CS 4215: Quantitative Performance Evaluation for Computing ystems

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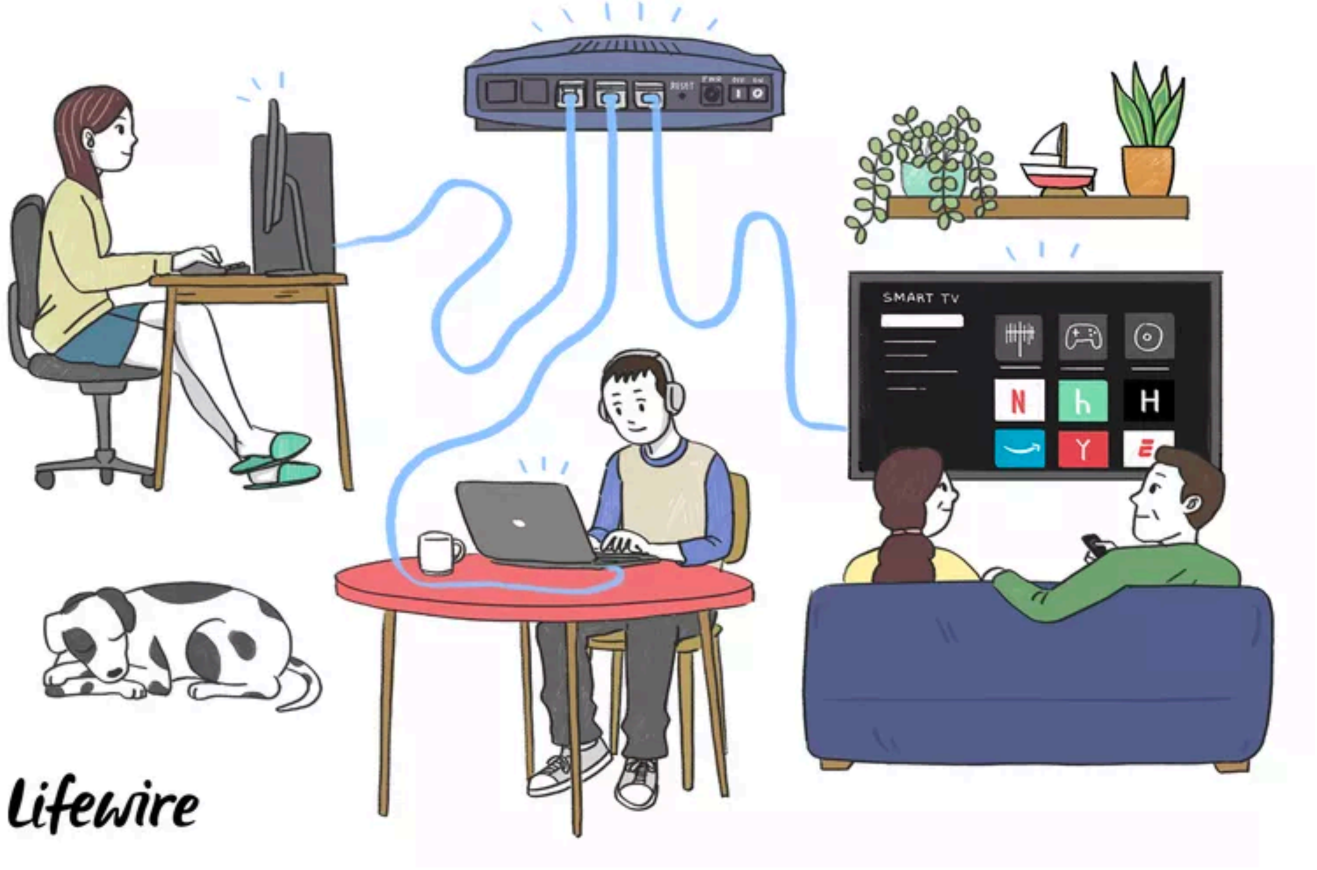
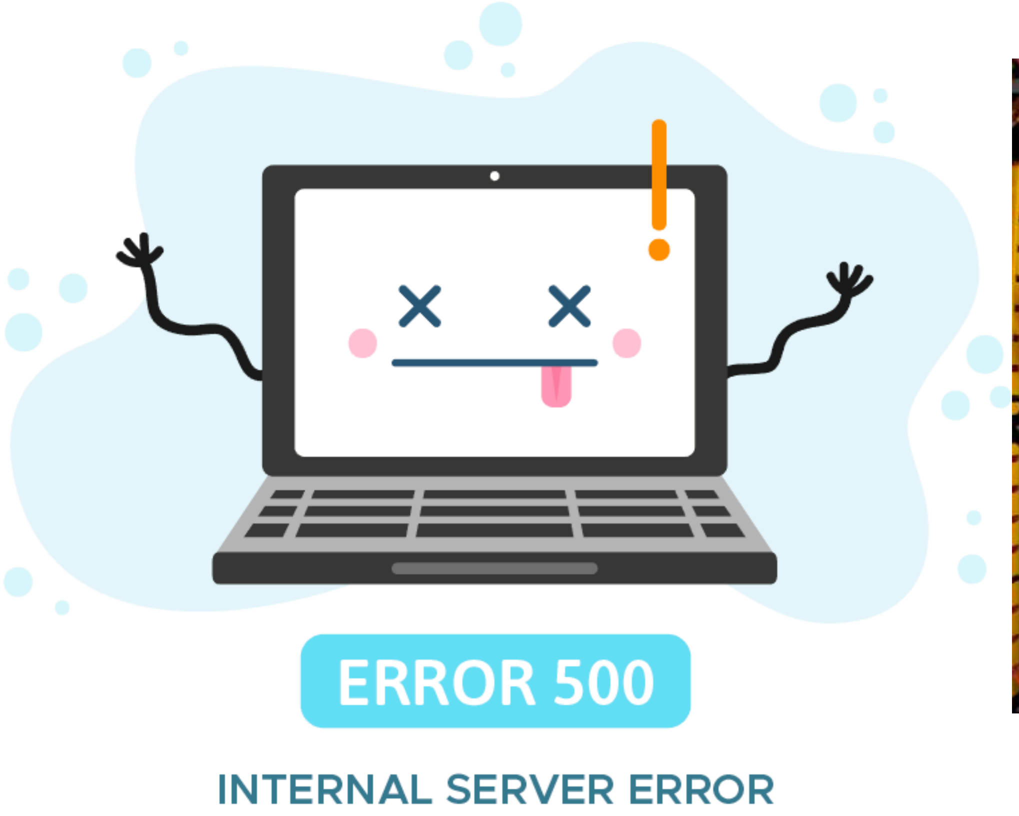
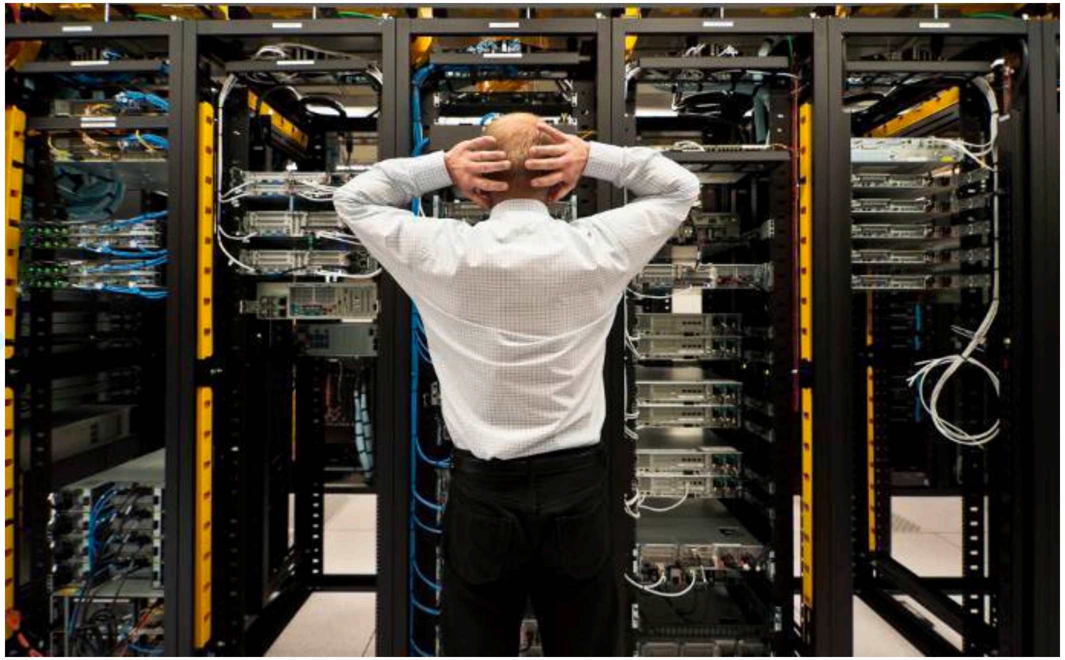
Ethernet



Server Failure





Energy Management for Cloud

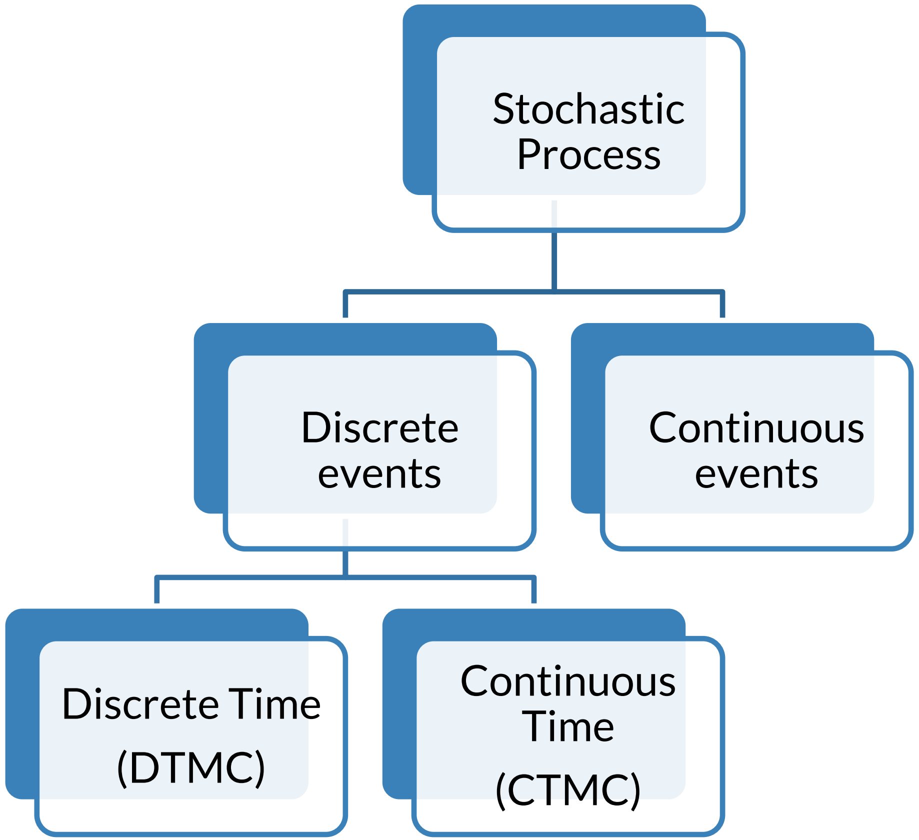
Last week
Operational Laws

Littles Law
Forced Law
Utilization Law
Asymptotic Bounds

Discrete-Time Markov Chain

Stochastic process

- Stochastic process:
  - Definition: a collection of random variables indexed by some set.
  - Category by the type of index
    - Continuous event
    - Discrete event
- Markov property: it is not necessary to know how long the process has been in the current state
- Markov chain
  - Discrete event stochastic process



Overview

Model
- State
- Transition probability

Solving
- Limiting v.s. stationary distribution
- Finite to infinite states

Theorem
- Erogodicity theorems

Discrete Time Markov Chains

Broken into a synchronized time steps: an event (arrival or departure) can only occur at the end of a time step.

- State at the time step n : X_n
- Transition probability matrix : $\mathbf{P}$, *where* (i, j) *entry is* $P_{i,j}$

is the probability of moving to state j from the state i in one step

Definition 8.1 A ***DTMC*** (discrete-time Markov chain) is a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$, where X_n denotes the state at (discrete) time step n and such that, $\forall n \geq 0$, $\forall i, j$, and $\forall i_0, \ldots, i_{n-1}$,

Markovian Property

$$\boxed{\mathbf{P}\left\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right\} = \mathbf{P}\left\{X_{n+1} = j \mid X_n = i\right\}}$$
$$= P_{ij} \text{ (by stationarity)},$$

where P_{ij} is independent of the time step and of past history.

Example of Repair Facility

A machine is either working or in the repair center. If it's working today, then there is a 95% chance it will be working tomorrow. If it's in the repair, there is a 40 % chance it will be working.

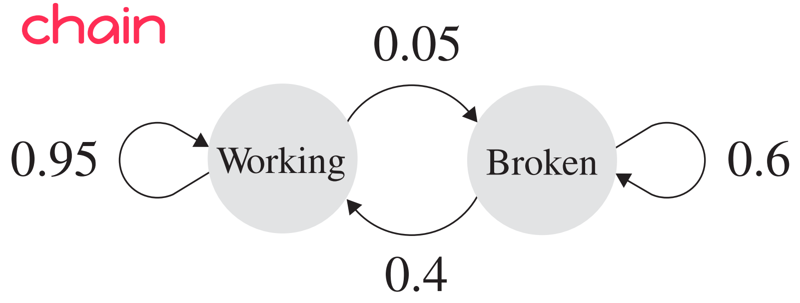
Q: what fraction of time does the machine spend in the repair shop?

Ans: two states of "working" and "broken".

Transition probability

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{cc} W & B \end{array} \\ \begin{array}{c} W \\ B \end{array} & \begin{bmatrix} 0.95 & 0.05 \\ 0.40 & 0.60 \end{bmatrix} \end{array}$$

Markov chain



Example of Program Analysis

A Program has three types of instructions: CPU (C), memory (M), and User interactions (U).

- C instruction with probability 0.7 is followed by another C instruction, with 0.2 probability by an M instruction, and with probability 0.1 by a U instruction.
- M instruction with probability 0.8 is followed by another C instruction, with 0.1 probability by an M instruction, and with probability 0.1 by a U instruction.
- U instruction with probability 0.9 is followed by another C instruction, with 0.1 probability by an M instruction, and with probability 0 by a U instruction.

Transition probability

Markov chain

$$\mathbf{P} = \begin{array}{c} \\ C \\ M \\ U \end{array} \begin{array}{c} \begin{array}{ccc} C & M & U \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.9 & 0.1 & 0 \end{bmatrix} \end{array}$$

Limiting probabilities

Observation:

All the rows become the same

$$\mathrm{P}^n = \mathrm{P} \cdot \mathrm{P} \cdots \mathrm{P}???$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 \end{bmatrix}, \quad \mathbf{P}^5 = \qquad , \quad \mathbf{P}^{30} = $$

Definition

$$\boxed{\lim_{n \to \infty} P_{ij}^n = \left(\lim_{n \to \infty} \mathbf{P}^n \right)_{ij}}$$

Q. What does it says about all rows are the same?

A: starting state (i) does not matter

Limiting probabilities

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Limiting Distribution

Multiplying P by itself multiple times is onerous !!!

$$\pi_j = \lim_{n \to \infty} P_{ij}^n .$$

π_j represents the ***limiting probability*** that the chain is in state j (independent of the starting state i). For an M-state DTMC, with states $0, 1, \ldots, M-1$,

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \text{where} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

represents the ***limiting distribution*** of being in each state.

- A probability distribution is said to be stationary for the Markov chain if

$$\vec{\pi} \cdot P = \vec{\pi} \quad \textit{and} \quad \sum_{i=0}^{M} \pi_i = 1$$

Limiting distribution = stationary distribution

Given a finite-state DTMC with M states, let

$$\pi_j = \lim_{n \to \infty} P_{ij}^n > 0$$

be the limiting probability of being in state j and let

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \textit{where} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

How can we assume so?

be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and no other *stationary distribution exists.*

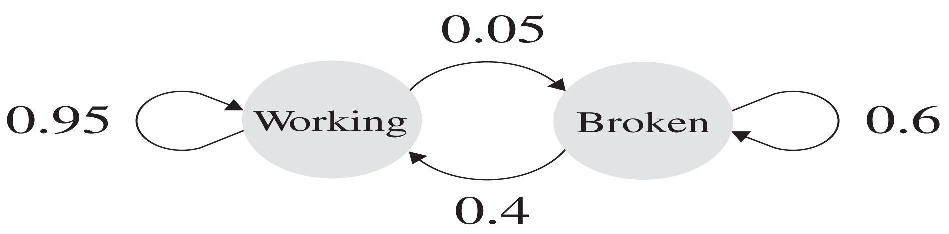
Proof: see chapter 8

Solving Limiting Distribution

Let $\vec{\pi} = (\pi_0, \pi_1 \cdots \pi_{M-1})$

$$\vec{\pi} \cdot \mathrm{P} = \vec{\pi} \quad and \quad \sum_{i=0}^{M} \pi_i = 1$$

Back to the example of repair



$$\vec{\pi} = \vec{\pi} \cdot \mathbf{P}, \text{ where } \mathbf{P} = \begin{pmatrix} .95 & .05 \\ .4 & .6 \end{pmatrix}$$

$$\pi_W + \pi_B = 1$$

Observation:
First two eq. are identical

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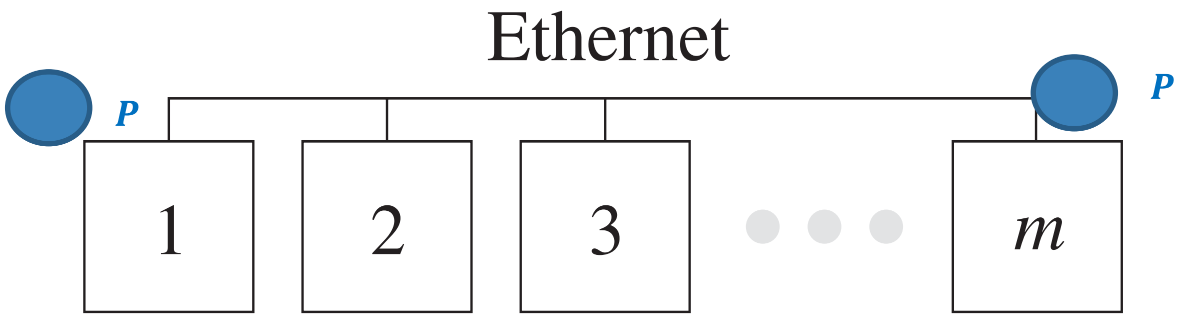
➔ Rank page according to the number of links to that page (so called backlinks of the page)

Example of Aloha Protocol

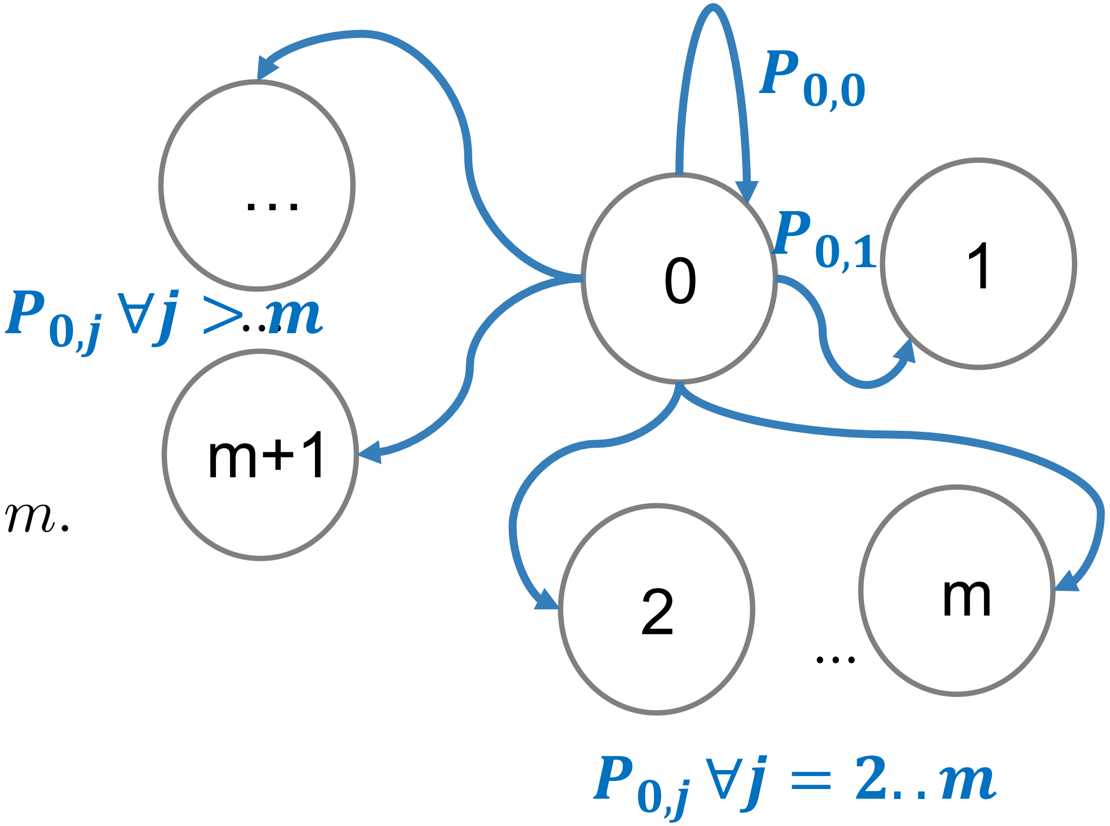
- Progenitor of the Ethernet protocol, a datalink-level protocol
- Allows m users to transmit a data frame along a single wire with a probability p
- Only one user can use a wire at a time
- Collision: more than one users tried at the same time, and all messages corrupted
- Retransmission: every unsuccessful message can retransmit with probability q



Modeling Aloha as DTMC

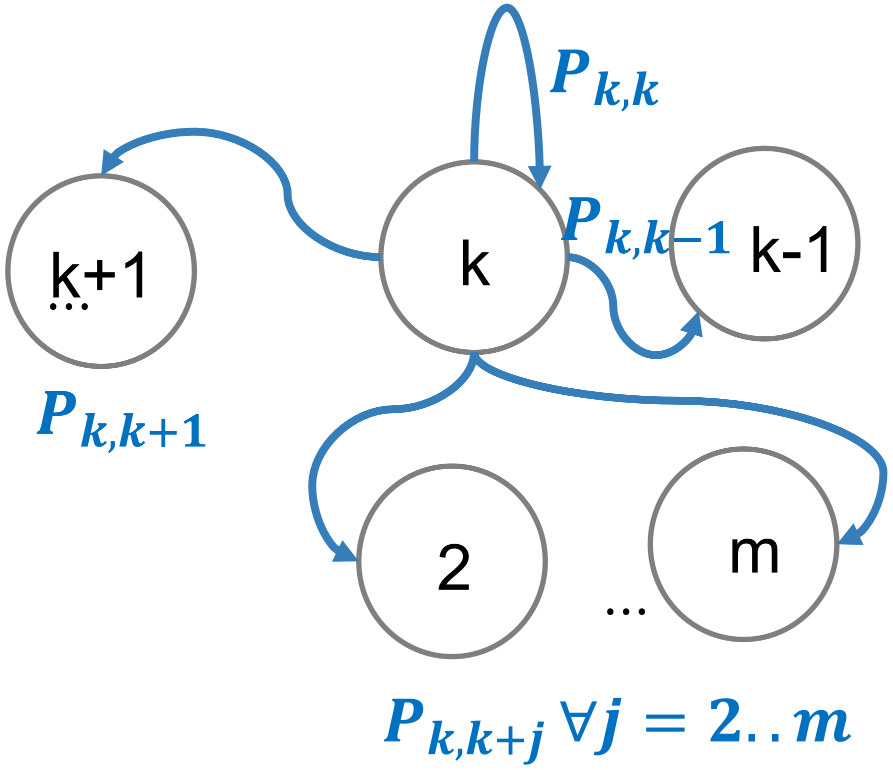
- States! Number of messages remaining in the system
- Transition probability!!

$$P_{0,0} = (1-p)^m + mp(1-p)^{m-1}.$$
$$P_{0,1} = 0.$$
$$P_{0,j} = \binom{m}{j} p^j (1-p)^{m-j} \quad ,\forall j = 2, \ldots, m.$$
$$P_{0,j} = 0 \quad ,\forall j > m.$$



Modeling Aloha as DTMC

- States! Number of messages remaining in the system
- Transition probability!!



- k➔k-1: no new transmission and 1 retransmission

$$(1-p)^m kq(1-q)^{k-1}$$

- k➔k:
  - One new transmission, no retransmissions $mp(1-p)^{m-1}(1-q)^k$
  - No new transmission and retransmission $(1-p)^m(1-q)^k$
  - No new transmission and at least 2 retransmissions

$$(1-p)^m \left(1-(1-q)^k - kq(1-q)^{k-1}\right)$$

- k➔k+1:one new transmission and at least 1 retransmission

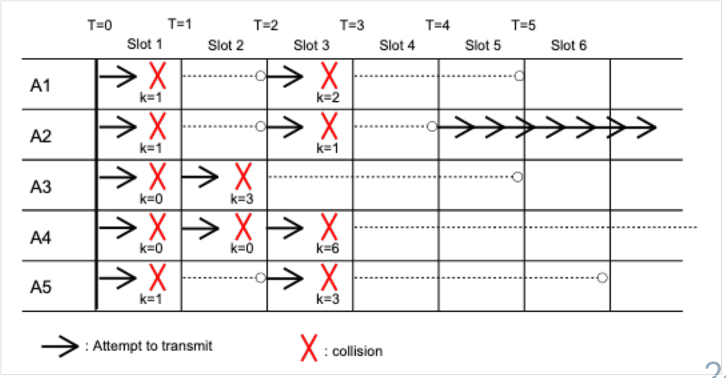
$$mp(1-p)^{m-1}\left(1-(1-q)^k\right)$$

- K➔k+j: j new transmission

$$\binom{m}{j} p^j (1-p)^{m-j}$$

Modeling Aloha as DTMC

- Does Aloha protocol work well?
  - The probability of going to lower state tends to zero
  - The probability of staying at the same state tends to a small constant
  - With high probability moving to a higher stte
- Can we improve it further?
  - Make q really small?
  - Today's Ethernet design: exponential backoff



Bonus quiz

- Write a numerical program that can model Aloha protocol for any given p, m, and q
- Try to be as efficient as possible.

“

Assuming that the limiting distribution exits,

Limiting distribution = Stationary distribution

When can we assume so?
What if such a condition does not hold?

A counter example

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Does it have limiting distribution?

$$\lim_{n\to\infty} P_{j,j}^{n} \quad v.s. \quad \lim_{n\to\infty} P_{j,j}^{2n}$$

- Doe it have stationary distribution?

$$\begin{aligned} \pi_0 &= \pi_1 \\ \pi_1 &= \pi_0 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

$\vec{\pi}$ =(.5,.5)

Ergocity theorem

Given a (recurrent), aperiodic, irreducible DTMC $\vec{\pi} = \lim_{n\to\infty} P_{i,j}^n$ exits and

$$\pi_j = \frac{1}{m_{j,j}}, \forall j$$

- $m_{j,j}$ denotes the expected number of steps between visit to state j
- Finite state DTMC: aperiodic + irreducible
- Infinite DTMC: recurrent + aperiodic + irreducible

Aperiodicity

- Aperiodic: a chain of its states are all aperiordic.

- The period of state j is the greatest common divisor (GCD) of the set of integers, such that $P_{j,j}^{n} > 0$. A state I aperiodic if it has period 1.

- Another example

What is period?

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Irreducible

- Irreducible: a MC is irreducible if all its states communicate with each other

- State j is accessible from state i if $P_{i,j}^{n} > 0 \ for \ some \ n > 0$, states I and j communicate if I is accessible from j and vice versa

What matrix is not irreducible?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Positive Recurrent

Definition

- Let f_j the probability that a chain starting in state j ever returns to state j
- A state j is either recurrent or transient
  - If f_j =1, then j is a recurrent state
  - If f_j <1, then j is a transient state
- Recurrent Markov chains
  - Positive recurrent: time between recurrences is finite
  - Null recurrent: time between recurrences is infinite

Theorem

- Theorem: With the probability 1 , the number of visits to a recurrent state is infinite. With probability 1, the number of visits to a transient state is finite.
- Theorem: for a irreducible MC, either all state are transient or all recurrent
- Theorem: for a transient Markov chain the limiting distribution does not exist

What if.....

Given a (~~positive recurrent), aperiodic, irreducible~~ DTMC $\vec{\pi} = \lim_{n\to\infty} P_{i,j}^{n}$ exits and

$$\pi_j = \frac{1}{m_{j,j}}, \forall j$$

- The chain is not positive recurrent?
- The chain is periodic?
- The chain is reducible?

Different Views of Limiting Distribution

Stationary probability for state j

$$\pi_j = \sum_i \pi_i P_{ij}$$

Limiting probability of being in state j

$$\pi_j = \lim_{n \to \infty} P_{ij}^n$$

Reciprocal of time between visits

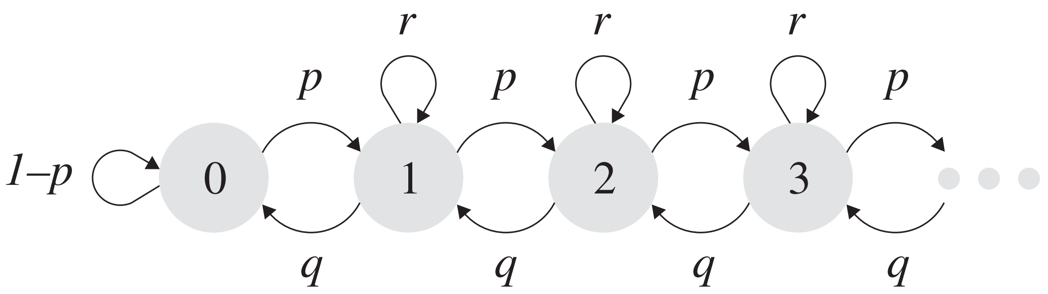
$$\pi_j = \frac{1}{m_{jj}}$$

Time–average fraction of time spent in state j

$$\pi_j = \lim_{t \to \infty} \frac{N_j(t)}{t} = p_j$$

Solving DTMC with Different Views

How to solve the following chain?



- Regular Stationary Equations

$$\pi_i = \pi_{i-1}p + \pi_i r + \pi_{i-1}q, and \sum_i \pi_i = 1$$

Which one is easier?
By hand?
By computer?

- Balanced Equations

$$\pi_i(1-r) = \pi_{i-1}p + \pi_{i+1}q, and \sum_i \pi_i = 1$$

- Time-Reversibility Equations

$$\pi_i p = \pi_{i+1}q, and \sum_i \pi_i = 1$$

Continuous Time Models (CTMC)

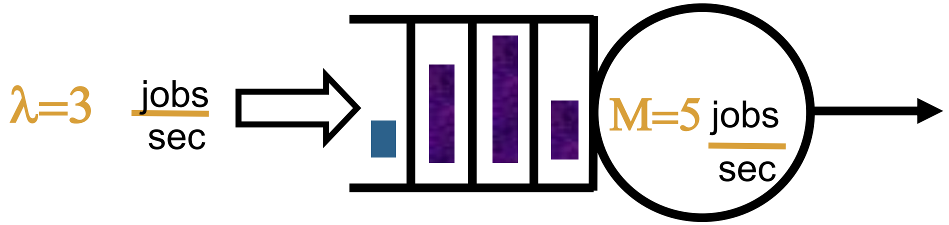
Definition

Definition 12.2 A ***Continuous-Time Markov Chain (CTMC)*** is a continuous-time stochastic process $\{X(t), t \geq 0\}$ s.t., $\forall s, t \geq 0$ and $\forall i, j, x(u)$,

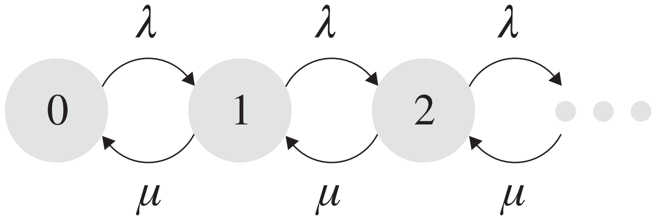
$$\begin{aligned}
&\mathbf{P}\{X(t+s) = j \mid X(s) = i, X(u) = x(u), 0 \leq u \leq s\} \\
&\quad = \mathbf{P}\{X(t+s) = j \mid X(s) = i\} \quad \text{(by M.P.)} \\
&\quad = \mathbf{P}\{X(t) = j \mid X(0) = i\} = P_{ij}(t) \quad \text{(stationarity).}
\end{aligned}$$

It's easier to build CTMC than DTMC! Why?

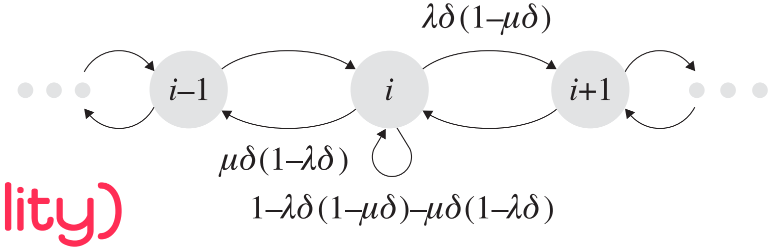
Build a CTMC model



- States: only track the state change



What is the difference against DTMC?

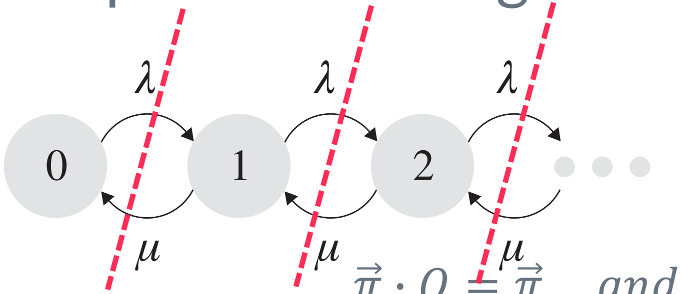


- Rate generator (Not transition probability)

$$Q = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -\lambda-\mu & \mu & & & \\ & & \mu & -\lambda-\mu & \lambda & \\ & & \ddots & & & \\ & & & & \mu & -\mu \end{bmatrix}$$

Solving CTMC

- Use balance equations through rate generator Q



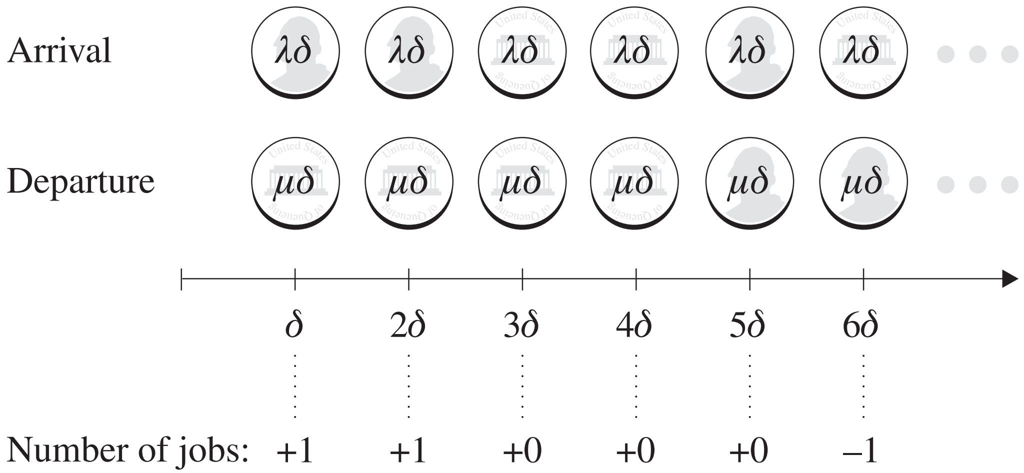
$$\vec{\pi} \cdot Q = \vec{\pi} \quad and \quad \sum_{i=0} \pi_i = 1$$

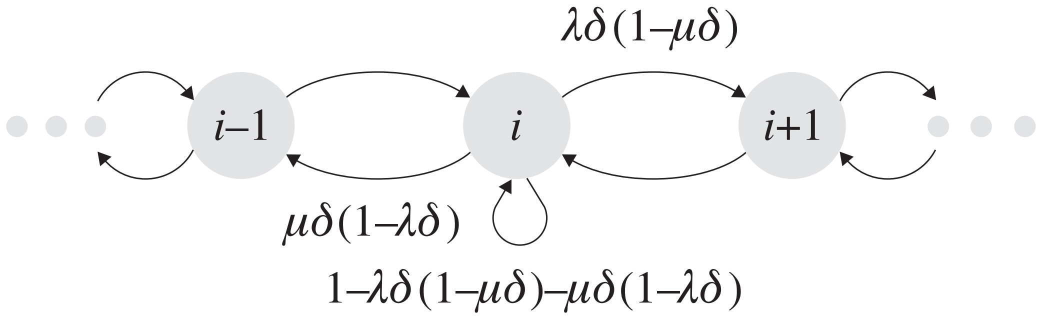
Any physical meaning?

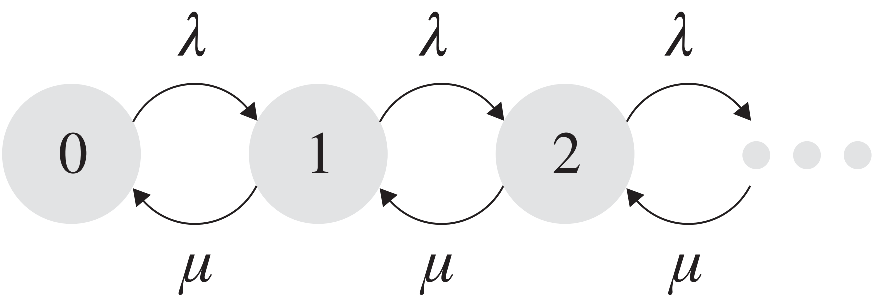
- CTMC and DTMC can be equivalent through uniformification, but
  - For DTMC, balance equations are equivalent to stationary equations
  - For CTMC, balance equations yield the limiting probabilities, while the stationary equations are meaningless

DTMC v.s. CTMC by Simple Example

- Discrete time step of δ







More Example of CTMC

- Machine Failure

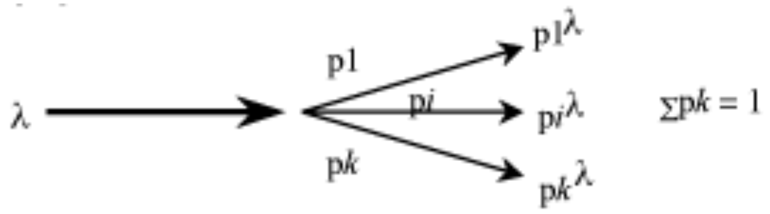
Poisson Process

Poisson Process

Poisson Processes

- Interarrival time *s* = IID and exponential
  $\Rightarrow$ number of arrivals *n* over a given interval *(t, t+x)* has a Poisson distribution
  $\Rightarrow$ arrival = Poisson process or Poisson stream
- Sum of Poisson processes with $\lambda_i \; \forall \; i$ is still Poisson process with $\lambda = \sum_i \lambda_{i_i}$

  Why? How to prove?
  Think of exponential distribution!
- Probabilistic splitting of Poisson process is also Poisson



Relationship Among Stochastic Processes

Bonus quiz

Next lecture:

Queueing Theories (a lot of CTMC)

Thanks!

Any questions?

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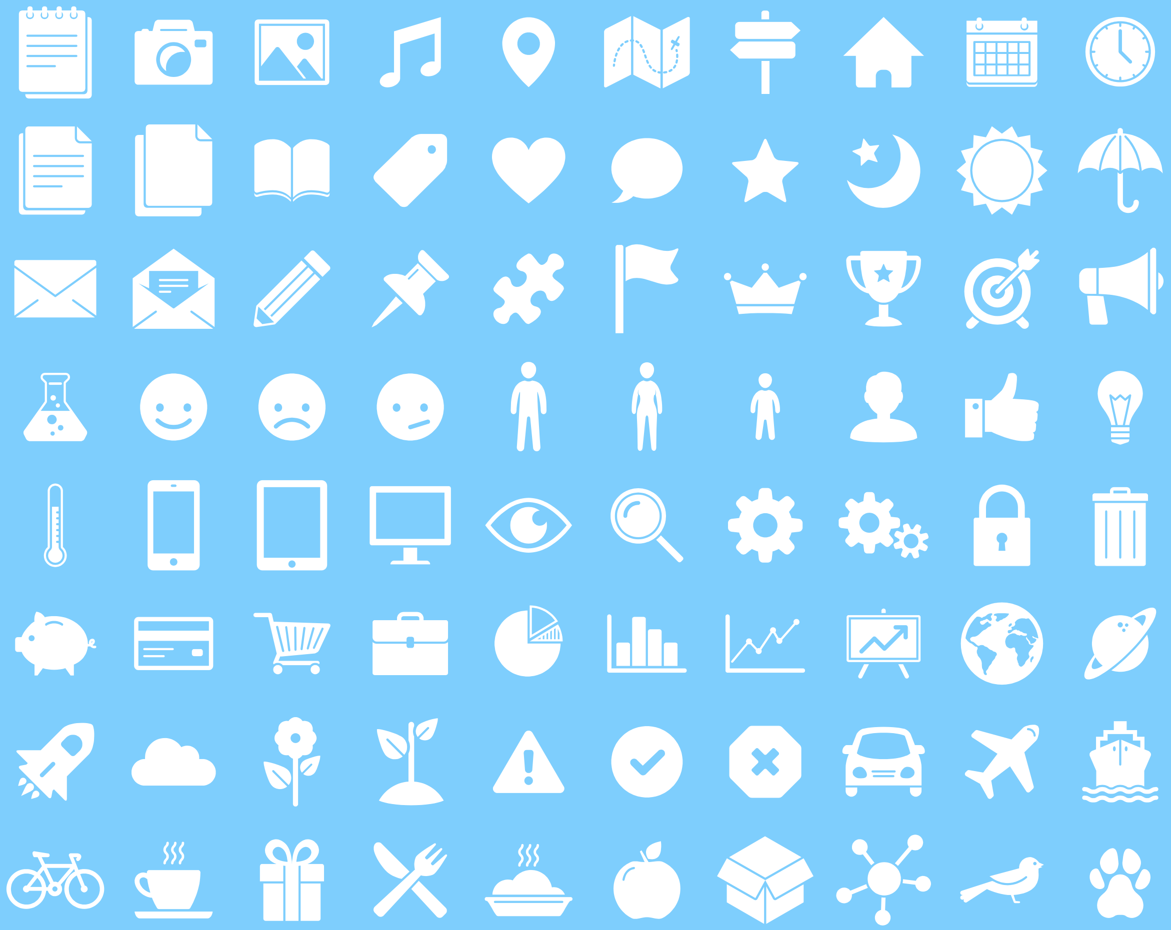

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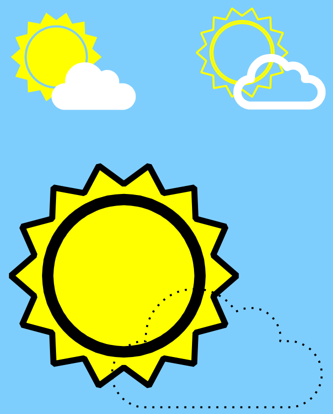

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