Problem Set 0

All parts are due Sunday, September 8 at 6PM.

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Problem 0-1.

(a) $A = \{2^0, 2^1, 2^2, 2^3, 2^4\} = \{1, 2, 4, 8, 16\}$
$B = \{0-1, 2-1, 4-1, 6-1\} = \{-1, 1, 3, 5\}$
$A \cap B = \{1\}$

(b) $|A \cup B| = |A| + |B| - |A \cap B| = 5 + 4 - 1 = 8$

(c) $A - B = \{2, 4, 8, 16\}$
$|A - B| = 4$

Problem 0-2.

(a) Possible outcomes: 2 blue balls, 1 blue ball (2 different ways), 0 blue balls:
$E[X] = 2(3/5)(1/2) + 1(3/5)(1/2) + 1(2/5)(3/4) + 0 = 6/5$

(b) Possible outcomes: 2 heads, 1 head (2 different ways), 0 heads:
$E[Y] = 2(1/2)(1/2) + 1(1/2)(1/2) + 1(1/2)(1/2) + 0 = 1$

(c) $E[X + Y] = E[X] + E[Y] = 11/5$

Problem 0-3.

(a) $A - B = 606 - 360 = 246$
$2|246$, **True**

(b) $3|246$, **True**

(c) $4 \nmid 246$, **False**

Problem 0-4.

Proof by induction that the inductive hypothesis $P(n) := \sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$ when $a \neq 1$ for any integer $n \geq 0$.

Base Case: $P(0) := \sum_{i=0}^{0} a^i = \frac{1-a^1}{1-a}, a \neq 1$
$a^0 = 1$ and $\frac{1-a}{1-a} = 1$, so $P(0)$ is true.

Inductive step: For any arbitrary integer n, assume that $P(n)$ is true. Use this assumption to prove that $P(n+1)$ is also true.
$P(n+1) := \sum_{i=0}^{n+1} a^i = \frac{1-a^{n+2}}{1-a}, a \neq 1$
$\sum_{i=0}^{n+1} a^i = \sum_{i=0}^{n} a^i + a^{n+1} = \frac{1-a^{n+1}}{1-a} + a^{n+1}$ (since $P(n)$ is true)
$= \frac{1-a^{n+1}}{1-a} + \frac{a^{n+1}-a^{n+2}}{1-a} = \frac{1-a^{n+2}}{1-a}$
Both sides are equivalent, so $P(n+1)$ is true and the inductive hypothesis is true for all integers $n \geq 0$.

Problem 0-5.

Proof by induction that the inductive hypothesis $P(n)$:= a tree with n vertices can be colored either red or blue such that no edge connects two vertices of the same color is true for all natural numbers n.

Base Case: $P(1)$:= a tree with 1 vertex can be colored either red or blue such that no edge connects two vertices of the same color, which is true because there is only one vertex that can be colored anything and no edges.
$P(2)$:= a tree with 2 vertices can be colored either red or blue such that no edge connects two vertices of the same color, which is true because there is one edge connecting the two vertices and each vertex can be colored a different color.

Inductive Step: Assume that $P(n)$ is true for any arbitrary natural number n and use this assumption to prove $P(n+1)$.
$P(n+1)$:= a tree with $n+1$ vertices can be colored either red or blue such that no edge connects 2 vertices of the same color. We already know that this statement is true for a tree with n vertices. We can then connect the $(n+1)^{th}$ vertex to any existing vertex in the tree with an edge and color it with the opposite color of the existing vertex that it was connected to. Since this new vertex is not connected to any other vertices except for the existing vertex with the opposite color and $P(n)$ is true, $P(n+1)$ holds true and the inductive hypothesis is true for all natural numbers n.

Problem 0-6. Submit your implementation to `alg.mit.edu`.

```python
def min_mod_tuple(A, k):
    i, j = 0, 1
    ##################
    # Your Code Here #
    ##################
    return (i, j)
```