MA 405

More Matrix Multiplication

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$$

n rows

k cols

these dimensions must match

$$B = \begin{bmatrix} C_1, C_2, \ldots, C_p \end{bmatrix}$$

k rows

p cols

$$(AB) = [A \cdot C_1, A \cdot C_2, \ldots, A \cdot C_p]$$

In terms of entries,

$$(AB)_{ij} = R_i \cdot C_j$$

in i-th row j-th column

row of A

column of B

inner product

$$= \sum_{l=1}^{k} a_{il} \cdot b_{lj}$$

ex) $A = \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 3 & 0 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

4×2 $\qquad\qquad$ 2×3

$$\underset{4 \times 3}{AB} = \left[A \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}, A \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}, A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$$

3 matrix-vector products

$$= \begin{bmatrix} 4(-1)+1(0) & 11 & 5 \\ 0(-1)+(-1)(0) & -3 & -1 \\ 3(-1)+0(0) & 6 & 3 \\ 2(-1)+5(0) & 19 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 11 & 5 \\ 0 & -3 & -1 \\ -2 & 19 & 7 \end{bmatrix}$$

ex] What about $B \cdot A$?

$$= \left[B \cdot \begin{bmatrix} 4 \\ 0 \\ 3 \\ 2 \end{bmatrix} \quad B \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 5 \end{bmatrix} \right]$$

sizes don't match up:

$$2 \times 3 \cdot 4 \times 2$$
$$3 \neq 4$$

can't do it.

ex] $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

2×2 2×2

Is matrix product same in both orders?

$$AB = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

$$BA = \begin{bmatrix} a\alpha + \beta c & b\alpha + d\beta \\ a\gamma + c\delta & b\gamma + d\delta \end{bmatrix}$$

Even if the sizes match, in general

$$AB \neq BA$$

Matrix multiplication is not commutative!

It is, however

associative: $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

distributive: $A(B+D) = A \cdot B + A \cdot D$

$(A+B)C = AC + BC$

$\neq CA + CB$!

Special (useful) kinds of matrices:

▷ <u>Identity Matrix</u>

- square $\mathbb{R}^{n \times n}$
- all 0 or 1
- 1's on diagonal,
- 0's everywhere else
- e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $I \cdot A = A$
- $A \cdot I = A$

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row vec $\times$ col vec

$$\begin{bmatrix} u_1 & \cdots & u_k \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$$

$$= \sum_{j=1}^{k} u_j v_j$$

col vec × row vec

$$\begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_k \end{bmatrix}$$

1 row
k cols

k rows
1 col

You get a k×k matrix

$$\underline{v} \otimes \underline{u} = \left[u_1 \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}, \ldots, u_k \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} \right]$$

$$= \left[u_1 \underline{v},\ u_2 \underline{v},\ \ldots,\ u_k \underline{v} \right]$$

$$=$$

column row

$(\underline{v} \cdot \underline{u}) = u_j v_j$

col row

col vec × row vec

"outer product"

(compare to row × col)

Elementary Matrices

A matrix representation of our elementary row operations

$E \cdot A = A$ after some row operation

$\boxed{\text{ex}}$ $R_1 \leftrightarrow R_2$, A is 3×3

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

swapped identity rows

ex)

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 8 \\ 1 & 4 & 7 \\ 3 & 6 & 9 \end{bmatrix}$$

- $R_1 \leftarrow$ [redacted] $5R_1$

$$E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R_2 \leftarrow R_2 - 3R_1$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ← coefficients of rows to add

ex)

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(*)

$$= \begin{bmatrix} 1 & 4 & 7 \\ -1 & -7 & -13 \\ 3 & 6 & 9 \end{bmatrix}$$

1) Compute the products

a) $\begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix}$

c) $\begin{bmatrix} 6 & -1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$

d) $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 6 \end{bmatrix}$

② Find product $A\underline{x}$ when

$$A = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

↳ What matrix equation does $\underline{x}$ solve?

↳ Can you find other solutions to this eqn?

3) Write down the elementary matrices corresponding to each row operation (3×3)

a) $R_2 \leftarrow R_2 - 5R_1$

b) $R_3 \leftarrow R_3 + 7R_2$

c) $R_1 \leftrightarrow R_2$ then $R_2 \leftrightarrow R_3$

1) $\begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

a) $\begin{bmatrix} 8-2+5 \\ 2 \\ 6-1+5 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 10 \end{bmatrix}$

b) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

c) $\begin{bmatrix} 6 & -1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix}$

$36+1+16 = 53$

d) $\begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} -3 & 1 & 6 \end{bmatrix}$

$\begin{bmatrix} -15 & 5 & 30 \\ -6 & 2 & 12 \\ -3 & 1 & 6 \end{bmatrix}$

2

2 Matrix eqn solves $A\underline{x} = \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3 a) $E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \leftarrow R_2 - 5R_1$

b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ $R_3 \leftarrow R_3 + 7R_2$

c) $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ then $R_2 \leftrightarrow R_3$

~~also any scalar multiple~~

$c\underline{x}$ of $\underline{x}$

$A(c\underline{x}) = c(A\underline{x})$

since $A\underline{x} = 0$,

$c(A\underline{x}) = 0$

$$\boxed{A\underline{x} = \underline{0}}$$

Set of all solutions to this matrix equation is called the <u>null space</u> of matrix A.