ASTRO C207 Radiative Processes in Astrophysics
Lydia Lee

Problem Set 3

1. Hyperfine Emission from Neutral Hydrogen

- $\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-\frac{h\nu}{k_B T_{\text{ex}}}}$
- $g_0 = 1$, $g_1 = 3$
- $\lambda = 21\text{cm}$

(1)

$$\begin{aligned} T_* &= \frac{h\nu}{k_B} \\ &= \frac{hc}{\lambda k_B} \end{aligned}$$

$$\boxed{T_* \approx .06\text{K}}$$

(2)

$$\begin{aligned} \alpha_\nu &= \frac{1}{4\pi} \cdot (n_0 B_{12} - n_1 B_{21}) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot B_{21}(n_0 \frac{g_1}{g_0} - n_1) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot B_{21} \left(n_0 \frac{g_1}{g_0} \right) \left(1 - e^{-\frac{h\nu}{k_B T_{\text{ex}}}} \right) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot B_{21} \left(n_0 \frac{g_1}{g_0} \right) \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot A_{21} \left(\frac{c^2}{2h\nu^3} \right) \left(n_0 \frac{g_1}{g_0} \right) \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{8\pi} A_{21} \lambda^2 n_0 \frac{g_1}{g_0} \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \frac{1}{\Delta\nu} \\ &= \frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \end{aligned}$$

$$\boxed{\alpha_\nu = \frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right)}$$

(3)

$$
\begin{aligned}
j_\nu &= h\nu \cdot A_{21} \cdot n_1 \cdot \frac{1}{4\pi} \cdot \frac{1}{\Delta\nu} \\
&= \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}
\end{aligned}
$$

$$\boxed{j_\nu = \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}}$$

(4)

$$
\begin{aligned}
S_\nu &= \frac{j_\nu}{\alpha_\nu} \\
&= \frac{\frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}}{\frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right)} \\
&= \frac{\frac{hc}{\lambda} \frac{3}{4\pi} \frac{A_{21}}{\Delta\nu} n_0 e^{-T_*/T_{\text{ex}}}}{\frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right)} \\
&= \frac{2hc}{\lambda^3} \left(\frac{e^{-T_*/T_{\text{ex}}}}{1 - e^{-T_*/T_{\text{ex}}}}\right) \\
&= \frac{2hc}{\lambda^3} \left(\frac{1}{e^{T_*/T_{\text{ex}}} - 1}\right) \\
&\approx \frac{2hc}{\lambda^3} \left(\frac{T_{\text{ex}}}{T_*}\right)
\end{aligned}
$$

$$\boxed{S_\nu \approx \frac{2hc}{\lambda^3} \left(\frac{T_{\text{ex}}}{T_*}\right)}$$

(5) Assuming no background,

$$
\begin{aligned}
I_\nu(\tau_\nu) &= S_\nu \left(1 - e^{-\tau_\nu}\right) \\
&\approx S_\nu \tau_\nu \\
I_\nu(L) &\approx S_\nu L \alpha_\nu \\
&= j_\nu L \\
&= L \cdot \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}
\end{aligned}
$$

$$\boxed{I_\nu(L) \approx L \cdot \tfrac{hc}{\lambda}\tfrac{A_{21}}{\Delta\nu} n_1 \tfrac{1}{4\pi}}$$

$$I_\nu(L) \approx L \cdot \frac{hc}{\lambda}\frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}$$

where λ and A_{21} are known leaves $\frac{Ln_1}{\Delta\nu}$, i.e. $\frac{N_1}{\Delta\nu}$

- N_1, the column density of atoms in the excited hyperfine level
- N_0, the column density of atoms in the ground hyperfine level (via the Boltzmann relationship between n_0 and n_1)

(6)

$$\begin{aligned}\tau_\nu &= \alpha_\nu L \\ &= \frac{3}{8\pi}\frac{A_{21}}{\Delta\nu}\lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right)\end{aligned}$$

Yes, the optical depth depends on T_{ex}

(7)

$$\begin{aligned}\tau_\nu = 1 &= \alpha_\nu L \\ &= L \cdot \frac{3}{8\pi}\frac{A_{21}}{\Delta\nu}\lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right) \\ &\approx L \cdot \frac{3}{8\pi}\frac{A_{21}}{\Delta\nu}\lambda^2 n_0 \left(\frac{T_*}{T_{\text{ex}}}\right) \\ &\approx \frac{3L}{8\pi}\frac{A_{21}}{\Delta\nu}\lambda^2 \frac{n}{4}\left(\frac{T_*}{T_{\text{ex}}}\right) \longleftarrow \begin{cases} n & = n_0 + n_1 \\ & = n_0 + n_0 \frac{g_1}{g_0} e^{-T_*/T_{\text{ex}}} \\ & \approx n_0 \left(1 + \frac{g_1}{g_0}\right) \\ & = 4n_0 \\ n_0 & \approx \frac{n}{4} \end{cases} \\ &\approx \frac{3L}{8\pi}\frac{A_{21}c}{\nu_0 \nu}\lambda^2 \frac{n}{4}\left(\frac{T_*}{T_{\text{ex}}}\right)\end{aligned}$$

$$\boxed{L \approx 2.4(10^{20})\text{cm}}$$

2. Hyperfine $^3\mathrm{He}^+$

(1)

$$
\begin{aligned}
a_{0,\mathrm{He}} &= \frac{0.52}{Z_{\mathrm{He}}}\mathrm{\mathring{A}} \\
\mathrm{Ryd}_{\mathrm{He}} &= \frac{Z_{\mathrm{He}} e^2}{a_{0,\mathrm{He}}} \\
&= \frac{Z_{\mathrm{He}}^2 e^2}{0.52} \\
&= 4 \cdot 13.6\mathrm{eV} \\
&= 54.4\mathrm{eV} \\
\mathrm{Ly}\alpha_{\mathrm{He}} &= \mathrm{Ryd}_{\mathrm{He}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \\
&= 4 \cdot 10.2\mathrm{eV} \\
&= 40.8\mathrm{eV} \\
\lambda &= \frac{121.6\mathrm{nm}}{4} \\
&= 30.4\mathrm{nm}
\end{aligned}
$$

$$\boxed{\lambda \approx 30.4\mathrm{nm}}$$

(2)
- nucleus charge: Ze, $Z = 2$
- nucleus mass: Cm_p, $C = 3$

$$\Delta E \approx \mu_e \cdot B_{\mathrm{nucleus}}$$

$$
\begin{aligned}
B_{\mathrm{nucleus},^3\mathrm{He}} &\propto \frac{\mu_{\mathrm{nucleus,He}}}{a_{0,\mathrm{He}}^3} \\
&= \frac{Ze\hbar}{2Cm_p c} \cdot \frac{1}{a_{0,\mathrm{H}}^3 / Z^3} \\
&= B_{\mathrm{nucleus,H}} \cdot \frac{Z^4}{C} \\
&= B_{\mathrm{nucleus,H}} \cdot \frac{16}{3} \\
\lambda &\propto \frac{1}{E} \\
&\propto \frac{1}{B_{\mathrm{nucleus}}} \\
\lambda_{\mathrm{He,hf}} &= \frac{3}{16} \lambda_{\mathrm{H,hf}} \\
&\approx 3.94\mathrm{cm}
\end{aligned}
$$

$$\boxed{\begin{gathered}\lambda_{\text{He,hf}} \approx 3.94\text{cm} \\ \text{Not too far off from 3.46cm}\end{gathered}}$$

3. Rotating Magnetic Dipole

(1) For electric dipoles:

$$P = \frac{2}{3}\frac{|\ddot{\vec{p}}|^2}{c^3}$$

For magnetic dipoles:

$$P = \frac{2}{3}\frac{|\ddot{\vec{m}}|^2}{c^3}$$

$\vec{m}$ is rotating about the axis of rotation

$$\begin{aligned}
\vec{m} &= \begin{bmatrix} m\sin(\alpha)\cos(\omega t) \\ m\sin(\alpha)\sin(\omega t) \\ m\cos(\alpha) \end{bmatrix} \\
\ddot{\vec{m}} &= \begin{bmatrix} -\omega^2 m\sin(\alpha)\cos(\omega t) \\ -\omega^2 m\sin(\alpha)\sin(\omega t) \\ 0 \end{bmatrix} \\
|\ddot{\vec{m}}|^2 &= \omega^4 m^2 \sin^2(\alpha)
\end{aligned}$$

Definitely had to look up how to derive the magnetic field from the magnetic dipole

$$\begin{aligned}
\vec{B} &= \frac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3} \longleftarrow \text{because CGS, that's why} \\
&= m\frac{2cos(\theta)\hat{r} + sin(\theta)\hat{\theta}}{r^3} \\
B_0 &= m\frac{2}{R^3} \longleftarrow \theta = 0 \\
m &= \frac{B_0 R^3}{2}
\end{aligned}$$

$$\begin{aligned}
P &= \frac{2}{3}\frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{4c^3} \\
&= \frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{6c^3}
\end{aligned}$$

$$\boxed{P = \frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{6c^3}}$$

(2)

$$
\begin{aligned}
E_{\text{rot}} &= \frac{1}{2} I\omega^2 \\
&= \frac{1}{5} MR^2\omega^2 \\
P &= \frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{6c^3} \\
\tau &= \frac{E}{P} \\
&= \frac{6}{5} \frac{Mc^3}{\omega^2 \sin^2(\alpha) B_0^2 R^4}
\end{aligned}
$$

$$
\boxed{\tau = \frac{6}{5} \frac{Mc^3}{\omega^2 \sin^2(\alpha) B_0^2 R^4}}
$$

(3) What fresh hell are these units

power

$$
\begin{aligned}
\frac{\mathrm{s}^{-4} \cdot \mathrm{G}^2 \cdot \mathrm{cm}^6}{\mathrm{cm}^3/\mathrm{s}^3} &= \frac{\mathrm{G}^2 \cdot \mathrm{cm}^3}{\mathrm{s}} \\
&= \frac{\mathrm{Mx}^2}{\mathrm{cm} \cdot \mathrm{s}} \\
&= \frac{\mathrm{cm}^2}{\mathrm{s}^2} \cdot \frac{\mathrm{g}}{\mathrm{s}} \\
&= \frac{\mathrm{erg}}{\mathrm{s}}
\end{aligned}
$$

time

$$
\begin{aligned}
\frac{\frac{\mathrm{cm}^3}{\mathrm{s}^3} \cdot \mathrm{g}}{\mathrm{s}^{-2} \cdot \mathrm{G} \cdot \mathrm{cm}^4} &= \frac{\mathrm{g}}{\mathrm{s} \cdot \mathrm{cm} \cdot \mathrm{G}^2} \\
&= \frac{\mathrm{g} \cdot \mathrm{cm}^3}{\mathrm{s}} \cdot \frac{\mathrm{s}^2}{\mathrm{cm}^3 \cdot \mathrm{g}} \\
&= \mathrm{s}
\end{aligned}
$$

ω (s^{-1})	P (erg/s)	τ (s)
10^4	$6.2(10^{43})$	$6.5(10^8)$
10^3	$6.2(10^{39})$	$6.5(10^{10})$
10^2	$6.2(10^{35})$	$6.5(10^{12})$