ASTRO C207 Radiative Processes in Astrophysics
Fall 2021 Problem Set 2

1. Blackbody Flux

$$
\begin{aligned}
B_\nu &= \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_BT}}-1} \\
\int_0^\infty\int_0^{2\pi}\int_0^{\pi/2} B_\nu \cos\theta\sin\theta d\theta d\phi d\nu &= 2\pi\int_0^{\pi/2}\cos\theta\sin\theta d\theta\int_0^\infty B_\nu d\nu \\
&= 2\pi\cdot\frac{1}{2}\cdot\int_0^\infty\frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_BT}}-1}d\nu \\
&= \pi\int_0^\infty\frac{2h\frac{c^3}{\lambda^3}}{c^2}\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1}d\nu \\
&= \pi\int_0^\infty\frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1}d\lambda \longleftarrow \begin{cases}\nu=\frac{c}{\lambda}\\ d\nu=-\frac{c}{\lambda^2}d\lambda\end{cases} \\
&= \pi\cdot 2hc^2\cdot\frac{hc}{k_BT}\cdot\left(\frac{k_BT}{hc}\right)^5\int_0^\infty\frac{1}{x\left(e^{1/x}-1\right)}dx \longleftarrow \begin{cases}x=\lambda\frac{k_BT}{hc}\\ dx=d\lambda\frac{k_BT}{hc}\end{cases} \\
&= \pi\cdot\frac{2k_B^4T^4}{h^3c^2}\cdot\frac{\pi^4}{15} \\
&= \frac{2\pi^5k_B^4}{15h^3c^2}T^4 \\
&= \sigma T^4
\end{aligned}
$$

2. Flat Disks

For an annulus in the ring at distance r, power in must equal power out

$$\begin{aligned} P_{\text{in}} &= P_{\text{out}} \\ F_{\text{in}} &= F_{\text{out}} \\ &= \sigma T^4 \end{aligned}$$

where T is the temperature in question. For the flux coming from the star where $\tan\theta_c = \frac{R_*}{r}$,

$$\begin{aligned} F_{\text{in}} &\approx F_* \cdot 4\pi R_*^2 \cdot \frac{1}{4\pi r^2} \cdot C \sin\theta_c \\ &\approx \sigma T_*^4 \frac{R_*^2}{r^2} \frac{R_*}{r} \end{aligned}$$

where C (roughly 1) accounts for error from $r - R_* \approx r$ and the fact that there should be an integral over θ going from 0 to $+\theta_c$.

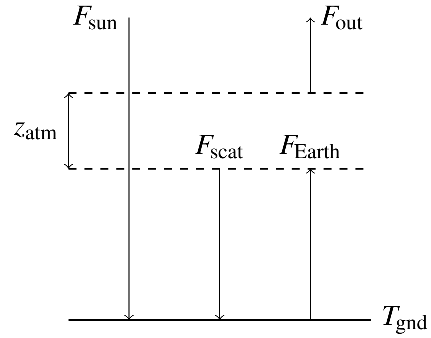
$$\begin{aligned} \sigma T^4 &\approx \sigma T_*^4 \frac{R_*^2}{r^2} \frac{R_*}{r} \\ T &\approx T_* \left(\frac{R_*}{r}\right)^{\frac{3}{4}} \end{aligned}$$

This feels weird, but I guess as you get further away a slice of the ring of the same area occupies a smaller solid angle from the view of some $z \neq 0$ point on the star?

$$\boxed{T \approx T_* \left(\frac{R_*}{r}\right)^{\frac{3}{4}}}$$

3. A Simplified Greenhouse Effect

I suppose we'll assume the atmosphere doesn't emit as a blackbody and Earth doesn't reflect anything coming off of the sun?



$$F_{\text{Earth}} = \sigma T_{\text{gnd}}^4 = F_{\text{scat}} + F_{\text{sun}}$$

$$\alpha = n\sigma_{\text{scat}} = \frac{N}{z_{\text{atm}}}\sigma_{\text{scat}}$$

We consider the amount of energy scattered back by a volume of scatterers depth z.

optically thin

$$z = z_{\text{atm}}$$

Here we treat some constant fraction of energy which doesn't reach depth $s \in (0, z_{\text{atm}}]$ can be as scattered and assume extinction along the back-path isn't significant given the short distance.

optically thick

$$z = \lambda_{\text{mfp}} = \frac{1}{n\sigma_{\text{scat}}} = \frac{z_{\text{atm}}}{N\sigma_{\text{scat}}}$$

This isn't an unreasonable assumption, given backscattering within the atmosphere necessarily deals with extinction with the decaying exponential on the path back out of the atmosphere.

Optically thin:

$$F_{\text{scat}} = F_{\text{Earth}}\left(1 - k_{\text{thin}} e^{-\alpha z_{\text{atm}}}\right), 0 < k_{\text{thin}} < 1$$

$$F_{\text{Earth}} = F_{\text{Earth}}\left(1 - k_{\text{thin}} e^{-\alpha z_{\text{atm}}}\right) + F_{\text{sun}}$$

$$\sigma T_{\text{gnd}}^4 \propto e^{N\sigma_{\text{scat}}}$$

Optically thick:

$$F_{\text{scat}} = F_{\text{Earth}}(1 - k_{\text{thick}}), 0 < k_{\text{thick}} < 1/e$$

$$F_{\text{Earth}}(k_{\text{thick}}) = F_{\text{sun}}$$

$$T_{\text{gnd}}^4 \propto \text{constant}$$

$$\boxed{T_{\text{gnd}} \propto \begin{cases} \sqrt[4]{e^{N\sigma_{\text{scat}}}} & \text{optically thin} \\ \text{constant} & \text{optically thick} \end{cases}}$$