MA 405
eigenproblems, Page Rank

Recall: eigenvals of a square matrix $A \in \mathbb{R}^{n \times n}$ are numbers λ s.t.

$$A\underline{x} = \lambda \underline{x}$$

has a nonzero solution.
Any nonzero solution to this for a particular λ is called an eigenvector.

① Find eigenvalues
Solve characteristic eqn $\det(A - \lambda I) = 0$ for λ

② Find eigenvectors corresponding to λ_1:

$$(A - \lambda I)\underline{x} = 0$$

eigenvectors are all the nullspace vectors of $A - \lambda I$. Just find a basis to get all the eigenvectors!

Special Cases

① A is diagonal
- λ's are all diagonal entries
- $\vec{v}$'s are columns of I

② A is triangular
- λ's are all diagonals
- $\vec{v}$'s are cols of I

③ Projection Matrices
- λ's are only 1's or 0's
- $\vec{v}$'s:

$\lambda = 1$	$\lambda = 0$
$A\underline{x} = \lambda\underline{x} = \underline{x}$ so $\underline{x} \in Col(A)$ (any vec in Col(A) is an eigenvector for $\lambda = 1$)	$A\underline{x} = \lambda\underline{x} = 0$ so $\underline{x} \in Nul(A)$ (any vec in Nul(A) is an eigenvector for $\lambda = 0$)

Sanity check

① Does $A\underline{x} = \lambda\underline{x}$?

② Fact: Sum of all n eigenvals of $A \in \mathbb{R}^{n \times n}$

$$\sum_{i=1}^{n} \lambda_i = tr(A) = \text{trace of } A = \sum_{i=1}^{n} a_{ii}$$

diagonal entries

* not the same as $\lambda_i = a_{ii}$!

③ Fact: product of all n eigenvals of a square matrix

$$\prod_{i=1}^{n} \lambda_i = \det(A)$$

Eigenvalues in the Wild

search engine:
1) find webpages w/ public access
2) index pages to sort/search by keyword
3) rate the importance of search results so that more-important results are listed first

Let x_j denote importance score of jth page

↳ just # of back-links?

$x_1 = 2 \quad x_2 = 1 \quad x_3 = 3 \quad x_4 = 2$
highest importance

↳ add complexity: back-links of more-important pages matter more

↳ back-links of pages w/ a bunch of outgoing links matter less

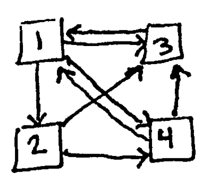
→ add up importance of linked pages:

$x_1 = x_3 + x_4$

Similarly for the rest

"importance score"
1) should be a non-neg real #
2) related to the # of other pages which link to this page (backlinks)

ex)



links go one way and might not be reciprocated ↳ directed graph

graphs are collections of:
pages ⟹ → nodes/vertices
links ⟹ → connecting edges

directed graphs are graphs where edges have some directions

with "only one" eigenvector, if you're lucky!

→ re-weight outgoing links by total # of links from the page:

$x_1 = 1 \cdot x_3 + \frac{1}{2} \cdot x_4$

$x_2 = \frac{1}{3} x_1$

$x_3 = \frac{1}{3} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_4$

$x_4 = \frac{1}{3} x_1 + \frac{1}{2} x_1$

$\lambda = 1$

Solve eqn ~~for eigenvals~~ to get pageranks!

let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & \cancel{0} & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \underline{x} = A\underline{x}$

↳ link matrix

What if ranking is not unique?
what if $\underline{x} = A\underline{x}$ has more than 1 nonzero, indep soln?

* what about "dangling nodes"

what about disconnected components?

The problem is huge!
need hard to solve on a