MTH 341 Notes 4/30/2019

CH 11

- Consider EX:

Ex 1 $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $x_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$x_3 = 2x_1 + x_2$

$\therefore 2x_1 + x_2 - x_3 = 0$

Ex 2 $y_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $y_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $y_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$y_1 \cdot c_1 + y_2 \cdot c_2 + y_3 \cdot c_3 = 0$

$$(y_1 y_2 y_3)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \; : \; \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0_{3\times 1}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} = Y_{3\times 3}$$

→ method 1: Bring $Y_{3\times 3}$ to Identity.

Read equations like you would for a sys of Eq's.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}$$

→ Method 2: Check Invertible, using det

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \neq 0 \; : \text{ yes, invert.}$$

Rearrange Equation:

$$\cancel{Y Y^{-1}}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot Y^{-1}}_{= 0_{3\times 1}}$$

Def:

- Consider vectors $x_1, x_2 \dots x_m$ in R^n
- These are linearly independent if the system

$$c_1 x_1 + c_2 x_2 \cdots c_m x_m = 0$$

only has a trivial solution.

- These are linearly dependent if they have nontrivial solutions

Linear Dependence

- If the coefficient matrix has $\det() = 0$, it is non-invertible, and thus has infinitely many solutions, which make it linearly dependent

Ex:

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} -5 \\ -2 \\ 4 \end{pmatrix} \begin{pmatrix} 3 \\ 8 \\ -6 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \\ -4 \end{pmatrix} \in \mathbb{R}^3$$

$$\begin{pmatrix} 2 & 5 & 3 & 2 \\ 1 & -2 & 8 & 7 \\ 3 & 4 & -6 & -4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0_{3\times 1}$$

- given its 3 equations and 4 variables, there is at least 1 free variable. Therefore, Linearly Dependent.

NEXT PG

Ex:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 2 & 4 & 7 \\ 1 & 1 & 5 \end{pmatrix} \text{ in } \mathbb{R}^4 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Independent}$$

- If rows > cols

  LINEARLY DEPENDENT

- If rows < cols

  often L. INDEP

Chapter 12

Recall:

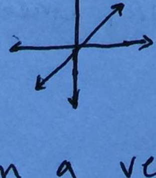
- $\text{span}(X_1 \dots X_m) =$

$$\underbrace{\left\{ c_1X_1 + \dots + c_mX_m \;\middle|\; \begin{matrix} c_i \in \mathbb{R} \\ i \text{ on } 1-m \end{matrix} \right\}}_{\text{this is a subspace of } \mathbb{R}^n}$$

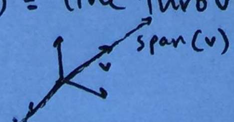
Ex: $\text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$$= \left\{ c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} c \\ c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; x = y \right\} \longrightarrow$$



- In general, if given a vector v in $\mathbb{R}^n$, span(v) = line through origin and v:



Mo' Spannin

Note: $\text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

Ex: $\text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) \longrightarrow \mathbb{R}^2$ (plane)

If vectors not same.

$$V_1 = \text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$$

$$= \left\{ s\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} = V_1$$

Ex: Add $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$V_2 = \text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$V_1 \subset V_2$ subset.

Claim that also $V_2 \subset V_1$

given $\frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then $V_2 = s\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix} + a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= s\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix} + a\left[\frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]$$

$$\underbrace{= \left(s + \frac{a}{5}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t + \left(\frac{2a}{5}\right)\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{}$$

Just a scalar mult of V_1 vectors. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is redundant

$\Rightarrow$ Any $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}2$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Prove it:

$$c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

then $\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

det $\neq 0$, invertible:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \left[\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \frac{1}{5}\right] \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

In general, If $e_1 \cdots e_n \in \mathbb{R}^n$ are linearly Dependent,

(i.e. $c_1e_1 + c_2e_2 \cdots c_ne_n = 0$ has nontrivial solutions)

then $e_m = -\frac{c_1}{c_m} e_1 - \cdots - \frac{c_{m-1}}{c_m} e_{m-1}$

then $\text{span}(e_1 \cdots e_m) = \text{span}(e_1 \cdots e_{m-1})$

continue this until you get a linearly independent set $(e_1 \cdots e_k)$.

↳ this set, $(e_1 \cdots e_k)$ is a basif for a subspace. V if:

1] $\text{span}(e_1 \cdots e_k) = V$

2] $e_1 \cdots e_k$ are linearly Independent

Standard Basis:

- $\mathbb{R}^2$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $\mathbb{R}^3$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Ex: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$\det\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \neq 0$

↳ they are inverstble.

therefore:

(1) linearly independent

(2) span $\mathbb{R}^3$

↳ any vector can be made from

$c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

- Generally given $x_1 \cdots x_n$ in $\mathbb{R}^n$ If $\det(x_1 \cdots x_n) \neq 0$, then $x_1 \cdots x_n$ form a basis for $\mathbb{R}^n$