2. Linear Algebra

1. *Injective* iff f(x1) = f(x2) implies that x1 = x2. Also, dim(NULL($\mathcal{A}$)) = 0
2. *Surjective* iff for all $y \in Y$ there exists $x \in X$ such that y = f(x). dim(RANGE($\mathcal{A}$)) = n.
3. *Bijective* iff it is both injective and surjective, i.e. for all $y \in Y$ there exists a unique $x \in X$ such that y = f(x).

1. f has a left inverse iff it is injective.
2. f has a right inverse iff it is surjective.
3. f is invertible iff it is bijective.
4. If f is invertible then any two inverses (left-, right- or both) coincide.

Group (G, *):
1. $Associative \forall a, b, c \in G, a * (b * c) = (a * b) * c.$
2. $Identity : \exists e \in G, \forall a \in G, a * e = e * a = a.$
3. $Inverse : \forall a \in G, \exists a^{-1} \in G, a * a^{-1} = a^{-1} * a = e.$

(G, *) is commutative (or Abelian) iff in addition to 1-3:
4. $Commutative : \forall a, b \in G, a * b = b * a.$

Ring (R, +, $\cdot$):
$+ : associative, identity, inverse, communtative$
$\cdot : associative, identity$
$distributive : a \cdot (b + c) = a \cdot b + a \cdot c and (b + c) \cdot a = b \cdot a + c \cdot a$

Field is a *communitative Ring* that in addition satisfies *Multiplication inverse.*

Linear Space $(V, F, \oplus, \odot)$:
$\oplus : associative, identity, inverse, commutative(V!!)$
$\odot : associative \forall a, b \in F, x \in V, a \odot (b \odot x) = (a \cdot b) \odot x$
$inverse \forall x \in V, 1 \odot x = x$
$Distributive : \forall a, b \in F, \forall x, y \in V, (a + b) \odot x =$
$(a \odot x) \oplus (b \odot x) and (a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$

Product Space $If (V, F, \oplus V, \odot V) and (W, F, \oplus W, \odot W)$ are linear spaces over the same field, the product space $(V \times W, F, \oplus, \odot)$ is the linear space comprising all pairs $(v, w) \in V \times W with \oplus\ defined\ by (v1, w1) \oplus (v2, w2) = (v1 \oplus v2, w1 \oplus w2), and \odot\ definedby a \odot (v, w) = (a \odot V v, a \odot W w).$

Subspace Let (V, F) be a linear space and $W \subseteq V$. (W, F) is a linear subspace of V iff it's a L.S. i.e. $\forall w_1, w_2 \in W, a_1, a_2 \in F$, we have $a_1 w_1 + a_2 w_2 \in W$.
*In $\mathbb{R}^3$, all subspaces are $\mathbb{R}^3$, 2D planes through the origin, 1D lines through the origin, $\{0\}$.
*Any finite-dimensional subspace W of a linear space $(V, F, \| \cdot \|)$ is a closed subset of V.

SPAN(S) $= \{\sum_{i=1}^{n} a_i v_i | a_i \in F, v_i \in S, i = 1...n\}$
Let (V, F) a L.S.. A set of vectors $S \subseteq V$ is a **basis** of (V, F) iff linearly independent and Span(S) = V .
If a basis of (V, F) with a finite number of elements exists, the number of elements of this basis is dimension of (V, F) and (V, F) is <u>finite dimensional</u>. Otherwise, infinite dimensional.

Linear Map: Given (U, F) and (V, F), the function $\mathcal{A} : U \to V$ is a linear map iff $\forall u_1, u_2 \in U, a_1, a_2 \in F$, we have $\mathcal{A}(a_1 u_1 + a_2 u_2) = a_1 \mathcal{A}(u_1) + a_2 \mathcal{A}(u_2)$.
Let $\mathcal{A} : U \to V$ linear.
NULL$(\mathcal{A}) = \{u \in U | \mathcal{A}(u) = \theta_V\} \subseteq U$ (Nullity)
RANGE$(\mathcal{A}) = \{v \in V | \exists u \in U : v = \mathcal{A}(u)\} \subseteq V$ (rank)
*1. A vector $u \in U$ s.t. $\mathcal{A}(u) = b$ exists iff $b \in RANGE(\mathcal{A}). \mathcal{A}$ is surjective iff $RANGE(\mathcal{A}) = V$.
*2. If $b \in RANGE(\mathcal{A})$ and for some $u_0 \in U$ we have that $\mathcal{A}(u_0) = b$ then for all $u \in U : \mathcal{A}(u) = b \Leftrightarrow u = u_0 + z$ with $z \in NULL(\mathcal{A})$
*3. $\mathcal{A}$ is injective iff $NULL(\mathcal{A}) = \{\theta_U\}$

Rank and **Nullity**: Let $\mathcal{A} \in F^{n \times m}$ and $B \in F^{m \times p}$.
1. $RANK(\mathcal{A}) + NULLITY(\mathcal{A}) = m.$
2. $0 \leq RANK(\mathcal{A}) \leq min\{m, n\}.$
3. $RANK(\mathcal{A}) + RANK(\mathcal{B}) - m \leq RANK(\mathcal{AB}) \leq min\{RANK(\mathcal{A}), RANK(\mathcal{B})\}.$
4. If $P \in F^{m \times m, Q \in F^{n \times n}}$ are <u>invertible</u>, $RANK(\mathcal{A}) = RANK(\mathcal{AP}) = RANK(\mathcal{QA}) = RANK(\mathcal{QAP})$ (also Nullity)
5. If $\mathcal{A}(x) = Ax, A \in F^{n \times n}$, we have $\mathcal{A}\ invertible \Leftrightarrow bijective \Leftrightarrow injective \Leftrightarrow surjective \Leftrightarrow RANK(A) = n.$

Eigenvector: 1. There exists $v \in \mathbb{C}^n$ s.t. $v \neq 0$ and $Av = \lambda v$. v is called **right eigenvector**.
2. There exists $\eta \in \mathbb{C}^n$ s.t. $\eta \neq 0$ and $\eta^T A = \lambda \eta^T$. η is called **left eigenvector**.
SPEC[A] $= \{\lambda_1, ..., \lambda_n\}$.

$$\begin{array}{ccccccc} (U,F) & \xrightarrow{1_U} & (U,F) & \xrightarrow{\mathcal{A}} & (V,F) & \xrightarrow{1_V} & (V,F) \\ \{\tilde{u}_j\}_{j=1}^{n} & \xrightarrow{Q \in F^{n \times n}} & \{u_j\}_{j=1}^{n} & \xrightarrow{A \in F^{m \times n}} & \{v_i\}_{i=1}^{m} & \xrightarrow{P \in F^{m \times m}} & \{\tilde{v}_i\}_{i=1}^{m}. \end{array}$$

Change of basis: $\mathcal{A}* = P \cdot A \cdot Q$

3. Analysis

Norm:1.$\forall v_1, v_2 \in V, \| v_1 + v_2 \| \leq \| v_1 \| + \| v_2 \|$
2.$\forall v \in V, \forall a \in F, \| av \| = |a| \cdot \| v \|$
3.$\| v \| = 0 \Leftrightarrow v = 0$

Normed Linear Space: $(V, F, \| \cdot \|)$
$\| x \|_1 = \sum_{i=1}^{n} |x_i|,$
$\| x \|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2},$
$\| x \|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}},$
$\| x \|_\infty = max|x_i|.$

Ball: Given $(V, F, \| \cdot \|)$, the **ball** of radius $r \in \mathbb{R}_+$ centered at $v \in V$ is $B(v, r) = \{v' \in V | \| v - v' \| \leq r\}$. $B(0, 1) is$ **unit ball**.
Bound: $S \subseteq V$ is **bounded** if $S \subseteq B(0, r)$ for some r.
Convergence: Let (V, F, $\| \cdot \|$) be a normed space. A function v : N $\to$ V is called a sequence in V. The sequence converges to a point $\overline{v} \in V$ iff
$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m \geq N, \| v(m) - \overline{v} \| < \epsilon$
In this case, $\overline{v}$ is the **limit** of the sequence.
Close: iff all a set contains all its limit points.
Open: iff V K is closed.
Compact: Closed + Bounded.
Continuous: f is continuous at $u \in U$ iff
$\forall \epsilon > 0\ \exists \delta > 0 s.t. \| u - u' \|_U < \delta \to \| f(u) - f(u') \|_U < \epsilon.$
f is continuous on U iff it's continuous everywhere.
*All linear functions between finite dimensional spaces are always continuous.
Equivalence:Consider a L.S. (V, F) with two norms, $\| \cdot \|_a$ and $\| \cdot \|_b$.. Th two norms are equivalent iff $\exists m_u \geq m_l \geq 0, \forall v \in V\ m_l \| v \|_a \leq \| v \|_b \leq m_u \| v \|_a.$
Weierstrass Theorem: If $f : S \to \mathbb{R}$ is <u>continuous</u> and set S is <u>compact</u>, then f attains a minimum on S.
Cauchy Inequality:
$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$
Any two norms on a finite-dimensional space V are equivalent.

Infinite-dimensional normed space:
$\| f(t) \|_1 = \int_{t_0}^{t_1} \| f(t) \|_2\ dt,$
$\| f(t) \|_2 = \sqrt{\int_{t_0}^{t_1} \| f(t) \|_2^2\ dt},$
$\| f(t) \|_p = (\int_{t_0}^{t_1} \| f(t) \|_2^p\ dt)^{\frac{1}{p}},$
$\| f(t) \|_\infty = max \| f(t) \|_2$.
*Replacing $\| f(t) \|_2$ by another norm on $\mathbb{R}^n$ in the $\int_{t_0}^{t_1} dt$ and the max are equivalent to the ones above.

Cauchy Sequence: $\{v_i\}_{i=0}^{\infty}$ is a C.S. iff
$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall m \geq N, \| v_m - n_N \| < \epsilon.$
*Every convergent Sequence is Cauchy. But Cauchy Sequence may not converge to a point!
Complete: The normed space $(V, F, \| \cdot \|)$ is complete (or **Banach**) iff every Cauchy sequence converges.
*Let $F = \mathbb{R}$ or $F = \mathbb{C}$ and if (V, F) is finite-dimensional. Then $(V, F, \| \cdot \|)$ is a Banach Space for any norm $\| \cdot \|$.
*Many function spaces might not be Banach, but $(C([t_0, t_1], \mathbb{R}^n), \mathbb{R}, \| \cdot \|_\infty)$ is a Banach space.

Induced Norm: $\| f \| = \sup_{u \neq 0} \frac{\| f(u) \|_V}{\| u \|_U}$
*$\| \mathcal{A} \| = \sup_{\| u \|_U = 1} \| \mathcal{A}(u) \|_V$
$\| A \|_1 = max_{j=1,...,n} \sum_{i=1}^{m} |a_{ij}|$ (max column sum)
$\| A \|_2 = max_{\lambda \in SPEC[A^T A]} \sqrt{\lambda}$ (max singular value)
$\| A \|_\infty = max_{i=1,...,m} \sum_{j=1}^{n} |a_{ij}|$ (max row sum)
* $\mathcal{A}$ is continuous $\Leftrightarrow$ $\mathcal{A}$ is continuous at 0 $\Leftrightarrow \sup_{\| u \|_U = 1} \| \mathcal{A}(u) \|_V < \infty$, the induced norm $\| \mathcal{A} \|$ is well defined.
Consider continuous linear functions $\mathcal{A}, \tilde{\mathcal{A}} : (V, F, \| \cdot \|_V) \to (W, F, \| \cdot \|_W),\ \mathcal{B} : (U, F, \| \cdot \|_U) \to (V, F, \| \cdot \|_V)$:
1. $\forall v \in V, \| (A)(v) \|_W \leq \| \mathcal{A} \| \cdot \| v \|_V$.
2. $\forall a \in F, \| a(A) \| = |a| \cdot \| \mathcal{A} \|$.
3. $\| \mathcal{A} + \tilde{\mathcal{A}} \| \leq \| \mathcal{A} \| + \| \tilde{\mathcal{A}} \|$.
4. $\| \mathcal{A} \| = 0 \Leftrightarrow \mathcal{A}(v) = 0 forall v \in V$.
5. $\| \mathcal{A} \circ \mathcal{B} \leq \| \mathcal{A} \| \cdot \| \mathcal{B} \|$.

A function is **piecewise continuous** iff it's continuous at all $t \in \mathbb{R}$ except those in a set of discontinuity points $D \subseteq \mathbb{R}$ that satisfiy:
1. $\forall \tau \in D$ left and right limits of u exist, i.e. $lim_{t \to \tau^+} u(t)$ and $lim_{t \to \tau^-} u(t)$ exist and are finite. Moreover, $u(\tau) = lim_{t \to \tau^+} u(t)$.
2. $\forall t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, $D \cap [t_0, t_1]$ contains a finite number of points.

The function $p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is **globally Lipschitz** in x iff there exists a piecewise continuous function $k : \mathbb{R} \to \mathbb{R}_+$ s.t.
$\forall x, x' \in \mathbb{R}^n, \forall t \in \mathbb{R}\ \| p(x, t) - p(x', t) \| \leq k(t) \| x - x' \|.$

Existence and uniqueness Assume $p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is piecewise continuous w.r.t. its second argument (with discontinuity set $D \subseteq \mathbb{R}$) and globally Lipschitz w.r.t. its first argument. Then for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique continuous function $\phi : \mathbb{R} \times \mathbb{R}^n$ s.t.:
1. $\phi(t_0) = x_0$.
2. $\forall t \in \mathbb{R} \backslash D, \frac{d}{dt} \phi(t) = p(\phi(t), t)$.

*Let $\| \cdot \|$ be any norm on $\mathbb{R}^n$. Then for all $t_0, t_1 \in \mathbb{R}$,
$\| \int_{t_0}^{t_1} f(t) dt \| \leq | \int_{t_0}^{t_1} \| f(t) \| dt|$
*1. $\forall m, k \in \mathbb{N}, (m + k)! \geq m!k!$.
*2. $\forall c \in \mathbb{R}, lim_{m \to \infty} \frac{c^m}{m!} = 0$.

Gronwall: Consider $u(.), k(.) : \mathbb{R} \to \mathbb{R}_+$ piecewice continuous, $c_1 \geq 0$, and $t_0 \in \mathbb{R}$. If for all $t \in \mathbb{R}$, we have $u(t) \leq c_1 + |int_{t_0}^{t} k(\tau) u(\tau) d\tau|$. Then for all $t \in \mathbb{R}$, $u(t) \leq c_1 exp|int_{t_0}^{t} k(\tau) d\tau|$.

Autonomous Systems: does not depends explicitly on time, $\dot{x}(t) = p(x(t))$.
*$s(t, t_0, x_0) = s(t - t_0, 0, x_0)$

4. Time varying linear systems

$\dot{x}(t) = f(x(t), u(t)) = A(t)x(t) + B(t)u(t) \quad (1)$
$y(t) = h(x(t), u(t)) = C(t)x(t) + D(t)u(t) \quad (2)$
where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, $x(t) \in \mathcal{R}^p$, $A(\cdot): \mathcal{R} \to \mathcal{R}^{n\times n}$, $B(\cdot): \mathcal{R} \to \mathcal{R}^{n\times m}$, $C(\cdot): \mathcal{R} \to \mathcal{R}^{p\times n}$, $D(\cdot): \mathcal{R} \to \mathcal{R}^{p\times m}$

Linearization perturbation
$x(t) = x^*(t) + e_x(t)$, $y(t) = x^*(t) + e_y(t)$

Taylor extension of LVT
$\dot{x}(t) = f(x^*(t) + e_x(t), u^*(t) + e_u(t)) = f(x^*(t), u^*(t)) + \frac{\partial f}{\partial x}(x^*(t), u^*(t))e_x(t) + \frac{\partial f}{\partial u}(x^*(t), u^*(t))e_u(t)$ + higher order terms
where $\frac{\partial f}{\partial x}(x^*(t), u^*(t)) =$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^*(t), u^*(t)) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*(t), u^*(t)) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^*(t), u^*(t)) & \cdots & \frac{\partial f_n}{\partial x_n}(x^*(t), u^*(t)) \end{bmatrix} =$$

$A(t)$,
$\frac{\partial f}{\partial u}(x^*(t), u^*(t)) =$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1}(x^*(t), u^*(t)) & \cdots & \frac{\partial f_1}{\partial u_m}(x^*(t), u^*(t)) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1}(x^*(t), u^*(t)) & \cdots & \frac{\partial f_n}{\partial u_m}(x^*(t), u^*(t)) \end{bmatrix} =$$

$B(t)$
$\frac{d}{dt}(e_x(t)) = A(t)e_x(t) + B(t)e_u(t)$

Existence and structure of solutions

	$(X, \mathbb{R})$	$(U, \mathbb{R})$	$(Y, \mathbb{R})$
base	$\{e_i\}_{i=1}^n$	$\{f_i\}_{i=1}^m$	$\{g_i\}_{i=1}^p$
dim.	n	m	p

Assump 4.1: $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are piecewise continuous. *Fact 4.1*: For all $u(\cdot): \mathbb{R} \to \mathbb{R}^m$ piecewise continuous and all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}_n$ there exists UNIQUE solution $x(\cdot) :\to \mathbb{R}^n$ and $y(\cdot) :\to \mathbb{R}^p$ for the system (1) and (2).
The unique solution of (1) and (2)
State transition matrix: $x(t) = s(t, t_0, x_0, u)$,
Output response map: $y(t) = \rho(t, t_0, x_0, u)$
Theorem 4.1

Theorem 4.1 *Assume that $u(\cdot)$ is piecewise continuous. Under Assumption 4.1, let D_x denote the union of the discontinuity sets of $A(\cdot)$, $B(\cdot)$ and $u(\cdot)$ and D_y the union of the discontinuity sets of $C(\cdot)$, $D(\cdot)$ and $u(\cdot)$.*

1. *For all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, $u(\cdot) \in PC(\mathbb{R}, \mathbb{R}^m)$*
   - *$x(\cdot) = s(\cdot, t_0, x_0, u) : \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable for all $t \in \mathbb{R} \setminus D_x$.*
   - *$y(\cdot) = \rho(\cdot, t_0, x_0, u) : \mathbb{R} \to \mathbb{R}^p$ is piecewise continuous with discontinuity set D_y.*
2. *For all $t, t_0 \in \mathbb{R}$, $u(\cdot) \in PC(\mathbb{R}, \mathbb{R}^m)$, $x(\cdot) = s(t, t_0, \cdot, u) : \mathbb{R}^n \to \mathbb{R}^n$ and $\rho(t, t_0, \cdot, u) : \mathbb{R}^n \to \mathbb{R}^p$ are continuous.*
3. *For all $t, t_0 \in \mathbb{R}$, $x_{01}, x_{02} \in \mathbb{R}^n$, $u_1(\cdot), u_2(\cdot) \in PC(\mathbb{R}, \mathbb{R}^m)$, $a_1, a_2 \in \mathbb{R}$*
$$s(t, t_0, a_1x_{01} + a_2x_{02}, a_1u_1 + a_2u_2) = a_1 s(t, t_0, x_{01}, u_1) + a_2 s(t, t_0, x_{02}, u_2)$$
$$\rho(t, t_0, a_1x_{01} + a_2x_{02}, a_1u_1 + a_2u_2) = a_1 \rho(t, t_0, x_{01}, u_1) + a_2 \rho(t, t_0, x_{02}, u_2).$$
4. *For all $t, t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $u \in PC(\mathbb{R}, \mathbb{R}^m)$,*
$$s(t, t_0, x_0, u) = s(t, t_0, x_0, 0) + s(t, t_0, 0, u)$$
$$\rho(t, t_0, x_0, u) = \rho(t, t_0, x_0, 0) + \rho(t, t_0, 0, u)$$

State transition matrix $\phi(t, t_0)$
$s(t, t_0, x_0, 0) = \phi(t, t_0)x_0$
Theorem 4.2 $\phi(t, t_0)$ has the following properties:
1. $\phi(\cdot, t_0) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is the UNIQUE solution of the linear matrix ODE:
$\frac{\partial}{\partial t}\phi(t, t_0) = A(t)\phi(t, t_0)$ with $\phi(t_0, t_0) = I$
Hence it is CONTINUOUS for all $t \in \mathbb{R}$ and DIFFERENTIABLE everywhere except at the discontinuity points of $A(t)$
2. $\phi(t, t_0) = \phi(t, t_1)\phi(t_1, t_0)$ for all t, t_0, t_1
3. $[\phi(t_1, t_0)]^{-1} = \phi(t0, t_1)$. $\phi(t_1, t_0)$ is invertible for all t, t_0, t_1
Fact: If A(t) and its integral commute, then
$\phi(t, t_0) = \exp^{\int_{t_0}^t A(\tau)d\tau}$

Following matrices commute with integral:
1.$A(t) = w(t) * \bar{A}$, $w : \mathbb{R} \to \mathbb{R}$ and $\bar{A}$ constant matrix
2.$A(t) \in \mathbb{R}$ scalar 3.$A(t)$ diagonal matrix 4.$A(t) = \bar{A}$ constant matrix
Theorem 4.3 for all t, $t_0 \in \mathbb{R}$, $u(\cdot) \in PC(\mathbb{R}, \mathbb{R}^m)$

$$\begin{array}{ccccc} s(t, t_0, x_0, u) & = & \Phi(t, t_0)x_0 & + & \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau \\ \text{state transition} & = & \text{zero input transition} & + & \text{zero state transition} \\ \\ \rho(t, t_0, x_0, u) & = & C(t)\Phi(t, t_0)x_0 & + & C(t)\int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t) \\ \text{output response} & = & \text{zero input response} & + & \text{zero state response.} \end{array}$$

The rule of Leibniz $\frac{d}{dt}[\int_{a(t)}^{b(t)} f(t, \tau)d\tau] = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t, \tau)d\tau + f(t, b(t))\frac{d}{dt}b(t) - f(t, a(t))\frac{d}{dt}a(t)$

5. Time invariant linear systems

$\dot{x}(t) = Ax(t) + Bu(t) \quad (3)$
$y(t) = Cx(t) + Du(t) \quad (4)$
$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} \in \mathbb{R}^{n\times n}$
Theorem 5.1 $\phi(t, t_0) = e^{A(t-t_0}$ for all t, $t_0 \in \mathbb{R}_+$
Corollary 5.1 The state transition matrix, solution, impulse transition,and impulse response of a time invariant linear system satisfying the following properties:

1. *For all $t, t_1, t_0 \in \mathbb{R}$, $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$ and $\left[e^{At}\right]^{-1} = e^{-At}$.*
2. *For all $t, t_0 \in \mathbb{R}$, $\Phi(t, t_0) = \Phi(t - t_0, 0)$.*
3. *For all $t, t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $u(\cdot) \in \mathrm{PC}(\mathbb{R}, \mathbb{R}^m)$,*
$$s(t, t_0, x_0, u) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$\rho(t, t_0, x_0, u) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$
4. *For all $t, \sigma \in \mathbb{R}$ the*
$$K(t, \sigma) = K(t - \sigma, 0) = \begin{cases} e^{A(t-\sigma)}B & \text{if } t \geq \sigma \\ 0 & \text{if } t < \sigma \end{cases}$$
$$H(t, \sigma) = H(t - \sigma, 0) = \begin{cases} Ce^{A(t-\sigma)}B + D\delta_0(t - \sigma) & \text{if } t \geq \sigma \\ 0 & \text{if } t < \sigma. \end{cases}$$

For $t_0 = 0$:
$x(t) = e^{At}x_0 + (K * u)(t)$, $y(t) = Ce^{At}x_0 + (H * u)(t)$
Semi-simple A matrix $A \in \mathbb{R}^{n\times n}$ is semi-simple if and only if there exists right eigenvectors $\{v_i\}_{i=1}^n \in \mathbb{C}^n$ that are linearly independent in the linear space $(\mathbb{C}^n, \mathbb{C})$
Theorem 5.2 Matrix $A \in \mathbb{R}^{n\times n}$ is semi-simple if and only if there exists a nonsingular matrix $T \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ such that $A = T^{-1}\Lambda T$ with $T = [v_1 v_2 \cdots v_n]^{-1}$
Fact 5.1 If A is semi-simple

$$e^{At} = T^{-1}e^{\Lambda t}T = T^{-1}\begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix} T$$

Simple A matrix $A \in \mathbb{R}^{n\times n}$ is simple if and only if its eigenvalues are distinct, i.e. $\lambda_i \neq \lambda_j$ for all $i \neq j$
Theorem 5.3 All simple matrices are semi-simple
Jordan form The algebraic multiplicity of an eigenvalue $\lambda \in \mathbb{C}$ of a matrix $A \in \mathbb{R}^{n\times n}$ is the number of times λ appears in the spectrum $SPEC[A]$. The geometric multiplicity of λ is the dimension of $NULL[A - \lambda I]$
Fact 5.2 $NULL[(A - \lambda I)^j] \subseteq NULL[(A - \lambda I)^(j + 1)]$ for any $j = 1, 2, \cdots$ and there exists $j < n$ such that $NULL[(A - \lambda I)^j] = NULL[(A - \lambda I)^(j + 1)]$

Jordan Chain

Definition 5.4 *A Jordan chain of length $\mu \in \mathbb{N}$ at eigenvalue $\lambda \in \mathbb{C}$ is a family of vectors $\{v^j\}_{j=1}^{\mu} \subseteq \mathbb{C}^n$ such that*

1. *$\{v^j\}_{j=1}^{\mu}$ are linearly independent, and*
2. *$[A - \lambda I]v^1 = 0$ and $[A - \lambda I]v^j = v^{j-1}$ for $j = 2, \ldots, \mu$.*

A Jordan chain $\{v^j\}_{j=0}^{\mu}$ is called maximal if it cannot be extended, i.e. there does not exist $v \in \mathbb{C}^n$ linearly independent from $\{v^j\}_{j=1}^{\mu}$ such that $[A - \lambda I]v = v^{\mu}$. The elements of all the maximal Jordan chains at λ are the generalized eigenvectors of λ.

Fact 5.3 *Let $\{v^j\}_{j=1}^{\mu} \subseteq \mathbb{C}^n$ be a Jordan chain of length μ at eigenvalue $\lambda \in \mathbb{C}$ of the matrix $A \in \mathbb{R}^{n\times n}$:*

1. *v^1 is an eigenvector of A with eigenvalue λ.*
2. *$v^j \in \mathrm{NULL}[(A - \lambda I)^j]$ for $j = 1, \ldots, \mu$.*

Lemma 5.1 *Assume that the matrix $A \in \mathbb{R}^{n\times n}$ has k linearly independent eigenvectors $v_1, \ldots, v_k \in \mathbb{C}^n$ with corresponding maximal Jordan chains $\{v_i^j\}_{j=1}^{\mu_i} \subseteq \mathbb{C}^n$, $i = 1, \ldots, k$. Then the collection of vectors $\{v_1^1, \ldots, v_1^{\mu_1}, \ldots, v_k^1, \ldots, v_k^{\mu_k}\}$ is linearly independent in $(\mathbb{C}^n, \mathbb{C})$ and $\sum_{i=1}^k \mu_i = n$.*

The proof of this fact is rather tedious and will be omitted, see [21]. Consider now a change of basis

$$T = \left[v_1^1 \ \ldots \ v_1^{\mu_1} \ \ldots \ v_k^1 \ \ldots \ v_k^{\mu_k}\right]^{-1} \in \mathbb{C}^{n\times n} \quad (5.5)$$

comprising the generalised eigenvectors as the columns of the matrix T^{-1}.

Theorem 5.4 (Jordan canonical form) *With the definition of T in equation (5.5), the matrix $A \in \mathbb{R}^{n\times n}$ can be written as $A = T^{-1}JT$ where $J \in \mathbb{C}^{n\times n}$ is block-diagonal*

$$J = \begin{bmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_k \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{\mu_i\times\mu_i}, \quad i = 1, \ldots, k$$

and $\lambda_i \in \mathbb{C}$ is the eigenvalue corresponding to the Jordan chain $\{v_i^j\}_{j=1}^{\mu_i}$.

Jordan canonical form and Jordan blocks The block diagonal matrix J in Theorem 5.4 is the Jordan canonical form of the matrix A and the matrices J_i are the Jordan blocks of the matrix A.
Theorem 5.5 $e^{At} = T^{-1}e^{Jt}T$, where

$$e^{Jt} = \begin{bmatrix} e^{J_1t} & 0 & \ldots & 0 \\ 0 & e^{J_2t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{J_kt} \end{bmatrix} \text{ and } e^{J_it} = \begin{bmatrix} e^{\lambda_it} & te^{\lambda_it} & \frac{t^2e^{\lambda_it}}{2!} & \ldots & \frac{t^{\mu_i-1}e^{\lambda_it}}{(\mu_i-1)!} \\ 0 & e^{\lambda_it} & te^{\lambda_it} & \ldots & \frac{t^{\mu_i-2}e^{\lambda_it}}{(\mu_i-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & e^{\lambda_it} \end{bmatrix}, \quad i = 1, \ldots, k.$$

$e^{At} = \sum_{\lambda \in SPEC[A]} \prod_{\lambda} \exp \lambda t$
Laplace Transformation $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ mapping the non-negative real numbers to the linear space of n × m real matrices:
$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt \in \mathbb{C}^{n\times m}$

Fact 5.4 *The Laplace transform (assuming that it is defined for all functions concerned) has the following properties:*

1. *It is linear, i.e. for all $A_1, A_2 \in \mathbb{R}^{p\times n}$ and all $f_1(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, $f_2(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$*
$$\mathcal{L}\{A_1f_1(t) + A_2f_2(t)\} = A_1\mathcal{L}\{f_1(t)\} + A_2\mathcal{L}\{f_2(t)\} = A_1F_1(s) + A_2F_2(s)$$
2. *$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0)$.*
3. *$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$ where $(f * g)(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{p\times m}$ denotes the convolution of $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{p\times n}$ and $g(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ defined by*
$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau.$$

Proof: Exercise, just use the definition and elementary calculus. ∎

Fact 5.5 *For all $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}_+$, $\mathcal{L}\left\{e^{At}\right\} = (sI - A)^{-1}$.*

$(sI - A)^{-1} = \frac{M(s)}{\chi_A(s)}$, where
$\chi_A(x) = s^n + \chi_1 s^{n-1} + \ldots + \chi_n$ and
$M(s) = M_0s^{n-1} + \ldots + M_{n-2}s + M_{n-1}$

Lemma 5.2 *The matrices M_i satisfy*

$$M_0 = I$$
$$M_i = M_{i-1}A + \chi_iI \quad \text{for } i = 1, \ldots, n-1$$
$$M_{n-1}A + \chi_nI = 0.$$

Theorem 5.6 (Cayley-Hamilton) *Every square matrix $A \in \mathbb{R}^{n\times n}$ satisfies its characteristic polynomial, i.e.*

$$\chi_A(A) = A^n + \chi_1A^{n-1} + \ldots + \chi_nI = 0 \in \mathbb{R}^{n\times n}.$$

Corollary 5.2 *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. For any $k \in \mathbb{N}$, A^k can be written as a linear combination of $\{I, A, A^2, \ldots, A^{n-1}\}$.*

Definition 5.6 *A matrix $A \in \mathbb{R}^{n\times n}$ is nilpotent if and only if $A^N = 0$ for some $N \in \mathbb{N}$.*

Theorem 5.7 *The following statements are equivalent:*

1. *$A \in \mathbb{R}^{n\times n}$ is nilpotent.*
2. *$A^n = 0$.*
3. *$\mathrm{SPEC}[A] = \{0, \ldots, 0\}$.*

Definition 5.7 *The function $G(\cdot) : \mathbb{C} \to \mathbb{C}^{p\times m}$ defined by*

$$G(s) = C(sI - A)^{-1}B + D$$

is called the transfer function of the system.

Definition 5.8 *The poles of the system are the values of $s \in \mathbb{C}$ are the roots of the denominator polynomial of $G(s)$.*