MA 341

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0$$

$F(x,y)$ such that

$$F_x = M, \quad F_y = N$$

$$F_{xy} = M_y \qquad F_{yx} = N_x$$

$$F_{xy} = F_{yx}$$

$M_y = N_x$ a condition for "exactness"

Condition 1

M and N are defined in a rectangle $a \leq x \leq b$, $c < y < d$, and M, N and their partial derivatives are continuous in this rectangle.

Condition 2

$M_y = N_x$

Then such $F(x,y)$ exists.

ex] $(3x^2+y)dx + (x^2y-x)dy = 0$

$M \qquad\qquad N$

can quickly check

$$M_y \neq N_x$$
$$=1 \qquad = 2xy-1$$

This equation is not exact.
But there is a way to make it exact.

Integrating Factors

$M(x,y)\,dx + N(x,y)\,dy = 0$

not exact.

Try to find function μ such that

$$\mu M(x,y)dx + \mu N(x,y)\,dy = 0$$

can "tune" eqn into becoming exact

$\rightarrow$ ask about physics applications

Special Cases

1) Assume $\mu = \mu(x)$ (just function of one of the two vars)

$$\mu(x)\, M(x,y)\, dx + \mu(y)\, N(x,y)\, dy = 0$$

$$(\mu M)_y = (\mu N)_x$$

$$\mu M_y = \mu_x N + \mu N_x$$

bc $\mu(x)$

product rule

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - M_x}{N}\mu}$$

IF $\dfrac{M_y - N_x}{N}$ does not depend on y, then ~~becomes~~ a simple ODE for μ.

ex] $\dfrac{M_y - N_x}{N}$

$M_y = 1$

$N_x = 2xy - 1$

$N = x^2y - x$

$$= \frac{2 - 2xy}{x^2y - x}$$

$$= \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x}\left(\cancel{\frac{1 - xy}{1 - xy}}^{\,1}\right)$$

$$= \boxed{-\frac{2}{x} = \frac{M_y - N_x}{N}}$$

$$\frac{d\mu}{dx} = -\frac{2}{x}\mu$$

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1) Find $\dfrac{d\mu}{dx} = \dfrac{M_y - N_x}{N}\mu$

2) Solve for μ.

$$\frac{d\mu}{dx} = p(x)\mu$$

$$\mu = e^{\int p(x)\,dx}$$

ex)

$$\mu = e^{\int -\frac{2}{x}\,dx}$$

$$= e^{-2\ln(x)} = x^{-2}$$

$$\mu = \boxed{\frac{1}{x^2}}$$

$\frac{1}{x^2}$ is an integrating factor.

3) Use I.F.

ex)

$$\left(3 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0$$

$$M_y = \frac{1}{x^2} \qquad N_x = \frac{1}{x^2}$$

exact (for $x \neq 0$)

now you can find $F(x,y)$

$$\begin{cases} F_x = 3 + \frac{y}{x^2} \\ F_y = y - \frac{1}{x} \end{cases}$$

from F_x: $F = \int F_x\,dx = 3x - \frac{y}{x} + g(y)$

$F_y = -\frac{1}{x} + g'(y) = y - \frac{1}{x}$

$g' = y \to g = \frac{y^2}{2}$

$$\boxed{F(x,y) = 3x - \frac{y}{x} + \frac{y^2}{2}}$$

$3x - \frac{y}{x} + \frac{y^2}{2} = C$ are solutions

$x \neq 0$

Now, if $\mu = \mu(y)$,

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu \qquad (*)$$

If this quantity $\dfrac{N_x - M_y}{M}$

depends on y and does not depend on x, the solution of $(*)$ is an integrating factor.

$$\mu = e^{\int \frac{N_x - M_y}{M}\,dy}$$

<u>ex</u> $(y^2 + 2xy)dx + (-x^2)dy = 0$

$$\underset{M_y}{2y + 2x} \neq \underset{N_x}{-2x} \qquad \text{not exact.}$$

$$N_x - M_y = -2x - 2x - 2y = -4x - 2y$$

$$\frac{N_x - M_y}{M} = \frac{-2(2x+y)}{y(y+2x)} = -\frac{2}{y}$$

$-\frac{2}{y}$ ← function only of y

$$\mu : e^{-\int \frac{2}{y}\,dy} = e^{-2\ln(y)} = \boxed{\frac{1}{y^2} = \mu}$$

$$\left(\text{bc } \frac{d\mu}{dy} = -\frac{2}{y}\mu\right)$$

Solve

$$\left(1 + \frac{2x}{y}\right)dx - \frac{x^2}{y^2}\,dy = 0$$

after multiplying by y

double check exact:

$$M_y = \frac{-2x}{y^2} = -\frac{2x}{y^2} = N_x$$

Now, solve:

$y \neq 0$

$$F_x = 1 + \frac{2x}{y} \longrightarrow F(x,y) = x + \frac{x^2}{y} + g(y)$$

$$F_y = -\frac{x^2}{y^2}$$

$F_y =$ [redacted] $\frac{x^2}{y^2} + g'(y) = -\frac{x^2}{y^2}$

$$g'(y) = 0$$

$$g(y) = c$$

$$F(x,y) = x + \frac{x^2}{y}$$

$$\boxed{x + \frac{x^2}{y} = C}$$

the solutions