ASTRO C207 Radiative Processes in Astrophysics
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Problem Set 7

1. Blowing Strömgren Bubbles

- Emitting $\eta \frac{\text{Lyman limit photons}}{\text{s}}$
- Infinite hydrogen gas, number density n

(1)

$$
\begin{aligned}
t_{\text{rec}} &= \frac{\lambda_{\text{mfp,rec}}}{v_{\text{ion-e,rel}}} && (1)\\
&\approx \frac{\lambda_{\text{mfp,rec}}}{v_e} \longleftarrow \text{roughly stationary ion} && (2)\\
\frac{1}{t_{\text{rec}}} &= \alpha n_e && (3)\\
&= \langle v\sigma_{\text{fb,e}}(v)\rangle \cdot n_e && (4)\\
&= n_e \left\langle v\sigma_{\text{bf}}(f)\frac{g_0}{g_+}\left(\frac{hf}{m_e c v}\right)^2\right\rangle \longleftarrow \text{Milne} && (5)\\
&\approx n_e\sigma_{\text{bf},\chi}\frac{g_0}{g_+}\frac{\chi^2}{m_e^2c^2}\left\langle\frac{1}{v}\right\rangle \longleftarrow f=\frac{\chi}{h} \Rightarrow \sigma_{\text{bf}} = \sigma_{\text{bf},\chi} \approx 6(10^{-18})\text{cm}^2 && (6)\\
&= n_e\sigma_{\text{bf},\chi}\frac{g_0}{g_+}\frac{\chi^2}{m_e^2c^2}\sqrt{\frac{2\pi m_e}{k_B T}} && (7)\\
&\approx n\sigma_{\text{bf},\chi}\frac{g_0}{g_+}\frac{\chi^2}{m_e^2c^2}\sqrt{\frac{2\pi m_e}{k_B T}} \longleftarrow n_e \approx n \text{ within the sphere (nearly fully ionized)} && (8)
\end{aligned}
$$

- $\frac{g_0}{g_+} = 2$
- $\chi = 13.6\text{eV}$
- $\sigma_{\text{bf},\chi} \approx 6 \cdot 10^{-8}\text{cm}^2$

In other words,

$$t_{\text{rec}} = \frac{1}{n_e\alpha} \approx \frac{1}{n\alpha}$$

where most everything above was just about solving for α.

$$\boxed{t_{\text{rec}} \approx \frac{1}{n\alpha} \approx 74,000 \text{ years}}$$

(2) The time it takes a photon at the edge of the bubble to encounter a neutral to ionize:

$$t_{\text{ion}} = \frac{\lambda_{\text{mfp,photon}}}{c} \tag{9}$$
$$= \frac{1}{cn_0\sigma_{\text{bf},\chi}} \tag{10}$$
$$\ll t_{\text{rec}} \tag{11}$$

Because the timescale of radiative recombination is long, we assume each photon ionizes an atom basically immediately after being emitted and that radiative recombination is negligible. That said, we're interested in how long it takes to emit enough photons to fill the sphere.

$$\eta t_{\text{bubble}} = n\frac{4}{3}\pi r^3 \tag{12}$$
$$t_{\text{bubble}} = \frac{n\frac{4}{3}\pi r^3}{\eta} \tag{13}$$

where r is the radius of the bubble, found by equating the photoionization rate to recombination within the sphere (after the bubble has formed)

$$\eta = \frac{4}{3}\pi r^3 \cdot n_e n_+ \alpha \tag{14}$$
$$r = \left(\frac{\eta}{\frac{4}{3}\pi n_e n_+ \alpha}\right)^{\frac{1}{3}} \tag{15}$$
$$\approx \left(\frac{\eta}{\frac{4}{3}\pi n^2 \alpha}\right)^{\frac{1}{3}} \tag{16}$$

which leaves us with

$$t_{\text{bubble}} \approx \frac{1}{n\alpha} \tag{17}$$

$$\boxed{t_{\text{bubble}} \approx \frac{1}{n\alpha}}$$

(3) Probably around one mean free path of a photon $\lambda_{\text{mfp,photon}} = \frac{1}{n_0\sigma_{\nu,\text{H}}} \approx \frac{1}{n_0\sigma_{\text{bf},\chi}}$, where $n_0 = kn, k < 1$ correction since a nontrivial portion of atoms are ionized

$$\boxed{\begin{gathered}\ell_{\text{boundary}} \approx \lambda_{\text{mfp,photon}} \approx \frac{1}{kn\sigma_{\text{bf},\chi}} \\ k \approx 0.5\end{gathered}}$$

2. Time to Relax in the Strömgren Sphere

(1)

$$
\begin{aligned}
t_e &= \frac{\lambda_{\text{mfp,e}}}{v} && (18)\\
&= \frac{1}{n_e \sigma_e v} && (19)\\
&\approx \frac{1}{n_e \pi \frac{e^4}{m_e^2 v^3}} \longleftarrow \begin{cases} \sigma_e \approx \pi r_e^2 \\ m_e v^2 \approx \frac{e^2}{r_e} \longleftarrow \text{Coulomb scattering} \end{cases} && (20)\\
&\approx \frac{m_e^2 \left(\frac{2k_B T_e}{m_e}\right)^{\frac{3}{2}}}{n_e \pi e^4} && (21)
\end{aligned}
$$

$$
\boxed{t_e \approx \frac{m_e^{\frac{1}{2}} (2k_B T_e)^{\frac{3}{2}}}{n \pi e^4}}
$$

(2)

$$
\boxed{t_p \approx \frac{m_p^{\frac{1}{2}} (2k_B T_p)^{\frac{3}{2}}}{n \pi e^4}}
$$

(3) $t_{\text{collision}} = \frac{t_{ep}}{N}$ as the time for an electron to collide with a proton

$$
\frac{1}{t_{\text{collision}}} \approx \frac{1}{t_e} + \frac{1}{t_p}
$$

Finding the energy a proton picks up from a single collision ΔE_p

$$
\begin{aligned}
\Delta E_p &= \frac{1}{2} m_p (v_{p,f}^2 - v_{p,i}^2)\\
&\approx \frac{1}{2} m_p v_{p,f}^2 \longleftarrow T_p \text{ is low}\\
&= \frac{1}{2} m_p \left(\frac{m_e}{m_p} \Delta v_e\right)^2 \longleftarrow m_e \Delta v_e = m_p \Delta v_p \approx m_p v_{p,f}\\
&\approx \frac{1}{2} \frac{m_e^2}{m_p} (\Delta v_e)^2\\
&\approx \frac{1}{2} \frac{m_e^2}{m_p} v_e^2 \longleftarrow \text{yay for order of magnitude}
\end{aligned}
$$

And comparing against the kinetic energy of a hot electron

$$
\begin{aligned}
t_{ep} &= t_{\text{collision}} \frac{\frac{1}{2} m_e v_e^2}{\Delta E_p} \\
&\approx \frac{m_p}{m_e} t_{\text{collision}}
\end{aligned}
$$

$$\boxed{t_{ep} \approx \frac{m_p}{m_e} t_{\text{collision}}, \text{ where } t_{\text{collision}} = \frac{t_e t_p}{t_e + t_p}}$$

(4)

$$
\begin{aligned}
\frac{t_e}{t_{\text{rec}}} &\approx 3(10^{-7}) \\
\frac{t_p}{t_{\text{rec}}} &\approx 1(10^{-5}) \\
\frac{t_{ep}}{t_{\text{rec}}} &\approx 5(10^{-4})
\end{aligned}
$$

All collision times are significantly shorter than the recombination time, so assuming a Maxwellian distribution for elecrtrons and protons is safe.