Recitation 2

15.076

Modeling with binary variables

Notation: x is binary or $x \in \{0,1\}$ or $0 \leq x \leq 1$ and $x \in \mathbf{Z}$

If x_1 is selected (i.e. $x_1 = 1$), then x_2 is selected. Equivalent to: if x_2 is not selected (i.e. $x_2 = 0$), then x_1 is not selected.

$$x_1 \leq x_2$$

Select x_1 or x_2 or both (i.e. at least one has to be 1).

$$x_1 + x_2 \geq 1$$

Select at least k of n

$$\sum_{i=1}^{n} x_i \geq k$$

Select at most k of n

$$\sum_{i=1}^{n} x_i \leq k$$

Consider x, y binary. If x is selected, then y is selected.

$$x \leq y$$

Now suppose $x \geq 0$ is not binary but we still want to say that if x is "selected" (i.e. we take a positive quantity of x), then y is selected. Equivalent to: if y is not selected, then x is not selected either, or $x = 0$.

$$x \leq My$$

with M being a large number, so that if $y = 1$, x is not practically restricted. The choice of M is usually up to us and, sometimes, could be infered by the context of the optimization problem. However, very large values for M tend to slow down the solvers.

A facility location problem

We have n facilities indexed by $j = 1, ..., n$.

We also have m customers indexed by $i = 1, ..., m$.

Notation: "for $i = 1, ..., m$" is the same as $\forall i \in [m]$.

Parameters (similar to what we saw in class)

1. c_j: cost of opening facility j
2. d_{ij}: cost of customer i being served by facility j (i.e. assigning customer i to facility j)

Goal: minimize costs such that all customers are served but each customer is served by exactly one facility (so x_{21} and x_{24}, for example, can't both be 1).

Formulation 1

$$\min \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \in [m]$$

$$x_{ij} \leq y_j \quad \forall i \in [m], j \in [n]$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in [m], j \in [n]$$

$$y_j \in \{0, 1\} \quad \forall j \in [n]$$

Formulation 2

$$\min \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \in [m]$$

$$\sum_{i=1}^{m} x_{ij} \leq m y_j \quad \forall j \in [n]$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in [m], j \in [n]$$

$$y_j \in \{0, 1\} \quad \forall j \in [n]$$

Why are these two formulations equivalent?

Show that every feasible (x, y) in A (i.e. that satisfies all constraints in the first formulation) belongs to B (satisfies all constraints in the second formulation).

Suppose (x, y) in A. Then

$$x_{ij} \leq y_j \quad \forall i \in [m], j \in [n]$$

Fix some facility j_0, sum over all customers $i \in [m]$. Then we have

$$\sum_{i=1}^{m} x_{ij_0} \leq m y_{j_0}$$

Repeating for each $j \in [n]$, we have

$$\sum_{i=1}^{m} x_{ij} \leq m y_j \quad \forall j \in [n]$$

.
All other constraints are the same, so if (x, y) in A, then (x, y) in B.

Show that every feasible (x, y) in B belongs to A.

Suppose that (x, y) in B. Then

$$\sum_{i=1}^{m} x_{ij} \leq m y_j \quad \forall j \in [n]$$

.
Fix some facility j_0 and consider two cases.

1. $y_{j_0} = 0$. Then $\sum_{i=1}^{m} x_{ij_0} \leq 0$ and x non-negative so $x_{ij_0} = 0, \forall i \in [m]$
2. $y_{j_0} = 1$. Then $\sum_{i=1}^{m} x_{ij_0} \leq m$. But x is binary so $\sum_{i=1}^{m} x_{ij_0} \leq m$ is redundant and, at the same time, $x_{ij} \leq y_{j_0}$.

In both cases,

$$x_{ij} \leq y_{j_0} \quad \forall i \in [m]$$

Repeating for each $j \in [n]$, we have

$$x_{ij} \leq y_j \quad \forall i \in [m], j \in [n]$$

So, if (x, y) belongs to A, then it also belongs to B. Also, if (x, y) belongs to B, then it belongs to A. Therefore the two sets contain exactly the same points.

A graph coloring problem

x_{ij} variables that will be true if and only if node i is assigned color j.

w_j variables that will be true if at least one node is assigned color j.

$$\min \sum_j w_j$$

$$\text{s.t.} \quad \sum_j x_{ij} = 1 \quad \forall i \in V$$

$$x_{uj} + x_{vj} \leq 1 \quad \forall u, v \in E, j \in C$$

$$x_{ij} \leq w_j \forall i \in V, j \in C$$

For a casual explanation have a look at:

https://manas.tech/blog/2010/07/22/partitioned-graph-coloring/

https://manas.tech/blog/2010/09/16/modelling-graph-coloring-with-integer-linear-programming/

Also, we can model the sudoku problem from class as a graph coloring problem and obtain a new integer optimization formulation.