Theorems

Invertible Matrix Theorem

The following are all equivalent:

1. $A \in \mathbb{R}^{n \times m}$ is invertible
2. A is row-equivalent to I
3. A has n pivots.
4. $A\underline{x} = \underline{b}$ has a unique solution for any $\underline{b} \in \mathbb{R}^n$
5. $A\underline{x} = \underline{0}$ has only the solution $\underline{x} = \underline{0}$.
6. The columns of A are linearly independent
7. $\mathrm{Col}(A)$ spans all of $\mathbb{R}^n$ $\rightarrow$ columns of A form a basis for $\mathbb{R}^n$
8. A^T is invertible.
9. A has a one-sided inverse
10. $\dim(\mathrm{col}(A)) = n$
11. $\mathrm{rank}(A) = \dim(\mathrm{col}(A)) = n$
12. $\mathrm{Null}(A) = \{\underline{0}\}$
13. $\dim(\mathrm{Null}(A)) = 0$

Miscellaneous

1. row rank A = Column rank A
2. n LI vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ and form a basis for it
3. Bases are never unique
4. \# of columns of $A = \dim(\mathrm{col}(A)) + \dim(\mathrm{Nul}(A))$
5. any vector $\underline{v}_\perp$ perpendicular to vector $\underline{c}$ satisfies $\underline{c}^T \underline{v}_\perp = 0$

Proof Tactics

- if $\mathrm{Nul}(A) = \emptyset$, piv in every column
- n LI vecs ~~span~~ basis $\mathbb{R}^n$
- if A cols are basis $\mathbb{R}^n$, any $\underline{b}$ in $\mathbb{R}^n$ can be made of LC's of A cols i.e. is in $\mathrm{Col}(A)$. since $A\underline{x}$ expands to a LC of A's cols, $A\underline{x} = \underline{b}$ has unique basis-vec-rep sol'n for every $\underline{b}$

Miscellaneous (II)

1. any vector in $\mathbb{R}^n$ can be represented as a unique linear combination of vectors in a basis for $\mathbb{R}^n$
2. If you show something to be true for each of the standard basis vectors of a vector space $\mathbb{R}^n$, you have shown it to be true for every vector $\underline{v} \in \mathbb{R}^n$

Important Subspaces

Dimensions

Column Space $(\underline{Col(A)})$

The span of all Linearly Independent columns of A

$\underline{dim(Col(A))}$ = # Pivot vars
= # pivot columns
= r

Null Space $(\underline{Null(A)})$

Span of all vecs $\underline{x}$ such that $A\underline{x} = \underline{0}$

$\underline{dim(Null(A))}$ = # free vars
= # cols A − # piv cols
= $n - r$

Row Space $(\underline{Col(A^T)})$

Span of all LI rows of A

$\underline{dim(Col(A^T))} = r$

Left Null $(\underline{Null(A^T)})$

$\underline{dim(Nul(A^T))} = m - r$

Short Version:

$dim(Col(A))$ = # Pivots = p

$dim(Col(A^T)) = p$

$dim(Nul(A))$ = # free = $n - p$

$dim(Nul(A^T)) = m - p$

Def

Linear Independence

A set of vectors $\underline{v}_1, \dots, \underline{v}_k$ are Linearly Independent if no vector can be written as a Linear Combination of the others. Otherwise, the set is Linearly Dependent.

Standard Basis Vectors

of space $\mathbb{R}^n$ are

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

and form most intuitive basis of $\mathbb{R}^n$

Basis

A basis for a subspace S of a vector space V is a set of vectors in S which

1. are Linearly Independent
2. span the subspace

And any vector in S can be written as a Linear Combination

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n$$

of the basis vectors, which is unique.

Proving Linear Independence

Vectors are LI if:

1. The REF matrix constructed from them has a pivot in every column
2. The eqn. $c_1\underline{v}_1 + \dots + c_n\underline{v}_n = \underline{0}$ has only solution all $\underline{c}_i = 0$
3. The determinant of the square matrix constructed from them $\neq 0$
4. The matrix constructed from them satisfies a condition of the Invertible Matrix Thm

Proving Basis Vector Representations Are Unique

- $\underline{u} \in S$, basis $= \{\underline{v}_1, \dots, \underline{v}_k\}$
- $\underline{u} = c_1\underline{v}_1 + \dots + c_k\underline{v}_k$
- if $\underline{u}$ also $= d_1\underline{v}_1 + \dots + d_k\underline{v}_k$, then $\underline{u} - \underline{u} = 0 = (c_1 - d_1)\underline{v}_1 \dots$
- and all coeffs $= 0$ bc $\{\underline{v}_1, \dots, \underline{v}_k\}$ are LI
- So $c_1 - d_1 = 0$
  $c_1 = d_1$
- $\therefore \underline{u}$ is unique

Col(A)

$= \text{span}\{\text{LI cols } A\}$

Nul(A)

$=$ solutions to $A\underline{x} = \underline{0}$

Dimension

of a subspace S with basis $\{\underline{v}_1, \dots, \underline{v}_k\}$ of $\underline{k}$ many vectors $= \#$ vectors in basis $= k$

Thm: dims of Col(A), Nul(A)

$A \in \mathbb{R}^{m \times n}$,

$\dim(\text{Col}(A)) + \dim(\text{Nul}(A))$
$=$ total # columns $= n$

Rank

$\text{rank}(A) = \dim(\text{col}(A))$
$= \#$ pivots

Thm: subspace dims

the only subspace of $\mathbb{R}^n$ w/ dimension n is $\mathbb{R}^n$

Bases From K Vectors (Thm)

given $S \subseteq V$,

1. if S is dim K, any set of $\underline{K}$ $\underline{\text{LI}}$ vectors forms a basis for S
2. " " " ", any set of $\underline{K}$ vecs that span S form a basis for it

Linear Dependent

$\alpha\underline{x} + \beta\underline{y} = 0$

Proving a Span Is a Subspace

If you can construct every basis vector from an LC of span vectors, then whole space is in span, and, bc of dimensions, span = space.

② Determine if a set of vectors is Linearly Independent

$\Rightarrow$ if $c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k = \underline{0}$ is only solvable for all $c = 0$, LI

$\Rightarrow$ Combine vectors into a matrix

$\Longrightarrow$ REF matrix

$\Longrightarrow$ if pivot in every column, vecs are LI.

$\Longrightarrow$ Calc determinant (if square)

$\Longrightarrow$ if det. $\neq 0$, vecs are LI.

③ Find Perpendicular Vector $\underline{V}_\perp$ to vec $\underline{u}$

solve

$$\underline{u}^T \underline{V}_\perp = \underline{0}$$

④ is $\underline{u}$ in Col(A) or Nul(A)?

construct $\left[A \;\vdots\; \underline{u} \right]$ and solve. If consistent,

it is:

Problems

① Compute bases and dimensions of each of the four fundamental subspaces of a matrix A.

Col(A)

① reduce to echelon form
② Identify pivots
③ identify pivot columns
④ Same columns in $\underline{A}$ form basis of column space
⑤ $\dim(\text{Col}(A))$ = # pivot variables

Col(A^T)

① Calc A^T
② RREF A^T
③ Pivot columns
④ basis is pivot columns from original $\underline{A}$
⑤ $\dim(\text{Col}(A^T)) = \dim(\text{Col}(A))$

Nul(A)

⓪ to find basis, just characterize the solution set to $A\underline{x} = \underline{0}$
① reduce to echelon form
② identify non-pivot columns (columns without pivots)
~~③ Form new matrix of non-pivot columns~~
~~④ construct one equation per row, = 0~~
~~⑤ Express pivot variables as functions of other variables; all "non-pivot-column" variables = free~~
⑤ go to reduced-row echelon form
⑥ pivot vars as functions of free vars
⑦ non-pivot vars = free
⑧ write solution set in vector form
⑨ the vectors you just constructed are the basis vectors for Nul(A)
⑩ $\dim(\text{Nul}(A))$ = # columns − # pivot columns = # vecs in basis

Nul(A^T)

① variable eqns from RREF A^T
② eliminate as many vars as possible
③ vecs in vector form of solution set form basis / = # rows − # pivs
④ $\dim(\text{Nul}(A)) = m - \#\text{pivs}$ = # vecs in basis

Definitions

Basis: if a set of vectors in $\mathbb{R}^n$

("The smallest span") (1) are Linearly Independent, and

(2) Span the space $\mathbb{R}^n$,

then the vectors form a basis of $\mathbb{R}^n$.

This means that any vector $\underline{v} \in \mathbb{R}^n$ can be constructed from a unique linear combination of vectors in the basis set.

Span: ~~a set of vectors spans $\mathbb{R}^n$~~

The span of a set of vectors is the set of all ~~linear combin~~ vectors that can be made from LC's of the span vectors

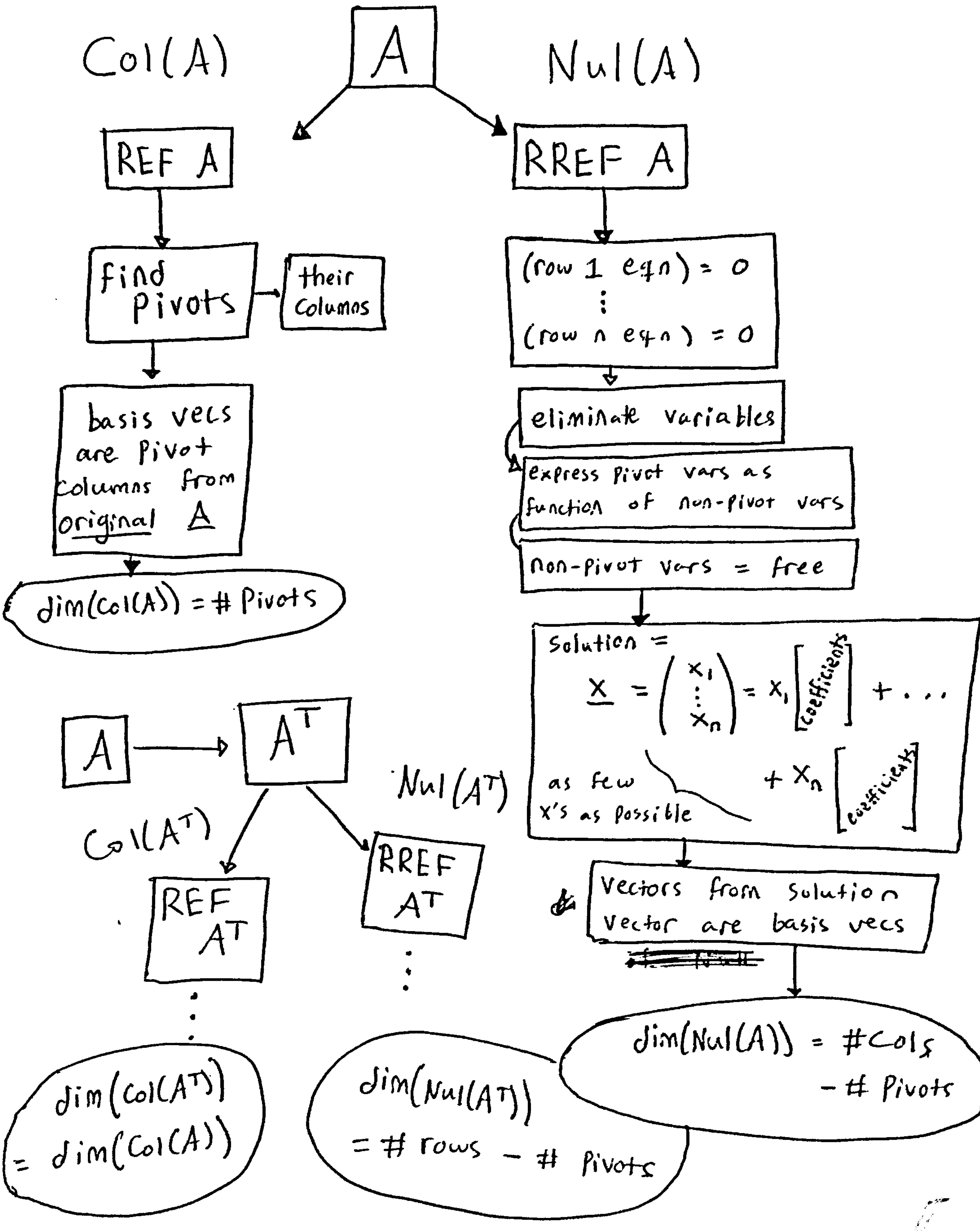
$$\text{span}\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$$

$$= \{\underline{x} : \underline{x} = c_1\underline{v}_1 + \ldots + c_n\underline{v}_n\}$$

- Spans are subspaces by definition

Standard Basis Vectors

For a vector space $\mathbb{R}^n$ there are n standard basis vectors $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ which form a basis of $\mathbb{R}^n$


Col(A)
A
Nul(A)
REF A
RREF A
find pivots
their columns
basis vecs are pivot columns from original A
dim(Col(A)) = # Pivots
(row 1 eqn) = 0
⋮
(row n eqn) = 0
eliminate variables
express pivot vars as function of non-pivot vars
non-pivot vars = free
Solution =
X = (x1 ⋮ xn) = x1 [coefficients] + . . .
as few x's as possible
+ xn [coefficients]
Vectors from solution vector are basis vecs
dim(Nul(A)) = #Cols − # Pivots
A
A^T
Nul(A^T)
Col(A^T)
REF A^T
RREF A^T
dim(Col(A^T)) = dim(Col(A))
dim(Nul(A^T)) = # rows − # Pivots

Geometric Interp of

- Pivots: we'll revisit
- dim(

- 2D subspc is plane
- 1D line
- 0D pt

- dim is most important for telling how many basis vectors you're looking for.

17) if $\underline{x}^T\underline{y}=0$, $y \in \mathbb{R}^n$, then $\underline{x}=\underline{0}$

* $\underline{x}^T\underline{y}=0 \longrightarrow$ two vecs are $\perp$.

Trick: if $\{b_1, \dots, b_n\}$ is a basis for $\mathbb{R}^n$,

and $\underline{x}^T\underline{b}=0$ for all basis vectors $\underline{b}$,

then $\underline{x}^T\underline{y}=0$ for all $y \in \mathbb{R}^n$

(*) To show true for all $\underline{y}$, show true for all basis vectors

$\to$ (bc any $y \in \mathbb{R}^n$ is a combination of the vectors $\underline{b}$ in a basis of $\mathbb{R}^n$.)

Proof of trick

$\hookrightarrow$ $y = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$

$$\underline{x}^T\underline{y} = x^T(c_1\underline{b}_1 + c_2\underline{b}_2 + \dots + c_n\underline{b}_n)$$

$$= x^T(c_1 b_1) + \dots + x^T(c_n b_n)$$

$$= c_1(x^T b_1) + \dots + c_n(x^T b_n)$$

If $x^T b_i = 0$, $= 0 + \dots + 0$

$$= \underline{x}^T\underline{y} = 0$$

17) Now, Prove lemma/precondition

$$\underline{x}^T \underline{b}_k = 0$$

So consider standard basis for $\mathbb{R}^n$

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{e_2}, \dots, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{e_n}$$

Since $\underline{x}^T \underline{y} = 0$ for any $\underline{y}$, this will be true if $\underline{y} = e_1, e_2$, etc.

$$x^T e_1 = 0 = x_1(1) + x_2(0) + \dots + 0 = x_1 \longrightarrow x_1 = 0$$

$$x^T e_2 = 0 \longrightarrow x_2 = 0$$

$$\vdots$$

$$x^T e_n = 0 \longrightarrow x_n = 0$$

for all x_i. Thus,

$$\underline{x} = \underline{0}$$

3d) big red flag: unique

26) false bc will always have $\underline{0}$