MA 341

4.2

Generalizing 2nd-Order Linear ODE's

ex

$y'' - y = 0$

$y(0) = 2$

$y'(0) = 0$

1) characteristic equation

$r^2 - 1 = 0$

$r_{1,2} = \pm 1$ (two roots!)

e^x, e^{-x}

2) General Solution

$C_1 e^x + C_2 e^{-x}$

Important: it is general because $\dfrac{e^{x}}{e^{-x}} \neq \text{const}$

3) Apply initial conditions

$\underbrace{C_1 e^0 + C_2 e^0 = 2}_{\text{first initial}} \xRightarrow{\text{derivative}} C_1 e^{x} - C_2 e^{-x} = 0$ second initial w/ $c_1 e^0 - c_2 e^0 = 0$ and $c_1 e^0 + c_2 e^0 = 2$,

Solve for c_1, c_2

$c_1 = c_2 = 1$

$$\boxed{y(x) = e^x + e^{-x}}$$

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ex| $y'' - 2y' + y = 0$

$y(0) = 2 \qquad y'(0) = 4$

important difference!

$r^2 - 2r + 1 = 0$ (characteristic eqn.)

$(r-1)^2 = 0$

$r_1 = r_2 = 1$ (multiplicity 2 root)

hella wrong: $c_1 e^x + c_2 e^x$ because $\frac{e^x}{e^x} = \text{const}$!

$\frac{e^x}{e^x} = \text{const}$, so no $c_1 e^x + c_2 e^x$!

instead, $e^x, xe^x \Longrightarrow \boxed{y(x) = c_1 e^x + c_2 x e^x}$ ✓

apply initial conditions:

$$C_1 \cdot 1 + C_2 \cdot 0 = 2 \quad \leftarrow (1)$$

$$4 = C_1 e^{(0)} + C_2 e^{(0)} + C_2 (0) e^{(0)} \quad \leftarrow (2)$$

$$4 = C_1 + C_2$$

from (1), $C_1 = 2$

(2), $C_2 = 2$

$$\boxed{y(x) = 2e^x + 2xe^x}$$

Let's get less specific

$$y'' + py' + q = 0$$

$$r^2 + pr + q = 0$$

(real roots, for now)

$y_1(x)$

$y_2(x)$

$$\underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{when will this be a general solution?}}$$

If $\dfrac{y_1(x)}{y_2(x)} \neq$ constant,

then $C_1 y_1(x) + C_2 y_2(x)$ is general solution.

$y_1(x)$ and $y_2(x)$ (functions)

are <u>linearly</u> <u>independent</u> if

$$\frac{y_1(x)}{y_2(x)} \neq \text{constant}$$

Thm

If $\det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \neq 0$,

$C_1 y_1(x) + C_2 y_2(x)$ is general solution

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ex $y'' - 2y' + y = 0$

$y_1 = e^x$, $y_2 = xe^x$

$y_1' = e^x \quad y_2' = e^x + xe^x$

$$\det\begin{bmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{bmatrix} = e^x \cdot e^x + e^x \cdot xe^x - e^x \cdot xe^x = e^{2x} \neq 0$$

If $D \neq 0$, then one can always solve the initial conditions for C_1 and C_2.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

If $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$,

then system has a unique solution

Initial Conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0$$

$$\left.\begin{array}{ll} 1) & c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 \\ 2) & c_1 y'_1(x_0) + c_2 y'_2(x_0) = y'_0 \end{array}\right\} \text{buncha \#'s}$$

this is a system of linear eqns

to make sure you can solve this uniquely,

$$\det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{bmatrix} \neq 0$$

if $\neq 0$, you can uniquely find c's.

Linear

order is...

1? → var coeffs? → Wronski

2? → const coeffs? → characteristic eqn → roots: unique?

$c_1 e^{r_1 x} + c_2 e^{r_2 x}$

$c_1 e^{r_1 x} + c_2 x e^{r_2 x}$

or any $y_1(x)$, $y_2(x)$

$\det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{bmatrix} \neq 0$?

→ unique sol'n

If $y_1(x)$ and $y_2(x)$ are solutions of

$$y'' + py' + qy = 0$$

$$\underbrace{\frac{y_1(x)}{y_2(x)} \neq \text{constant}}_{\substack{\text{quicker,} \\ \text{(general)} \\ \text{solution}}} \quad \text{if and only if} \quad \underbrace{\det[\ ] \neq 0}_{\substack{\text{unique } c\text{'s} \\ \text{(IVP)} \\ \text{uniqueness}}}$$

quicker, unless you want unique c's (IVP)

(general) solution — uniqueness

$$a(x)y'' + b(x)y' + c(x)y = 0$$

- homogeneous, non-constant coefficients
- can sometimes be solved

ex] $x^2y'' - 3xy' + 4 = 0$

can find solutions

- in general, though, w/ variable solutions, we're out of luck for general solution.
- Verifying $\det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \neq 0$ implicitly.

That condition survives w/ variable coefficient; and, you can compute determinant w/out $y_1(x)$, $y_2(x)$

Wronskian

Process

$$y'' + py' + qy = 0$$

if const coeffs,

$$r^2 + pr + q = 0$$

If roots real,

we see exponentials

and sometimes x-exponential

and nothing else

if complex roots,

a different story

What if the roots of $r^2 + pr + q = 0$ are complex?

$z = x + iy \quad (x, y \in \mathbb{R})$

i: imaginary unit

$i^2 = -1$

x: real part of z

$x = \mathrm{Re}(z)$

y: imaginary part of z

$y = \mathrm{Im}(z)$

Complex #

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z_1, z_2

$z_1 = 1 + 2i$

$z_2 = 2 - i$

"which is bigger?" makes no sense

"bigger"?

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

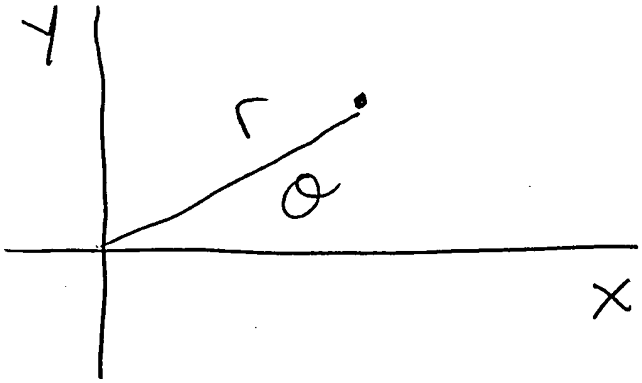
$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

since $i^2 = -1$

Complex arithmetic



Polar
Coordinates