Compute the Bayes Rule: Example 1

$$\begin{aligned} Y &\sim \text{Bern}(p), \\ X \mid Y = 0 &\sim \text{N}(\mu_0, \sigma^2 \mathbf{I}_p), \\ X \mid Y = 1 &\sim \text{N}(\mu_1, \sigma^2 \mathbf{I}_p). \end{aligned}$$

The joint dist can be factorized as $P(Y, X) = P(Y) \times P(X \mid Y)$. All the calculation on the next slide is to use Bayes' theorem to compute $P(Y \mid X)$.

Note that $P(Y, X)$ is neither a pmf (for discrete r.v.) nor a pdf (for continuous r.v.). It will get a little technical if I tend to rigorously justify my calculation on the next slide. Let's ignore the technicality, and just treat X as discrete with pmf same as its density function. Trust me, the result is correct.

$$P(Y=1 \mid X=x) = \frac{P(Y=1, X=x)}{P(X=x)}$$

$$= \quad \frac{P(Y=1, X=x)}{P(Y=1, X=x) + P(Y=0, X=x)}$$

$$= \quad \frac{P(Y=1)P(X=x|Y=1)}{P(Y=1)P(X=x|Y=1) + P(Y=0)P(X=x|Y=0)}$$

$$= \quad \frac{(p)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\|x-\mu_1\|^2}{2\sigma^2}\right\}}{(p)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\|x-\mu_1\|^2}{2\sigma^2}\right\} + (1-p)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\|x-\mu_0\|^2}{2\sigma^2}\right\}}$$

$$= \quad \left[1 + \exp\left\{\frac{1}{2\sigma^2}(\|x-\mu_1\|^2 - \|x-\mu_0\|^2) - \log\frac{p}{1-p}\right\}\right]^{-1}$$