Analytics for a Better World

Recitation 3

Schedule

1. Homework
2. Quick Topic Review
3. Working through an application

Homework

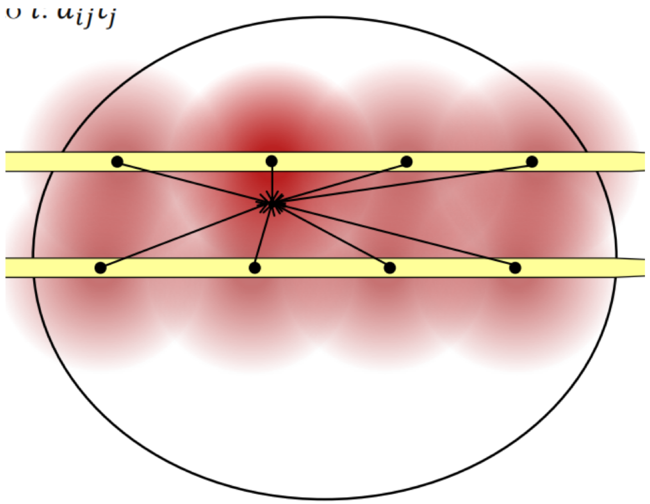
- Homework 1 feedback and solutions are out
  - Everyone was able to get correct optimal solutions
  - Some folks had issues with their drawings. Gradients would point in the wrong direction. (Often the second gradient was in the opposite direction of the first when it should have been perpendicular). By looking at the geometry you should be able to tell these gradients would lead to a different optimal solution than the one computed with software.

- Homework 2 is released
  - Focuses on modeling and coding for two problems
    - Vaccination clinic location
    - Radiotherapy treatment
  - Due Feb 28th

Topic Review

- Past Week:
  - Lecture 6 HDR Brachytherapy Treatment
  - Lecture 5 Cattle Feed for Small Farmers
- Earlier:
  - Lecture 4 Integer Optimization
  - Lecture 3 World Food Programme
  - Lecture 2 Linear Optimization
  - Lecture 1 Introduction

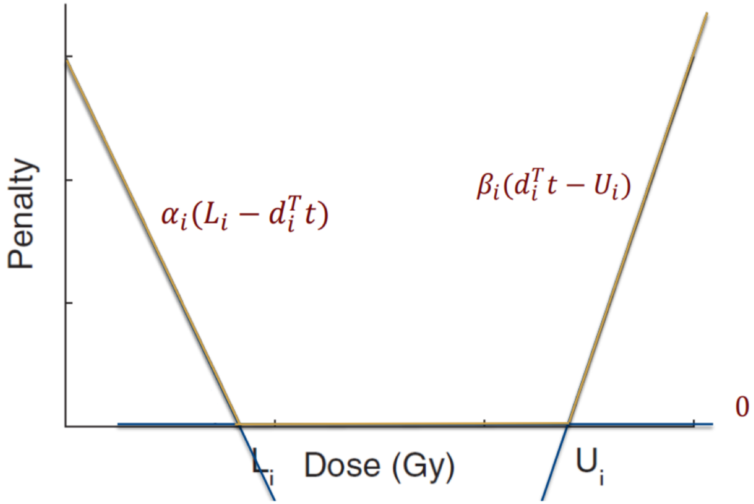
Lecture 6 HDR Brachytherapy Treatment



- Problem: Deliver lethal amount of radiation to cancerous cells and nonlethal amount to healthy cells.
- Approach:
  - Define a discrete set of target locations $i \in I$ with radiation bounds L_i, U_i, a discrete set of catheters $j \in J$ in some pre-defined location
  - Given catheter locations we can use physics models to precompute the radiation delivery rate d_{ij} at location i from emitter j
  - Use a penalty function rather than constraints. Why? How do we choose α_i and β_i?
- Model:

$$\min_{t} \sum_{i \in I} \max\{0, \alpha_i(L_i - d_i' t), \beta_i(d_i' t - U_i)\}$$

where $d_i' t = \sum_{j \in J} d_{ij} t_j$ the total dose at location i



Lecture 6 HDR Brachytherapy Treatment cont.

Also several tricks:

1. Conversion to a linear optimization problem:

$$\min_t \sum_{i \in I} \max\{0, \alpha_i(L_i - d_i' t), \beta_i(d_i' t - U_i)\} = \begin{array}{lcr} \displaystyle\min_{y,t} & \displaystyle\sum_{i \in I} y_i & \\ \text{s.t.} & y_i \geq 0 & \forall i \\ & y_i \geq \alpha_i(L_i - d_i' t) & \forall i \\ & y_i \geq \beta_i(d_i' t - U_i) & \forall i \end{array}$$

Why does this work? Draw a simple example and its reformulation: $\min_x \max\{x, -x\}$

In general, this is conversion of optimizing a piecewise-linear convex objective function with linear constraints to linear optimization

Lecture 6 HDR Brachytherapy Treatment cont.

Also several tricks:

2. Optimizing catheter positions (less than N used):

Define $z_j = \begin{cases} 1 & \text{if catheter } j \text{ is used} \\ 0 & \text{if catheter } j \text{ is not used} \end{cases}$

$$\begin{array}{lll} \min\limits_{y,t,z\in\{0,1\}^{|J|}} & \sum\limits_{i\in I} y_i & \\ \text{s.t.} & y_i \geq 0 & \forall i \\ & y_i \geq \alpha_i(L_i - d_i' t) & \forall i \\ & y_i \geq \beta_i(d_i' t - U_i) & \forall i \\ & \sum\limits_{j\in J} z_j \leq N & \\ & t_j \leq M z_j & \forall j \end{array}$$

In general, this is picking less than N

Lecture 6 HDR Brachytherapy Treatment cont.

Also several tricks:

3. Optimizing catheter positions (no neighbors):

Define $z_j = \begin{cases} 1 & \text{if catheter } j \text{ is used} \\ 0 & \text{if catheter } j \text{ is not used} \end{cases}$

Ratio of maximal to minimal dose

$$\begin{array}{lcc}
\min\limits_{y,t,z\in\{0,1\}^{|J|}} & \sum\limits_{i\in I} y_i & \\
\text{s.t.} & y_i \geq 0 & \forall i \\
 & y_i \geq \alpha_i(L_i - d_i' t) & \forall i \\
 & y_i \geq \beta_i(d_i' t - U_i) & \forall i \\
 & z_i + z_k \leq 1 & \forall i, k \text{ nbhrs} \\
 & \mathrm{t_j \leq M z_j} & \forall \mathrm{j}
\end{array}$$

In general, this is an “or”

Lecture 6 HDR Brachytherapy Treatment cont.

Also several tricks:

4. Doses to locations shouldn't differ too much:

$$
\begin{array}{lcr}
\min\limits_{y,t} & \sum\limits_{i \in I} y_i & \\
\text{s.t.} & y_i \geq 0 & \forall i \\
 & y_i \geq \alpha_i (L_i - d_i' t) & \forall i \\
 & y_i \geq \beta_i (d_i' t - U_i) & \forall i \\
 & d_i' t - d_k' t \leq B & \forall i, k
\end{array}
$$

In general, this is a reformulation of an absolute

Lecture 6 HDR Brachytherapy Treatment cont.

Also several tricks:

5. Dwelling times should be multiples of 0.1 [s]:

   Define $\tau = 10t$

$$
\begin{aligned}
\min_{y,\tau \in \mathbb{Z}^{|J|}} \quad & \sum_{i \in I} y_i & \\
\text{s.t.} \quad & y_i \geq 0 & \forall i \\
& y_i \geq \alpha_i \left(L_i - d_i' \left(\frac{\tau}{10} \right) \right) & \forall i \\
& y_i \geq \beta_i \left(d_i' \left(\frac{\tau}{10} \right) - U_i \right) & \forall i
\end{aligned}
$$

In general, this is an “integer-like”

Lecture 6 HDR Brachytherapy Treatment cont.

Also several tricks:

6. Max dose to min dose ratio bound:

$$
\begin{array}{lll}
\min\limits_{y,t} & \sum\limits_{i \in I} y_i & \\
\text{s.t.} & y_i \geq 0 & \forall i \\
& y_i \geq \alpha_i (L_i - d_i' t) & \forall i \\
& y_i \geq \beta_i (d_i' t - U_i) & \forall i \\
& M \geq d_i' t & \forall i \\
& m \leq d_i' t & \forall i \\
& M/m \leq K &
\end{array}
$$

Be careful! Why wouldn't this work if we flipped the inequality direction?

Lecture 6 HDR Brachytherapy Treatment cont.

Also several tricks:

6. Max dose to min dose ratio bound version 2:

$$
\begin{aligned}
\min_{y,t} \quad & \sum_{i \in I} y_i & \\
\text{s.t.} \quad & y_i \geq 0 & \forall i \\
& y_i \geq \alpha_i (L_i - d_i' t) & \forall i \\
& y_i \geq \beta_i (d_i' t - U_i) & \forall i \\
& d_i' t \leq K d_k' t & \forall i, k
\end{aligned}
$$

Lecture 5 Cattle Feed for Small Farmers

- Problem: Minimize cost of 1kg of feed mix to meet nutritional needs of farm animals
- Approach:
  - Define a set of ingredients $i \in I$, a set of nutrients $j \in J$.
  - Collect data on ingredient costs, ingredient nutrition, animal nutrient needs, ingredient environmental costs, maximum number of ingredients, and minimal ingredient quantity per kg, and constraints on any individual ingredients
- Model:

$$
\begin{array}{lll}
\min\limits_{x,y\in\{0,1\}^{|I|}} & \sum\limits_{i\in I} c_i x_i + e_i x_i & \\
\text{s.t.} & d_j^{lb} \le \sum\limits_{i\in I} v_{ij} x_i \le d_j^{ub} & \forall j \in N \\
 & b_i^{lb} \le x_i \le b_i^{ub} & \forall i \in I \\
 & \sum\limits_{i\in I} x_i = 1 & \\
 & x_i \ge 0 & \\
 & \sum\limits_{i\in I} y_i \le \Gamma & \\
 & y_i \ge x_i & \forall i \in I \\
 & x_i \ge \delta y_i & \forall i \in I
\end{array}
$$

Working through an application

- We will implement the brachytherapy with no neighbors problem in the attached Jupyter notebook:

$$
\begin{array}{lll}
\min\limits_{y,t\geq 0, z\in\{0,1\}^{|J|}} & \sum\limits_{i\in I} y_i & \\
\text{s.t.} & y_i \geq 0 & \forall i \\
 & y_i \geq \alpha_i(L_i - d_i' t) & \forall i \\
 & y_i \geq \beta_i(d_i' t - U_i) & \forall i \\
 & z_j + z_k \leq 1 & \forall j, k \text{ nbhrs} \\
 & \mathrm{t_j} \leq \mathrm{M} \mathrm{z_j} & \forall \mathrm{j} \\
 & \sum\limits_{j\in J} z_j \leq N &
\end{array}
$$