Matrix Multiplication

Not Reversible: generally, $AB \neq BA$

Not Commutative!

But it is:

- associative: $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
- distributive: $A(B+D) = AB + AD$

Identity Matrix

- square $\mathbb{R}^{n \times n}$
- all 0 or 1
- 1's on diagonal,
- 0's everywhere else

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- $I \cdot A = A$
- $A \cdot I = A$

(row vec) $\cdot$ (col vec) = "inner product"

(col vec) $\cdot$ (row vec) = "outer product"

Elementary Matrices: a matrix representation of our elementary row operations

ex) $R_1 \leftrightarrows R_2$, $A \in \mathbb{R}^{3 \times 2}$

$$\Longrightarrow \quad E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ex

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ -1 & -7 & -13 \\ 3 & 6 & 9 \end{bmatrix}$$

Null space: all solutions to matrix eqn.

$$A \underline{x} = \underline{0}$$

Matrix - Matrix Multiplication

ex]

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \qquad B = [C_1, C_2, \ldots, C_p]$$

n rows   k rows

k cols   p cols

$$(AB)_{ij} = R_i(C_j)$$

$$\boxed{(AB)_{ij} = \sum_{l=1}^{k} a_{il} \cdot b_{lj}}$$

$$\boxed{AB = [A \cdot C_1, A \cdot C_2, \ldots, A \cdot C_p]}$$

ex]

$$A = \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 3 & 0 \\ 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

4×2 ✓ 2×3

$$\underset{4\times 3}{AB} = \left[A \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} + A \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$$

$$= \left[\begin{array}{c|c|c} 4(-1) + 0 & -11 & 5 \\ 0(-1) + (-1)(0) & -3 & -1 \\ 3(-1) + 0(0) & 6 & 3 \\ 2(-1) + 5(0) & 19 & 7 \end{array}\right]$$

Matrix Multiplication (Alt)

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \\ 3 & 5 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

▷ Matrix multiplication forms a linear combination of the columns of A.

$$A\underline{x} = \begin{bmatrix} x - y \\ y \\ 2x \\ 3x + 5y \end{bmatrix}$$

$$A\underline{x} = x \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \\ 5 \end{bmatrix}$$

Representing A Linear System

Original Linear System

Augmented matrix $\longleftrightarrow$ Linear combinations of column vectors

Fact: matrix eqn. $A\underline{x} = \underline{b}$ has solution $\underline{x}$ if and only if $\underline{b}$ is a linear combination of the columns of matrix A.

Theorem given generic $m \times n$ matrix A, the following are equivalent:

1. $A\underline{x} = \underline{b}$ has solution for any vector $\underline{b} \in \mathbb{R}^m$
2. Any vector $\underline{b} \in \mathbb{R}^m$ is a linear combination of columns of A
3. The matrix A has a pivot in every row.

$$\boxed{A \cdot \underline{v}} = \begin{bmatrix} R_1 \cdot \underline{v} \\ R_2 \cdot \underline{v} \\ \vdots \\ R_n \cdot \underline{v} \end{bmatrix}$$

← a column vectors with n rows

$$\boxed{A\,\underline{x} \text{ is } n \times 1 \text{ column vector}}$$

$$\boxed{(A\,\underline{x})_i = (R_i \cdot \underline{x})}$$

← row i of A times vector $\underline{x}$

$$\boxed{(A\,\underline{x})_i = \sum_{j=1}^{K} (R_i)_j \, x_j}$$

jth column in ith row (a_{ij})

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

← matrix indexing

$$\boxed{(A\,\underline{x})_i = \sum_{i=1}^{K} a_{ij} \, x_j}$$

☆ Linear Combinations' representations are not necessarily unique.

$\mathbb{R}^{3\times 2}$: all matrices which are 3×2 and whose entries are all real #'s

Square Matrix: A matrix w/ same # of rows and columns ($\in \mathbb{R}^{n\times n}$)

☆ Like vectors, matrices can also be added and scaled entrywise

↳ can only add matrices of same size

Multiplication of $\underline{u} \cdot \underline{v}$ (dot product)

$\underline{u} \cdot \underline{v} = u_1 v_1 + \dots + u_k v_k$ ← a scalar (just a #)

$$\boxed{\underline{u} \cdot \underline{v} = \sum_{i=1}^{k} u_i v_i}$$ "inner"/"dot" product

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_p \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

n rows — k rows

k cols — 1 col

Order Matters!

→ A vector is basically an <u>ordered</u> list of numbers.

Vector Equation:

- Find a <u>linear</u> <u>combination</u> of vectors to solve the system.

$$3x_1 + 2x_2 + x_3 = -1$$
$$x_1 + 0x_2 + 5x_3 = 6$$

$$x_1\begin{bmatrix}3\\1\end{bmatrix} + x_2\begin{bmatrix}2\\0\end{bmatrix} + x_3\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$

Scaling and Adding Vectors:

- both happen <u>entrywise</u>.
- Parallelogram rule

Singular System: equations are scalar multiples of each other on left side

Intersection of Planes

n vars, p eqns.

$\cong$

Linear Combo of Column Vectors

n vectors w/ p rows

ex)

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & 0 \\ 4 & 6 & -1 & 0 \\ -1 & 3 & -8 & 0 \end{array}\right] \iff x_1\begin{bmatrix}0\\4\\-1\end{bmatrix} + x_2\begin{bmatrix}1\\6\\3\end{bmatrix} + x_3\begin{bmatrix}5\\-1\\-8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Existence And Uniqueness

determining Consistence of a linear system

① A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column — i.e., if and only if an echelon form of the augmented matrix has no row of the form $[0 \ldots 0 \quad b]$, w/ $b \neq 0$

determining # (uniqueness) of solutions

② if a linear system is consistent, then the solution set contains either

(i) a unique solution, when there are no free variables, or

(ii) infinitely many solutions, where there is at least one free variable.

A **Column vector** is a matrix with only one column.

A **row vector** is a matrix with only one row

$\boxed{\mathbb{R}^n}$: Set of all vectors w/ n rows, 1 column

A **pivot position** is the location of a leading 1 in the reduced echelon form of a matrix.

A **pivot column** is any column containing a pivot position.

A **pivot** is a nonzero number in a pivot position.

(A) Pivot positions and pivot columns do not change under row operations.

(A) In echelon form, non-pivot columns correspond to free variables in your solution (if your system is consistent)

why only multiple? Couldn't just one always ensure infinite solutions?

▷ If multiple non-pivot columns, you'll have infinite solutions, but each extra free variable adds an extra dimension to the solution set.

Gaussian Elimination: row reduction

→ obtain an equivalent augmented matrix which is in a special form:

row echelon form, or

row reduced echelon form.

row echelon form

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

- staircase structure

1. all nonzero rows are above any all-zero rows
2. each row's leading entry (leftmost nonzero number) is to the right of the rows above it
3. all entries in a column below a leading entry are zero.

reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

4. each nonzerow row's leading entry is 1
5. each leading 1 is the <u>only</u> nonzero entry in its column

Three Row Operations

- Replacement
- Interchange
- Scaling

A linear system is **consistent** if it has at least one solution.

A linear system is **inconsistent** if it has no solutions.

A **matrix** is a rectangular array of ~~numbers~~.

Each **entry** is a number (or variable).

Augmented Matrix: corresponds to a linear system

- rows = eqns in system
- cols = variable coefficients and right-hand side

Goal: transform matrix and solve linear system.

Row Operations

- reversible
- can produce any # of "equivalent" matrices

Two Fundamental Questions about Linear Systems:

1. Does a solution exist?
2. Is the solution unique?

Linear Cheat Sheet

The Two Types of Problems

$A\underline{x} = \underline{b}$ Solve for $\underline{x}$

1.1 – 1.3

definitions

linear equation: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

w/ variables $(x_1, x_2, \dots, x_n)$

System of linear equations: a collection of two or more linear equations with the same variables

a **Solution** to a system of linear equations is $(s_1, s_2, \dots, s_n)$ a set of numbers which can be plugged into all equations from the system and have each be true.

For any linear system, the **Solution set** of all possible solutions is either

- an infinite set (line, plane, etc.)
- a single point (two lines intersecting)
- or there are no points in the solution set.

Linear Algebra M1

Linear Systems

- Solution set can have 0 solutions, 1 solution, or ∞ solutions (A)
- Consistent: at least 1 solution
- Augmented matrix:

$$\left[\begin{array}{c|c} \text{variable coefficients} & \text{right hand sides} \end{array}\right]$$

Matrices

- Singular: row of all 0's (or missing pivot)
- upper triangular

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

- entrywise operations:

$A + B$,

cA

Matrix Multiplication

- $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times k}$

$AB \in \mathbb{R}^{n \times k}$

- AB =

(i) $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

(ii) $AB = [A \cdot b_1, \ldots, A \cdot b_k]$

with $b_1, \ldots, b_n$ columns of B

(iii) $AB = \begin{bmatrix} a_1 \cdot B, \\ \ldots, \\ a_n \cdot B \end{bmatrix}$

with $a_1, \ldots, a_n$ rows of A

Transpose

For $A \in \mathbb{R}^{n \times m}$,

A^T where

$t_{ij} = a_{ji}$

("flip over diagonal!")

A^T Properies

- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$
- A^T always exists

Symmetric

A is symmetric if $A = A^T$

- $UT \cdot UT = UT$
- $LT \cdot LT = LT$

Symm. Properties

- if $A, B \in \mathbb{R}^{m \times n}$, symm.,

$A \pm B$ symmetric,

cA ,

A^T

- AA^T symmetric
- A^{-1} symmetric

Invertible

A^{-1}

- (★) $A \in \mathbb{R}^{n \times n}$ is invertible iff A has n pivot positions
- invertible = nonsingular
- $A \cdot A^{-1} = A^{-1} \cdot A = I$
- (★) If A is invertible, then $A\underline{x} = \underline{b}$ has a unique solution for any $\underline{b}$

Existence / Uniqueness

- If linear system is consistent,
  - if free variables (non-pivot columns), Solutions are infinite
  - if <u>no</u> free variables, <u>unique</u> solution

Consistent

- A linear system is <u>consistent</u> <u>iff</u> rightmost column of augmented matrix is <u>not</u> a pivot column.

LU Decomposition

- If A can be reduced to echelon form <u>w/out row exchanges</u>, then it can be factored into a product of an upper and lower Δ matrices:

$$A = LU$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

row multipliers (inverted row ops)

L

$$L = E_{net}^{-1}$$

- to find:
  1. find steps to reduce A to echelon form
  2. undo them
  3. $= L$

D

- contains pivots

$$\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}$$

$\downarrow$

divide all rows by diagonal coefficients to create the new U

U

- echelon form A, w/ #'s or 1 diagonals

P

- permutation matrix
- $PA = LU$
  - when u need row exchg's
- P always invertible

LU for $A\underline{x} = \underline{b}$

- if $A\underline{x} = \underline{b}$,

1. $L\underline{y} = \underline{b}$

$$\begin{bmatrix} * & 0 \\ \ell_{21} & * \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

back-sub to find $\underline{y}$

2. $U\underline{x} = \underline{y}$

$$\begin{bmatrix} * & u_{12} \\ 0 & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

back-sub for $\underline{x}$

Find A^{-1}

- <u>Gauss – Jordan</u>

$$[A \mid I]$$

$\downarrow$

$$[I \mid A^{-1}]$$

Elementary Matrices

- Can sorta "add" them, unless one of entries isn't 0. Then, multiply
- all invertible

Symmetric LDU

If $A \in \mathbb{R}^{n \times n}$, symmetric, then

$$A = LDU = LDL^T$$

LDU Decomposition

- unique

Vectors

- **Vector space:** (1) collection of vecs and (2) field of scalars (3) ops (+) and (·)
- Addition:
  - inverse: $\underline{u} + (-\underline{u}) = \underline{0}$
  - identity element $\underline{0}$
- Multiplication (Scalar):
  - identity element 1

vector spaces are closed under their two operations!

- Always includes @ least 1 object: $\underline{0}$ (additive identity)

Subspace

- A subspace of a vector space V is a smaller (nonempty) subset of objects which also satisfies the requirements of a vector space.

$$S \subseteq V$$

- V, $\{\emptyset\}$ always $\subseteq V$

$\text{Span}\{\underline{V}_1, \underline{V}_2, \underline{V}_3, \ldots, \underline{V}_k\}$ = set of all vectors $\underline{u} \in \mathbb{R}^n$ s.t.

$$\underline{u} = c_1 \underline{V}_1 + c_2 \underline{V}_2 + \ldots + c_k \underline{V}_k$$

- all spans are subspaces!

Column Space

- given $A \in \mathbb{R}^{n \times m}$, column space of A is the span of the columns → vecs that are linear combo of columns of A
- $\text{Col}(A) \subseteq \mathbb{R}^n$ (A)

$= \text{span}\{\underline{a}_1, \ldots \underline{a}_m\}$

Span

ex) $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

= all vecs $\underline{u} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \mathbb{R}^2$, set $c_3 = 0$ and any c_1, c_2

(A) If A is invertible, and $A\underline{x} = \underline{0}$, $\underline{x} = \underline{0}$

(A) if $A\underline{x} = \underline{b}$ is consistent, $\underline{b} \in \text{Col}(A)$

Null Space

- Subspace of all solutions to $A\underline{x} = \underline{0} \longrightarrow N(A) \subseteq \mathbb{R}^m$ (A)

(A) if $\underline{b} \in \text{Col}(A)$, $A\underline{x} = \underline{b}$

Problems

"Is It A Subspace?"

① Is $\underline{0}$ in it?

"in it": follows same rules

② Are linear combinations in it?

(i) check element-wise scaling
(ii) check element-wise addition

do these satisfy rules?

If ①, ②, it's a subspace!

Solve $A\underline{x} = \underline{b}$ w/ $A = LU$

① $L\underline{y} = \underline{b}$

▸ Solve for $\underline{y}$

② $U\underline{x} = \underline{y}$

∗ Solve for $\underline{x}$

if necessary, factor $A = LU$ or $A = LDU$

① $L : E^{-1}$

→ reduce A

→ $L = E_1^{-1} E_2^{-1} \dots E_k^{-1}$

② U = A in ech form

Find A^{-1}

① Gauss-Jordan

$$[\, A \mid I \,]$$

reduce to ↓ ⇓ ↓ and this becomes...

$$[\, I \mid A^{-1} \,]$$

(Scary Proof)