CSC418 A1 — David Ly — 1001435501 — lydavid1

1. Curves parametric: $x(t) = 2\sin t$, $y(t) = 5\sin t\cos t$, $0 \leq t \leq 2\pi$

$$\boxed{\text{implicit: } \frac{1}{4}x^2 + \frac{4}{25}\frac{y^2}{x^2} - 1 = 0}$$

$$\sin^2 t + \cos^2 t = 1$$

$$x = 2\sin t \qquad y = 5\left(\frac{x}{2}\right)\cos t$$

$$\frac{x}{2} = \sin t \qquad \frac{2}{5}y = x\cos t$$

$$\frac{2}{5}\frac{y}{x} = \cos t$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{2}{5}\frac{y}{x}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{4}{25}\frac{y^2}{x^2} - 1 = 0$$

tangent vector:

$$\vec{r}(t) = 2\sin t\,\vec{i} + 5\sin t\cos t\,\vec{j}$$

$$\boxed{\vec{r}\,'(t) = 2\cos t\,\vec{i} + (5\cos^2 t - 5\sin^2 t)\vec{j}}$$

$$* \ \frac{d}{dt} 5\sin t\cos t = 5\cos^2 t - 5\sin^2 t$$

normal vector:

~~first find unit tangent vector~~

$$\cancel{T(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}} = \cancel{\frac{2\cos t\,\vec{i} + (5\cos^2 t - 5\sin^2 t)\vec{j}}{\sqrt{(2\cos t)^2 + (5\cos^2 t - 5\sin^2 t)^2}}}$$

~~then unit normal vector is~~

$$\cancel{N(t) = \frac{T'(t)}{|T'(t)|}}$$

tangent $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow$ normal $\begin{bmatrix} -y \\ x \end{bmatrix}$

$$\boxed{\vec{n}(t) = -(5\cos^2 t - 5\sin^2 t)\vec{i} + 2\cos t\,\vec{j}}$$

test symmetry:

symmetric about x-axis? $f(x,-y) = f(x,y)$

$$f(x,y) = \frac{1}{4}x^2 + \frac{4}{25}\frac{y^2}{x^2} - 1 = 0$$

$$f(x,-y) = \frac{1}{4}x^2 + \frac{4}{25}\frac{(-y)^2}{x^2} - 1 = 0$$

$$= \frac{1}{4}x^2 + \frac{4}{25}\frac{y^2}{x^2} - 1 = 0$$

symmetric about y-axis? $f(-x,y) = f(x,y)$

$$f(-x,y) = \frac{1}{4}(-x)^2 + \frac{4}{25}\frac{y^2}{(-x)^2} - 1 = 0$$

$$= \frac{1}{4}x^2 + \frac{4}{25}\frac{y^2}{x^2} - 1 = 0$$

$\boxed{\therefore \text{ curve is symmetric about both x- and y-axis.}}$

enclosed area:

$$A = \int_a^b y\,dx$$

first change implicit to explicit

$$\frac{x^2}{4} + \frac{4y^2}{25x^2} = 1$$

$$\frac{25x^4 + 16y^2}{100x^2} = 1$$

$$25x^4 + 16y^2 = 100x^2$$

$$y^2 = \frac{100}{16}x^2 - 25x^4$$

$$y = \pm\sqrt{\left(\frac{25}{16}\right)(4x^2 - x^4)}$$

$$y = \pm\frac{5}{4}\sqrt{4x^2 - x^4}$$

$\frac{5}{4}\sqrt{4x^2-x^4}$ is top half, $-\frac{5}{4}\sqrt{4x^2-x^4}$ is bottom

we will exploit symmetry of x- and y- axis

$$A = 2\int_{-2}^{2} \frac{5}{4}\sqrt{4x^2-x^4}\,dx = 2\cdot 2\cdot \frac{5}{4}\int_0^2 \sqrt{4x^2-x^4}\,dx = 5\int_0^2 \sqrt{4x^2-x^4}\,dx$$

$$= 5\int_0^2 \sqrt{4-x^2}\cdot x\,dx$$

let $u = 4 - x^2$ $\quad -\frac{1}{2}du = x\,dx$

$$A = 5\int_{4-0^2}^{4-(2)^2} -\frac{1}{2}\sqrt{u}\,du = 5\int_4^0 -\frac{1}{2}\sqrt{u}\,du$$

$$= 5\left[-u^{3/2}\cdot\frac{1}{3}\right]_4^0$$

$$= 5\left[0 - \left(-(\sqrt{4})^3\cdot\frac{1}{3}\right)\right]$$

$$= 5\left(\frac{8}{3}\right)$$

$$= \boxed{\frac{40}{3}}$$

we can piecewise linearly approx the perimeter of a bowtie but measuring the lengths of small lines ~~from~~ between two nearby points on the curve and summing them up

```
def linear_approx():
    perimeter = 0
    for i in range(1, 360):  * using smaller increments will improve accuracy
        perimeter += get_length(bowtie_f(i-1), bowtie_f(i))}
    perimeter += get_length(bowtie_f(360), bowtie_f(0))  * between last and first coords
    return perimeter

def bowtie_f(t):          * returns parametric coords given radian t
    t = math.radians(t)
    x = 2 * sin(t)
    y = 5 * sin(t) * cos(t)
    return [x, y]

def get_length(p0, p1):
    return sqrt((p1[0] - p0[0]) ** 2 + (p1[1] - p0[1]) ** 2)  * distance between p0 and p1
```

2. Transformations

we will show that two matrices T1 and T2 commute by merging them into a single matrix in both orders and comparing them

let $a, b, c, d \in \mathbb{R}$ and arbitrary, let $\phi \in [0, 2\pi]$, let $p0, p1 \in \mathbb{R}$, $p0 \neq p1$

a) translation, translation

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & c+a \\ 0 & 1 & d+b \\ 0 & 0 & 1 \end{bmatrix}$$

these two are equal because $a+c = c+a$, $b+d = d+b$ since basic addition commutative

$\Rightarrow \boxed{\text{commutative}}$

b) translation, rotation

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & a\cos\phi - b\sin\phi \\ \sin\phi & \cos\phi & a\sin\phi + b\cos\phi \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{bmatrix}$$

these are not the same

$\Rightarrow \boxed{\text{not commutative}}$

c) scaling, rotation, different fixed pt

scaling relative to a point: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p0.x & -p0.y & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p0.x & p0.y & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ p0.x(1-a) & p0.y(1-b) & 1 \end{bmatrix}$

rotation relative to a pt: $\begin{bmatrix} 1 & 0 & p1.x \\ 0 & 1 & p1.y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -p1.x \\ 0 & 1 & -p1.y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & -p1.x\cos\phi + p1.y\sin\phi + p1.x \\ \sin\phi & \cos\phi & -p1.x\sin\phi - p1.y\cos\phi + p1.y \\ 0 & 0 & 1 \end{bmatrix}$

actually, just show a counterexample:

let $a=2, b=2$, $p0 = [0,0]$, $p1 = [1,1]$, $\phi = \frac{\pi}{2}$ $(\cos\phi, \sin\phi) = (0, 1)$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -0+(-1)+1 \\ -1 & 0 & 1-0+1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 4 \\ -2 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} = M1$$

$$\begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = M2$$

$M1 \neq M2$, counterexample exists

$\Rightarrow \boxed{\text{not commutative}}$

2.d) scaling, scaling, same fixed point

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ po.x(1-a) & po.y(1-b) & 1 \end{bmatrix}\begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ po.x(1-c) & po.y(1-d) & 1 \end{bmatrix} = \begin{bmatrix} ac & 0 & 0 \\ 0 & bd & 0 \\ c\cdot po.x(1-a) + po.x(1-c) & d(po.y(1-b)) + po.y(1-d) & 1 \end{bmatrix}$$

$$\begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ po.x(1-c) & po.y(1-d) & 1 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ po.x(1-a) & po.y(1-b) & 1 \end{bmatrix} = \begin{bmatrix} \cancel{ac}\, ca & 0 & 0 \\ 0 & db & 0 \\ a(po.x(1-c)) + po.x(1-a) & b(po.y(1-d)) + po.y(1-b) & 1 \end{bmatrix}$$

$ac = ca$

$bd = db$

$\cancel{cpo.x} - ac\,po.x + po.x \cancel{- cpo.x} = \cancel{apo.x} - acpo.x + po.x \cancel{- apo.x}$

$\cancel{dpo.y} - bdpo.y + po.y \cancel{- dpo.y} = \cancel{bpo.y} - bdpo.y + po.y \cancel{- bpo.y}$

since multiplication commutative

$\Rightarrow$ $\boxed{\text{commutative}}$

3. Homography

Derive Affine transformation

- we notice that if you reflect~~s~~ about the y-axis, then translate right 7 units, up 2 units, all 4 pts ends up in place

translate by (7, 2) ② reflect about y-axis ①

$$\underbrace{\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{as our transformation}}$$

$$\begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix} \rightarrow (5,7)$$

$\therefore$ (2,5) becomes (5,7)

we can confirm this ourselves: ① flip on y-axis: (−2,5)

② translate (7,2): (5,7)

4. Polygons

Procedure O (q, v0, v1, v2):

* we will use the Barycentric Coordinate method to determine whether q is inside/outside/on the edge of
* a triangle w̄ vertices v0, v1, v2
* we note that $\alpha = A_a/A$, $\beta = A_b/A$, $\gamma = A_c/A$
* where α, β, γ are our Barycentric coords, A is area of triangle, A_i is area of subtriangle i
* that is the triangle formed off q and the two vertices that are not i
* in our case: a = v0, b = v1, c = v2

$$A = \frac{1}{2}\begin{vmatrix} v1.x - v0.x & v2.x - v0.x \\ v1.y - v0.y & v2.y - v0.y \end{vmatrix}$$ * area of triangle given 3 points

$$A_a = \frac{1}{2}\begin{vmatrix} v1.x - \cancel{v0.x}\, q.x & v2.x - q.x \\ v1.y - q.y & v2.y - q.y \end{vmatrix}$$

$A_b = \ldots$

$A_c = \ldots$ * done similarly to A_a

$\alpha = A_a / A$

$\beta = A_b / A$

$\gamma = A_c / A$

* now we can use these Barycentric coords to check the status of q

if $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$:
    then q is inside
else if exactly one of α, β, γ is equal to 0, and other two equal to 1:
    or exactly two equal to 0, and other equal to 1:
        then q is on edge
else:
    q is outside

One can compute the area of a triangle as shown above.
The centroid of a triangle can be computed as:

$x = (a.x + b.x + c.x) / 3$
$y = (a.y + b.y + c.y) / 3$
return [x, y]
for a triangle w̄ vertices a, b, c

since the centroid is a cumulation of the medians of sides of the triangle.