MA 405

Inverse and Transpose

ex) Find $PA = LU$ factorization

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$
$R_1 \leftrightarrow R_3$

$$PA = \cancel{\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

~~$R_3 \leftarrow -\frac{1}{2}R_1$~~

$$P = \cancel{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow -\frac{1}{3}R_2$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix}$$

$$L = E_{tot}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$PA = LU$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & \cancel{1/3}\ 1/3 \end{bmatrix}$$

DIFF

Not NECESSARILY
~~(I.E. WRONG!)~~

Test:

$$\begin{bmatrix} 1 & 0 & 01 \\ 2 & 3 & 4 \end{bmatrix}$$ ~~NOT CORRECT!~~ works

ex) teach:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad R_2 \leftrightarrow R_1 \quad \underset{\sim}{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_1$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Then:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - 3R_2$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Then:

$$\underset{\sim}{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Midterm 1: Next Wed.

- bring blue book by next Monday

Mostly 1.1–1.6

and some 2.1

From HW:

$A^2 = AA, \; A^3 = A \cdot A \cdot A$

$$(A^2)_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

Recall: inverse matrix A^{-1} "undo" A by multiplication

↳ all elementary matrices are invertible

Computing Inverse Matrices

→ special case for 2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant (can't = 0!)

2×2 matrix invertible iff determinant $\neq 0$

→ algorithmic approach

$\boxed{\text{Theorem}}$ $\underbrace{A \in \mathbb{R}^{n \times n} \text{ is invertible iff } A \text{ has } n \text{ pivot positions}}$

essentially, this matrix can be reduced down to identity matrix.

→ to produce inverse, Do This ↑

ex:

$$E_k \cdots \underbrace{E_3 E_2 E_1}_{A^{-1}} (A) = I$$

Two ways to form this product:

1) Construct each E_i and multiply them
2) Gauss-Jordan Method

$$\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

☺ whaaaaaa

Why does this work?

$$(E_k \cdots E_3 E_2 E_1) I = E_k \cdots E_1 = A^{-1}$$

$$(E_k \cdots E_3 E_2 E_1) A = I$$

ex] Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

G-J!

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{array}\right. \quad R_3 \leftarrow R_3 - 4R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \quad R_3 \leftarrow R_3 + 3R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \quad R_3 \leftarrow \frac{R_3}{2}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \quad R_2 \leftarrow R_2 - 2R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \quad R_1 \leftarrow R_1 - 3R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \rightarrow \boxed{A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}}$$

Note: each col of A^{-1} can be found by reducing

$$\left[A \;\middle|\; c_r\right]$$

c_r: rth column of I

row reduction will give you rth col of A^{-1}

Note: unless you need entries of A^{-1} explicitly, for a practical problem

$$\underline{x} = A^{-1}\underline{b}$$

not an efficient way to solve linear systems

often, only care about yes/no: does A^{-1} exist?

→ many ways to answer;

→ n piv pos

$\boxed{\text{def}}$ Transpose of a matrix $A \in \mathbb{R}^{n \times m}$ is $B \in \mathbb{R}^{m \times n}$

where $b_{ij} = a_{ji}$ ("flip across the diagonal") rows <—> cols

denote $B = A^T$

ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ transposes always exist

Properties

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$:) "A^{-T}"

we will prove

MA 405

LU decomp Practice

$A = LU$

$PA = LU$

ex)

1) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

$R_2 \leftarrow R_2 - 4R_1$

$\cancel{R_1 \leftarrow R_1 - 4R_2}$

$= LU$

$E_1 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

2   1

8 - 2 8

$\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$ 2

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$EA = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = U$

2)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$$

reduce: $R_2 \leftarrow R_2 - R_3$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & 4 & 8 \end{matrix} \quad R_2 \leftarrow R_2 - R_1 \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{matrix} \quad R_3 \leftarrow R_3 - R_1 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\boxed{U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}} \quad R_3 \leftarrow R_3 - R_2 \qquad E_3 = E_{tot} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$E^{-1}_{total} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \Big[$$

ex] $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

to reduce to ech form,

row exchanges:

$\Bigg[\qquad\qquad \Bigg]$ $R_1 \leftrightarrow R_3$

$R_2 \leftrightarrow R_3$

$PA = LU$?

Recall

A square matrix is invertible if $B \cdot A = I$

$A \cdot B = I$

(there's a B that's both a left and right inverse)

then $B = A^{-1}$

$\rightarrow (A^{-1})^{-1} = A$ $\bigcirc\!\!\!\star$

if A is invertible, then $A\underline{x} = \underline{b}$ has a unique sol'n for any vector $\underline{b}$.

if nonzero $\underline{x}$ s.t. $A\underline{x} = \underline{0}$

then A can't be invertible

bc if A^{-1} existed,

$$A^{-1}A\underline{x} = A^{-1}\underline{0}$$

$$\underline{x} = \underline{0}$$

no way for inverse to exist.

$\boxed{\text{Fact}}$ if A, B are invertible (and $n \times n$),

then AB is also invertible,

$$(AB)^{-1} = \underbrace{B^{-1}A^{-1}}_{\text{products in reverse order}}$$