MA 405 (Day 3)

before: - Matrices

- forms: $\Rightarrow$ echelon form

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ reduced echelon

wow. Apparently
→ You can have non-1's
→ You can have all-0 rows

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{\text{def}}$ A <u>pivot position</u> is the location of a leading 1 in the reduced echelon form of a matrix.

$\boxed{\text{def}}$ A <u>pivot column</u> is any column containing a pivot position

$\boxed{\text{def}}$ A <u>pivot</u> is a (nonzero) # in a pivot position ex

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Pivot positions

pivot columns:
C1, C3
Pivots: 3, 1

equivalent

ex

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 2 & 10 \end{array}\right]$$

pivot columns:
C1, C3
Pivots: 3, 2

just a scale on R_2: Same reduced echelon form as above.
(same p. cols, p. pos.'s, diff pivs.)

equivalent matrices will have same piv. cols and pos's, but maybe diff piv's.

(★) Pivot positions and columns <u>do</u> <u>not</u> <u>change</u> under row operations.

<u>ex</u>

$$3x_1 + 2x_2 + 3x_3 = 4$$
$$x_3 = 5$$

<u>Sol'ns:</u> Solution set is a line; infinitely many solutions:

$$3x_1 + 2x_2 = -11$$

we can see it's the eqn. for a line.

(★) In echelon form, non-pivot columns correspond to free variables in your solution (if system is consistent).
or solt'n

<u>ex</u> In above, $x_1 = -2x_2 - 11$

$x_2 = \text{free}$

$x_3 = 5$

If you have multiple non-pivot columns, you'll have infinite solutions, but each extra free var adds an extra dimension to your solution set.

ex)
$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

- already in <u>reduced</u> echelon form
- pivot columns: C1, C2
  Pivots: 1, 1

Solution?

- C3: non-pivot column
  - x_3 is a free variable

Solution Set
$$\begin{cases} x_1 = -5x_3 \\ x_2 = -2x_3 + 1 \\ x_3 \text{ free} \end{cases}$$

What kind of geometric object is this?

- 3 vars, but only <u>1</u> free var: a <u>line</u> in <u>3D-space</u>.
- Matrix looks like two planes intersecting; of course it would be a line!

<u>ex</u>

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \left\{\begin{array}{l} x_2 \text{ free} \\ x_3 \text{ free} \\ \\ x_1 = -5x_3 \\ x_4 = 0 \\ \hline \text{Sol't'n Set} \\ \hline \end{array}\right.$$

reduced echelon form

Pivot col: C1, C4

(Pivots: 1, 1)

Non-pivot col: C2, C3

Geometrically, this is a <u>plane</u> (two free variables) ex

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

red. ech. form

piv col: C1, C4

np col: C2, C3

Solution set

$x_1 = -5x_3$ $\quad$ x_2 = free

x_3 = free

But! No solution:

R2: $0 = 1$!

→ inconsistent linear system!

- can see from last column is a pivot column
- to avoid inconsistency, pivot columns can't be the last column of augmented matrix.

Thm: If an (row) echelon form of an augmented matrix contains a row of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & | & b \end{bmatrix}$$

$b \neq 0$

then the associated linear system is inconsistent.

① For which #'s h is system consistent?

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 2 & 10 & 4 \end{array}\right]$$

② A linear system w/ 4 variables, 3 equations can have what possible solution sets?

a) nothing b) single point c) line d) 2D-plane
e) 3D-plane f) 4D-plane

① $h \neq 5$ (if $h = 5$, $R_2 = \begin{bmatrix} 0 & 0 & 1 & -4 \end{bmatrix}$

if $R_1 \leftarrow 2R_1$,

$$\left[\begin{array}{cc|c} 2 & 2h & 8 \end{array}\right]$$

then $h \neq 5$. For $R_1 \leftarrow 3R_1$,

$$\left[\begin{array}{cc|c} 3 & h & 12 \end{array}\right]$$

meh. $h \neq 5$!

$$\left[\begin{array}{cc|c} 4 & 4h & 16 \end{array}\right]$$

$-2R_2$

$4 \quad -20$

I'm pretty sure it's $h \in \mathbb{Z}$, $\rightarrow h \in \mathbb{N}$

$h \neq 5$

(2) one free var. Can always be a), 3 4-D planes is sorta like 2 3-D planes: a line (c)).

I'll say a), c)

(1) Let's check (1) again.

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 1 & 5 & 2 \end{array}\right]$$

makes it a bit more obvious.

(2) $$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

Can have minimum 1 piv col, 3 np cols, so 3 free vars

$\longrightarrow$ line, 2D, 3D

Teach

1) Reduce. $R_2 \leftarrow R_2 - 2R_1$

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 10-2h & -4 \end{array}\right]$$

$$10 - 2h \neq 0$$

$$h \neq 5$$

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 1 & \dfrac{-4}{10-2h} \end{array}\right]$$

Can't divide by 0:

$$10 - 2h \neq 0$$

$$h \neq 5$$

Teach

2) a)

✓

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

b) more variables than equations:

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & \underbrace{*}_{\text{non-}} & * \end{array}\right]$$

▷ always at <u>least</u> 1 non-pivot column,
always at least 1 free variable

[redacted]

c) 1 non-pivot column: You can get 1 free var.

✓

d)

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

d) just use two non-pivot columns

e) just use <u>three</u> non-pivot columns:

$$\begin{array}{c} \begin{array}{ccccc} & \downarrow & \downarrow & \downarrow & \end{array} \\ \left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

f) Can't have 4/4 np cols:
Matrix would be all 0's
(trivial linear system)