Projections

1-dimensional (onto a line): $p = c\,\underline{a}$, $c = \dfrac{x^T \underline{a}}{\underline{a}^T \underline{a}}$

Projection of $\underline{x}$ onto $\underline{a}$

$P = \dfrac{aa^T}{a^Ta}$

$$\boxed{p_{\underline{x}\text{ onto }\underline{a}} = \frac{x^Ta}{a^Ta}\,\underline{a}}$$

Onto Column Space: Proj ($\underline{x}$ onto col(A)) $= \boxed{P\underline{x} = A(A^TA)^{-1}A^T\underline{x}}$

① Find $A^T\underline{x}$

② Find A^TA

③ Find $(A^TA)^{-1}A^T\underline{x}$

Solve linear system

i) $(A^TA)^{-1}A^T\underline{x} = \underline{y}$

ii) $A^T\underline{x} = A^TA\,\underline{y}$

iii) $[A^TA]\,\underline{y} = [A^T\underline{x}]$

iv) $\left[\begin{array}{c|c} A^TA & A^T\underline{x} \end{array}\right]$

or

construct inverse directly

i) if $A^TA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

ii) $(A^TA)^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Constructing Orthonormal Basis Vectors

1) make $q_1, q_2, \ldots$ using Gram-Schmidt

2) check $q_1^Tq_2 = 0$

Factorize A=QR

1) given $A = [a_1\ a_2 \ldots]$

$Q = [q_1, q_2 \ldots]$ ← from G-S

2) find R s.t. $\boxed{A = QR}$ ← Invertible

3) $R = \begin{bmatrix} q_1^Ta_1 & q_1^Ta_2 \\ 0 & q_2^Ta_2 \end{bmatrix}$ ← upper triangular

a_i

q_i^T

Gram-Schmidt

1) given a_1, a_2, a_3

Make orthogonal

2) $b_1 = a_1$

$b_2 = a_2 - \text{Proj}(a_2 \text{ onto } b_1)$

$b_3 = a_3 - \text{proj}(a_3 \text{ onto } b_2) - \text{proj}(a_3 \text{ onto } b_1)$

Normalize

3) $\vec{q}_1 = \dfrac{\vec{b}_1}{b_1}$, $\vec{q}_2 = \dfrac{\vec{b}_2}{b_2}$, $\vec{q}_3 = \dfrac{\vec{b}_3}{b_3}$

Checking Linearity of a Transformation

Transformations and their Matrices

To disprove,

- show $T(k\underline{x}) \neq kT(\underline{x})$
- show $T(\underline{x}+\underline{y}) \neq T(\underline{x})+T(\underline{y})$

To prove,

AND
- show $T(k\underline{x}) = kT(\underline{x})$ (scalar multiplication)
- show $T(\underline{x}+\underline{y}) = T(\underline{x})+T(\underline{y})$ (vector addition)

OR show $T(a\underline{x}+b\underline{y}) = aT(\underline{x})+bT(\underline{y})$

Standard Matrix of a Transformation

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \dots \end{bmatrix}$$

where $\{b_1 \dots b_n\}$ are the standard basis vectors fed through the transformation

Change of Basis Matrix

To change from basis V to basis W, CoB matrix is $\boxed{WV^{-1}}$

Order of Transformation Matrices

[third] [second] [first] target

Rotation Matrix

to rotate θ,

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

Inner Prod

$\underline{a}_2^T\underline{a}_1 = \underline{a}_1^T\underline{a}_2$

Outer Prod

$\underline{a}_2\underline{a}_1^T$

Normal Equations

$$A^T A \hat{x} = A^T B$$

wooo

LLS (1-D)

$$\hat{x} = \frac{a^T b}{a^T a} ,$$

Props, II

Onto

transformation matrix

A has a pivot in every column

one-to-one

transformation matrix

A has a pivot in every row

Superposition of orthogonal projections

if $\underline{a} \perp \underline{b}$, then $P_1 \underline{x} =$ proj onto $\underline{a}$

$P_2 \underline{x} =$ proj onto $\underline{b}$

and proj onto span$\{\underline{a}, \underline{b}\}$

$= P_1 \underline{x} + P_2 \underline{x}$

Mutual orthogonality implies Linear Independence

LLS (multivar)

① construct A:

$$A = \begin{bmatrix} 1 & t_{min} \\ 1 & \vdots \\ 1 & t_{max} \end{bmatrix}$$

① Construct B:

$$B = \begin{bmatrix} x(t_{min}) \\ \vdots \\ x(t_{max}) \end{bmatrix}$$

or

$$A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y(x_1) \\ \vdots \\ y(x_n) \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} m \\ b \end{bmatrix} \qquad B = \begin{bmatrix} y(x_1) \\ \vdots \\ y(x_n) \end{bmatrix}$$

② Plug into

$$\hat{x} = (A^T A)^{-1} A^T B$$

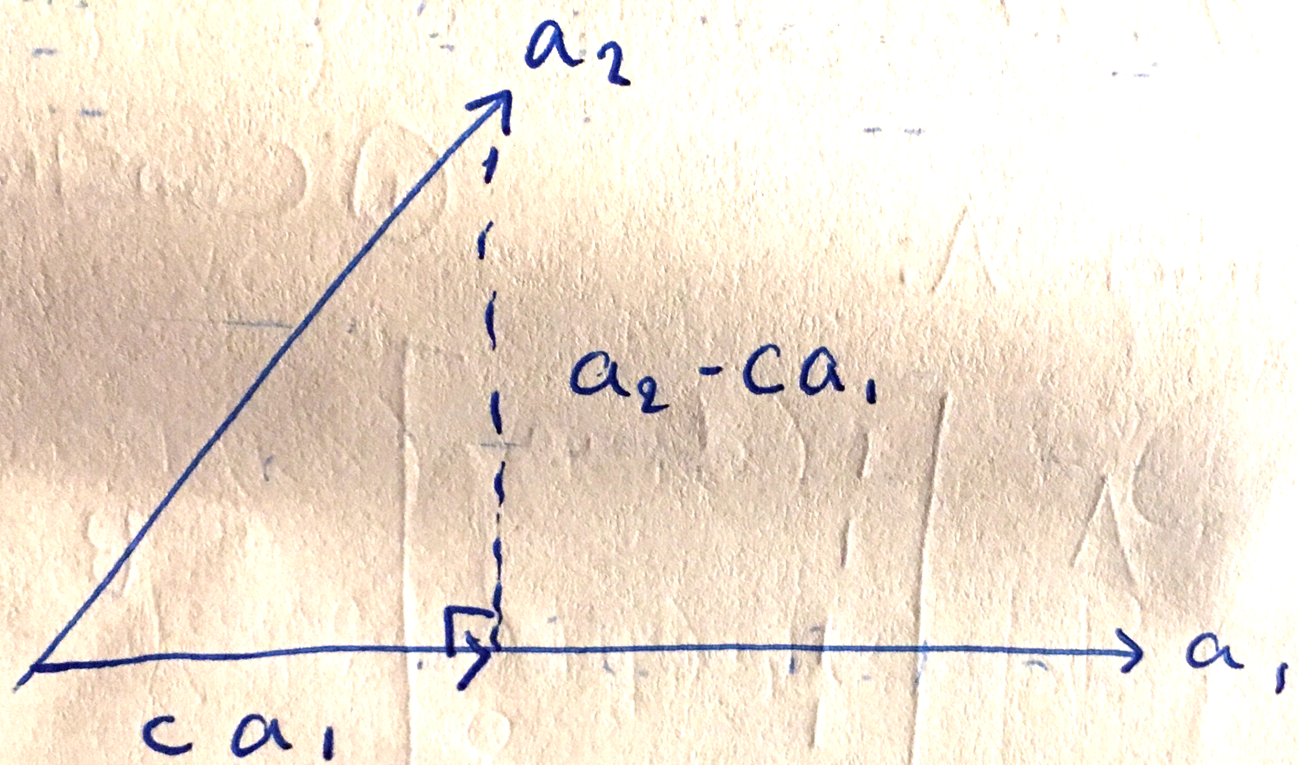
i) find $A^T B$

ii) find $A^T A$

iii) find $(A^T A)^{-1}$

③ Projection (optional)

$$Proj = A \hat{x}$$


a_2
$a_2 - ca_1$
a_1
ca_1

Props

$V^{\perp}$

Orthogonal Subspaces

Two subspaces V + W are orthogonal if

$v^T w = 0$ for all $v \in V$ and all $w \in W$

Orthogonality

$\underline{v}, \underline{w}$ are orthogonal if

$\underline{v}^T \underline{w} = 0$

Orthogonal Complement

given $V \subseteq \mathbb{R}^n$ the orthogonal complement of V is the space of all vectors orthogonal to V.

Fundamental Thm of Linear Algebra

$\text{Null}(A) \perp_c \text{Col}(A^T)$

$\text{Null}(A^T) \perp_c \text{Col}(A)$

where $\perp_c$ = ortho complement

Proj Matrix Props

① $P^T = P$

② $P^2 = P$

Linear Transformation

any transformation whose transformation matrix A satisfies

$A(cx + dy) = cAx + dAy$

is linear.

Types of Linear Transformations

① stretch

$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

② rotation

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

e.g. $(x, y) \to (-y, x)$

③ reflection

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

e.g. over $y = x$

④ Projection

$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

e.g. (covered extensively elsewhere)

Linear T's when $T(\underline{b_i})$ is known for all basis vectors $\underline{b}_i$

if Ax $(T(x))$ is known for every vector in a basis, then we know Ax for all x.

If $\underline{x} = \underbrace{c_1 \underline{x}_1 + \dots + c_n \underline{x}_n}_{\text{basis vectors}}$ then $A\underline{x} = c_1(A\underline{x}_1) + \dots + c_n(A\underline{x}_n)$

1D proj $P = \frac{aa^T}{a^Ta}$

$p = Px = \frac{aa^T}{a^Ta}x$

Col(A) Proj $P = A(A^TA)^{-1}A^T$

$p = A(A^TA)^{-1}A^T x$

① find A^Tx

② A^TA

③ $(A^TA)^{-1}$

1D

$P = \frac{aa^T}{a^Ta}$

Col(A)

$P = A(A^TA)^{-1}A^T$

$P = A(A^TA)^{-1}A^T$

$p = Px$

① A^Tx

② A^TA

③ $(A^TA)^{-1}$

→ G-J

$[A^TA \mid A^Tx]$

G-S

a_1, a_2, a_3

$b_1 = a_1$

$b_2 = a_2 - proj_{b_1} a_2$

$b_3 = a_3 - proj_{b_2} a_3 - proj_{b_1} a_3$

$q_i = \frac{b_i}{\|b_i\|}$

check $q_1^T q_2 = 0$

$P^2 = P$

$P^T = P$

$A^TA\hat{x} = A^TB$

Mat mult is linear

$A^TA\hat{x} = A^TB$

$A = QR$

$Q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$

P

$P^2 = P$

$P^T = P$

R

$R^T = R$

R^{-1} exists

Symm, Inv

$R = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & q_1^T \\ 0 & q_2^T a_2 & q_2^T \\ \vdots & 0 & q_3^T \\ \vdots & \vdots & \end{bmatrix}$

$R = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$

$R^{-1} = R^T$

$P = A(A^TA)^{-1}A^T$

$p = A(A^TA)^{-1}A^T x$

$p = Px$

Change of basis

WV^{-1}

$Q_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

$b_1 = a_1$

$b_2 = a_2 - proj_{b_1} a_2$

$b_3 = a_3 - proj_{b_2} a_3 - proj_{b_1} a_3$

LLS

Norm eqns $A^TA\hat{x} = A^TB$

$A = \begin{bmatrix} x\text{'s}, & 1\text{'s} \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = B = \begin{bmatrix} y\text{-vals} \end{bmatrix}$

$\boxed{\hat{x} = (A^TA)^{-1}A^T B}$

Linearity

$T(ax + by) = aT(x) + bT(y)$

$Q_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

CoB mat $= WV^{-1}$

$V \to W$

$P = \frac{aa^T}{a^Ta}$ $P = A(A^TA)^{-1}A^T$

onto

Tmat has pivs in every col

1-to-1

Tmat has pivs in every row

Spaces V, W are orthogonal if

all $v^T w = 0$

for basis vecs

orthogonal complement

all vecs $\perp$ V

$\text{Null}(A) \perp_c \text{Col}(A^T)$

$\text{Null}(A^T) \perp_c \text{Col}(A)$

$\underline{x} = c_1 \underline{x}_1 + \dots + c_n \underline{x}_n$, $A\underline{x} = c_1(A\underline{x}_1) + \dots + c_n(A\underline{x}_n)$

basis vecs

If you know T of all basis vecs — you know T of all vecs

— columns of standard mat

orthogonality implies independence

LLS

$\hat{x} = (A^T A)^{-1} A^T B$

$$\begin{bmatrix} x\text{-values} & 1 \\ & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y\text{-values} \end{bmatrix}$$

invert, UT

$$R = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$

if $\underline{b} \perp$

proj $\underline{b}$ onto Col(A)

also $\underline{b}$ − proj $\underline{b}$ onto Null(A^T)

$\hat{x} = (A^T A)^{-1} A^T B$

where $A = \begin{bmatrix} x\text{'s} & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$ $B = \begin{bmatrix} y\text{'s} \end{bmatrix}$

$P^T = P$

$P^2 = P$

$P_{1D} = \dfrac{a a^T}{a^T a}$ onto a

onto A

$P_{\text{col}(A)} = A(A^T A)^{-1} A^T$

$p = P\underline{x}$