MA 341 (3)

Explicit Solution: $y = \phi(x)$

ex $\frac{dy}{dx} = y$

$y = e^x$ ← a solution
← an explicit solution

Implicit Solution:

$F(x, y, y') = 0$
first-order ODE

$\Phi(x, y) = 0$

this equation defines y as a function of x.

$\frac{\partial \Phi}{\partial y}(x_0, y_0) \neq 0$

If this implicit function solves our equation, we call $\varphi(x,y)$ the implicit solution.

[redacted] Idea| $\frac{dy}{dx} = y$

$$dy = y\,dx$$

generically,

$$\frac{dy}{dx} = f(x,y)$$

$$dy = f(x,y)\,dx$$

differential form of eqn.

ex| $x\,dx + y\,dy = 0 \longrightarrow \varphi(x,y) = x^2 + y^2$

$y\,dy = -x\,dx$

$\frac{dy}{dx} = -\frac{x}{y}$

$$x^2 + y^2 = 4$$

$$2x\,dx + 2y\,dy = 0$$

$$x\,dx + y\,dy = 0$$

$\frac{d}{dx}(x^2+y^2)$

$\frac{d}{d\cancel{y}}(x^2+y^2)$

$\underline{\text{ex}}$ 2

take $\Phi(x,y) = x^2+y^2$

$\varphi(x,y)$

$d\varphi = \varphi_x\,dx + \varphi_y\,dy$

$x^2+y^2 = \text{const}$] any formula like this solves eqn.

For any positive C, the formula $x^2+y^2=C$ is an implicit solution of $x\,dx+y\,dy=0$

Existence / Uniqueness

$$\frac{dy}{dx} = f(x,y)$$

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

depends on $n+1$ variables $\Rightarrow$ ~~If $f(x, u_1, \dots, u_n)$~~

Over

Initial Value Problem for $(*)$ is stated like this:

Solve $(*)$ subject to the initial conditions

$$\left.\begin{array}{l} y(x_0) = y_0, \\ y'(x_0) = y'_0, \\ \vdots \\ y^{(n-1)}(x_0) = y_0^{(n-1)} \end{array}\right\} n \text{ conditions}$$

If $f(x, u_1, u_2, \ldots, u_n)$

Partial derivative and $f_{u_1}, f_{u_2}, f_{u_3}, \ldots, f_{u_n}$

are continuous around

$(x_0, y_0, y'_0, \ldots, y_0^{(n-1)})$,

then this problem can be solved uniquely.

ex) $$\frac{d^2y}{dx^2} + \sin y = 0$$

assume $g = 1$, $L = 1$:
oscillations of swing set

$y(x_0) = y_0$ initial position
$y'(x_0) = y_0'$ velocity

$$\frac{d^2y}{dx^2} = -\sin y$$

$f(x, y, y')$
(happens only to depend on y)

$$\left.\begin{aligned} f &= -\sin y \\ f_y &= -\cos y \\ f_{y'} &= 0 \end{aligned}\right]$$

$\triangleright$ continuous everywhere
(will be cont. around *any* initial conditions)

Thus, we know solutions *exist*, even if currently we don't have the tools to find them.

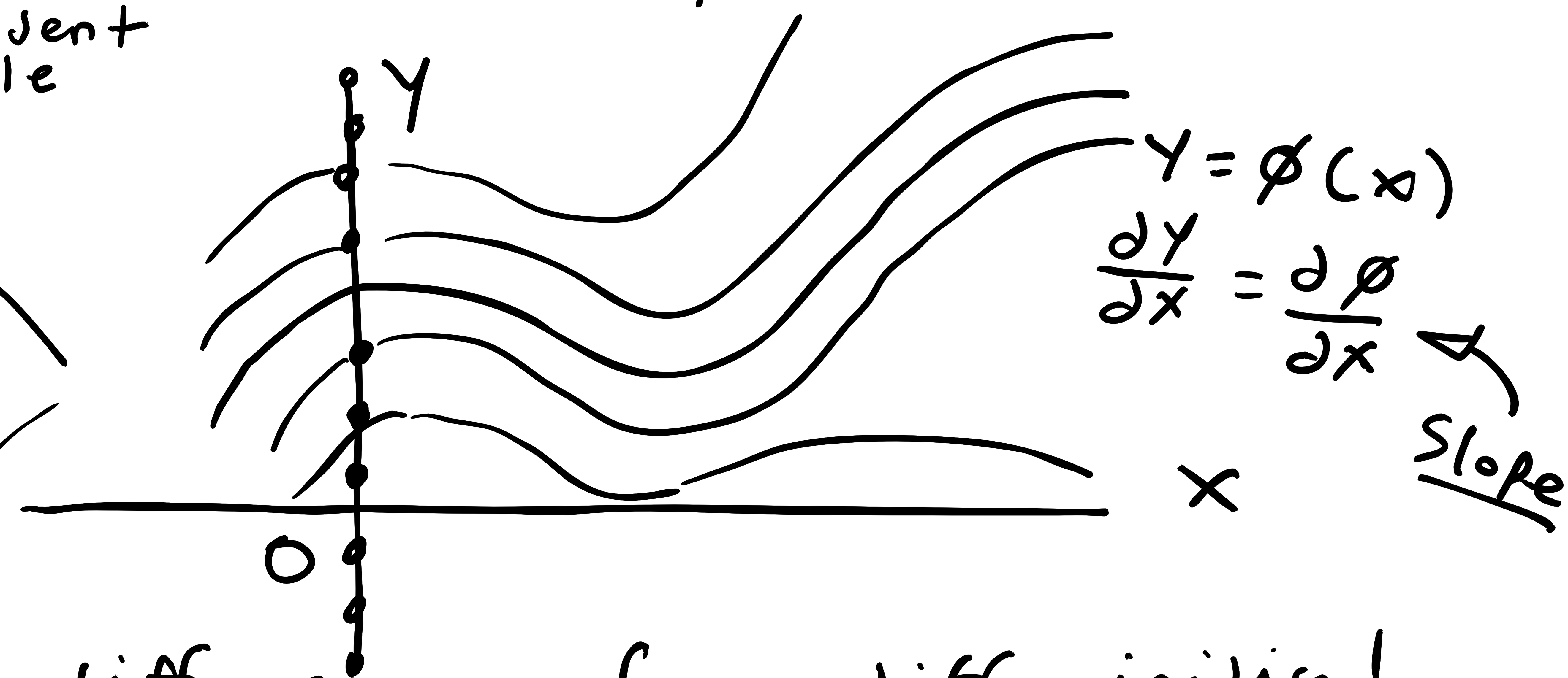
1.3 Direction Fields

$$\frac{dy}{dx} = f(x, y)$$

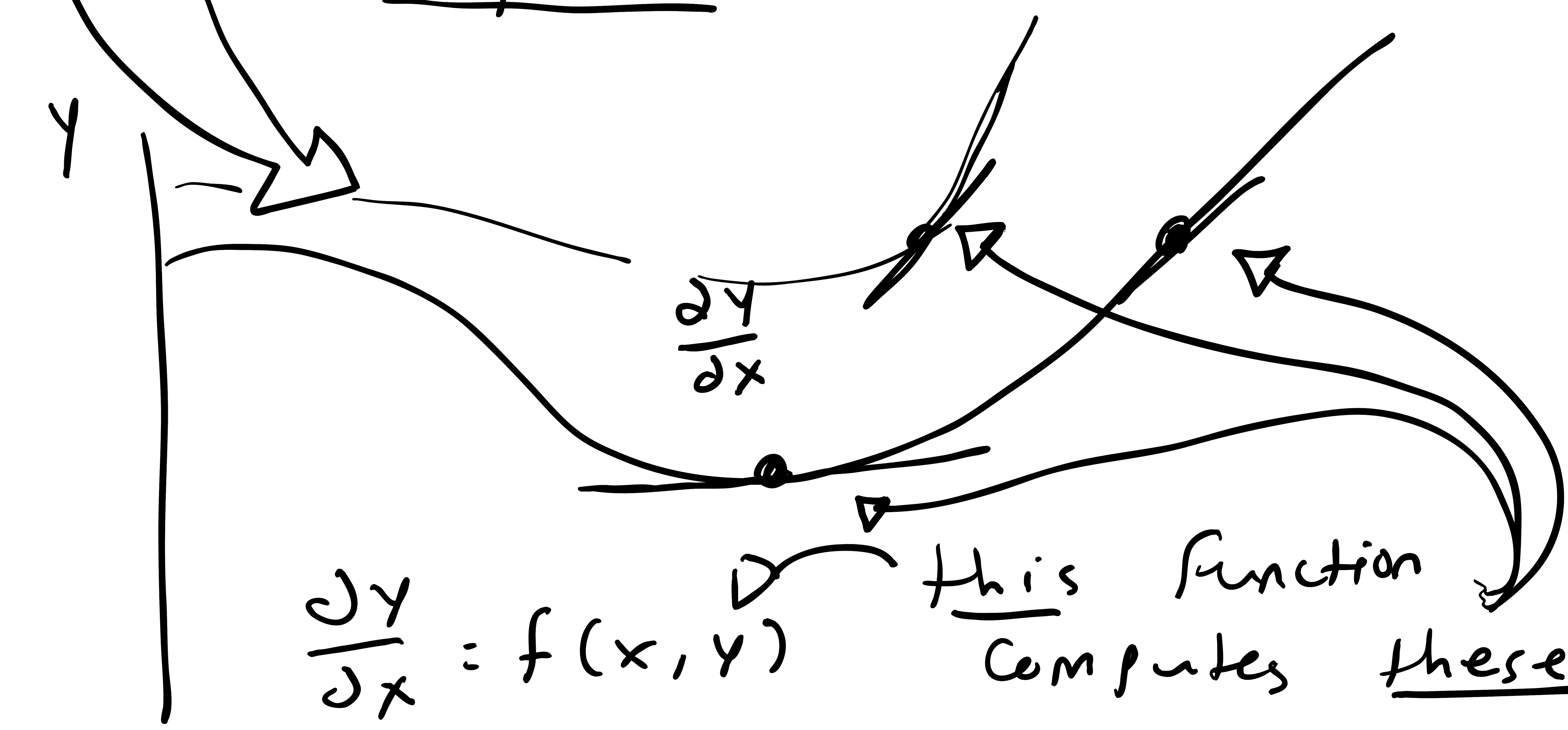
$y = \phi(x)$ $\qquad (x_0, y_0)$

output, or dependent variable

input, or independent variable



diff. curves from diff initial conditions (x_0, y_0) — don't intersect (uniqueness assumed)



$\frac{dy}{dx} = f(x, y)$ this function computes these slopes.

The collection of these slopes:
the direction field.
("slope field")