# MA 405



Which of the following singular systems are also consistent?

1) $x_1 + x_2 = 3$
   $x_1 + x_2 = 0$

2) $x_1 + x_2 = 1$
   $2x_1 + 2x_2 = 5$

3) $x_1 + x_2 = 2$
   $3x_1 + 3x_2 = 6$

Me 1, nope, 2, nope, 3, yes

Singular system: eqns. are scalar multiples of each other on left side

Consistent ~~solution~~ system: @ least one solution

---

Anyway,

ex| Vectors

Solving linear systems –

- Intersection of planes

equivalent $n$ vars, $p$ eqns.

- Linear Combination of Column Vectors

$n$ vectors w/ $p$ rows

ex|

$$x_2 + 5x_3 = 0$$

$$4x_1 + 6x_2 - x_3 = 0$$

$$-x_1 + 3x_2 - 8x_3 = 0$$

Augmented Matrix Form:

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & 0 \\ 4 & 6 & -1 & 0 \\ -1 & 3 & -8 & 0 \end{array}\right]$$ ← Do Row Reduction

Vector Equation Form:

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ex/ $u = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

$$u + v = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$\underline{u} - 2\underline{v} = \begin{bmatrix} -1 \\ 4 \\ -8 \end{bmatrix}$$

- vectors $\underline{v}$
- matrices $A$

ex/ ../ is $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ a

linear combination of column vectors of coefficient matrix?

ex

$$\underline{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \quad \text{(vectors)}$$

of $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$ ?

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

aug matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right]$$

$R_2 \leftarrow R_2 + 2R_1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution set:

geometrically: a line $\left\{\begin{array}{l} x_3 \text{ free} \\ x_2 = 3 - 4x_3 \\ x_1 = 2 - 5x_3 \end{array}\right.$

hence $\underline{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination,

$$\underline{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \underset{\triangle}{0}\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$$

is a solution

this is okay!

Key Fact

(☆) Linear Combinations' representations are not necessarily unique

# Matrices

Recall: a matrix is a rectangular array of numbers arranged in rows and columns

ex)

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 0 \end{bmatrix} \quad \begin{matrix} 3 \text{ rows} \\ 2 \text{ cols} \end{matrix} \qquad \text{"}3 \times 2\text{"}$$

Set of all matrices which are $3 \times 2$ and whose entries are all real numbers is $\mathbb{R}^{3 \times 2}$

If a matrix has same number of rows as columns, it's a <u>square matrix</u>

A

like vectors, matrices can be added and scaled:

ex)

$$A = \begin{bmatrix} 3 & 7 & 3 & 2 \\ 4 & 0 & 8 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & 7 & -6 \end{bmatrix}$$

both "$2 \times 4$"

Thus

$$A + B = \begin{bmatrix} 4 & 9 & 2 & 2 \\ 4 & 5 & 8 & 2 \end{bmatrix} \quad \text{"entrywise"}$$

ex] $-2 \cdot A = \begin{bmatrix} -6 & -14 & -6 & -4 \\ -8 & 0 & -2 & -16 \end{bmatrix}$

(★) Adding matrices is only possible if they have the same size.

Multiplying vectors and Matrices: size matters!

ex] Multiplying a row and column vector

$\underline{u} = [u_1 \ u_2 \ \ldots \ u_k]$

1 row, k cols

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_k \end{bmatrix}$ k rows, 1 col

$\text{rows}_u, \text{cols}_u = \text{cols}_v, \text{rows}_v$

$\boxed{\text{def}}$ Multiplication $\underline{u} \cdot \underline{v}$ (dot prod)

$$= u_1 v_1 + \ldots + u_k v_k$$

This is a scalar (just a number)

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{k} u_i v_i$$

★ This is also called the <u>inner</u> <u>product</u> of $\underline{u}$ and $\underline{v}$

ex

$$\begin{bmatrix} 1 & 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x + 5y - 2z$$

to generalize:

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_n \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_k \end{bmatrix}$$

n rows, k cols

k rows, 1 col

Then $A \cdot \underline{v} = \begin{bmatrix} R_1 \cdot \underline{v} \\ R_2 \cdot \underline{v} \\ \vdots \\ R_n \cdot \underline{v} \end{bmatrix}$

a column vector w/ n rows

$A$ is "$n \times k$", $v$ is "$k \times 1$"

$A \underline{v}$ gives "$n \times 1$"

<u>ex</u>

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 5 \\ 0 & -1 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$3 \times 2$ $\qquad$ $2 \times 1$ can be multiplied

$$A \underline{v} = \begin{bmatrix} [-2 \quad 1] \cdot [x \quad y] \\ [3 \quad 5] \cdot [x \quad y] \\ [0 \quad -1] \cdot [x \quad y] \end{bmatrix}$$

$$= \begin{bmatrix} -2x + y \\ 3x + 5y \\ -y \end{bmatrix} = A\underline{v}$$