

2T2: The Discrete Fourier Transform (2 of 2)

Xavier Serra

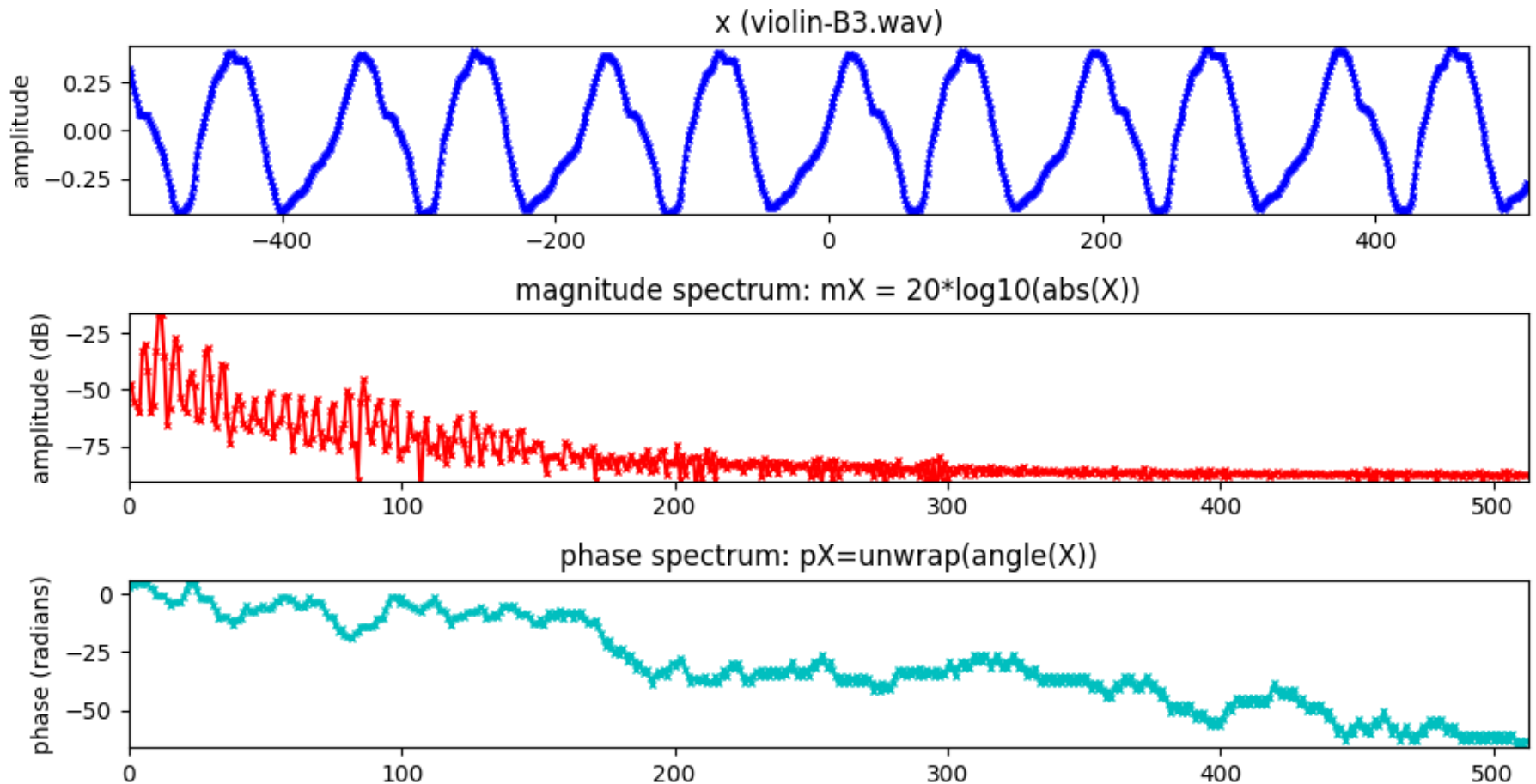
Universitat Pompeu Fabra, Barcelona

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Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \quad k=0, \dots, N-1$$



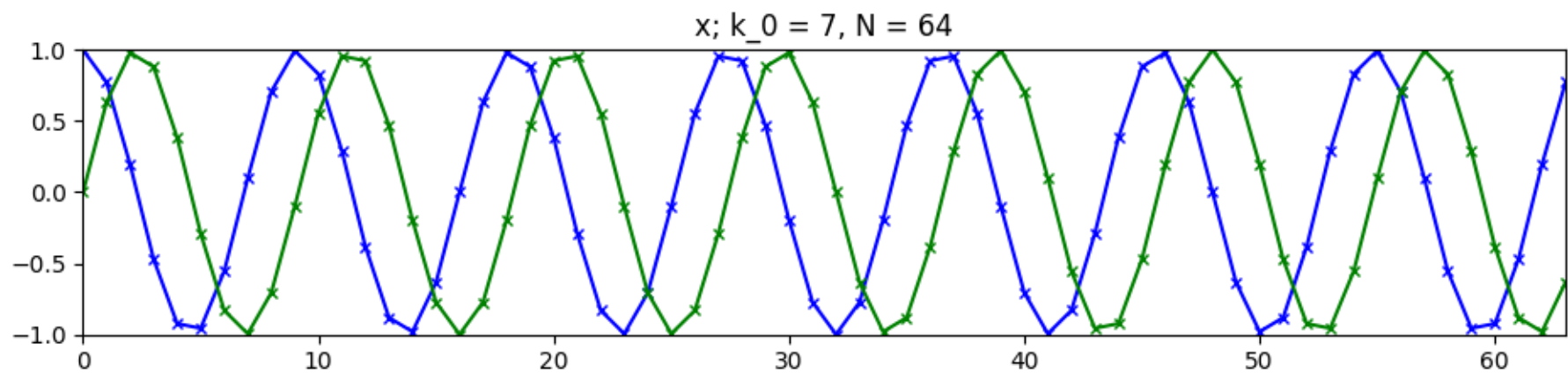
DFT of complex sinusoid

$$x_1[n] = e^{j2\pi k_0 n/N} \quad \text{for } n=0, \dots, N-1$$

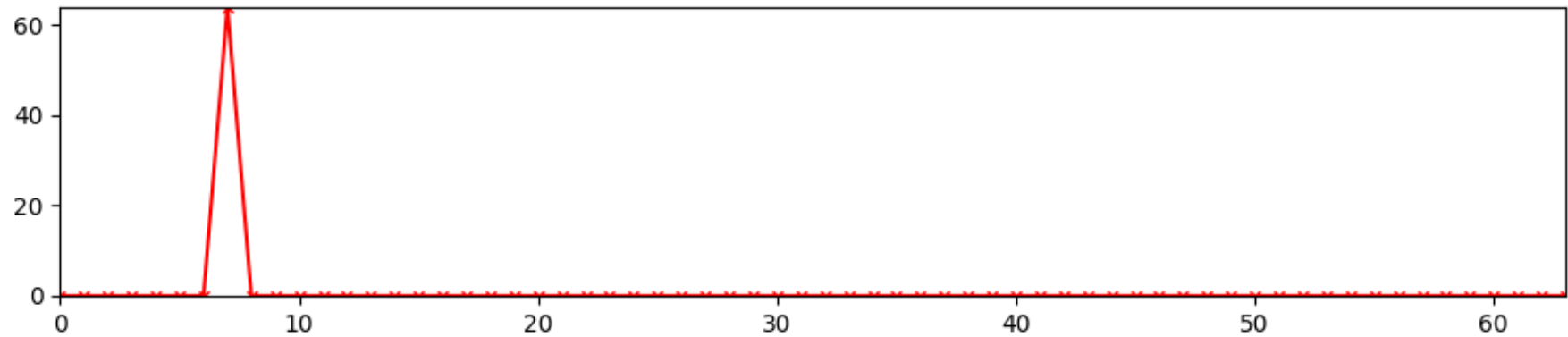
$$\begin{aligned} X_1[k] &= \sum_{n=0}^{N-1} x_1[n] e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N} e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} e^{-j2\pi(k-k_0)n/N} \\ &= \frac{1 - e^{-j2\pi(k-k_0)}}{1 - e^{-j2\pi(k-k_0)/N}} \quad (\text{sum of a geometric series}) \end{aligned}$$

if $k \neq k_0$, denominator $\neq 0$ and numerator $= 0$

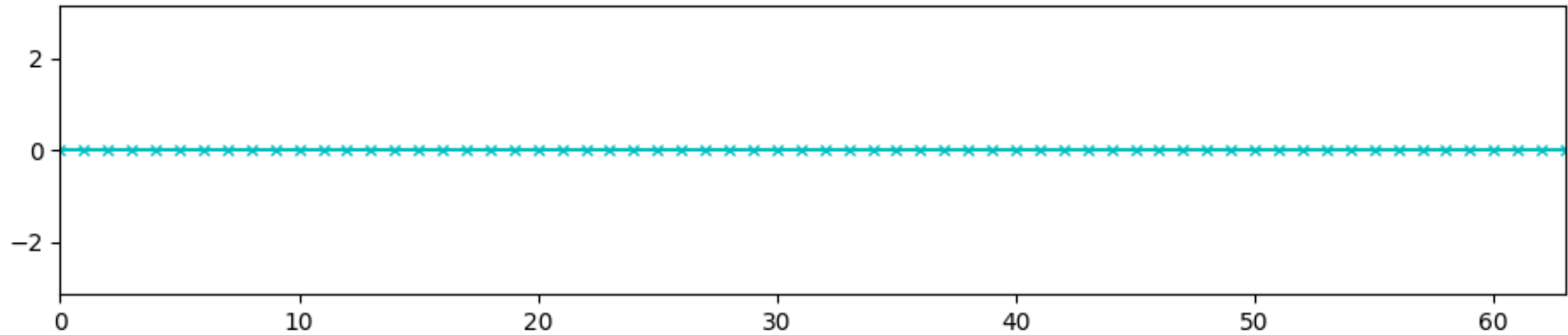
thus $X_1[k] = N$ for $k = k_0$ and $X_1[k] = 0$ for $k \neq k_0$



magnitude spectrum: $\text{abs}(X)$



phase spectrum: $\text{angle}(X)$

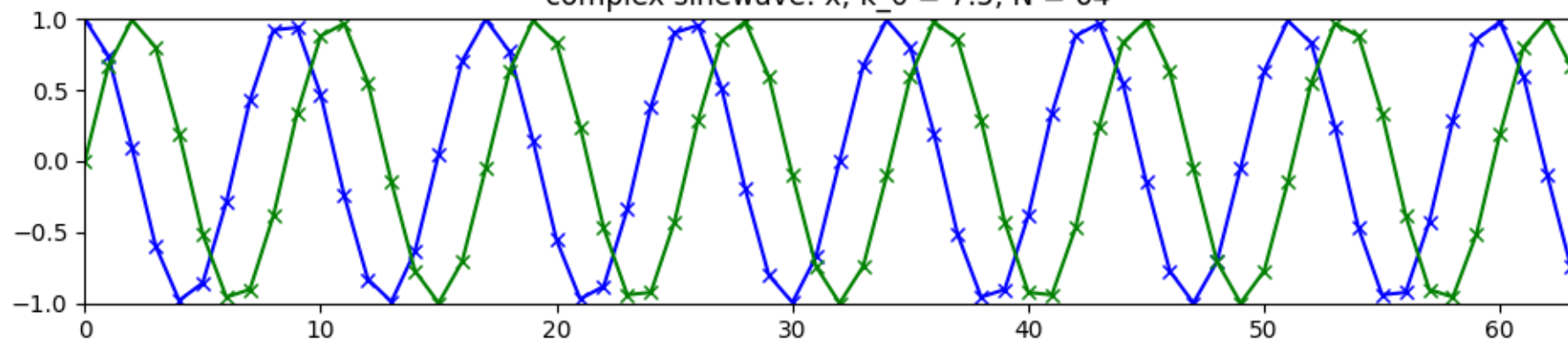


DFT of any complex sinusoid

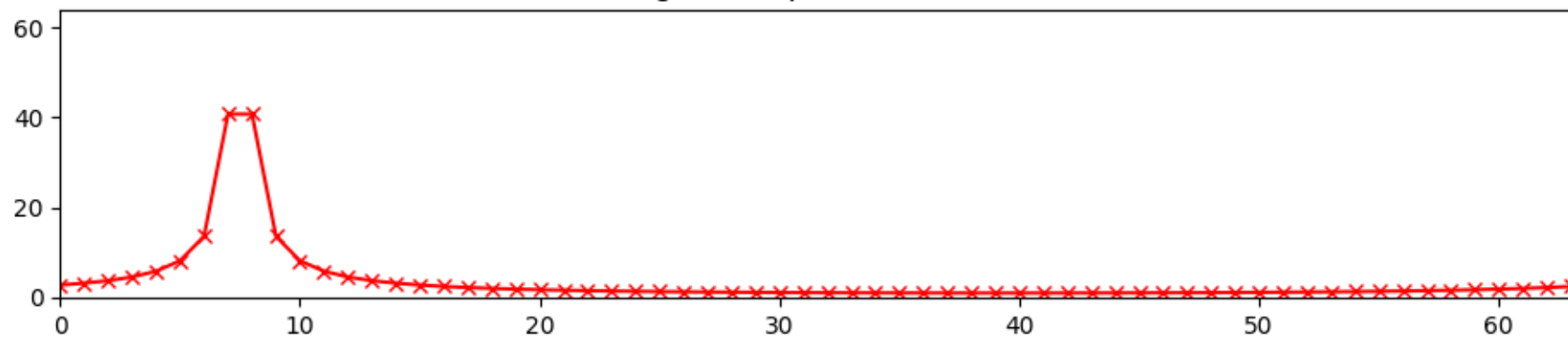
$$x_2[n] = e^{j(2\pi f_0 n + \phi)} \quad \text{for } n = 0, \dots, N-1$$

$$\begin{aligned} X_2[k] &= \sum_{n=0}^{N-1} x_2[n] e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} e^{j(2\pi f_0 n + \phi)} e^{-j2\pi kn/N} \\ &= e^{j\phi} \sum_{n=0}^{N-1} e^{-j2\pi(k/N - f_0)n} \\ &= e^{j\phi} \frac{1 - e^{-j2\pi(k/N - f_0)N}}{1 - e^{-j2\pi(k/N - f_0)}} \end{aligned}$$

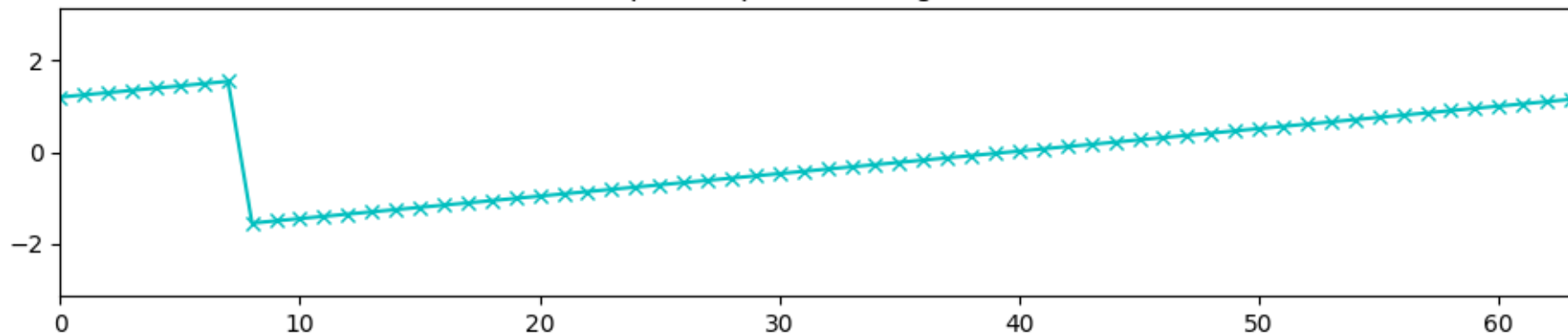
complex sinewave: x ; $k_0 = 7.5$, $N = 64$



magnitude spectrum: $\text{abs}(X)$



phase spectrum: $\text{angle}(X)$



DFT of real sinusoids

$$x_3[n] = A_0 \cos(2\pi k_0 n/N) = \frac{A_0}{2} e^{j2\pi k_0 n/N} + \frac{A_0}{2} e^{-j2\pi k_0 n/N}$$

$$X_3[k] = \sum_{n=-N/2}^{N/2-1} x_3[n] e^{-j2\pi kn/N}$$

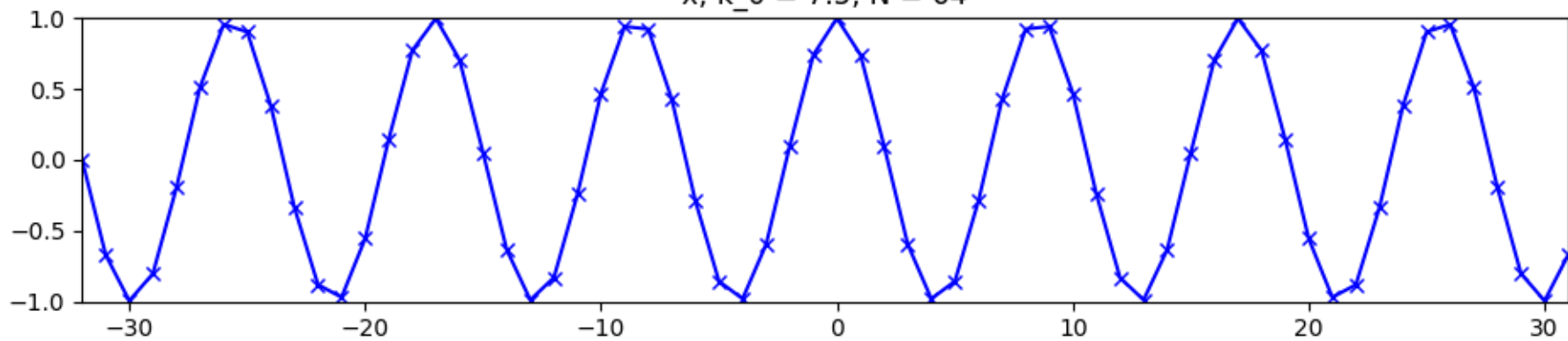
$$= \sum_{n=-N/2}^{N/2-1} \left(\frac{A_0}{2} e^{j2\pi k_0 n/N} + \frac{A_0}{2} e^{-j2\pi k_0 n/N} \right) e^{-j2\pi kn/N}$$

$$= \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{j2\pi k_0 n/N} e^{-j2\pi kn/N} + \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi k_0 n/N} e^{-j2\pi kn/N}$$

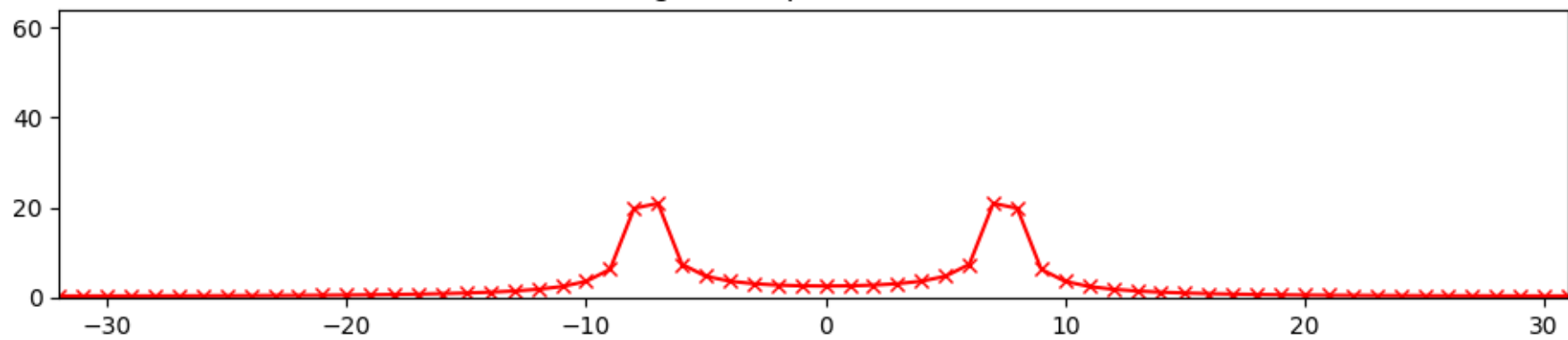
$$= \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi(k-k_0)n/N} + \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi(k+k_0)n/N}$$

$$= N \frac{A_0}{2} \text{ for } k = k_0, -k_0; 0 \text{ for rest of } k$$

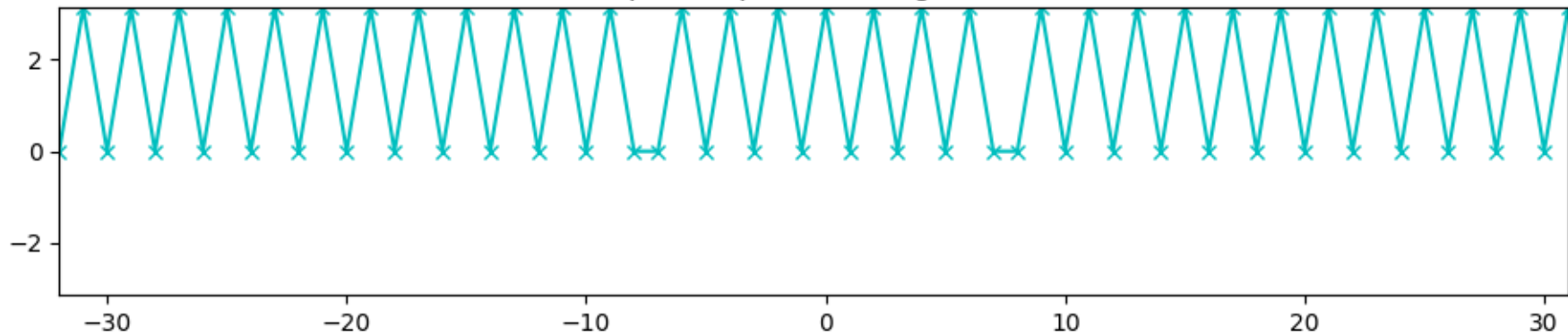
$x; k_0 = 7.5, N = 64$



magnitude spectrum: $\text{abs}(X)$



phase spectrum: $\text{angle}(X)$



Inverse DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] s_k[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \quad n=0,1,\dots,N-1$$

Example:

$$X[k] = [0, 4, 0, 0]; N = 4$$

$$x[0] = \frac{1}{4} (X * s)[n=0] = \frac{1}{4} (0*1 + 4*1 + 0*1 + 0*1) = 1$$

$$x[1] = \frac{1}{4} (X * s)[n=1] = \frac{1}{4} (0*1 + 4*j + 0*(-1) + 0*(-j)) = j$$

$$x[2] = \frac{1}{4} (X * s)[n=2] = \frac{1}{4} (0*1 + 4*(-1) + 0*1 + 0*(-1)) = -1$$

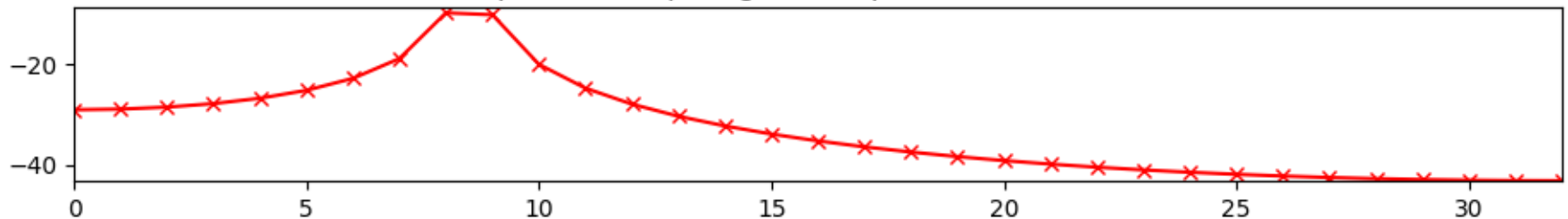
$$x[3] = \frac{1}{4} (X * s)[n=3] = \frac{1}{4} (0*1 + 4*(-j) + 0*(-1) + 0*j) = -j$$

Inverse DFT for real signals

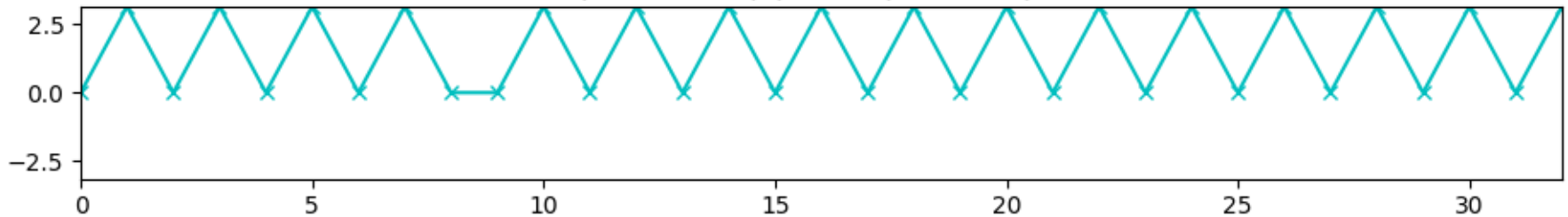
$$X[k] = |X[k]| e^{j\angle X[k]} \quad \text{and} \quad X[-k] = |X[k]| e^{-j\angle X[k]}$$

for $k=0,1,\dots,N/2$

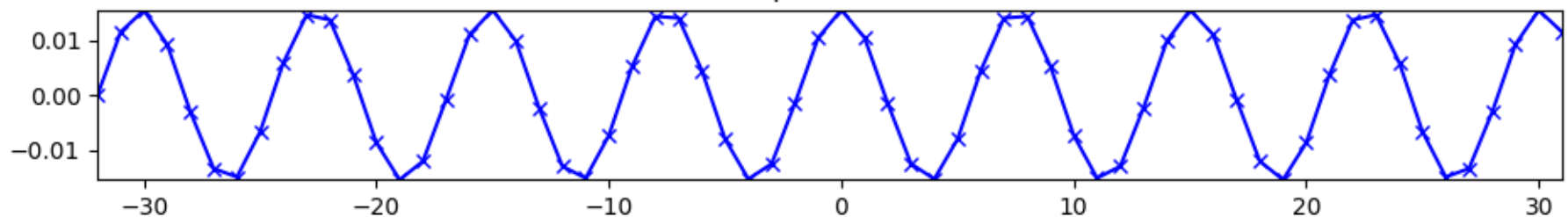
positive freq. magnitude spectrum in dB: mX



positive freq. phase spectrum: pX



inverse spectrum: IDFT(X)



References and credits

- More information in:
https://en.wikipedia.org/wiki/Discrete_Fourier_transform
- Reference on the DFT by Julius O. Smith: <https://ccrma.stanford.edu/~jos/mdft/>
- Sounds from: <http://www.freesound.org/people/xserra/packs/13038/>
- Slides released under CC Attribution-Noncommercial-Share Alike license and code under Affero GPL license; available from <https://github.com/MTG/sms-tools>

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