SI 211: Numerical Analysis
Homework 3

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Deadline: Oct 28, 2019

1. Apply the Gram-Schmidt algorithm to construct three linearly independent polynomials p_1, p_2, p_3 with order ≤ 2, which satisfy

$$\int_{-1}^{1} p_1(x)p_2(x)e^x \, \mathrm{d}x = \int_{-1}^{1} p_2(x)p_3(x)e^x \, \mathrm{d}x = \int_{-1}^{1} p_1(x)p_3(x)e^x \, \mathrm{d}x = 0 \, .$$

2. Solve the least-squares optimization problem

$$\min_{p \in P_2} \int_{1}^{2} \left[f(x) - p(x) \right]^2 \, \mathrm{d}x$$

   for $f(x) = e^x$ by using Legendre polynomials. Here, P_2 denotes the set of polynomials of order 2.

3. Prove that the Legendre polynomials

$$P_n = \frac{1}{2^n n!} \frac{\partial^n}{\partial x^n} (x^2 - 1)^n$$

   are orthogonal with respect to the L_2-scalar product on the interval $[-1, 1]$.

4. Implement the closed Newton Cotes formulas for $n = 2, 3, 4$ in JULIA, Matlab, Python, or C++ and compute the integral

$$I = \int_{1}^{2} \frac{1}{x^2} \mathrm{d}x$$

   with all three methods and compare your results with the exact value for the integral.

5. In order to integrate the function $f(x)$ on the interval $[a, b]$, one writes the computer code

```
function integrate(f,a,b,N)
    h = (b-a)/N;
    I = 0.5*f(a);
    for i=1:N-1;
         I = I+f(a+i*h);
    end
    return (I+0.5*f(b))*h;
end
```

In order to test the approximation accuracy of this code we try to integrate the function $f(x) = x^2$ on the interval $[0, 1]$,

```
function f(x)
    return x*x;
end
```

- What would happen if we run the command

  `integrate(f,0.0,1.0,10)` ?

  Explain in words / formulas how the above code works.

- Let $K = \int_0^1 x^2 \, dx$ denote the exact value of the integral. Can you find an error bound of the form

$$\text{abs}(\text{integrate}(f, 0, 1, N) - K) \leq \mathbf{O}(N^{-q}) \quad ?$$

  What is the largest possible convergence order q in this estimate?