MA 405

Recall elementary matrices (way to represent row operations as matrix multiplications)

ex

$$E = \underset{3\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}$$ row exchange

$R_1 \leftrightarrows R_2$, then $R_2 \leftrightarrows R_3$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (we can see where Identity rows are "sent to")

⊛ Permutation Matrix

- very similar to I
- square
- binary (entries 0 or 1)
- only one 1 per row and column
- rearranged

ex

$$A = \begin{bmatrix} 1 & 2 & 7 & -1 \\ 0 & -1 & -4 & 5 \\ 1 & 3 & 0 & 2 \end{bmatrix} \quad 3\times 4$$

$$B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 7 \end{bmatrix} \quad 2\times 3$$

▷ vs $\underset{(3\times 4)}{E \cdot A}$:

$$= \begin{bmatrix} 0 & -1 & -4 & 5 \\ 1 & 3 & 0 & 2 \\ 1 & 2 & 7 & -1 \end{bmatrix}$$

rows are exchanged.

▷ vs $\underset{2\times 3}{B \cdot E}$:

$$= \begin{bmatrix} 0 & 1 & 3 \\ 7 & 2 & 4 \end{bmatrix}$$

columns are exchanged

▷ vs $\underset{(2\times 3)}{B \cdot A}$:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 7 & -1 \\ 0 & -1 & -4 & 5 \\ 1 & 3 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 5 \\ 9 & 21 & -2 & \end{bmatrix}$$

▷ Since row operations are reversible, so is multiplying by an elementary matrix.

ex) undo $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ with $F = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

→ compare to $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

~~$R_1 \leftrightarrow R_2,\ R_2 \leftrightarrow R_3$~~

$R_3 \leftrightarrow R_2,\ R_2 \leftrightarrow R_1$

ex)

$F \cdot (EA)$

$$= \begin{bmatrix} 1 & 2 & 7 & -1 \\ 0 & -1 & -4 & 5 \\ 1 & 3 & 0 & 2 \end{bmatrix} = A$$

Can also check entrywise

(*) The matrix F "undoes" multiplication by E ⇒ so they are like <u>multiplicative inverses</u>. (product = identity matrix)

<u>def</u>: A square matrix A in $\mathbb{R}^{n\times n}$ is called <u>invertible</u> if there exists some other B in $\mathbb{R}^{n\times n}$ so that

- $B \cdot A = I$, and
- $A \cdot B = I$

Then B is called the <u>inverse</u> of A, denoted A^{-1}.

★ Inverses are not defined for rectangular matrices

A is $\mathbb{R}^{m \times n}$

- $\underset{m \times n}{B \cdot A} = I_{n \times n}$ → would need B in $\mathbb{R}^{n \times m}$.

$n \times m$
(to get square matrix)

But then,

- $A \cdot B = I_{n \times n}$

$m \times n$ ↓ $n \times m$ would need a different size of B.

▷ Nevertheless, if $B \cdot A = I$, B is called "left inverse",
and if $A \cdot B = I$, B is called "right inverse"

▷ For A to have an inverse, B must $=$ l. inverse $=$ r inverse

(*) Do all square matrices have inverses?

- Nope.

ex| Any square matrix w/ a row or column of all $\emptyset$'s:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

if A was invertible, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{bmatrix}$$

$\neq I$ b/c of row of all $\emptyset$'s.

(*) Are inverses unique? Yes!

- Proof: let A be $\mathbb{R}^{n \times n}$ and B, C be inverses of A

- Consider $B(AC) = B \cdot I$

$(BA)C = I \cdot C$

also, $B(AC) = (BA)C$

$$B \cdot I = I \cdot C \xrightarrow[I]{\text{def. of}} B = C$$

:)

Inverses of Elementary Matrices

▷ row exchange: permutation matrix

→ Inverse: another row exchange

ex] $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \leftrightarrows R_2$

▷ row scaling:

→ inverse: another row scaling

ex] $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

▷ (row) elimination:

→ inverse: another elimination

ex] $E = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \leftarrow R_2 + l \cdot R_1$

▷ what's the point?

— Matrix Factorization!

▷ The main idea of these elementary matrices is <u>Matrix</u> <u>Factorization</u>:

$$\begin{tikzcd} A \in \mathbb{R}^{n\times n} \arrow[r, squiggly, "\text{elimination}"] & U \in \mathbb{R}^{n\times n} \arrow[l, squiggly, bend left=40, "\text{inverses of elementary matrices}"] \end{tikzcd}$$

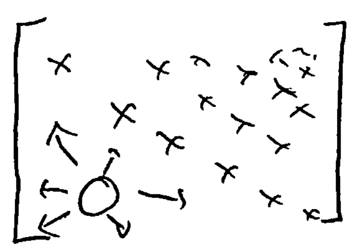
in echelon form

(Useful for solving $A\underline{x} = \underline{b}$)

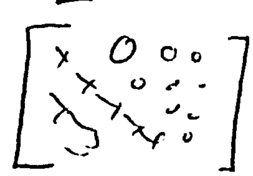
▷ To get echelon form,

$$U = \underbrace{E_k \cdots\cdots E_2 E_1}_{\text{matrix representations of elimination steps}} A$$

in particular, U is <u>upper</u> <u>triangular</u>



(<u>lower</u> <u>triangular</u>:



These can all be undone:

▷ $E_k^{-1}(U) = E_k^{-1}(E_k \cdots E_2 E_1 A)$

- $E_k^{-1} E_k = I$
- keep multiplying by inverses:

$$\underbrace{\left(E_1^{-1} E_2^{-1} \cdots E_k^{-1} \right)}_{\text{Call these } L} U = A$$

"LU decomposition"

$$\boxed{A = LU}$$

a product of two structured matrices:

upper triangular

~~a product~~ comes from only ~~many~~ elementary matrices

- In practice: $A \underline{x} = \underline{b}$

Factor $A = L \cdot U$

Then

$$\left.\begin{aligned} A \underline{x} = \underline{b} \iff L \underline{y} &= b \\ U \underline{x} &= \underline{y} \end{aligned}\right\} \text{Two, } \underline{\text{easier}} \text{ systems to solve.}$$

$$L(U\underline{x}) = \underline{b}$$

ex| of LU decomposition:

▷ $A = \begin{bmatrix} 2 & 3 \\ 6 & 3 \end{bmatrix}$

▷ Reduce to echelon form:

$R_2 \leftarrow R_2 - 3R_1$

$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, so $EA = \begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix}$

echelon form matrix U

▷ Apply $E^{-1}U = A$:

$E^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Matrix L
(note lower triangular structure)

▷ Check:

$LU = A$