2.2 MA 341

Separable Equations

$$\frac{dy}{dx} = f(x,y)$$

$$\frac{dy}{dx} = g(x)\,h(y)$$

separate into related parts.

$$dy = g(x)\,h(y)\,dx$$ differential form

$$\frac{dy}{h(y)} = g(x)\,dx$$ separated

Something may be lost when you assume $h(y) \neq 0$

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

implicit solution

ex1 $\quad \dfrac{dy}{dx} = x(y^2+1)$

$$\int \frac{dy}{y^2+1} = \int x \, dx$$

$$\boxed{\arctan y = \frac{x^2}{2} + C}$$

$C = C_2 - C_1$

general solution

implicit solution.

For explicit solution:

$$\boxed{y = \tan\left(\frac{x^2}{2} + C\right)}$$

explicit solution.

- subject to $y(0) = 1$
  (now it's an IVP!)
  - Find C s.t. $y(0) = 1$

$x_0 = 0, \; y_0 = 1$

Plug in: $\arctan(1) = C$

$C = \pi/4$

$$\boxed{\arctan y = \frac{1}{2}x^2 + \frac{\pi}{4}}$$

solution to IVP

ex1 $\quad \dfrac{dy}{dx} = xy^2 \qquad$ separable

$$\frac{dy}{y^2} = x\,dx \quad , \quad \underline{\underline{y \neq 0}}$$

investigate this later

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

implicit solution

$$y = -\frac{1}{(x^2/2) + C}$$

explicit solution

- What if $y = 0$? Did we lose any solutions?
  - check if $y=0$ is a solution.

$y = 0$, $\dfrac{dy}{dx} = 0 \quad \longrightarrow \quad \dfrac{dy}{dx} = xy^2$

$$0 = 0(0) \checkmark$$

Thus $y=0$ is a solution, even though ~~is~~ is not here.

$y = 0$ is a "special solution"

▷ Now, $\dfrac{dy}{dx} = x y^2, \qquad \underbrace{y(0) = 1}_{\text{IVP!}}$

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

$$C = -1$$

$$\boxed{-\frac{1}{y} = \frac{x^2}{2} - 1}$$

or

$$\boxed{y = \frac{-1}{\left(\frac{x^2}{2}\right) - 1}}$$

▷ If $y(0) = 0$,

$$\frac{-1}{0} = c$$

However, the $y=0$ from earlier is, of course, a solution to this IVP.

2.3 Linear Eqns.

$$\frac{dy}{dx} + p(x)\,y = q(x)$$

ex)

$$\frac{dy}{dx} + xy = e^x$$

This is a linear equation w/ $p(x) = x$, $q(x) = e^x$

Linear Eqns. can always be solved, so long as the integrals involved can be.

ex) $\frac{dy}{dx} = xy$

$\frac{dy}{dx} - xy = 0$ $p(x) = -x,$
$q(x) = 0$

$$\frac{dy}{dx} + P(x)\,y = q(x)$$

Integrating Factor

ex) $\frac{dy}{dx} + \frac{1}{x}y = 0$

$$x\frac{dy}{dx} + y = 0$$

This is better

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

$$\frac{d}{dx}(xy) = 0$$

$$xy = C$$ ← all thanks to product rule earlier

Integrating Factor

$$\boxed{\mu(x)\frac{dy}{dx} + \overbrace{\mu(x)\,p(x)}^{d\mu/dx}\,y} = \mu(x)\,q(x)$$

Tune $\mu(x)$ s.t. this can be rewritten as

$$\boxed{\frac{d}{dx}\Big(\mu(x)\,y(x)\Big)} = \mu(x)\,q(x)$$

ex)

$$\mu(x) = e^{\int p(x)\,dx}$$

$$\frac{d\mu}{dx} = e^{\int p(x)\,dx}\,p(x)$$

$$\frac{d\mu}{dx} = \mu(x)\,p(x)$$