Optimization Methods in Business Analytics

MIT 15.053, Spring 2019

Practice Midterm 1A

- Calculators are **not** permitted.
- You can use crib notes written on one-half side of an 11×8.5 sheet of paper.
- Answer all questions in the spaces that are marked. Nothing will be graded outside of those spaces. If you want to use scrap paper, please use the backside of papers on the exam.
- Justifications are only needed for those problems for which we ask it. For all other problems, justifications need not be given, nor will they result in partial credit.
- Write your full name below, and write the your last name in the top right corner of every subsequent page.
- Note that the total is 100 points.

Full Name: ____

1. (Total: 20 points) **Linear Programming Formulation**

You are mixing juices to make an optimal drink for yourself to bring to lunch. Below are the various base ingredients you have, each with a sugar content (g/mL), calorie content (cal/mL), and total amount you have (mL). The numbers in the table are chosen for mathematical convenience and do not correspond to true nutritional facts.

Base Juice	Sugar Density (g/mL)	Calorie Density (cal/mL)	Total Available (mL)
Apple Juice	2	10	100
Blueberry Juice	1	20	200
Carrot Juice	4	30	300
Date Juice	3	40	400
Elderberry Juice	5	80	500

In this problem we formulate a linear program to optimize your juice mix.

(a) (6 points) Your objective is to minimize the total number of calories in your drink. Clearly state your *decision variables*, and write the *objective*.

(b) (6 points) You would like to fill your 500mL bottle *exactly*, using the *juices available to you*. Write the constraint on filling your bottle as well as constraints on the limits of available juices.

(c) (3 points) You have a sweet tooth; so you would like to ensure that the sugar density of your final mixture is at least 3. The sugar density is measured as grams of sugar per mL of mixture. For example, if you mix 100mL of apple juice with 100mL of carrot juice, then the number of grams of sugar is $2 \times 100 + 4 \times 100$, and the volume is 200mL. So, the sugar density is $600/200 = 3$.

Formulate your sugar requirement and then transform it into a linear constraint (if necessary).

(d) (5 points) On the other hand, you would like to ensure that the sugar density as restricted to the date and elderberry juices (and ignoring the other three juices) is at most 4. Formulate this constraint, and then transform it into a linear constraint.

2. (Total: 20 points) **Regression and SVM**

In this question we investigate the relationship between an individual's Resting Heart Rate (RHR), and his/her characteristics, such as exercise and eating habits. Below we have data on 5 individuals.

Resting Heart Rate (beats/min)	**Exercise Frequency** (hrs/week)	**Eating Out Frequency** (meals/week)
80	4	14
75	2	9
74	7	11
70	10	8
65	7	5

First, we use linear regression, and aim to write the Resting Heart Rate in the following form:

$$\text{RHR} = \beta_0 + \beta_1 \cdot \text{ExerciseFrequency} + \beta_2 \cdot \text{EatingOutFrequency}$$

(a) (6 points) Given values of β_0, β_1, and β_2, write an expression for the *sum of squared residuals* (SSR). (Your expression may use the numbers in the table. Alternatively, you may use notation to refer to the data in the table. For example, you can let index j stand for row j of the table. You can let y_j refer to the RHR, x_{1j} refer to the exercise frequency, and x_{2j} refer to the eating out frequency, for $j = 1, \ldots, 5$.)

(b) (6 points) Formulate the problem of finding the $\beta_0, \beta_1, \beta_2$ which *minimize* the *sum of absolute deviations* (MSAD).

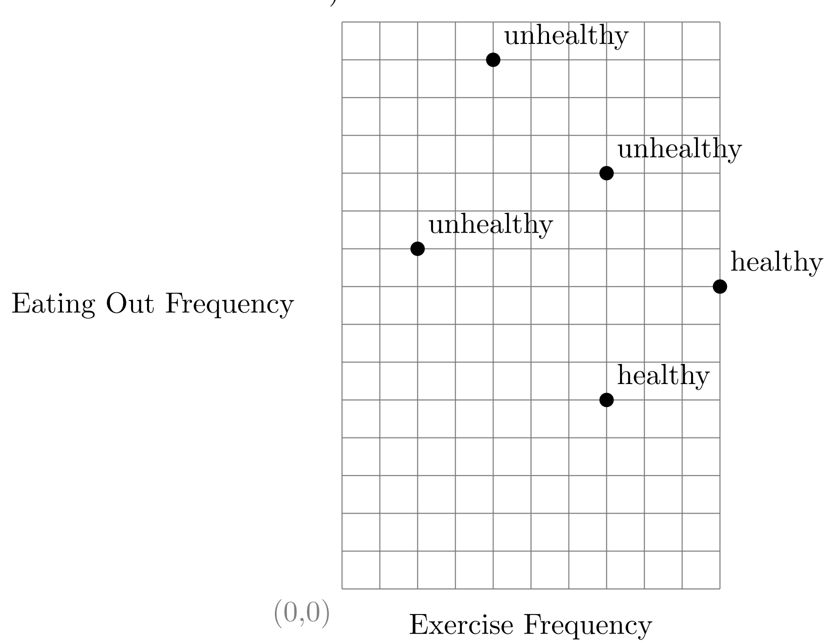
(c) (6 points) Now, instead of using regression to predict the exact RHR, we will use Support Vector Machines (SVM's) to classify RHR's into

- *healthy*: below the benchmark of 72; and
- *unhealthy*: above the benchmark of 72.

If we think of each of the five individuals as a point in the 2-dimensional plane, based on their exercise frequency and eating out frequency, we want to find the best line for separating healthy RHR's from unhealthy RHR's.

Formulate this SVM problem of finding the "widest street" as a convex optimization problem. As before, you may use notation to refer to the data in the table.

(d) (2 points) In this RHR example, one can obtain the optimal solution by visual inspection of the diagram below. On the diagram, draw the widest street as two parallel lines. (HINT: there are two ways of drawing so that the parallel lines include three of the points. One of these ways is wider than the other.)

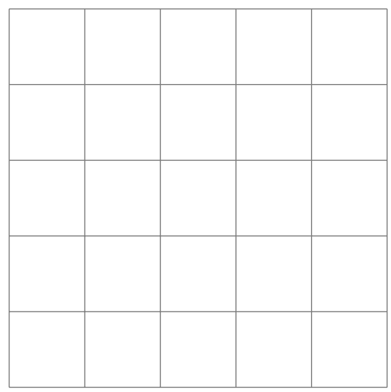


3. (Total: 31 points) **Geometry and Sensitivity Analysis**

Consider the following optimization problem:

$$\begin{aligned} \max y \\ s.t.\ x + y &\leq 6 \\ -x + y &\leq 0 \\ x &\leq 4 \\ y &\geq 0 \end{aligned}$$

(a) (6 points) Graph the feasible region on the diagram below.



(0,0)

(b) (4 points) Identify an optimum solution. Is there a unique optimal solution?

(c) (3 points) Is there a redundant constraint? If so, identify it.

(d) (6 points) What is the *shadow price* of the constraint $x + y \leq 6$? (Hint: first determine the optimal solution if the right-hand side of the constraint is increased from 6 to 7.)

(e) (6 points) What is the *allowable increase* in the right-hand side of the constraint $x + y \leq 6$ so that the shadow price remains valid?

(f) (6 points) The simplex algorithm is used to solve the optimization problem. The algorithm starts at the point of coordinates $(4, 0)$. Which points is the algorithm going through before it terminates?

4. (Total: 14 points) **Multi-criteria optimization**
The following question is on multi-criteria optimization and finding the pareto-optimal solutions. In this problem, a smaller objective value is preferable for both objectives. We will refer to the first objective value as a cost, and to the second objective value as a "loss."

Suppose that there are 7 feasible solutions. The solutions and their objective values are as follows:

Solution	Cost	Loss
S1	0	8
S2	2	5
S3	3	5
S4	4	4
S5	5	2
S6	6	3
S7	8	1

Cost and loss of each solution.

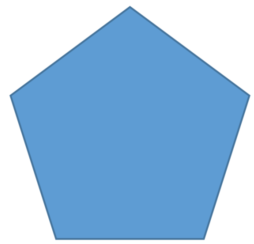
(a) (4 points) Graph the solutions.

(b) (5 points) Which solutions are Pareto optimal?

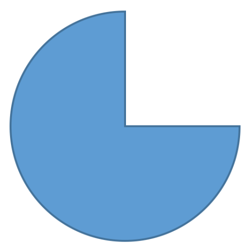
(c) (5 points) If one uses the technique of "weighting the objectives", one will not obtain all of the Pareto optimal solutions. Which Pareto optimal solutions can be obtained using this approach?

5. (Total: 15 points) **Convexity**

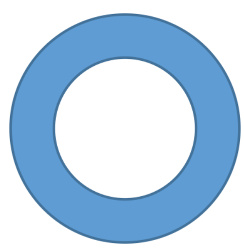
   For parts (a), (b), and (c), answer whether the corresponding **set** is convex or not. For parts (d) and (e), answer whether the corresponding **function** is convex or not. (Each part is worth 3 points.)



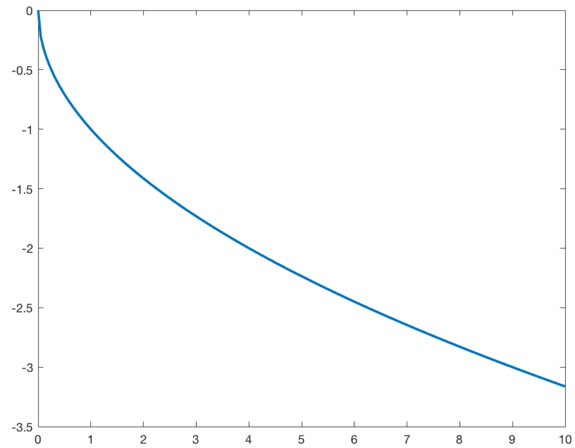
(a)



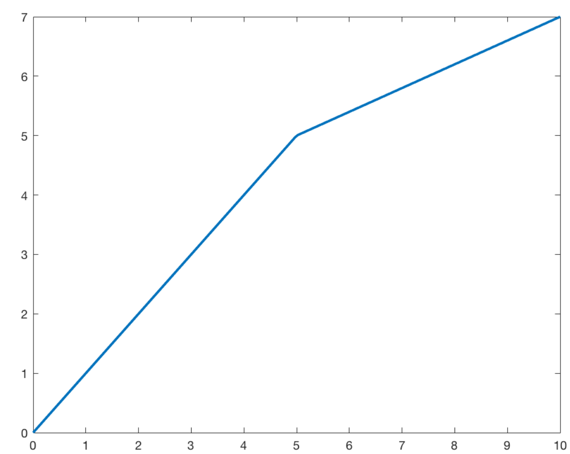
(b)



(c)



(d)



(e)

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