SI 211: Numerical Analysis

Homework 2

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1. Solution:

Given f (0) = 1, f (1) = 3, and f (2) = 19, so the three points are:

$$\begin{cases} x_0 = 0 \\ y_0 = 1 \end{cases}, \begin{cases} x_1 = 1 \\ y_1 = 3 \end{cases}, \begin{cases} x_2 = 2 \\ y_2 = 19 \end{cases}$$

Construct a lagrange polynomial to pass through these three points. The corresponding lagrange polynomials are:

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} = \frac{(x-1)(x-2)}{2}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = -x(x-2)$$

$$L_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{x(x-1)}{2}$$

Thus, $p(x) = y_0 L_0(X) + y_1 L_1(X) + y_2 L_2(X)$

$$= \frac{(x-1)(x-2)}{2} - 3x(x-2) + \frac{19x(x-1)}{2}$$

$$= 1 - 5x + 7x^2$$

The form p(x) = $a_0 + a_1x + a_2x^2$ such that p interpolates f at x ∈ {0, 1, 2}. So, the results are $a_0 = 1, a_1 = -5, a_2 = 7$.

2. Solution:

Given f (0, 0) = 1, f (0, 1) = 3, f (0, 2) = 19, f (1, 0) = 3, f (2, 0) = 19, f (1, 1) = 0, so the six points are:

$$\begin{cases} x_{1,0} = 0 \\ x_{2,0} = 0 \\ y_0 = 1 \end{cases}, \begin{cases} x_{1,0} = 0 \\ x_{2,1} = 1 \\ y_1 = 3 \end{cases}, \begin{cases} x_{1,0} = 0 \\ x_{2,2} = 2 \\ y_2 = 19 \end{cases}, \begin{cases} x_{1,1} = 1 \\ x_{2,0} = 0 \\ y_3 = 3 \end{cases}, \begin{cases} x_{1,2} = 2 \\ x_{2,0} = 0 \\ y_4 = 19 \end{cases}, \begin{cases} x_{1,1} = 1 \\ x_{2,1} = 1 \\ y_5 = 0 \end{cases}$$

Use the Newton's polynomials:

$$p(x) = \sum_{i=0}^{n} b_i N_i(x_2) \left(b_i = p_i(x_1), N_i(x_2) = \prod_{j=0}^{i-1} (x_2 - x_{2,j}) \right)$$

$$= p_0(x_1) + p_1(x_1)(x_2 - x_{2,0}) + p_2(x_1)(x_2 - x_{2,0})(x_2 - x_{2,1})$$

a)When $x_2 = 0$, we have three points $\begin{cases} x_{1,0} = 0 \\ x_{2,0} = 0 \\ y_0 = 1 \end{cases}, \begin{cases} x_{1,1} = 1 \\ x_{2,0} = 0 \\ y_3 = 3 \end{cases}, \begin{cases} x_{1,2} = 2 \\ x_{2,0} = 0 \\ y_4 = 19 \end{cases}.$

$$p(x) = p_0(x_1) + p_1(x_1)(x_2 - x_{2,0}) + p_2(x_1)(x_2 - x_{2,0})(x_2 - x_{2,1})$$

$$= p_0(x_1) + p_1(x_1)(0-0) + p_2(x_1)(0-0)(0-1)$$

$$= p_0(x_1)$$

So, we can interpolate the $p_0(x_1)$ at (0,1), (1,3), (2,19). It's easy to use the divided differences to get that:

$$p_0(x_1) = 1 - 5x_1 + 7x_1^{\ 2}$$

b) When $x_2 = 1$, we have two points $\begin{cases} x_{1,0} = 0 \\ x_{2,1} = 1 \\ y_1 = 3 \end{cases}$, $\begin{cases} x_{1,1} = 1 \\ x_{2,1} = 1 \\ y_5 = 0 \end{cases}$.

$$p(x) = 1 - 5x_1 + 7x_1^{\ 2} + p_1(x_1)(x_2 - x_{2,0}) + p_2(x_1)(x_2 - x_{2,0})(x_2 - x_{2,1})$$

$$\begin{aligned} &= 1 - 5x_1 + 7x_1^{\ 2} + p_1(x_1)(1-0) + p_2(x_1)(1-0)(1-1) \\ &= 1 - 5x_1 + 7x_1^{\ 2} + p_1(x_1) \end{aligned}$$

Due to $p(x) \neq p_1(x_1)$, we need to calculate the $p_1(x_1)$ at $x_1 = 0,1$.

$$\begin{cases} 1 - 5\times 0 + 7 \times 0^2 + p_1(0) = 3 \\ 1 - 5 \times 1 + 7 \times 1^2 + p_1(1) = 0 \end{cases} \Rightarrow \begin{cases} p_1(0) = 2 \\ p_1(1) = -3 \end{cases}$$

So, we can interpolate the $p_1(x_1)$ at (0,0), (1, -3). It's easy to get that:

$$p_1(x_1) = 2 - 5x_1$$

c) When $x_2 = 2$, we have one point $\begin{cases} x_{1,0} = 0 \\ x_{2,2} = 2 \\ y_2 = 19 \end{cases}$.

$$p(x) = 1 - 5x_1 + 7x_1^{\ 2} + (2 - 5x_1)(x_2 - x_{2,0}) + p_2(x_1)(x_2 - x_{2,0})(x_2 - x_{2,1})$$

$$\begin{aligned} &= 1 - 5x_1 + 7x_1^{\ 2} + (2 - 5x_1)(2-0) + p_2(x_1)(2-0)(2-1) \\ &= 5 - 15x_1 + 7x_1^{\ 2} + 2p_2(x_1) \end{aligned}$$

In a similar way as above, we can calculate the $p_2(x_1)$ at $x_1 = 0$.

$$5 - 15 \times 0 + 7 \times 0^2 + 2p_2(x_1) = 19 \Rightarrow p_2(0) = 7$$

So, we can interpolate the $p_2(x_1) = 7$.

Finally, we get the $p(x) = p_0(x_1) + p_1(x_1)(x_2 - x_{2,0}) + p_2(x_1)(x_2 - x_{2,0})(x_2 - x_{2,1})$

$$\begin{aligned} &= 1 - 5x_1 + 7x_1^{\ 2} + (2 - 5x_1)(x_2 - 0) + 7(x_2 - 0)(x_2 - 1) \\ &= 1 - 5x_1 + 7x_1^{\ 2} - 5x_2 + 7x_2^{\ 2} - 5x_1x_2 \end{aligned}$$

Thus, $a_0 = 1, a_1 = -5, a_2 = 7, a_3 = -5, a_4 = 7, a_5 = -5.$

3. Solution:

(a) The plot of the function $f(x) = \sin(x)$ and its interpolating polynomial are shown in Figure 1.

For the function $f(x) = \sin(x)$ all derivatives are uniformly bounded by 1 on the interval $[\bar{x}, \underline{x}]$. Thus, we have

$$\left|f(x)-p(x)\right| \leq \frac{1}{(n+1)!}\prod_{j=1}^{n}(x-x_j) \leq \frac{1}{(n+1)!}\left[\overline{x}-\underline{x}\right]^{n} = \frac{1}{(10+1)!}\left[5-(-5)\right]^{11} = 2.50521083854\times 10^{3}$$

So the approximation error is $2.50521083854\times 10^{3}$ and we also can see the error in Figure 2 which is approximately equal to 6.5×10^{-3}.

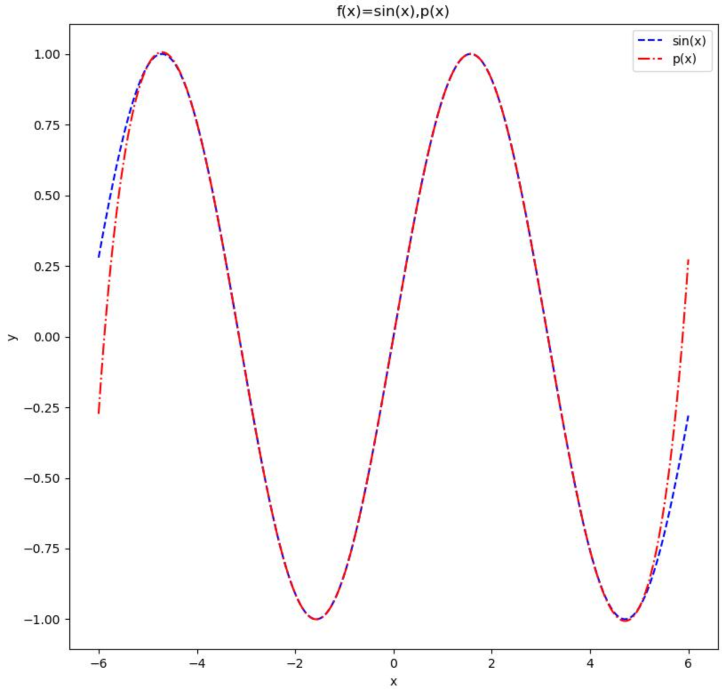

Figure 1

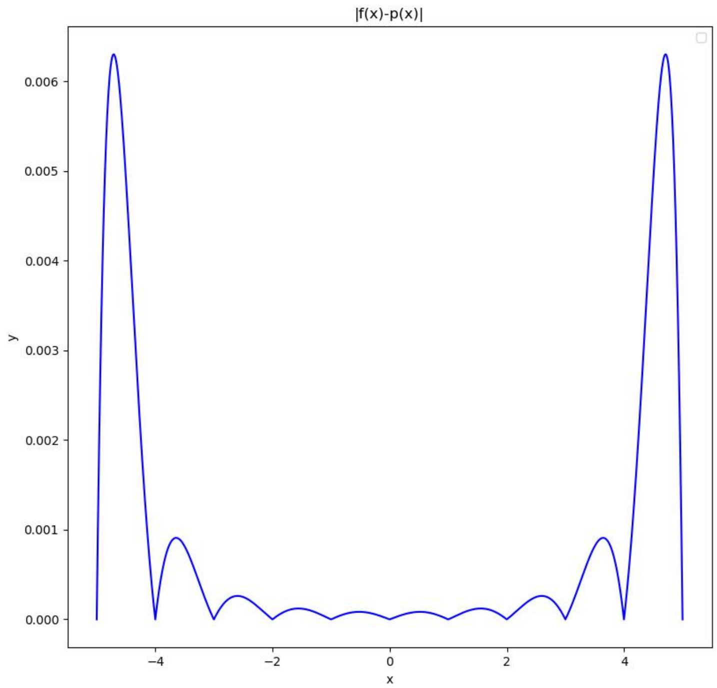

Figure 2

(b) The plot of the function $g(x)=\dfrac{1}{1+x^2}$ and its interpolating polynomial are shown in Figure 3. The approximation error is 1.9 as shown in Figure 4.

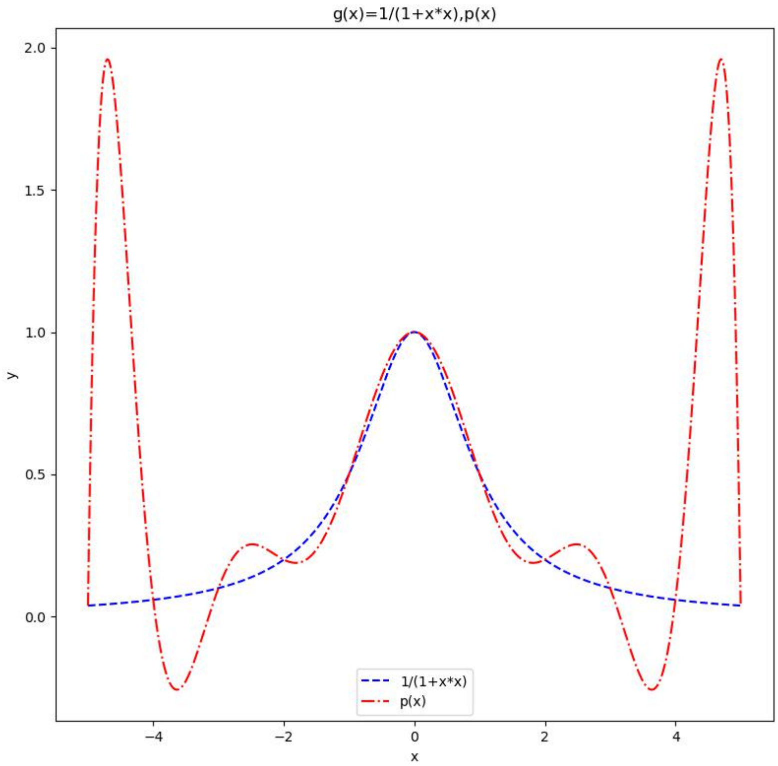

Figure 3

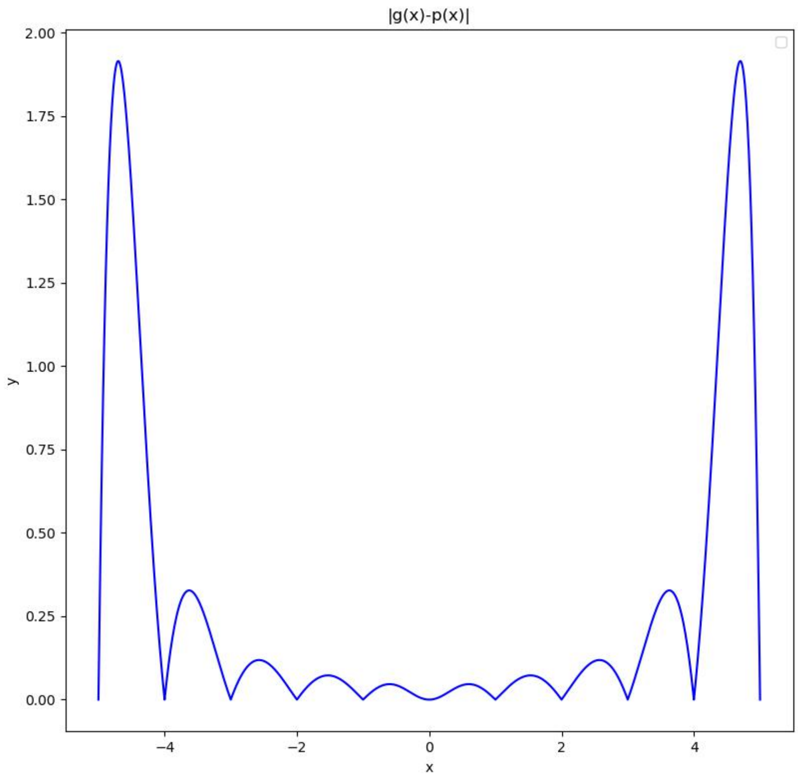

Figure 4

Here is the code of Julia.

```
julia> mutable struct DDTable
       x
       d
       end

     function addData!(T,x,y)
       n = length(T.x);
       append!(T.x,x);
       append!(T.d,y);
       for i=1:n
         nom = T.d[end]-T.d[end-n];
         den = T.x[end]-T.x[end-i];
         append!(T.d,nom/den);
         end
         end

     T = DDTable(zeros(0),zeros(0));

     function getNewtonCoefficients(T)
       n = length(T.x);
       c = zeros(0);
       k = 0;
       for i=1:n
       k+=i;
       append!(c,T.d[k]);
       end
       return c;
       end

     function getPolynomial(T)
       n = length(T.x);
       c = getNewtonCoefficients(T);
       return function p(x)
         b = c[end];
         for i=1:n-1
           b = c[n-i]+b*(x-T.x[n-i]);
         end
         return b;
         end
         end

     p = getPolynomial(T)
```

```
        function interpolate(range,f)
           table = DDTable(zeros(0),zeros(0));
           for i=1:length(range)
             addData!(table,range[i],f(range[i]));
           end
           return getPolynomial(table);
           end

        function f(x)
           return sin(x);
           end

  range = [-5,-4,-3,-2,-1,0,1,2,3,4,5]
  p = interpolate(range,f);

  x=(-6:0.01:6);
  y1=zeros(0);
  y2=zeros(0);
  for i=1:length(x)
           append!(y1,f(x[i]));
           append!(y2,p(x[i]));
           end

  using PyPlot
  plot(x,y1,"b--")
  plot(x,y2,"r-.")

x=(-5:0.01:5);
  y1=zeros(0);
  y2=zeros(0);
  y=zeros(0);
  for i=1:length(x)
           append!(y1,f(x[i]));
           append!(y2,p(x[i]));
           append!(y,abs(f(x[i])-p(x[i])));
           end

  using PyPlot
  plot(x,y,"b")

        function g(x)
           return 1/(1+x*x);
           end
```

```
range = [-5,-4,-3,-2,-1,0,1,2,3,4,5]
p = interpolate(range,g);

x=(-5:0.01:5);
y1=zeros(0);
y2=zeros(0);
y=zeros(0);
for i=1:length(x)
        append!(y1,g(x[i]));
        append!(y2,p(x[i]));
        append!(y,abs(g(x[i])-p(x[i])));
        end

using PyPlot
plot(x,y1,"b--")
plot(x,y2,"r-.")
plot(x,y,"b")
```