MTH341 FINAL REVIEW

- pre-notecard phase -

- **Solving Linear Systems**: place Equations into a matrix. Solve "variable part" to I (preferred) or Reduced Echelon form:

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 - x_2 + x_3 = 3 \\ -x_1 + 2x_2 + 3x_3 = 7 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

- **Matrix Multiplication**: See example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad AB = \begin{pmatrix} ae + bg & \cdots \\ \cdots & cf + dh \end{pmatrix}.$$

for $A_{(x \cdot y)}$, $B_{(z \cdot w)} \rightarrow$ ~~$AB_{(x \cdot w)}$~~ ~~$BA_{(y \cdot z)}$~~ ~~$x \neq w$, $y = z$~~ ~~Not commutative~~

$\underbrace{y = z}$ to work, output of $x \cdot w$

- **Matrix Inverses** : $AB = I = BA \Rightarrow B = A^{-1}$, $A = B^{-1}$
  - $\rightarrow$ Matrix Must be square matrix!
  - $\rightarrow$ Not every square matrix has an inverse.
  - $\rightarrow$ If $\text{Det}(A) = 0$, then there is no inverse

$\downarrow$ Shortcut (Must Check)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

AKA: $\left\{ \left(\begin{array}{c} \nearrow\swarrow \ * = -1 \\ \nwarrow\searrow \ \text{SWAP} \end{array} \right) \right.$

$\downarrow$ Formal Way (should check)

consider $(A \mid I)_{n \times 2n}$

Solve to $(I \mid \underbrace{A^{-1}}_{\text{inverse found!}})_{n \times 2n}$

- **Determinants** :
  - $\rightarrow$ Minor: M_{ij} is resultant Matrix when row i and col j are eliminated: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $M_{21} = (b)$

→ cofactor $A_{ij} = \underbrace{-1^{(i+j)}}_{\text{"sign determiner"}} \det(\underbrace{M_{ij}}_{\text{"Minor"}})$

→ $\det(A) = \underbrace{a_{ij}}_{\text{value}} \underbrace{A_{ij}}_{\text{cofactor}}$ for a row or column.

TIPS about Determinants

→ Choose the row, or column that includes the most 0's

→ $\text{Det}(A) = \text{Det}(A^T)$ transpose

→ $\text{Det}(A) \mid A$ is a triangular matrix is the product of all the values on the diagonal ↘.

→ Two identical rows or columns will form a determinant = 0

→ Any row or column of all zeroes will make det = 0

Determinant and Row Operations

→ when swapping two rows, swap sign of determinant

$$A \xrightarrow{(i) \leftrightarrow (j)} \tilde{A} \rightarrow \det(\tilde{A}) = -\det(A).$$

→ when multiplying a row by a constant multiple, multiply determinant by that multiple.

$$A \xrightarrow{(i) \cdot \alpha} \tilde{A} \rightarrow \det(\tilde{A}) = \alpha \det(A)$$

→ when replacing a row with some arithmetic operation no change occurs

$$A \xrightarrow[\alpha(i) + 3(j)]{\text{replace } (i) \text{ w/}} \tilde{A} \rightarrow A = \tilde{A}$$

- **Subspaces**

  - W is a subspace of $\mathbb{R}^n$ if

    1] $W \neq \emptyset$, i.e. W is not empty

    2] for $x, y \in W$, $x+y \in W$ (closed under addition)

    3] if $a \in \mathbb{C}$, $x \in W$, $ax \in W$ (constant multiple closure)

  unless shown to be a subspace of $\mathbb{R}^n$, W remains a sub<u>set</u> of $\mathbb{R}^n$

- **Linear Dependence and Independence**

  → to determine if vectors are linearly dependent, try to find the determinant. Det = 0 : Linearly Dependent.

  → Linear Dependent Vector Elimination:

  NOTE: SEE ALTERNATE METHOD BELOW

  for $\left\{ \begin{pmatrix} v_1 \end{pmatrix} \begin{pmatrix} v_2 \end{pmatrix} \cdots \begin{pmatrix} v_n \end{pmatrix} \right\}$, repeat while $\det \left(v_1 | v_2 | \cdots | v_n \right) = 0$:

  "starter set"

  $$\left(v_1 | v_2 | \cdots | v_n \right) \xrightarrow{} \text{reduced Echelon form} \rightsquigarrow \begin{pmatrix} * & * & \cdots & c_1 \\ * & * & \cdots & c_2 \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & c_b \end{pmatrix}$$

  $v_1 c_1 + v_2 c_2 \cdots v_n c_b$ should = another vector "x" from the starter set.

  redefine the starter set to be all vectors except x.

- **Basis and Dimensions** : apply the "Linear Dependent Vector Ellimination" to the set. Dimension = # of elements in resultant set. For vectors $e_1 \cdots e_n \in V$, $e_1 \cdots e_n$ are basis if
  - they span = V
  - they are lin-dep

- **Vector Linearity Elimination** METHOD #2

  → for $\left\{ \begin{pmatrix} v_1 \end{pmatrix}, \begin{pmatrix} v_2 \end{pmatrix}, \cdots, \begin{pmatrix} v_n \end{pmatrix} \right\}$, form $A = \left(v_1 | v_2 | \cdots | v_n \right)$

NEXT PAGE

Turn A into R.E.F, and only keep the vectors that "correspond" with leading variables. i.e.

$$A = \left(v_1 | v_2 | \dots | v_n \right) \longrightarrow \begin{array}{ccc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ v_1 \quad v_2 \quad v_n \end{array} \qquad \text{Basis} = \{ v_1 , v_n \}$$

Dim = # of elements in Basis.

- **Null Space, Row Space, Col space, Nullity**

Consider $A = (*)_{3\times 2}$.

Null Space: $A \rightsquigarrow A_{\text{Echelon}}$. solve into form $\begin{pmatrix} 3s + 2t \\ s \\ t \end{pmatrix}$ for example.

$= \left\{ s \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} \right\}$. Null $= \text{span}\left(\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} \right)$.

Col space: is the span of the columns of A, reduced until they are linearly dependent.

Row Space: span of the ~~col~~ rows of A, reduced until linearly dependent.

→ Col space dimension = row space dimension = leading variables

→ Null space dimension = free variables

→ Nullity = dim (null (A)) → Rank = dim (row)

- **Linear transformations**

→ can go "up", "down" or "same"

$\mathbb{R}^2 \to \mathbb{R}^3 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+y \\ x-y \\ x \end{pmatrix}$ $\quad \mathbb{R}^3 \to \mathbb{R}^1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to x+y+z$ $\quad \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x^2 + 2y^2 \\ x - y^2 \end{pmatrix}$

→ To check that a linear transformation is linear,

1] $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$, $c \in \mathbb{R}$

2] $f(cx) = c \cdot f(x)$

• <u>Kernel and Range</u>

If $A = f_A : \mathbb{R}^n \to \mathbb{R}^m$, then:

→ ~~Domain~~(A) = $\mathbb{R}^n$ → Kernel(A) = Null Space (A)

→ Range (A) = Col space (A)

→ Note: Dim(domain) = dim(ker(A)) + dim (Range)

• <u>Coordinates ("with respect to")</u>

→ find the coordinates of $x = [x]$ w/rt $U = \{U_1, U_2, U_3\}$

$$x = (U_1\ U_2\ U_3)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \to \left(\begin{array}{ccc|c} U_1 & U_2 & U_3 & \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \end{array}\right) \to \text{Reduced Echelon form}$$

solve for c_1, c_2, c_3. which are coordinates.

$$[x]_U = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

• <u>Matrix Representation</u>

ex: for $T : \mathbb{R}^A \to \mathbb{R}^B$ take Basis for $\mathbb{R}^A = \{e_1 \cdots e_A\}$ and

calculate T for $\{e_1 \cdots e_n\}$

$$\Rightarrow \left(T_{e_1} \middle| T_{e_2} \middle| \cdots \middle| T_{en} \right)$$

- **Change of Basis**
  - Basis $B = \{V_1 \ldots V_K\}$ $\underset{\mathbb{R}^K}{\underbrace{}}$

$[a]_B$ = coordinates of a w/rt B $= \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$

$$a = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

See "withrespect to" / "coordinates"

- **Eigen Values & Eigen Vectors & Diagonalization**

→ solve $\det(A - \lambda I) = 0$ to find eigenvalues.

→ for each eigenvalue λ_n, solve $(A - \lambda_n I) = 0$ to get eigenvector(s)

→ Let $Q = (V_1 | V_2 | \cdots | V_n)$

→ $Q^{-1} A Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

Hermition, Symmetric, Similar Matricies

— Hermition: $\bar{A}^t = A$
  ↳ conjugate (complex)

— A, B are similar when $B = Q^{-1} A Q$.

— symmetric $A^t = A$

Invertability

- A is invertible if $\det(A) = 0$ (at least one Eigenvalue = 0)
- If A has eigenvalues $(\lambda_1 \cdots \lambda_n)$, then A^{-1} has $\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right)$

Exponents:

solve to $\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} = A \cdot Q^{-1} e^A = \begin{pmatrix} e^\lambda & & \\ & \ddots & \\ & & e^\lambda \end{pmatrix} Q$