MA 341

Variation of Parameters

$y'' + py' + qy = f(t)$ — we want solution to in the form $u_1(x)y_1(x) + u_2(x)y_2(x)$

corresponding homogeneous equation

$y'' + py' + qy = 0$

general solution is

$C_1 y_1(x) + C_2 y_2(x)$

(y_1, y_2 independent) 1

find $y_1(x)$, $y_2(x)$

$u_1(x)$ and $u_2(x)$ satisfy the system:

$$\boxed{\begin{aligned} y_1(x)u_1'(x) + y_2(x)u_2'(x) &= 0 \\ y_1'(x)u_1'(x) + y_2'(x)u_2'(x) &= 0 \end{aligned}}$$

$$u_1' = \frac{-y_2(x)f(x)}{y_1y_2' - y_2y_1'}$$

$$u_2' = \frac{y_1(x)f(x)}{y_1y_2' - y_2y_1'}$$

integrate: 2

$$u_1 = \int \frac{-y_2 f}{y_1y_2' - y_2y_1'}\,dx$$

$$u_2 = \int \frac{y_1 f}{y_1y_2' - y_2y_1'}\,dx$$

3 Plug into

$u_1(x)y_1(x) + u_2(x)y_2(x)$

4 if IVP, solve for constants

Notes

- use undetermined coefficients if integrals are too difficult
- use this for e.g. $\tan(x) = f(x)$

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ex $2y'' - 2y' - 4y = 2e^{2x}$

$y'' - y' - 2y = e^{2x}$

homo eqn 1 $y'' - y' - 2y = 0$

char eqn

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

roots $r_1 = 2,\ r_2 = -1$

$y_h = C_1 e^{2x} + C_2 e^{-x}$

indep (e^{2x}, e^{-x} independent)

funcs $y_1 = e^{2x}$ $\quad y_1' = 2e^{2x}$

$y_2 = e^{-x}$ $\quad y_2' = -e^{-x}$

calc denominator $y_1y_2' - y_2y_1'$

$= e^{2x}(-e^{-x})$

$= -3e^{x}$

calc u_1, u_2

2

$$u_1 = \int \frac{-e^{x}e^{2x}}{-3e^{x}}\,dx$$

$$= \int \frac{e^{x}}{3e^{x}}\,dx = \int \frac{dx}{3} = \frac{1}{3}x + C_1$$

$$u_2 = \int \frac{e^{2x}e^{2x}}{-3e^{x}}\,dx = \int \frac{e^{4x}}{-3e^{x}}\,dx = \int -\frac{1}{3}e^{3x}\,dx$$

$$= -\frac{1}{9}e^{3x} + C_2$$

plug y_1, y_2, u_1, u_2 into gen. eqn.

3

$$\boxed{y = \left(\tfrac{1}{3}x + C_1\right)e^{2x} + \left(-\tfrac{1}{9}e^{3x} + C_2\right)e^{-x}}$$

$$y = \underbrace{\left(\tfrac{1}{3}x - \tfrac{1}{9}\right)e^{2x}}_{y_p} + \underbrace{C_1e^{2x} + C_2e^{-x}}_{y_h}$$

ex] $y'' + y = \tan x$ ← Not a quasipolynomial: use V.o.P.

① $y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

y_1, y_2, y_1', y_2'

$y_1 = \cos x \qquad y_1' = -\sin x$

$y_2 = \sin x \qquad y_2' = \cos x$

denom for u's

$y_1 y_2' - y_2 y_1' = 1$

② $u_1 = \int -\sin x \tan x \, dx = -\int \frac{\sin^2 x}{\cos x} dx$ ← 242!

substitution: $v = \sin x$

$= -\int \frac{1 - \cos^2 x}{\cos x} dx$

$u_2 = \int \cos x \tan x \, dx$

$= \int \sin x \, dx$

$\boxed{u_2 = -\cos x + C_2}$

$= -\int \left(\frac{1}{\cos x} - \cos x \right) dx$

$= \int -\frac{dx}{\cos x} + \int \cos x \, dx$

$= \sin - \int \frac{1}{\cos x} dx$

Attempt 1 $\quad x = \arccos(t)$

$\frac{dx}{dt} = -\frac{1}{\sqrt{1 - t^2}}$

$= \sin x - \int \frac{2t \, dt}{2t^2 \sqrt{1 - t^2}}$?

sub: $v = 1 - t^2$

$v' = -2t \, dt$

$= \int \frac{dv}{2\sqrt{v}\,(1 - v)}$

sub: $v = z^2$:

$(v' = 2z \, dz)$

$= \int \frac{dv}{2\sqrt{v}\,(1-v)} = \int \frac{2z \, dz}{2z(1 - z^2)}$

Partial fractions: $\int \frac{dz}{1 - z^2} = \frac{A}{1 - z} + \frac{B}{1 + z}$

$= \int \frac{1}{2} \left(\frac{1}{1 - z} + \frac{1}{1 + z} \right) dz$

$= -\frac{1}{2} \ln|1 - z|$

$+ \frac{1}{2} \ln|1 + z|$

$= \frac{1}{2} \ln|1 + \sqrt{v}| - \frac{1}{2} \ln|1 - \sqrt{v}|$

$= \frac{1}{2} \ln|1 + \sqrt{1 - t^2}|$

$- \frac{1}{2} \ln|1 - \sqrt{1 - t^2}|$

$$u_1 = \sin x - \left(\frac{1}{2} \ln|1 + \sin x| - \frac{1}{2} \ln|1 - \sin x| \right) + C_1$$

$$y = \left(\sin x - \frac{1}{2} \left(\ln|1 + \sin x| - \ln|1 - \sin x| \right) + C_1 \right) \cos x + \left(-\cos x + C_2 \right) \sin x$$