MA 405, 4/7/17

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Chapter 4: Determinants (4.1, 4.2)

Recall: The determinant is a number which is computed from a matrix $A \in \Re^{n\times n}$ which can be used to determine:
1. Is A an invertible matrix?
2. What is the "scaling factor" of a linear transformation?
3. Pivots of A?

ex:

For a 2×2 matrix,
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
and
$$\det(A) = ad - bc$$
1. How does this tell me about invertibility?
$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
Now we can see how the determinant shows when the inverse exists and when it doesn't: you can't divide by $\frac{1}{\det(A)}$ if $\det(\mathrm{A}) = 0$. So, if the determinant is 0, then A^{-1} is not defined, and A is not invertible.
2. How does it tell you about scaling factor?
change of variables:
$$(x,y) \leftarrow (r,\theta) \tag{1}$$
where (x,y) are normal coordinates for $\Re^2$ and (r,θ) are polar coordinates.
To convert, you'd do
$$\int\int g(x,y)dxdy = \int\int g(r\cos\theta, r\sin\theta)(?) \leftarrow (rdrd\theta)$$
To find the remaining factor in a change of variables, construct the Jacobian matrix using all of the partial derivatives:
To go from (x,y) to (r,θ), Jacobian matrix is
$$J = \begin{bmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} \end{bmatrix}$$
The determinant of the Jacobian matrix tells you the "scaling factor" you get when you change variables. Using $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is
$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$
and
$$\det(J) = \cos\theta \cdot r\cos\theta - (-r\sin\theta)\sin\theta \quad = r\cos^2\theta + r\sin^2\theta = r(1)$$
which makes sense, because the polar change is indeed $dxdy \to rdrd\theta$

Invertibility and determinants will be used mostly for eigenvalue problems! (One of the two main kinds of linear algebraic problems:
1. $A\mathbf{x} = \mathbf{b}$, solve for $\mathbf{x}$
2. $A\mathbf{x} = \lambda\mathbf{x}$, solve for $\mathbf{x}$ and λ)

If I know the matrix A and λ, solve for $\mathbf{x}$:
$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} - \lambda\mathbf{x} &= 0 \\ A\mathbf{x} - \lambda I\mathbf{x} &= 0 \\ (A-\lambda I)\mathbf{x} &= 0 \\ (\text{matrix - a number})\mathbf{x} &= 0 \end{aligned}$$
Are there non-trivial solutions to this? I.e. non-zero?
Is $A - \lambda I$ invertible? More importantly, where is it singular?
Answer it with det!

"Defining" determinant in terms of its properties

There are two different ways to think about the determinant.

Facts about determinants

1. $\det(\mathrm{I}) = 1$, for any identity matrix $I \in \Re^{n\times n}$
   (a)
$$\left|\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right| = 1^2 - 0 = 1$$
2. Row exchanges change the sign of the determinant
   (a)
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -(ad - bc)$$
3. Determinans are linear in the first row.
   (a)
$$\begin{aligned} \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} &= tad - tbc \\ &= t(ad-bc) \\ &= t\begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \text{(scalar multiplication)} \end{aligned}$$
   That's one half of proving linearity. Let's prove the other half:
$$\begin{aligned} \begin{vmatrix} a+\alpha & b+\beta \\ c & d \end{vmatrix} &= (a+\alpha)d - (b+\beta)c \\ &= ad + \alpha d - bc - \beta c \\ &= (ad - bc) + (\alpha d - \beta c) \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \alpha & \beta \\ c & d \end{vmatrix} \qquad \text{(addition)} \end{aligned}$$

Warning: this does not mean
$$det(tA) \neq t \cdot det(A)$$
But, since
$$tA = \Big(\text{all rows of A get scaled by t}\Big) \Rightarrow \text{so there are n many rows means}$$
So, the true version is
$$\det(tA) = t^n \cdot \det(A)$$
Warning: this is also not true:
$$\det(A+B) \neq \det(A) + \det(B)$$

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(Using 1 & 2)

Let P be a permutation matrix. (Like I but with rows exchanged.)

Then $\det(P) = \pm 1$ depending on the number of row exchanges.

Other interesting facts (from main 3)

4. If A has 2 identical rows,
$$\det(A) = 0$$
8. (a) If A is a singular matrix, det(A)=0.
   (b) If A is invertible, det(A)$\neq$0
5. Row elimination does <u>not</u> change det!
   (a) ex: $R2 \leftarrow R2 - 3R1$
10. $\det(A) = \det(A^T)$
9. $\det(A^{-1}) = \frac{1}{\det(A)}$
7. If A is triangular, then det(A)= product of <u>all</u> the diagonal entries of A.

We can use these properties to compute det without formulas!

ex:

Find det(B) for
$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{bmatrix}$$

We want to transform B into a triangular form (via row ops) so we can get det from product of diagonals.

First, exchange R1 & R2:
$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 5 & 9 & 1 \end{bmatrix} \qquad -\det(\text{B})$$

$R2 \leftarrow R2 - 2R1$:
$$[whatever] \qquad -\det(\text{B})$$

$R3 \leftarrow R3 - 5R1$
$$[whatever] \qquad -\det(\text{B})$$

$R3 \leftarrow R3 + 4R2$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = u = \qquad -\det(\text{B})$$

So
$$\det(u) = -\det(B)$$

and since
$$\det(u) = 1 \cdot -1 \cdot 1 = -1$$

then
$$-1 = -\det(B) \Rightarrow \det(B) = 1$$