

ASTRO C207 Radiative Processes in Astrophysics

Fall 2021

Problem Set 3

1. Hyperfine Emission from Neutral Hydrogen

- $\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-\frac{h\nu}{k_B T_{\text{ex}}}}$
- $g_0 = 1, g_1 = 3$
- $\lambda = 21\text{cm}$

(1)

$$T_* = \frac{h\nu}{k_B}$$

$$= \frac{hc}{\lambda k_B}$$

$$T_* \approx .06\text{K}$$

(2)

$$\begin{aligned} \alpha_\nu &= \frac{1}{4\pi} \cdot (n_0 B_{12} - n_1 B_{21}) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot B_{21} \left(n_0 \frac{g_1}{g_0} - n_1 \right) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot B_{21} \left(n_0 \frac{g_1}{g_0} \right) \left(1 - e^{-\frac{h\nu}{k_B T_{\text{ex}}}} \right) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot B_{21} \left(n_0 \frac{g_1}{g_0} \right) \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{4\pi} \cdot A_{21} \left(\frac{c^2}{2h\nu^3} \right) \left(n_0 \frac{g_1}{g_0} \right) \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \cdot \frac{1}{\Delta\nu} \cdot h\nu \\ &= \frac{1}{8\pi} A_{21} \lambda^2 n_0 \frac{g_1}{g_0} \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \frac{1}{\Delta\nu} \\ &= \frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right) \end{aligned}$$

$$\alpha_\nu = \frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}} \right)$$

(3)

$$\begin{aligned}
 j_\nu &= h\nu \cdot A_{21} \cdot n_1 \cdot \frac{1}{4\pi} \cdot \frac{1}{\Delta\nu} \\
 &= \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}
 \end{aligned}$$

$$j_\nu = \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}$$

(4)

$$\begin{aligned}
 S_\nu &= \frac{j_\nu}{\alpha_\nu} \\
 &= \frac{\frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}}{\frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right)} \\
 &= \frac{\frac{hc}{\lambda} \frac{3}{4\pi} \frac{A_{21}}{\Delta\nu} n_0 e^{-T_*/T_{\text{ex}}}}{\frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right)} \\
 &= \frac{2hc}{\lambda^3} \left(\frac{e^{-T_*/T_{\text{ex}}}}{1 - e^{-T_*/T_{\text{ex}}}} \right) \\
 &= \frac{2hc}{\lambda^3} \left(\frac{1}{e^{T_*/T_{\text{ex}}} - 1} \right) \\
 &\approx \frac{2hc}{\lambda^3} \left(\frac{T_{\text{ex}}}{T_*} \right)
 \end{aligned}$$

$$S_\nu \approx \frac{2hc}{\lambda^3} \left(\frac{T_{\text{ex}}}{T_*} \right)$$

(5) Assuming no background,

$$\begin{aligned}
 I_\nu(\tau_\nu) &= S_\nu (1 - e^{-\tau_\nu}) \\
 &\approx S_\nu \tau_\nu \\
 I_\nu(L) &\approx S_\nu L \alpha_\nu \\
 &= j_\nu L \\
 &= L \cdot \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}
 \end{aligned}$$

$$I_V(L) \approx L \cdot \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}$$

$$I_V(L) \approx L \cdot \frac{hc}{\lambda} \frac{A_{21}}{\Delta\nu} n_1 \frac{1}{4\pi}$$

where λ and A_{21} are known leaves $\frac{L n_1}{\Delta\nu}$, i.e. $\frac{N_1}{\Delta\nu}$

- N_1 , the column density of atoms in the excited hyperfine level
- N_0 , the column density of atoms in the ground hyperfine level (via the Boltzmann relationship between n_0 and n_1)

(6)

$$\begin{aligned} \tau_V &= \alpha_V L \\ &= \frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right) \end{aligned}$$

Yes, the optical depth depends on T_{ex}

(7)

$$\begin{aligned} \tau_V = 1 &= \alpha_V L \\ &= L \cdot \frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(1 - e^{-\frac{T_*}{T_{\text{ex}}}}\right) \\ &\approx L \cdot \frac{3}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 n_0 \left(\frac{T_*}{T_{\text{ex}}}\right) \\ &\approx \frac{3L}{8\pi} \frac{A_{21}}{\Delta\nu} \lambda^2 \frac{n}{4} \left(\frac{T_*}{T_{\text{ex}}}\right) \leftarrow \begin{cases} n &= n_0 + n_1 \\ &= n_0 + n_0 \frac{g_1}{g_0} e^{-T_*/T_{\text{ex}}} \\ &\approx n_0 \left(1 + \frac{g_1}{g_0}\right) \\ &= 4n_0 \\ n_0 &\approx \frac{n}{4} \end{cases} \\ &\approx \frac{3L}{8\pi} \frac{A_{21}c}{\nu_0 \nu} \lambda^2 \frac{n}{4} \left(\frac{T_*}{T_{\text{ex}}}\right) \end{aligned}$$

$$L \approx 2.4(10^{20})\text{cm}$$

2. Hyperfine $^3\text{He}^+$

(1)

$$\begin{aligned}
 a_{0,\text{He}} &= \frac{0.52 \text{ \AA}}{Z_{\text{He}}} \\
 \text{Ryd}_{\text{He}} &= \frac{Z_{\text{He}}^2 e^2}{a_{0,\text{He}}} \\
 &= \frac{Z_{\text{He}}^2 e^2}{0.52} \\
 &= 4 \cdot 13.6 \text{ eV} \\
 &= 54.4 \text{ eV} \\
 \text{Ly}\alpha_{\text{He}} &= \text{Ryd}_{\text{He}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \\
 &= 4 \cdot 10.2 \text{ eV} \\
 &= 40.8 \text{ eV} \\
 \lambda &= \frac{121.6 \text{ nm}}{4} \\
 &= 30.4 \text{ nm}
 \end{aligned}$$

$$\lambda \approx 30.4 \text{ nm}$$

- (2)
- nucleus charge: Ze , $Z = 2$
 - nucleus mass: Cm_p , $C = 3$

$$\begin{aligned}
 \Delta E &\approx \mu_e \cdot B_{\text{nucleus}} \\
 B_{\text{nucleus}, ^3\text{He}} &\propto \frac{\mu_{\text{nucleus, He}}}{a_{0,\text{He}}^3} \\
 &= \frac{Ze\hbar}{2Cm_p c} \cdot \frac{1}{a_{0,\text{H}}^3 / Z^3} \\
 &= B_{\text{nucleus, H}} \cdot \frac{Z^4}{C} \\
 &= B_{\text{nucleus, H}} \cdot \frac{16}{3} \\
 \lambda &\propto \frac{1}{E} \\
 &\propto \frac{1}{B_{\text{nucleus}}} \\
 \lambda_{\text{He, hf}} &= \frac{3}{16} \lambda_{\text{H, hf}} \\
 &\approx 3.94 \text{ cm}
 \end{aligned}$$

$\lambda_{\text{He,hf}} \approx 3.94\text{cm}$
Not too far off from 3.46cm

3. Rotating Magnetic Dipole

(1) For electric dipoles:

$$P = \frac{2}{3} \frac{|\ddot{\vec{p}}|^2}{c^3}$$

For magnetic dipoles:

$$P = \frac{2}{3} \frac{|\ddot{\vec{m}}|^2}{c^3}$$

\vec{m} is rotating about the axis of rotation

$$\vec{m} = \begin{bmatrix} m \sin(\alpha) \cos(\omega t) \\ m \sin(\alpha) \sin(\omega t) \\ m \cos(\alpha) \end{bmatrix}$$

$$\ddot{\vec{m}} = \begin{bmatrix} -\omega^2 m \sin(\alpha) \cos(\omega t) \\ -\omega^2 m \sin(\alpha) \sin(\omega t) \\ 0 \end{bmatrix}$$

$$|\ddot{\vec{m}}|^2 = \omega^4 m^2 \sin^2(\alpha)$$

Definitely had to look up how to derive the magnetic field from the magnetic dipole

$$\vec{B} = \frac{3\hat{r}(\vec{m} \cdot \hat{r}) - \vec{m}}{r^3} \leftarrow \text{because CGS, that's why}$$

$$= m \frac{2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}}{r^3}$$

$$B_0 = m \frac{2}{R^3} \leftarrow \theta = 0$$

$$m = \frac{B_0 R^3}{2}$$

$$P = \frac{2}{3} \frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{4c^3}$$

$$= \frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{6c^3}$$

$$P = \frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{6c^3}$$

(2)

$$\begin{aligned}
 E_{\text{rot}} &= \frac{1}{2} I \omega^2 \\
 &= \frac{1}{5} M R^2 \omega^2 \\
 P &= \dot{E}_{\text{rot}} \\
 \frac{\omega^4 \sin^2(\alpha) B_0^2 R^6}{6c^3} &= \frac{1}{5} M R^2 \cdot 2\omega \dot{\omega} = \\
 \dot{\omega} &= \frac{\omega}{\tau} \\
 \tau &= \frac{6c^3 M \cdot \frac{2}{5}}{\omega^2 \sin^2(\alpha) B_0^2 R^4}
 \end{aligned}$$

$$\tau = \frac{12c^3 M}{5\omega^2 \sin^2(\alpha) B_0^2 R^4}$$

(3) What fresh hell are these units

power

$$\begin{aligned}
 \frac{\text{s}^{-4} \cdot \text{G}^2 \cdot \text{cm}^6}{\text{cm}^3/\text{s}^3} &= \frac{\text{G}^2 \cdot \text{cm}^3}{\text{s}} \\
 &= \frac{\text{Mx}^2}{\text{cm} \cdot \text{s}} \\
 &= \frac{\text{cm}^2}{\text{s}^2} \cdot \frac{\text{g}}{\text{s}} \\
 &= \frac{\text{erg}}{\text{s}}
 \end{aligned}$$

time

$$\begin{aligned}
 \frac{\frac{\text{cm}^3}{\text{s}^3} \cdot \text{g}}{\text{s}^{-2} \cdot \text{G} \cdot \text{cm}^4} &= \frac{\text{g}}{\text{s} \cdot \text{cm} \cdot \text{G}^2} \\
 &= \frac{\text{g} \cdot \text{cm}^3}{\text{s}} \cdot \frac{\text{s}^2}{\text{cm}^3 \cdot \text{g}} \\
 &= \text{s}
 \end{aligned}$$

ω (s^{-1})	P (erg/s)	τ (s)
10^4	$6.2(10^{43})$	$1.3(10^9)$
10^3	$6.2(10^{39})$	$1.3(10^{11})$
10^2	$6.2(10^{35})$	$1.3(10^{13})$