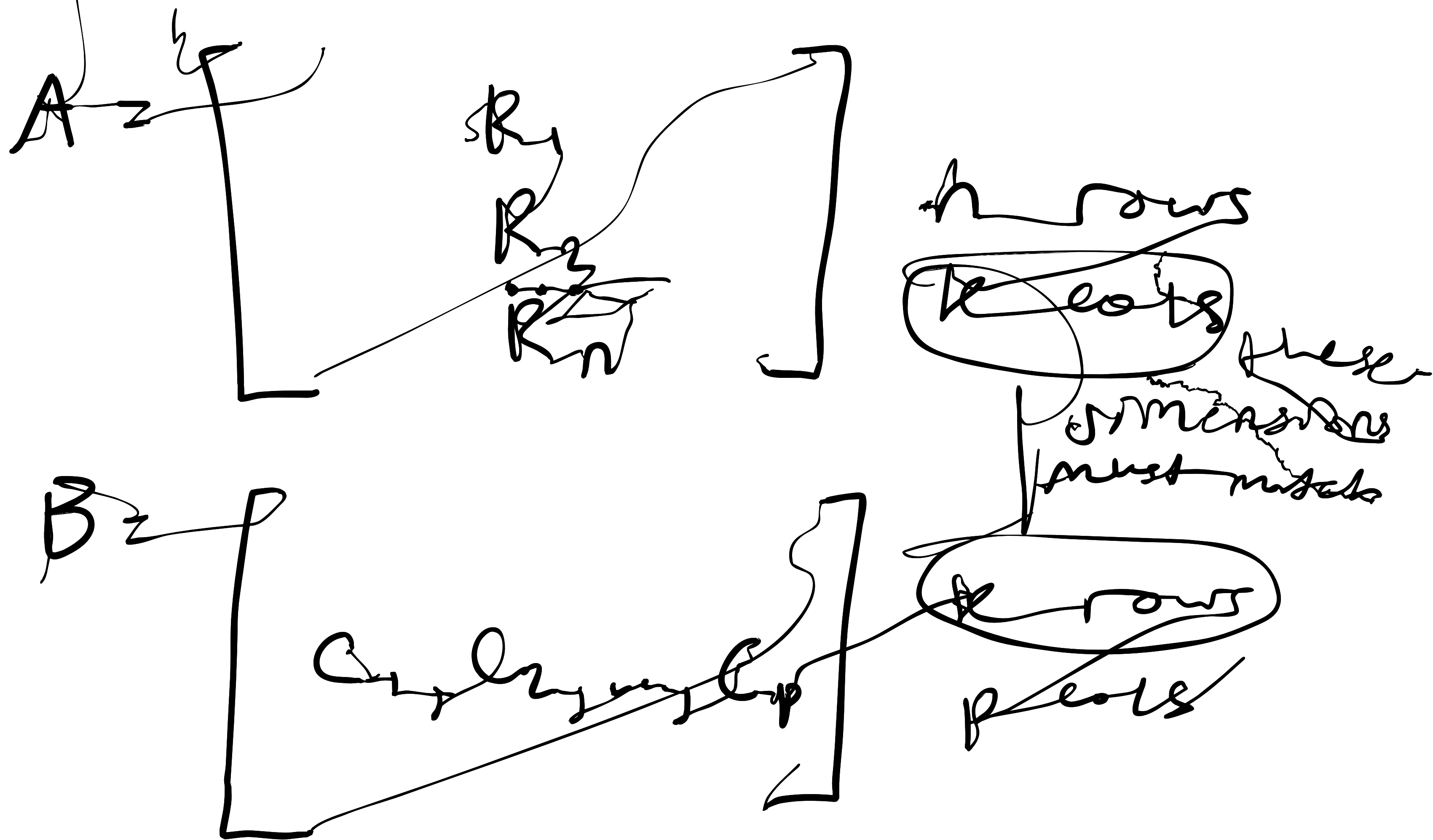


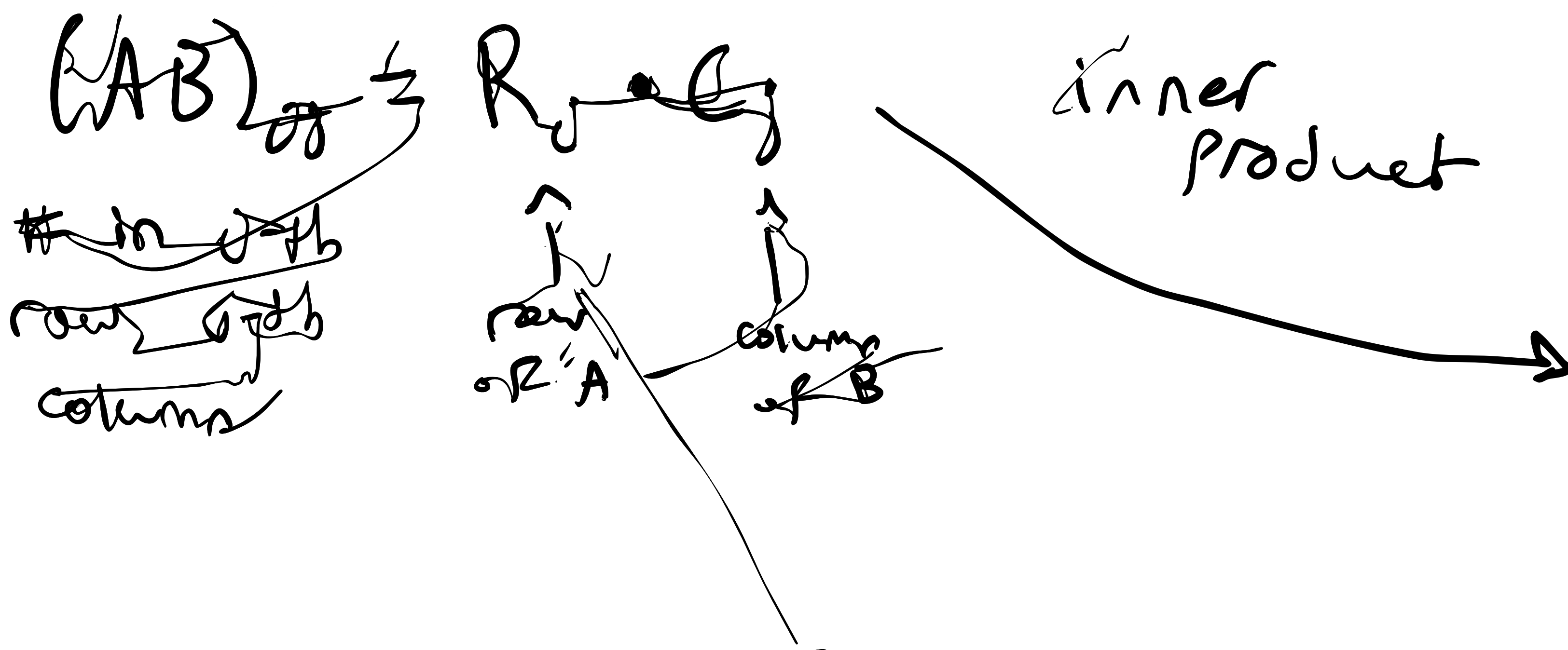
# MA 405

## More Matrix Multiplication



$$(AB) = \begin{bmatrix} A \cdot C_1 & A \cdot C_2 & \dots & A \cdot C_p \end{bmatrix}$$

In terms of entries



$$z = \sum_{i=1}^n a_i \cdot b_i$$

ex]  $A = \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 3 & 0 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

$4 \times 2 \qquad \qquad 2 \times 3$

$AB =$   $\left[ \cancel{A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \cancel{A \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}}, \cancel{A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right]$

$\underbrace{\hspace{10em}}_{\text{matrix-vector products}}$

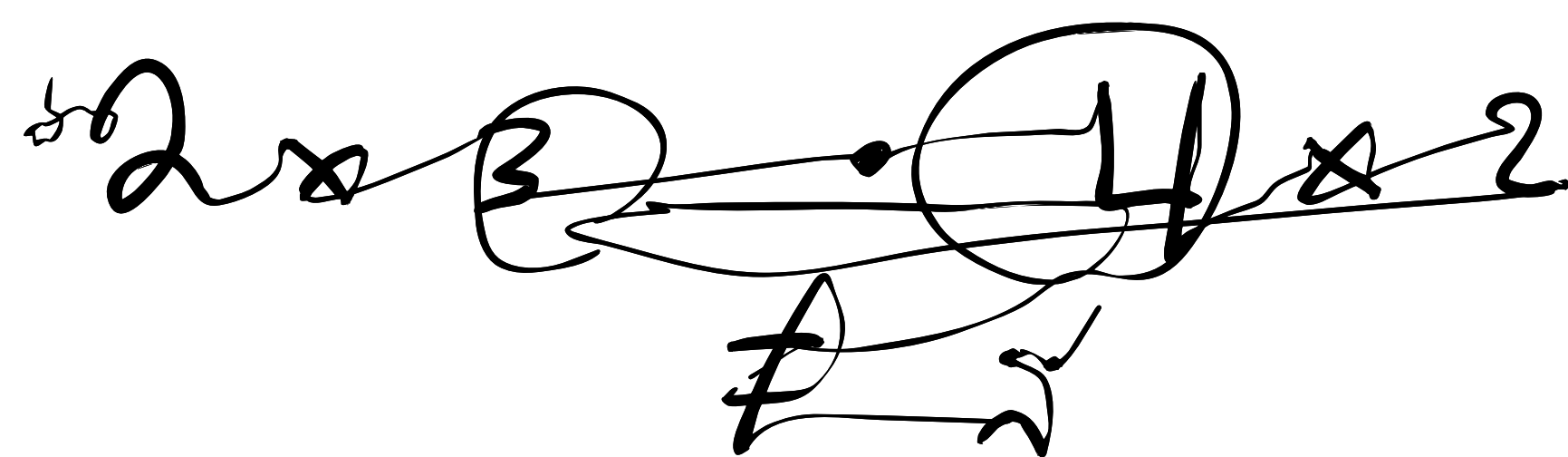
$=$   $\begin{bmatrix} 4(-1) + 1(0) & 4(2) + 1(3) & 4(1) + 1(1) \\ 0(-1) + (-1)(0) & 0(2) + (-1)(3) & 0(1) + (-1)(1) \\ 3(-1) + 0(0) & 3(2) + 0(3) & 3(1) + 0(1) \\ 2(-1) + 5(0) & 2(2) + 5(3) & 2(1) + 5(1) \end{bmatrix}$

$=$   $\begin{bmatrix} -4 & 11 & 5 \\ 0 & -3 & -1 \\ 3 & 10 & 7 \\ -2 & 17 & 7 \end{bmatrix}$

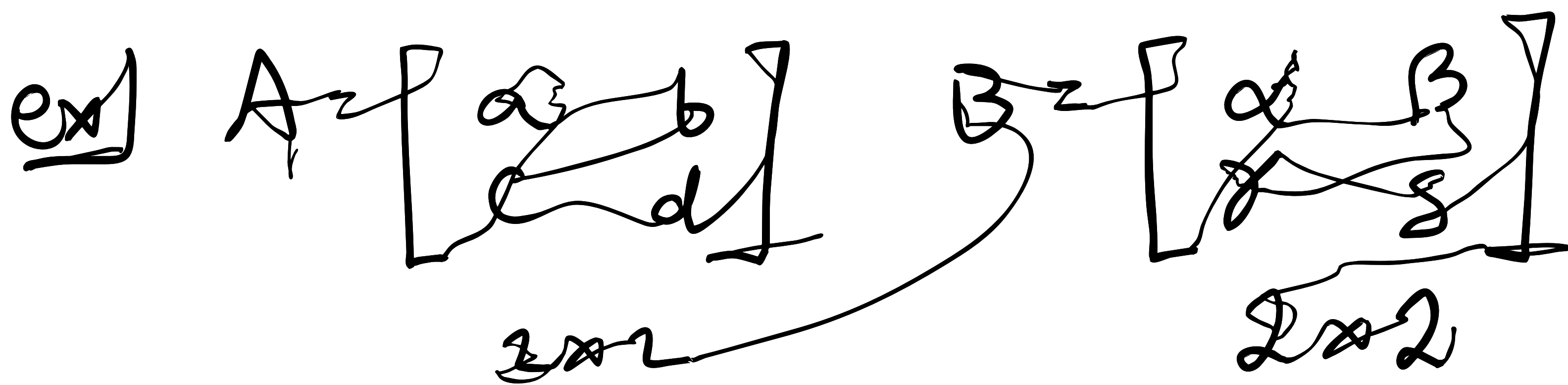
ex] What about  $B \cdot A$ ?



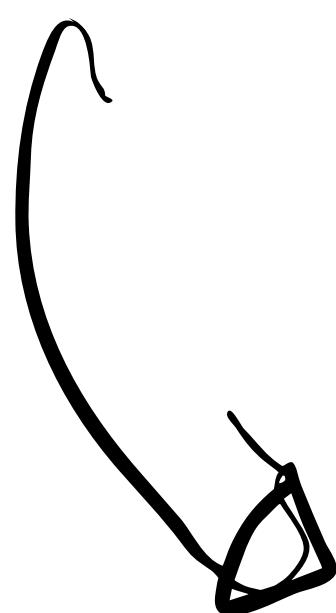
sizes don't match up



can't do it.



the matrix product same in both orders



$$AB = \begin{bmatrix} a\alpha + b\beta & a\beta + b\delta \\ c\alpha + d\beta & c\beta + d\delta \end{bmatrix}$$

$$BA = \begin{bmatrix} a\alpha + b\beta & b\alpha + d\beta \\ c\alpha + d\beta & b\beta + d\delta \end{bmatrix}$$

Ⓐ Even if the sizes match, in general

$$AB \neq BA$$

Matrix Multiplication is  
Not commutative!

It is however

associative:  $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

Distributive:  $A(B + D) = A \cdot B + A \cdot D$

$$(A + B)C = AC + BC, \\ \neq CA + CB!$$

Special (useful) kinds of matrices:

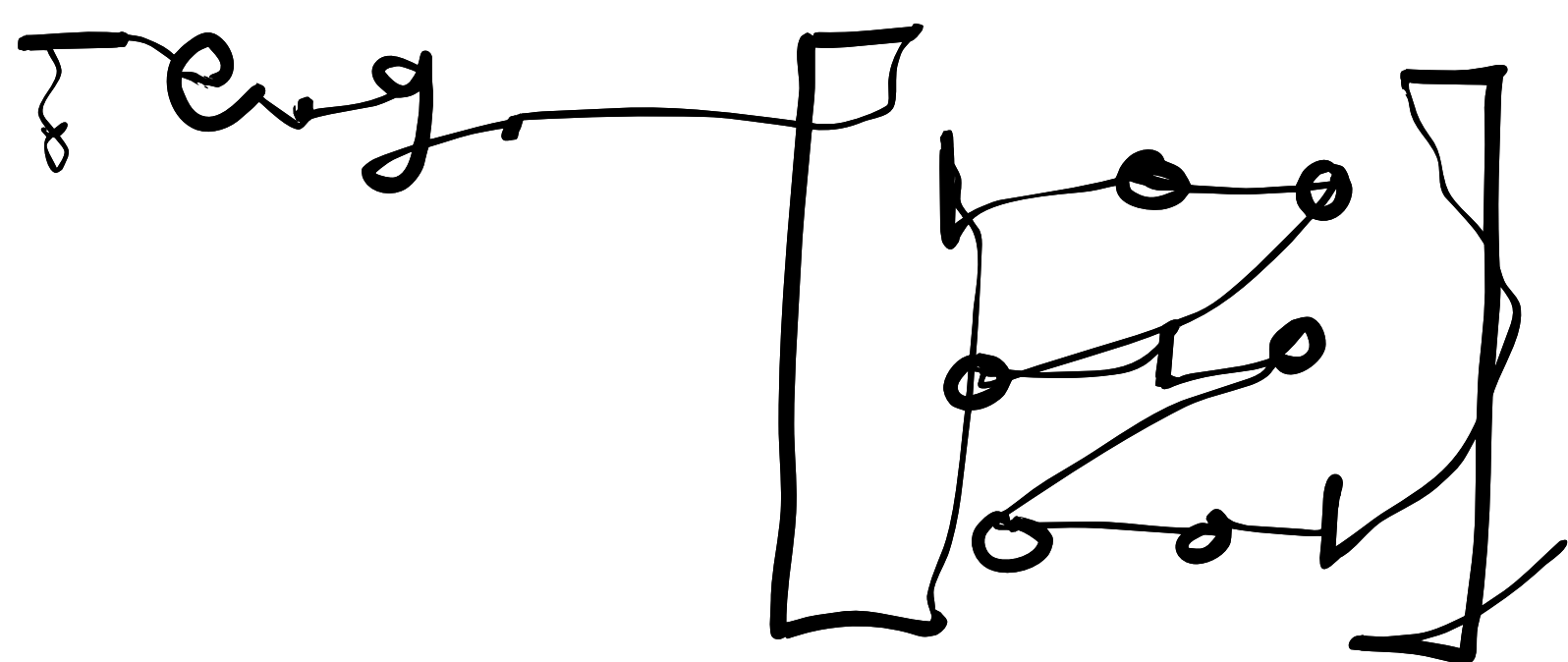
▷ Identity Matrix

↳ square  $\mathbb{R}^{n \times n}$

↳ all 0 or 1

↳ 1s on diagonal

↳ 0s everywhere else



$$\text{— } I \cdot A = A$$

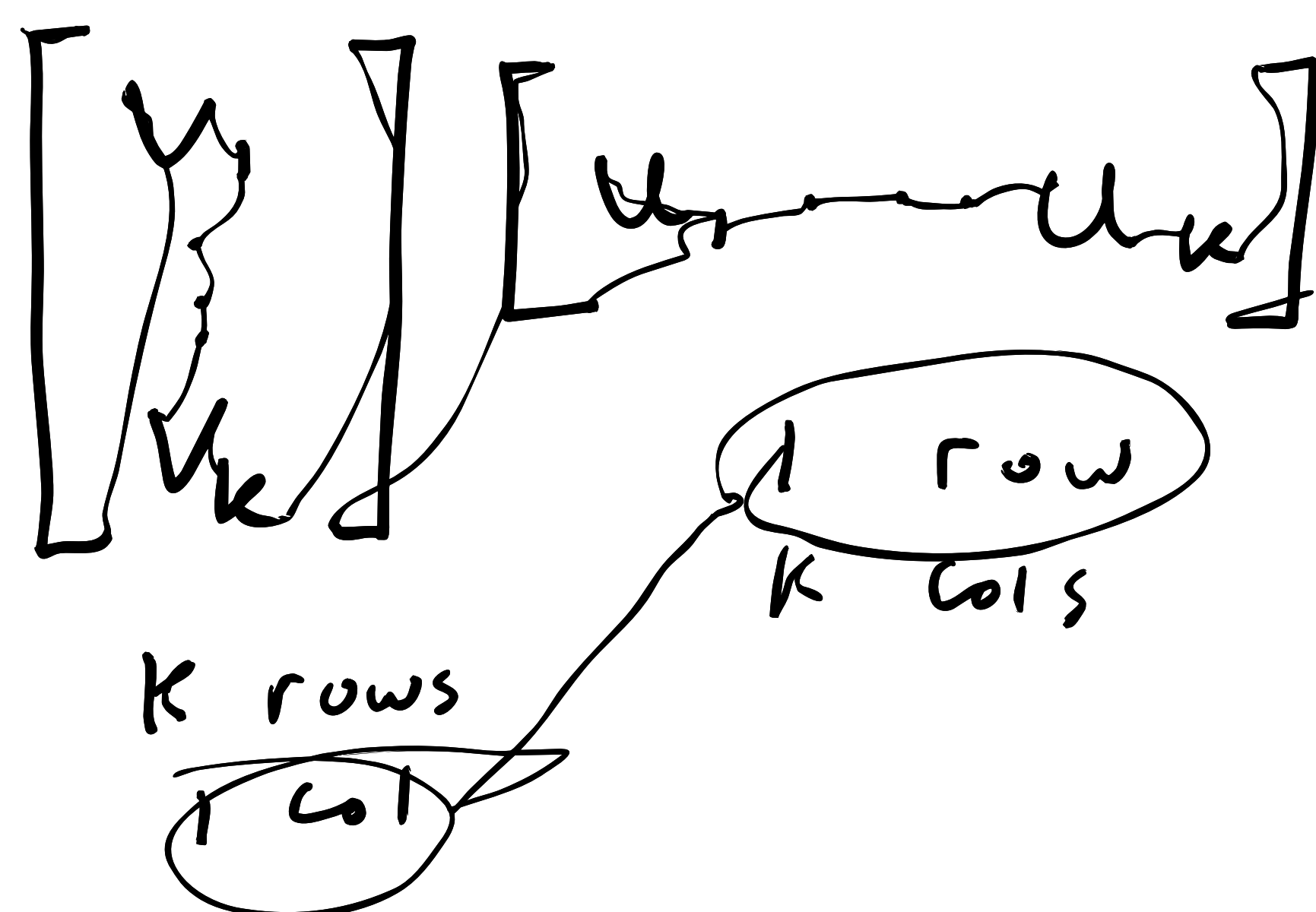
$$\text{— } A \cdot I = A$$

row vec  $\times$  col vec

$$[u_1 \dots u_k] \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$$

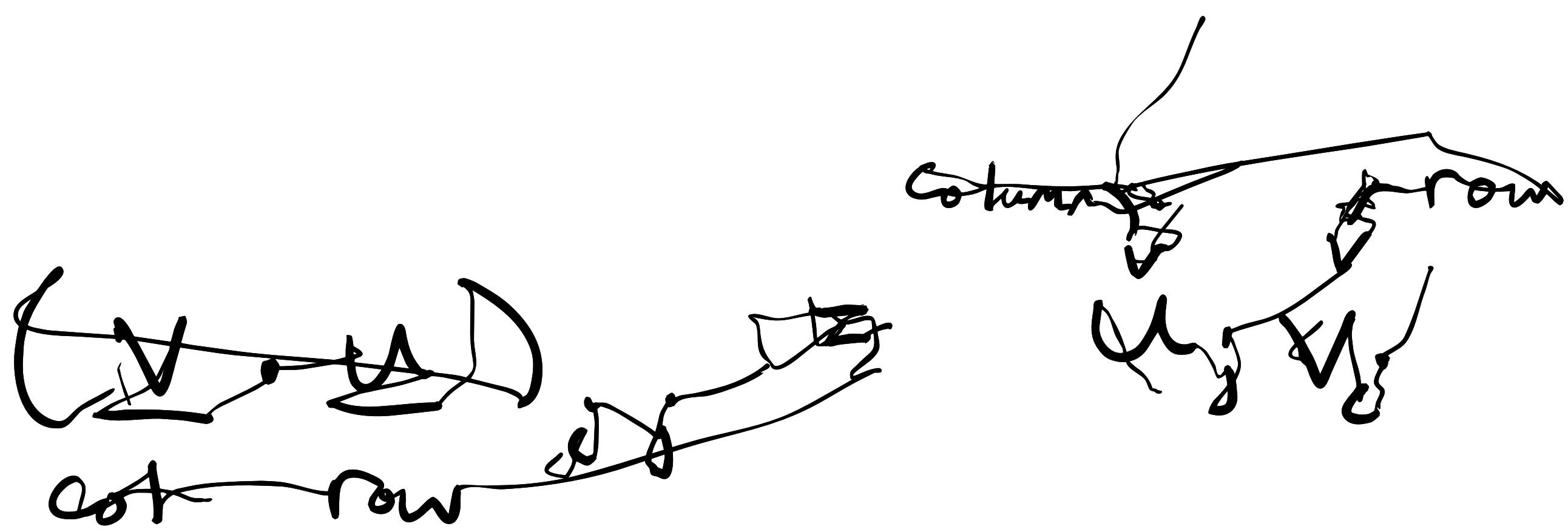
$$= \sum_{j=1}^k u_j v_j$$

col vec ~~x~~ row vec



You get a  $K \times K$  matrix

$$\begin{aligned}
 \underline{V} \underline{U} &= \begin{bmatrix} u_1 \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}, \dots, u_k \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} u_1 \underline{V}, u_2 \underline{V}, \dots, u_k \underline{V} \end{bmatrix}
 \end{aligned}$$



col vec  $\times$  row vec  
Outer Product

(compare to row  $\times$  col)

## Elementary Matrices

A matrix representation of our elementary row operations

$E \cdot A = A$  after some row operation

ex)  $R_1 \leftrightarrow R_2, A$  is  $3 \times 2$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

swapped identity rows

ex)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 4 & 7 \\ 3 & 6 & 9 \end{bmatrix}$$

$$- R_1 \leftarrow \text{[scribbled out]} 5 R_1$$

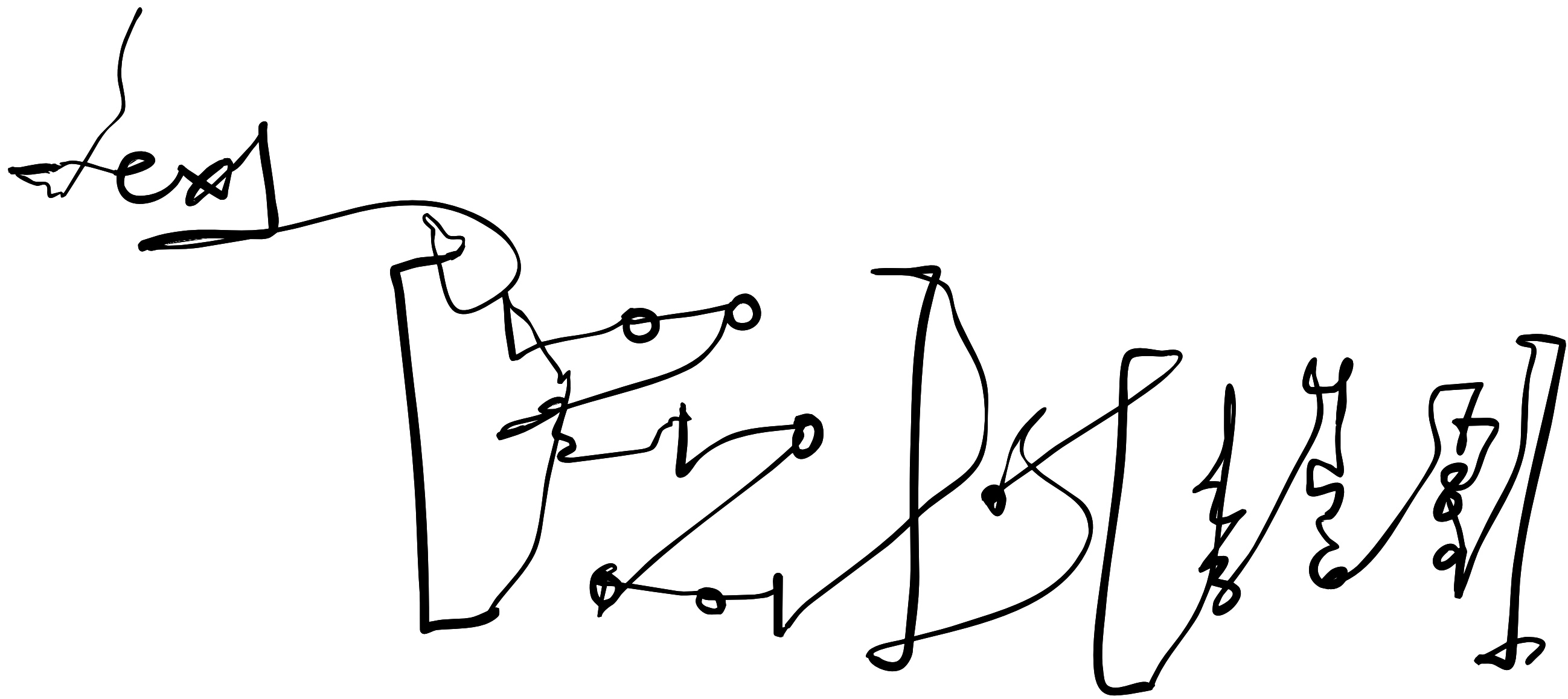
$$E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$- R_2 \leftarrow R_2 + 3 R_1$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

← coefficients of rows to add



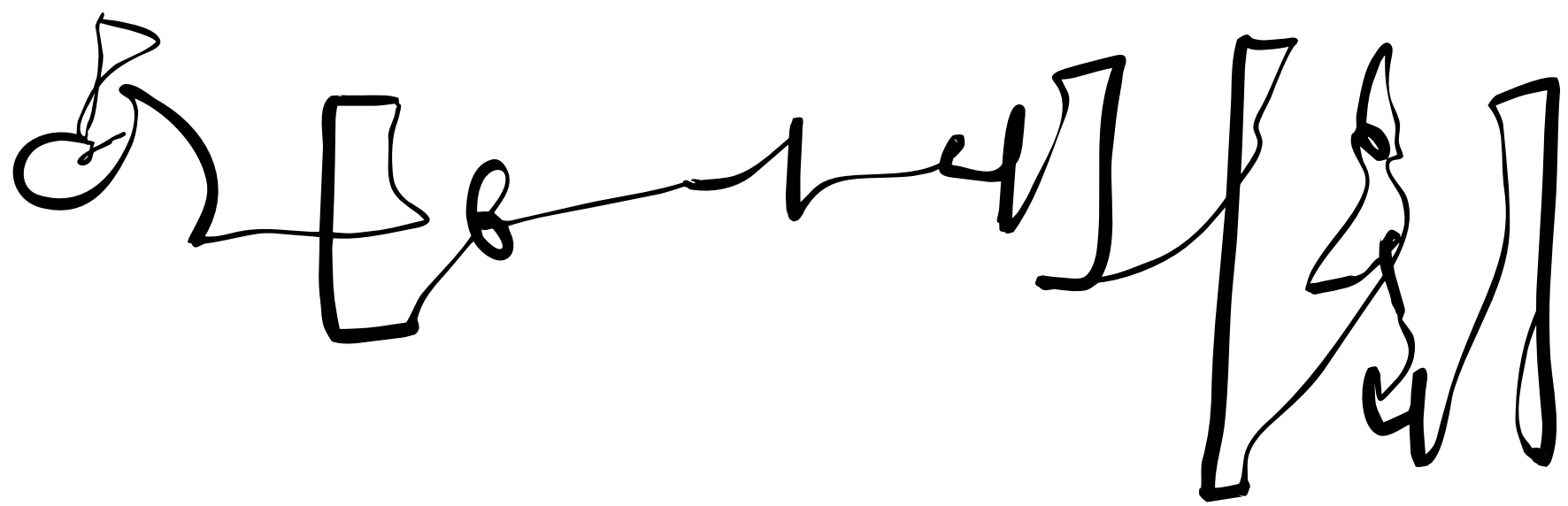
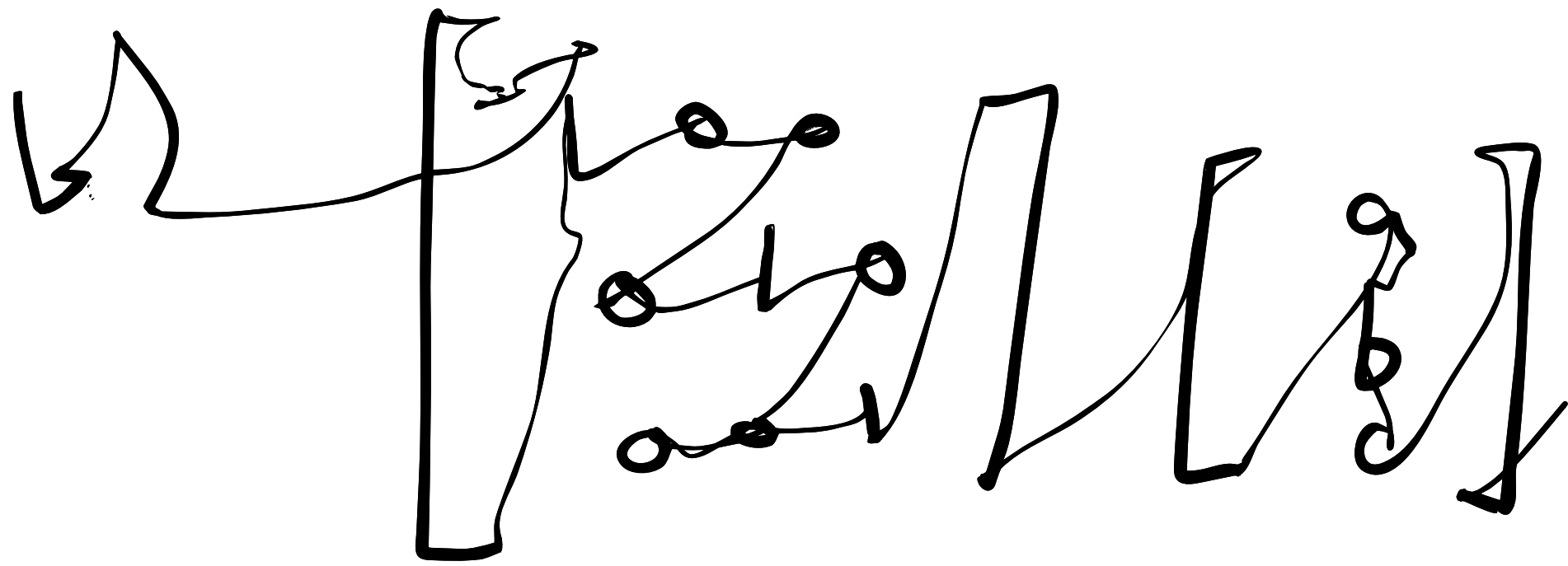


$\odot$

$$\begin{bmatrix} 1 & 4 & 7 \\ -1 & 5 & -13 \\ 3 & 6 & 9 \end{bmatrix}$$

① Compute the products

a)  $\begin{bmatrix} 4 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$



$$d) \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 6 \end{bmatrix}$$

② Find product  $Ax$  when

$$A = \begin{bmatrix} 3 & 6 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

What matrix equation does  $x$  solve?

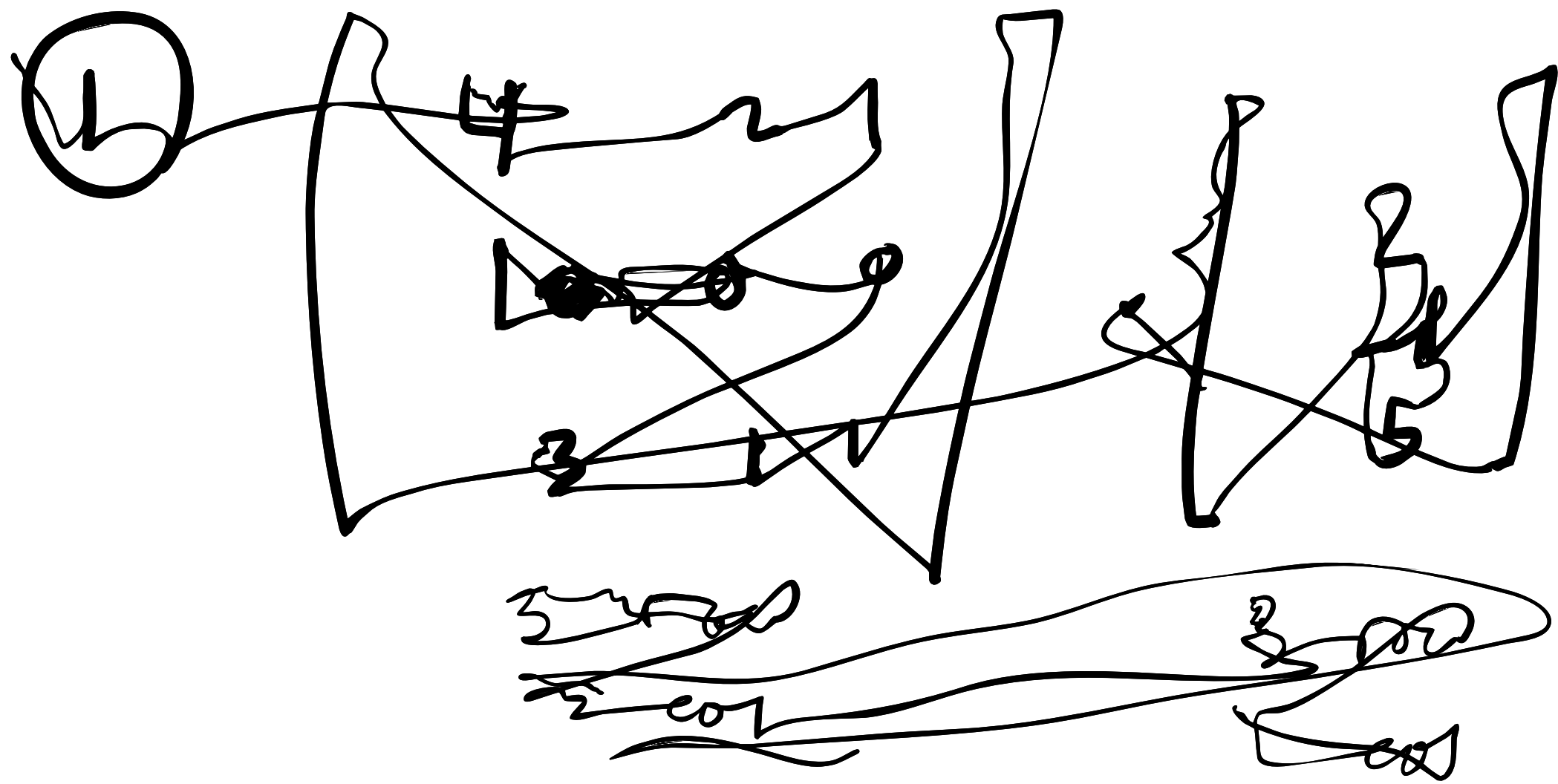
Can you find other solutions to this eqn?

③ write down the elementary matrices corresponding to each row operation (3x3)

a)  $R_2 \leftarrow R_2 + 5R_1$

b)  $R_3 \leftarrow R_3 + 7R_1$

c)  $R_1 \leftrightarrow R_2$  then  $R_2 \leftrightarrow R_3$



a)

$$\begin{bmatrix} 8 & -2 & +5 \\ 2 \\ 6 & -1 & +5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 \\ 1 & 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

c)

$$\begin{bmatrix} 6 & -1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix}$$

$$36 + 1 + 16 = 53$$

d)

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 10 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 5 & 30 \\ 6 & 2 & 20 \\ 3 & 1 & 6 \end{bmatrix}$$

②

② Matrix eqn  
 $\text{solve } Ax = b \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

③

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   $R_2 \leftrightarrow R_3$

b)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   $R_3 \leftarrow R_3 + R_2$

c)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$  then  $R_2 \leftrightarrow R_3$

2) ~~suppose  $x=0$~~

~~also any scalar multiple~~

$Cx$  of  $x$

$$A(Cx) = c(Ax)$$

$$\text{since } Ax = 0,$$

$$c(Ax) = 0$$

$$Ax = 0$$

Set of all solutions to this matrix equation is called the null space of matrix  $A$ .

④