2. Linear Algebra

1. *Injective* iff f(x1) = f(x2) implies that x1 = x2.
2. *Surjective* iff for all $y \in Y$ there exists $x \in X$ such that y = f(x).
3. *Bijective* iff it is both injective and surjective, i.e. for all $y \in Y$ there exists a unique $x \in X$ such that y = f(x).
1. f has a left inverse iff it is injective.
2. f has a right inverse iff it is surjective.
3. f is invertible iff it is bijective.
4. If f is invertible then any two inverses (left-, right- or both) coincide.

Group (G, *):
1. $Associative \forall a, b, c \in G, a * (b * c) = (a * b) * c.$
2. $Identity : \exists e \in G, \forall a \in G, a * e = e * a = a.$
3. $Inverse : \forall a \in G, \exists a^{-1} \in G, a * a^{-1} = a^{-1} * a = e.$

(G, *) is commutative (or Abelian) iff in addition to 1-3:
4. $Commutative : \forall a, b \in G, a * b = b * a.$

Ring (R, +, ·):
$+ : associative, identity, inverse, communtative$
$\cdot : associative, identity$
$distributive : a\cdot(b+c) = a\cdot b+a\cdot c and (b+c)\cdot a = b\cdot a+c\cdot a$

Field is a *communitative Ring* that in addition satisfies *Multiplication inverse.*

Linear Space $(V, F, \oplus, \odot)$:
$\oplus : associative, identity, inverse, communtative(V!!)$
$\odot : associative \forall a, b \in F, x \in V, a \odot (b \odot x) = (a \cdot b) \odot x$
$inverse \forall x \in V, 1 \odot x = x$
$Distributive : \forall a, b \in F, \forall x, y \in V, (a + b) \odot x = (a \odot x) \oplus (b \odot x) and (a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$

Product Space $If (V, F, \oplus V, \odot V) and (W, F, \oplus W, \odot W)$ are linear spaces over the same field, the product space $(V \times W, F, \oplus, \odot)$ is the linear space comprising all pairs $(v, w) \in V \times W with \oplus defined by (v1, w1) \oplus (v2, w2) = (v1 \oplus v2, w1 \oplus w2), and \odot defined by a \odot (v, w) = (a \odot V v, a \odot W w)$.

Subspace Let (V, F) be a linear space and $W \subseteq V$. (W, F) is a linear subspace of V iff it's a L.S. i.e. $\forall w_1, w_2 \in W, a_!, a_2 \in F$, we have $a_1 w_1 + a_2 w_2 \in W$.
*In $\mathbb{R}^3$, all subspaces are $\mathbb{R}^3$, 2D planes through the origin, 1D lines through the origin, $\{0\}$.
*Any finite-dimensional subspace W of a linear space $(V, F, \| \cdot \|)$ is a closed subset of V.

SPAN(S) $= \{\sum_{i=1}^{n} a_i v_i | a_i \in F, v_i \in S, i = 1...n\}$
Let (V, F) a L.S.. A set of vectors $S \subseteq V$ is a **basis** of (V, F) iff linearly independent and Span(S) = V .
If a basis of (V, F) with a finite number of elements exists, the number of elements of this basis is dimension of (V, F) and (V, F) is finite dimensional. Otherwise, infinite dimensional.

Linear Map: Given (U, F) and (V, F), the function $\mathcal{A} : U \to V$ is a linear map iff $\forall u_1, u_2 \in U, a_1, a_2 \in F$, we have $\mathcal{A}(a_1 u_1 + a_2 u_2) = a_1 \mathcal{A}(u_1) + a_2 \mathcal{A}(u_2)$.
Let $\mathcal{A} : U \to V$ linear.
NULL$(\mathcal{A}) = \{u \in U | \mathcal{A}(u) = \theta_V\} \subseteq U$ (Nullity)
RANGE$(\mathcal{A}) = \{v \in V | \exists u \in U : v = \mathcal{A}(u)\} \subseteq V$ (rank)
*1. A vector $u \in U$ s.t. $\mathcal{A}(u) = b$ exists iff $b \in RANGE(\mathcal{A}).\mathcal{A}$ is surjective iff $RANGE(\mathcal{A}) = V$.
*2. If $b \in RANGE(\mathcal{A})$ and for some $u_0 \in U$ we have that $\mathcal{A}(u_0) = b$ then for all $u \in U : \mathcal{A}(u) = b \Leftrightarrow u = u_0 + z$ with $z \in NULL(\mathcal{A})$
*3. $\mathcal{A}$ is injective iff $NULL(\mathcal{A}) = \{\theta_U\}$

Rank and **Nullity**: Let $\mathcal{A} \in F^{n\times m}$ and $B \in F^{m\times p}$.
1. $RANK(\mathcal{A}) + NULLITY(\mathcal{A}) = m.$
2. $0 \le RANK(\mathcal{A}) \le min\{m, n\}.$
3. $RANK(\mathcal{A}) + RANK(\mathcal{B}) - m \le RANK(\mathcal{AB}) \le min\{RANK(\mathcal{A}), RANK(\mathcal{B})\}.$
4. If $P \in F^{m\times m, Q \in F^{n\times n}}$ are invertible, $RANK(\mathcal{A}) = RANK(\mathcal{AP}) = RANK(\mathcal{QA}) = RANK(\mathcal{QAP})$ (also Nullity)
5. If $\mathcal{A}(x) = Ax, A \in F^{n\times n}$, we have $\mathcal{A}\ invertible \Leftrightarrow bijective \Leftrightarrow injective \Leftrightarrow surjective \Leftrightarrow RANK(A) = n.$

Eigenvector: 1. There exists $v \in \mathbb{C}^n$ s.t. $v \ne 0$ and $Av = \lambda v$. v is called **right eigenvector**.
2. There exists $\eta \in \mathbb{C}^n$ s.t. $\eta \ne 0$ and $\eta^T A = \lambda \eta^T$. η is called **left eigenvector**.
SPEC[A] $= \{\lambda_1, ..., \lambda_n\}$.

$$\begin{array}{ccccccc} (U,F) & \xrightarrow{1_U} & (U,F) & \xrightarrow{\mathcal{A}} & (V,F) & \xrightarrow{1_V} & (V,F) \\ \{\tilde{u}_j\}_{j=1}^n & \xrightarrow{Q\in F^{n\times n}} & \{u_j\}_{j=1}^n & \xrightarrow{A\in F^{m\times n}} & \{v_i\}_{i=1}^m & \xrightarrow{P\in F^{m\times m}} & \{\tilde{v}_i\}_{i=1}^m. \end{array}$$

Change of basis: $\mathcal{A}* = P \cdot A \cdot Q$

3. Analysis

Norm:1.$\forall v_1, v_2 \in V, \| v_1 + v_2 \| \le \| v_1 \| + \| v_2 \|$
2.$\forall v \in V, \forall a \in F, \| av \| = |a| \cdot \| v \|$
3.$\| v \| = 0 \Leftrightarrow v = 0$

Normed Linear Space: $(V, F, \| \cdot \|)$
$\| x \|_1 = \sum_{i=1}^n |x_i|,$
$\| x \|_2 = \sqrt{\sum_{i=1}^n |x_i|^2},$
$\| x \|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}},$
$\| x \|_\infty = max|x_i|.$

Ball: Given $(V, F, \| \cdot \|)$, the **ball** of radius $r \in \mathbb{R}_+$ centered at $v \in V$ is $B(v, r) = \{v' \in V | \| v - v' \| \le r\}$. $B(0, 1) is$ **unit ball**.
Bound: $S \subseteq V$ is **bounded** if $S \subseteq B(0, r)$ for some r.
Convergence: Let (V, F, $\| \cdot \|$) be a normed space. A function v : N → V is called a sequence in V. The sequence converges to a point $\overline{v} \in V$ iff $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m \ge N, \| v(m) - \overline{v} \| < \epsilon$
In this case, $\overline{v}$ is the **limit** of the sequence.
Close: iff all a set contains all its limit points.
Open: iff V K is closed.
Compact: Closed + Bounded.
Continuous: f is continuous at $u \in U$ iff $\forall \epsilon > 0\ \exists \delta > 0 s.t. \| u - u' \|_U < \delta \to \| f(u) - f(u') \|_U < \epsilon.$
f is continuous on U iff it's continuous everywhere.
*All linear functions between finite dimensional spaces are always continuous.
Equivalence:Consider a L.S. (V, F) with two norms, $\| \cdot \|_a$ and $\| \cdot \|_b$.. Th two norms are equivalent iff $\exists m_u \ge m_l \ge 0, \forall v \in V\ m_l \| v \|_a \le \| v \|_b \le \| v \|_a.$
Weierstrass Theorem: If $f : S \to \mathbb{R}$ is continuous and set S is compact, then f attains a minimum on S.
Cauchy Inequality:
$(\sum_{i=1}^n a_i b_i)^2 \le (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$
Any two norms on a finite-dimensional space V are equivalent.
Infinite-dimensional normed space:
$\| f(t) \|_1 = \int_{t_0}^{t_1} \| f(t) \|_2\ dt,$
$\| f(t) \|_2 = \sqrt{\int_{t_0}^{t_1} \| f(t) \|_2^2\ dt},$
$\| f(t) \|_p = (\int_{t_0}^{t_1} \| f(t) \|_2^p\ dt)^{\frac{1}{p}},$
$\| f(t) \|_\infty = max \| f(t) \|_2 .$
*Replacing $\| f(t) \|_2$ by another norm on $\mathbb{R}^n$ in the $\int_{t_0}^{t_1} dt$ and the max are equivalent to the ones above.
Cauchy Sequence: $\{v_i\}_{i=0}^\infty$ is a C.S. iff $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall m \ge N, \| v_m - n_N \| < \epsilon.$
*Every convergent Sequence is Cauchy.
Complete: The normed space $(V, F, \| \cdot \|)$ is complete (or **Banach**) iff every Cauchy sequence converges.
*Let $F = \mathbb{R}$ or $F = \mathbb{C}$ and if (V, F) is finite-dimensional. Then $(V, F, \| \cdot \|)$ is a Banach Space for any norm $\| \cdot \|$.
*Many function spaces might not be Banach, but $(C([t_0, t_1], \mathbb{R}^n), \mathbb{R}, \| \cdot \|_\infty)$ is a Banach space.

Induced Norm: $\| f \| = \sup_{u \ne 0} \frac{\| f(u) \|_V}{\| u \|_U}$
*$\| \mathcal{A} \| = \sup_{\| u \|_U = 1} \| \mathcal{A}(u) \|_V$
$\| A \|_1 = max_{j=1,...,n} \sum_{i=1}^m |a_{ij}|$ (max column sum)
$\| A \|_2 = max_{\lambda \in SPEC[A^T A]} \sqrt{\lambda}$ (max singular value)
$\| A \|_\infty = max_{i=1,...,m} \sum_{j=1}^n |a_{ij}|$ (max row sum)
* $\mathcal{A}$ is continuous $\Leftrightarrow$ $\mathcal{A}$ is continuous at 0 $\Leftrightarrow \sup_{\| u \|_U = 1} \| \mathcal{A}(u) \|_V < \infty$, the induced norm $\| \mathcal{A} \|$ is well defined.
Consider continuous linear functions $\mathcal{A}, \tilde{\mathcal{A}} : (V, F, \| \cdot \|_V) \to (W, F, \| \cdot \|_W),\ \mathcal{B} : (U, F, \| \cdot \|_U) \to (V, F, \| \cdot \|_V)$:
1. $\forall v \in V, \| (A)(v) \|_W \le \| \mathcal{A} \| \cdot \| v \|_V .$
2. $\forall a \in F, \| a(A) \| = |a| \cdot \| \mathcal{A} \|.$
3. $\| \mathcal{A} + \tilde{\mathcal{A}} \| \le \| \mathcal{A} \| + \| \tilde{\mathcal{A}} \|.$
4. $\| \mathcal{A} \| = 0 \Leftrightarrow \mathcal{A}(v) = 0 for all v \in V.$
5. $\| \mathcal{A} \circ \mathcal{B} \le \| \mathcal{A} \| \cdot \| \mathcal{B} \|.$

A function is **piecewise continuous** iff it's continuous at all $t \in \mathbb{R}$ except those in a set of discontinuity points $D \subseteq \mathbb{R}$ that satisfiy:
1. $\forall \tau \in D$ left and right limits of u exist, i.e. $lim_{t\to\tau^+} u(t)$ and $lim_{t\to\tau^-} u(t)$ exist and are finite. Moreover, $u(\tau) = lim_{t\to\tau^+} u(t)$.
2. $\forall t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, $D \cap [t_0, t_1]$ contains a finite number of points.

The function $p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is **globally Lipschitz** in x iff there exists a piecewise continuous function $k : \mathbb{R} \to \mathbb{R}_+$ s.t.
$\forall x, x' \in \mathbb{R}^n, \forall t \in \mathbb{R}\ \| p(x, t) - p(x', t) \| \le k(t) \| x - x' \|.$

Existence and uniqueness Assume $p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is piecewise continuous w.r.t. its second argument (with discontinuity set $D \subseteq \mathbb{R}$) and globally Lipschitz w.r.t. its first argument. Then for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique continuous function $\phi : \mathbb{R} \times \mathbb{R}^n$ s.t.:
1. $\phi(t_0) = x_0.$
2. $\forall t \in \mathbb{R} \backslash D, \frac{d}{dt} \phi(t) = p(\phi(t), t).$

*Let $\| \cdot \|$ be any norm on $\mathbb{R}^n$. Then for all $t_0, t_1 \in \mathbb{R}$, $\| \int_{t_0}^{t_1} f(t)dt \| \le | \int_{t_0}^{t_1} \| f(t) \| dt|$
*1. $\forall m, k \in \mathbb{N}, (m + k)! \ge m!k!.$
*2. $\forall c \in \mathbb{R}, lim_{m\to\infty} \frac{c^m}{m!} = 0.$

Gronwall: Consider $u(.), k(.) : \mathbb{R} \to \mathbb{R}_+$ piecewice continuous, $c_1 \ge 0$, and $t_0 \in \mathbb{R}$. If for all $t \in \mathbb{R}$, we have $u(t) \le c_1 + |int_{t_0}^t k(\tau)u(\tau)d\tau|$. Then for all $t \in \mathbb{R}$, $u(t) \le c_1 exp|int_{t_0}^t k(\tau)d\tau|.$

Autonomous Systems: does not depends explicitly on time, $\dot{x}(t) = p(x(t))$.
*$s(t, t_0, x_0) = s(t - t_0, 0, x_0)$