MA 341, 4/7/17

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Midterm Review

Target: systems of differential equations.

Outline of topics (similar to last time):

1. Homogeneous systems
$$\frac{dx}{dt} = Ax$$
Solve for:
   (a) gen solution
   (b) IVP
   (c) fundamental matrix
   (d) how many eigenvectors

   Won't have to worry about edge cases (e.g. insufficient number of eigenvectors)
2. Non-Homogeneous Systems
$$\frac{dx}{dt} = Ax + \rho(t)$$
Using
   (a) Undetermined coefficients
   (b) Variation of parameters

   Assumes you can handle homogeneous systems and that you can *integrate by parts*
3. Eigenvector Properties
   (a) given eigenvalues $r_1, \ldots, r_n$, all distinct, multiplicity 1, the corresponding eigenvectors are independent
   (b) multiple eigenvalues: if sufficiently many eigenvectors exist, they can be selected as independent ones.
4.

Examples

1. Inverting a matrix (Gauss-Jordan)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & \frac{-1}{3} & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{-1}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} \\ \frac{-1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

2. Multiple Eigenvalues

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

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$$\begin{aligned} A - rI &= \begin{vmatrix} 1-r & 2 & -1 \\ 0 & 1-r & 0 \\ 0 & 1 & 1-r \end{vmatrix} \\ &= (1-r)\left[(1-r)^2\right] \\ &= (1-r)^3 \\ r &= 1 \qquad \text{multiplicity 3} \end{aligned}$$

Plug r into the matrix

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and extract equations:

$$\begin{aligned} 2u_2 - u_3 &= 0 \\ u_2 &= 0 \\ 0 &= 0 \end{aligned}$$

which is an insufficient number of eigenvectors.

$$\begin{aligned} u_2 = u_3 &= 0 \\ 0 &= 0 \\ &\Rightarrow \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} \\ &= s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

3. Multiple Eigenvalues (II)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A - rI) &= \begin{vmatrix} -1-r & 1 & 0 \\ 1 & 2-r & 1 \\ 0 & 3 & 1-r \end{vmatrix} \\ &= -(1+r)[(2-r)(-1-r)-3] \\ &= -1[(-1-r)-1-0] \\ &= (1+r)\left[r^2 - r - 2 - 3\right] + (1+r) \\ &= -(1+r)(r-3)(r+2) \\ r_1 &= -1 \\ r_2 &= -2 \\ r_3 &= 3 \end{aligned}$$

Plug each into A-rI:

$$\boxed{r=-1}:$$

$$A-rI=\begin{bmatrix}0&1&0\\1&3&1\\0&3&0\end{bmatrix}$$

$$u_2=0$$

$$u_1+u_3=0$$

$$u=\begin{bmatrix}-s\\0\\s\end{bmatrix}$$

$$s=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$\boxed{r=-2}:$$

$$A-rI=\begin{bmatrix}1&1&0\\1&4&1\\0&3&1\end{bmatrix}$$

$$u_1+u_2=0$$

$$u_1+4u_2+u_3=0$$

$$3u_2+u_3=0$$

$$u=s\begin{bmatrix}-1\\1\\-3\end{bmatrix}$$

$$\boxed{r=3}:$$

$$4u_1+u_2=0$$

$$u_1-u_2+u_3=0$$

$$3u_2-4u_3=0$$

$$u_1-u_2+u_3=0$$

$$-3u_2+4u_3=0$$

$$u_2=4s$$

$$u_3=3s$$

$$u_1-4s+3s=0$$

$$u_1=s$$

$$u=\begin{bmatrix}s\\4s\\3s\end{bmatrix}$$

$$=s\begin{bmatrix}1\\4\\3\end{bmatrix}$$

The system of algebraic equations will reduce to a single equation.

If multiple:

$$u_1-u_2+u_3=0$$

$$u_2=s_1$$

$$u_3=s_2$$

$$u_1=s_1-s_2$$

$$u=\begin{bmatrix}s_1-s_2\\s_1\\s_2\end{bmatrix}$$