

2.2

MA 341

## Separable Equations

$$\frac{dy}{dx} = f(x, y)$$

$$\frac{dy}{dx} = g(x) h(y)$$

separate into related parts.

$$dy = g(x) h(y) dx$$

differential  
form

$$\frac{dy}{h(y)} = g(x) dx$$

separated

Something  
may be lost  
when you  
assume  
 $h(y) \neq 0$

$$\int \frac{dy}{h(y)} = \int g(x) dx$$

implicit solution

ex |  $\frac{dy}{dx} = x(y^2 + 1)$

$$\int \frac{dy}{y^2 + 1} = \int x dx$$

$$\arctan y = \frac{x^2}{2} + C$$

$C = C_2 - C_1$

↗ implicit solution.

For explicit solution: ↘

$$y = \tan\left(\frac{x^2}{2} + C\right)$$

↗ explicit solution.

general  
solution

▷ subject to  $y(0) = 1$

(now it's an IVP!)

- Find  $C$  s.t.  $y(0) = 1$

$$x_0 = 0, y_0 = 1$$

plug in:  $\arctan(1) = C$

$$C = \pi/4$$

$$\arctan y = \frac{1}{2}x^2 + \frac{\pi}{4}$$

solution  
to IVP

ex

$$\frac{dy}{dx} = xy^2$$

separable

$$\frac{dy}{y^2} = x dx$$

$$, \quad \underline{y \neq 0}$$

investigate this later

$$\frac{1}{y} = \frac{x^2}{2} + C$$

implicit solution

$$y = - \frac{1}{(x^2/2) + C}$$

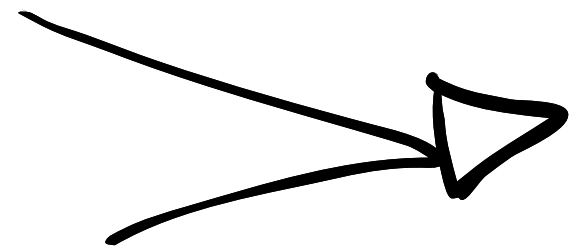
explicit solution

▷ What if  $y = 0$ ? Did we lose any solutions?

— check if  $y = 0$  is a solution.

$$y = 0$$

$$\frac{dy}{dx} = 0$$



$$\frac{dy}{dx} = xy^2$$

$$0 = 0(0) \quad \checkmark$$

Thus  $y = 0$  is a here solution! even though

$y=0$  is a "special solution"

▷ Now,  $\frac{dy}{dx} = xy^2$ ,  $y(0) = 1$   
↓  
IVP!

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

$$C = -1$$

$$-\frac{1}{y} = \frac{x^2}{2} - 1$$

or

$$y = \frac{-1}{\left(\frac{x^2}{2}\right) - 1}$$

$$\Delta \text{ if } y(0) = 0,$$

$$\frac{-1}{0} = C$$

However, the  $y=0$  from earlier is, of course, a solution to this IVP.

### 2.3 Linear Eqns.

$$\frac{dy}{dx} + p(x)y = q(x)$$

ex

$$\frac{dy}{dx} + xy = e^x$$

This is a linear equation  
w/  $p(x)=x$ ,  $q(x)=e^x$

Linear Eqns. can always be solved, so long as the integrals involved can be.

ex)  $\frac{dy}{dx} = xy$

$$\frac{dy}{dx} - xy = 0$$

$$p(x) = -x,$$

$$q(x) = 0$$

$$\frac{dy}{dx} + p(x)y = q(x)$$

## Integrating Factor

ex)  $\frac{dy}{dx} + \frac{1}{x}y = 0$

This is better  $\left[ x \frac{dy}{dx} + y = 0 \right]$

$$\rightarrow \frac{d}{dx}(xy) = x \frac{dy}{dx} + y$$

$$\frac{d}{dx}(xy) = 0$$

$$XY = C \quad \leftarrow \text{all thanks to product rule earlier}$$

Integrating Factor  $\frac{d\mu}{dx}$

$$\mu(x) \frac{dy}{dx} + \mu(x) p(x) y$$

Tune  $\mu(x)$   $= \mu(x) q(x)$

s.t. this can be rewritten as

$$\frac{d}{dx} (\mu(x) y(x))$$

$$= \mu(x) q(x)$$

ex

$$\mu(x) = e^{\int p(x) dx}$$

$$\frac{d\mu}{dx} = e^{\int p(x) dx} p(x)$$

$$\frac{d\mu}{dx} = \mu(x) p(x)$$