MA 405

Practice

1) $\begin{bmatrix} 1 & 3 & -2 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$

2) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} =$

$3 \times 2 \qquad 2 \times 1$

1) $2 - 3 - 6 + 7 = 0$

2) $6 - 4 =$
$-6 + 2 =$
$18 + 0 =$
$\begin{bmatrix} 2 \\ 2 \\ 18 \end{bmatrix}$

$A\underline{x}$ is $n \times 1$ column vector

$$(A\underline{x})_i = (R_i \cdot \underline{x})$$

row i of
A times vector
$\vec{x}$

jth column
in ith row
(a_{ij})

$$(A\underline{x})_i = \sum_{j=1}^{k} (R_i)_j \, x_j$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

a_{ij} is the
"i j"th entry

i = row #
j = col #

$$(A\underline{x})_i = \sum_{i=1}^{k} a_{ij} x_j$$

ex)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

col index

row index

$$A\underline{x} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \quad \begin{matrix} \sum_{j=1}^{3} a_{1j} x_j \\ \sum_{j=1}^{3} a_{2j} x_j \\ \sum_{j=1}^{3} a_{3j} x_j \end{matrix}$$

different iterators

ex)

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \cdot \underline{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3 × 2 ö 3 × 1

sizes don't match! Can't do.

Another way to represent matrix multiplication:

ex)

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \\ 3 & 5 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad A\underline{x} = \begin{bmatrix} x - y \\ y \\ 2x \\ 3x + 5y \end{bmatrix}$$

or:

$$A\underline{x} = \begin{bmatrix} x-y \\ y \\ 2x \\ 3x+5x \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \\ 0 \\ 5y \end{bmatrix}$$

$$A\underline{x} = x\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 5 \end{bmatrix}$$

Matrix multiplication forms a <u>linear combination</u> of the columns of A.

$$\begin{tikzcd}
 & \text{Original Linear System} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Augmented Matrix} \arrow[rr, leftrightarrow] & & \text{Linear combinations of column vectors}
\end{tikzcd}$$

$\boxed{\text{Fact}}$ The matrix equation $A\underline{x} = \underline{b}$ has solution $\underline{x}$ if and only iff $\underline{b}$ is a <u>linear combination</u> of the columns of Matrix A.

resultant from that fact,

Theorem Let A be a generic $m \times n$ matrix. Then the following statements are all equivalent:

1) The eqn. $A\underline{x} = \underline{b}$ has a solution for any vector $\underline{b} \in \mathbb{R}^m$
2) Any vector $\underline{b} \in \mathbb{R}^m$ is a linear combination of columns of A.
3) The matrix A has a pivot in every row.

▷ Let's prove it!

Know: A is $m \times n$ matrix

Equivalence means if one is true, so are the rest, and vice versa.

Start: assume one statement is true, then use it to show the others are too.

goal: Show 1) and 2) are equivalent.

$\boxed{1) \Rightarrow 2)}$

Assume that 1) is true.
So, pick an arbitrary $\underline{b} \in \mathbb{R}^m$
— Want to show it is a linear
Combination of columns of A:

$$x_1 \cdot C_1 + x_2 C_2 + x_n C_n = \underline{b}$$

are there x's which make this work?

1): $A \underline{x} = \underline{b}$

$$(A\underline{x})_i = (R_i)\,\underline{x}$$

$$= \sum_{j=1}^{n} a_{ij} x_j$$

$$= b_i$$

<u>visualize:</u>

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ & & \cdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\text{product} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= \underline{b}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}X \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \underline{b}$$

a linear combination of vectors

each column vector is a <u>different</u> co<u>lumn</u> of my matrix A !

$$x_1 \begin{bmatrix} \text{first} \\ \text{column} \\ \\ \end{bmatrix} + \dots + x_n \begin{bmatrix} n\text{th} \\ \text{column} \\ \\ \end{bmatrix}$$

So, since $A\underline{x} = \underline{b}$ has a solution, this gives exactly the coefficients for the L.C. we need for Statement 2.

Second half of showing equivalence : $\boxed{2) \Rightarrow 1)}$

→ assume 2) true

→ show that 1) true

Any arbitrary $\underline{b} \in \mathbb{R}^m$, we can write

$$\underline{b} = x_1 c_1 + x_2 c_2 + \ldots + x_n c_n$$

for some #'s x_1, x_2, etc.

So,

$$\underline{b} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + \ldots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n$$

$$\underline{b} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \end{bmatrix}$$

So, $b_i = \sum_{j=1}^{n} a_{ij} x_j$

$= (A\underline{x})_i$

So, that means that the matrix eqn. $A\underline{x} = \underline{b}$ has solution

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

shown 1) = 2)

Now, if you prove 1) => 3) and 3) => 1), (or 2 w/ 3)) they'll all be shown equivalent.

Properties of Multiplication

▷ $\underline{u}\,(\underline{v} + \underline{w}) = \underline{u}\underline{v} + \underline{u}\underline{w}$

row vector   column vectors

▷ $\underline{u}\,(c\,\underline{v}) = c\,(\underline{u}\,\underline{v})$

row   column $= (c\,\underline{u})\,\underline{v}$

for $c =$ a #

▷ $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v}$

▷ $A(c\,\underline{u}) = c(A\,\underline{u})$

$= cA(\underline{u})$

Matrix-Matrix Multiplication

ex) $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$ $B = [C_1, C_2, \ldots, C_p]$

n rows
k cols

k rows
p cols

$$AB = \left[A \cdot C_1 , A \cdot C_2 , \ldots , A \cdot C_p \right]$$

n rows
p columns

$$(AB)_{ij} = R_i \cdot C_j$$

ith row, jth column

ith row A

jth col B

$$= \sum_{l=1}^{k} (R_i)_l (C_j)_l$$

$$\boxed{= \sum_{l=1}^{k} a_{il} \cdot b_{lj}}$$

Matrix Multiplication in Sigma notation