ASTRO C207 Radiative Processes in Astrophysics
Fall 2021 Problem Set 2

1. Blackbody Flux

$$
\begin{aligned}
B_\nu &= \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_BT}}-1} \\
\int_0^\infty\int_0^{2\pi}\int_0^{\pi/2} B_\nu \cos\theta\sin\theta d\theta d\phi d\nu &= 2\pi\int_0^{\pi/2}\cos\theta\sin\theta d\theta\int_0^\infty B_\nu d\nu \\
&= 2\pi\cdot\frac{1}{2}\cdot\int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_BT}}-1}d\nu \\
&= \pi\int_0^\infty \frac{2h\frac{c^3}{\lambda^3}}{c^2}\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1}d\nu \\
&= \pi\int_0^\infty \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1}d\lambda \longleftarrow \begin{cases} \nu = \frac{c}{\lambda} \\ d\nu = -\frac{c}{\lambda^2}d\lambda \end{cases} \\
&= \pi\cdot 2hc^2\cdot\frac{hc}{k_BT}\cdot\left(\frac{k_BT}{hc}\right)^5\int_0^\infty \frac{1}{x\left(e^{1/x}-1\right)}dx \longleftarrow \begin{cases} x = \lambda\frac{k_BT}{hc} \\ dx = d\lambda\frac{k_BT}{hc} \end{cases} \\
&= \pi\cdot\frac{2k_B^4T^4}{h^3c^2}\cdot\frac{\pi^4}{15} \\
&= \frac{2\pi^5k_B^4}{15h^3c^2}T^4 \\
&= \sigma T^4
\end{aligned}
$$

2. Flat Disks

For an annulus in the ring at distance r, power in must equal power out

$$\begin{aligned} P_{\text{in}} &= P_{\text{out}} \\ F_{\text{in}} &= F_{\text{out}} \\ &= \sigma T^4 \end{aligned}$$

where T is the temperature in question. For the flux coming from the star where $\tan\theta_c = \frac{R_*}{r}$,

$$\begin{aligned} F_{\text{in}} &\approx F_* \cdot 4\pi R_*^2 \cdot \frac{1}{4\pi r^2} \cdot C \sin\theta_c \\ &\approx \sigma T_*^4 \frac{R_*^2}{r^2} \frac{R_*}{r} \end{aligned}$$

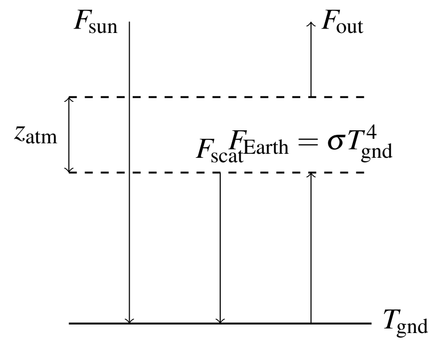
where C (roughly 1) accounts for error from $r - R_* \approx r$ and the fact that there should be an integral over θ going from 0 to $+\theta_c$.

$$\begin{aligned} \sigma T^4 &\approx \sigma T_*^4 \frac{R_*^2}{r^2} \frac{R_*}{r} \\ T &\approx T_* \left(\frac{R_*}{r}\right)^{\frac{3}{4}} \end{aligned}$$

This feels weird, but I guess as you get further away a slice of the ring of the same area occupies a smaller solid angle from the view of some $z \neq 0$ point on the star?

3. A Simplified Greenhouse Effect

I suppose we'll assume the atmosphere doesn't emit as a blackbody and Earth doesn't reflect anything coming off of the sun?



$$\begin{aligned} F_{\rm Earth} &= F_{\rm scat} + F_{\rm sun} \\ F_{\rm out} &= F_{\rm sun} \end{aligned}$$

For scattering, we consider the volume of scatterers with some arbitrary column area and depth z.

optically thin atmosphere

$$z = z_{\rm atm}$$

optically thick atmosphere

$$z = \lambda_{\rm mfp} = \frac{1}{n\sigma_\nu} = \frac{z_{\rm atm}}{N\sigma_\nu}$$

Considering $j_{\nu,\rm scat} = n\sigma_{\rm scat} J_\nu = \frac{N}{z_{\rm atm}}\sigma_{\rm scat} J_\nu$,

$$j_{\nu,\rm scat} V_{\rm scat} \propto \begin{cases} NJ_\nu & \text{optically thin} \\ J_\nu & \text{optically thick} \end{cases}$$

where $J_\nu = KF_{\rm Earth}$ for some scalar K.

$$\begin{aligned} F_{\rm Earth} &= F_{\rm scat} + F_{\rm sun} \\ \sigma T_{\rm gnd}^4 &= F_{\rm scat} + F_{\rm sun} \\ \sigma T_{\rm gnd}^4 &= \begin{cases} \sigma T_{\rm gnd}^4 C_{\rm thin} N + F_{\rm sun} & \text{optically thin} \\ \sigma T_{\rm gnd}^4 C_{\rm thick} + F_{\rm sun} & \text{optically thick} \end{cases} \end{aligned}$$

where $C_{\rm thick/thin}$ are constants.

$$\boxed{T_{\rm gnd} \propto \begin{cases} \sqrt[4]{\frac{1}{1 - C_{\rm thin} N}} & \text{optically thin} \\ \text{constant} & \text{optically thick} \end{cases}}$$