MTH 341 NOTES FROM 5/7/2019

MIDTERM REVIEW

- Solve these linear equations:

$$\begin{cases} 2x_1 + 8x_2 + 3x_3 = 2 \\ x_1 + 3x_2 + 2x_3 = 5 \\ 2x_1 + 7x_2 + 4x_3 = 8 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 2 & 8 & 3 & 2 \\ 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 8 \end{array}\right) \xrightarrow[3-2(2)]{1-3} \begin{pmatrix} 0 & 1 & -1 & -6 \\ 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow{2 \leftrightarrow 1} \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & -1 & -6 \\ 0 & 1 & 0 & 2 \end{pmatrix} \xrightarrow[1-3(3)]{2-3} \begin{pmatrix} 1 & 0 & 2 & 11 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 0 & -2 \end{pmatrix} \xrightarrow{2 \leftrightarrow 3} \begin{pmatrix} 1 & 0 & 2 & 11 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -4 \end{pmatrix}$$

$$\xrightarrow[-1(3)]{1+2(3)} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \rightarrow \text{Unique solution} \begin{cases} x_1 = 3 \\ x_2 = -2 \\ x_3 = 4 \end{cases}$$

- Solve this set of linear equations

$$\begin{cases} x_1 - x_2 + 3x_3 + 2x_4 = 1 \\ -x_1 + x_2 - 2x_3 + x_4 = -2 \\ 2x_1 - 2x_2 + 7x_3 + 7x_4 = 1 \end{cases} \longrightarrow \begin{pmatrix} 1 & -1 & 3 & 2 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & -2 & 7 & 7 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Δ Δ free vars

$\hookrightarrow$ $x_2 = s$, $x_4 = t$ $\rightarrow$ $x_1 = s - 7t + 4$, $x_3 = -3t - 1$. has infinitely many solutions... vars > eqs

- if $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix}_{2\times3}$ and $B = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 6 & 5 \end{pmatrix}_{3\times2}$. find AB and BA

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} \overset{11}{21} & \overset{12}{15} \\ \underset{21}{35} & \underset{22}{0} \end{pmatrix}_{2\times2} \quad \Bigg| \quad \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{pmatrix}_{3\times3}$$

- find the inverse of the Matrix $\begin{pmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{pmatrix}$

NOTE: can use formula $A^{-1} = \frac{1}{\det A} \cdot \underbrace{(A_{ji})}_{\text{not } ij!}$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & 1 & \cancel{0}1 & 0 & 1 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\cdots]{} () \xrightarrow[\cdots]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right)$$

cont'd

Check problem using $A^{-1} \cdot A = I$

• find determinant of $\begin{pmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & 3 & 1 \end{pmatrix}$ and state about the invertibility.

$$\begin{pmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & 3 & 1 \end{pmatrix} \xrightarrow[③-5②]{③+①} \begin{pmatrix} 0 & 1 & 3 \\ 1 & 5 & -3 \\ 0 & -22 & 16 \end{pmatrix}$$

ad-bc

$$\det\begin{pmatrix} 1 & 3 \\ -22 & 16 \end{pmatrix} = -82$$

↳ is invertible as det $\neq 0$

• Let A, B be 3x3 matricies. $\det(A) = 4$, $\det(B) = 6$. E = elementary matrix of type I. find:

•• (a) find $\det\left(\frac{1}{2}A\right) = \left(\frac{1}{2}\right)^3 \cdot 4 = \cancel{4} = \frac{1}{2}$

NOTE: $\det(cA) = c^n \det(A)$ where n = rows.

•• (b) find $\det(B^{-1}A^t) = \det(B^{-1}) \cdot \det(A^t)$

$= \frac{1}{\det B} \cdot \det A = \frac{4}{6}$

•• (c) find $\det(EA^2) = \det(E) \cdot \det(A) \cdot \det(A)$

$\quad - \quad 4 \quad 4 \quad = -16$

• which of the following vectors are linearly Dependent

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right.$ —— NOTE: $\mathbb{R}^3$

If $m > n$, then m vectors in $\mathbb{R}^n$ are linearly Dependent

• Are these vectors linearly inDependent?

$$\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & -2 \end{pmatrix} \xrightarrow[\text{echelon form}]{\text{try to get into}} \underbrace{\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{pmatrix}}$$

Independent

this will easily become REF. therefore there is a unique soln

METHOD 2:

$$\det\begin{pmatrix} -1 & 1 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & -2 \end{pmatrix} \xrightarrow[①+②]{\text{col: } ③+②} \begin{pmatrix} 0 & 1 & 0 \\ 4 & 1 & 0 \\ 1 & -1 & 3 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}$$

$= 12 \neq 0$. Therefore vectors are Independent

NOTE: if det = 0, then vecs are lin dep.

• Are these linearly Dependent ?

$$\begin{pmatrix}2\\2\\2\\2\end{pmatrix}\begin{pmatrix}2\\2\\0\\2\end{pmatrix}\begin{pmatrix}2\\0\\2\\2\end{pmatrix}\Bigg\}\mathbb{R}^4 \longrightarrow \begin{pmatrix}2&2&2\\2&2&0\\2&0&2\\2&2&2\end{pmatrix}$$

NOTE: No det() b/c !square

$$\xrightarrow[\text{by 2}]{\text{DIV}} \begin{pmatrix}1&1&1\\1&1&0\\1&0&1\\\cancel{1}&\cancel{1}&\cancel{1}\end{pmatrix} \longrightarrow \begin{pmatrix}1&1&0\\0&0&1\\0&-1&0\\0&0&0\end{pmatrix} \longrightarrow \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

no nontrivial solutions to $c_1\begin{pmatrix}2\\2\\2\\2\end{pmatrix} + c_2\begin{pmatrix}2\\2\\0\\2\end{pmatrix} + c_3\begin{pmatrix}2\\0\\2\\2\end{pmatrix} = 0$, $\therefore$ Indep

NOTE: If u, v in $\mathbb{R}^n$, are linearly Dependent, then $\begin{pmatrix}u\\0\end{pmatrix}, \begin{pmatrix}v\\0\end{pmatrix} \in \mathbb{R}^{n+1}$ are still linearly Dependent

• for matrix $A = \begin{pmatrix}1&-2&3&2\\-1&2&-2&-1\\2&-4&5&3\end{pmatrix}$, find $\begin{cases}\text{dimension}\\\text{basis}\end{cases}$ for $\begin{cases}\text{row space}\\\text{column space}\\\text{null space}\end{cases}$

$$A \rightsquigarrow \begin{array}{c}\begin{pmatrix}1&-2&0&-1\\0&0&1&1\\0&0&0&0\end{pmatrix}\\ \begin{matrix}\Delta&\;F&\;\Delta&\;F\end{matrix}\end{array}$$

$\underbrace{R = \text{RANK}(A) = 2}_{\text{leading vars}}$ $\underbrace{N = \text{NULL}(A) = 2}_{\text{free vars}}$

$\downarrow$ = Rank = dim of col space = dim row space

$\downarrow$ = dim null space

basis for row space is the nonzero rows

$$\begin{pmatrix}1\\-2\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

basis of col space is the columns that have leading variables, but TAKE FROM ORIGINAL MATRIX!!

$$\begin{pmatrix}-2\\2\\-4\end{pmatrix}, \begin{pmatrix}2\\-1\\3\end{pmatrix}$$

basis for null space

$x_2 = s$ $\quad x_1 = 2s + t$

$x_4 = t$ $\quad x_3 = -t$

$$\hookrightarrow \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}2s+t\\s\\-t\\t\end{pmatrix} = s\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

$\hookrightarrow$ Basis: $\begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$

• If $S = \left\{ \text{all } \begin{pmatrix} x \\ y \end{pmatrix} \text{ satisfying } x^2 + y = 0 \right\} \subset \mathbb{R}^2$

AKA is S a subspace of $\mathbb{R}^2$

NOTE: For true subspace, $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in S$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in S$, $s\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in S$

↳ If we choose $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, both $\in S$,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \notin S \longrightarrow \therefore \text{ NOT a subspace.}$$

• If $S = \left\{ \begin{pmatrix} 4u + v - 5w \\ 21u + 6v - 6w \\ 4u + 4v + 4w \end{pmatrix} \middle| \, u, v, w \in \mathbb{R} \right\} = \text{span}\left(\begin{pmatrix} 4 \\ 21 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ -6 \\ 4 \end{pmatrix} \right)$

$$S = \left\{ \underbrace{u \begin{pmatrix} 4 \\ 21 \\ 4 \end{pmatrix} + v \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix} + w \begin{pmatrix} -5 \\ -6 \\ 4 \end{pmatrix}} \right\}.$$

Dimension? 3 because 3v in $\mathbb{R}3$

Basis? those 3 vec's

$$\longrightarrow \begin{pmatrix} 4 & 1 & -5 \\ 21 & 6 & -6 \\ 4 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 1 & -5 \\ 21 & 6 & -6 \\ 1 & 1 & 1 \end{pmatrix} \cdot 4$$

$$↳ 4 \cdot \det \begin{pmatrix} 3 & 1 & 6 \\ 15 & 6 & -12 \\ 0 & & 0 \end{pmatrix} = 4 \cdot \det \begin{pmatrix} 3 & 6 \\ 15 & -12 \end{pmatrix} \neq 0$$

↳ Linearly Independent