SI 211: Numerical Analysis

Homework 4

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1. Solution:

a) According to the Gram-Schmidt algorithm for function on the interval [-1,1],

$$q_0(x)=\sqrt{\frac{1}{2}},\ q_1(x)=\sqrt{\frac{3}{2}}x,\ q_2(x)=\sqrt{\frac{5}{8}}(3x^2-1)$$

Then, we use the interval transform function as below.

$$y=a+\frac{x+1}{2}(b-a)\Rightarrow x=\frac{2(y-a)}{b-a}-1,\frac{dx}{dy}=\frac{2}{b-a}$$

$$\Rightarrow \hat{q}_i(y)=q_i(x)\cdot\sqrt{\frac{dx}{dy}}=q_i\left[\frac{2(y-a)}{b-a}-1\right]\cdot\sqrt{\frac{2}{b-a}}=\sqrt{2}\cdot q_i(2y-1)$$

(Here, $a=0,b=1$.)

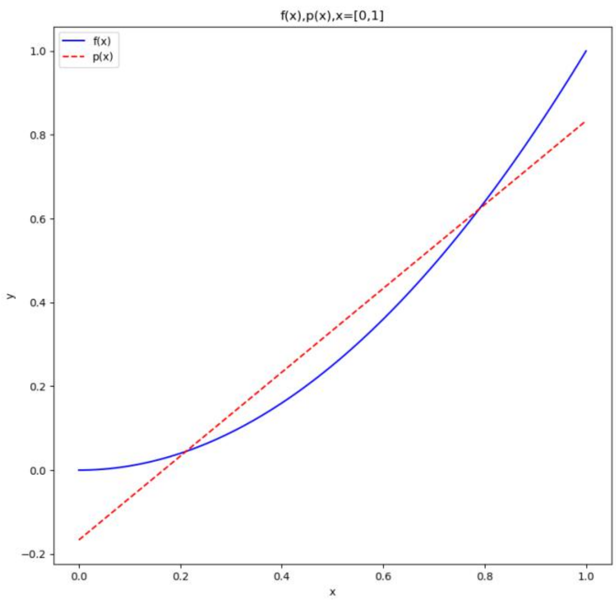
Thus, we can construct the orthogonal polynomials $q_0,q_1\in P_1$.

$$q_0(x)=\sqrt{2}\cdot\sqrt{\frac{1}{2}}=1,q_1(x)=\sqrt{2}\cdot\sqrt{\frac{3}{2}}(2x-1)=\sqrt{3}(2x-1)$$

$$C_0=\langle f,q_0\rangle=\int_0^1 x^2\cdot\sqrt{\frac{3}{2}}dx=\frac{1}{3},C_1=\langle f,q_1\rangle=\int_0^1 x^2\cdot\sqrt{3}\cdot(2x-1)dx=\frac{\sqrt{3}}{6}$$

$$\Rightarrow p(x)=C_0q_0(x)+C_1q_1(x)=\frac{1}{3}+\frac{1}{2}(2x-1)=-\frac{1}{6}+x$$

We can use the "PyPlot" to plot the function $f(x)=x^2$ and its polynomial $p(x)$ which are shown in 1.a. Figure 1.



1.a. Figure 1 (The code is named hw4_zjh_1a.)

b) In a similar way as above, $q_0(x)=1, q_1(x)=\sqrt{3}(2x-1), q_2(x)=6\sqrt{5}(x^2-x+\frac{1}{6})$

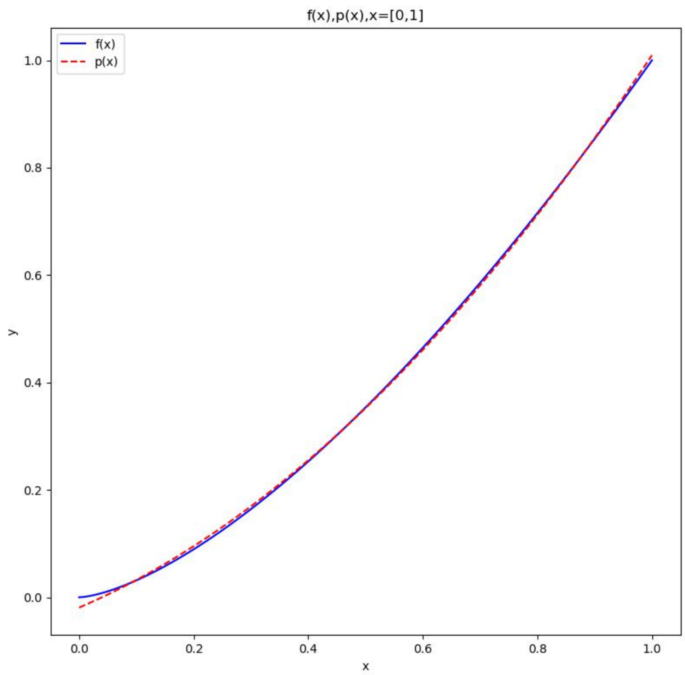
$$C_0=\langle f,q_0\rangle=\int_0^1 x^{\frac{3}{2}}\cdot\sqrt{\frac{3}{2}}dx=\frac{2}{5},$$

$$C_1=\langle f,q_1\rangle=\int_0^1 x^{\frac{3}{2}}\cdot\sqrt{3}(2x-1)dx=\frac{6\sqrt{3}}{35},$$

$$C_2=\langle f,q_2\rangle=\int_0^1 x^{\frac{3}{2}}\cdot 6\sqrt{5}(x^2-x+\frac{1}{6})dx=\frac{2\sqrt{5}}{105},$$

$$\Rightarrow p(x)=C_0q_0(x)+C_1q_1(x)=\frac{4}{7}x^2+\frac{16}{35}x-\frac{2}{105}$$

Then, we can use the "PyPlot" to plot the function $f(x)=x^{\frac{3}{2}}$ and its polynomial $p(x)$ which are shown in 1.b. Figure 2.



1.b. Figure 2 (The code is named hw4_zjh_1b.)

b) We use the interval transform function again.

$$\hat{q}_i(y)=q_i(x)\cdot\sqrt{\frac{dx}{dy}}=q_i\left[\frac{2(y-a)}{b-a}-1\right]\cdot\sqrt{\frac{2}{b-a}}=\sqrt{\frac{1}{5}}\cdot q_i(\frac{1}{5}y)$$

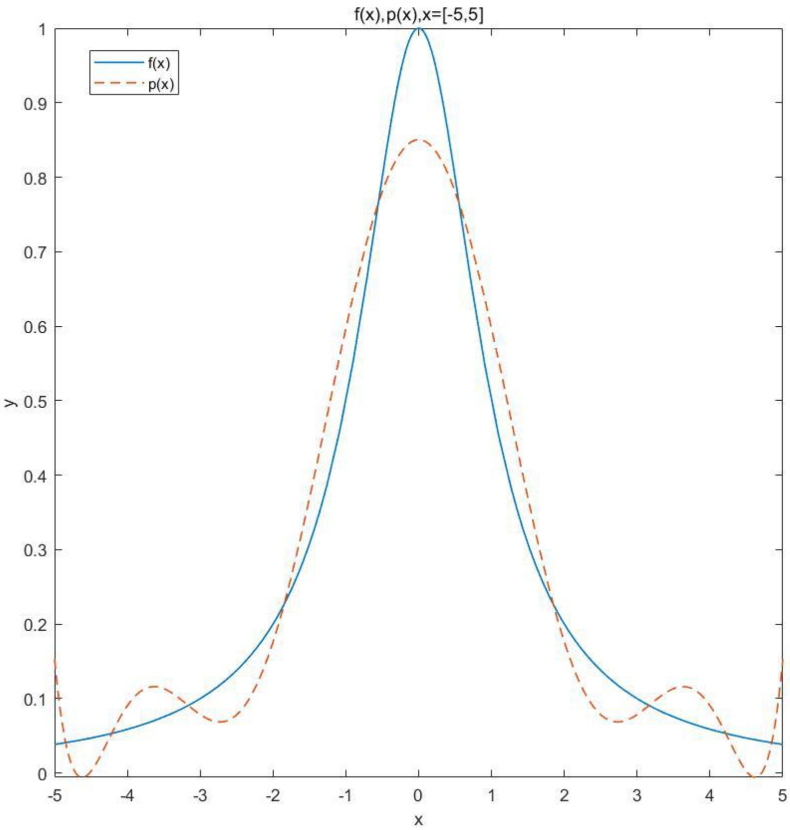
(Here, $a=-5, b=5$.)

The Legendre polynomials is $L_n(x)=\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}(x^2-1)^n \Rightarrow L_n(y)=\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}(\frac{y}{25}^2-1)^n$,

$$q_i(y)=\sqrt{\frac{2n+1}{b-a}}L_n(y), c_i=\langle f,q_i\rangle=\int_a^b f(y)q_i(y)dy$$

Thus, we can use the Matlab to compute the $p(y)$ and plot the function $f(y)=\dfrac{1}{1+y^2}$ and its polynomial $p(y)$ which are shown in 1.c. Figure 3.

$p(y)$ =atan(5)/5-((1949411013750059779*atan(5))/1407374883553280000-1350066230468774987/844424930131968000)*((144418883051629*y^2*(y^2/25-1)^2)/562949953421312+(4813962768387633*y^4*(y^2/25-1))/3518437208883200000+(205395744784539*y^6)/2814749767106560000+(1604654256129211*(y^2/25-1)^3)/4503599627370496)+((1775753551343302 11*atan(5))/1759218604441600000-98801325413085983/105553116266496000)*((6408734621389361*y^2*(y^2/25-1))/56294995342131200+(4272489747592907*y^4)/2814749767106560000+(6408734621389361*(y^2/25-1)^2)/18014398509481984)+((8715608092114314820 89*atan(5))/439804651110400000000-11625052680123685800 11/461794883665920000000)*((3211236604058819*y^2*(y^2/25-1)^3)/7036874417766400+(4816854906088229*y^4*(y^2/25-1)^2)/87960930222080000+(5137978566494111*y^6*(y^2/25-1))/4398046511104000000+(2935987752282349*y^8)/879609302220800000000+(3211236604058819*(y^2/25-1)^4)/9007199254740992)-(2^(1/2)*((14*2^(1/2)*atan(5))/25-(3*2^(1/2))/10)*((12*y^2)/25 - 4))/16



1.c. Figure 3 (The code is named hw4_zjh_1c.m.)

2. Solution:

Firstly,we can caculate the integrals as below.

$$\int_0^1 x^3 dx = \frac{1}{4}, \int_0^1 e^x dx = e-1, \int_0^{10} e^x dx = e^{10}-1, \int_0^{\pi} \sin(x)dx = 2$$

According to the Simpson's rule,

$$\int_a^b f(x)dx \approx \frac{H}{3}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right], H=\frac{b-a}{2}$$

Then, we use the Julia to program the numerical approximations of the integrals. (see the code in file hw4_zjh_2.)

a) The numerical integration error is 0.
b) The numerical integration error is $0.5793234175479611\times10^{-3}$.
c) The numerical integration error is $1.5676398257221656\times10^{4}$.
d) The numerical integration error is $0.9439510239319526\times10^{-1}$.

3. Solution:

$$\int_1^{\infty} e^{-x^2}dx = \int_0^{\infty} e^{-x^2}dx - \int_0^1 e^{-x^2}dx \quad , \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \Rightarrow \int_1^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2} - \int_0^1 e^{-x^2}dx$$

According to the Simpson's rule,

$$\int_1^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2} - \int_0^1 e^{-x^2}dx \approx \frac{\sqrt{\pi}}{2} - \frac{1}{6}\left[1+4e^{-\frac{1}{4}}+e^{-1}\right] = 0.13904649654324763$$

When $f(x)=e^{-x^2}$,we can use Simpson's rule to get the numerical integration error.

$$\int_0^1 e^{-x^2}dx - \left(\frac{1}{n}\sum_{i=0}^{n} f(x_i)\partial_i\right) = \int_0^1 f[x0,x1,...,xn,x]\prod_{j=0}^{n}(x-x_j)dx$$

The error of Simpson's formula is bounded by:

$$\int_0^1 f[x0,x1,...,xn,x]\prod_{j=0}^{n}(x-x_j)dx = \int_0^1 f[0,\frac{1}{2},1,x](x-0)(x-\frac{1}{2})(x-1)dx$$

$$= \int_0^1 \frac{f[0,\frac{1}{2},1,x]-f[0,\frac{1}{2},1,\frac{1}{2}]}{x-\frac{1}{2}}\underbrace{(x-0)(x-\frac{1}{2})^2(x-1)dx}_{\le 0} + f[0,\frac{1}{2},1,\frac{1}{2}]\underbrace{\int_0^1 (x-0)(x-\frac{1}{2})(x-1)dx}_{=0}$$

$$\le \frac{\max_{x\in[0,1]}\left|f^{(4)}(x)\right|}{4!}\underbrace{\left|\int_0^1 (x-0)(x-\frac{1}{2})^2(x-1)dx\right|}_{=\frac{1}{120}}$$

Thus, the numerical integration error's bound of $\int_1^{\infty} e^{-x^2}dx$ is as below (see the code in file hw4_zjh_3.m.).

$$\frac{\max\limits_{x\in[0,1]}\left|f^{(4)}(x)\right|}{2880}=\frac{12}{2880}=4.16666667\times 10^{-3}$$