MA 405

Test back

badbadnotgood

teach blames herself

Trick: A is almost triangular. $R_1 \leftrightarrow R_3$, it's upper-Δ.

Permutation matrix, orthonormal, would do that.

$$PA = R$$

undo permutation (w/ itself)

$$A = PR$$

Solutions

① a) Find QR decomp of $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Construct Q by Gram-Schmidt

$$b_1 = v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$b_2 = v_2 - \underset{\text{onto } b_1}{\text{proj of } v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{b_1^T v_2}{b_1^T b_1} b_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$b_3 = v_3 - \underset{\text{onto } b_1}{\text{proj } v_3} - \underset{\text{onto } b_2}{\text{proj } v_3}$$

$$= v_3 - \frac{b_1^T v_3}{b_1^T b_1} b_1 - \frac{b_2^T v_3}{b_2^T b_2} b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} & & \end{bmatrix} \to = \begin{bmatrix} q_1^T v_1 & q_1^T v_2 & q_1^T v_3 \\ 0 & q_2^T v_2 & q_2^T v_3 \\ 0 & 0 & q_3^T v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

② a) two ways to project onto a vector:

i) construct projection matrix

$$P = \frac{a a^T}{a^T a}$$

and multiply $P\underline{b}$

or

ii) find the scaling factor

$$c = \frac{a^T b}{a^T a}$$

and multiply $c\underline{a}$

$$c = \frac{2-3+0+8}{17} = \frac{7}{7}$$

Proj is

$$\frac{7}{17} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

b) Project onto 3D plane:

create a matrix A whose col(A) is the plane

$$P = A(A^TA)^{-1}A^T$$

b) use normal eqns

$$A^TA\,\hat{\underline{x}} = A^T \underline{b}$$

or

$$\hat{\underline{x}} = (A^TA)^{-1}A^T\underline{b}$$

i) $A^T\underline{b} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

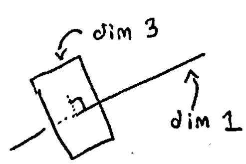
ii) $(A^TA)^{-1} = \frac{1}{36}\begin{bmatrix} 10 & -2 \\ -2 & 4 \end{bmatrix}$

iii) $\hat{x} = \frac{1}{36}\begin{bmatrix} 10 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 0 \end{bmatrix}$

$$\underline{b} = P\underline{b} + (\qquad)$$

$$\underline{b} - P\underline{b} = (\underline{\qquad\qquad})$$

Part of $\underline{b}$ which is on the plane



part a): $P\underline{b}$, $c\underline{a}$

or

use orthogonal complements: ⟹

the line and 3D plane are orthogonal. Since the dimensions add up to $\dim(\mathbb{R}^4)$, they are orthogonal complements.

So, any vector $\underline{b} \in \mathbb{R}^4$ has a part on the line and a part on the plane.

② b) hard way:

$Col(A) \perp Null(A^T)$

Find A s.t.

A^T has null space $= \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

③ given $S \subset V$

a) and $S, V \subseteq \mathbb{R}^n$,

want $V^\perp \subset S^\perp$

Pick any $\underline{x} \in V^\perp$.

Prove $\underline{x} \in S^\perp$

Since $\underline{x} \in V^\perp$, $\underline{x}$ perpendic to any vec in V

$\underline{x}^T \underline{v} = 0$ for any $\underline{v} \in V$

if $\underline{x} \in S^\perp$, then

$\underline{x}^T \underline{s} = 0$ for any $\underline{s} \in S$

since $S \subset V$, then

$\underline{x}^T \underline{s} = 0$ for any $\underline{s} \in S$

So, $\underline{x} \in S^\perp$

So $V^\perp \subset S^\perp$

b) if Q_1 and Q_2 are orthonormal columns,

(A) $Q_1^T Q_1 = I$

$Q_2^T Q_2 = I$

we can show $Q_1 Q_2$ has orthonormal cols if we prove

$$(Q_1 Q_2)^T (Q_1 Q_2) = I$$

$$= Q_2^T \underbrace{Q_1^T Q_1}_{I} Q_2 \longrightarrow \text{since } Q_1 \text{ has orthonormal columns}$$

$$= Q_2^T Q_2 = I \longrightarrow \text{since } Q_2 \text{ has orthonormal columns}$$

c) $P = \dfrac{a a^T}{a^T a} = \dfrac{1}{a^T a} \begin{bmatrix} a_1 \cdot a_1 & a_1 a_2 & \cdots & a_1 a_n \\ \vdots & & \ddots & \vdots \\ a_n a_1 & \cdots & & a_n a_n \end{bmatrix}$

So, $(P)_{ij} = \dfrac{a_i \cdot a_j}{a^T a}$

(A) The trace of a matrix is the sum of the diagonal entries:

$$tr(P) = \sum_{i=1}^{n} P_{ii}$$

$$= \sum_{i=1}^{n} \frac{a_i^2}{a^T a}$$

$$= \frac{1}{a^T a} \left(\sum_{i=1}^{n} a_i^2 \right)$$

c) $Q \in \mathbb{R}^{n \times n}$ square w/ orthonormal cols.

If Q is also upper Δ, show Q is diagonal.

(A) → Q is square and orthonormal — since there are n cols, they form a basis for $\mathbb{R}^n$.

(A) ↳ By IMT, Q is invertible

↳ since Q has orthonormal columns,

$$Q^T Q = I$$

(A) For invertible, orthonormal matrices,

$$Q^{-1} = Q^T$$

since, for invertible matrices, a one-sided inverse is the inverse.

→ Q is also upper-Δ, so Q^T is lower-Δ

(A) → since Q is upper-Δ, Q^{-1} is also upper-Δ ($= Q^T$)

→ $Q^{-1} = Q^T$ is both upper-Δ and lower-Δ

↳ Q^T is diagonal

↳ Q is diagonal

③ a) No, not necessarily I!

Just be diagonal and orthonormal . . .

$$Q = \begin{bmatrix} \boxed{\pm 1} & 0 & 0 \\ 0 & \boxed{\pm 1} & 0 \\ 0 & 0 & \boxed{\pm 1} \end{bmatrix}$$

m still allows orthonormality (still ortho, still length = 1)