MA 341, 4/5/17

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Variation of Parameters

$$\frac{dx}{dt} = Ax + \rho(t) \tag{1}$$

elaboration:

$$\frac{dx}{dt} = Ax + \rho(t),$$
$$A \in \Re^{n \times n},$$
$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$
$$\rho(t) = \begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_n(t) \end{bmatrix}$$

then

$$\frac{dx}{dt} = Ax \tag{2}$$

1) Find fundamental matrix X(t)

For $X(t)$, a fundamental matrix that solves

$$\frac{dX}{dt} = AX \tag{3}$$
$$(\frac{dX}{dt} - AX)C = 0 \tag{4}$$

then

$$x(t) = X(t)C(\text{general solution}) \tag{5}$$

where

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

Find solutions of (1) in the form of

$$x(t) = X(t) \cdot C(t), C(t) = \begin{bmatrix} C_1(t) \\ \vdots \\ C_n(t) \end{bmatrix}$$

substitute in:

$$\frac{dx}{dt} = \boxed{\frac{dX}{dt} \cdot C} + X \cdot \frac{dC}{dt} = \boxed{AXC} + \rho(t)$$

which leaves

2) Find X^{-1}

3) Find $C(t)$

$$\frac{dC}{dt} = \rho(t)$$
$$\frac{dC}{dt} = X^{-1}(t)\rho(t)$$
$$C(t) = \int X^{-1}(t)\rho(t)dt$$

so you have your coefficient column vector $C(t)$.

4) General Solution

$$x(t) = X(t) \cdot C(t) \tag{6}$$

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EXAMPLE

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{7}$$

$$\rho(t) = \begin{bmatrix} 8\sin(t) \\ 0 \end{bmatrix} \tag{8}$$

1)

$$\frac{dx}{dt} = Ax$$
$$det(A - rI) = 0$$
$$det(\begin{bmatrix} -r & 1 \\ -1 & -r \end{bmatrix}) = r^2 + 1 = 0$$
$$r \pm i$$

Extract linear equations from matrix:

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \tag{9}$$

$$-iu_1 + u_2 = 0$$
$$-u_1 - iu_2 = 0$$

If you don't drop an equation, one of your eigen values was wrong.

Multiply first eqn by i:

$$u_1 + iu_2 = 0$$
$$-u_1 - iu_2 = 0$$

and simplify:

$$\boxed{u_1 + iu_2 = 0}$$

$$u_1 = -is$$
$$u_2 = s$$

so

and

$$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{10}$$

general solt'n

$$\left(e^{\alpha t}\cos(\beta t)a - e^{\alpha t}\sin(\beta t)b\right)C1 + \left(e^{\alpha t}\cos(\beta t)a + e^{\alpha t}\sin(\beta t)b\right)C2 \tag{11}$$

$$x(t) = \left(\cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)C_1 + \left(\sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)C_2$$

$$x_1(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, x_2(t) = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

construct fundamental matrix $X(t)$

$$X(t) = \begin{bmatrix} \sin t & -\cos t \\ cost & \sin t \end{bmatrix} \tag{12}$$

find X^{-1}

For 2 by 2,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$X^{-1} = \frac{1}{1} \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}$$
$$\begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} 8\sin t \\ 0 \end{bmatrix} = \begin{bmatrix} 8\sin^2 t \\ -8\sin t\cos t \end{bmatrix}$$

integrate to find $C(t)$

$$\int \begin{bmatrix} 8\sin^2 t \\ -8\sin t\cos t \end{bmatrix} dt = \begin{bmatrix} \int 8\sin^2 t dt \\ \int -8\sin t\cos t dt \end{bmatrix}$$

integrate (i)

Use $\sin^2 t = \frac{1-\cos 2t}{2}$ and $\cos^2 t = \frac{1+\cos 2t}{2}$

$$\int 8\sin^2 t dt = \int 4(1 - \cos 2t)dt = \int (4 - 4\cos 2t)dt = 4t - 2\sin 2t + C_1$$

integrate (ii)

$$\int -8\sin t\cos t dt = 2\cos 2t + C_2$$

integrate (net)

$$C(t) = \begin{bmatrix} 4t - 2\sin 2t + C_1 \\ 2\cos 2t + C_2 \end{bmatrix}$$

general solution

$$x(t) = X(t) \cdot C(t)$$

In sum:

1. find $X(t)$
2. find X^{-1}
3. find $C(t)$
4. gen solt'n

special fundamental matrix

$$\frac{dx}{dt} = Ax$$

$$\frac{dX}{dt} = AX \leftarrow m \times n \text{matrices}$$

$$x(t) = X(t)C$$

special fundamental matrix: instead of generic C vals in col, you see initial conditions:

$$x(t) = X(t)x(0)$$

an easy way to incorporate initial conditions, and computing this special $X(t)$ from the generic $X(t)$ isn't even hard.

start with $Y(t)$, then, special $X(t)$ can be found by

$$X(t) = Y(t)\Big(Y(0)\Big)^{-1}$$