MA 341 — M1 Review

Bernoulli

$$\frac{\partial y}{\partial x} + y = e^t \boxed{y^{-2}}$$

(general: $\frac{\partial y}{\partial x} + p(x)y = q(x)y^a$

where $a \neq 0$, $a \neq 1$,
in which cases it's linear.)

$a = -2$

$p(x) = 1$

$q(x) = e^t$

① divide by this! (y^a)

② introduce $u = y^{1-a}$

$\frac{du}{dx} + \ldots$ will be linear

ex

$u = y^{1-a} = y^{1-(-2)} = y^3$

$\frac{du}{dx} = \frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx}$

$$y^2 \frac{dy}{dt} + y^3 = e^t$$

assume all t's are meant to be x's.

③ Plug in u:

$$y^2 \frac{dy}{dt} + u = e^t$$

$$\frac{1}{3}\frac{du}{dx} + u = e^t \text{ - linear}$$

Linear

$$\frac{1}{3}\frac{du}{dx} + u = e^{x}$$

$$\frac{du}{dx} + 3u = 3e^{x}$$

1. integrating factor

$$\mu(x) = e^{\int 3\,dx}$$

$$\boxed{\mu(x) = e^{3x}}$$

2. multiply by IF

$$\underbrace{e^{3x}\frac{du}{dx} + 3e^{3x}u}_{\frac{d}{dx}\left(e^{3x}u\right)} = 3e^{4x}$$

(2.5)

$$\frac{d}{dx}\left(e^{3x}u\right) = 3e^{4x}$$

we know from IF that this is $\mu(x)\cdot u$, differentiated.

3. integrate

$$\int \frac{d}{dx}\left(e^{3x}u\right) = \int 3e^{4x}$$

$$e^{3x}u = \frac{3}{4}e^{4x} + C$$

(4) simplify:

$$u = \frac{3}{4} e^{x} + Ce^{-3x}$$

solves linear eqn.

(5) back to y:

$$y = u^{1/3} = \sqrt[3]{u}$$

$$y = \left(\frac{3}{4} e^{x} + Ce^{-3x} \right)^{1/3}$$

(6) check for lost solutions!

otherwise, some IVPs will give you hard times,

(check notes, HW, text for examples)

Homogeneous Equations with Substitution

$$\left(x^2+y^2\right)dx + 2xy\,dy = 0$$

→ homogeneous, exact,

① $\boxed{\text{rearrange to } \dfrac{dy}{dx} = f(x,y)}$

① divide by dx, then by $2xy$, and move:

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy} = -\frac{1+\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

② check if homogeneous:

(2.1) Check structure, or

(2.2) replace x, y w/ ax, ay: $\dfrac{(ax)^2+(ay)^2}{2(ax)(ay)}$

(2.2) Check that $f(ax, ay) = f(x,y)$

$$= \frac{a^2\left(x^2+y^2\right)}{a^2(x\,y)(2)} = \frac{x^2+y^2}{2xy}$$

if right-hand side unchanged, it's homogeneous.

This one is.

③ use $y = ux$, u is new dependent variable:

right-hand: $f(x, ux) = -\dfrac{1+u^2}{2u}$

left-hand: $\dfrac{dy}{dx} = x\dfrac{du}{dx} + u$

$$x\frac{du}{dx} + u = -\frac{1+u^2}{2u}$$

(4) Simplify: will yield separable equation

$$x \frac{du}{dx} = - \frac{1 + u^2 + 2u^2}{2u}$$

$$x \frac{du}{dx} = - \frac{1 + 3u^2}{2u}$$ ← separable!

(5) Separate:

$$\frac{2u}{1 + 3u^2} \, du = - \frac{1}{x} \, dx$$

(6) Integrate:

$$\int \frac{2u \, du}{1 + 3u^2} = - \int \frac{dx}{x}$$

(7) u-substitution:

$$v = 1 + 3u^2$$

$$v' = 6u \qquad dv = 6u \, du$$

$$\int \frac{\frac{1}{3} \, dv}{v} = - \ln|x| + C$$

$$\frac{1}{3} \ln|v| = - \ln|x| + C$$

(8) Return to u:

$$\frac{1}{3}\ln\left(1+3u^2\right) = -\ln|x| + C$$

(9) Return to y:

$$\boxed{\frac{1}{3}\ln\left(1+3\left(\frac{y}{x}\right)^2\right) = -\ln|x| + C}$$

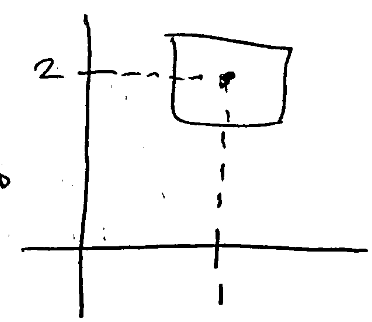
existence / uniqueness

always about IVP's...

(0) Draw rectangle

$$\frac{dy}{dx} = y^3 + x^2 \qquad y(1) = 2$$

$$(x_0, y_0) = (1, 2) \rightarrow$$



Does problem have solution, and is solution unique?

(1) rectangle surrounding initial ~~value~~ point:

(2) r-h side: $F(x,y) = y^3 + x^2$

$$\frac{\partial F}{\partial y} = 3y^2$$

(2) Are $f(x,y)$ and $\frac{\partial f}{\partial y}$ continuous in a vicinity of $(1,2)$?

(2) Find $f(x,y)$ and $\frac{\partial f}{\partial y}$

(3) Find continuity in vicinity of initial coordinate

Polynomials are continuous everywhere.

$f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous everywhere, including near solution to IVP

→ proven by thm

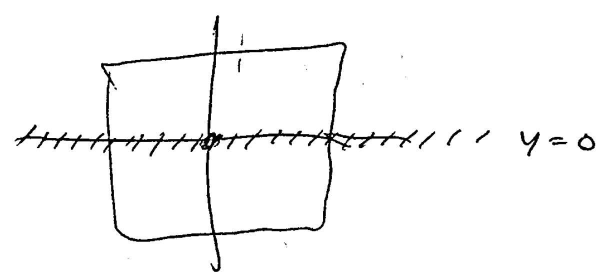
Alt

Consider $\frac{dy}{\partial x} = 3y^{2/3}$ $y(0) = 0$ or $(x_0, y_0) = (0, 0)$

(2) $f(x,y) = 3y^{2/3}$

$\frac{\partial f}{\partial y} = \frac{2}{\sqrt[3]{y}}$

(1) Draw rectangle



(3) $f(x,y)$ cont. everywhere, but

$\frac{\partial f}{\partial y}$ not continuous at $y=0$ (at solution!) (not even defined at $y=0$), which is a line.

→ No matter how much you shrink rectangle around solution (origin), you'll always have places where $y=0$

Thus, ex/unq thm can't state if Solution exists or if it is unique.

(4) Construct extra solutions, to verify (if possible):

$$y=0 \quad \text{and} \quad y=x^3$$

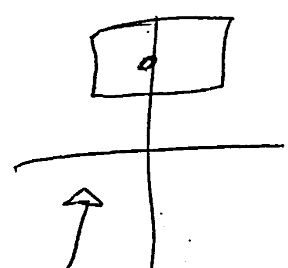
are solutions.

$y=0$, then, and $y=x^3$ are exists, but ~~is~~ not unique.

alt

$$\frac{dy}{dx}=3y^{2/3}$$

$$y(0)=1$$



$3y^{2/3}$ and $\frac{2}{\sqrt[3]{y}}$

in this case, you can avoid $y=0$.

{Do recommended problems!}

6 Questions, ±

{general solutions, explicit vs implicit} {exst, unique, and eqns, IVP, identify all types on eqn, is}

ODE'S:

Check Conditions for

- homogeneous? $\left(\text{if } \frac{\partial y}{\partial x} = F(x,y), \; f(ax,ay) = f(x,y) \right)$
- exact?
- linear?
- bernoulli? $\left(\frac{\partial y}{\partial x} + p(x)y = q(x)y^a \right)$
- separable?

homogeneous ✓	exact	linear ✓	bernoulli ✓	separable: ✓

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ex/unitue thm ✓