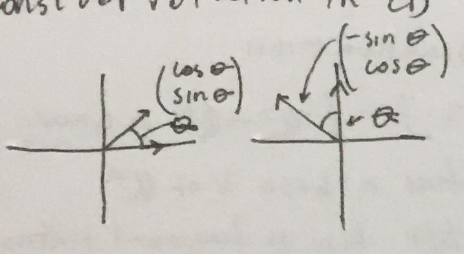
MTH 341 NOTES FROM 5/16/2019

Recall

- If $f: \mathbb{R}^n \to \mathbb{R}^m$ then there exists a matrix A such that

$$f(x) = Ax$$

Consider rotation in 2D



- $f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$
- $f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$

$\to f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Ex: $x = \begin{pmatrix}3\\5\end{pmatrix} = 3\cdot\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix} = 3e_1 + 5e_2$

coordinates w/rt $\{e_1, e_2\}$

$V = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$ is a basis of $\mathbb{R}^2$

$$\begin{pmatrix}3\\5\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1&0&1\\0&1&2\end{array}\right)$$

$$\boxed{\left[\begin{pmatrix}1\\2\end{pmatrix}\right]_V}$$

$\begin{pmatrix}3\\5\end{pmatrix} = 1\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}$ w/RT V (basis)

Ex: $\left[\begin{pmatrix}x_1\\x_2\\ \vdots\\ x_n\end{pmatrix}\right]_V \in \mathbb{R}^n$

$$f\left(\left[\begin{pmatrix}x_1\\x_2\\ \vdots\\ x_n\end{pmatrix}\right]_V\right) = \left[A_{vw}\begin{pmatrix}x_1\\x_2\\ \vdots\\ x_n\end{pmatrix}\right]_W$$

Ex: $P: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \to \begin{pmatrix}x_1\\x_2\\0\end{pmatrix}$$

over $\{e_1, e_2, e_3\}$

$$P\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad P\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad P\begin{pmatrix}0\\0\\1\end{pmatrix} = \emptyset$$

$\hookrightarrow A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ w/Rt standard basis

$\hookrightarrow P\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

Let $V = W = \left\{v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}\ v_2 = \begin{pmatrix}1\\2\\2\end{pmatrix}\ v_3 = \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

$P(v_1) = \begin{pmatrix}1\\1\\0\end{pmatrix} = c_1v_1 + c_2v_2 + c_3v_3$
$= (v_1 + v_2 - v_3) = (v_1\ v_2\ v_3)\begin{pmatrix}1\\1\\-1\end{pmatrix}$

$P(v_2) = \begin{pmatrix}1\\2\\0\end{pmatrix} = 3v_2 - 2v_3 = (v_1\ v_2\ v_3)\begin{pmatrix}0\\3\\-2\end{pmatrix}$

$P(v_3) = \begin{pmatrix}1\\2\\0\end{pmatrix}$ " " " "

$$A_{vw} = \begin{pmatrix}1&0&0\\1&3&3\\-1&-2&-2\end{pmatrix}$$

$\hookrightarrow f\left(\left[\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right]_V\right) = \left[A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right]$

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$$f : \mathbb{R}^n \to \mathbb{R}^m \quad \begin{cases} A \text{ w/rt } \{e_1 \cdots e_n\} \text{ of } \mathbb{R}^n \\ \text{(st. basis) } \{e_1 \cdots e_m\} \text{ of } \mathbb{R}^m \\ A_{vw} \text{ w/rt } \{v_1 \cdots v_n\} \text{ of } \mathbb{R}^n \\ \{V_1 \cdots V_m\} \text{ of } \mathbb{R}^m \end{cases}$$

Matrix Similarities

- for $n \times n$ matrices A, B

A and B are similar if:

$$B = Q^{-1} A Q$$

for some invertible matrix Q

Considering ...

$$f\left(\left[\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right]_v\right) = \left[A_{vw} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right]$$

$$\downarrow \qquad\qquad \downarrow$$

$$f(x_1 v_1 \cdots x_n v_n) \qquad = \underline{(w_1 \cdots w_m) A_{vw} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$$

$$\downarrow$$

$$f\left(\begin{pmatrix} x_1 & \cdots & x_n \\ v & & v \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)$$

$$\downarrow$$

$$\underline{A (v_1 \cdots v_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$$

must EQUAL

hence:

$$A (v_1 \cdots v_n) = (w_1 \cdots w_m) A_{vw}$$

$$\boxed{(w_1 \cdots w_m)^{-1} A (V_1 \cdots V_n) = A_{vw}}$$

Ex

P projection

$$A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ standard basis}$$

$$A_{vw} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$W = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

we know $W^{-1} \cdot A \cdot V = A_{vw}$

Diagonalization

- ?: let $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

find a basis V of $\mathbb{R}^n$ s/t A_{vv} is diagonal matrix

↳ let $fx = Ax$. find V s/t

$$V^{-1} A V = \text{Diagonal}$$

Ex: $A = \begin{pmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{pmatrix}$

$f(x) = Ax$: $\quad V = \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\left.\begin{aligned} f(v_1) &= A \cdot v_1 = v_1 \\ f(v_2) &= A v_2 = -3 v_2 \\ f(v_3) &= A v_3 = -3 v_3 \end{aligned}\right\} \text{eigen vectors/vals}$$

$$\hookrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Diagonal!!