_3\}}{P\{X_2 > X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(d) $P\{X_1 > X_2 \mid X_2 < X_3\} = \dfrac{P\{X_2 = \min(X_1, X_2, X_3)\}}{P\{X_2 < X_3\}} = \dfrac{1/3}{1/2} = 2/3$

20. $$\begin{aligned} P\{U > s \mid U > a\} &= P\{U > s\}/P\{U > a\} \\ &= \frac{1-s}{1-a},\ a < s < 1 \end{aligned}$$

$$\begin{aligned} P\{U < s \mid U < a\} &= P\{U < s\}/P\{U < a\} \\ &= s/a,\ 0 < s < a \end{aligned}$$

Hence, $U \mid U > a$ is uniform on $(a, 1)$, whereas $U \mid U < a$ is uniform over $(0, a)$.

21. $$\begin{aligned} f_{W|N}(w \mid n) &= \frac{P\{N = n \mid W = w\} f_W(w)}{P\{N = n\}} \\ &= Ce^{-w}\frac{w^n}{n!}\beta e^{-\beta w}(\beta w)^{t-1} \\ &= C_1 e^{-(\beta+1)w} w^{n+t-1} \end{aligned}$$

where C and C_1 do not depend on w. Hence, given $N = n$, W is gamma with parameters $(n + t, \beta + 1)$.

22. $$f_{W|X_i,i=1,\ldots,n}(w|x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_n|w)f_w(w)}{f(x_1,\ldots,x_n)}$$
$$= C\prod_{\mathrm{i=1}}^{\mathrm{n}} we^{-wx_i}e^{-\beta w}(\beta w)^{t-1}$$
$$= Ke^{-w\left(\beta+\sum_1^n x_i\right)}w^{n+t-1}$$

23. Let X_{ij} denote the element in row i, column j.

$P\{X_{ij}$ is s saddle point$\}$

$$= P\left\{\min_{k=1,\ldots,m} X_{ik} > \max_{k\neq i} X_{kj}, X_{ij} = \min_k X_{ik}\right\}$$

$$= P\left\{\min_k X_{ik} > \max_{k\neq i} X_{kj}\right\}P\left\{X_{ij} = \min_k X_{ik}\right\}$$

where the last equality follows as the events that every element in the i^{th} row is greater than all elements in the j^{th} column excluding X_{ij} is clearly independent of the event that X_{ij} is the smallest element in row i. Now each size ordering of the $n + m - 1$ elements under consideration is equally likely and so the probability that the m smallest are the ones in row i is $1\Big/\binom{n+m-1}{m}$. Hence

$$P\{X_{ij} \text{ is a saddlepoint}\} = \frac{1}{\binom{n+m-1}{m}}\frac{1}{m} = \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

and so

$$P\{\text{there is a saddlepoint}\} = P\left(\bigcup_{i,j}\{X_{ij} \text{ is a saddlepoint}\}\right)$$
$$= \sum_{\mathrm{i,j}} P\{X_{ij} \text{ is a saddlepoint}\}$$
$$= \frac{m!n!}{(n+m-1)!}$$

24. For $0 < x < 1$

$$P([X] = n, X - [X] < x) = P(n < X < n + x) = e^{-n\lambda} - e^{-(n+x)\lambda} = e^{-n\lambda}(1 - e^{-x\lambda})$$

Because the joint distribution factors, they are independent. $[X] + 1$ has a geometric distribution with parameter $p = 1 - e^{-\lambda}$ and $x - [X]$ is distributed as an exponential with rate λ conditioned to be less than 1.

25. Let $Y = \max(X_1, \ldots, X_n)$, $Z = \min(X_1, \ldots, X_n)$

$$P\{Y \le x\} = P\{X_i \le x, i = 1, \ldots, n\} = \prod_1^n P\{X_i \le x\} = F^n(x)$$

$$P\{Z > x\} = P\{X_i > x, i = 1, \ldots, n\} = \prod_1^n P\{X_i > x\} = [1 - F(x)]^n.$$

26. (a) Let $d = D/L$. Then the desired probability is

$$n! \int_0^{1-(n-1)d} \int_{x_1+d}^{1-(n-2)d} \cdots \int_{x_{n-3}+d}^{1-2d} \int_{x_{n-2}+d}^{1-d} \int_{x_{n-1}+d}^{1} dx_n dx_{n-1} \ldots dx_2 dx_1$$
$$= [1 - (n-1)d]^n.$$

(b) 0

27. $$F_{x_{(j)}}(x) = \sum_{i=j}^n \binom{n}{i} F^i(x)[1 - F(x)]^{n-i}$$

$$f_{X_{(j)}}(x) = \sum_{i=j}^n \binom{n}{i} iF^{i-1}(x) f(x)[1 - F(x)]^{n-i}$$
$$- \sum_{i=j}^n \binom{n}{i} F^i(x)(n-i)[1 - F(x)]^{n-i-1} f(x)$$

$$= \sum_{i=j}^n \frac{n!}{(n-i)!(i-1)!} F^{i-1}(x) f(x)[1 - F(x)]^{n-i}$$
$$- \sum_{k=j+1}^n \frac{n!}{(n-k)!(k-1)!} F^{k-1}(x) f(x)[1 - F(x)]^{n-k} \text{ by } k = i + 1$$
$$= \frac{n!}{(n-j)!(j-1)!} F^{j-1}(x) f(x)[1 - F(x)]^{n-j}$$

28. $$f_{X(n+1)}(x) = \frac{(2n+1)!}{n!n!} x^n (1-x)^n$$

29. In order for $X_{(i)} = x_i$, $X_{(j)} = x_j$, $i < j$, we must have

(i) $i - 1$ of the X's less than x_i

(ii) 1 of the X's equal to x_i

(iii) $j - i - 1$ of the X's between x_i and x_j

(iv) 1 of the X's equal to x_j

(v) $n - j$ of the X's greater than x_j

Hence,

$$f_{X_{(i)},X_{(j)}}(x_i, x_j)$$
$$= \frac{n!}{(i-1)!1!(j-i-1)!1!(n-j)!} F^{i-1}(x_i) f(x_i)[F(x_j) - F(x_i)]^{j-i-1} f(x_j) \times [1 - F(x_j)]^{n-j}$$

31. Let $X_1, \ldots, X_n$ be n independent uniform random variables over $(0, a)$. We will show by induction on n that

$$P\{X_{(k)} - X_{(k-1)} > t\} = \begin{cases} \left(\dfrac{a-t}{a}\right)^n & \text{if } t < a \\ 0 & \text{if } t > a \end{cases}$$

It is immediate when $n = 1$ so assume for $n - 1$. In the n case, consider

$$P\{X_{(k)} - X_{(k-1)} > t \mid X_{(n)} = s\}.$$

Now given $X_{(n)} = s$, $X_{(1)}, \ldots, X_{(n-1)}$ are distributed as the order statistics of a set of $n - 1$ uniform $(0, s)$ random variables. Hence, by the induction hypothesis

$$P\{X_{(k)} - X_{(k-1)} > t \mid X_{(n)} = s\} = \begin{cases} \left(\dfrac{s-t}{s}\right)^{n-1} & \text{if } t < s \\ 0 & \text{if } t > s \end{cases}$$

and thus, for $t < a$,

$$P\{X_{(k)} - X_{(k-1)} > t = \int_t^a \left(\frac{s-t}{s}\right)^{n-1} \frac{ns^{n-1}}{a^n} ds = \left(\frac{a-t}{a}\right)^n$$

which completes the induction. (The above used that $f_{X_{(n)}}(s) = n\left(\dfrac{s}{a}\right)^{n-1} \dfrac{1}{a} = \dfrac{ns^{n-1}}{a^n}$).

32. (a) $P\{X > X_{(n)}\} = P\{X$ is largest of $n + 1\} = 1/(n + 1)$

(b) $P\{X > X_{(1)}\} = P\{X$ is not smallest of $n + 1\} = 1 - 1/(n + 1) = n/(n + 1)$

(c) This is the probability that X is either the $(i + 1)^{\text{st}}$ or $(i + 2)^{\text{nd}}$ or … j^{th} smallest of the $n + 1$ random variables, which is clearly equal to $(j - 1)/(n + 1)$.

35. The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 1/y \\ 0 & -x/y^2 \end{vmatrix} = -x/y^2$$

Hence, $|J|^{-1} = y^2/|x|$. Therefore, as the solution of the equations $u = x$, $v = x/y$ is $x = u$, $y = u/v$, we see that

$$f_{u,v}(u, v) = \frac{|u|}{v^2} f_{X,Y}(u, u/v) = \frac{|u|}{v^2} \frac{1}{2\pi} e^{-(u^2+u^2/v^2)/2}$$

Hence,

$$\begin{aligned} f_{V(u)} &= \frac{1}{2\pi v^2} \int_{-\infty}^{\infty} |u| e^{-u^2(1+1/v^2)/2} du \\ &= \frac{1}{2\pi v^2} \int_{-\infty}^{\infty} |u| e^{-u^2/2\sigma^2} du \text{ , where } \sigma^2 = v^2/(1+v^2) \\ &= \frac{1}{\pi v^2} \int_0^{\infty} u e^{-u^2/2\sigma^2} du \\ &= \frac{1}{\pi v^2} \sigma^2 \int_0^{\infty} e^{-y} dy \\ &= \frac{1}{\pi(1+v^2)} \end{aligned}$$

Chapter 7

Problems

1. Let $X = 1$ if the coin toss lands heads, and let it equal 0 otherwise. Also, let Y denote the value that shows up on the die. Then, with $p(i, j) = P\{X = i, Y = j\}$

$$E[\text{return}] = \sum_{j=1}^{6} 2jp(1, j) + \sum_{j=1}^{6} \frac{j}{2} p(0, j)$$
$$= \frac{1}{12}(42 + 10.5) = 52.5/12$$

2. (a) $6 \cdot 6 \cdot 9 = 324$

(b) $X = (6 - S)(6 - W)(9 - R)$

(c)
$$\begin{aligned} E[X] = {} & 6(6)(6)P\{S = 0, W = 0, R = 3\} + 6(3)(9)P\{S = 0, W = 3, R = 0\} \\ & + 3(6)(9)P\{S = 3, W = 0, R = 0\} + 6(5)(7)P\{S = 0, W = 1, R = 2\} \\ & + 5(6)(7)P\{S = 1, W = 0, R = 2\} + 6(4)(8)P\{S = 0, W = 2, R = 1\} \\ & + 4(6)(8)P\{S = 2, W = 0, R = 1\} + 5(4)(9)P\{S = 1, W = 2, R = 0\} \\ & + 4(5)(9)P\{S = 2, W = 1, R = 0\} + 5(5)(8)P\{S = 1, W = 1, R = 1\} \end{aligned}$$

$$= \frac{1}{\binom{21}{3}}\left[216\binom{9}{3} + 324\binom{6}{3} + 420 \cdot 6\binom{9}{2} + 384\binom{6}{2}9 + 360\binom{6}{2}6 + 200(6)(6)(9)\right]$$
$$\approx 198.8$$

3. If the first win is on trial N, then the winnings is $W = 1 - (N - 1) = 2 - N$. Thus,

(a) $P(W > 0) = P(N = 1) = 1/2$
(b) $P(W < 0) = P(N > 2) = 1/4$
(c) $E[W] = 2 - E[N] = 0$

4.
$$E[XY] = \int_0^1 \int_0^y xy \frac{1}{y} dxdy = \int_0^1 y^2 / 2dy = 1/6$$
$$E[X] = \int_0^1 \int_0^y x \frac{1}{y} dxdy = \int_0^1 y / 2dy = 1/4$$
$$E[Y] = \int_0^1 \int_0^y y \frac{1}{y} dxdy = \int_0^1 ydy = 1/2$$

5. The joint density of the point (X, Y) at which the accident occurs is

$$f(x, y) = \frac{1}{9}, \; -3/2 < x, y < 3/2$$
$$= f(x)\, f(y)$$

where

$$f(a) = 1/3, \; -3/2 < a < 3/2.$$

Hence we may conclude that X and Y are independent and uniformly distributed on $(-3/2, 3/2)$ Therefore,

$$E[\,|X| + |Y|\,] = 2\int_{-3/2}^{3/2} \frac{1}{3} x\, dx = \frac{4}{3}\int_{0}^{3/2} x dx = 3/2\,.$$

6. $E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = 10(7/2) = 35.$

8. $E[\text{number of occupied tables}] = E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i]$

Now,

$E[X_i] = P\{i^{\text{th}}$ arrival is not friends with any of first $i - 1\}$
$= (1 - p)^{i-1}$

and so

$$E[\text{number of occupied tables}] = \sum_{i=1}^{N} (1-p)^{i-1}$$

7. Let X_i equal 1 if both choose item i and let it be 0 otherwise; let Y_i equal 1 if neither A nor B chooses item i and let it be 0 otherwise. Also, let W_i equal 1 if exactly one of A and B choose item i and let it be 0 otherwise. Let

$$X = \sum_{i=1}^{10} X_i\,, \quad Y = \sum_{i=1}^{10} Y_i\,, \quad W = \sum_{i=1}^{10} W_i$$

(a) $E[X] = \sum_{i=1}^{10} E[X_i] = 10(3/10)^2 = .9$

(b) $E[Y] = \sum_{i=1}^{10} E[Y_i] = 10(7/10)^2 = 4.9$

(c) Since $X + Y + W = 10$, we obtain from parts (a) and (b) that

$$E[W] = 10 - .9 - 4.9 = 4.2$$

Of course, we could have obtained $E[W]$ from

$$E[W] = \sum_{i=1}^{10} E[W_i] = 10(2)(3/10)(7/10) = 4.2$$

9. Let X_j equal 1 if urn j is empty and 0 otherwise. Then

$E[X_j] = P\{\text{ball } i \text{ is not in urn } j,\ i \ge j\} = \prod_{i=j}^{n}(1-1/i)$

Hence,

(a) $E[\text{number of empty urns}] = \sum_{j=1}^{n}\sum_{i=j}^{n}(1-1/i)$

(b) $P\{\text{none are empty}\} = P\{\text{ball } j \text{ is in urn } j, \text{ for all } j\}$

$$= \prod_{j=1}^{n} 1/j$$

10. Let X_i equal 1 if trial i is a success and 0 otherwise.

(a) .6. This occurs when $P\{X_1 = X_2 = X_3\} = 1$. It is the largest possible since $1.8 = \sum P\{X_i = 1\} = 3P\{X_i = 1\}$. Hence, $P\{X_i = 1\} = .6$ and so

$$P\{X = 3\} = P\{X_1 = X_2 = X_3 = 1\} \le P\{X_i = 1\} = .6.$$

(b) 0. Letting

$$X_1 = \begin{cases} 1 \text{ if } U \le .6 \\ 0 \text{ otherwise} \end{cases}, \qquad X_2 = \begin{cases} 1 \text{ if } U \le .4 \\ 0 \text{ otherwise} \end{cases}, \qquad X_3 = \begin{cases} 1 \text{ if } U \le .3 \\ 0 \text{ otherwise} \end{cases}$$

Hence, it is not possible for all X_i to equal 1.

11. Let X_i equal 1 if a changeover occurs on the i^{th} flip and 0 otherwise. Then

$$\begin{aligned} E[X_i] &= P\{i-1 \text{ is } H,\ i \text{ is } T\} + P\{i-1 \text{ is } T,\ i \text{ is } H\} \\ &= 2(1-p)p,\ \ i \ge 2. \end{aligned}$$

$E[\text{number of changeovers}] = E\left[\sum X_i\right] = \sum_{i=1}^{n} E[X_i] = 2(n-1)(1-p)$

12. (a) Let X_i equal 1 if the person in position i is a man who has a woman next to him, and let it equal 0 otherwise. Then

$$E[X_i] = \begin{cases} \dfrac{1}{2}\dfrac{n}{2n-1}, & \text{if } i = 1, 2\text{n} \\ \dfrac{1}{2}\left[1 - \dfrac{(n-1)(n-2)}{(2n-1)(2n-2)}\right], & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E\left[\sum_{i=1}^{n} X_i\right] &= \sum_{i=1}^{2n} E[X_i] \\ &= \frac{1}{2}\left(\frac{2n}{2n-1} + (2n-2)\frac{3n}{4n-2}\right) \\ &= \frac{3n^2 - n}{4n-2} \end{aligned}$$

(b) In the case of a round table there are no end positions and so the same argument as in part (a) gives the result

$$n\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right] = \frac{3n^2}{4n-2}$$

where the right side equality assumes that $n > 1$.

13. Let X_i be the indicator for the event that person i is given a card whose number matches his age. Because only one of the cards matches the age of the person i

$$E\left[\sum_{i=1}^{1000} X_i\right] = \sum_{i=1}^{1000} E[X_i] = 1$$

14. The number of stages is a negative binomial random variable with parameters m and $1 - p$. Hence, its expected value is $m/(1 - p)$.

15. Let $X_{i,j}$, $i \neq j$ equal 1 if i and j form a matched pair, and let it be 0 otherwise.

Then

$$E[X_{i,j}] = P\{i, j \text{ is a matched pair}\} = \frac{1}{n(n-1)}$$

Hence, the expected number of matched pairs is

$$E\left[\sum_{i<j} X_{i,j}\right] = \sum_{i<j} E[X_{i,j}] = \binom{n}{2}\frac{1}{n(n-1)} = \frac{1}{2}$$

16. $$E[X] = \int_{y>x} y\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

17. Let I_i equal 1 if guess i is correct and 0 otherwise.

(a) Since any guess will be correct with probability $1/n$ it follows that

$$E[N] = \sum_{i=1}^{n} E[I_i] = n/n = 1$$

(b) The best strategy in this case is to always guess a card which has not yet appeared. For this strategy, the i^{th} guess will be correct with probability $1/(n - i + 1)$ and so

$$E[N] = \sum_{i=1}^{n} 1/(n-i+1)$$

(c) Suppose you will guess in the order 1, 2, …, n. That is, you will continually guess card 1 until it appears, and then card 2 until it appears, and so on. Let J_i denote the indicator variable for the event that you will eventually be correct when guessing card i; and note that this event will occur if among cards 1 thru i, card 1 is first , card 2 is second, …, and card i is the last among these i cards. Since all $i!$ orderings among these cards are equally likely it follows that

$$E[J_i] = 1/i! \text{ and thus } E[N] = E\left[\sum_{i=1}^{n} J_i\right] = \sum_{i=1}^{n} 1/i!$$

18. $$\begin{aligned} E[\text{number of matches}] &= E\left[\sum_{1}^{52} I_i\right], \quad I_i = \begin{cases} 1 & \text{match on card } i \\ 0 & \text{---} \end{cases} \\ &= 52\frac{1}{13} = 4 \quad \text{since } E[I_i] = 1/13 \end{aligned}$$

19. (a) E[time of first type 1 catch] − 1 = $\dfrac{1}{p_1} - 1$ using the formula for the mean of a geometric random variable.

(b) Let

$$X_j = \begin{cases} 1 & \text{a type } j \text{ is caught before a type 1} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E\left[\sum_{j\neq 1} X_j\right] &= \sum_{j\neq 1} E[X_j] \\ &= \sum_{j\neq 1} P\{\text{type } j \text{ before type } 1\} \\ &= \sum_{j\neq 1} P_j/(P_j + P_1), \end{aligned}$$

where the last equality follows upon conditioning on the first time either a type 1 or type j is caught to give.

$$P\{\text{type } j \text{ before type } 1\} = P\{j \mid j \text{ or } 1\} = \frac{P_j}{P_j + P_1}$$

20. Similar to (b) of 19. Let

$$X_j = \begin{cases} 1 & \text{ball } j \text{ removed before ball 1} \\ 0 & \text{---} \end{cases}$$

$$\begin{aligned} E\left[\sum_{j\neq 1} X_j\right] = \sum_{j\neq 1} E[X_j] &= \sum_{j\neq 1} P\{\text{ball } j \text{ before ball } 1\} \\ &= \sum_{j\neq 1} P\{j \mid j \text{ or } 1\} \\ &= \sum_{j\neq 1} W(j)/W(1) + W(j) \end{aligned}$$

21. (a) $365\binom{100}{3}\left(\frac{1}{365}\right)^3\left(\frac{364}{365}\right)^{97}$

(b) Let $X_j = \begin{cases} 1 & \text{if day } j \text{ is someones birthday} \\ 0 & \text{---} \end{cases}$

$$E\left[\sum_1^{365} X_j\right] = \sum_1^{365} E[X_j] = 365\left[1-\left(\frac{364}{365}\right)^{100}\right]$$

22. From Example 3g, $1 + \frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+6$

23. $E\left[\sum_1^5 X_i + \sum_1^8 Y_i\right] = \sum_1^5 E[X_i] + \sum_1^8 E(Y_i)$

$$= 5\frac{2}{11}\frac{3}{20} + 8\frac{3}{120} = \frac{147}{110}$$

24. Number the small pills, and let X_i equal 1 if small pill i is still in the bottle after the last large pill has been chosen and let it be 0 otherwise, $i = 1, \ldots, n$. Also, let Y_i, $i = 1, \ldots, m$ equal 1 if the i^{th} small pill created is still in the bottle after the last large pill has been chosen and its smaller half returned.

Note that $X = \sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i$. Now,

$E[X_i] = P\{\text{small pill } i \text{ is chosen after all } m \text{ large pills}\}$
$= 1/(m + 1)$

$E[Y_i] = P\{i^{\text{th}} \text{ created small pill is chosen after } m - i \text{ existing large pills}\}$
$= 1/(m - i + 1)$

Thus,

(a) $E[X] = n/(m + 1) + \sum_{i=1}^{m} 1/(m-i+1)$

(b) $Y = n + 2m - X$ and thus

$E[Y] = n + 2m - E[X]$

25. $P\{N \ge n\}\ P\{X_1 \ge X_2 \ge \ldots \ge X_n\} = \frac{1}{n!}$

$$E[N] = \sum_{n=1}^{\infty} P\{N \ge n\} = \sum_{n=1}^{\infty}\frac{1}{n!} = e$$

26. (a) $E[\max] = \int_0^1 P\{\max > t\}dt$

$$= \int_0^1 (1 - P\{\max \le t)\}dt$$

$$= \int_0^1 (1 - t^n / dt = \frac{n}{n+1}$$

(b) $\text{E}[\min] = \int_0^1 p\{\min > t\}4t$

$$= \int_0^1 (1-t)^n dt = \frac{1}{n+1}$$

27. Let X denote the number of items in a randomly chosen box. Then, with X_i equal to 1 if item i is in the randomly chosen box

$$E[X] = E\left[\sum_{i=1}^{101} X_i\right] = \sum_{i=1}^{101} E[X_i] = \frac{101}{10} > 10$$

Hence, X can exceed 10, showing that at least one of the boxes must contain more than 10 items.

28. We must show that for any ordering of the 47 components there is a block of 12 consecutive components that contain at least 3 failures. So consider any ordering, and randomly choose a component in such a manner that each of the 47 components is equally likely to be chosen. Now, consider that component along with the next 11 when moving in a clockwise manner and let X denote the number of failures in that group of 12. To determine $E[X]$, arbitrarily number the 8 failed components and let, for $i = 1, \ldots, 8$,

$$X_i = \begin{cases} 1, & \text{if failed component } i \text{ is among the group of 12 components} \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$X = \sum_{i=1}^{8} X_i$$

and so

$$E[X] = \sum_{i=1}^{8} E[X_i]$$

Because X_i will equal 1 if the randomly selected component is either failed component number i or any of its 11 neighboring components in the counterclockwise direction, it follows that $E[X_i] = 12/47$. Hence,

$$E[X] = 8(12/47) = 96/47$$

Because $E[X] > 2$ it follows that there is at least one possible set of 12 consecutive components that contain at least 3 failures.

29. Let X_{ii} be the number of coupons one needs to collect to obtain a type i. Then

$$E[X_{i)}] = 8, \quad i = 1, 2$$
$$E\{X_i] = 8/3, \quad i = 3, 4$$
$$E[\min(X_1, X_2)] = 4$$
$$E[\min(X_i, X_j)] = 2, \quad i = 1, 2, \quad j = 3, 4$$
$$E[\min(X_3, X_4)] = 4/3$$
$$E[\min(X_1, X_2, X_j)] = 8/5, \quad j = 3, 4$$
$$E[\min(X_i, X_{3,} X_4)] = 8/7, \quad i = 1, 2$$
$$E[\min(X_1, X_2, X_3, X_4] = 1$$

(a) $E[\max X_i] = 2 \cdot 8 + 2 \cdot 8/3 - (4 + 4 \cdot 2 + 4/3) + (2 \cdot 8/5 + 2 \cdot 8/7) - 1 = \dfrac{437}{35}$

(b) $E[\max(X_1, X_2)] = 8 + 8 - 4 = 12$

(c) $E[\max(X_3, X_4)] = 8/3 + 8/3 - 4/3 = 4$

(d) Let $Y_1 = \max(X_1, X_2)$, $Y_2 = \max(X_3, X_4)$. Then

$$E[\max(Y_1, Y_2)] = E[Y_1] + E[Y_2] - E[\min(Y_1, Y_2)]$$

giving that

$$E[\min(Y_1, Y_2)] = 12 + 4 - \frac{437}{35} = \frac{123}{35}$$

30. $E[(X - Y)]^2 = \mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(-Y) = 2\sigma^2$

31. $\mathrm{Var}\left(\sum_{i=1}^{10} X_i\right) = 10\,\mathrm{Var}(X_1)$. Now

$$\begin{aligned}\mathrm{Var}(X_1) &= E[X_1^2] - (7/2)^2 \\ &= [1 + 4 + 9 + 16 + 25 + 36]/6 - 49/4 \\ &= 35/12\end{aligned}$$

and so $\mathrm{Var}\left(\sum_{i=1}^{10} X_i\right) = 350/12.$

32. Use the notation in Problem 9,

$$X = \sum_{j=1}^{n} X_j$$

where X_j is 1 if box j is empty and 0 otherwise. Now, with

$E[X_j] = P\{X_j = 1\} = \prod_{i=j}^{n}(1-1/i)$, we have that

$$\text{Var}(X_j) = E[X_j](1 - E[X_j]).$$

Also, for $j < k$

$$E[X_jX_k] = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i)$$

Hence, for $j < k$,

$$\text{Cov}(X_j, X_k) = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i) - \prod_{i=j}^{n}(1-1/i)\prod_{i=k}^{n}(1-1/i)$$

$$\text{Var}(X) = \sum_{j=1}^{n} E[X_j](1-E[X_j]) + 2\text{Cov}(X_j, X_k)$$

33. (a) $E[X^2 + 4X + 4] = E[X^2] + 4E[X] + 4 = \text{Var}(X) + E^2[X] + 4E[X] + 4 = 14$

(b) $\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 45$

34. Let $X_j = \begin{cases} 1 & \text{if couple } j \text{ are seated next to each other} \\ 0 & \text{otherwise} \end{cases}$

(a) $E\left[\sum_{1}^{10} X_j\right] = 10\frac{2}{19} = \frac{20}{19}$; $P\{X_j = 1\} = \frac{2}{19}$ since there are 2 people seated next to wife j and so the probability that one of them is her husband is $\frac{2}{19}$.

(b) For $i \neq j$, $E[X_iX_j] = P\{X_i = 1, X_j = 1\}$

$$= P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\}$$

$$= \frac{2}{19}\frac{2}{18}$$ since given $X_i = 1$ we can regard couple i as a single entity.

$$\text{Var}\left(\sum_{j=1}^{10} X_j\right) = 10\frac{2}{19}\left(1-\frac{2}{19}\right) + 10\cdot 9\left[\frac{2}{19}\frac{2}{18} - \left(\frac{2}{19}\right)^2\right]$$

35. (a) Let X_1 denote the number of nonspades preceding the first ace and X_2 the number of nonspades between the first 2 aces. It is easy to see that

$$P\{X_1 = i,\ X_2 = j\} = P\{X_1 = j,\ X_2 = i\}$$

and so X_1 and X_2 have the same distribution. Now $E[X_1] = \frac{48}{5}$ by the results of Example 3j and so $E[2 + X_1 + X_2] = \frac{106}{5}$.

(b) Same method as used in (a) yields the answer $5\left(\frac{39}{14}+1\right)=\frac{265}{14}$.

(c) Starting from the end of the deck the expected position of the first (from the end) heart is, from Example 3j, $\frac{53}{14}$. Hence, to obtain all 13 hearts we would expect to turn over

$52 - \frac{53}{14} + 1 = \frac{13}{14}(53)$.

36. Let $X_i = \begin{cases} 1 & \text{roll } i \text{ lands on 1} \\ 0 & \text{otherwise} \end{cases}$, $\quad Y_i = \begin{cases} 1 & \text{roll } i \text{ lands on 2} \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{Cov}(X_i, Y_j) &= E[X_i Y_j] - E[X_i]E[Y_j] \\ &= \begin{cases} -\frac{1}{36} & i = j \text{ (since } X_i Y_j = 0 \text{ when } i = j \\ \frac{1}{36} - \frac{1}{36} = 0 & i \neq j \end{cases} \end{aligned}$$

$$\begin{aligned} \text{Cov} \sum_i X_i, \sum_j Y_j &= \sum_i \sum_j \text{Cov}(X_i, Y_j) \\ &= -\frac{n}{36} \end{aligned}$$

37. Let W_i, $i = 1, 2$, denote the i^{th} outcome.

$$\begin{aligned} \text{Cov}(X, Y) &= \text{Cov}(W_1 + W_2\,,\ W_1 - W_2) \\ &= \text{Cov}(W_1, W_1) - \text{Cov}(W_2, W_2) \\ &= \text{Var}(W_1) - \text{Var}(W_2) = 0 \end{aligned}$$

38. $E[XY] = \int_0^\infty \int_0^x y2e^{-2x} dydx$

$$= \int_0^\infty x^2 e^{-2x} dx = \frac{1}{8}\int_0^\infty y^2 e^{-y} dy = \frac{\Gamma(3)}{8} = \frac{1}{4}$$

$E[X] = \int_0^\infty xf_x(x)dx,\ f_x(x) = \int_0^x \frac{2e^{-2x}}{x} dy\ = 2e^{-2x}$

$$= \frac{1}{2}$$

$E[Y] = \int_0^\infty yf_Y(y)dy,\ f_Y(y) = \int_0^\infty \frac{2e^{-2x}}{x} dx$

$$= \int_0^\infty \int_y^\infty y \frac{2e^{-2x}}{x} dxdy$$

$$= \int_0^\infty \int_0^x y \frac{2e^{-2x}}{x} dydx$$

$$= \int_0^\infty xe^{-2x} dx = \frac{1}{4}\int ye^{-2} dy = \frac{\Gamma(2)}{4} = \frac{1}{4}$$

$\mathrm{Cov}(X, Y) = \frac{1}{4} - \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

39. $\mathrm{Cov}(Y_n, Y_n) = \mathrm{Var}(Y_n) = 3\sigma^2$
$\mathrm{Cov}(Y_n, Y_{n+1}) = \mathrm{Cov}(X_n + X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2} + X_{n+3})$
$= \mathrm{Cov}(X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2}) = \mathrm{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2$
$\mathrm{Cov}(Y_n, Y_{n+2}) = \mathrm{Cov}(X_{n+2}, X_{n+2}) = \sigma^2$
$\mathrm{Cov}(Y_n, Y_{n+j}) = 0$ when $j \geq 3$

40. $f_Y(y) = e^{-y}\int \frac{1}{y} e^{-x/y} dx = e^{-y}$. In addition, the conditional distribution of X given that $Y = y$ is exponential with mean y. Hence,

$E[Y] = 1,\ E[X] = E[E[X \mid Y]] = E[Y] = 1$

Since, $E[XY] = E[E[XY \mid Y]] = E[YE[X \mid Y]] = E[Y^2] = 2$ (since Y is exponential with mean 1, it follows that $E[Y^2] = 2$). Hence, $\mathrm{Cov}(X, Y) = 2 - 1 = 1$.

41. The number of carp is a hypergeometric random variable.

$$E[X] = \frac{60}{10} = 6$$

$$\text{Var}(X) = \frac{20(80)}{99}\frac{3}{10}\frac{7}{10} = \frac{336}{99} \text{ from Example 5c.}$$

42. (a) Let $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a man and a woman} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = P\{X_i = 1\} = \frac{10}{19}$$

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1\}P\{X_j = 1 \mid X_2 = 1\} = \frac{10}{19}\frac{9}{17}, \ i \neq \text{j}$$

$$E\left[\sum_1^{10} X_i\right] = \frac{100}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{10}{19}\left(1-\frac{10}{19}\right) + 10\cdot 9\left[\frac{10}{19}\frac{9}{17} - \left(\frac{10}{19}\right)^2\right] = \frac{900}{(19)^2}\frac{18}{17}$$

(b) $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a married couple} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = \frac{1}{19}, \ E[X_iX_j] = P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\} = \frac{1}{19}\frac{1}{17}, \ i \neq j$$

$$E\left[\sum_1^{10} X_i\right] = \frac{10}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{1}{19}\frac{15}{19} + 10\cdot 9\left[\frac{1}{19}\frac{1}{17} - \left(\frac{1}{19}\right)^2\right] = \frac{180}{(19)^2}\frac{18}{17}$$

43. $E[R] = n(n + m + 1)/2$

$$\text{Var}(R) = \frac{nm}{n+m-1}\left[\frac{\sum_{i=1}^{n+m} i^2}{n+m} - \left(\frac{n+m+1}{2}\right)^2\right]$$

The above follows from Example 3d since when $F = G$, all orderings are equally likely and the problem reduces to randomly sampling n of the $n + m$ values 1, 2, …, $n + m$.

44. From Example 8l $\dfrac{n}{n+m}+\dfrac{nm}{n+m}$. Using the representation of Example 2l the variance can be computed by using

$$E[I_1 I_{l+j}] = \begin{cases} 0 & , \quad j=1 \\ \dfrac{n}{n+m}\dfrac{m}{n+m-1}\dfrac{n-1}{n+m-2} & , \quad n-1 \le j < 1 \end{cases}$$

$$E[I_i I_{i+j}] = \begin{cases} 0 & , \quad j=1 \\ \dfrac{mn(m-1)(n-1)}{(n+m)(n+m-1)(n+m-2)(n+m-3)} & , \quad n-1 \le j < 1 \end{cases}$$

45. (a) $\dfrac{\text{Cov}(X_1+X_2, X_2+X_3)}{\sqrt{\text{Var}(X_1+X_2)}\sqrt{\text{Var}(X_2+X_3)}} = \dfrac{1}{2}$

(b) 0

46.
$$\begin{aligned} E[I_1 I_2] &= \sum_{i=2}^{12} E[I_1 I_2 \mid \text{bank rolls } i]\text{P}\{\text{bank rolls } i\} \\ &= \sum_i (P\{\text{roll is greater than } i\})^2 P\{\text{bank rolls } i\} \\ &= E[I_1^2] \\ &\ge (E[I_1])^2 \\ &= E[I_1]\,E[I_2] \end{aligned}$$

47. (a) It is binomial with parameters $n-1$ and p.

(b) Let $x_{i,j}$ equal 1 if there is an edge between vertices i and j, and let it be 0 otherwise. Then, $D_i = \sum_{k \ne i} X_{i,k}$, and so, for $i \ne j$

$$\begin{aligned} \text{Cov}(D_i, D_j) &= \text{Cov}\left(\sum_{k \ne i} X_{i,k}, \sum_{r \ne j} X_{r,j}\right) \\ &= \sum_{k \ne i}\sum_{r \ne j} Cov(X_{i,k}, X_{r,j}) \\ &= \text{Cov}(X_{i,j}, X_{i,j}) \\ &= \text{Var}(X_{i,j}) \\ &= p(1-p) \end{aligned}$$

where the third equality uses the fact that except when $k = j$ and $r = i$, $X_{i,k}$ and $X_{r,j}$ are independent and thus have covariance equal to 0. Hence, from part (a) and the preceding we obtain that for $i \ne j$,

$$\rho(D_i, D_j) = \frac{p(1-p)}{(n-1)p(1-p)} = \frac{1}{n-1}$$

48. (a) $E[X] = 6$

(b) $E[X \mid Y = 1] = 1 + 6 = 7$

(c) $1\frac{1}{5} + 2\frac{4}{5}\frac{1}{5} + 3\left(\frac{4}{5}\right)^2 \frac{1}{5} + 4\left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^4 (5+6)$

49. Let C_i be the event that coin i is being flipped (where coin 1 is the one having head probability .4), and let T be the event that 2 of the first 3 flips land on heads. Then

$$P(C_1 \mid T) = \frac{P(T \mid C_1)P(C_1)}{P(T \mid C_1)P(C_1) + P(T \mid C_2)P(C_2)}$$

$$= \frac{3(.4)^2(.6)}{3(.4)^2(.6) + 3(.7)^2(.3)} = .395$$

Now, with N_j equal to the number of heads in the final j flips, we have

$$E[N_{10} \mid T] = 2 + E[N_7 \mid T]$$

Conditioning on which coin is being used, gives

$$E[N_7 \mid T] = E[N_7 \mid TC_1]P(C_1T) + E[N_7TC_2]P(C_2 \mid T) = 2.8(.395) + 4.9(.605) = 4.0705$$

Thus, $E[N_{10} \mid T] = 6.0705$.

50. $$f_{X|Y}(x \mid y) = \frac{e^{-x/y}e^{-y}/y}{\int_0^\infty e^{-x/y}e^{-y}/y\,dx} = \frac{1}{y}e^{-x/y}, \qquad 0 < x < \infty$$

Hence, given $Y = y$, X is exponential with mean y, and so

$$E[X^2 \mid Y = y] = 2y^2$$

51. $$f_{X|Y}(x \mid y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y\,dx} = \frac{1}{y}, \quad 0 < x < y$$

$$E[X^3 \mid Y = y] = \int_0^y x^3 \frac{1}{y}dx = y^3/4$$

52. The average weight, call it $E[W]$, of a randomly chosen person is equal to average weight of all the members of the population. Conditioning on the subgroup of that person gives

$$E[W] = \sum_{i=1}^r E\{W \mid \text{member of subgroup } i]\, p_i = \sum_{i=1}^r w_i p_i$$

53. Let X denote the number of days until the prisoner is free, and let I denote the initial door chosen. Then

$$E[X] = E[X \mid I = 1](.5) + E[X \mid I = 2](.3) + E[X \mid I = 3](.2)$$
$$= (2 + E[X])(.5) + (4 + E[X])(.3) + .2$$

Therefore,
$$E[X] = 12$$

54. Let R_i denote the return from the policy that stops the first time a value at least as large as i appears. Also, let X be the first sum, and let $p_i = P\{X = i\}$. Conditioning on X yields

$$E[R_5] = \sum_{i=2}^{12} E[R_5 \mid X = i\} p_i$$
$$= E[R_5](p_2 + p_3 + p_4) + \sum_{i=5}^{12} ip_i - 7p_7$$
$$= \frac{6}{36} E[R_5] + 5(4/36) + 6(5/36) + 8(5/36) + 9(4/36) + 10(3/36) + 11(2/36) + 12(1/36)$$
$$= \frac{6}{36} E[R_5] + 190/36$$

Hence, $E[R_5] = 19/3 \approx 6.33$. In the same fashion, we obtain that

$$E[R_6] = \frac{10}{36} E[R_6] + \frac{1}{36}[30 + 40 + 36 + 30 + 22 + 12]$$

implying that

$$E[R_6] = 170/26 \approx 6.54$$

Also,

$$E[R_8] = \frac{15}{36} E[R_8] + \frac{1}{36}(140)$$

or,

$$E[R_8] = 140/21 \approx 6.67$$

In addition,

$$E[R_9] = \frac{20}{26} E[R_9] + \frac{1}{36}(100)$$

or

$$E[R_9] = 100/16 = 6.25$$

And

$$E[R_{10}] = \frac{24}{36}E[R_{10}] + \frac{1}{36}(64)$$

or

$$E[R_{10}] = 64/12 \approx 5.33$$

The maximum expected return is $E[R_8]$.

55. Let N denote the number of ducks. Given $N = n$, let $I_1, \ldots, I_n$ be such that

$$I_i = \begin{cases} 1 & \text{if duck } i \text{ is hit} \\ 0 & \text{otherwise} \end{cases}$$

$$E[\text{Number hit} \mid N = n] = E\left[\sum_{i=1}^{n} I_i\right]$$
$$= \sum_{i=1}^{n} E[I_i] = n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right], \text{ since given}$$

$N = n$, each hunter will independently hit duck i with probability $.6/n$.

$$E[\text{Number hit}] = \sum_{n=0}^{\infty} n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right] e^{-6}6^n/n!$$

56. Let $I_i = \begin{cases} 1 & \text{elevator stops at floor } i \\ 0 & \text{otherwise} \end{cases}$. Let X be the number that enter on the ground floor.

$$E\left[\sum_{i=1}^{N} I_i \mid X = k\right] = \sum_{i=1}^{N} E[I_i \mid X = k] = N\left[1 - \left(\frac{N-1}{N}\right)^k\right]$$

$$E\left[\sum_{i=1}^{N} I_i\right] = N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k e^{-10}\frac{(10)^k}{k!}$$
$$= N - Ne^{-10/N} = N(1 - e^{-10/N})$$

57. $$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X] = 12.5$$

58. Let X denote the number of flips required. Condition on the outcome of the first flip to obtain.

$$\begin{aligned} E[X] &= E[X \mid \text{heads}]p + E[x \mid \text{tails}](1-p) \\ &= [1 + 1/(1-p)]p + [1 + 1/p](1-p) \\ &= 1 + p/(1-p) + (1-p)/p \end{aligned}$$

59. (a) E[total prize shared] = P\{someone wins\} = $1 - (1 - p)^{n+1}$

(b) Let X_i be the prize to player i. By part (a)

$$E\left[\sum_{i=1}^{n+1} X_i\right] = 1 - (1 - p)^{n+1}$$

But, by symmetry all $E[X_i]$ are equal and so

$$E[X] = [1 - (1 - p)^{n+1}]/(n + 1)$$

(c) $E[X] = p\, E[1/(1 + B)]$ where B, which is binomial with parameters n and p, represents the number of other winners.

60. (a) Since the sum of their number of correct predictions is n (one for each coin) it follows that one of them will have more than $n/2$ correct predictions. Now if N is the number of correct predictions of a specified member of the syndicate, then the probability mass function of the number of correct predictions of the member of the syndicate having more than $n/2$ correct predictions is

$$\begin{aligned} P\{i \text{ correct}\} &= P\{N = i\} + P(N = n - i\} \quad i > n/2 \\ &= 2P\{N = i\} \\ &= P\{N = i \mid N > n/2\} \end{aligned}$$

(b) X is binomial with parameters m, 1/2.

(c) Since all of the $X + 1$ players (including one from the syndicate) that have more than $n/2$ correct predictions have the same expected return we see that

$$(X + 1) \cdot \text{Payoff to syndicate} = m + 2$$

implying that

$$E[\text{Payoff to syndicate}] = (m + 2)\, E[(X + 1)^{-1}]$$

(d) This follows from part (b) above and (c) of Problem 56.

61. (a) $P(M \le x) = \sum_{n=1}^{\infty} P(M \le x \mid N = n)P(N = n) = \sum_{n=1}^{\infty} F^n(x)p(1 - p)^{n-1} = \dfrac{pF(x)}{1 - (1 - p)F(x)}$

(b) $P(M \le x \mid N = 1) = F(x)$

(c) $P(M \le x \mid N > 1) = F(x)P(M \le x)$

(d) $$\begin{aligned} P(M \le x) &= P(M \le x \mid N = 1)P(N = 1) + P(M \le x \mid N > 1)P(N > 1) \\ &= F(x)p + F(x)P(M \le x)(1 - p) \end{aligned}$$

again giving the result

$$P(M \le x) = \frac{pF(x)}{1 - (1 - p)F(x)}$$

62. The result is true when $n = 0$, so assume that

$$P\{N(x) \geq n\} = x^n/(n-1)!$$

Now,

$$\begin{aligned}
P\{N(x) \geq n+1\} &= \int_0^1 P\{N(x) \geq n+1 \mid U_1 = y\}dy \\
&= \int_0^x P\{N(x-y) \geq n\}dy \\
&= \int_0^x P\{N(u) \geq n\}du \\
&= \int_0^x u^{n-1}/(n-1)!\ du \text{ by the induction hypothesis} \\
&= x^n/n!
\end{aligned}$$

which completes the proof.

(b) $E[N(x)] = \sum_{n=0}^{\infty} P\{N(x) > n = \sum_{n=0}^{\infty} P\{N(x) \geq n+1\} = \sum_{n=0}^{\infty} x^n / n! = e^x$

63. (a) Number the red balls and the blue balls and let X_i equal 1 if the i^{th} red ball is selected and let it by 0 otherwise. Similarly, let Y_j equal 1 if the j^{th} blue ball is selected and let it be 0 otherwise.

$$\text{Cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$

Now,

$$E[X_i] = E[Y_j] = 12/30$$

$$E[X_iY_j] = P\{\text{red ball } i \text{ and blue ball } j \text{ are selected}\} = \binom{28}{10}\Big/\binom{30}{12}$$

Thus,

$$\text{Cov}(X, Y) = 80\left[\binom{28}{10}\Big/\binom{30}{12} - (12/30)^2\right] = -96/145$$

(b) $E[XY|X] = XE[Y|X] = X(12 - X)8/20$

where the above follows since given X, there are 12-X additional balls to be selected from among 8 blue and 12 non-blue balls. Now, since X is a hypergeometric random variable it follows that

$E[X] = 12(10/30) = 4$ and $E[X^2] = 12(18)(1/3)(2/3)/29 + 4^2 = 512/29$

As $E[Y] = 8(12/30) = 16/5$, we obtain

$$E[XY] = \frac{2}{5}(48 - 512/29) = 352/29,$$

and

$$\text{Cov}(X, Y) = 352/29 - 4(16/5) = -96/145$$

64. (a) $E[X] = E[X|\text{type 1}]p + E[X|\text{type 2}](1 - p) = p\mu_1 + (1 - p)\mu_2$

(b) Let I be the type.

$$E[X|I] = \mu_I,\ \text{Var}(X|I) = \sigma_I^2$$

$$\text{Var}(X) = E[\sigma_I^2] + \text{Var}(\mu_I)$$
$$= p\sigma_1^2 + (1-p)\sigma_2^2 + p\mu_1^2 + (1-p)\mu_2^2 - [p\mu_1 + (1-p)\mu_2]^2$$

65. Let X be the number of storms, and let $G(B)$ be the events that it is a good (bad) year. Then

$$E[X] = E[X|G]P(G) + E[X|B]P(B) = 3(.4) + 5(.6) = 4.2$$

If Y is Poisson with mean λ, then $E[Y^2] = \lambda + \lambda^2$. Therefore,

$$E[X^2] = E[X^2|G]P(G) + E[X^2|B]P(B) = 12(.4) + 30(.6) = 22.8$$

Consequently,

$$\text{Var}(X) = 22.8 - (4.2)^2 = 5.16$$

66. $$E[X^2] = \frac{1}{3}\{E[X^2|Y=1] + E[X^2|Y=2] + E[X^2|Y=3]\}$$
$$= \frac{1}{3}\{9 + E[(5+X)^2] + E[(7+X)^2]\}$$
$$= \frac{1}{3}\{83 + 24E[X] + 2E[X^2]\}$$
$$= \frac{1}{3}\{443 + 2E[X^2]\} \quad \text{since } E[X] = 15$$

Hence,

$$\text{Var}(X) = 443 - (15)^2 = 218.$$

67. Let F_n denote the fortune after n gambles.

$$\begin{aligned} E[F_n] = E[E[F_n \mid F_{n-1}]] &= E[2(2p-1)F_{n-1}p + F_{n-1} - (2p-1)F_{n-1}] \\ &= (1 + (2p-1)^2)E[F_{n-1}] \\ &= [1 + (2p-1)^2]^2 E[F_{n-2}] \\ &\vdots \\ &= [1 + (2p-1)^2]^n E[F_0] \end{aligned}$$

68. (a) $.6e^{-2} + .4e^{-3}$

(b) $.6e^{-2}\dfrac{2^3}{3!} + .4e^{-3}\dfrac{3^3}{3!}$

(c) $P\{3 \mid 0\} = \dfrac{P\{3,0\}}{P\{0\}} = \dfrac{.6e^{-2}e^{-2}\dfrac{2^3}{3!} + .4e^{-3}e^{-3}\dfrac{3^3}{3!}}{.6e^{-2} + .4e^{-3}}$

69. (a) $\displaystyle\int_0^\infty e^{-x}e^{-x}dx = \frac{1}{2}$

(b) $\displaystyle\int_0^\infty e^{-x}\frac{x^3}{3!}e^{-x}dx = \frac{1}{96}\int_0^\infty e^{-y}y^3dy = \frac{\Gamma(4)}{96} = \frac{1}{16}$

(c) $\dfrac{\displaystyle\int_0^\infty e^{-x}e^{-x}\frac{x^3}{3!}e^{-x}dx}{\displaystyle\int_0^\infty e^{-x}e^{-x}dx} = \dfrac{2}{3^4} = \dfrac{2}{81}$

70. (a) $\displaystyle\int_0^1 p\,dp = 1/2$

(b) $\displaystyle\int_0^1 p^2dp = 1/3$

71. $$\begin{aligned} P\{X = i\} = \int_0^1 P\{X = i \mid p\}dp &= \int_0^1 \binom{n}{i} p^i(1-p)^{n-i}dp \\ &= \binom{n}{i}\frac{i!(n-i)!}{(n+1)!} = 1/(n+1) \end{aligned}$$

72. (a) $P\{N \geq i\} = \int_0^1 P\{N \geq i | p\}dp = \int_0^1 (1-p)^{i-1}dp = 1/i$

(b) $P\{N = i\} = P\{N \geq i\} - P\{N \geq i+1\} = \dfrac{1}{i(i+1)}$

(c) $E[N] = \sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty} 1/i = \infty$.

73. (a) $E[R] = E[E[R | S]] = E[S] = \mu$

(b) $\text{Var}(R | S) = 1$, $E[R | S] = S$
$\text{Var}(R) = 1 + \text{Var}(S) = 1 + \sigma^2$

(c) $f_R(r) = \int f_S(s)F_{R|S}(r | s)ds$

$$= C\int e^{-(s-\mu)^2/2\sigma^2} e^{-(r-s)^2/2} ds$$

$$= K\int \exp\left\{-\left(S - \frac{\mu + r\sigma^2}{1+\sigma^2}\right)\Big/ 2\left(\frac{\sigma^2}{1+\sigma^2}\right)\right\} ds \exp\{-(ar^2 + br)\}$$

Hence, R is normal.

(d) $E[RS] = E[E[RS | S]] = E[SE[R | S]] = E[S^2] = \mu^2 + \sigma^2$

$\text{Cov}(R, S) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

75. X is Poisson with mean $\lambda = 2$ and Y is Binomial with parameters 10, 3/4. Hence

(a) $P\{X + Y = 2\} = P\{X = 0)P\{Y = 2\} + P\{X = 1\}P\{Y = 1\} + P\{X = 2\}P\{Y = 0\}$

$$= e^{-2}\binom{10}{2}(3/4)^2(1/4)^8 + 2e^{-2}\binom{10}{1}(3/4)(1/4)^9 + 2e^{-2}(1/4)^{10}$$

(b) $P\{XY = 0\} = P\{X = 0\} + P\{Y = 0\} - P\{X = Y = 0\}$
$= e^{-2} + (1/4)^{10} - e^{-2}(1/4)^{10}$

(c) $E[XY] = E[X]E[Y] = 2 \cdot 10 \cdot \dfrac{3}{4} = 15$

77. The joint moment generating function, $E[e^{tX+sY}]$ can be obtained either by using

$$E[e^{tX+sY}] = \int\int e^{tX+sY} f(x,y)dy\,dx$$

or by noting that Y is exponential with rate 1 and, given Y, X is normal with mean Y and variance 1. Hence, using this we obtain

$$E[e^{tX+sY} \mid Y] = e^{sY}E[E^{tX} \mid Y] = e^{sY}e^{Yt+t^2/2}$$

and so

$$\begin{aligned} E[e^{tX+sY}] &= e^{t^2/2}E[e^{(s+t)Y}] \\ &= e^{t^2/2}(1-s-t)^{-1},\ s+t<1 \end{aligned}$$

Setting first s and then t equal to 0 gives

$$E[e^{tX}] = e^{t^2/2}(1-t)^{-1},\ \ t<1$$
$$E[e^{sY}] = (1-s)^{-1},\ s<1$$

78. Conditioning on the amount of the initial check gives

$$\begin{aligned} E[\text{Return}] &= E[\text{Return} \mid A]/2 + E[\text{Return} \mid \text{B}]/2 \\ &= \{AF(A) + B[1-F(A)]\}/2 + \{BF(B) + A[1-F(B)]\}/2 \\ &= \{A + B + [B-A][F(B) - F(A)]\}/2 \\ &> (A+B)/2 \end{aligned}$$

where the inequality follows since $[B-A]$ and $[F(B) - F(A)$ both have the same sign.

(b) If $x < A$ then the strategy will accept the first value seen: if $x > B$ then it will reject the first one seen; and if x lies between A and B then it will always yield return B. Hence,

$$E[\text{Return of } x\text{-strategy}] = \begin{matrix} B & \text{if } A < x < B \\ (A+B)/2 & \text{otherwise} \end{matrix}$$

(c) This follows from (b) since there is a positive probability that X will lie between A and B.

79. Let X_i denote sales in week i. Then

$$\begin{aligned} E[X_1 + X_2] &= 80 \\ \text{Var}(X_1 + X_2) &= \text{Var}(X_1) + \text{Var}(X_2) + 2\,\text{Cov}(X_1, X_2) \\ &= 72 + 2[.6(6)(6)] = 93.6 \end{aligned}$$

(a) With Z being a standard normal

$$\begin{aligned} P(X_1 + X_2 > 90) &= P\left(Z > \frac{90-80}{\sqrt{93.6}}\right) \\ &= P(Z > 1.034) \approx .150 \end{aligned}$$

(b) Because the mean of the normal $X_1 + X_2$ is less than 90 the probability that it exceeds 90 is increased as the variance of $X_1 + X_2$ increases. Thus, this probability is smaller when the correlation is .2.

(c) In this case,

$$\begin{aligned} P(X_1 + X_2 > 90) &= P\left(Z > \frac{90-80}{\sqrt{72+2[.2(6)(6)]}}\right) \\ &= P(Z > 1.076) \approx .141 \end{aligned}$$

Theoretical Exercises

1. Let $\mu = E[X]$. Then for any a

$$\begin{aligned} E[(X-a)^2 &= E[(X-\mu+\mu-a)^2] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2E[(x-\mu)(\mu-a)] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2(\mu-a)E[(X-\mu)] \\ &= E[(X-\mu)^2 + (\mu-a)^2 \end{aligned}$$

2. $$\begin{aligned} E[\,|X-a|\,] &= \int_{x<a} (a-x)f(x)dx + \int_{x>a} (x-a)f(x)dx \\ &= aF(a) - \int_{x<a} xf(x)dx + \int_{x>a} xf(x)dx - a[1-F(a)] \end{aligned}$$

Differentiating the above yields
derivative = $2af(a) + 2F(a) - af(a) - af(a) - 1$
Setting equal to 0 yields that $2F(a) = 1$ which establishes the result.

3. $$\begin{aligned} E[g(X,Y)] &= \int_0^\infty P\{g(X,Y) > a\}da \\ &= \int_0^\infty \int_{\substack{x,y:\\ g(x,y)>a}} \int f(x,y)dydxda = \iint \int_0^{g(x,y)} daf(x,y)dydx \\ &= \iint g(x,y)dydx \end{aligned}$$

4. $$\begin{aligned} g(X) &= g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\frac{(X-\mu)^2}{2} + \ldots \\ &\approx g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\frac{(X-\mu)^2}{2} \end{aligned}$$

Now take expectations of both sides.

5. If we let X_k equal 1 if A_k occurs and 0 otherwise then

$$X = \sum_{k=1}^n X_k$$

Hence,

$$E[X] = \sum_{k=1}^n E[X_k] = \sum_{k=1}^n P(A_k)$$

But

$$E[X] = \sum_{k=1}^n P\{X \geq k\} = \sum_{k=1}^n P(C_k)\,.$$

6. $X = \int_0^\infty X(t)dt$ and taking expectations gives

$$E[X] = \int_0^\infty E[X(t)]\,dt = \int_0^\infty P\{X > t\}dt$$

7. (a) Use Exercise 6 to obtain that

$$E[X] = \int_0^\infty P\{X > t\}dt \geq \int_0^\infty P\{Y > t\}dt \ = E[Y]$$

(b) It is easy to verify that

$$X^+ \geq_{\text{st}} Y^+ \text{ and } Y^- \geq_{\text{st}} X^-$$

Now use part (a).

8. Suppose $X \geq_{\text{st}} Y$ and f is increasing. Then

$$\begin{aligned} P\{f(X) > a\} &= P\{X > f^{-1}(a)\} \\ &\geq P\{Y > f^{-1}(a)\} \text{ since } x \geq_{st} Y \\ &= P\{f(Y) > a\} \end{aligned}$$

Therefore, $f(X) \geq_{st} f(Y)$ and so, from Exercise 7,
$E[f(X)] \geq E[f(Y)]$.

On the other hand, if $E[f(X)] \geq E[f(Y)]$ for all increasing functions f, then by letting f be the increasing function

$$f(x) = \begin{matrix} 1 & \text{if } x > t \\ 0 & \text{otherwise} \end{matrix}$$

then

$$P\{X > t\} = E[f(X)] \geq E[f(Y)] = P\{Y > t\}$$

and so X $>_{\text{st}}$ Y.

9. Let

$$I_j = \begin{cases} 1 & \text{if a run of size } k \text{ begins at the } j^{\text{th}} \text{ flip} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\text{Number of runs of size } k = \sum_{j=1}^{n-k+1} I_j$$

$$E[\text{Number of runs of size } k = E\left[\sum_{j=1}^{n-k+1} I_j\right]$$

$$= P(I_1 = 1) + \sum_{j=2}^{n-k} P(I_j = 1) + P(I_{n-k+1} = 1)$$

$$= p^k(1-p) + (n-k-1)p^k(1-p)^2 + p^k(1-p)$$

10. $$1 = E\left[\sum_1^n X_i \middle/ \sum_1^n X_i\right] = \sum_1^n E\left[X_i \middle/ \sum_1^n X_i\right] = nE\left[X_1 \middle/ \sum_1^n X_i\right]$$

Hence,

$$E\left[\sum_1^k X_i \middle/ \sum_1^n X_i\right] = k/n$$

11. Let

$$I_j = \begin{cases} 1 & \text{outcome } j \text{ never occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then $X = \sum_1^r I_j$ and $E[X] = \int_{j=1}^{r} (1-p_j)^n$

12. For X having the Cantor distribution, $E[X] = 1/2$, $\text{Var}(X) = 1/8$

13. Let

$$I_j = \begin{cases} 1 & \text{record at } j \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_1^n I_j\right] = \sum_1^n E[I_j] = \sum_1^n P\{X_j \text{ is largest of } X_1, ..., X_j\} = \sum_1^n 1/j$$

$$\text{Var}\left(\sum_1^n I_j\right) = \sum_1^n \text{Var}(I_j) = \sum_1^n \frac{1}{j}\left(1 - \frac{1}{j}\right)$$

15. $\mu = \sum_{i=1}^{n} p_i$ by letting Number $= \sum_{i=1}^{n} X_i$ where $X_i = \begin{cases} 1 & i \text{ is success} \\ 0 & \text{---} \end{cases}$

Var(Number) $= \sum_{i=1}^{n} p_i(1 - p_i)$

maximization of variance occur when $p_i \equiv \mu/n$

minimization of variance when $p_i = 1$, $i = 1, \ldots, [\mu]$, $p_{[\mu]+1} = \mu - [\mu]$

To prove the maximization result, suppose that 2 of the p_i are unequal—say $p_i \neq p_j$. Consider a new p-vector with all other p_k, $k \neq i, j$, as before and with $\overline{p}_i = \overline{p}_j = \frac{p_i + p_j}{2}$. Then in the variance formula, we must show

$$2\left(\frac{p_i + p_j}{2}\right)\left(1 - \frac{p_i + p_j}{2}\right) \geq p_i(1 - p_i) + p_j(1 - p_j)$$

or equivalently,

$$p_i^2 + p_j^2 - 2p_ip_j = (p_i - p_j)^2 \geq 0.$$

The maximization is similar.

16. Suppose that each element is, independently, equally likely to be colored red or blue. If we let X_i equal 1 if all the elements of A_i are similarly colored, and let it be 0 otherwise, then $\sum_{i=1}^{r} X_i$ is the number of subsets whose elements all have the same color. Because

$$E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E\left[X_i\right] = \sum_{i=1}^{r} 2(1/2)^{|A_i|}$$

it follows that for at least one coloring the number of monocolored subsets is less than or equal to $\sum_{i=1}^{r} (1/2)^{|A_i|-1}$

17. $\mathrm{Var}\left(\lambda X_1 + (1-\lambda)X_2\right) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2$

$$\frac{d}{d\lambda}(\quad) = 2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2 = 0 \Rightarrow \lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

As Var($\lambda X_1 + (1 - \lambda)X_2$) $= E\left[(\lambda X_1 + (1-\lambda)X_2 - \mu)^2\right]$ we want this value to be small.

18. (a. Binomial with parameters m and $P_i + P_j$.

(b) Using (a) we have that $\text{Var}(N_i + N_j) = m(P_i + P_j)(1 - P_i - P_j)$ and thus

$$m(P_i + P_j)(1 - P_i - P_j) = mP_i(1 - P_i) + mP_j(1 - P_j) + 2\,\text{Cov}(N_i, N_j)$$

Simplifying the above shows that

$$\text{Cov}(N_i, N_j) = -mP_iP_j.$$

19. $$\begin{aligned}\text{Cov}(X + Y, X - Y) &= \text{Cov}(X, X) + \text{Cov}(X, -Y) + \text{Cov}(Y, X) + \text{Cov}(Y, -Y)\\ &= \text{Var}(X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Var}(Y)\\ &= \text{Var}(X) - \text{Var}(Y) = 0.\end{aligned}$$

20. (a) $\text{Cov}(X, Y|Z)$

$$\begin{aligned}&= E[XY - E[X|Z]Y - XE[Y|Z] + E[X|Z]E[Y|Z]\ [Z]\\ &= E[XY|Z] - E[X|Z]\ E[Y|Z] - E[X|Z]E[Y|Z] + E[X|Z]E[Y|Z]\\ &= E[XY|Z] - E[X|Z]E[Y|Z]\end{aligned}$$

where the next to last equality uses the fact that given Z, $E[X|Z]$ and $E[Y|Z]$ can be treated as constants.

(b) From (a)

$$E[\text{Cov}(X, Y|Z)] = E[XY] - E[E[X|Z]E[Y|Z]]$$

On the other hand,

$$\text{Cov}(E[X|Z], E[Y|Z] = E[E[X|Z]E[Y|Z]] - E[X]E[Y]$$

and so

$$\begin{aligned}E[\text{Cov}(X, Y|Z)] + \text{Cov}(E[X|Z], E[Y|Z]) &= E[XY] - E[X]E[Y]\\ &= \text{Cov}(X, Y)\end{aligned}$$

(c) Noting that $\text{Cov}(X, X|Z) = \text{Var}(X|Z)$ we obtain upon setting $Y = Z$ that

$$\text{Var}(X) = E[\text{Var}(X|Z)] + \text{Var}(E[X|Z])$$

21. (a) Using the fact that f integrates to 1 we see that

$c(n, i) \equiv \int_0^1 x^{i-1}(1-x)^{n-i}\,dx = (i-1)!(n-i)!/n!$. From this we see that

$E[X_{(i)}] = c(n+1, i+1)/c(n, i) = i/(n+1)$

$E[X_{(i)}^2] = c(n+2, i+2)/c(n, i) = \dfrac{i(i+1)}{(n+2)(n+1)}$

and thus

$$\text{Var}(X_{(i)}) = \frac{i(n+1-i)}{(n+1)^2(n+2)}$$

(b) The maximum of $i(n+1-i)$ is obtained when $i = (n+1)/2$ and the minimum when i is either 1 or n.

22. $\text{Cov}(X, Y) = b\,\text{Var}(X)$, $\text{Var}(Y) = b^2\,\text{Var}(X)$

$$\rho(X,Y) = \frac{b\,\text{Var}(X)}{\sqrt{b^2}\,\text{Var}(X)} = \frac{b}{|b|}$$

26. Follows since, given X, $g(X)$ is a constant and so

$E[g(X)Y \mid X] = g(X)E[Y \mid X]$

27. $E[XY] = E[E[XY \mid X]]$
$\qquad = E[XE[Y \mid X]]$

Hence, if $E[Y \mid X] = E[Y]$, then $E[XY] = E[X]E[Y]$. The example in Section 3 of random variables uncorrelated but not independent provides a counterexample to the converse.

28. The result follows from the identity

$E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]]$ which is obtained by noting that, given X, X may be treated as a constant.

29. $$x = E[X_1 + \ldots + X_n \mid X_1 + \ldots + X_n = x] = E\left[X_1 \middle| \sum X_i = x\right] + \ldots + E\left[X_n \middle| \sum X_i = x\right]$$
$$= nE\left[X_1 \middle| \sum X_i = x\right]$$

Hence, $E[X_1 \mid X_1 + \ldots + X_n = x] = x/n$

30. $E[N_iN_j \mid N_i] = N_iE[N_j \mid N_i] = N_i(n - N_i)\dfrac{p_j}{1-p_i}$ since each of the $n - N_i$ trials no resulting in outcome i will independently result in j with probability $p_j/(1 - p_i)$. Hence,

$$E[N_iN_j] = \frac{p_j}{1-p_i}\left(nE[N_i] - E\left[N_i^2\right]\right) = \frac{p_j}{1-p_i}\left[n^2p_i - n^2p_i^2 - np_i(1-p_i)\right]$$
$$= n(n-1)p_ip_j$$

and

$$\text{Cov}(N_i, N_j) = n(n-1)p_ip_j - n^2p_ip_j = -np_ip_j$$

31. By induction: true when $t = 0$, so assume for $t - 1$. Let $N(t)$ denote the number after stage t.

$$E[N(t) \mid N(t-1)] = N(t-1) - E[\text{number selected}]$$
$$= N(t-1) - N(t-1)\frac{r}{b+w+r}$$
$$E[N(t) \mid N(t-1)] = N(t-1)\frac{b+w}{b+w+r}$$
$$E[N(t)] = \left(\frac{b+w}{b+w+r}\right)^t w$$

32.
$$E[XI_A] = E[XI_A \mid A]P(A) + E[XI_A \mid A^c]P(A^c)$$
$$= E[X \mid A]P(A)$$

34. (a) $E[T_r \mid T_{r-1}] = T_{r-1} + 1 + (1 - p)E[T_r]$

(b) Taking expectations of both sides of (a) gives

$$E[T_r] = E[T_{r-1}] + 1 + (1 - p)E[T_r]$$

or

$$E[T_r] = \frac{1}{p} + \frac{1}{p}E[T_{r-1}]$$

(c) Using the result of part (b) gives

$$
\begin{aligned}
E[T_r] &= \frac{1}{p} + \frac{1}{p}E[T_{r-1}] \\
&= \frac{1}{p} + \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p}E[T_{r-2}]\right) \\
&= 1/p + (1/p)^2 + (1/p)^2 E[T_{r-2}] \\
&= 1/p + (1/p)^2 + (1/p)^3 + (1/p)^3 E[T_{r-3}] \\
&= \sum_{i=1}^{r} (1/p)^i + (1/p)^r E[T_0] \\
&= \sum_{i=1}^{r} (1/p)^i \quad \text{since } E[T_0] = 0.
\end{aligned}
$$

35. $$
\begin{aligned}
P(Y > X) &= \sum_j P(Y > X \mid X = j) p_j \\
&= \sum_j P(Y > j \mid X = j) p_j \\
&= \sum_j P(Y > j) p_j \\
&= \sum_j (1-p)^j p_j
\end{aligned}
$$

36. Condition on the first ball selected to obtain

$$M_{a,b} = \frac{a}{a+b} M_{a-1,b} + \frac{b}{a+b} M_{a,b-1},\ a, b > 0$$

$$M_{a,0} = a, \qquad M_{0,b} = b, \qquad M_{a,b} = M_{b,a}$$

$$M_{2,1} = \frac{4}{3}, \qquad M_{3,1} = \frac{7}{4}, \qquad M_{3,2} = 3/2$$

37. Let X_n denote the number of white balls after the n^{th} drawing

$$E[X_{n+1} \mid X_n] = X_n \frac{X_n}{a+b} + (X_n + 1)\left(1 - \frac{X_n}{a+b}\right) = \left(1 - \frac{1}{a+b}\right) X_n + 1$$

Taking expectations now yields (a).

To prove (b), use (a) and the boundary condition $M_0 = a$

(c) $P\{(n+1)\text{st is white}\} = E[P\{(n+1)\text{st is white} \mid X_n\}]$

$$= E\left[\frac{X_n}{a+b}\right] = \frac{M_n}{a+b}$$

40. Let I equal 1 if the first trial is a success and 0 if it is a failure. Now, if $I = 1$, then $X = 1$; because the variance of a constant is 0, this gives

$$\text{Var}\left(X \mid I = 1\right) = 0$$

On the other hand, if $I = 0$, then the conditional distribution of X given that $I = 0$ is the same as the unconditional distribution of 1 (the first trial) plus a geometric with parameter p (the number of additional trials needed for a success).

Therefore,

$$\text{Var}\left(X \mid I = 0\right) = \text{Var}(1 + X) = \text{Var}(X)$$

Consequently,

$$\begin{aligned} E\left[\text{Var}\left(X \mid I\right)\right] &= \text{Var}\left(X \mid I = 1\right)P(I = 1) + \text{Var}\left(X \mid I = 0\right)P(I = 0) \\ &= (1 - p)\text{Var}(X) \end{aligned}$$

By the same reasoning used to compute the conditional variances, we have

$$E\left[X \mid I = 1\right] = 1, \qquad E\left[X \mid I = 0\right] = 1 + E[X] = 1 + \frac{1}{p}$$

which can be written as

$$E\left[X \mid I\right] = 1 + \frac{1}{p}(1 - I)$$

yielding that

$$\text{Var}\left(E\left[X \mid I\right]\right) = \frac{1}{p^2}\text{Var}(I) = \frac{1}{p^2}p(1 - p) = \frac{1 - p}{p}$$

The conditional variance formula now gives

$$\begin{aligned} Var(X) &= E\left[\text{Var}\left(X \mid I\right)\right] + Var\left(E\left[X \mid I\right]\right) \\ &= (1 - p)\text{Var}(X) + \frac{1 - p}{p} \end{aligned}$$

or

$$\text{Var}(X) = \frac{1 - p}{p^2}$$

41. (a) No

(b) Yes, since $f_Y(x \mid I = 1) = f_X(x) = f_X(-x) = f_Y(x \mid I = 0)$

(c) $f_Y(x) = \frac{1}{2} f_X(x) + \frac{1}{2} f_X(-x) = f_X(x)$

(d) $E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]] = 0$

(e) No, since X and Y are not jointly normal.

42. If $E[Y \mid X]$ is linear in X, then it is the best linear predictor of Y with respect to X.

43. Must show that $E[Y^2] = E[XY]$. Now

$$\begin{aligned} E[XY] &= E[XE[X \mid Z]] \\ &= E[E[XE[X \mid Z] \mid Z]] \\ &= E[E^2[X \mid Z]] = E[Y^2] \end{aligned}$$

44. Write $X_n = \sum_{i=1}^{X_{n-1}} Z_i$ where Z_i is the number of offspring of the ith individual of the $(n-1)$st generation. Hence,

$$E[X_n] = E[E[X_n \mid X_{n-1}]] = E[\mu X_{n-1}] = \mu E[X_{n-1}]$$

so,

$$E[X_n] = \mu E[X_{n-1}] = \mu^2 E[X_{n-2}] \ldots = \mu^n E[X_0] = \mu^n$$

(c) Use the above representation to obtain

$$E[X_n \mid X_{n-1}] = \mu X_{n-1},\ \mathrm{Var}(X_n \mid X_{n-1}) = \sigma^2 X_{n-1}$$

Hence, using the conditional Variance Formula,

$$\mathrm{Var}(X_n) = \mu^2\ \mathrm{Var}(X_{n-1}) + \sigma^2 \mu^{n-1}$$

(d) $\pi = P\{\text{dies out}\}$

$$= \sum_j P\{\text{dies out} \mid X_i = j\} p_j$$

$= \sum_j \pi^j p_j$, since each of the j members of the first generation can be thought of as starting their own (independent) branching process.

46. It is easy to see that the n^{th} derivative of $\sum_{j=0}^{\infty} (t^2/2)^j / j!$ will, when evaluated at $t = 0$, equal 0 whenever n is odd (because all of its terms will be constants multiplied by some power of t). When $n = 2j$ the n^{th} derivative will equal $\frac{d^n}{dt^n}\{t^n\}/(j!2^j)$ plus constants multiplied by powers of t. When evaluated at 0, this gives that

$$E[Z^{2j}] - (2j)!/(j!2^j)$$

47. Write $X = \sigma Z + \mu$ where Z is a standard normal random variable. Then, using the binomial theorem,

$$E[X^n] = \sum_{i=0}^{n} \binom{n}{i} \sigma^i E[Z^i] \mu^{n-i}$$

Now make use of theoretical exercise 46.

48. $\phi_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = e^{tb}E[e^{taX}] = e^{tb}\phi_X(ta)$

49. Let $Y = \log(X)$. Since Y is normal with mean μ and variance σ^2 it follows that its moment generating function is

$$M(t) = E[e^{tY}] = e^{\mu t + \sigma^2 t^2/2}$$

Hence, since $X = e^Y$, we have that

$$E[X] = M(1) = e^{\mu + \sigma^2/2}$$

and

$$E[X^2] = M(2) = e^{2\mu + 2\sigma^2}$$

Therefore,

$$\text{Var}(X) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

50. $\psi(t) = \log \phi(t)$

$\psi'(t) = \phi'(t)/\phi(t)$

$$\psi''(t) = \frac{\phi(t)\phi''(t) - (\phi'(t))^2}{\phi^2(t)}$$

$$\psi''(t)\Big|_{t=0} = E[X^2] - (E[X])^2 = \text{Var}(X).$$

51. Gamma (n, λ)

52. Let $\phi(s, t) = E[e^{sX+tY}]$

$$\frac{\partial^2}{\partial s \partial t}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[XYe^{sX+tY}]\Big|_{\substack{s=0\\t=0}} = E[XY]$$

$$\frac{\partial}{\partial s}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[X], \quad \frac{\partial}{\partial t}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[Y]$$

53. Follows from the formula for the joint moment generating function.

54. By symmetry, $E[Z^3] = E[Z] = 0$ and so $\text{Cov}(Z,Z^3) = 0$.

55. (a) This follows because the conditional distribution of $Y + Z$ given that $Y = y$ is normal with mean y and variance 1, which is the same as the conditional distribution of X given that $Y = y$.

(b) Because $Y + Z$ and Y are both linear combinations of the independent normal random variables Y and Z, it follows that $Y + Z$, Y has a bivariate normal distribution.

(c)
$$\mu_x = E[X] = E[Y + Z] = \mu$$
$$\sigma_x^2 = \text{Var}(X) = \text{Var}(Y + Z) = \text{Var}(Y) + \text{Var}(Z) = \sigma^2 + 1$$
$$\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(Y+Z, Y)}{\sigma\sqrt{\sigma^2+1}} = \frac{\sigma}{\sqrt{\sigma^2+1}}$$

(d) and (e) The conditional distribution of Y given $X = x$ is normal with mean

$$E[Y \mid X = x] = \mu + \rho\frac{\sigma}{\sigma_x}(x - \mu_x) = \mu + \frac{\sigma^2}{1+\sigma^2}(x - \mu)$$

and variance

$$\text{Var}(Y \mid X = x) = \sigma^2\left(1 - \frac{\sigma^2}{\sigma^2+1}\right) = \frac{\sigma^2}{\sigma^2+1}$$

Chapter 8

Problems

1. $P\{0 \le X \le 40\} = 1 - P\{|X - 20| > 20\} \ge 1 - 20/400 = 19/20$

2. (a) $P\{X \ge 85\} \le E[X]/85 = 15/17$

   (b) $P\{65 \le X \le 85) = 1 - P\{|X - 75| > 10\} \ge 1 - 25/100$

   (c) $P\left\{\left|\sum_{i=1}^{n} X_i/n - 75\right| > 5\right\} \le \dfrac{25}{25n}$ so need $n = 10$

3. Let Z be a standard normal random variable. Then,

$$P\left\{\left|\sum_{i=1}^{n} X_i/n - 75\right| > 5\right\} \approx P\{|Z| > \sqrt{n}\} \le .1 \text{ when } n = 3$$

4. (a) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} \le 20/15$

   (b)
$$\begin{aligned} P\left\{\sum_{i=1}^{20} X_i > 15\right\} &= P\left\{\sum_{i=1}^{20} X_i > 15.5\right\} \\ &\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} \\ &= P\{Z > -1.006\} \\ &\approx .8428 \end{aligned}$$

5. Letting X_i denote the i^{th} roundoff error it follows that $E\left[\sum_{i=1}^{50} X_i\right] = 0$,

$\text{Var}\left(\sum_{i=1}^{50} X_i\right) = 50 \text{ Var}(X_1) = 50/12$, where the last equality uses that $.5 + X$ is uniform (0, 1) and so $\text{Var}(X) = \text{Var}(.5 + X) = 1/12$. Hence,

$$\begin{aligned} P\left\{\left|\sum X_i\right| > 3\right\} &\approx P\{|N(0, 1)| > 3(12/50)^{1/2}\} \text{ by the central limit theorem} \\ &= 2P\{N(0, 1) > 1.47 = .1416 \end{aligned}$$

6. If X_i is the outcome of the i^{th} roll then $E[X_i] = 7/2$ $\text{Var}(X_i) = 35/12$ and so

$$\begin{aligned} P\left\{\sum_{i=1}^{79} X_i \le 300\right\} &= P\left\{\sum_{i=1}^{79} X_i \le 300.5\right\} \\ &\approx P\left\{N(0,1) \le \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\right\} = P\{N(0,1) \le 1.58\} = .9429 \end{aligned}$$

7. $$P\left\{\sum_{i=1}^{100} X_i > 525\right\} \approx P\left\{N(0,1) > \frac{525-500}{\sqrt{(100\times 25)}}\right\} = P\{N(0,1) > .5\} = .3085$$

where the above uses that an exponential with mean 5 has variance 25.

8. If we let X_i denote the life of bulb i and let R_i be the time to replace bulb i then the desired probability is $P\left\{\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \le 550\right\}$. Since $\sum X_i + \sum R_i$ has mean $100 \times 5 + 99 \times .25 =$ 524.75 and variance 2500 + 99/48 = 2502 it follows that the desired probability is approximately equal to $P\{N(0, 1) \le [550 - 524.75]/(2502)^{1/2}\} = P\{N(0, 1) \le .505\} = .693$
It should be noted that the above used that

$$\text{Var}(R_i) = \text{Var}\left(\frac{1}{2}\text{Unif}[0,1]\right) = 1/48$$

9. Use the fact that a gamma (n, 1) random variable is the sum of n independent exponentials with rate 1 and thus has mean and variance equal to n, to obtain:

$$\begin{aligned}
P\left\{\left|\frac{X-n}{n}\right| > .01\right\} &= P\left\{|X-n|/\sqrt{n} > .01\sqrt{n}\right\} \\
&\approx P\left\{|N(0,1)| > .01\sqrt{n}\right\} \\
&= 2P\left\{N(0,1) > .01\sqrt{n}\right\}
\end{aligned}$$

Now $P\{N(0, 1) > 2.58\} = .005$ and so $n = (258)^2$.

10. If W_n is the total weight of n cars and A is the amount of weight that the bridge can withstand then $W_n - A$ is normal with mean $3n - 400$ and variance $.09n + 1600$. Hence, the probability of structural damage is

$$P\{W_n - A \ge 0\} \approx P\left\{Z \ge (400-3n)/\sqrt{.09n+1600}\right\}$$

Since $P\{Z \ge 1.28\} = .1$ the probability of damage will exceed .1 when n is such that

$$400 - 3n \le 1.28\sqrt{.09n+1600}$$

The above will be satisfied whenever $n \ge 117$.

12. Let L_i denote the life of component i.

$$E\left[\sum_{i=1}^{100} L_i\right] = 1000 + \frac{1}{10}50(101) \; = 1505$$

$$\text{Var}\left(\sum_{i=1}^{100} L_i\right) = \sum_{i=1}^{100}\left(10+\frac{i}{10}\right)^2 = (100)^2 + (100)(101) + \frac{1}{100}\sum_{i=1}^{100} i^2$$

Now apply the central limit theorem to approximate.

13. (a) $P\{\overline{X} > 80\} = P\left\{\dfrac{\overline{X}-74}{14/5} > 15/7\right\} \approx P\text{PZ} > 2.14\} \approx .0162$

(b) $P\{\overline{Y} > 80\} = P\left\{\dfrac{\overline{Y}-74}{14/8} > 24/7\right\} \approx P\{Z > 3.43\} \approx .0003$

(c) Using that $SD(\overline{Y}-\overline{X}) = \sqrt{196/64 + 196/25} \approx 3.30$ we have

$$\begin{aligned} P\{\overline{Y}-\overline{X} > 2.2\} &= P\{\overline{Y}-\overline{X}\}/3.30 > 2.2/3.30\} \\ &\approx P\{Z > .67\} \approx .2514 \end{aligned}$$

(d) same as in (c)

14. Suppose n components are in stock. The probability they will last for at least 2000 hours is

$$p = P\left\{\sum_{i=1}^{n} X_i \geq 2000\right\} \approx P\left\{Z \geq \frac{2000-100n}{30\sqrt{n}}\right\}$$

where Z is a standard normal random variable. Since

$.95 = P\{Z \geq -1.64\}$ it follows that $p \geq .95$ if

$$\frac{2000-100n}{30\sqrt{n}} \leq -1.64$$

or, equivalently,

$$(2000 - 100n)/\sqrt{n} \leq -49.2$$

and this will be the case if $n \geq 23$.

15. $P\left\{\sum_{i=1}^{10,000} X_i > 2,700,000\right\} \approx P\{Z \geq (2,700,000 - 2,400,000)/(800 \cdot 100)\} = P\{Z \geq 3.75\} \approx 0$

16. (a) Number AJ's jobs, let X_i be the time it takes to do job i, and let $X_A = \sum_{i=1}^{20} X_i$ be the time that it takes AJ to finish all 20 jobs. Because

$$E[X_A] = 20(50) = 1000, \qquad \text{Var}(X_A) = 20(100) = 2000$$

the central limit theorem gives that

$$\begin{aligned} P\{X_A \le 900\} &= P\left\{\frac{X_A - 1000}{\sqrt{2000}} \le \frac{900 - 1000}{\sqrt{2000}}\right\} \\ &\approx P\{Z \le -2.236\} \\ &= 1 - \Phi(2.236) = .013 \end{aligned}$$

(b) Similarly, if we let X_M be the time that it takes MJ to finish all of her 20 jobs, then by the central limit theorem X_M is approximately normal with mean and variance

$$E[X_M] = 20(52) = 1040, \qquad \text{Var}(X_M) = 20(225) = 4500$$

Thus

$$\begin{aligned} P\{X_M \le 900\} &= P\left\{\frac{X_M - 1040}{\sqrt{4500}} \le \frac{900 - 1040}{\sqrt{4500}}\right\} \\ &\approx P\{Z \le -2.087\} \\ &= 1 - \Phi(2.087) = 0.18 \end{aligned}$$

(c) Because the sum of independent normal random variables is also normal, $D \equiv X_M - X_A$ is approximately normal with mean and variance

$$E[D] = 1040 - 1000 = 40, \qquad \text{Var(D)} = 4500 + 2000 = 6500$$

Hence,

$$\begin{aligned} P\{D > 0\} &= P\left\{\frac{D - 40}{\sqrt{6500}} \ge \frac{-40}{\sqrt{6500}}\right\} \\ &\approx P\{Z \ge -.4961\} \\ &= \Phi(.4961) = .691 \end{aligned}$$

Thus even though AJ is more likely than not to finish earlier than MJ, MJ has the better chance to finish within 900 minutes.

19. Let Y_i denote the additional number of fish that need to be caught to obtain a new type when there are at present i distinct types. Then Y_i is geometric with parameter $\frac{4-i}{4}$.

$$E[Y] = E\left[\sum_{i=0}^{3} Y_i\right] = 1 + \frac{4}{3} + \frac{4}{2} + 4 = \frac{25}{3}$$

$$\text{Var}[Y] = \text{Var}\left(\sum_{i=0}^{3} Y_i\right) = \frac{4}{9} + 2 + 12 = \frac{130}{9}$$

Hence,

$$P\left\{\left|Y - \frac{25}{3}\right| > \frac{25}{3}\sqrt{\frac{1300}{9}}\right\} \le \frac{1}{10}$$

and so we can take $a = \dfrac{25 - \sqrt{1300}}{3}$, $b = \dfrac{25 + \sqrt{1300}}{3}$.

Also,

$$P\left\{Y - \frac{25}{3} > a\right\} \le \frac{130}{130 + 9a^2} = \frac{1}{10} \text{ when } a = \frac{\sqrt{1170}}{3}.$$

Hence $P\left\{Y > \dfrac{25 + \sqrt{1170}}{3}\right\} \le .1.$

21. $g(x) = x^{n(n-1)}$ is convex. Hence, by Jensen's Inequality

$E[Y^{n/(n-1)}] \ge E[Y])^{n/(n-1)}$ Now set $Y = X^{n-1}$ and so
$E[X^n] \ge (E[X^{n-1}])^{n/(n-1)}$ or $(E[X^n])^{1/n} \ge (E[X^{n-1}])^{1/(n-1)}$

22. No

23. (a) $20/26 \approx .769$

(b) $20/(20 + 36) = 5/14 \approx .357$

(d) $p \approx P\{Z \ge (25.5 - 20)/\sqrt{20}\} \approx P\{Z \ge 1.23\} \approx .1093$

(e) $p = .112184$

Theoretical Exercises

1. This follows immediately from Chebyshev's inequality.

2. $P\{D > \alpha\} = P\{\,|X - \mu| > \alpha\mu\} \le \dfrac{\varsigma^2}{\alpha^2\mu^2} = \dfrac{1}{\alpha^2 r^2}$

3. (a) $\dfrac{\lambda}{\sqrt{\lambda}} = \sqrt{\lambda}$

   (b) $\dfrac{np}{\sqrt{np(1-p)}} = \sqrt{np/(1-p)}$

   (c) answer = 1

   (d) $\dfrac{1/2}{\sqrt{1/12}} = \sqrt{3}$

   (e) answer = 1

   (d) answer = $|\mu|/\sigma$

4. For $\varepsilon > 0$, let $\delta > 0$ be such that $|g(x) - g(\mathrm{c})| < \varepsilon$ whenever $|x - c| \le \delta$. Also, let B be such that $|g(x)| < B$. Then,

$$\begin{aligned} E[g(Z_n)] &= \int_{|x-c|\le\delta} g(x)dF_n(x) + \int_{|x-c|>\delta} g(x)dF_n(x) \\ &\le (\varepsilon + g(\mathrm{c}))P\{\,|Z_n - c| \le \delta\} + BP\{\,|Z_n - c| > \delta\} \end{aligned}$$

In addition, the same equality yields that

$$E[g(Z_n)] \ge (g(c) - \varepsilon)P\{\,|Z_n - c| \le \delta\} - BP\{\,|Z_n - c| > \delta\}$$

Upon letting $n \to \infty$, we obtain that

$$\begin{aligned} &\limsup E[g(Z_n)] \le g(c) + \varepsilon \\ &\liminf\; E[g(Z_n)] \ge g(c) - \varepsilon \end{aligned}$$

The result now follows since ε is arbitrary.

5. Use the notation of the hint. The weak law of large numbers yields that

$$\lim_{n\to\infty} P\{|(X_1 + \ldots + X_n)/n - c| > \varepsilon\} = 0$$

Since $X_1 + \ldots + X_n$ is binomial with parameters n, x, we have

$$E\left[f\left(\frac{X_1 + \ldots + X_n}{n}\right)\right] = \sum_{k=1}^{n} f(k/n)\binom{n}{k} x^k (1-x)^{n-k}$$

The result now follows from Exercise 4.

6. $$E[X] = \sum_{i=1}^{k} i\,P\{X = i\} + \sum_{i=k+1}^{\infty} i\,P\{X = i\}$$
$$\geq \sum_{i=1}^{k} i\,P\{X = k\}$$
$$= P\{X = k\}^{k(k+1)/2}$$
$$\geq \frac{k^2}{2} P\{X = k\}$$

7. Take logs and apply the central limit theorem

8. It is the distribution of the sum of t independent exponentials each having rate λ.

9. 1/2

10. Use the Chernoff bound: $e^{-ti}M(t) = e^{\lambda(e^t - 1) - ti}$ will obtain its minimal value when t is chosen to satisfy

$\lambda e^t = i$, and this value of t is negative provided $i < \lambda$.

Hence, the Chernoff bound gives

$$P\{X \leq i\} \leq e^{i-\lambda}(\lambda/i)^i$$

11. $e^{-ti}M(t) = (pe^t + q)^n e^{-ti}$ and differentiation shows that the value of t that minimizes it is such that

$$npe^t = i(pe^t + q) \text{ or } e^t = \frac{iq}{(n-i)p}$$

Using this value of t, the Chernoff bound gives that

$$P\{X \geq i\} \leq \left(\frac{iq}{n-i} + q\right)^n (n-i)^i\, p^i\,/(iq)^i$$
$$= \frac{(nq)^n (n-i)^i\, p^i}{i^i q^i (n-i)^n}$$

12. $1 = E[e^{\theta X}] \geq e^{\theta E[X]}$ by Jensen's inequality.

Hence, $\theta E[X] \leq 0$ and thus $\theta > 0$.

Chapter 9

Problems and Theoretical Exercises

1. (a) $P(2 \text{ arrivals in } (0, s) \mid 2 \text{ arrivals in } (0, 1)\}$

$=P\{2 \text{ in } (0, s), 0 \text{ in } (s, 1)\}/e^{-\lambda}\lambda^2/2)$
$= [e^{-\lambda s}(\lambda s)^2/2][e^{-(1-s)\lambda}]/(e^{-\lambda}\lambda^2/2) = s^2 = 1/9$ when $s = 1/3$

(b) $1 - P\{\text{both in last 40 minutes}) = 1 - (2/3)^2 = 5/9$

2. $e^{-3s/60}$

3. $e^{-3s/60} + (s/20)e^{-3s/60}$

8. The equations for the limiting probabilities are:

$$\begin{aligned}
\Pi_c &= .7\Pi_c + .4\Pi_s + .2\Pi_g \\
\Pi_s &= .2\Pi_x + .3\Pi_s + .4\Pi_g \\
\Pi_g &= .1\Pi_c + .3\Pi_s + .4\Pi_g \\
\Pi_c &+ \Pi_s + \Pi_g = 1
\end{aligned}$$

and the solution is: $\Pi_c = 30/59$, $\Pi_s = 16/59$, $\Pi_g = 13/59$. Hence, Buffy is cheerful 3000/59 percent of the time.

9. The Markov chain requires 4 states:

0 = RR = Rain today and rain yesterday

1 = RD = Dry today, rain yesterday

2 = DR = Rain today, dry yesterday

3 = DD = Dry today and dry yesterday

with transition probability matrix

$$\underline{P} = \begin{vmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .3 & .7 \\ .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \end{vmatrix}$$

The equations for the limiting probabilities are:

$$\Pi_0 = .8\Pi_0 + .4\Pi_2$$
$$\Pi_1 = .2\Pi_0 + .6\Pi_2$$
$$\Pi_2 = .3\Pi_1 + .2\Pi_3$$
$$\Pi_3 = .7\Pi_1 + .8\Pi_3$$
$$\Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 = 1$$

which gives

$$\Pi_0 = 4/15,\ \Pi_1 = \Pi_2 = 2/15,\ \Pi_3 = 7/15.$$

Since it rains today when the state is either 0 or 2 the probability is 2/5.

10. Let the state be the number of pairs of shoes at the door he leaves from in the morning. Suppose the present state is i, where $i > 0$. Now after his return it is equally likely that one door will have i and the other $5 - i$ pairs as it is that one will have $i - 1$ ant the other $6 - i$. Hence, since he is equally likely to choose either door when he leaves tomorrow it follows that

$$P_{i,i} = P_{i,5-i} = P_{i,i-1} = P_{i,6-i} = 1/4$$

provided all the states i, $5 - i$, $i - 1$, $6 - i$ are distinct. If they are not then the probabilities are added. From this it is easy to see that the transition matrix P_{ij}, $i, j = 0, 1, \ldots, 5$ is as follows:

$$\underline{P} = \begin{matrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \end{matrix}$$

Since this chain is doubly stochastic (the column sums as well as the row sums all equal to one) it follows that $\Pi_i = 1/6$, $i = 0, \ldots, 5$, and thus he runs barefooted one-sixth of the time.

11. (b) 1/2

(c) Intuitively, they should be independent.

(d) From (b) and (c) the (limiting) number of molecules in urn 1 should have a binomial distribution with parameters (M, 1/2).

Chapter 10

1. (a) After stage k the algorithm has generated a random permutation of 1, 2, …, k. It then puts element $k + 1$ in position $k + 1$; randomly chooses one of the positions 1, …, $k + 1$ and interchanges the element in that position with element $k + 1$.

   (b) The first equality in the hint follows since the permutation given will be the permutation after insertion of element k if the previous permutation is $i_1, \ldots, i_{j-1}, i, i_j, \ldots, i_{k-2}$ and the random choice of one of the k positions of this permutation results in the choice of position j.

2. Integrating the density function yields that that distribution function is

$$F(x) = \begin{matrix} e^{2x}/2 \quad , \quad x > 0 \\ 1 - e^{-2s}/2, \quad x > 0 \end{matrix}$$

which yields that the inverse function is given by

$$F^{-1}(u) = \begin{matrix} \log(2u)/2 & \text{if } u < 12 \\ -\log(2[1-u])/2 & \text{if } u > 1/2 \end{matrix}$$

Hence, we can simulate X from F by simulating a random number U and setting $X = F^{-1}(U)$.

3. The distribution function is given by

$$F(x) = \begin{matrix} x^2/4 - x + 1, & 2 \le x \le 3, \\ x - x^2/12 - 2, & 3 \le x \le 6 \end{matrix}$$

Hence, for $u \le 1/4$, $F^{-1}(u)$ is the solution of

$$x^2/4 - x + 1 = u$$

that falls in the region $2 \le x \le 3$. Similarly, for $u \ge 1/4$, $F^{-1}(u)$ is the solution of

$$x - x^2/12 - 2 = u$$

that falls in the region $3 \le x \le 6$. We can now generate X from F by generating a random number U and setting $X = F^{-1}(U)$.

4. Generate a random number U and then set $X = F^{-1}(U)$. If $U \le 1/2$ then $X = 6U - 3$, whereas if $U \ge 1/2$ then X is obtained by solving the quadratic $1/2 + X^2/32 = U$ in the region $0 \le X \le 4$.

5. The inverse equation $F^{-1}(U) = X$ is equivalent to

or

$$1 - e^{-\alpha X^{\beta}} = U$$
$$X = \{-\log(1 - U)/\alpha\}^{1/\beta}$$

Since $1 - U$ has the same distribution as U we can generate from F by generating a random number U and setting $X = \{-\log(U)/\alpha\}^{1/\beta}$.

6. If $\lambda(t) = ct^n$ then the distribution function is given by

$$1 - F(t) = \exp\{-kt^{n+1}\},\ t \ge 0 \quad \text{where } k = c/(n + 1)$$

Hence, using the inverse transform method we can generate a random number U and then set X such that

$$\exp\{-kX^{n+1}\} = 1 - U$$

or

$$X = \{-\log(1 - U)/k\}^{1/(n+1)}$$

Again U can be used for $1 - U$.

7. (a) The inverse transform method shows that $U^{1/n}$ works.

(b) $$\begin{aligned} P\{\text{Max}U_i \le v\} &= P\{U_1 \le x, \ldots, U_n \le x\} \\ &= \prod P\{U_i \le x\} \quad \text{by independence} \\ &= x^n \end{aligned}$$

(c) Simulate n random numbers and use the maximum value obtained.

8. (a) If X_i has distribution F_i , $i = 1, \ldots, n$, then, assuming independence, F is the distribution of $\text{Max}X_i$. Hence, we can simulate from F by simulating X_i, $i = 1, \ldots, n$ and setting $X = \text{Max}X_i$.

(b) Use the method of (a) replacing Max by Min throughout.

9. (a) Simulate X_i from F_i, $i = 1, 2$. Now generate a random number U and set X equal to X_1 if $U < p$ and equal to X_2 if $U > p$.

(b) Note that

$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

where

$$F_1(x) = 1 - e^{-3x}, \quad x > 0, \;\; F_2(x) = x, \quad 0 < x < 1$$

Hence, using (a) let U_1, U_2, U_3 be random numbers and set

$$X = \begin{matrix} -\log(U_1)/3 \text{ if } U_3 < 1/3 \\ U_2 \qquad\qquad \text{if } U_3 > 1/3 \end{matrix}$$

where the above uses that $-\log(U_1)/3$ is exponential with rate 3.

10. With $g(x) = \lambda e^{-\lambda x}$

$$\frac{f(x)}{g(x)} = \frac{2e^{-x^2/2}}{\lambda(2\pi)^{1/2}e^{-\lambda x}} = \frac{2}{\lambda(2\pi)^{1/2}}\exp\{-[(x-\lambda)^2 - \lambda^2]/2\}$$
$$\frac{2e^{\lambda^2/2}}{\lambda(2\pi)^{1/2}}\exp\{-(x-\lambda)^2/2\}$$

Hence, $c = 2e^{\lambda^2/2}/[\lambda(2\pi)^{1/2}]$ and simple calculus shows that this is minimized when $\lambda = 1$.

11. Calculus yields that the maximum value of $f(x)/g(x) = 60x^3(1-x)^2$ is attained when $x = 3/5$ and is thus equal to 1296/625. Hence, generate random numbers U_1 and U_2 and set $X = U_1$ if $U_2 \le 3125U_1^3(1-U_1)^2/108$. If not, repeat.

12. Generate random numbers $U_1, \ldots, U_n$, and approximate the integral by $[k(U_1) + \ldots + k(U_n)]/n$. This works by the law of large numbers since $E[k(U)] = \int_0^1 k(x)dx$.

16. $E[g(X)/f(X)] = \int [g(x)/f(x)]f(x)dx = \int g(x)dx$