MTH 341 notes 5/2/2019

Recall

- for $e_1, e_2 \cdots e_k$ in V

  $e_1 \cdots e_k$ form a basis for V if

  (1) they span = V

  (2) They are linearly independent

- In general, for $x_1 \ldots x_n$ in $\mathbb{R}^n$ these form a basis for $\mathbb{R}^n$ iff

(1) $\det(x_1 \ldots x_n) \neq 0$

↳ thereby, $x_1 \cdots x_n$ are lin indep

Any vector in $\mathbb{R}^n$, b, can be expressed using:

$$c_1 x_1 + \cdots + c_n x_n = b$$

$$= (x_1\ x_2\ \ldots\ x_n)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = b$$

A quick note about Bases:

- The base has an equivalent # of elements as its degree of freedom.
- $\{e_1 \cdots e_m\}$, $\{\bar{e}_1 \cdots \bar{e}_x\}$ both bases of V, m=l.
- If $V \neq \{0\}$, is a subspace of $\mathbb{R}^n$: The <u>dimension</u> of V is the number of elements in a basis for V

$\longrightarrow \dim(\mathbb{R}^n) = n$

→ dim(plane through origin) = 2

→ dim(Line through origin) = 1

Consider the Plane...

- not the 737 max... not MH17...

- $P = \{ x, y, z \mid ax + by + cz = 0 \}$

  → through origin

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, (a, b, c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$

$= \text{Null}((abc))$

Assume that $a \neq 0$:

$$(a\ b\ c) \to (1, \tfrac{b}{a}, \tfrac{c}{a})$$

let $y = s$

let $z = t$

$$\underbrace{x = -\frac{b}{a}s - \frac{c}{a}t}$$

$$\underbrace{s\begin{pmatrix} -b/a \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -c/a \\ 0 \\ 1 \end{pmatrix}}$$

P = span

<u>Chapter 13</u>: Rank Nullity theorem

Recall..

- given Matrix $A_{m \times n}$, There are 3 subspaces:

① Null Space = Null(A) = {solutions Ax=0}

② Row Space = Span(rows of A)

③ Col Space = span(cols of A)

Ex: Consider $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$

① Reduce to Echelon = $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

let $y = s$, $z = t$

$x = -2s - 3t$. Null = $\left\{ \begin{pmatrix} -2s-3t \\ s \\ t \end{pmatrix} \right\}$

$$= \left\{ s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}-3\\0\\1\end{pmatrix} \right\}$$

$$= \text{span}\left(\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right)$$

$\dim(\text{Null}(A)) = 2$

$\text{basis} = \begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}$

② Row space: 2 rows

$\text{row space} = \text{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}\right)$

$= \text{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right)$ as $(2\ 4\ 6)$ is linear dependent

$2\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix}$

dim row space $= 1$

basis: $\begin{pmatrix}1\\2\\3\end{pmatrix}$

③ Col space. 3 cols

$\text{row space} = \text{span}\left(\begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}2\\4\end{pmatrix} \begin{pmatrix}3\\6\end{pmatrix}\right)$

$= \text{span}\begin{pmatrix}1\\2\end{pmatrix}$

$\dim = 1$

$\text{basis} = \begin{pmatrix}1\\2\end{pmatrix}$

- Note col space dimension = row space dimension = # leading variables
- Note Null space dimension = # of free variables.

↳ dim(col space) + dim(Null (A)) = # variables = # cols in A = n

↳ <u>Rank Nullity Theorem</u>

Nullity

- the nullity of A = dim(Null(A))
- the rank of A = dim(row space).

Ex.

$$A = \begin{pmatrix}1 & 1 & 1 & 1\\1 & 2 & 0 & 5\\3 & 2 & 4 & -1\end{pmatrix}$$

① reduce to E.F.

$$\begin{pmatrix}1 & 1 & 1 & 1\\0 & 1 & -1 & 4\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\downarrow$$

$$\begin{array}{cccc} & & & \\ \end{array}\begin{pmatrix}1 & 0 & 2 & -3\\0 & 1 & -1 & 4\\0 & 0 & 0 & 0\end{pmatrix}$$

$x_1 \quad x_2 \quad x_3 \quad x_4$

let $x_3 = s$ let $x_4 = t$

then $x_1 = -2s + 3t$

$x_2 = s - 4t$

$$\text{Null}(A) = \left\{ \begin{pmatrix}-2s+3t\\s-4t\\s\\t\end{pmatrix} \right\}$$

$$= \left\{ s\begin{pmatrix}-2\\1\\1\\0\end{pmatrix} + t\begin{pmatrix}3\\-4\\0\\1\end{pmatrix} \right\}$$

$$= \text{Span}\left(\begin{pmatrix}-2\\1\\1\\0\end{pmatrix} \begin{pmatrix}3\\-4\\0\\1\end{pmatrix}\right)$$

$\dim(\text{Null}(A)) = N(A) = 2$

Note there are 2 free variables yeet!

② because in ①, we canceled the 3rd row with the other rows (3rd is lin comb of first two).

Therefore, we are left with two vectors / rows:

$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\0\\5\end{pmatrix}$ which are linearly indep.

↳ $R(A)$ = dim (rows) = 2

|| Note there are 2 leading variables
Yeet aswell

Some Notes

- if $A \rightarrow E$ in echelon form

  null (A) = null (E)

  row space (A) = row space (E)

  col space (A) $\neq$ col space (E)

- If question asks "find basis for span"

  → solve for Red. EF.

③ col space

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 5 \\ 3 & 2 & 4 & -1 \end{pmatrix} \xrightarrow{\text{Ech}} \begin{pmatrix} 1 & 0 & \cancel{2} & \cancel{3} \\ 0 & 1 & \cancel{-1} & \cancel{4} \\ 0 & 0 & \cancel{0} & \cancel{6} \end{pmatrix}$$

lin comb of cols 1, 2

Col space = span $\left(\begin{pmatrix}1\\1\\3\end{pmatrix} \begin{pmatrix}1\\2\\2\end{pmatrix} \right)$

= 2

Another Theorem

- $R(A)$ = row space dimension
  = col space dimension
  = num. leading variables

- $N(A)$ = free variables

$R(A) + N(A) = n$

|| NOTE: CAN ONLY FIND LEADING, FREE VARS FROM ECHELON FORM (REDUCED)