Computational Number Theory August 6, 2013

Lecture Notes

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Riemann ζ function Define Riemann zeta function $\zeta(s)$ in this case:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, Re(s) > 1$$

Riemann zeta function is a meromorphic function on the whole complex s-plane, which is holomorphic everywhere except for a simple pole at s = 1 with residue 1.

The connection between the zeta function and prime numbers :

$$\zeta(s) = \prod_{p}^{\infty} \frac{1}{1 - p^{-s}}$$

Riemann's functional equation :

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \gamma(1-s) \zeta(1-s)$$

This equation relates values of the Riemann zeta function at the points s and 1 − s. The functional equation (owing to the properties of the sine function) implies that $\zeta(s)$ has a simple zero at each even negative integer s = −2n — these are known as the trivial zeros of (s). For s an even positive integer, the product $sin(\pi^{s/2})\gamma(1-s)$ is regular and the functional equation relates the values of the Riemann zeta function at odd negative integers and even positive integers.

Riemann found a symmetric version of the functional equation, given by first defining :

$$\xi(s) = \frac{1}{2} \pi^{-\frac{s}{2}} s(s-1) \gamma(\frac{s}{2}) \zeta(s)$$

Riemann found an explicit formula for the number of primes $\pi(x)$ less than a given number x. His formula was given in terms of the related function :

$$\Pi(x) = \sum_{p, p^k < x} \frac{1}{k} = \sum n \frac{\pi(x^{\frac{1}{n}})}{n}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\ln \zeta(s) = -\sum_p \ln(1 - p^{-s}) = \sum_p \sum_{n \geq 1} \frac{p^{-ns}}{n}$$

$$= \int_0^{+\infty} x^{-s} d\Pi(x) = s \int_0^{+\infty} \Pi(x) x^{-s-1} dx$$

$$\Pi(x) = \frac{1}{2\pi} \int_{a-\infty}^{a+\infty} \frac{\ln \zeta(z)}{z} x^z dz, a > 1$$

Riemann hypothesis The Riemann zeta function $\zeta(s)$ is defined for all complex numbers $s \neq 1$ with a simple pole at $s = 1$. It has zeros at the negative even integers (i.e. at s = -2, -4, -6, ...). These are called the trivial zeros. The Riemann hypothesis is concerned with the non-trivial zeros, and states that:

1. For $0 < Im(s) < T$, $\zeta(s)$ has about $\frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi}$ zeros. 2. For $0 < Im(s) < T, Re(s) = \frac{1}{2}$, $\zeta(s)$ has about $\frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi}$ zeros. 3. The real part of any non-trivial zero of the Riemann zeta function is 1/2.

Only the first part of RH has been proved.

Generalized Riemann hypothesis (GRH) A Dirichlet character is a completely multiplicative arithmetic function such that there exists a positive integer k with (n + k) = (n) for all n and (n) = 0 whenever gcd(n, k) > 1. If such a character is given, we define the corresponding Dirichlet L-function by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi}{n^s}$$

for every complex number s with real part > 1. By analytic continuation, this function can be extended to a meromorphic function defined on the whole complex plane. The generalized Riemann hypothesis asserts that for every Dirichlet character and every complex number s with L(,s) = 0: if the real part of s is between 0 and 1, then it is actually 1/2.
The case (n) = 1 for all n yields the ordinary Riemann hypothesis.

Extended Riemann hypothesis (ERH) Suppose K is a number field (a finite-dimensional field extension of the rationals Q) with ring of integers O_K (this ring is the integral closure of the integers Z in K). If a is an ideal of O_K, other than the zero ideal we denote its norm by Na. The Dedekind zeta-function of K is then defined by

$$\zeta_K(s) = \sum_a \frac{1}{(Na)^s}$$

for every complex number s with real part > 1. The sum extends over all non-zero ideals a of O_K.
The Dedekind zeta-function satisfies a functional equation and can be extended by analytic continuation to the whole complex plane. The resulting function encodes important information about the number field K. The extended Riemann hypothesis asserts that for every number field K and every complex number s with K(s) = 0: if the real part of s is between 0 and 1, then it is in fact 1/2.

Grand Riemann hypothesis The grand Riemann hypothesis is a generalisation of the Riemann hypothesis and Generalized Riemann hypothesis. It states that the nontrivial zeros of all automorphic L-functions lie on the critical line 1/2 + it with t a real number variable and i the imaginary unit.

Critical Line Theorem $\exists K > 0, T_0 > 0, \forall T > T_0$, $\zeta(s)$ has at least $KT \ln T$ nontrivial zeros on $Re(s) = \frac{1}{2}, 0 \leq Im(s) \leq T$