MA 405

Recall: Matrix Factorizations

$$A \xrightarrow{\text{row reduction}} U$$

$A \in \mathbb{R}^{m \times n}$, $\underbrace{E_k \cdot \ldots \cdot E_2 \cdot E_1}$

U ← upper triangular

undo elimination

Since $U = E_k \cdots E_1 \cdot A$,

$$E_k^{-1} U = E_{k-1} \cdots E_1 \cdot A$$

$$\ldots$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_k^{-1}}_{\text{row ops undone in reverse order}} U$$

Put $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

L is lower triangular

$$\boxed{A = L \cdot U}$$ "LU Decomposition"

ex $A = \begin{bmatrix} 2 & 3 \\ 6 & 3 \end{bmatrix}$

reduce: $R_2 \leftarrow R_2 - 3 \cdot R_1$

$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ ← only row op, in this case.

$E \cdot A = \begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix}$ ← "U", echelon form

Since $E^{-1} = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L}$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix}$$

↑ the LU-decomposition

Notice:

1) There weren't any row exchanges – these only give lower-triangular L if you do all row exchanges first.

2) U contains pivots of system on its diagonal.

3) L only has 1's on diagonal, and it contains a "history" of reduction steps.

4)

LU-decomp is useful for repeatedly solving linear systems:

$$A\underline{x} = \underline{b},\quad A\underline{x} = \underline{b}_2,\ \ldots,\ A\underline{x} = \underline{b}_k$$

Solving $A\underline{x} = \underline{b}$ $\Rightarrow$ solving 2 linear systems:

$$A = L \cdot U$$

$$A\underline{x} = L\underbrace{(U\underline{x})}_{= \underline{y}} = \underline{b}$$

(1) $L\underline{y} = \underline{b}$ then (2) $U\underline{x} = \underline{y}$

ex) $A = \begin{bmatrix} 2 & 3 \\ 6 & 3 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Solve $A\underline{x} = \underline{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$!

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(1) Let $L\underline{y} = \underline{b}$ then solve

$\longrightarrow$ $y_1 = 1$

$\longrightarrow$ $3y_1 + y_2 = 2$

$y_2 = -1$

read off y's from top to bottom

(exploit special structure)

ex cont

then,

$$\begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad (2)\ U\underline{x} = \underline{y}$$

simple:

$$-6x_2 = -1$$

$$x_2 = \frac{1}{6}$$

$$2x_1 + 3x_2 = 1$$

$$x_1 = 1/4$$

read off x's from bottom to top (exploit special structure)

thus $\underline{x} = \begin{bmatrix} 1/4 \\ 1/6 \end{bmatrix}$

Since L has only 1's on diagonal, but U doesn't, you can further factor U to resemble L:

ex) $U = \begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix}$

$$\left.\begin{array}{l} R_1 \leftarrow R_1 \cdot 1/2 \\ R_2 \leftarrow R_2 \cdot -1/6 \end{array}\right\} \quad E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/6 \end{bmatrix}$$

New $U = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$

$$E_1 \cdot E_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/6 \end{bmatrix}$$

$= D \cdot U$

scaling factors along diagonal

$= \underline{\underline{D}}$

This is over

▷ $A = L \cdot U$

$\downarrow$

$A = L \cdot D \cdot U$, "LDU" Factorization

only 1's on diagonal

Proof (use for HW!)

Theorem

Let A be $\mathbb{R}^{n \times n}$ square matrix, and suppose A is invertible. Then if A has an LDU factorization, this is unique.

Proof HW

(hints: $\overbrace{A = L_1 D_1 U_2 \qquad A}^{\text{assume}} = L_2 D_2 U_2$

$\underbrace{L_1 D_1 U_2 \qquad A = L_2}_{\text{two different LDU's}}$

show $L_1 = L_2$,
$D_1 = D_2$,
$U_1 = U_2$

But first! We need more tools...

To prove this, we need

$\boxed{\text{Facts}}$ about upper and lower triangular matrices

1) If U_1 & U_2 upper Δ, then $U_1 + U_2$ is also upper Δ (or lower / lower)

2) $c \cdot U_1$ is also upper triangular (constant c)

* 3) If U_1 and U_2 are upper Δ, then $U_1 \cdot U_2$ is also upper Δ.

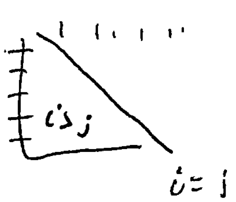
4) If U_1 is invertible and upper Δ, then U_1^{-1} is also upper Δ.

(1), (2) easy to verify.

(3), (4) not as obvious.

<u>Proof</u> of (3):

Let ~~U_1, U_2~~ U, V be $\mathbb{R}^{n \times n}$ square and upper Δ.



$U_{ij} = 0$ as long as $i > j$

also true for V.

Show that it's also true for UV!

$$(U \cdot V)_{ij} = (i\text{th row } U)(j\text{th col } V)$$
$$= \sum_{k=1}^{n} U_{ik} V_{kj}$$

Now, $U_{ik} = 0$ as long as $i > k$

→ so, we can ignore the ~~sums~~ entries involving $k < i$

$$\sum_{k=1}^{n} U_{ik} V_{kj} = \underbrace{\sum_{k=1}^{i-1} U_{ik} + V_{kj}}_{k\text{'s} < i} + \underbrace{\sum_{k=i}^{n} U_{ik} V_{kj}}_{k\text{'s} \geq i}$$

↓

Since U is upper Δ, these are all 0.

So, $$(UV)_{ij} = \sum_{k=i}^{n} U_{ik} V_{kj}$$

Now use fact that V is <u>also</u> upper-Δ.

Since V is upper Δ, $V_{kj} = 0$ when $k > j$

And, to show $(UV)_{ij} = 0$ if $i > 0$, →

if $i > j$, and $k > i$, then $k > j$

(assume condition)

Since $k > j$, and V is upper Δ,

$$V_{kj} = 0$$

and $$(UV)_{ij} = \sum_{k=i}^{n} U_{ik} V_{kj} = 0$$

$$(UV)_{ij} = 0 \text{ if } i > j$$

Thus product matrix $U \cdot V$ is upper Δ

(prove result)

▷ Works similar for lower Δ

$(U_{ij} = 0$ as long as $i < j)$

⟶ on HW

Argument For 4)

Let $U \in \mathbb{R}^{n \times n}$ square, invertible, upper Δ

want to see why U^{-1} is upper Δ!

$$U^{-1} = [c_1, c_2, \ldots, c_n]$$

$$U \cdot U^{-1} = I$$

so, $U \cdot c_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$U \cdot c_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ etc.

U is upper Δ, so

across first row U,

0's down " col U^{-1}

$$U_{11} \cdot c_1 + U_{12} \cdot c_2 + \ldots + U_{1n} \cdot c_n = 1$$

$$U_{22} \cdot c_2 + \ldots + U_{2n} \cdot c_n = 0$$

$$\vdots$$

$$\boxed{U_{nn} \cdot c_n = 0} \rightarrow c_n = 0$$

$$\boxed{U_{n-1\,n-1}\, c_{n-1} + U_{n-1\,n}\, c_n = 0} \rightarrow c_{n-1} = 0$$

thus $U^{-1} = \left[\begin{matrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{matrix}, \ldots, C_n \right]$

Facts, B

Back to our LU factorization:

Since $L = E_1^{-1} \cdot E_2^{-1} \cdots E_k^{-1}$

<u>without</u> row exchanges,

each of these is lower Δ!

<u>ex</u>

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ← row exchange: <u>not</u> lower Δ (and neither is E^{-1})

So row exchanges would mess up L's structure

A → circumvent this by doing row exchanges <u>first</u>!

When row exchanges are necessary in reduction process, you can still get LU or LDU factorization if you do row exchanges first, then use matrix that doesn't need any.

P = Product of all row exchange matrices

then $P \cdot A$ can be factorized

$$P \cdot A = L \cdot U$$

or

$$P \cdot A = L \cdot D \cdot U$$, just like before.