MTH 231 LECTURE 9

Proofs by Contradiction

Ex: Prove $p \to q$ by contradiction

1] Proof by Contradiction

2] Suppose $p \wedge \neg q$ is true

3] Unpack defenitions of p and $\neg q$

4] Argue that r $\wedge$ $\neg r$ are true for some proposition r. A contradiction

5] $\therefore p \to q$ must be true as $\neg(p \to q) \equiv p \wedge \neg q$ $\square$

Ex: $\underline{x \in \mathbb{Q} \text{ and } y \text{ is irrational}}_{p}$, then $\underline{x+y \text{ is irrational.}}_{q}$

1] Proof by Contradiction

2] suppose $x \in \mathbb{Q}$, $y \notin \mathbb{Q}$ and $x+y \in \mathbb{Q}$

3] So, $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$

And $x+y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ and $d \neq 0$

4] $y = \frac{cb-ad}{bd}$ || missed some calculations...

Since $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$

$\hookrightarrow cb - ad \in \mathbb{Z}$

$\hookrightarrow b \cdot d \neq 0$

$\hookrightarrow b \cdot d \in \mathbb{Z}$

We have found that y is rational. but we supposed a contradiction $(\neg q)$

5] $\therefore p \to q$ is TRUE, as $p \wedge \neg q$ is false

$\square$

Ex: Prove by contradiction that p is true

1] Proof by contradiction

2] Suppose $\neg p$ is true

3] Unpack defenitions for $\neg p$

4] show r and $\neg r$ is the same for prop r

5] Therefore p is true $\square$

Ex: Prove $\sqrt{2} \notin \mathbb{Q}$

1] Proof by Contradiction

2] suppose $\sqrt{2} \in \mathbb{Q}$

3] then, $\sqrt{2} = \frac{a}{b}$ for $a, b \in \mathbb{Z}$, $b \neq 0$ | assume $\frac{a}{b}$ is in reduced form

4] Then $2 = a^2/b^2$

$2b^2 = a^2$

Since $b^2 \in \mathbb{Z}$, a^2 is even

[If n^2 is even, n is even, then a is even

$\hookrightarrow a = 2k$, $k \in \mathbb{Z}$

$2b^2 = 2k^2 = 4k^2$

$b^2 = 4k^2$. Since k is int b^2 is even, meaning b is even.

$\hookrightarrow b = 2L$, $L \in \mathbb{Z}$

thus $\sqrt{2} = a/b = 2k/2L$

Which contradicts that $\frac{a}{b}$ was in reduced form

$\therefore \sqrt{2} \notin \mathbb{Q}$ $\square$

Other Proof Types

- Note that Direct, contra position, contradiction are the 3 <u>main</u> proof types. most others appear less.
- Existance Proofs
  - ↳ $\exists x\, P(x)$: just give an example to prove there exists an x

Note that proofs can end with any of the following: '□', 'Δ', '◇', '', '<u>Boom!</u>'*

*NOTE "<u>Boom!</u>" would lose points as it is arrogant

- Proof by Counterexample
  - ↳ essentially an existence proof where you show $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$
- Equivalence proof
  - ↳ to show that $p \leftrightarrow q$
  - ↳ 1] prove $p \rightarrow q$
    - 2] prove $p \leftarrow q$
    - 3] conclude $p \leftrightarrow q$ □

Ex: Prove $\underbrace{\text{n is even}}_{p}$ iff $\underbrace{n+1 \text{ odd}}_{q}$

1] 1] Direct proof

2] suppose n is even

3] then $n = 2k$ for $k \in \mathbb{Z}$

4] so $n+1 = 2k+1$

Since $k \in \mathbb{Z}$, $n+1$ is odd

5] If n is even, $n+1$ is odd

2] 1] Proof by Contradiction

2] suppose $n+1$ is odd and n is odd $(p \wedge \neg q)$

3] $n+1 = 2k+1$ for some int k

$n = 2\ell+1$ for some int ℓ

4] $$\begin{aligned} 1 &= (n+1) - n \\ &= (2k+1) - (2\ell+1) \\ &= 2k - 2\ell \\ &= 2(k-\ell) \end{aligned}$$

Since $k, \ell \in \mathbb{Z}$ ~~and~~, $k - \ell \in \mathbb{Z}$, so 1 is even. A contradiction

5] ∴ If $n+1$ is odd, then n is even

3] ∴ If n is even, $n+1$ is odd.