MIDTERM EXAM
MTH 341 LINEAR ALGEBRA I
SPRING 2019

- Show your work, in a reasonably neat and coherent way. All answers must be justified by valid mathematical reasoning. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- Mysterious or unsupported answers will not receive full credit. Your work should be mathematically correct and carefully and legibly written.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem 1 (10 points). Solve the following linear system:

$$\begin{aligned} 2x_1 + 8x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + 2x_3 &= 5 \\ 2x_1 + 7x_2 + 4x_3 &= 8. \end{aligned}$$

Problem 2 (10 points). Let

$$A = \begin{pmatrix} 3 & 5 & 2 \\ -2 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 3 & 7 \\ 1 & 4 \end{pmatrix},$$

Compute AB and BA.

Problem 3 (10 points). Find the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{pmatrix}.$$

Problem 4 (10 points). Determine whether the following vectors are linearly independent in $\mathbb{R}^3$.

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

Problem 5 (10 points). Find the dimension and a basis for the column space of the matrix A, where

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}.$$

Problem 6 (10 points). Find the dimension and a basis for the null space of the matrix A in problem 5.

10

10

10

10

10

8

58

Problem 1

$$\begin{cases} 2x_1 + 8x_2 + 3x_3 = 2 \\ x_1 + 3x_2 + 2x_3 = 5 \\ 2x_1 + 7x_2 + 4x_3 = 8 \end{cases} \xrightarrow[\text{Mtrx}]{\text{Aug}} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 8 \end{pmatrix} \xrightarrow{2②} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 2 & 6 & 4 & 10 \\ 2 & 7 & 4 & 8 \end{pmatrix}$$

$$\xrightarrow[③-①]{②-①} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & -2 & 1 & 8 \\ 0 & -1 & 1 & 6 \end{pmatrix} \xrightarrow{2③} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & -2 & 1 & 8 \\ 0 & -2 & 2 & 12 \end{pmatrix} \xrightarrow{③-②} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & -2 & 1 & 8 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{②-③} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{-1②} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{①-3③} \begin{pmatrix} 2 & 8 & 0 & -10 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{①-4②} \begin{pmatrix} 2 & 0 & 0 & 6 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$-4 \cdot 4 = -16$ $-10 - -16 = 6$

$$\longrightarrow \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \\ x_1 \quad x_2 \quad x_3 \end{array} \longrightarrow \boxed{1 \text{ solution} \begin{cases} x_1 = 3 \\ x_2 = -2 \\ x_3 = 4 \end{cases}}$$

✓

Check that!

$x_1 + 3x_2 + 2x_3 = 5$

$3 + 3(-2) + 2(4) =$

$3 \cancel{-6} + 8 = 5$ ✓

$+2$

$2x_1 + 8x_2 + 3x_3 = 2$

$2(3) - 8(2) + 3(4) = 2$

$6 \quad -16 + 12$

$-10 + 12 = 2$ ✓

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Problem 2

$$A = \underset{2\times 3}{\begin{pmatrix} 3 & 5 & 2 \\ -2 & 0 & 2 \end{pmatrix}} \qquad B = \underset{3\times 2}{\begin{pmatrix} 2 & 1 \\ 3 & 7 \\ 1 & 4 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} 23 & 46 \\ -2 & 6 \end{pmatrix}}$$

$$AB = \begin{pmatrix} 3 & 5 & 2 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 7 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 6+15+2 & 3+35+8 \\ -4+2 & -2+8 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 \\ 3 & 7 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 5 & 2 \\ -2 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 6-2 & 10 & 4+2 \\ 9-14 & 15 & 6+14 \\ 3-8 & 5 & 2+8 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 4 & 10 & 6 \\ -5 & 15 & 20 \\ -5 & 5 & 10 \end{pmatrix}}$$

$$\begin{pmatrix} 23 & 46 \\ -2 & 6 \end{pmatrix}$$

CHECK SPACE

BA

$$\begin{pmatrix} 2 & 1 \\ 3 & 7 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 5 & 2 \\ -2 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 10 & 6 \\ -5 & 15 & 20 \\ -5 & 5 & 10 \end{pmatrix} \leftarrow \begin{pmatrix} 6-2 & 10 & 4+2 \\ 9-14 & 15 & 6+14 \\ 3-8 & 5 & 2+8 \end{pmatrix}$$

AB

$$\begin{pmatrix} 3 & 5 & 2 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 7 \\ 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 6+15+2 & 3+35+8 \\ -4+2 & -2+8 \end{pmatrix}$$

Problem 3

$$A = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{pmatrix} \quad (A|I) = \left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right)$$

Now, we bring $(A|I) \rightarrow (I|A^{-1})$...

$$\xrightarrow{②-2①} \left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{③-2①} \left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[③\cdot 5]{②\cdot 3} \left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -15 & -6 & -6 & 3 & 0 \\ 0 & -15 & -5 & -10 & 0 & 5 \end{array}\right) \xrightarrow{③-②} \left(\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -15 & -6 & -6 & 3 & 0 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right)$$

$$\xrightarrow{①-③} \left(\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & -15 & -6 & -6 & 3 & 0 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right) \xrightarrow{②+6③} \left(\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & -15 & 0 & -30 & -15 & 30 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right)$$

$-5 - -6$

$$\xrightarrow{②/-15} \left(\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right) \xrightarrow{①-5②} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right)$$

$-5 - -10$

$-4 \cdot 6 = -24$

$3 + (-3 \cdot 6)$

$3 - 18 = -15$

$$\hookrightarrow A^{-1} = \begin{pmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{pmatrix}.$$

CHECK THAT! ✓

$$A \cdot A^{-1} \stackrel{?}{=} I$$

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{pmatrix} \begin{pmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{pmatrix} = \begin{pmatrix} -5+10-4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$-2+5-3$

Check Invert.

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{pmatrix} ③-①$$

$$\begin{pmatrix} \cancel{1\ 5\ 1} \\ 2\ 5\ 0 \\ 1\ 2\ 0 \end{pmatrix}$$

$1 \cdot \det\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}$

$4 - 5 = -1$

$\det \neq 0$

$\hookrightarrow$ invertible

Problem 4

$$\begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix} \rightarrow \begin{pmatrix}2&3&2\\1&2&2\\-2&-2&0\end{pmatrix} \xrightarrow{① - ②} \begin{pmatrix}1&1&0\\1&2&2\\-2&-2&0\end{pmatrix}$$

$$2 \cdot \det\begin{pmatrix}1&1\\-2&-2\end{pmatrix} \quad (1)(-2) - (1)(-2)$$
$$-2 \;-\; -2 = \boxed{0}$$

→ determinant is zero. therefore these are linearly dependent in $\mathbb{R}^3$.

CHECK SPACE (FIND LCD)

$$\begin{pmatrix}2&3&2\\1&2&2\\-2&-2&0\end{pmatrix} \xrightarrow{③+①} \begin{pmatrix}2&3&2\\1&2&2\\0&1&0\end{pmatrix} \rightarrow \begin{pmatrix}2&3&2\\1&2&2\\0&3&0\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}0&-1&-2\\1&2&2\\0&1&0\end{pmatrix} \begin{pmatrix}0&-1&-2\\1&2&2\\0&0&-2\end{pmatrix} \begin{pmatrix}0&-2&-4\\1&2&2\\0&0&-2\end{pmatrix}$$

$$\begin{pmatrix}1&0&-2\\1&2&2\\0&0&-2\end{pmatrix} \rightarrow \begin{pmatrix}\ldots\end{pmatrix} \ldots$$

likely they are lin dep

Problem 5

find dim, basis for col. space of A

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix} \xrightarrow{③-①} \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 0 & 7 & 1 & -2 \end{pmatrix} \xrightarrow{②\cdot 3} \begin{pmatrix} -3 & 1 & 3 & 4 \\ 3 & 6 & -3 & -6 \\ 0 & 7 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow{②+①} \begin{pmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 7 & 1 & -2 \end{pmatrix} \xrightarrow{③-②} \begin{pmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{①-3③} \begin{pmatrix} -3 & 1 & 0 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{①\cdot 7} \begin{pmatrix} -21 & 7 & 0 & 28 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{①-②} \begin{pmatrix} -21 & 0 & 0 & 30 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[②/7]{①/-21} \begin{pmatrix} 1 & 0 & 0 & \frac{-30}{21} \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies$$

$\Delta \quad \Delta \quad \Delta$

3 leading, 0 free vars.

$\boxed{\text{Dimension of Col space = \# Leading vars = 3}}$

Dimension of Col space =
Leading vars = 3
Basis of Col space = cols from original where leading values are in echelon matrix
$\hookrightarrow \left\{ \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\}$

Check on scratch page

Problem 6

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix} \longrightarrow \sim A \sim \begin{pmatrix} 1 & 0 & 0 & -\frac{30}{21} \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Delta \quad \Delta \quad \Delta$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{30}{21}s \\ -2/7\, s \\ 0 \\ s \end{pmatrix}$$ ✓

Dimension of null space # free variables

↳ Dim(Null) = $\cancel{\emptyset}$ 1

Basis = { }, as Dim = $\emptyset$.

✗ ✗