MA 341

goal: transform eqn. so it becomes exact.

$$(y^2+2xy)\,dx - x^2\,dy = 0$$

Compute integrating factors

$$\mu(y) = \frac{1}{y^2}$$

$$\frac{1}{y^2}(y^2+2xy)\,dx - \frac{1}{y^2}(x^2\,dy) = 0$$

$$\left(1+\frac{2x}{y}\right)dx - \frac{x^2}{y^2}\,dy = 0$$

$$M_y = \frac{-2x}{y^2} \qquad N_x = \frac{-2x}{y^2}$$

Exact.

$$F_x = \qquad F_y =$$

$$1+\frac{2x}{y} \qquad -\frac{x^2}{y^2}$$

$f(x) = x + \frac{x^2}{2} + g(x)$

$\frac{x^2}{2} + g(x) = \frac{x^2}{2}$

$g(x) = 0$

$g = 0$

pick simplest possible solution

$$\boxed{f(x) = x + \frac{x^2}{2}}$$

$x + \frac{x^2}{2} = c$

$c \neq 0$

ex) IVP
eqn subject to
$y(1) = -1$
$(x_0, y_0) = (1, -1)$

$L + \frac{1}{4} = c$

$c = 0$

Solution $\Rightarrow x + \frac{1}{4} = 0$

ex) Solve that eqn
subject to $(1, 0)$

Not allowed. Remember
$y \neq 0$.

even tho $y \neq 0$,

in original form

$y=0$ is a solution

$C(0+0)=0$

Thus by making eqn

exact we lose solution

where $y=0$!

end of 2.5

2.6) Homogeneous Equations

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

ex $$\frac{dy}{dx} = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2$$

Substitution

$$y = ux$$

new dependent variable

requirements:

▷ $\dfrac{dy}{dx} = x\dfrac{du}{dx} + u$

▷ $f\left(\dfrac{y}{x}\right) = f(u)$

$$(x^2+y^2)\,dx + x\,dy = 0$$

$$\frac{dy}{dx} = -\left(\frac{x^2+y^2}{x^2}\right)$$

$$= -\left[1 + \left(\frac{y}{x}\right)^2\right]$$

not always easy to re-express as $\dfrac{y}{x}$

$$f(ax, ay)$$

$$= f(x, y)$$

here, $$\left(\frac{(ax)^2 + (ay)^2}{(ax)^2}\right)(-1)$$

$$= -\frac{\cancel{a^2}\,(x^2+y^2)}{\cancel{a^2}\,x^2}$$

$$= -\left(\frac{x^2+y^3}{x^2}\right)$$

another way to see if you're dealing with homogenous equations.

$$\frac{dy}{dx} = -\frac{(x^2+xy)}{x^2}$$

$$= -\left(1+\left(\frac{y}{x}\right)^2\right)$$

$y = ux$

$$= -\left(1+\left(\frac{ux}{x}\right)^2\right)$$

$$x\frac{du}{dx} + u = -(1+u^2)$$

$$x\frac{du}{dx} = -1-u-u^2$$

Seperable!

$$\frac{du}{1+u+u^2} = -\frac{dx}{x}$$

$$\int \frac{du}{1+u^2+u} = \int \frac{dx}{x}$$

$$\int \frac{du}{1+u+u^2}$$

roots

$1+u+u^2=0$

complex roots.

$$\left(u+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \int \frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$v = u+\frac{1}{2}$

$$= \int \frac{du}{v^2+\frac{3}{4}} \xrightarrow[\text{inverse tangent}]{\text{close to}} \left(\int \frac{d\theta}{1+\theta^2}\right)$$

(Correction on left: replace dv with du)

$$v = \frac{\sqrt{3}}{2} w$$

$$dv = \frac{\sqrt{3}}{2}\, dw$$

$$\int \frac{dv}{v^2 + \frac{3}{4}} = \int \frac{\frac{\sqrt{3}}{2}\, dw}{\frac{3}{4} w^2 + \frac{3}{4}}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{4}{3} \int \frac{dw}{1+w^2}$$

$$\int \frac{du}{(u+\frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \arctan w$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}} v\right)$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(u + \frac{1}{2}\right)\right)$$

$$\frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(u+\frac{1}{2}\right)\right)$$

$$= -\ln|x| + C$$

$$u = \frac{y}{x}$$

$$\frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(\frac{y}{x}+\frac{1}{2}\right)\right) = -\ln|x| + C$$

$$\boxed{\frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(\frac{y}{x}+\frac{1}{2}\right)\right) = -\ln|x| + C}$$

implicit solution

▷ <u>Bernoulli Equations</u>

$$\frac{dy}{dx} + p(x)y = q(x)y^{a}$$

if $a > 1$, not linear

There's a substitution we can use,

Substitution

divide the equations by y^a

($y = 0$ — a solution of $a > 0$)

So we may be losing a 0 solution.

$y^{-a} = \frac{1}{y^a}$

$$y^{-a}\frac{dy}{dx} + P(x)\underline{y^{1-a}} = q(x)$$

$y^{1-a} = u$ ← new dependent variable

$P(x)u = q(x)$

$$\frac{du}{dx} = \frac{d}{dx} y^{1-a} = (1-a) y^{-a} \frac{dy}{dx}$$

$$y^{-a} \frac{dy}{dx} = \frac{1}{1-a} \frac{du}{dx}$$

$$\boxed{\frac{1}{1-a} \frac{du}{dx} + p(x) u = q(x)}$$

ex)

$$\frac{dy}{dx} - y = e^{2x} y^3$$

$$p(x) = -1$$

$$q(x) = e^{2x}$$

$p(x) = -1$

$q(x) = e^{2x}$

$$\frac{dy}{dx} - y = e^{2x} y^3$$

$a = 3$

$u = y^{-2}$

$(u = y^{1-a})$

$$\frac{1}{-2} \frac{du}{dx} - u = e^{2x}$$

linear

$$\frac{du}{dx} + 2u = -2e^{2x}$$

→ integrating factor

$$e^{\int p(x)dx} \rightarrow e^{\int 2 dx} = \mu(x) = e^{2x}$$

$$\underbrace{e^{2x}\frac{du}{dx} + 2e^{2x}u}_{\frac{d}{dx}\left(e^{2x}u\right)} = -2e^{4x}$$

$$\frac{d}{dx}\left(e^{2x}u\right) \text{ as}$$

lets you rewrite left-hand side as differentiation of single ~~term~~ quantity

$$\frac{d}{dx}\left(e^{2x}u\right) = -2e^{4x}$$

$$e^{2x}u = \int -2e^{4x}\,dx$$

$$e^{2x}u = -\frac{e^{4x}}{2} + c$$

$$\boxed{u = -\frac{1}{2}e^{2x} + ce^{-2x}}$$

now return to y:

$$u = y^{-2}$$

$$u^{-1/2} = y$$

$$\boxed{y = \left(-\frac{1}{2}e^{2x} + Ce^{-2x}\right)^{-1/2}}$$

and $y=0$ (special solution)

All of above will be on test!