Optimization Methods in Business Analytics

MIT 15.053, Spring 2019

Practice Midterm 1B

- Calculators are not permitted.
- You can use crib notes written on one-half side of an 11×8.5 sheet of paper.
- Answer all questions in the spaces that are marked. Nothing will be graded outside of those spaces. If you want to use scrap paper, please use the backside of papers on the exam.
- Justifications are only needed for those problems for which we ask it. For all other problems, justifications need not be given, nor will they result in partial credit.
- Write your full name below, and write the your last name in the top right corner of every subsequent page.

Full Name: ______________________________

1. (Total: 15 points) **Linear Regression** Consider the linear regression problem of finding the line of best fit through the points (1,1), (2,2), (3,3), (4,4), (5,5), and (6,10) in the plane.

   (a) (6 points) Formulate the problem of minimizing the sum of absolute deviations (MSAD) as a linear program.

   (b) (6 points) Formulate the optimization problem of minimizing the sum of squared residuals (MSSR).

   (c) (3 points) Note that the line $y = x$ fits perfectly through the points (1,1), (2,2), (3,3), (4,4), (5,5). However, due to the *outlier* data point (6,10), the line of best fit has to be adjusted from $y = x$. **Answer true or false**: the formulation in part a) adjusts the line of best fit by *more*, to accommodate the outlier point (6,10).

2. (Total: 12 points) **Convex Analysis**

The symbol "|" in this notation means "such that". That is, $\{(x, y)|x^2 + y^2 = 1\}$ means "the set of pairs of points (x, y) such that $x^2 + y^2 = 1$."

For parts (a), (b), (c) and (d) answer whether the statement must be true or whether it can be false. (Each part is worth 3 points.)

(a) When $f(x)$ is a convex function, $\{x|0 \leq f(x) \leq 3\}$ is always a convex set.

(b) The set $\{(x, y)|\max(x, y) \geq 1, x \geq 0, y \geq 0\}$ is a convex set.

(c) The set $\{(x, y)|\max(x, y) \leq 1, x \geq 0, y \geq 0\}$ is a convex set.

(d) When $f(x)$ is a convex function, $f(-x)$ is a convex function as well.

3. (Total: 12 points, 6 points per part) **Two more models**

   (a) Write $\frac{\min(2x,3y)}{x+y} \geq 1, x+y \geq 2$ as linear constraints without adding any additional variables.

   (b) Suppose we have four points: (a_j, b_j) for $j = 1, 2, 3, 4$. Write an LP whose solution shows how to express (a_4, b_4) as a convex combination of the other three points.

4. (Total: 11 points) **Multi-criteria optimization**
The following question is on multi-criteria optimization and finding the pareto-optimal solutions. In this problem, a higher objective value is preferable for both objectives. We will refer to the first objective value as a utility, and to the second objective value as a "benefit."

Suppose that there are 8 feasible solutions. The solutions and their objective values are as follows:

Solution	Utility	Benefit
S1	0	10
S2	2	6
S3	2	6
S4	5	5
S5	5	4
S6	5	3
S7	6	2
S8	9	1

Utility and benefits of the solutions.

(a) (3 points) Which solutions are Pareto optimal?

(b) (4 points) Graph the Pareto frontier.

(c) (4 points) If one uses the technique of "weighting the objectives", one will not obtain all of the Pareto optimal solutions. Which Pareto optimal solutions can be obtained using this approach?

5. (Total: 20 points) **Geometry**

   Consider the following linear program, for some constant α:

$$\left.\begin{array}{ll} \min & x - \alpha y \\ \text{s.t.:} & \\ & 2x - y \geq -1 \\ & x - y \leq 2 \\ & x + y \leq 4 \\ & x, y \geq 0. \end{array}\right\} \qquad \text{(LP)}$$

   (a) (5 points) Graph the feasible region.

   (b) (5 points) What is the range of α such that the point $(0, 1)$ is an optimal solution?

   (c) (5 points) Remove **one** constraint from the region in (a) such that the number of extreme points reduces **by** two. Which constraint would that be?

   (d) (5 points) Suppose you added the restriction to the region in (a) that x, y are integers. Now, what would the range of α be such that the point $(0, 1)$ is an optimal solution to minimize $x - \alpha y$ over this new region?