MA 341

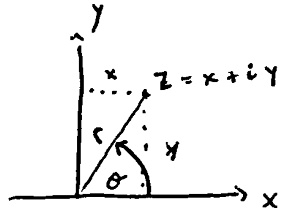
2nd-Order ODE (linear) when char. eqn has complex roots (Plus IVP)

$$y'' + py' + qy = 0$$

$$r^2 + pr + q = 0$$

→ when this has complex roots:

- complex #s:



(r, θ) defines where your complex # is located

- ▷ r can't ≤ 0 $(r \geq 0)$
- ▷ $\theta = \theta + 2\pi n$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = |z| \cos\theta$$

$$y = |z| \sin\theta$$

$$\theta = \operatorname{Arg} z \quad (0 \leq \theta \leq 2\pi)$$

$$\arg z \quad (\text{all } \theta)$$

e^{iy}, y is real.

$$\boxed{e^{iy} = \cos y + i \sin y}$$

so

$$\boxed{z = |z| e^{i\theta}}$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1 x_2 + x_1 i y_2 + x_2 i y_1 - y_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

$$z_1 z_2 = |z_1| e^{i\theta_1} \left(|z_2| e^{i\theta_2}\right)$$

$$= |z_1||z_2| e^{i(\theta_1 + \theta_2)}$$

$x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$

$$\left.\begin{array}{l} z = x + iy \\ \bar{z} = x - iy \end{array}\right\} \quad \boxed{\cancel{z(\bar{z}) = z^2 e^0 = \bar{z}}}$$

complex conjugate → $\bar{z} = x - iy$

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

ex

$$y'' + 4y' + 5y = 0$$

$$r^2 + 4r + 5 = 0$$

$$(r+2)^2 + 1$$ CTS!

$$(r+2)^2 + 1 = 0$$

$$r + 2 = \sqrt{-1}$$

$$r + 2 = \pm i$$

$$r_1 = -2 + i$$

and

$$r_2 = -2 - i$$

note: $r_2 = \overline{r_1}$

(complex solutions always come as a conjugat pair for real p, q)

Think of $y'' + py' + qy = 0$ as if y were a complex-valued dependent variable:

solutions

$$e^{r_1 t} = e^{(a+ib)t} = e^{at} e^{ibt}$$

$$e^{r_2 t} = e^{(a-ib)t} = e^{at} e^{-ibt}$$

$$\left(e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x}(\cos y + i \sin y)\right)$$

$$r_1, r_2 = a \pm ib$$

ex| continued: $(y'' + 4y' + 5y = 0)$

$$r_{1,2} = -2 \pm i$$

$y_1 = e^{-2t} e^{it}$ are complex-valued, linearly independent

$y_2 = e^{-2t} e^{-it}$ Solutions of $(*)$

$$\frac{y_2}{y_1} = \frac{e^{it}}{e^{-it}} = e^{2it} \neq 0$$

$$\boxed{y = \tilde{C}_1 e^{r_1 t} + \tilde{C}_2 e^{r_2 t}}$$

general complex-valued solution

$y(t) \rightarrow$ a solution

Q: will $\bar{y}(t)$ be a solution?

A: Yes.

Consider $\boxed{y(t) + \bar{y}(t) = \ldots}$ general solution with real values

$$y + \bar{y} = \dots$$

algebra

$$\boxed{y = C_1 e^{at} \cos bt + C_2 e^{at} \sin bt}$$

general real-valued solution

can jump here from $r = a \pm bi$

$$y_1(t) = e^{at} \cos bt$$

$$y_2(t) = e^{at} \sin bt$$

$$\frac{y_2}{y_1} \neq \text{const}$$

$$\det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \neq 0$$

Lin

I-o Lin. ODE → old notes

II-o Lin. ODE → const coefficients? → Char eqn → CTS

Char eqn → real → last notes

Char eqn → complx? → $r = a \pm ib$ → $y(t) = C_1 e^{at} \cos bt + C_2 e^{at} \sin bt$

ex| continued:

$a = -2, \quad b = 1$

$$\left.\begin{aligned} y_1(t) &= e^{-2t}\cos t \\ y_2(t) &= e^{-2t}\sin t \end{aligned}\right\} \; y(t) = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t$$

general solution

ex|

$$y'' - 4y' + 13y = 0$$

$$r^2 - 4r + 13 = 0$$

$$(r-2)^2 + 9 = 0$$

$$r - 2 = \pm\sqrt{-9}$$

$$r - 2 = \pm 3i$$

$$r_{1,2} = 2 \pm 3i$$

$$y_1(t) = e^{2t}\cos 3t$$
$$y_2(t) = e^{2t}\sin 3t$$

$$y(t) = C_1 e^{2t}\cos 3t + C_2 e^{2t}\sin 3t$$

Initial - Value Problem

Same eqn, subject to $y(0) = 2$
$y'(0) = 0$

1) Obtain general solution
2) Find the C-values

   (i) Apply $y(0)=2$ first

   $$y(0) = c_1(1) + c_2(0)$$
   $$c_1 = 2$$

   (ii) Now $y(t) = 2e^{2t}\cos 3t + c_2 e^{2t}\sin 3t$

   (iii) Now apply $y'(0)=0$

   iiia) differentiate $y(t)$

   $$y'(t) = 4e^{2t}\cos 3t - 6e^{2t}\sin 3t + 2c_2 e^{2t}\sin 3t + 3c_2 e^{2t}\cos(3t)$$

   iiib) Plug $y'(0)=0$;

   $$0 = 4 + 3c_2$$
   $$c_2 = -3/4$$

3) Plug in c_1, c_2:

$$y(t) = 2e^{2t}\cos 3t - \frac{4}{3}e^{2t}\sin 3t$$