Linear Equations

MA 341

5?

$$\frac{dy}{dx} + p(x)\, y = q(x)$$

$$\mu(x)\frac{dy}{dx} + \mu(x)\, p(x)\, y = \mu(x)\, q(x)$$

integrating factor,

$$\mu(x) = e^{\int p(x)\,dx}$$

$$\frac{d}{dx}\left(\mu(x)\, y\right) = \mu(x)\, q(y)$$

ex

$$\frac{dy}{dx} + ay = 0 \quad , \quad a = \text{const}$$

$$p(x) = a \quad , \quad q(x) = 0$$

$$\mu(x) = e^{\int p(x)\,dx} = e^{ax}$$

$$e^{ax}\frac{dy}{dx} + a e^{ax} y = 0$$

$$\frac{d}{dx}\left(e^{ax}y\right) = e^{ax}\frac{dy}{dx} + ae^{ax}y$$

$$\frac{d}{dx}\left(e^{ax}y\right) = 0$$

$e^{ax}y = C$, C = constant

$$\boxed{y = \frac{C}{e^{ax}} = Ce^{-ax}}$$

- general solution

More generally,

$$\frac{dy}{dx} + ay = x$$

$p(x) = a$

$q(x) = x$

μ still $= e^{ax}$

<u>but</u>

$$e^{ax}\frac{dy}{dx} + a e^{ax} y = e^{ax} x$$

$$\frac{d}{dx}\left(e^{ax} y\right) = e^{ax} x$$

$$e^{ax} y = \int e^{ax} x \, dx$$

integration by parts!

$$\int u \, dv = uv - \int v \, du$$

$u = x \qquad v' = e^{ax}$

$u' = 1 \qquad v = \frac{1}{a} e^{ax}$

$$= \frac{1}{a} x e^{ax} - \int \frac{1}{a} e^{ax} dx$$

$$= \frac{1}{a} x e^{ax} - \frac{1}{a^2} e^{ax} + C$$

$$e^{ax} y = \frac{1}{a} x e^{ax} - \frac{1}{a^2} e^{ax} + C$$

$$\boxed{y = \frac{x}{a} - \frac{1}{a^2} + Ce^{-ax}}$$

- general solution of

$$\frac{\partial y}{\partial x} + ay = x$$

ex) You see $\frac{\partial y}{\partial x} + ay = 0$ in

1) Radioactive Decay

y = mass of decaying chemical

t = time

$$\frac{\partial y}{\partial t} + ay = 0$$

$\frac{\partial y}{\partial t}$ prop. to $y(t)$

$$y = Ce^{-at}$$

$y(0) = C$

$t = 0$

$$\boxed{y(t) = y_0 e^{-at}}$$

half-life: t for $Y_f = \frac{Y_0}{2}$ (T)

$$\frac{Y_0}{2} = e^{-aT}(Y_0)$$

a: <u>decay</u> <u>const</u>

$$\frac{1}{2} = e^{-aT}$$

$$-aT = \ln(1/2)$$

$$aT = \ln(2)$$

$$T = \frac{\ln(2)}{a}$$

$Y_0 = 16$ lbs

$T = 10$ days

how long until sample decays to 4 lbs?

$\frac{Y_0}{4}$

$$\underline{\underline{Y(t)}} \over 4 = \frac{Y_0}{16} e^{\underline{\underline{-a}}t} \left(\frac{\ln(2)}{T}\right)$$

$$4 = 16e^{-\frac{\ln(2)}{10}t}$$

solve for t

$$\frac{1}{4} = e^{\left(-\frac{\ln(2)}{10}\right)t}$$

$$e^{\ln\left(\frac{1}{4}\right)} = e^{\left(-\frac{\ln(2)}{10}t\right)}$$

$$\ln\frac{1}{4} = \frac{\ln(2)}{-10}t$$

$$t = -\frac{10\ln(2^{-2})}{\ln(2)}$$

$$= -10(-2)\left(\frac{\ln(2)}{\ln(2)}\right)$$

$$= \boxed{20}$$

2) Newton's Law of Cooling

y - temp of object
y_0 - room temperature

the rate of change of y
is proportional to $(y - y_0)$

$$\frac{dy}{dt} = -a(y - y_0)$$

$a > 0$, add minus

(radiactive decay: $\frac{dy}{dt} = -ay$; almost identical)

$$\frac{d(y - y_0)}{dt} = -a(y - y_0)$$

2.4: Exact Equations

$$\frac{dy}{dx} = f(x)g(y)$$

$$dy = f(x)g(y)\,dx$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M(x,y)$, $N(x,y)$ – some given functions

differential forms

A differential form is exact if there is a function $F(x,y)$ s.t.

$$M = F_x$$
$$N = F_y$$

$$M\,dx + N\,dy = F_x\,dx + F_y\,dy$$
$$= dF$$

differential of $F(x,y)$

If $M\,dx + N\,dy$ is exact,
and $F(x,y)$ is known, eqn.
can be solved quickly.

$\triangleright$ $dF = 0$

this is only possible
if $F(x,y) = C$

($F(x,y) = C$ is an
implicit solution)

Solve for F

$$F_x = M$$
$$F_y = N$$

ex) $x\,dx + y\,dy = 0$

it's exact. (we'll see why later.)

$F_x = x$

$F_y = y$

$F = \int F\,dx$

$= \frac{x^2}{2} + g(y) + C$

$\left(\frac{x^2}{2} + g(y)\right)_y = y$

$g' = y \longrightarrow g = \frac{y^2}{2}$

$F = \frac{x^2}{2} + \frac{y^2}{2}$

$\boxed{\frac{x^2}{2} + \frac{y^2}{2} = C}$ ← implicit solution