SI 211: Numerical Analysis
Homework 2

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Deadline: October 21, 2019

1. Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 1$, $f(1) = 3$, and $f(2) = 19$. Construct a polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2$ such that p interpolates f at $x \in \{0, 1, 2\}$. What are a_0, a_1, a_2?

2. Assume that a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$f(0,0) = 1\,, \quad f(0,1) = 3\,, \quad f(0,2) = 19\,, \quad f(1,0) = 3\,, \quad f(2,0) = 19\,, \quad f(1,1) = 0$$

   Construct a polynomial $p : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$p(x) = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_2 + a_4 x_2^2 + a_5 x_1 x_2$$

   such that p interpolates f at all 6 points. What are $a_0, a_1, a_2, a_3, a_4, a_5$?

3. Implement a computer program that interpolates a function $f(x)$ at the points

$$x_1 = -5,\ x_2 = -4,\ x_3 = -3,\ \ldots,\ x_{10} = 4,\ x_{11} = 5$$

   with a polynomial p of order 10. Test your program for

   (a) the function $f(x) = \sin(x)$ and

   (b) the function $f(x) = \frac{1}{1+x^2}$.

   Plot the functions as well as their interpolating polynomials. How big are the approximation errors?

4. Write a computer code in JULIA, Matlab, Python, or C++, which returns a natural spline that intepolates the function $f : [x_0, x_N] \to \mathbb{R}$ at the equidistant points

$$x_i = x_0 + hi \qquad \text{with} \qquad h = \frac{x_N - x_0}{N}\ .$$

5. Use your computer code from the first exercise in order to compute a natural spline of the function

$$f(x) = \frac{1}{1+x^2}$$

   on the interval $[x_0, x_N] = [-5, 5]$. You may set $N = 10$. Plot the function f as well as the natural spline that interpolates f.

6. Use your compute code to compute a natural spline of the function

$$f(x) = x^2$$

on the interval $[x_0, x_N] = [0, 1]$ with $N = 10$. What is the exact value for the integral

$$\int_0^1 \left[f''(x)\right]^2 \, \mathrm{d}x \ = ?$$

Also compute the value

$$\int_0^1 \left[p''(x)\right]^2 \, \mathrm{d}x \ = ?$$

for the interpolating spline. Explain how you compute this integral numerically. Which value is bigger, $\int_0^1 \left[f''(x)\right]^2 \, \mathrm{d}x$ or $\int_0^1 \left[p''(x)\right]^2 \, \mathrm{d}x$?