MTH341 NOTES 5/21

Eigen (vectors | values) - Chapter 16

- Question: given $A_{n\times n}$, find an invertible Matrix, Q, s.t. $Q^{-1}AQ$ is Diagonal

A K A

- find a linear transformation

$$f: \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto Ax$$

find a basis of $\mathbb{R}^n$ s.t. matrix representation of f is diagonal.

Ex: $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$Ax = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$Ax = 3x$

Eigen Value (3), Eigen Vector (x)

Let A be an $n\times n$ matrix. If there exists a number, λ, and nonzero vector x, s.t.

$$Ax = \lambda x$$

λ is an eigenvalue of A

x is an eigen vector corresponding to λ

- To find Eigen (vector | value), solve

$Ax = \lambda x$ for λ and x

NEXT COLUMN

$$Ax = \lambda x$$
$$Ax = (\lambda I)x$$
$$x(\underbrace{A - \lambda I}) = 0$$

non invertible $\equiv \det(A - \lambda I) = 0$

$\underbrace{\qquad\qquad}$

Characteristic Equation

- Ex (find Eigen value)

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$0 = \det\begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix}$$

$(4-\lambda)(1-\lambda) \cancel{(-2)} + 2$

$\hookrightarrow \lambda^2 - 5\lambda + 6$

$= (\lambda - 2)(\lambda - 3) \Rightarrow \begin{cases} 2 \\ 3 \end{cases}$

- Ex (find Eigen Vector)

$\lambda = 2 \quad A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

$A - 2I$

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$$

solve: $\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

let $x_2 = s$, $x_1 = s$ $\longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

NXT PG

Eigenspace $\lambda = 2$ $= \text{Span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- Ex (more of the same)

$\lambda = 3$

$$A - 3I = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

$x_2 = s$
$x_1 = 2s$ $\Rightarrow \begin{pmatrix} 2s \\ s \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Eigenspace is $(\text{span}\begin{pmatrix} 2 \\ 1 \end{pmatrix})$

Solving problem from start of lec

$Q = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

Claim: $Q^{-1}AQ = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ } Eigen-vals found earlier

- Another Problem...

$A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

1] solve det ~~(A)~~ $(A - \lambda I) = 0$ to get eigen values

2] for eigenvalue λ, solve the homogenious linear system $(A - \lambda I)x = 0$ to get Eigen vector X

3] let $Q = (v_1, v_2 \cdots v_n)$

↳ $Q^{-1}AQ = \begin{pmatrix} \lambda & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}$

Ex: $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$ $\lambda_1 = -3$ $\lambda_2 = 4$

$\lambda = -3: \rightarrow \begin{pmatrix} 1 & 1/3 \\ 0 & 0 \end{pmatrix}$ $x_2 = s$, $x_1 = -1/3\, s$ $=$ $x_2 = 3$, $x_1 = -1$

$\lambda = 4: \rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$ $x_2 = s$, $x_1 = 2s$

$$Q^{-1}AQ = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} \leftarrow Q = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$$

* If you find repeated eigen values, then there might not be a result (diagonal)

for $A = \begin{pmatrix} -3 & 1 & 3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$

and $\lambda = -2$, $\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

↑ free

$s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \leftarrow$ $x_3 = s$, $x_1 = -s$, $x_2 = 2s$