# MA 341

Linear 2nd Order w/ Constant Coefficients

$$y'' + py' + qy = 0$$

Characteristic Equations

$$r^2 + pr + q = 0$$

1) $r_1 \neq r_2$, ($r_1, r_2$ – real) $\Rightarrow$ $e^{r_1 x}, e^{r_2 x}$

2) $r_1 = r_2$ (real, multiplicity 2) $\Rightarrow$ $e^{rx}, xe^{rx}$

3) $r_2 = \bar{r}_1$ (complex pair) $\Rightarrow$ $e^{ax}\cos bx, e^{ax}\sin bx$

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1) Higher-Order Equations (homogeneous)

2) Non-homogeneous Equations

→ Linear 2nd Order

(method 1 of 2)

1) Higher-Order Homogeneous Equations

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$$r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

Solve!

multiplicity 1

1) If $r \in \mathbb{R}$ is a simple root, $e^{rx}$

2) If $r \in \mathbb{R}$ is a repeated root, multiplicity $k$,

$$e^{rx}, xe^{rx}, \dots, x^{k-1} e^{rx}$$

3) $r_2 = \bar{r}_1$, simple complex pair, $e^{ax} \cos bx$, $e^{ax} \sin bx$

$r_1 = a + ib$

4) $r_2 = \bar{r}_1$, multiplicity $k$, $e^{ax} \cos bx$, $e^{ax} \sin bx$

$$x e^{ax} \cos bx, \quad x e^{ax} \sin bx$$

$$\vdots \qquad \vdots$$

$$x^{k-1} e^{ax} \cos bx, \quad x^{k-1} e^{ax} \sin bx$$

$\boxed{ex}$

$$y''' - 3y'' + 3y' - y = 0$$

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0$$

$r = 1$, multiplicity 3
($k = 3$)

$$\Downarrow$$

$e^x,\ xe^x,\ x^2e^x$

$$\Downarrow$$

$$\underline{y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x}.$$

general solution

ex $y^{(4)} - y = 0$

$$r^4 - 1 = 0$$

$\Big\downarrow$ factor

$$(r^2-1)(r^2+1) = 0$$

$$(r-1)(r+1)(r^2+1) = 0$$

| | | |
|---|---|---|
| $r_1 = 1$<br>$r_2 = -1$ | two real simple roots | $e^x$<br>$e^{-x}$ |
| $r_3 = i$<br>$r_4 = -i$ | complex conjugate pair | $\cos x$<br>$\sin x$ |

$$C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

general Solution

ex

$$y^{(4)} + 2y'' + y = 0$$

$$r^4 + 2r^2 + 1 = 0$$

$$(r^2+1)^2 = 0$$

$r^2+1 = 0$, twice

$r = \pm i$, $k = 2$

$$\overbrace{\begin{matrix} \cos x & \sin x \\ x\cos x & x\sin x \end{matrix}}$$

$$\underline{C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x = y}$$

general
Solution

# Non Homogeneous 2nd-Order Eqns.

▷ $y'' + p y' + q y = f(x)$

constants function

(or)

▷ $a y'' + b y' + c y = f(x)$ for Physics

1) Solve homogeneous version

$$y'' + p y' + q y = 0$$

2) Two major ways to solve nonhomogeneous eqns:

Use if you can

i) method of undetermined coefficients (algebraic)

ii) variation of parameters (requires integrals) i)

# Method of Undetermined Coefficients

$$y'' + py' + qy$$

Func types that preserve structure when differentiated

Polynomials: $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ $\left(\frac{d}{dx} \to \text{new polynomial}\right)$

exponentials: $e^{ax} \to \frac{d}{dx} \to \text{multiple}$

trigonometrics: $\cos bx,\ \sin bx \to \frac{d}{dx} \to \text{superposition}$

But: $P(x)e^{ax}$

$\left(P(x)\cos bx + Q(x)\sin bx\right) e^{ax}$ — quasi-polynomials

etc.

also preserve structure when differentiated.

Hw: verify!

Now, if

$$y'' + py' + qy = \text{some quasi-polynomial},$$

one of the above function types might do the job.

1) Solve $y'' + py' + qy = 0$

$y_h$ = general solution of the associated homogeneous equation

2) Find just one solution of the given equation, $y_p(x)$

3) $y_h(x) + y_p(x) \longrightarrow$ general solution of the homogeneous equation

In-depth: Finding $y_p(x)$:

ex) $y'' + 3y' + 2y = x^2$ — Match degree of ~~Solution~~ eqn. quasi-polynomial

Look for $y_p(x) = ax^2 + bx + c$

$y_p'(x) = 2ax + b$

$y_p''(x) = 2a$

Plug in: $2a + 3(2ax+b) + 2(ax^2+bx+c) = x^2$

Solution

rearrange: (group w/ coefficients) $2ax^2 + (6a + 2b)x + (2a+3b+2c) = x^2$

infer values:

$$2a = 1$$

$$6a + 2b = 0$$

$$2a + 3b + 2c = 0$$

Solve: $a = 1/2$, $b = -3/2$, $c = 7/4$

$$y_p(x) = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$$

find gen soltn:

$$y'' + 3y' + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2)$$

$$r_1 = -1, \; r_2 = -2$$

$$y_h(x) = C_1 e^{-x} + C_2 e^{-2x}$$

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$$

general solution

ex

$$y'' + 3y' + 2y = \underbrace{x e^{x}}_{\text{most general first-degree polynomial}}$$

Conjecture $y_p$ $\longrightarrow$ $y_p(x) = (ax+b)e^{x}$

$$y_p' = ae^{x} + (ax+b)e^{x} = (a+b)e^{x} + axe^{x}$$

$$y_p'' = (a+b)e^{x} + ae^{x} + axe^{x}$$

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$$3\left[(a+b)e^{x} + \quad + axe^{x}\right] \quad y'$$

$$(2a+b)e^{x} + \quad + axe^{x} \quad y''$$

$$2\left[b e^{x} \quad + \quad axe^{x}\right] \quad y$$

Sub in $y_p(x)$:

$$(5a+6b)e^{x} + 6axe^{x} = xe^{x}$$

$$5a+6b=0 \quad , \quad 6a=1$$

solve for a, b

$$\boxed{b = \frac{-5}{36}} \quad \boxed{a = 1/6}$$

So, $y_p(x) = \frac{1}{6} x e^{x} - \frac{5}{36}$