MTH231 Lecture 10

Proof by Cases

- Say you're trying to prove $p \to q$, and you notice that it splits into cases $P \equiv P_1 \vee P_2 \dot{\vee} P_n$
- In this case, you can prove that $P_1 \Rightarrow q$. and $P_2 \Rightarrow q$ ... $P_n \Rightarrow q$
- $P \equiv P_1 \vee P_2$

$$p \to q \equiv (P_1 \vee P_2) \to q$$
$$\equiv \neg P_1 \vee p2) \vee q$$
$$\cdots$$
$$\equiv P_1 \to q \wedge P_2 \to q$$

Ex: DEF: $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$

Show for all real x, y, that $|xy| = |x| \cdot |y|$

1] Direct Proof Dead-Ends!

$P \equiv x, y \in \mathbb{R}$

1] Proof by Cases

Case 1: $x, y \geq 0$

1] Direct

2] Suppose $x, y \geq 0$

3] N/A

4] Then, $|xy| = xy = |x||y|$

5] Therefore if $x, y \geq 0$, $|xy| = |x||y|$

Case 2: $x < 0, y \geq 0$

1] Direct Proof

2] Suppose $x < 0, y \geq 0$

3] N/A

4] Then $|xy| = -xy$

5] $\therefore$ If $x<0$ $y\geq 0$ then ... $= (-x)(y)$
$= |x||y|$

Case 3: $x < 0$ $y < 0$

1] Direct

2] Suppose $x < 0$ and $y < 0$

3] N/A

4] Then $|xy| = xy$
$= (-x)(-y)$
$= |x||y|$

5] $\therefore$ If $x < 0, y < 0$, $|xy| = |x||y|$

Case 4: $x \geq 0$ $y < 0$

1] Direct Proof

2] Suppose $x \geq 0$ $y < 0$

3] N/A

4] Then $|xy| = -(xy)$
$|xy| = (-y)(x)$
$|xy| = |y||x|$

5] $\therefore$ If $x \geq 0$ and $y < 0$, then $|xy| = |x||y|$

Problems on a finite domain

- Cases can be used to exhaust all posibilities- this becomes a <u>proof by exhaustion</u>.

WLOG

- without loss of generality
- is stated when some symmetry in the problem, usually given by commutivi leads to redundant cases. All redundant cases are considered proven after "WL

- WLOG could be applied to the Cases #2, #4 on the example above.
- WLOG can't be used until after the Midterm !

Ex: suppose $x \neq y$ and $x, y \in \mathbb{R}$ and $x, y \geq 0$

Then $\frac{x+y}{2} = \sqrt{xy}$

1] Direct Proof

2] Suppose WLOG $x > y$

3] NA

4] If $\cancel{x} > y \rightarrow \sqrt{x} > \sqrt{y}$

$\rightarrow \sqrt{x} - \sqrt{y} > 0$

$\rightarrow (\sqrt{x} - \sqrt{y})^2 > 0$

Then

$(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{x}\sqrt{y} + y$

$= x - 2\sqrt{xy} + y$

$x + y = (\sqrt{x} - \sqrt{y})^2 + 2\sqrt{xy}$

$x + y > 2\sqrt{xy}$ as $(\sqrt{x} - \sqrt{y})^2 > 0$

5] This means $\sqrt{xy} < \frac{x+y}{2}$ | If $x \neq y$ and $x, y \geq 0$

then $\sqrt{xy} \leq \frac{x+y}{2}$ ☒

Sets

- containers of elements.
- Properties
  - order of elements does not matter
  - repetition of elements does not matter