ASTRO C207 Radiative Processes in Astrophysics
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Problem Set 6

1. Flipping Spins at the Epoch of Reionization

When $T \gg T_*$ (i.e. $h\nu_{fs} \ll k_B T$, the Rayleigh-Jeans limit)

$$I_\nu = \frac{2k_B T_b}{\lambda^2} = \frac{2\nu_{fs}^2}{c^2} k_B T_b$$

- T_b brightness temperature
- T_K kinetic temperature
- T_s spin temperature
- T_γ radiation temperature

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-\frac{h\nu_{fs}}{k_B T_s}}$$

- T_b may not be related to T_K
- LTE $\Rightarrow T_s = T_K$

$$J_\nu(\nu_{fs}) = B_\nu(T_\gamma, \nu_{fs})$$

- CMB: $T_\gamma = 2.7(1+z)$

(1)

For the blackbody CMB, $T_{b,\text{CMB}} = T_{\gamma,\text{CMB}} (\equiv T_\gamma)$. For the fluctuation in the 21cm line's brightness temperature

$$\begin{aligned} I_\nu(s) &= I_{\nu 0} e^{-\tau} + S_\nu(1 - e^{-\tau}) \\ &= B_\nu(T_\gamma, \nu_{fs}) e^{-\tau} + B_\nu(T_s, \nu_{fs})(1 - e^{-\tau}) \end{aligned}$$

Assuming $h\nu_{fs} \ll k_B T$ for all temperatures of interest and placing the intensity relative to the background CMB

$$\begin{aligned} I_\nu(s) - B_\nu(T_\gamma, \nu_{fs}) &= [B_\nu(T_s, \nu_{fs}) - B_\nu(T_\gamma, \nu_{fs})](1 - e^{-\tau}) \\ T_b - T_\gamma &= (T_s - T_\gamma)(1 - e^{-\tau}) \end{aligned}$$

(2)

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-\frac{h\nu_{fs}}{k_B T_s}} \approx \frac{g_1}{g_0}\left(1 - \frac{T_*}{T_s}\right)$$
$$\frac{A_{10}}{B_{10}} = \frac{2h\nu_{fs}^3}{c^2}$$
$$B_{01} = B_{10}\frac{g_1}{g_0}$$

A quick definition of the temperature T_K in a similar vein as the spin temperature–the temperature at which the LTE quantity of exciting and de-exciting collisions per time matches:

$$n_0 C_{01} = n_1 C_{10}$$
$$\frac{n_1}{n_0} = \frac{C_{01}}{C_{10}} = \frac{g_1}{g_0} e^{-T_*/T_K} \approx \frac{g_1}{g_0}\left(1 - \frac{T_*}{T_K}\right)$$

Now back to the more general statistical equilibrium:

$$n_0(B_{01}\bar{J} + C_{01}) = n_1(A_{10} + B_{10}\bar{J} + C_{10})$$
$$\frac{n_1}{n_0} = \frac{B_{01}\bar{J} + C_{01}}{A_{10} + B_{10}\bar{J} + C_{10}}$$

Getting back to the definition of spin temperature:

$$\frac{n_1}{n_0} \approx \frac{g_1}{g_0}\left(1 - \frac{T_*}{T_s}\right)$$
$$\frac{g_1}{g_0}\left(1 - \frac{T_*}{T_s}\right) \approx \frac{B_{01}\bar{J} + C_{01}}{A_{10} + B_{10}\bar{J} + C_{10}}$$
$$\frac{T_*}{T_s} = 1 - \frac{g_0}{g_1}\frac{B_{01}\bar{J} + C_{01}}{A_{10} + C_{10} + B_{10}\bar{J}}$$
$$\frac{1}{T_s} = \frac{1}{T_*}\frac{g_1(A_{10} + C_{10} + B_{10}\bar{J}) - g_0(B_{01}\bar{J} + C_{01})}{g_1(A_{10} + C_{10} + B_{10}\bar{J})}$$

Looking at the denominator:

$$\begin{aligned}
A_{10} + C_{10} + B_{10}\bar{J} &= A_{10}\left(1 + \frac{c^2}{2h\nu_{fs}^3}\bar{J}\right) + C_{10} \\
&= A_{10}\left(1 + \frac{c^2}{2h\nu_{fs}^3}\frac{2\nu_{fs}^2}{c^2} k_B T_\gamma\right) + C_{10} \\
&= A_{10}\left(1 + \frac{T_\gamma}{T_*}\right) + C_{10} \\
&\approx A_{10}\frac{T_\gamma}{T_*} + C_{10} \\
&= A_{10}\frac{T_\gamma}{T_*}(1 + x_c)
\end{aligned}$$

And going back to the full expression

$$
\begin{aligned}
\frac{1}{T_s} &= \frac{g_1(A_{10}+C_{10}+B_{10}\bar{J}) - g_0(B_{01}\bar{J}+C_{01})}{g_1 A_{10} T_\gamma (1+x_c)} \\
&= \frac{g_1(A_{10}+C_{10}) - g_0 C_{01}}{g_1 A_{10} T_\gamma (1+x_c)} \\
&= \frac{1}{1+x_c}\left(\frac{1}{T_\gamma} + \frac{C_{10}}{A_{10}T_\gamma}\left[1 - \frac{g_0}{g_1}\frac{C_{01}}{C_{10}}\right]\right) \\
&\approx \frac{1}{1+x_c}\left(\frac{1}{T_\gamma} + \frac{x_c}{T_*}\left[1 - \left\{1 - \frac{T_*}{T_K}\right\}\right]\right) \\
&= \frac{1}{1+x_c}\left(\frac{1}{T_\gamma} + \frac{x_c}{T_K}\right)
\end{aligned}
$$

What is with this alignment? Ragged columns why are you failing me

$$\lim_{x_c \ll 1} T_s \approx T_\gamma$$

which is consistent with excitation and de-excitation being dominated by the A_{10} rather than collisions C_{10}.

$$\lim_{x_c \gg 1} T_s \approx T_K$$

which is consistent with excitation and de-excitation being collisionally dominated.

(3) • $\sigma_{10} \approx \pi a_0^2$

$$\begin{aligned} x_c = \frac{C_{10}}{A_{10}} \frac{T_*}{T_\gamma} \approx 1 \\ C_{10} = n_1 \sigma_{10} v \approx A_{10} \frac{T_\gamma}{T_*} \\ n_1 \approx \frac{A_{10}}{\sigma_{10} v} \frac{T_\gamma}{T_*} \\ \approx \frac{A_{10}}{\sigma_{10} \sqrt{\frac{2k_B T_K}{m_H}}} \frac{T_\gamma}{T_*} \end{aligned}$$

where A_{10} for the 21cm line $\approx 2.85(10^{-15})\text{s}^{-1}$

$$\boxed{n_{1,\text{crit}} \approx 1.1(10^{-2})\text{cm}^3}$$

(4)

$$\begin{aligned}\frac{P_{01}}{P_{10}} &= \frac{g_1}{g_0} e^{-\frac{h\nu_{fs}}{k_B T_K}} \\ &\approx \frac{g_1}{g_0}\left(1 - \frac{h\nu_{fs}}{k_B T_K}\right)\end{aligned}$$

Inserting an additional term to account for Ly-α photons driving transitions

$$n_0(B_{01}\bar{J} + C_{01} + P_{01}) = n_1(A_{10} + B_{10}\bar{J} + C_{10} + P_{10})$$

Through the same algebra from before, we get

$$\begin{aligned}\frac{1}{T_s} &= \frac{1}{1 + x_c + x_\alpha}\left(\frac{1}{T_\gamma} + \frac{x_c + x_\alpha}{T_K}\right) \\ x_\alpha &= \frac{P_{10}}{A_{10}} \frac{T_*}{T_\gamma}\end{aligned}$$

$$\boxed{x_\alpha = \frac{P_{10}}{A_{10}} \frac{T_*}{T_\gamma}}$$

(5)

#	z	Info	δT_b
1	$200 \leq z \leq 1100$	• $T_s \approx T_K$: from $x_c \gg 1$ because high densty • $T_K = T_\gamma$: the gas and CMB radiation are thermally coupled	≈ 0
2	$40 \leq z \leq 200$	• $T_s \approx T_K$: from $x_c \gg 1$ • $T_K < T_\gamma$: because T_K drops with z quadratically—faster than T_γ's linear relationship	< 0
3	$30 \leq z \leq 40$	• $T_s \approx T_\gamma$	≈ 0
4	$15 \leq z \leq 30$	• $T_s \approx T_K$: from $x_\alpha \gg 1$ • $T_K < T_\gamma$	< 0
5	$7 \leq z \leq 15$	• $T_K > T_\gamma$ • $T_s \approx T_K$: from $x_\alpha \gg 1$	> 0
6	$z \leq 7$	• $x_\alpha \ll 1$: With essentially all the neutral hydrogen ionized, there are fewer Lyman-α photons because the hydrogen simply doesn't have electrons to excite. • $x_c \ll 1$: As the universe expands, the rate of excitations/de-excitation caused by collisions relative to the various Einstein coefficients goes down. • $T_s \approx T_\gamma$: from $x_\alpha \ll 1$ and $x_c \ll 1$	≈ 0