2. Linear Algebra

1. *Injective* iff f(x1) = f(x2) implies that x1 = x2.
2. *Surjective* iff for all $y \in Y$ there exists $x \in X$ such that y = f(x).
3. *Bijective* iff it is both injective and surjective, i.e. for all $y \in Y$ there exists a unique $x \in X$ such that y = f(x).

1. f has a left inverse iff it is injective.
2. f has a right inverse iff it is surjective.
3. f is invertible iff it is bijective.
4. If f is invertible then any two inverses (left-, right- or both) coincide.

Group (G, *):
1. $Associative \forall a, b, c \in G, a * (b * c) = (a * b) * c.$
2. $Identity : \exists e \in G, \forall a \in G, a * e = e * a = a.$
3. $Inverse : \forall a \in G, \exists a^{-1} \in G, a * a^{-1} = a^{-1} * a = e.$

(G, *) is commutative (or Abelian) iff in addition to 1-3:
4. $Commutative : \forall a, b \in G, a * b = b * a.$

Ring (R, +, $\cdot$):
$+ : associative, identity, inverse, communtative$
$\cdot : associative, identity$
$distributive : a \cdot (b+c) = a \cdot b + a \cdot c and (b+c) \cdot a = b \cdot a + c \cdot a$

Field is a $communitative\ Ring$ that in addition satisfies $Multiplication\ inverse.$

Linear Space $(V, F, \oplus, \odot)$:
$\oplus : associative, identity, inverse, communtative(V!!)$
$\odot : associative \forall a, b \in F, x \in V, a \odot (b \odot x) = (a \cdot b) \odot x$
$inverse \forall x \in V, 1 \odot x = x$
$Distributive : \forall a, b \in F, \forall x, y \in V, (a + b) \odot x =$
$(a \odot x) \oplus (b \odot x) and (a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$

Product Space $If (V, F, \oplus V, \odot V) and (W, F, \oplus W, \odot W)$ are linear spaces over the same field, the product space $(V \times W, F, \oplus, \odot)$ is the linear space comprising all pairs $(v, w) \in V \times W with \oplus defined\ by (v1, w1) \oplus (v2, w2) = (v1 \oplus v2, w1 \oplus w2), and \odot definedby a \odot (v, w) = (a \odot V v, a \odot W w).$

Subspace Let (V, F) be a linear space and $W \subseteq V$. (W, F) is a linear subspace of V iff it's a L.S. i.e. $\forall w_1, w_2 \in W, a_!, a_2 \in F$, we have $a_1 w_1 + a_2 w_2 \in W$.
*In $\mathbb{R}^3$, all subspaces are $\mathbb{R}^3$, 2D planes through the origin, 1D lines through the origin, {0}.

SPAN(S) $= \{\sum_{i=1}^{n} a_i v_i | a_i \in F, v_i \in S, i = 1...n\}$

Let (V, F) a L.S.. A set of vectors $S \subseteq V$ is a **basis** of (V, F) iff linearly independent and Span(S) = V .
If a basis of (V, F) with a finite number of elements exists, the number of elements of this basis is dimension of (V, F) and (V, F) is <u>finite dimensional</u>. Otherwise, infinite dimensional.

Linear Map: Given (U, F) and (V, F), the function $\mathcal{A} : U \to V$ is a linear map iff $\forall u_1, u_2 \in U, a_1, a_2 \in F$, we have $\mathcal{A}(a_1 u_1 + a_2 u_2) = a_1 \mathcal{A}(u_1) + a_2 \mathcal{A}(u_2)$.
Let $\mathcal{A} : U \to V$ linear.
NULL$(\mathcal{A}) = \{u \in U | \mathcal{A}(u) = \theta_V\} \subseteq U$ (Nullity)
RANGE$(\mathcal{A}) = \{v \in V | \exists u \in U : v = \mathcal{A}(u)\} \subseteq V$ (rank)
*1. A vector $u \in U$ s.t. $\mathcal{A}(u) = b$ exists iff $b \in RANGE(\mathcal{A}) . \mathcal{A}$ is surjective iff $RANGE(\mathcal{A}) = V.$
*2. If $b \in RANGE(\mathcal{A})$ and for some $u_0 \in U$ we have that $\mathcal{A}(u_0) = b$ then for all $u \in U : \mathcal{A}(u) = b \Leftrightarrow u = u_0 + z$ with $z \in NULL(\mathcal{A})$
*3. $\mathcal{A}$ is injective iff $NULL(\mathcal{A}) = \{\theta_U\}$

Rank and **Nullity**: Let $\mathcal{A} \in F^{n \times m}$ and $B \in F^{m \times p}$.
1. $RANK(\mathcal{A}) + NULLITY(\mathcal{A}) = m.$
2. $0 \leq RANK(\mathcal{A}) \leq min\{m, n\}.$
3. $RANK(\mathcal{A}) + RANK(\mathcal{B}) - m \leq RANK(\mathcal{AB}) \leq min\{RANK(\mathcal{A}), RANK(\mathcal{B})\}.$
4. If $P \in F^{m \times m, Q \in F^{n \times n}}$ are <u>invertible</u>, $RANK(\mathcal{A}) = RANK(\mathcal{AP}) = RANK(\mathcal{QA}) = RANK(\mathcal{QAP})$ (also Nullity)
5. If $\mathcal{A}(x) = Ax, A \in F^{n \times n}$, we have $\mathcal{A}\ invertible \Leftrightarrow bijective \Leftrightarrow injective \Leftrightarrow surjective \Leftrightarrow RANK(A) = n.$

Eigenvector: 1. There exists $v \in \mathbb{C}^n$ s.t. $v \neq 0$ and $Av = \lambda v$. v is called **right eigenvector**.
2. There exists $\eta \in \mathbb{C}^n$ s.t. $\eta \neq 0$ and $\eta^T A = \lambda \eta^T$. η is called **left eigenvector**.
SPEC[A] $= \{\lambda_1, ..., \lambda_n\}$.

$$\begin{array}{ccccccc} (U, F) & \xrightarrow{1_U} & (U, F) & \xrightarrow{\mathcal{A}} & (V, F) & \xrightarrow{1_V} & (V, F) \\ \{\tilde{u}_j\}_{j=1}^n & \xrightarrow{Q \in F^{n \times n}} & \{u_j\}_{j=1}^n & \xrightarrow{A \in F^{m \times n}} & \{v_i\}_{i=1}^m & \xrightarrow{P \in F^{m \times m}} & \{\tilde{v}_i\}_{i=1}^m. \end{array}$$

Change of basis: $\mathcal{A}* = P \cdot A \cdot Q$

3. Analysis

Norm:1.$\forall v_1, v_2 \in V, \| v_1 + v_2 \| \leq \| v_1 \| + \| v_2 \|$
2.$\forall v \in V, \forall a \in F, \| av \| = |a| \cdot \| v \|$
3.$\| v \| = 0 \Leftrightarrow v = 0$

Normed Linear Space: $(V, F, \| \cdot \|)$
$\| x \|_1 = \sum_{i=1}^n |x_i|,$
$\| x \|_2 = \sqrt{\sum_{i=1}^n |x_i|^2},$
$\| x \|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}},$
$\| x \|_\infty = max|x_i|.$

Ball: Given $(V, F, \| \cdot \|)$, the **ball** of radius $r \in \mathbb{R}_+$ centered at $v \in V$ is $B(v, r) = \{v' \in V | \| v - v' \| \leq r\}$. $B(0, 1) is$ **unit ball**.
Bound: $S \subseteq V$ is **bounded** if $S \subseteq B(0, r)$ for some r.
Convergence: Let (V, F, $\| \cdot \|$) be a normed space. A function v : N $\to$ V is called a sequence in V. The sequence converges to a point $\overline{v} \in V$ iff
$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m \geq N, \| v(m) - \overline{v} \| < \epsilon$
In this case, $\overline{v}$ is the **limit** of the sequence.
Close: iff all a set contains all its limit points.
Open: iff V K is closed.
Compact: Closed + Bounded.