MA405

Eigenvectors and Eigenvectors (5.1)

An <u>eigenvalue problem</u> is a linear algebra problem in which you look for special numbers, the <u>eigenvalues</u> of a square matrix $A \in \mathbb{R}^{n\times n}$, so that the linear system

$$A\underline{x} = \lambda \underline{x}$$

has at least one <u>nonzero</u> solution $\underline{x}$.

For any number λ which allows this

$$A\underline{x} = \lambda \underline{x}$$

to be solvable, λ is an <u>eigenvalue</u> of A and the solutions $\underline{x}$ are called the <u>eigenvectors</u> of A, associated with λ.

* eigenvalues and eigenvectors go together!
  1. find the eigenvalues first
  2. then look for the eigenvectors

ex) Google's PageRank uses eigenvals/vecs

Solving

If $A\underline{x} = \lambda\underline{x}$ has a nonzero solution,

then $A\underline{x} - \lambda\underline{x} = 0$ will <u>also</u> have a nonzero solution

$A\underline{x} - \lambda I\underline{x} = \underline{0}$

and $\boxed{(A-\lambda I)\,\underline{x} = \underline{0}}$ also has a nontrivial solution.

So: finding the eigenvalues λ of the matrix A can be done by finding when this matrix has a <u>non-trivial null space</u>.

How to tell when A has a nontrivial null space?

$A \in \mathbb{R}^{n\times n}$

$(A-\lambda I) \in \mathbb{R}^{n\times n}$

<u>Recall</u>: Invertible Matrix Theorem:

$A-\lambda I \in \mathbb{R}^{n\times n}$ will have $\text{Null}(A-\lambda I) = \{\underline{0}\}$ if $(A-\lambda I)$ is <u>invertible</u>.

So, to find the eigenvalues, find λ such that $A-\lambda I$ is <u>non-invertible</u>.

* i.e., if $\boxed{\det(A-\lambda I) = 0}$ *

Solve the above to find the eigenvalues

The "Characteristic Equation"

ex| find the eigenvalues of $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$

*Fact: $A \in \mathbb{R}^{n \times n}$ has n eigenvalues, but they may be <u>repeated</u>

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{so} \quad A - \lambda I = \begin{bmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}$$

So, $\det(A - \lambda I) = (5-\lambda)(-1-\lambda) + 8$

$= -5 - 5\lambda + \lambda + \lambda^2 + 8$

$= \lambda^2 - 4\lambda + 3$ ← Polynomial of degree n; roots give you eigenvals

we want $\det = 0 \longrightarrow 0 = (\lambda - 3)(\lambda - 1)$

$\boxed{\lambda = 1, 3}$ eigenvalues

After finding the eigenvalues, how to get the eigenvectors?

$$(A - \lambda I)\underline{x} = \underline{0}$$

so $\underline{x} \in \mathrm{Null}(A - \lambda I)$

to get <u>eigenvectors</u>, find a basis for $\mathrm{Null}(A - \lambda I)$.

ex| with $\lambda = 1, 3$, $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$

find all associated eigenvectors

<u>$\lambda = 3$|</u> $\mathrm{Null}(A - 3I)$

$A - 3I = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}$ * notice the structure in this matrix

To find Null space,

① reduce: $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

free variable

② so $\begin{matrix} x_1 = x_2 \\ x_2 = \text{free} = x_2 \end{matrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

③ $\mathrm{Null}(A - 3I) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ ← anything in this span is an eigenvector associated with the particular eigenvalue $\lambda = 3$

$\underline{\lambda = 1}$ $A - I = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix}$

Null$(A - I)$

$$\begin{array}{cc} \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} \\ \quad\quad\uparrow \text{free var} \end{array} \qquad \begin{aligned} 2x_1 &= x_2 \\ x_1 &= \frac{x_2}{2} \\ x_2 &= x_2 \end{aligned} \Rightarrow x_2 \cdot \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$\text{Null}_{(A-I)} = \underline{\text{span}}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$ ← anything in here is an eigenvector associated with $\lambda = 1$

⊛ Sanity check:

$$A\underline{x} = \lambda \underline{x}$$

In some cases, eigenvalue problems are simple!

ex| Diagonal Matrices

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ To Find eigenvalues,

$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$ and $\det(A - \lambda I)$

$$= (1-\lambda)(2-\lambda)(3-\lambda)$$

$$\lambda = 3, 2, 1$$

$\boxed{\text{eigenvalues are the diagonal entries}}$

$\boxed{\text{eigenvectors for diagonal matrices: are all columns of the } I\text{!}}$

$\underline{\lambda = 1}$ $A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\bar{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

free var | $x_2 = 0$ | $2x_3 = 0$, $x_3 = 0$

Same for other eigenvalues

ex| Projection Matrices

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Projecting onto First two coordinates of $\mathbb{R}^4$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}$

$\det(A - \lambda I) = (1-\lambda)^2 (-\lambda)^2$

$\lambda = 1, 0$ $\quad k = 2$

$\boxed{\text{all eigenvalues can only be } 0 \text{ or } 1}$

* repeated eigenvalues and 0 as an eigenvalue are totally okay

Zero as an eigenvalue

$A\underline{x} = 0\underline{x}$ has a non-trivial solution

$A\underline{x} = \underline{0}$ has a non-trivial solution

(by IMT)

new addition to IMT (A) [$A \in \mathbb{R}^{n\times n}$ is square, so A is not invertible if 0 is an eigenvalue of A.

ex| Triangular Matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

$\boxed{\text{all eigenvalues are just the diagonal entries}}$

$$\det(A-\lambda I) = (1-\lambda)(4-\lambda)(6-\lambda) = 0$$

* eigenvectors?