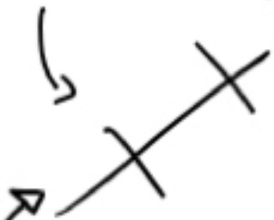
$\boxed{\text{MA 405}}$ **Solution Set**: points of intersection

consistent linear system: @ least one solution.

inconsistent " " if no solutions.



inconsistent system: nowhere where all three planes intersect at once. Parallel means no solutions.

$\boxed{\text{Correction}}$

Recall 3 Basic Operations:

1) Replacement (+)
2) Interchange ($\circlearrowleft$)
3) Scaling (multiplying by constant)

also Operations on **Matrices**!

$\boxed{\text{def}}$ A **matrix** is a rectangular array.
Each **entry** is a number (or variable)

$$\begin{array}{cc} & \text{rows} \\ \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 3 & 1 & 3 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \\ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} & \\ \text{columns} & \end{array}$$

$\boxed{\text{def}}$ **Augmented Matrix**: corresponds to linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \quad \longleftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

Augmented Matrix

- rows = eqns. in system
- cols = variable coefficients and right-hand side

How to do operations on matrix?

Goal: transform matrix and solve the linear system. [My guess: you obv. need to get to identity matrix.]

Step 1: Combine first row in 2nd and 3rd rows to eliminate x_1 coefficients in those rows.

$$A: \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \leftarrow \text{augmented matrix}$$

$$\Downarrow$$

$$B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$$

operations:

$$\underbrace{\begin{array}{l} R_2 \longleftarrow R_2 - 2 \cdot R_1 \\ R_3 \longleftarrow R_2 - 3 \cdot R_1 \end{array}}_{\text{replacement steps}}$$

$$B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$$

Step 2 | Switch Row 2 and Row 3

$$\cancel{C = \left[\begin{array}{ccc|c} 0 & 1 & 2 & \end{array}\right]} \quad C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right]$$

$$R_2 \leftrightarrow R_3$$

Step 3 | Divide Row 2 by -5:

$$D = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right] \quad R_2 \leftarrow R_2 \cdot -\frac{1}{5}$$

Step 4 | Eliminate x_2 from R_3 by adding $R_2 \cdot 7$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right] \quad R_3 \leftarrow R_3 + 7 \cdot R_2$$

This doesn't resemble original
(it's the reduced stage)

▷ combining rows <u>reduces</u> the augmented matrix. From the reduced augmented matrix, it's easy to back-substitute.

Three Row Operations

1) Replacement
2) Interchange
3) Scaling

These transform the linear system / aug matrix but maintain solution set.

★ reversible
★ any number of "equivalent" matrices

Two Fundamental Q's about Linear Systems:

1) Does a solt'n exist? (Is system <u>consistent</u>?)
2) Is the solt'n <u>unique</u>? (Is it the only one?)

* Using matrices (reduced or structured) is more efficient for computations.

Algorithmic Approach: Gaussian Elimination (Row Reduction)

▷ goal: obtain an equivalent augmented matrix which is in a special form:

row echelon form, or

row reduced echelon form

ex) row echelon form

(note staircase structure)

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

ex) row reduced echelon form

(usually requires extra row operations)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

echelon form

A rectangular matrix

1) all nonzero rows are above any all-zero rows
2) each row's leading entry (leftmost nonzero number) is to the right of the row's above it.
3) all entries in a column below a leading entry are zeros.

ex)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

ex)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

not in echelon form (2)

inconsistent system

not necessarily related

ex]

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

consistent and in echelon form

ex

$$\begin{bmatrix} 3 & 8 & 6 \\ 0 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

leading entries — not echelon form

▷ switch R_2 and R_3

$$\begin{bmatrix} 3 & 8 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

now in echelon form.

Reduced Echelon Form

also need:

4) each nonzero row's leading entry is a 1
5) each leading 1 is the only nonzero entry in its column

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

ex

reduced echelon form

- Any (nonzero) matrix can have multiple different echelon forms

<u>but</u>

- only one unique reduced echelon form

ex $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 6 \end{array}\right]$ & $\left[\begin{array}{ccc|c} 2 & 0 & 6 & 8 \\ 0 & 2 & 1 & 6 \end{array}\right]$

both in echelon form,
describe same linear system.