

2. Linear Algebra 1. *Injective* iff $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Also, $\dim(\text{NULL}(A)) = 0$
2. *Surjective* iff for all $y \in Y$ there exists $x \in X$ such that $y = f(x)$. $\dim(\text{RANGE}(A)) = n$.
3. *Bijective* iff it is both injective and surjective, i.e. for all $y \in Y$ there exists a unique $x \in X$ such that $y = f(x)$.

- f has a left inverse iff it is injective.
- f has a right inverse iff it is surjective.
- f is invertible iff it is bijective.
- If f is invertible then any two inverses (left-, right- or both) coincide.

Group (G, *):

- Associative* $\forall a, b, c \in G, a * (b * c) = (a * b) * c$.
 - Identity* : $\exists e \in G, \forall a \in G, a * e = e * a = a$.
 - Inverse* : $\forall a \in G, \exists a^{-1} \in G, a * a^{-1} = a^{-1} * a = e$.
- (G, *) is commutative (or Abelian) iff in addition to 1-3:
4. *Commutative* : $\forall a, b \in G, a * b = b * a$.

Ring (R, +, ·):

+ : *associative, identity, inverse, communtative*
· : *associative, identity distributive* : $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$
Field is a *communitative Ring* that in addition satisfies *Multiplication inverse*.

Linear Space (V, F, ⊕, ⊙):

⊕ : *associative, identity, inverse, communtative* (V!!)
⊙ : *associative* $\forall a, b \in F, x \in V, a \odot (b \odot x) = (a \cdot b) \odot x$
inverse $\forall x \in V, 1 \odot x = x$

Distributive : $\forall a, b \in F, \forall x, y \in V, (a + b) \odot x = (a \odot x) \oplus (b \odot x)$ and $(a \odot (x \oplus y)) = (a \odot x) \oplus (a \odot y)$
Product Space If $(V, F, \oplus V, \odot V)$ and $(W, F, \oplus W, \odot W)$ are linear spaces over the same field, the product space $(V \times W, F, \oplus, \odot)$ is the linear space comprising all pairs $(v, w) \in V \times W$ with *defined by* $(v1, w1) \oplus (v2, w2) = (v1 \oplus v2, w1 \oplus w2)$, and *defined by* $a \odot (v, w) = (a \odot Vv, a \odot Ww)$.

Subspace Let (V, F) be a linear space and $W \subseteq V$. (W, F) is a linear subspace of V iff it's a L.S. i.e. $\forall w_1, w_2 \in W, a_1, a_2 \in F$, we have $a_1w_1 + a_2w_2 \in W$.
*In \mathbb{R}^3 , all subspaces are \mathbb{R}^3 , 2D planes through the origin, 1D lines through the origin, {0}.

*Any finite-dimensional subspace W of a linear space $(V, F, \parallel \cdot \parallel)$ is a closed subset of V.

SPAN(S) = $\{\sum_{i=1}^n a_i v_i | a_i \in F, v_i \in S, i = 1...n\}$
Let (V, F) a L.S.. A set of vectors $S \subseteq V$ is a **basis** of (V, F) iff linearly independent and $\text{Span}(S) = V$.
If a basis of (V, F) with a finite number of elements exists, the number of elements of this basis is dimension of (V, F) and (V, F) is finite dimensional. Otherwise, infinite dimensional.

Linear Map: Given (U, F) and (V, F) , the function $\mathcal{A} : U \rightarrow V$ is a linear map iff $\forall u_1, u_2 \in U, a_1, a_2 \in F$, we have $\mathcal{A}(a_1u_1 + a_2u_2) = a_1\mathcal{A}(u_1) + a_2\mathcal{A}(u_2)$.
Let $\mathcal{A} : U \rightarrow V$ linear.

NULL(A) = $\{u \in U | \mathcal{A}(u) = \theta_V\} \subseteq U$ (Nullity)
RANGE(A) = $\{v \in V | \exists u \in U : v = \mathcal{A}(u)\} \subseteq V$ (rank)
*1. A vector $u \in U$ s.t. $\mathcal{A}(u) = b$ exists iff $b \in \text{RANGE}(\mathcal{A})$. \mathcal{A} is surjective iff $\text{RANGE}(\mathcal{A}) = V$.
*2. If $b \in \text{RANGE}(\mathcal{A})$ and for some $u_0 \in U$ we have that $\mathcal{A}(u_0) = b$ then for all $u \in U : \mathcal{A}(u) = b \Leftrightarrow u = u_0 + z$ with $z \in \text{NULL}(\mathcal{A})$
*3. \mathcal{A} is injective iff $\text{NULL}(\mathcal{A}) = \{\theta_U\}$

Rank and Nullity: Let $\mathcal{A} \in F^{n \times m}$ and $B \in F^{m \times p}$.

- $\text{RANK}(\mathcal{A}) + \text{NULLITY}(\mathcal{A}) = m$.
- $0 \leq \text{RANK}(\mathcal{A}) \leq \min\{m, n\}$.
- $\text{RANK}(\mathcal{A}) + \text{RANK}(\mathcal{B}) - m \leq \text{RANK}(\mathcal{A}\mathcal{B}) \leq \min\{\text{RANK}(\mathcal{A}), \text{RANK}(\mathcal{B})\}$.
- If $P \in F^{m \times m}, Q \in F^{n \times n}$ are invertible, $\text{RANK}(\mathcal{A}) = \text{RANK}(\mathcal{A}P) = \text{RANK}(Q\mathcal{A}) = \text{RANK}(Q\mathcal{A}P)$ (also Nullity)

5. If $\mathcal{A}(x) = Ax, A \in F^{n \times n}$, we have \mathcal{A} *invertible* \Leftrightarrow *bijective* \Leftrightarrow *injective* \Leftrightarrow *surjective* $\Leftrightarrow \text{RANK}(A) = n$.
Eigenvector: 1. There exists $v \in \mathbb{C}^n$ s.t. $v \neq 0$ and $Av = \lambda v$. v is called **right eigenvector**.
2. There exists $\eta \in \mathbb{C}^n$ s.t. $\eta \neq 0$ and $\eta^T A = \lambda \eta^T$. η is called **left eigenvector**.
SPEC[A] = $\{\lambda_1, ..., \lambda_n\}$.

$$\begin{array}{ccccc} (U, F) & \xrightarrow{1_U} & (U, F) & \xrightarrow{\mathcal{A}} & (V, F) & \xrightarrow{1_V} & (V, F) \\ \{ \tilde{u}_j \}_{j=1}^n \xrightarrow{Q \in F^{n \times n}} & \{ u_j \}_{j=1}^n & \xrightarrow{A \in F^{m \times n}} & \{ v_i \}_{i=1}^m & \xrightarrow{P \in F^{m \times m}} & \{ \tilde{v}_i \}_{i=1}^m. \end{array}$$

Change of basis: $A* = P \cdot A \cdot Q$

3. Analysis

Norm: 1. $\forall v_1, v_2 \in V, \| v_1 + v_2 \| \leq \| v_1 \| + \| v_2 \|$
2. $\forall v \in V, \forall a \in F, \| av \| = |a| \cdot \| v \|$
3. $\| v \| = 0 \Leftrightarrow v = 0$

Normed Linear Space: $(V, F, \parallel \cdot \parallel)$

$\| x \|_1 = \sum_{i=1}^n |x_i|$,
 $\| x \|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$,

$\| x \|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$,
 $\| x \|_\infty = \max |x_i|$.

Ball: Given $(V, F, \parallel \cdot \parallel)$, the **ball** of radius $r \in \mathbb{R}_+$ centered at $v \in V$ is $B(v, r) = \{v' \in V | \| v - v' \| \leq r\}$.
 $B(0, 1)$ is **unit ball**.

Bound: $S \subseteq V$ is **bounded** if $S \subseteq B(0, r)$ for some r.

Convergence: Let $(V, F, \parallel \cdot \parallel)$ be a normed space. A function $v : \mathbb{N} \rightarrow V$ is called a sequence in V. The sequence converges to a point $\bar{v} \in V$ iff $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m \geq N, \| v(m) - \bar{v} \| < \epsilon$

In this case, \bar{v} is the **limit** of the sequence.

Close: iff all a set contains all its limit points.

Open: iff V \ K is closed.

Compact: Closed + Bounded.

Continuous: f is continuous at $u \in U$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\| u - u' \|_U < \delta \Rightarrow \| f(u) - f(u') \|_U < \epsilon$.
f is continuous on U iff it's continuous everywhere.

*All linear functions between finite dimensional spaces are always continuous.

Equivalence: Consider a L.S. (V, F) with two norms, $\parallel \cdot \parallel_a$ and $\parallel \cdot \parallel_b$.. Th two norms are equivalent iff $\exists m_u \geq m_l \geq 0, \forall v \in V \ m_u \| v \|_a \leq \| v \|_b \leq m_u \| v \|_a$.
Weierstrass Theorem: If $f : S \rightarrow \mathbb{R}$ is continuous and set S is compact, then f attains a minimum on S.

Cauchy Inequality:

$(\sum_{i=1}^n a_i b_i)^2 \leq (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$
Any two norms on a finite-dimensional space V are equivalent.

Infinite-dimensional normed space:

$\| f(t) \|_1 = \int_{t_0}^{t_1} \| f(t) \|_2 \ dt,$

$\| f(t) \|_2 = \sqrt{\int_{t_0}^{t_1} \| f(t) \|_2^2 \ dt},$

$\| f(t) \|_p = (\int_{t_0}^{t_1} \| f(t) \|_2^p \ dt)^{\frac{1}{p}},$

$\| f(t) \|_\infty = \max \| f(t) \|_2$.

*Replacing $\| f(t) \|_2$ by another norm on \mathbb{R}^n in the $\int_{t_0}^{t_1} dt$ and the *max* are equivalent to the ones above.

Cauchy Sequence: $\{v_i\}_{i=0}^\infty$ is a C.S. iff $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m \geq N, \| v_m - n_N \| < \epsilon$.
*Every convergent Sequence is Cauchy. But Cauchy Sequence may not converge to a point!

Complete: The normed space $(V, F, \parallel \cdot \parallel)$ is complete (or **Banach**) iff every Cauchy sequence converges.
*Let $F = \mathbb{R}$ or $F = \mathbb{C}$ and if (V, F) is finite-dimensional. Then $(V, F, \parallel \cdot \parallel)$ is a Banach Space for any norm $\parallel \cdot \parallel$.
*Many function spaces might not be Banach, but $(C([t_0, t_1], \mathbb{R}^n), \mathbb{R}, \parallel \cdot \parallel_\infty)$ is a Banach space.

Induced Norm: $\| f \| = \sup_{u \neq 0} \frac{\| f(u) \|_V}{\| u \|_U}$

* $\| \mathcal{A} \| = \sup_{\| u \|_U = 1} \| \mathcal{A}(u) \|_V$

$\| A \|_1 = \max_{j=1, ..., n} \sum_{i=1}^m |a_{ij}|$ (max column sum)
 $\| A \|_2 = \max_{\lambda \in \text{SPEC}[A^T A]} \sqrt{\lambda}$ (max singular value)
 $\| A \|_\infty = \max_{i=1, ..., m} \sum_{j=1}^n |a_{ij}|$ (max row sum)
* \mathcal{A} is continuous $\Leftrightarrow \mathcal{A}$ is continuous at 0 $\Leftrightarrow \sup_{\| u \|_U = 1} \| \mathcal{A}(u) \|_V < \infty$, the induced norm $\| \mathcal{A} \|$ is well defined.
Consider continuous linear functions $\mathcal{A}, \tilde{\mathcal{A}} : (V, F, \parallel \cdot \|_V) \rightarrow (W, F, \parallel \cdot \|_W), \ \mathcal{B} : (U, F, \parallel \cdot \|_U) \rightarrow (V, F, \parallel \cdot \|_V)$:
1. $\forall v \in V, \| (\mathcal{A})(v) \|_W \leq \| \mathcal{A} \| \cdot \| v \|_V$.
2. $\forall a \in F, \| a(\mathcal{A}) \| = |a| \cdot \| \mathcal{A} \|$.
3. $\| \mathcal{A} + \tilde{\mathcal{A}} \| \leq \| \mathcal{A} \| + \| \tilde{\mathcal{A}} \|$.
4. $\| \mathcal{A} \| = 0 \Leftrightarrow \mathcal{A}(v) = 0$ for all $v \in V$.
5. $\| \mathcal{A} \circ \mathcal{B} \| \leq \| \mathcal{A} \| \cdot \| \mathcal{B} \|$.

A function is **piecewise continuous** iff it's continuous at all $t \in \mathbb{R}$ except those in a set of discontinuity points $D \subseteq \mathbb{R}$ that satisfy:

- $\forall \tau \in D$ left and right limits of u exist, i.e.

$\lim_{t \rightarrow \tau^+} u(t)$ and $\lim_{t \rightarrow \tau^-} u(t)$ exist and are finite.

Moreover, $u(\tau) = \lim_{t \rightarrow \tau^+} u(t)$.

- $\forall t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1, D \cap [t_0, t_1]$ contains a finite number of points.

The function $p : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is **globally Lipschitz** in x iff there exists a piecewise continuous function

$k : \mathbb{R} \rightarrow \mathbb{R}_+$ s.t.

$\forall x, x' \in \mathbb{R}^n, \forall t \in \mathbb{R} \ \| p(x, t) - p(x', t) \| \leq k(t) \| x - x' \|$.

Existence and uniqueness Assume $p : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is piecewise continuous w.r.t. its second argument (with discontinuity set $D \subseteq \mathbb{R}$) and globally Lipschitz w.r.t. its first argument. Then for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique continuous function $\phi : \mathbb{R} \times \mathbb{R}^n$ s.t.:

- $\phi(t_0) = x_0$.
- $\forall t \in \mathbb{R} \setminus D, \frac{d}{dt} \phi(t) = p(\phi(t), t)$.

*Let $\parallel \cdot \parallel$ be any norm on \mathbb{R}^n . Then for all $t_0, t_1 \in \mathbb{R}$,

$\| \int_{t_0}^{t_1} f(t) dt \| \leq | \int_{t_0}^{t_1} \| f(t) \| \ dt |$

*1. $\forall m, k \in \mathbb{N}, (m + k)! \geq m!k!$.

*2. $\forall c \in \mathbb{R}, \lim_{m \rightarrow \infty} \frac{c^m}{m!} = 0$.

Gronwall: Consider $u(\cdot), k(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$ piecewise continuous, $c_1 \geq 0$, and $t_0 \in \mathbb{R}$. If for all $t \in \mathbb{R}$, we have $u(t) \leq c_1 + |\int_{t_0}^t k(\tau) u(\tau) d\tau|$. Then for all $t \in \mathbb{R}$,

$u(t) \leq c_1 \exp[\int_{t_0}^t k(\tau) d\tau]$.

Autonomous Systems: does not depends explicitly on time, $\dot{x}(t) = p(x(t))$.

* $s(t, t_0, x_0) = s(t - t_0, 0, x_0)$

4. Time varying linear systems

ẋ(t) = f(x(t), u(t)) = A(t)x(t) + B(t)u(t) (1)

y(t) = h(x(t), u(t)) = C(t)x(t) + D(t)u(t) (2)

where x(t) ∈ ℝⁿ, u(t) ∈ ℝ^m, x(t) ∈ ℝ^p,

A(·) : ℝ → ℝ^{n×n}, B(·) : ℝ → ℝ^{n×m},

C(·) : ℝ → ℝ^{p×n}, D(·) : ℝ → ℝ^{p×m}

Linearization perturbation

x(t) = x*(t) + e_x(t), y(t) = x*(t) + e_y(t)

Taylor extension of LVT

ẋ(t) = f(x*(t) + e_x(t), u*(t) + e_u(t)) = f(x*(t), u*(t)) +

∂f/∂x (x*(t), u*(t))e_x(t) + ∂f/∂u (x*(t), u*(t))e_u(t) + higher

order terms

where ∂f/∂x (x*(t), u*(t)) =

[∂f1/∂x1 (x*(t), u*(t)) ... ∂f1/∂xn (x*(t), u*(t)) ; ∂fn/∂x1 (x*(t), u*(t)) ... ∂fn/∂xn (x*(t), u*(t))] =

A(t),

∂f/∂u (x*(t), u*(t)) =

[∂f1/∂u1 (x*(t), u*(t)) ... ∂f1/∂um (x*(t), u*(t)) ; ∂fn/∂u1 (x*(t), u*(t)) ... ∂fn/∂um (x*(t), u*(t))] =

B(t)

d/dt (e_x(t)) = A(t)e_x(t) + B(t)e_u(t)

Existence and structure of solutions

	(X, ℝ)	(U, ℝ)	(Y, ℝ)
base	{e _i } _{i=1} ⁿ	{f _i } _{i=1} ^m	{g _i } _{i=1} ^p
dim.	n	m	p

Assump 4.1: A(·), B(·), C(·), D(·) are piecewise continuous. Fact 4.1: For all u(·) : ℝ → ℝ^m piecewise continuous and all (t₀, x₀) ∈ ℝ × ℝⁿ there exists UNIQUE solution x(·) :→ ℝⁿ and y(·) :→ ℝ^p for the system (1) and (2).

The unique solution of (1) and (2)

State transition matrix: x(t) = s(t, t₀, x₀, u),

Output response map: y(t) = ρ(t, t₀, x₀, u)