# MA 405

More determinants (4.2, 4.3)

## Some Properties:

- $\det(I) = 1$
- $\det(P) = \pm 1$ for permutations
- $\det(A)$ = product of diagonal entries ($A$ triangular)
- $\det(AB) = \det(A) \cdot \det(B)$
  - Not for addition!

## Standard (Recursive) method of determinant calculation

**def:** $A \in \mathbb{R}^{n \times n}$

The cofactors of $A$ are numbers defined by using determinants of submatrices:

$$C_{ij} = -1^{(i+j)} \det(M_{ij})$$

where $M_{ij}$ is the submatrix obtained by discarding the $i$th row and $j$th col of $A$.

**Idea**

Compute cofactors along entire row, then combine them using that row of $A$:

**Formula**

Let $A \in \mathbb{R}^{n \times n}$ then $\det(A)$ is a linear combination of cofactors:

$$\boxed{\det(A) = \underbrace{a_{i1}}_{\text{entries in row } i} C_{i1} + \dots + a_{in} \underbrace{C_{in}}_{\text{cofactors along row } i}}$$

**Recall:** $\det(A)$ is a number which can be used to determine invertibility (if $\det \neq 0$)

Finding $\det(A)$ from $LDU$ decomposition

**Fact** If $A \in \mathbb{R}^{n \times n}$ is invertible, then

$$\underbrace{P}_{\text{Permutation}}A = \underbrace{L}_{LA}\underbrace{D}_{\text{diagonal}}\underbrace{U}_{UA}, \quad \text{and}$$

$$\det(PA) = \det(LDU)$$

$$\underbrace{\det(P)}_{\pm 1} \det(A) = \underbrace{\det(L)}_{1} \underbrace{\det(D)}_{\text{Product of diagonal entries}} \underbrace{\det(U)}_{1}$$

Product of diagonals → 1 (bc you have D matrix)

So,
$$\boxed{\begin{aligned}\det(A) &= \pm 1 \cdot \det(D) \\ &= \pm(\text{Product of diagonal entries in } D)\end{aligned}}$$

(★) LDU is useful for computing the determinant!

**ex**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- $C_{12} = -1^3 \det \underbrace{\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}_{M_{12}}$
  $= -1(36 - 42)$
  $= 6$
- $C_{33} = -1^6 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$
  $= 5 - 8 = -3$

---

**ex** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

we know that $\det(A) = ad - bc$

Using cofactors, expand using first row:

$C_{11} = -1^2 |d|$ $\qquad$ $C_{12} = -1(c)$
$= d$ $\qquad\qquad\qquad$ $= -c$

Now,

$\det(A)$
$= a_{11} C_{11} + a_{12} C_{12}$
$= ad + b(-c)$
$= ad - bc$ ✓

Next, w/ row 2,

$C_{21} = -1(b)$ and $C_{22} = 1(a)$

$\det(A) = -bc + ad$ ✓
$= ad - bc$ ✓

ex For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Expand using any row:

or, just use recursion for Helms method!) → same thing

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Strategy

ex $A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$\det(A) = ?$

Strategy: when using cofactor expansion, pick the row of A with the most 0's! Because those 0's will be coefficients, and you have to compute fewer cofactors.

If $a_{ij} = 0$, the $ij$ cofactor won't be needed.

use row 4!

$$\det(A) = a_{41}C_{41} + \cancel{a_{42}}^{0} C_{42} + \cancel{a_{43}}^{0} C_{43} + a_{44}C_{44}$$

$$= a_{41}C_{41} + a_{44}C_{44}$$

$$= 2C_{41} + C_{44}$$

▸ $C_{41} = -1 \begin{vmatrix} 0 & 3 & 2 \\ 3 & 4 & 5 \\ 4 & 0 & 3 \end{vmatrix}$

↓

using row 1,

$\det = 0(12) - 3(9-20) + 2(-16)$

$= -3(-11) - 32$

$= 33 - 32 = 1$

so $C_{41} = -1$

▸ $C_{44} = 1 \begin{vmatrix} 1 & 0 & 3 \\ 0 & 3 & 4 \\ 5 & 4 & 0 \end{vmatrix}$

↓

$\det = -16 - 0 + 3(-15)$

$= -16 - 45 = -61$

so $C_{44} = -61$

So, $\det(A) = 2(-1) + (-61)$

$= -61 - 2 = \boxed{-63}$

## Things to remember for computing dets

→ Look for "tricks"

- structure?
- factorization?
- look for lots of $\phi$'s

## Cofactors can be used to compute $A^{-1}$!

If $A \in \mathbb{R}^{n \times n}$ is invertible, and

$C_{ij}$ = cofactors of A.

$= (-1)^{i+j} \det(M_{ij})$,

then

$$\boxed{A^{-1} = \frac{C^T}{\det(A)}}$$

Matrix of all cofactors

In other words,

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}$$

shortcut for a single element

In general, however, Gauss-Jordan is quicker for finding $A^{-1}$

Next up: applications!