Optimization Methods in Business Analytics

MIT 15.053, Spring 2019

Problem Set 3, Due: March 7, 2019

Problem Set Rules:

1. Each student should upload an individual problem set.
2. You may interact with fellow students when preparing your homework solutions. However, at the end, you must write up solutions on your own. Duplicating a solution that someone else has written (verbatim or edited) is not accepted.
3. Late assignments will *not* be accepted. However, each student has three "flex days" during the semester. Students may use a flex day to obtain a 24-hour extension for a problem set. A flex day must be requested by sending a note to TA15053@mit.edu before the original due date of the assignment.
4. The solutions should be submitted electronically on the website prior to the beginning of class on the day the problem set is due. Any handwritten solutions should be scanned so that they are easily read by the TAs and graders, and then submitted electronically. If the scans are not sufficiently readable, we may request that they be rescanned. For Excel submission questions, only the Excel spreadsheet will be graded. For Julia/JuMP questions, the Julia/JuMP code (and comments on the code) as well as the answers will be graded.

Problem 1

Nonlinear Portfolio Optimization (25 points in total)

Consider the Portfolio Optimization Problem introduced on MITx. Given a portfolio of five companies, an investor wants to determine how much of each stock they should purchase to be as successful as possible in the future, by balancing estimated financial return with risk. The five companies are (1) Apple Inc. (AAPL), (2) Adobe Systems Incorporated (ADBE), (3) Amazon.com, Inc. (AMZN), (4) Cisco Systems, Inc. (CSCO), and (5) Microsoft Corporation (MSFT). For $i \in \{1, \dots, 5\}$, we denote x_i the fraction of portfolio invested in company i.

Part A: In this part, we assume that the stock returns are independent of each other. In `yearSummary.csv`, you will find the list of the companies in the first column, their corresponding expected yearly return can be found in the second column, and their variance is given in the third column.

(a) (3 points) We assume for this question that the goal is to create a portfolio that maximizes the expected return. Using Julia/JuMP, formulate this problem as a linear program and solve it. What is the optimal solution? What is the optimal value? What is the optimal portfolio's variance?

(b) (3 points) We now assume that the investor is interested in creating a portfolio so that the risk, represented by the variance of the portfolio, is minimized. Using Julia/JuMP, formulate this problem as a nonlinear program and solve it. What is the optimal solution? What is the optimal value? What is the optimal portfolio's expected return?

(c) (1 point) Compare the portfolios obtained in (a) and (b).

(d) (5 points) We now consider a goal programming version of the problem, where the investor wants to minimize the variance of the portfolio while making sure that its expected return is above a target value. Using Julia/JuMP, formulate this problem as a nonlinear program, and solve it for target values equal to $0.02, 0.04, 0.06, \ldots, 0.24$ (12 optimization problems need to be solved). For each optimization problem i that is solved, compute the expected return, denoted e_i, and the variance, denoted v_i, of the optimal portfolio. Then plot the variance of the optimal portfolio with respect to its expected return, i.e., plot the points of coordinates (e_i, v_i). Give an intuition regarding the relationship between v_i and e_i.

Hint: Define two empty lists and use a for loop that will solve 12 optimization problems.

Part B: For the rest of this problem, we assume that the stock returns are correlated. In `covariance.csv`, you will find the corresponding covariance matrix. The covariance between stock i and stock j can be read in cell (i, j). For e.g., the covariance between the stock of company 2 (ABDE) and the stock of company 5 (MSFT) is equal to 0.11163398.

(a) (4 points) We assume that the investor is only interested in creating a portfolio that minimizes the risk (which now takes into account the covariance between stocks). Using Julia/JuMP, formulate this problem as a nonlinear program and solve it. What is the optimal value? Compare this value with the optimal value obtained in Part A (b).

(b) (4 points) We now consider a goal programming version of the problem, where the investor wants to minimize the risk of the portfolio while making sure that its expected return is above a target value. Using Julia/JuMP, formulate this problem as a nonlinear program, and solve it for the same target values, equal to $0.02, 0.04, 0.06, \ldots, 0.24$. For each optimization problem that is solved, compute the expected return and the variance of the optimal portfolio. Then plot the variance of the optimal portfolio with respect to its expected return.

(c) (1 point) Compare this plot to that from Part A. Which method performs better? Do you have an intuitive reason why this answer makes sense? If so, briefly state your reason.

(d) (4 points) The investor recently learned about the $SharpeRatio = \frac{ExpectedPortfolioReturn}{PortfolioVariance}$ and desires to maximize this ratio. Using Julia/JuMP, formulate this problem as a nonlinear program and solve it. What is the optimal Sharpe Ratio? How much of each stock should be part of the portfolio?

Problem 2

Origami Optimization and Analysis (10 points in total)

The 15.053 class has decided to make fancy origami to sell at the next origaMIT convention. The class is capable of making paper cranes, dogs, beavers, and whales with a 4, 6, 10, and 9 dollar sales price. Each paper animal can be made by either of the two sections-A or B. Section

Figure 1: Origami Beaver

A takes 3 minutes to make a crane, 4 to make a dog, 8 to make a beaver, and 6 to make a whale. Section B takes 6 minutes to make a crane, 2 to make a dog, 5 to make a beaver, and 8 to make a whale. Each section has agreed to dedicate up to 400 total minutes to make the origami.

Assume that everything produced will be sold and that all materials were obtained for free. Use Excel when formulating and solving the linear program. Be sure to generate a sensitivity report.

(a) (2 points) How many of each paper animal should be produced to maximize profit?

(b) (1 point) What is the minimum and maximum price a paper crane can be sold at without changing the current optimal profit? Use the sensitivity report to justify your answer.

(c) (1 point) What is the minimum and maximum price a paper dog can be sold at without changing the current optimal profit? Use the sensitivity report to justify your answer.

(d) (2 points) What is the marginal value of increasing the number of minutes Section A has dedicated to production? Section B? Use the sensitivity report to justify your answer.

(e) (2 points) What is the minimum and maximum number of minutes dedicated to production that Section A can change to without changing the optimal profit? Use the sensitivity report to justify your answer.

(f) (2 points) What is the minimum and maximum number of minutes dedicated to production that Section A can change to without changing the optimal profit? Use the sensitivity report to justify your answer.

Problem 3

Sensitivity Analysis (15 points in total)

Part A: For the following questions, explain why the answer is true or false.

(a) (1 point) A non-binding constraint must have a shadow price of 0.

(b) (1 point) A binding constraint can possibly have a shadow price of 0.

(c) (1 point) A non-binding constraint is an inequality constraint that does not hold with equality at the optimum solution.

Part B: A wood furniture manufacturing firm asked your help in analyzing their production schedule. The firm can produce four types of furniture: Chairs, Desks, Tables, Wardrobes. Each item requires a certain number of man-hours in three departments: Cutting, Sanding, Finishing. A solution report is given in Figure 2, where Revenue is expressed in dollars. A sensitivity report is given in Figure 3. Using these two exhibits, answer the following questions. Note that three values, identified by "A", "B" and "C", are missing from Figure 3. Two of the questions will ask you to compute "A" and "B". The missing values are not needed for the remaining questions.

Objective Cell (Max)

Cell	Name	Original Value	Final Value
C11	Revenue	0	18000

Variable Cells

Cell	Name	Original Value	Final Value
C2	Chairs	0	0
D2	Desks	0	0
E2	Tables	0	140
F2	Wardrobes	0	25

Constraints

Cell	Name	Cell Value	Status	Slack
C15	Cutting	480	Binding	0
C16	Sanding	800	Binding	0
C17	Finishing	800	Not Binding	100

Figure 2: Linear Solution report for Problem 2.

(a) (1 point) By how much should the revenue for selling one unit of Chairs increase, before Chairs could be considered for production in the optimal production mix?

(b) (2 points) The firm is considering the possibility of buying equipment to increase the current Sanding capacity. What is the maximum amount that the firm should be willing to pay to increase the number of available Sanding hours from 800 to 880?

(c) (3 points) Suppose the profit of Tables decreased from 100 to 90. What would be the new optimal solution? What would be the new profit? (If there is not enough information in the problem to give the answer, please state so.)

(d) (2 points) To cut costs, the firm wants to reduce the number of hours in the Finishing department from 900 to 700. Which of the following statements is the most accurate?

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C2	Chairs	0	-130	40	**B**	1E+30
D2	Desks	0	**A**	90	**C**	1E+30
E2	Tables	140	0	100	100	10
F2	Wardrobes	25	0	160	240	80

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C15	Cutting	480	12.5	480	1120	160
C16	Sanding	800	15	800	100	560
C17	Finishing	800	0	900	1E+30	100

Figure 3: Linear Sensitivity report for Problem 2.

i) Reducing the number of Finishing hours by 200 will not affect total revenue.

ii) Reducing the number of Finishing hours by 200 could affect total revenue, but we do not have enough information to compute the magnitude of the change.

iii) Reducing the number of Finishing hours by 200 will decrease revenue by \$100.

(e) (3 points) The contribution of each unit of Desks to Revenue is 90. Producing one unit of Desks requires 2 hours of Cutting, 5 hours of Sanding, and 7 hours of Finishing. Compute the missing value "A" in Figure 3, that is, the reduced costs of Desks. Show your work.

(f) (1 point) Compute the missing value "B" in Figure 3, that is, the allowable increase in the row labeled "Chairs".

Problem 4

Cafe 53 Optimization (10 points in total)

Cafe 53 is famous for their buttery fruit tarts and their fancy decorated sponge cakes. Key ingredients for either cake is their respective dough. A pound of butter dough costs 16 dollars while a pound of sponge dough costs 12 dollars. Demands D_s and D_b for sponge dough and butter dough respectively depend on their respective prices P_s and P_b: $D_s = 230 - 10P_s$ and $D_b = 440 - 20P_b$

Production is mainly constrained by the available mixer time (a pound of butter dough requires 6 minutes of mixing, whereas a pound of sponge dough requires only 4.5 minutes) and the manual labor associated with each type of cake (3 minutes per pound of butter dough, 6 minutes per pound of sponge dough). Each day, 420 minutes of labor are available and 465 minutes of Mixer time.

(a) (2 points) Using your favorite optimization software, find the optimal production quantities.

(b) (2 points) What constraint(s) are binding? What constraint(s) are not?

(c) (3 points) If Cafe 53 hires a new baker, it will be able to double the amount of labor time. How much should Cafe 53 pay the new employee? Answer the question without re-optimizing the problem.

(d) (3 points) If Cafe 53 buys a second mixer to double mixing time, how much would the Cafe be willing to pay? Answer the question without re-optimizing the problem.

Problem 5

Shortest Path Problem (20 points in total)

The network in Figure 1 is for a shortest path problem. The goal is to find the shortest path from vertex a to vertex g.

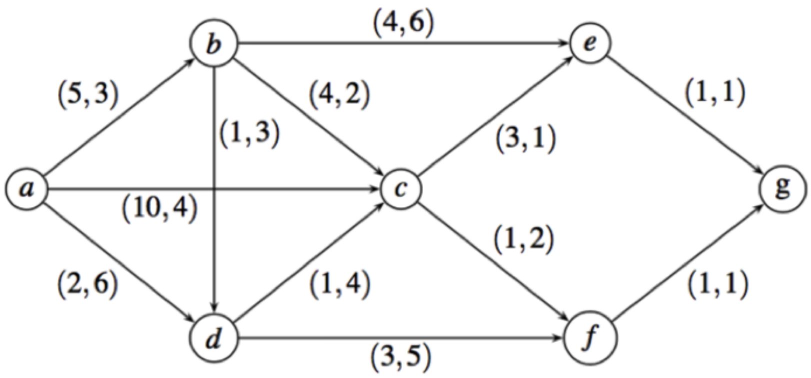

Figure 4: A network for the shortest path problem.

1. Each edge of the graph has two costs. The first cost is the length of the edge. The second cost is the time it takes to traverse the edge. There are 11 paths from a to g in the network, the first 9 of which are given in Figure 2.

Path	*Vertices*	*Costs*
1	(a,b,e,g)	(10,10)
2	(a,b,c,e,g)	(13,7)
3	(a,b,c,f,g)	(11,8)
4	(a,b,d,c,e,g)	(11,12)
5	(a,b,d,c,f,g)	(9,13)
6	(a,b,d,f,g)	(10,12)
7	(a,c,e,g)	(14,6)
8	(a,c,f,g)	(12,7)
9	(a,d,c,e,g)	(7,12)

Figure 5: The first 9 paths from a to g and their two costs.

The first cost is the length of the path. The second cost is the time it takes to get from a to g. For a given path p, let length(p) denote its length, and let time(p) denote the time it takes to traverse the path.

Suppose that one is interested in minimizing both the length and the time.

(a) (4 points) Let (a, d, c, f, g) and (a, d, f, g) be paths 10 and 11, respectively. What are the costs?

(b) (4 points) What are the Pareto optimal paths?

(c) (2 points) Graph the Pareto frontier.

(d) (5 points) For each integer value of b from 5 to 14, let $P(b)$ denote the problem of minimizing the time(p) of a path p from a to g subject to length(p) $\leq b$. List every optimal solution to $P(b)$ for all integer values of b from 5 to 14.

(e) (5 points) For each value λ in $[0, 1]$, Let $Q(\lambda)$ be the problem of finding a path p that minimizes $(1-\lambda)\times$ length(p)$+\lambda\times$ time(p). Solve $Q(\lambda)$ for all λ in $[0, 1]$ using bins of width 0.1, and plot $Q(\lambda)$. You may assume that $Q(\lambda)$ is linear between the discrete values of λ. HINT: there will only be a small number of paths that are optimal for some value of λ. Each of these paths will be optimal in an interval, which you should specify. You may use Excel or JuMP or any other technique to come up with the answer.

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Problem 6

Ticket Purchase Optimization (10 points in total)
Three consumers are looking at flights from Boston to Hawaii. There are currently 50 plane flights available with different times and costs associated with them. The ticket information for each is found in tickets.csv. The first column is the ticket number; second is total flight time in hours; third is number of layovers; and fourth is cost is dollars. Each layover lasts 45 minutes.

(a) (2 points) The first consumer is very price sensitive. Formulate the problem and find his/her optimal flight. Is there only one optimal solution?

(b) (2 points) The second consumer is willing to pay anything, but wants the shortest total time possible. Formulate the problem and find the optimal flight. Is there only one optimal solution?

(c) (3 points) The third consumer desires to minimize both cost and total time. Formulate the problem and find his/her optimal flight.

(d) (3 points) For the third consumer, plot the cost vs total time for each ticket type. What flights are Pareto-optimal?

Problem 7

Minimization of a Convex Piecewise Linear Function (10 points in total)

Consider the problem of minimizing a convex piecewise linear function:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.:} \quad & 0 \leq x \leq 10, \end{aligned} \tag{1}$$

where f is given as follows:

$$f(x) = \begin{cases} 30 - 5x, & \text{if } x \leq 3 \\ 24 - 3x, & \text{if } 3 \leq x \leq 5 \\ 9, & \text{if } 5 \leq x. \end{cases}$$

Use two different approaches to express the optimization problem (1) as a linear program. Explicitly write the two formulations. (5 points for each)