Chapter 4

-Test: Check original eqn for "lost" solutions (constraints, vals you can't have later, like $y=0$)

Second Order Linear Equations

$$a(x)\,y'' + b(x)y'(x) + c(x)\,y = f(x)$$

x-independent y-dependent

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)\,x = f(t)$$

t-ind x-dep

1) IVP

-> # of initial conditions = order of eqn.

$y(x_0) = y_0, \quad y'(x_0) = y_{0_2}$

2) Existence/Uniqueness

for $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$

Check continuity of $\boxed{F,\ F_y,}\ F_{y'}, \ldots, F_{y^{n-1}}, F_{y^{(n)}}$

Second order

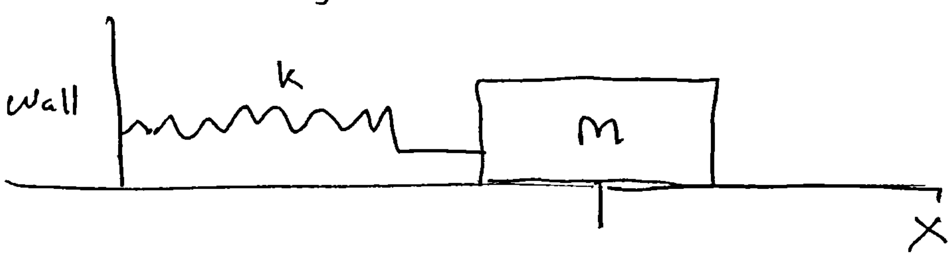
3) $\left(\dfrac{b(x)}{a(x)},\ \dfrac{c(x)}{a(x)}\right)$ If $a(x)$, $b(x)$, $c(x)$, and $f(x)$ are continuous, and $a(x) \neq 0$ (around initial value of x), then the IVP will have unique solution

4) If $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are continuous on $\alpha < x < \beta$, the solutions will exist throughout $\alpha < x < \beta$

Motivation:

1) Simple Mechanics

Mass-spring oscillator



$$\frac{\partial^2 x}{\partial t^2} = x'' = \text{acceleration}$$

$$-kx = \text{spring force}$$

$$mx'' = -kx$$

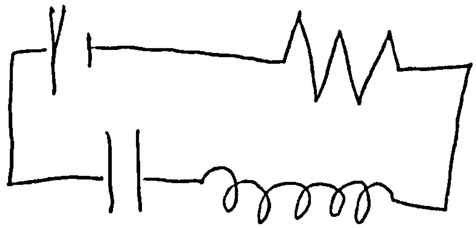
$$mx'' + kx = 0 \quad \leftarrow \text{2nd order ODE}$$

- Add dissipation

$$\hookrightarrow mx'' + bx' + kx = 0$$

- Add ext force $f(t)$ $\longrightarrow mx'' + bx' + kx = f(t)$

2) Simple circuits



similar 2nd-order ODE's (inductivity instead of mass, etc.)

electrical and mechanical oscillations captured by same mechanism.

<u>Start w/</u>

$$a(x)y'' + b(x)y' + c(x)y = 0$$

Zip(list[self.cml], list[self.cml[1::]])

<u>homogeneous</u>

ignore rhs at first

<u>linear second-order equation</u>

Tough. But if $a(x)$, $b(x)$, and $c(x)$ are <u>constants</u>, we get so-called linear homogeneous 2nd-order eqns with constant coefficients.

$$ay'' + by' + cy = 0$$

a, b, c = constants, $a \neq 0$

1 divide by a :

$\frac{b}{a} = p$ $\quad\quad$ $\frac{c}{a} = q$

$$\boxed{y'' + py' + qy = 0}$$

remark: $a(x)y'' + b(x)y' + c(x)y = 0$

$$\left.\begin{matrix} y_1(x) \\ y_2(x) \end{matrix}\right\} \text{ two solutions}$$

Linear Equation

$C_1 y_1(x) + C_2 y_2(x)$ will also solve eqn.

IF you know 2 solutions, and they're "good enough", you can construct more . . .

HW: verify

(2) Try finding solutions of form

$$y = e^{rx}$$

(3) substitute into eqn: $r^2 e^{rx} + pre^{rx} + qe^{rx} = 0$

(4) factor out e^x: $(r^2 + q + pr)\underbrace{e^{rx}}_{\text{never } 0} = 0$

(5) solve for roots: (quadratic) $r^2 + pr + q = 0$

(6) if r is real solution of the quadratic, e^{rx} solves differential equation.

- the characteristic eqn. for (*)

ex) $y'' + 3y' + 2y = 0$

1) $r^2 + 3r + 2 = 0$

2) Solve for r: $(r+1)(r+2) = 0$

$r = -1, -2$ (roots $r_1 = -2$, $r_2 = -1$)

3) Solutions e^{-2x}, e^{-x}

solve original equation

4) check for more:

$c_1 e^{-x} + c_2 e^{-2x}$

Since $\dfrac{e^{-2x}}{e^{-x}} \neq$ const,

$$\boxed{c_1 e^{-x} + c_2 e^{-2x}}$$

lists all solutions of (*)

and is thus the general solution.

Order

Linear

2: → Constant Coefficients! → $y = e^{rx}$ → quadratic (roots) → distinct? / real? / multiplicity? / complex? → tomorrow

$c_1 e^{r_1 x} + c_2 e^{r_2 x} = 0$

$c_1 e^{rx} + c_2 x e^{rx} = 0$

general solution

1: → IF

Consider $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$ (characteristic eqn)

$(r+1)^2 = 0$

$r = -1$ only 1 solution

-1 is a multiplicity 2 root.

$r_1 = -1, \quad r_2 = -1$?

$\cancel{c_1 e^{-x} + c_2 e^{-x}}$ Meaningless

We see that roots must be distinct for that method:

$$\frac{e^{-x}}{e^{-x}} = 1 = \text{constant} \leftarrow \text{not allowed!}$$

The way out!

Direct Calculation shows that xe^{-x} solves the equation.

$$C_1 e^{rx} + C_2 x e^{rx}$$

$$\frac{xe^{-x}}{e^{-x}} = x \neq \text{constant}$$

$\boxed{C_1 e^{-x} + C_2 x e^{-x}}$ is general solution ex] $y'' + 2.1 y' + y = 0$

$r_1 \neq r_2$

but they are nearby numbers

$e^{r_1 x} \qquad e^{r_2 x}$

and r_2 almost $= r_1$,

as $r_2 \to r_1$, $\dfrac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1}$ ... rate of difference

That's how Euler/Lagrange figured out the $x e^{r}$

TO Summarize:

$y'' + py' + ry = 0$

if $r^2 + pr + q = 0$ has a multiple real root, r,

the two basic solutions are e^{rx} and xe^{rx};

$$\boxed{\frac{xe^{rx}}{e^{rx}} \neq \text{const}}$$

$c_1 e^{rx} + c_2 x e^{rx}$ is a general solution