

## Chapter 4: Determinants (4.1, 4.2)

**Recall:** The determinant is a number which is computed from a matrix  $A \in \mathbb{R}^{n \times n}$  which can be used to determine:

1. Is  $A$  an invertible matrix?
2. What is the “scaling factor” of a linear transformation?
3. Pivots of  $A$ ?

**ex:**

For a  $2 \times 2$  matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$\det(A) = ad - bc$$

1. How does this tell me about invertibility?

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Now we can see how the determinant shows when the inverse exists and when it doesn't: you can't divide by  $\frac{1}{\det(A)}$  if  $\det(A) = 0$ . So, if the determinant is 0, then  $A^{-1}$  is not defined, and  $A$  is not invertible.

2. How does it tell you about scaling factor?

change of variables:

$$(x, y) \leftarrow (r, \theta) \quad (1)$$

where  $(x, y)$  are normal coordinates for  $\mathbb{R}^2$  and  $(r, \theta)$  are polar coordinates.

To convert, you'd do

$$\int \int g(x, y) dx dy = \int \int g(r \cos \theta, r \sin \theta) (?) \leftarrow (r dr d\theta)$$

To find the remaining factor in a change of variables, construct the Jacobian matrix using all of the partial derivatives:

To go from  $(x, y)$  to  $(r, \theta)$ , Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

The determinant of the Jacobian matrix tells you the “scaling factor” you get when you change variables. Using  $x = r \cos \theta$  and  $y = r \sin \theta$ , the Jacobian is

$$J = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

and

$$\det(J) = \cos \theta \cdot r \cos \theta - (-r \sin \theta) \sin \theta = r \cos^2 \theta + r \sin^2 \theta = r(1)$$

which makes sense, because the polar change is indeed  $dx dy \rightarrow r dr d\theta$

Invertibility and determinants will be used mostly for eigenvalue problems! (One of the two main kinds of linear algebraic problems:

1.  $A\mathbf{x} = \mathbf{b}$ , solve for  $\mathbf{x}$
2.  $A\mathbf{x} = \lambda\mathbf{x}$ , solve for  $\mathbf{x}$  and  $\lambda$ )

If I know the matrix  $A$  and  $\lambda$ , solve for  $\mathbf{x}$ :

$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A\mathbf{x} - \lambda\mathbf{x} = 0$$

$$A\mathbf{x} - \lambda I\mathbf{x} = 0$$

$$(A - \lambda I)\mathbf{x} = 0$$

$$(\text{matrix} - \text{a number})\mathbf{x} = 0$$

Are there non-trivial solutions to this? I.e. non-zero?

Is  $A - \lambda I$  invertible? More importantly, where is it singular?

Answer it with  $\det$ !

## “Defining” determinant in terms of its properties

There are two different ways to think about the determinant.

### Facts about determinants

1.  $\det(I) = 1$ , for any identity matrix  $I \in \mathbb{R}^{n \times n}$   
(a)

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1^2 - 0 = 1$$

2. Row exchanges change the sign of the determinant  
(a)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -(ad - bc)$$

3. Determinants are linear in the first row.  
(a)

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = tad - tbc$$

$$= t(ad - bc)$$

$$= t \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad (\text{scalar multiplication})$$

That's one half of proving linearity. Let's prove the other half:

$$\begin{vmatrix} a + \alpha & b + \beta \\ c & d \end{vmatrix} = (a + \alpha)d - (b + \beta)c$$

$$= ad + \alpha d - bc - \beta c$$

$$= (ad - bc) + (\alpha d - \beta c)$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \alpha & \beta \\ c & d \end{vmatrix} \quad (\text{addition})$$

Warning: this does not mean

$$\det(tA) \neq t \cdot \det(A)$$

But, since

$tA = (\text{all rows of } A \text{ get scaled by } t) \Rightarrow$  so there are  $n$  many rows means  
So, the true version is

$$\det(tA) = t^n \cdot \det(A)$$

Warning: this is also not true:

$$\det(A + B) \neq \det(A) + \det(B)$$

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(Using 1 & 2)

Let  $P$  be a permutation matrix. (Like  $I$  but with rows exchanged.)

Then  $\det(P) = \pm 1$  depending on the number of row exchanges.

### Other interesting facts (from main 3)

4. If  $A$  has 2 identical rows,

$$\det(A) = 0$$

8. (a) If  $A$  is a singular matrix,  $\det(A) = 0$ .

(b) If  $A$  is invertible,  $\det(A) \neq 0$

5. Row elimination does not change  $\det$ !

(a) ex:  $R2 \leftarrow R2 - 3R1$

10.  $\det(A) = \det(A^T)$

9.  $\det(A^{-1}) = \frac{1}{\det(A)}$

7. If  $A$  is triangular, then  $\det(A) =$  product of all the diagonal entries of  $A$ .

We can use these properties to compute  $\det$  without formulas!

**ex:**

Find  $\det(B)$  for

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{bmatrix}$$

We want to transform  $B$  into a triangular form (via row ops) so we can get  $\det$  from product of diagonals.

First, exchange  $R1$  &  $R2$ :

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 5 & 9 & 1 \end{bmatrix} \quad -\det(B)$$

$R2 \leftarrow R2 - 2R1$ :

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & 9 & 1 \end{bmatrix} \quad -\det(B)$$

$R3 \leftarrow R3 - 5R1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad -\det(B)$$

$R3 \leftarrow R3 + 4R2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = u \quad -\det(B)$$

So

$$\det(u) = -\det(B)$$

and since

$$\det(u) = 1 \cdot -1 \cdot 1 = -1$$

then

$$-1 = -\det(B) \Rightarrow \det(B) = 1$$