Consistent System (has solution(s)) $\Longleftrightarrow$ Solution column isn't pivot column

free variables $\cong$ non-pivot columns

unique solution $\Longleftrightarrow$ no non-pivot columns

Singular system $\Longleftrightarrow$ 1+ row(s) of all 0's

Pivot: leading entry in a matrix of echelon form

Pivot Column: Column containing a pivot position (leading 1's in RREF)

ex|

$$\left[\begin{array}{cc|c} * & * & * \\ 0 & * & * \end{array}\right]$$

Pivots

Pivot columns

Diagonal: $a_{ij} = 0$ if $i \neq j$

Symmetric: $a_{ij} = a_{ji}$ for all i, j

upper triangular: $a_{ij} = 0$ if $i > j$

Skew-Symmetric: $a_{ij} = -a_{ji}$ for all i, j

is $\underline{b}$ a L.C. of Columns of A?

Yes if $A\underline{x} = \underline{b}$ has a solution; i.e.,

$$[A \mid \underline{b}]$$

has a solution.

(Implies A has a pivot in every row)

Matrix multiplication

LC of columns of A

$$\underset{\substack{n\times k \\ \parallel \\ k \times p}}{AB} = \left[Ab_1 \;\vdots\; Ab_2 \;\vdots\; \cdots \;\vdots\; Ab_p \right] \quad (i)$$

columns b_i of B

$$(AB)_{ij} = \sum_{l=1}^{K} a_{il} \cdot b_{lj} \quad (ii)$$

LC of rows of B

$$AB = \begin{bmatrix} r_1 B \\ \hdashline r_2 B \\ \hdashline \vdots \\ \hdashline r_n B \end{bmatrix} \quad (iii)$$

rows r_i of A

$U_1 U_2 = U_3$ (Products of upper-Δ matrices are upper-Δ)

$L_1 L_2 = L_3$ (Products of lower-Δ matrices are lower-Δ)

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

⇑ Matrix multiplication is distributive, but not commutative

$$(AB)^2 = ABAB$$
$$\neq A^2 B^2$$

$$(2A)^{-1} \neq 2A^{-1} \; !$$

⇑ Treat parameters as unique matrices!

Prove $L_1 L_2 = L_3$, L_1, L_2, L_3 are lower-Δ

$L_1 \in \mathbb{R}^{n \times m}$ for $j > i$: $(L_1)_{ij} = 0, \; i > j$

$L_2 \in \mathbb{R}^{m \times k}$ $(L_1 L_2)_{ij} = \sum_{\ell=1}^{m} L_{1_{i\ell}} L_{2_{\ell j}}$ $(L_2)_{ij} = 0, \; i > j$

$L_1 L_2 \in \mathbb{R}^{n \times k}$

$$= \sum_{\ell=1}^{i} L_{1_{i\ell}} L_{2_{\ell j}} + \underbrace{\sum_{\ell=1+i}^{m} L_{1_{i\ell}} L_{2_{\ell j}}}_{\text{adding all 0's}}$$

$j = \ell > i : L_{1_{i\ell}} = 0$

$$= \underbrace{\sum_{\ell=1}^{i} L_{1_{i\ell}} L_{2_{\ell j}}}_{\text{all } 0} + 0$$

$j > i = \ell : L_{2_{\ell j}} = 0$

$= 0$ for $j > i$

Lower-Triangular

if A commutes with B,

then $AB = BA$

(True for all square B if $A = aI$, $a \in \mathbb{Z}$)

matrix multiplication

$$AB = \underbrace{c_1}_{\text{column of } A} \underbrace{r_1}_{\text{row of } B} + c_2 r_2 + \underbrace{c_3 r_3}_{\text{"simple matrix"}}$$

("outer product" method)

involution

$$A = A^{-1}$$

nilpotent

$$A^K = 0 \quad (\text{some } K)$$

Factorizations

$$\underbrace{P}_{\text{Permutations}}A = \underbrace{L}_{\text{lower } \Delta}\,\underbrace{D}_{\text{diagonal scalars}}\,\underbrace{U}_{\text{upper } \Delta}$$

① check if permutations needed for reduction. IF so, do them, and $A \Rightarrow PA_{\text{permuted}}$

② reduce A to upper-Δ form.

L = reversed E = undoing row ops performed

U = upper-Δ RE form of A

③ Factor out diagonal scalars of U into new matrix D:

$$\underset{U}{\begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}} \Rightarrow \underset{D}{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}} \underset{U_{new}}{\begin{bmatrix} 1 & */a & */a \\ 0 & 1 & */b \\ 0 & 0 & 1 \end{bmatrix}}$$

example $A = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$

elim: $R2 \leftarrow R2 - 4R1$
$R3 \leftarrow R3 + R2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $A\underline{x} = \underline{b}$ with $A = LU$

① Solve $L\underline{c} = \underline{b}$

② Solve $U\underline{x} = \underline{c}$

$$(A^{-1})^T = (A^T)^{-1}$$

for Permutation matrices,

$$P^{-1} = P^T$$

▷ A one-sided inverse of a square matrix is automatically two-sided

▷ if matrix is invertible, then $\exists$ some A^{-1} s.t.

$$AA^{-1} = A^{-1}A = I$$

▷ For B to be inverse A^{-1} of A, it must be both left- and right- inverses.

$$BA = I \qquad AB = I$$

Gauss-Jordan

$$[\, I \mid A \,]$$

$$\big\downarrow \text{reduce}$$

$$[\, A^{-1} \mid I \,]$$

▷ invertible LDU factorizations

if A is invertible, it has a unique factorization

$A = LDU$

▷ only diagonal matrices are both upper- and lower-triangular

example | solve $A\underline{x} = \underline{b}$ w/ $\underline{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$

$$A = LU$$

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 0 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

① Solve $L\underline{c} = \underline{b}$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 0 & 2 \end{array}\right] \end{array} \Rightarrow \underline{c} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$$

② Solve $U\underline{x} = \underline{c}$

$$\left[\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{array}\right] \Rightarrow \underline{x} = \begin{bmatrix} 3 \\ 4 \\ \end{bmatrix}$$

IMT The following are equivalent:

1. $A \in \mathbb{R}^{n \times m}$ is invertible
2. A is row-equivalent to I (can be reduced to I)
3. A has $\underline{n}$ pivots (a "full set")
4. $A\underline{x} = \underline{b}$ has a unique solution for any $\underline{b} \in \mathbb{R}^n$
5. $A\underline{x} = \underline{0}$ has only solution $\underline{x} = \underline{0}$
6. columns of A are LI
7. Col(A) spans $\mathbb{R}^n$ and forms a basis for $\mathbb{R}^n$
8. A^T is invertible
9. $\dim(\text{Col}(A)) = n$
10. $\text{rank}(A) = \dim(\text{Col}(A)) = n$
11. $\text{Null}(A) = \{\underline{0}\}$
12. $\dim(\text{Null}(A)) = 0$