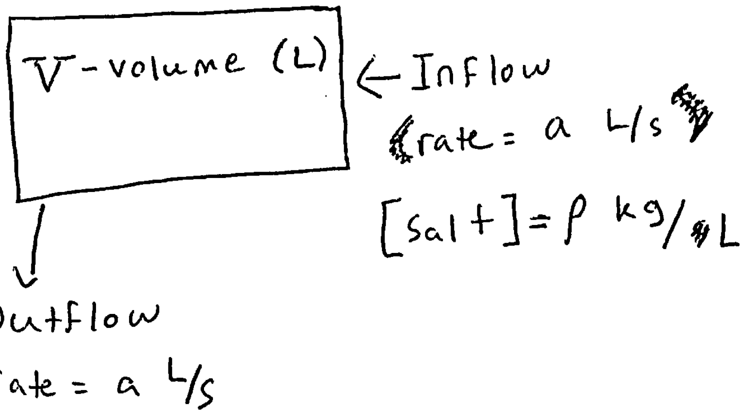
MA 341

(Applications)

① Mixing Problems



▷ Consider Mass of solvent in tank:

$x(t)$ = mass of salt in tank

▷ our principal quantity:

$$\frac{x(t)}{V} = \text{concentration of salt in tank}$$

▷ examine dx/dt :

$$\frac{dx}{dt} = \text{r.o.c. of mass of salt in tank}$$

$$= \underbrace{a\rho}_{\text{inflow mass}} - \underbrace{a\left(\frac{x(t)}{V}\right)}_{\text{outflow mass}}$$

▷ Add numbers:

$V = 1000$ L

$a = 10$ L/s

$\rho = 10$ kg/L

$$\frac{dx}{dt} = 10(10) - \frac{10x}{1000}$$

▷ IV:

$x(0) = 0$ ◁– initial amt of salt in tank

▷ Solve IVP

$$\frac{dx}{dt} = 100 - \frac{x}{100}, \quad x(0) = 0$$

→ linear equation!

▷
$$\frac{dx}{dt} + 0.01\,x = 100$$

▷ Integrating Factor:

$$\mu(t) = e^{\int 0.01\,dt} = e^{0.01t} = e^{t/100}$$

▷ Multiply eqn by IF:

$$e^{t/100}\left(\frac{dx}{dt}\right) + e^{t/100}(0.01\,x) = 100\,e^{t/100}$$

▷ Integrate:

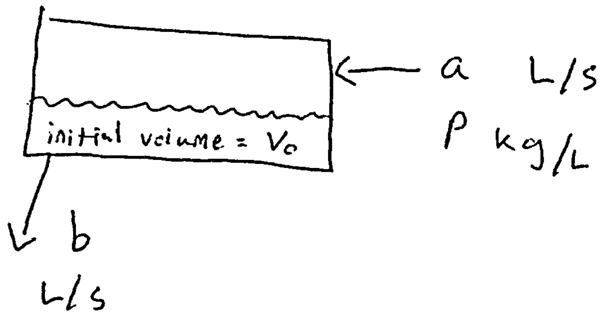
$$e^{t/100}\,x = \cancel{100}\,e^{-t/100} \cdot \frac{1}{\cancel{100}} + C$$

$$\boxed{x(t) = 1 + Ce^{-t/100}}$$

▷ Plug in IV:

$$x(0) = 1 + C = 0$$
$$C = -1 \implies \boxed{x(t) = 1 - e^{-t/100}}$$

▷ Let's diversify Problem:



▷ revisit eqn:

now b

$$\frac{\partial x}{\partial t} = a\rho - \frac{a \cdot x}{V}$$

variable quantity

▷ modify:

$$\frac{\partial x}{\partial t} = a\rho - \frac{b \cdot x}{V(t)}$$

▷ find $V(t)$:

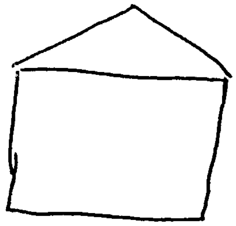
$$V(t) = (a-b)t + V_0$$

▷ Plug into eqn:

$$\frac{\partial x}{\partial t} = a\rho - \frac{b\,x}{(a-b)t + V_0}$$

more sophisticated, but still linear, so still solvable.

② Heating/cooling Problems



T = temperature

- Newton's Law of Cooling

$$\frac{dT}{dt} = -k\left(T - T_0\right)$$

measure of insulation

outside temperature

- Tweak NLoC!

  → T_0 is not really a constant

  $T_0(t)$

- eqn. w/ $T_0(t)$:

$$\frac{dT}{dt} = -k\left(T - T_0(t)\right)$$

- Linear equation: solvable!
- Tweak NLoC again!

  → House produces heat

  $Q(t)$ = rate of heat production, $\frac{\text{degrees}}{s}$

- eqn:

$$\frac{dT}{dt} = \underbrace{-k\left(T - T_0(t)\right)}_{\text{heat exchange w/ surroundings}} + \underbrace{Q(t)}_{\text{internal heat production}}$$

▷ Add numbers:

→ Outdoor

$$\cancel{T_o(t) = 50 + 10\cos\left(\frac{t}{24}\right)}$$

$$T_o(t) = 50 + \underbrace{10\sin\left(\pi\frac{t}{24}\right)}_{\text{hottest at midday}} \quad \text{with } t \text{ in hours}$$

→ oops.

$$T_o(t) = 50 + 10\sin\left(\pi\frac{\cancel{t}}{\cancel{2}4}\right) \quad \begin{cases}\text{hottest at midday}\\ \text{coldest at midnight}\end{cases}$$

$$T_o(t) = 50 + 10\sin\left(\frac{\pi}{12}t - \frac{\pi}{2}\right)$$

▷ revisit eqn. $(k = 0.1)$

$$\frac{dT}{dt} = -0.1\left(T - \left(50 + 10\sin\left(\frac{\pi}{12}t - \frac{\pi}{2}\right)\right)\right) + Q(t)$$

▷ numerical $Q(t)$:

$$Q(t) = 100$$

▷ eqn:

$$\boxed{\frac{dT}{dt} = -0.1\left(T - \left(50 + 10\sin\left(\frac{\pi}{12}t - \frac{\pi}{2}\right)\right)\right) + 100}$$

Linear!

▷ Solve:

$$\frac{dT}{dt} + 0.1\,T = 100 + \left(5 - \cos\left(\frac{\pi}{12}t\right)\right)$$

▷ Integrating Factor:

$$u(t) = e^{\int 0.1\, dt} = e^{\frac{t}{10}}$$

$$e^{\frac{t}{10}}\left(\frac{dT}{dt}\right) + 0.1\, e^{t/10}(T) = e^{\frac{t}{10}}\left(100 + 5 - \cos\left(\frac{\pi t}{12}\right)\right)$$

$$\frac{d}{dt}\left(e^{\frac{t}{10}} T\right) = e^{t/10}\left(105 - \cos\left(\frac{\pi t}{12}\right)\right)$$

▷ Integrate:

$$e^{\frac{t}{10}} T = \int e^{t/10}\left(105 - \cos\left(\frac{\pi}{12} t\right)\right) dt$$

$$= (105)(10)\, e^{t/10} - \int e^{t/10} \cos\left(\frac{\pi}{12} t\right) dt$$

▷ Integrate by Parts:

$$-\int \underbrace{e^{t/10}}_{u}\, \underbrace{\cos\left(\frac{\pi}{12} t\right)}_{v'} dt$$

$$\left.\begin{array}{ll} u = e^{t/10} & v' = \cos\left(\frac{\pi}{12} t\right) \\ u' = \dfrac{e^{t/10}}{10} & v = \dfrac{12}{\pi} \sin\left(\frac{\pi}{12} t\right) \end{array}\right] = -\left[\frac{12}{\pi} e^{t/10} - \int \frac{12}{\pi} \sin\left(\frac{\pi}{12} t\right) \cdot \frac{e^{t/10}}{10}\, dt\right]$$

▸ Int. by P., cont.:

→ again:

$$-\int \frac{12}{\pi} \sin\left(\frac{\pi}{12} t\right) \frac{e^{t/10}}{10} \, dt = -\frac{12}{10\pi} \int \sin\left(\frac{\pi t}{12}\right) e^{t/10} \, dt$$

$$u = e^{t/10} \qquad v' = \sin\left(\frac{\pi t}{12}\right)$$

$$u' = \frac{e^{t/10}}{10} \qquad v = -\cos\left(\frac{\pi t}{12}\right)\left(\frac{12}{\pi}\right)$$

$$= -\left[\frac{12}{10\pi} e^{\frac{t}{10}} \sin\left(\frac{\pi t}{12}\right) - \frac{12}{10\pi}\left(-\frac{12}{\pi} e^{\frac{t}{10}} \cos\left(\frac{\pi t}{12}\right)\right.\right.$$

$$\left.\left. - \int -\frac{12}{10\pi} e^{t/10} \cos\left(\frac{\pi t}{12}\right)\right)\right]$$

$$= -\int e^{t/10} \cos\left(\frac{\pi t}{12}\right) dt$$

Solve for

$$\int \frac{12}{10\pi} e^{t/10} \cos\left(\frac{\pi t}{12}\right)$$