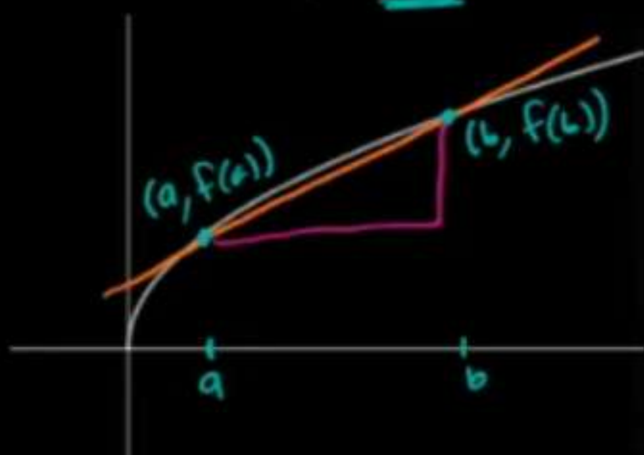
For a function $y = f(x)$,



Definition. A secant line is a line between two points on the graph of a function

Definition. The average rate of change for $f(x)$ on the interval $[a, b]$ is the slope of the secant line between the two points $(a, f(a))$ and $(b, f(b))$

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \boxed{\frac{f(b) - f(a)}{b - a}}$$

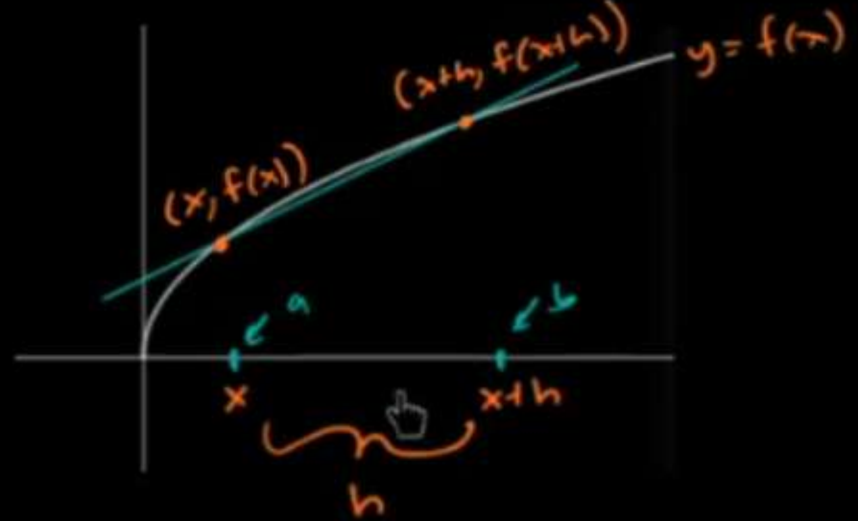
Ex: $f(x)$ is height of tree (inches)
x is time in years

$$m = \frac{10 \text{ in}}{2 \text{ yr}} = 5 \frac{\text{in}}{\text{yr}}$$

Example. The average rate of change for $f(x) = \sqrt{x}$ on the interval $[1, 4]$ is

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{4} - \sqrt{1}}{3} = \frac{2 - 1}{3} = \frac{1}{3}$$

Definition. A difference quotient represents the average rate of change of a fn $f(x)$ on the interval $[x, x+h]$



i.e. the slope of the secant line btwn the points $(x, f(x))$ and $(x+h, f(x+h))$

$$m = \frac{f(b)-f(a)}{b-a} = \frac{f(x+h)-f(x)}{(x+h)-(x)}$$

$$= \frac{f(x+h)-f(x)}{\cancel{x}+h-\cancel{x}}$$

$$= \boxed{\frac{f(x+h)-f(x)}{h}}$$

Example. Find and simplify the difference quotient for $f(x) = 2x^2 - x + 3$

difference quotient $= \frac{f(x+h) - f(x)}{h}$

$$\begin{aligned}
f(x+h) &= 2(x+h)^2 - (x+h) + 3 \\
&= 2(x+h)(x+h) - x - h + 3 \\
&= 2(x^2 + xh + hx + h^2) - x - h + 3 \\
&= 2x^2 + \underbrace{2xh + 2hx} + 2h^2 - x - h + 3 \\
&= \underbrace{2x^2 + 4xh + 2h^2 - x - h + 3}
\end{aligned}$$

$$\begin{aligned}
f(x+h) - f(x) &= 2x^2 + 4xh + 2h^2 - x - h + 3 - (2x^2 - x + 3) \\
&= \cancel{2x^2} + 4xh + 2h^2 \cancel{- x} - h \cancel{+ 3} \cancel{- 2x^2} \cancel{+ x} \cancel{- 3} \\
&= 4xh + 2h^2 - h
\end{aligned}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2 - h}{h} = \frac{\cancel{h}(4x + 2h - 1)}{\cancel{h}}$$

$$= 4x + 2h - 1$$

Example. Julia's sushi and salad buffet costs \$10 per pound, but if you get exactly one pound, your meal is free. Let $y = f(x)$ represent the price of your lunch in dollars as a function of its weight x in pounds.

Write an equation $y = f(x)$ as a piecewise defined function.

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 10x & \text{if } x \neq 1 \end{cases}$$

Draw a graph of $y = f(x)$.

Describe the behavior of $f(x)$ when x is near 1 but not equal to 1.

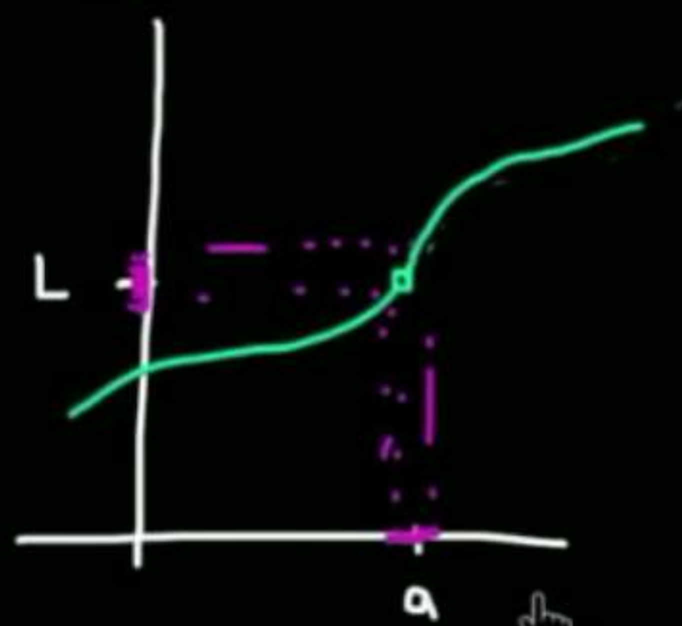
$$\lim_{x \to 1} f(x) = 10 \qquad \text{as } x \to 1,\ f(x) \to 10$$

not equal

$$f(1) = 0$$

Definition. (Informal definition) For real numbers a and L, $\lim_{x \to a} f(x) = L$ means

f(x) gets arbitrarily close to L
as x gets arbitrarily close to a.

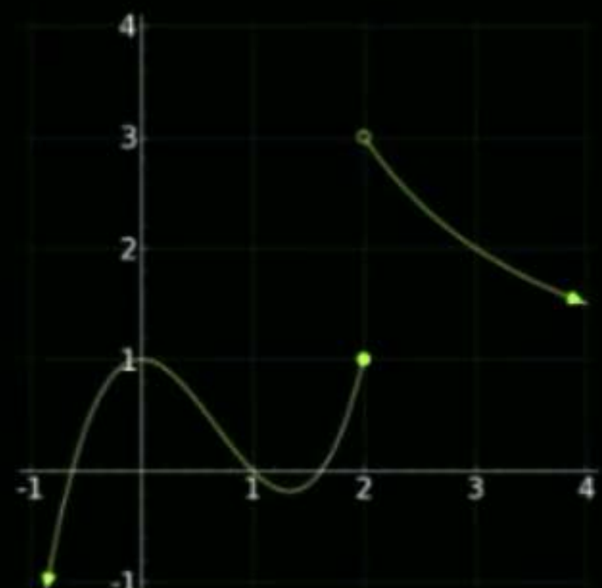


$$\lim_{x \to a} f(x) = L$$

when it approach
from both side, form
left and right

ONE SIDED LIMITS

Example. Describe the behavior of $y = g(x)$ when x is near 2.



$\lim_{x \to 2} g(x)$ DNE

"left-sided limit"

$\boxed{\lim_{x \to 2^-} g(x) = 1}$ $\quad x < 2$, $x \to 2$

"right-sided limits"

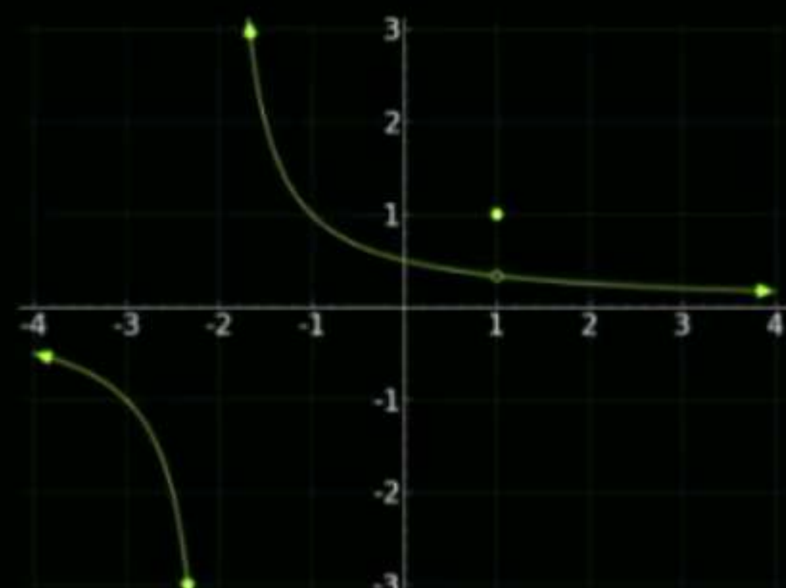
$\boxed{\lim_{x \to 2^+} g(x) = 3}$ $\quad x > 2$, $x \to 2$

Definition. (Informal definition) For real numbers a and L,

$\lim_{x \to a^-} f(x) = L$ means f(x) approaches L as x approaches a from the left

$\lim_{x \to a^+} f(x) = L$ means f(x) approaches L as x approaches a from the right

Example. Describe the behavior of $y = h(x)$ when x is near -2 in terms of limits.



$$\lim_{x \to -2^+} h(x) = \infty$$

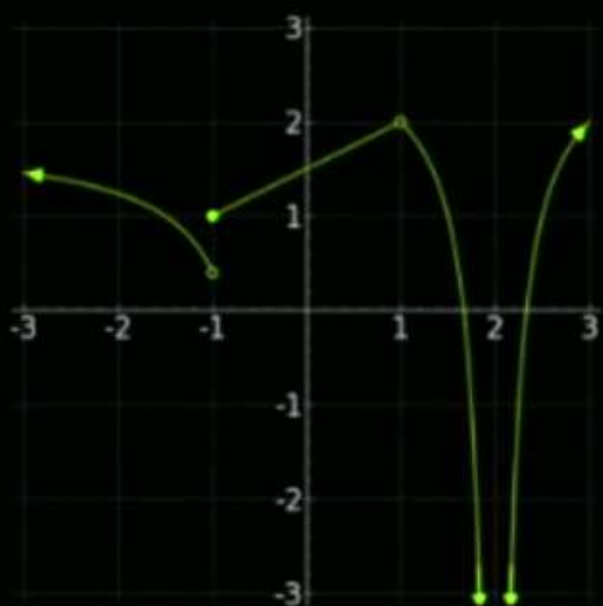
DNE as a finite number

$$\lim_{x \to -2^-} h(x) = -\infty$$

DNE as a finite number

$$\lim_{x \to -2} h(x) \text{ DNE}$$

Example. For the function $f(x)$ graphed below,



What is

$\lim_{x\to-1^-} f(x) = \frac{1}{2}$ $\qquad$ $\lim_{x\to-1^+} f(x) = 1$ $\qquad$ $\lim_{x\to-1} f(x)$ DNE

$\lim_{x\to1^-} f(x) = 2$ $\qquad$ $\lim_{x\to1^+} f(x) = 2$ $\qquad$ $\lim_{x\to1} f(x) = 2$ $\qquad$ Note: $f(1)$ DNE

$\lim_{x\to2^-} f(x) = -\infty$ $\qquad$ $\lim_{x\to2^+} f(x) = -\infty$ $\qquad$ $\lim_{x\to2} f(x) = -\infty$ $\qquad$ OR $\lim_{x\to2} f(x)$ DNE

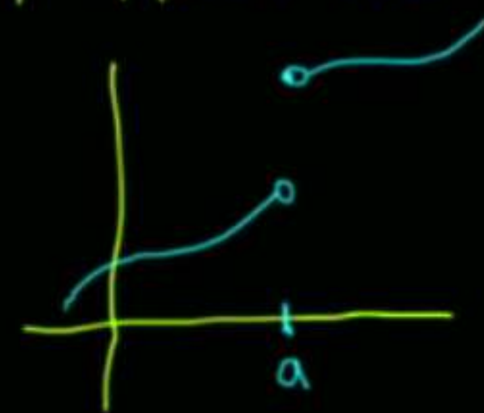
At what values of a does $\lim_{x\to a} f(x)$ fail to exist?

$x = -1, 2$

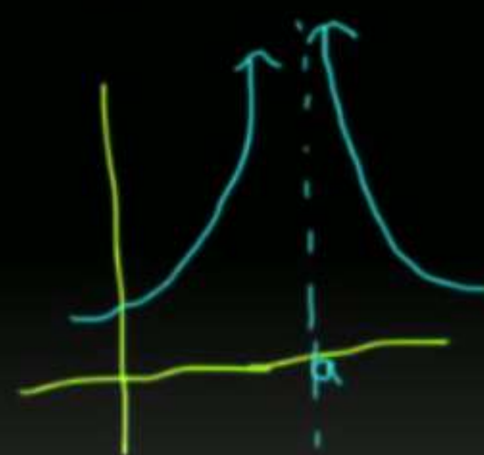
I mean, at what value of a does the limit as x -->a fail to exist.

Note. Ways that limits can fail to exist:

1) limit from the left ≠ limit from the right

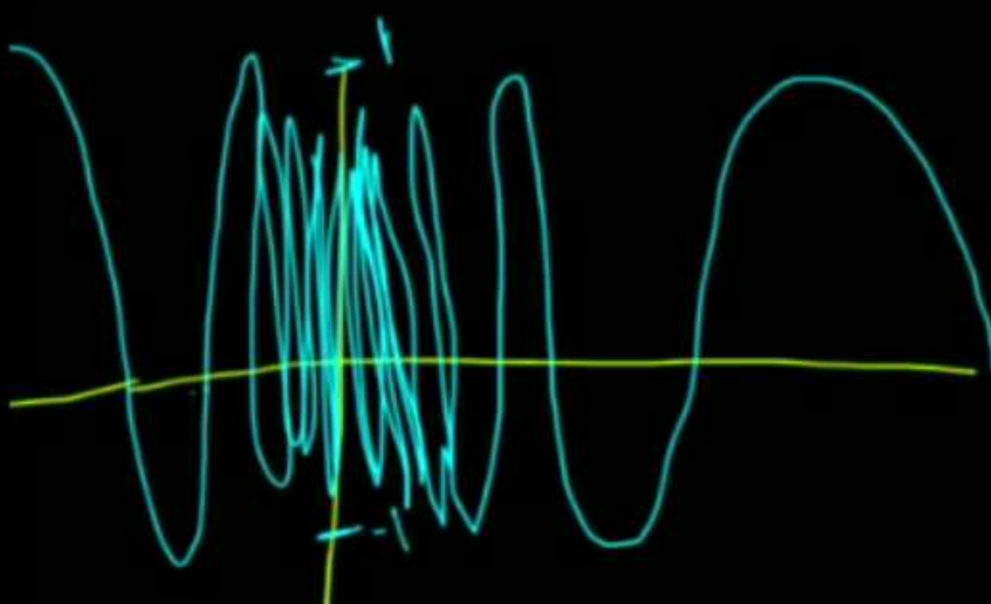


2) vertical asymptotes



3) wild behavior

Example. What is $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$? Use a graph or a table of values for evidence.



$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) \text{ DNE}$$

eg. of "Wild behaivour"

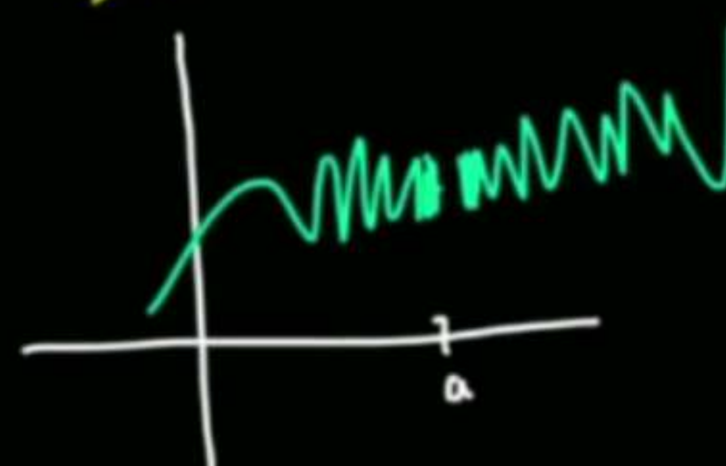
note: wild behaivour is
not technical term.

§2.2.2 When Limits Fail to Exist

1) one-sided limits on the left and on the right are not equal

2) vertical asymptotes

3) wild behavior



Example. Find $\lim_{x\to 2} \frac{x^2+3x+6}{x+9}$

$$= \frac{\lim_{x\to 2} x^2+3x+6}{\lim_{x\to 2} x+9}$$

$$= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} 3x + \lim_{x\to 2} 6}{\lim_{x\to 2} x + \lim_{x\to 2} 9}$$

$$\lim_{x\to 2} x = 2$$

$$= \frac{\left(\lim_{x\to 2} x\right)^2 + 3\lim_{x\to 2} x + \lim_{x\to 2} 6}{\lim_{x\to 2} x + \lim_{x\to 2} 9}$$

$$\lim_{x\to 2} 6 = 6$$

$$= \frac{2^2+3\cdot 2+6}{2+9} = \frac{16}{11}$$

§2.3.1 Limit Laws

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist (as finite numbers, not just ∞ or $-\infty$)

$$\lim_{x\to a} f(x)+g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

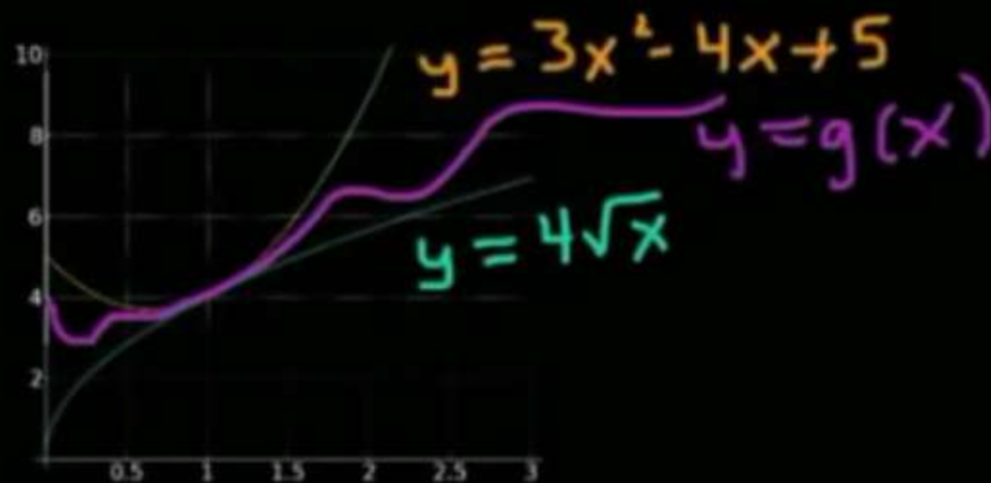
$$\lim_{x\to a} f(x)-g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$\vdots$

etc.

Example. For the function $g(x)$ suppose that, for x-values near 1,

$$4\sqrt{x} \leq g(x) \leq 3x^2 - 4x + 5$$



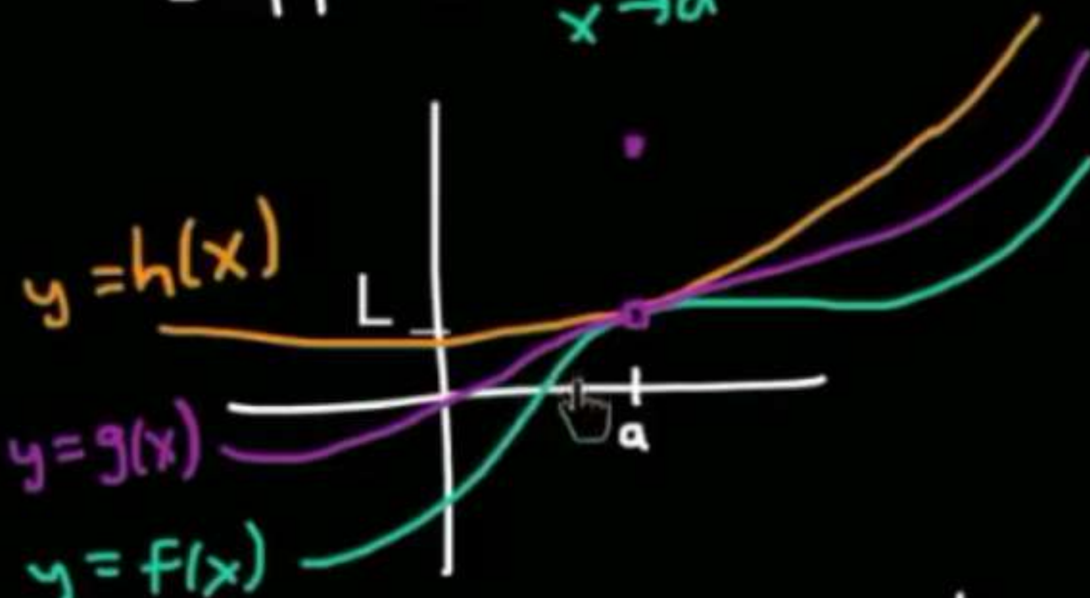
What can we say about $\lim_{x \to 1} g(x)$?

$$\lim_{x \to 1} g(x) = 4$$

Theorem. *(The Squeeze Theorem)*

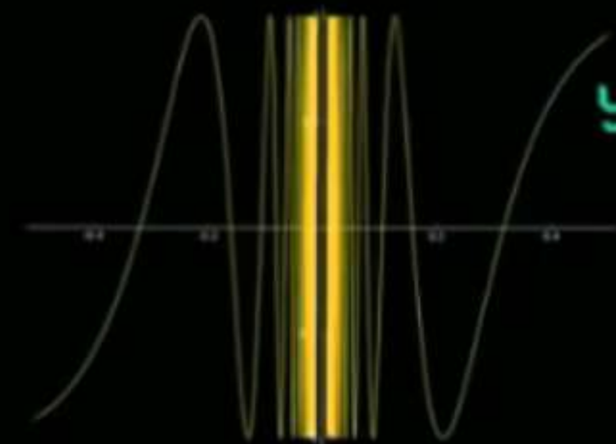
Suppose $f(x) \le g(x) \le h(x)$ for x-values near a. (not nec for $x = a$)

Suppose $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$



Then $\lim_{x \to a} g(x) = L$.

Example. Find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.



$y = \sin\left(\frac{1}{x}\right)$

Tempting:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ DNE

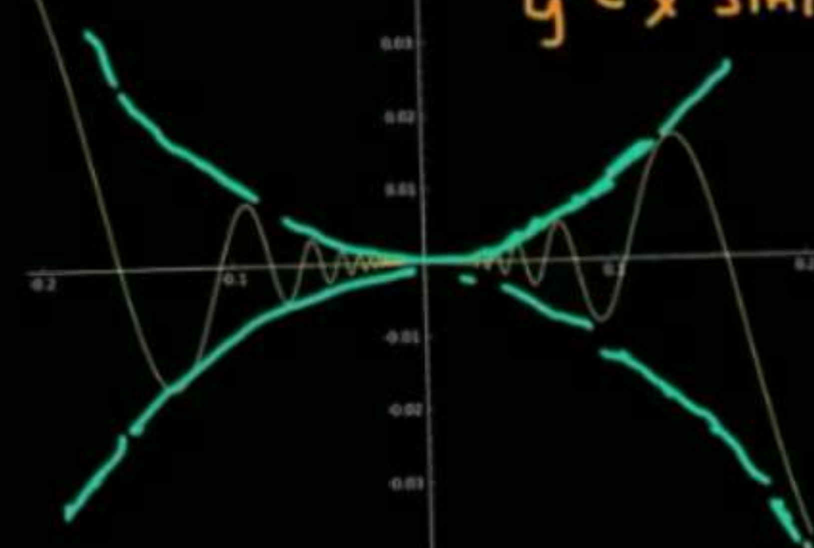
Squeeze Thm

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$\lim_{x\to 0} x^2 = 0$ $\quad$ $\lim_{x\to 0} -x^2 = 0$

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$



$y = x^2 \sin\left(\frac{1}{x}\right)$

Use algebra to find the following limits. If the limit DNE, decide if it is ∞, $-\infty$, or neither.

1. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$ → factor
2. $\lim_{z \to 0} \frac{(5-z)^2 - 25}{z}$ → multiply out
3. $\lim_{r \to 0} \frac{\frac{1}{r+3} - \frac{1}{3}}{r}$ → add together fractions & simplify
4. $\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1}$ → conjugate
5. $\lim_{x \to -5} \frac{2x + 10}{|x + 5|}$ → one-sided limits & cases

$\frac{0}{0}$ indeterminate form

$\lim_{x \to a} \frac{f(x)}{g(x)}$ where $\lim_{x \to a} f(x) = 0$

$\lim_{x \to a} g(x) = 0$

1. $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)}$$

$$= \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \boxed{\frac{3}{2}}$$

2. $\lim_{z\to 0} \frac{(5-z)^2 - 25}{z}$

$$= \lim_{z\to 0} \frac{\cancel{25} - 10z + z^2 - \cancel{25}}{z}$$

$$= \lim_{z\to 0} \frac{-10z + z^2}{z}$$

$$= \lim_{z\to 0} \frac{\cancel{z}(-10+z)}{\cancel{z}}$$

$$= \lim_{z\to 0} \frac{-10+z}{1} = \boxed{-10}$$

3. $\lim_{r\to 0} \frac{\frac{1}{r+3}-\frac{1}{3}}{r} = \lim_{r\to 0} \frac{\frac{1}{r+3}\cdot\frac{3}{3}-\frac{1}{3}\cdot\frac{r+3}{r+3}}{r}$ $\quad (r+3)3$

$$= \lim_{r\to 0} \frac{\frac{3-(r+3)}{(r+3)3}}{r}$$

$$= \lim_{r\to 0} \frac{\frac{\cancel{3}-r-\cancel{3}}{(r+3)3}}{\frac{r}{1}}$$

$$= \lim_{r\to 0} \frac{-\cancel{r}}{(r+3)3}\cdot\frac{1}{\cancel{r}}$$

$$= \lim_{r\to 0} \frac{-1}{(r+3)3} = \frac{-1}{(0+3)3} = \boxed{-\frac{1}{9}}$$

4. $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} =$

$a-b$
$a+b$

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$\lim_{x\to 1} \frac{(\sqrt{x+3})^2 + \cancel{2\sqrt{x+3}} - \cancel{2\sqrt{x+3}} - 4}{x\sqrt{x+3} + 2x - \sqrt{x+3} - 2}$$

$$= \lim_{x\to 1} \frac{x+3-4}{x\sqrt{x+3} + 2x - \sqrt{x+3} - 2}$$

$$= \lim_{x\to 1} \frac{x-1}{\sqrt{x+3}\,(x-1) + 2(x-1)}$$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \boxed{\frac{1}{4}}$$

5. $\lim_{x\to -5} \frac{2x+10}{\underbrace{|x+5|}}$

$$|x+5| = \begin{cases} x+5 & x+5>0 \quad (x>-5) \\ -(x+5) & x+5<0 \quad (x<-5) \end{cases}$$

$$\lim_{x\to -5^-} \frac{2x+10}{-(x+5)}$$

$$= \lim_{x\to -5^-} \frac{2\cancel{(x+5)}}{-\cancel{(x+5)}}$$

$$= \lim_{x\to -5^-} \frac{2}{-1} = \underline{-2}$$

$$\lim_{x\to -5^+} \frac{2x+10}{x+5}$$

$$= \lim_{x\to -5^+} \frac{2\cancel{(x+5)}}{\cancel{x+5}}$$

$$\underline{= 2}$$

$$\lim_{x\to -5} \frac{2x+10}{|x+5|} \quad \text{DNE}$$

Example. Find $\lim_{x \to 3} \frac{-4x}{x-3}$

$\lim_{x\to 3} -4x = -12 \qquad \lim_{x\to 3} x-3 = 0$

$$\boxed{\lim_{x\to 3^-} \frac{-4x}{x-3} = \infty}$$

x	$\frac{-4x}{x-3}$
2.9	116
2.99	1196
2.999	11996

$\approx \frac{-12}{-0.1} = 120$

$\approx \frac{-12}{-0.01} = 1200$

$\approx \frac{-12}{-0.001} = 12{,}000$

$\frac{-4x}{x-3} \Rightarrow \frac{-}{-} = +$

$$\boxed{\lim_{x\to 3^+} \frac{-4x}{x-3} = -\infty}$$

x	$\frac{-4x}{x-3}$
3.1	-124
3.01	-1204
3.001	$-12{,}004$

$\approx \frac{-12}{0.1} = -120$

$\frac{-4x}{x-3} \Rightarrow \frac{-}{+} \Rightarrow -$

$$\boxed{\lim_{x\to 3} \frac{-4x}{x-3} \text{ DNE}}$$

Example. Find $\lim_{x \to -4} \frac{5x}{|x+4|}$

$\lim_{x \to -4} 5x = -20$

$\lim_{x \to -4} |x+4| = 0$

$$|x+4| = \begin{cases} x+4 & x > -4 \\ -(x+4) & x < -4 \end{cases}$$

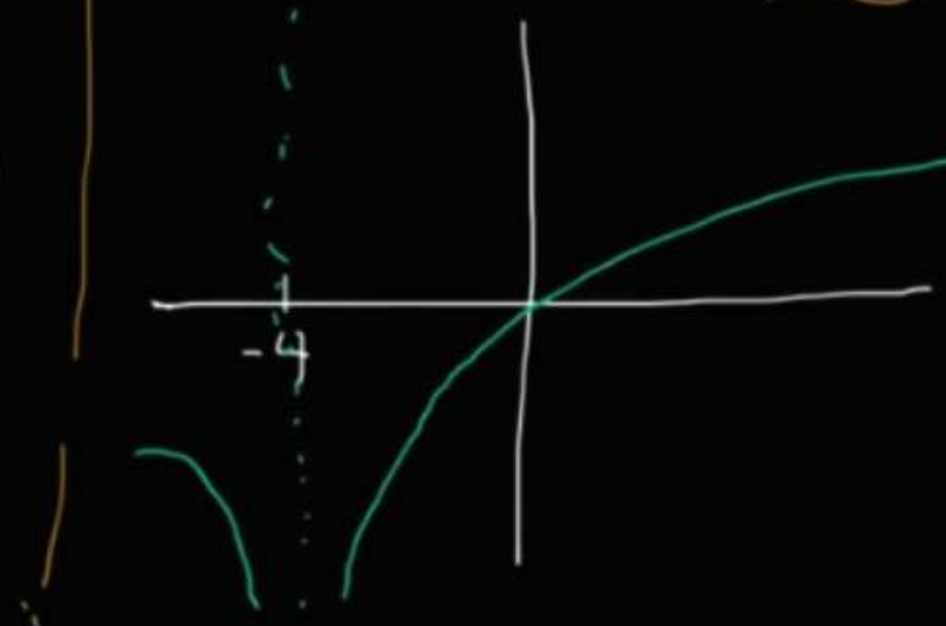
$$\lim_{x \to -4^-} \frac{5x}{|x+4|} = -\infty$$

$$\lim_{x \to -4^-} \frac{5x}{-(x+4)} \Rightarrow \frac{-}{+} \Rightarrow -$$

$$\lim_{x \to -4^+} \frac{5x}{|x+4|} = -\infty$$

$$\lim_{x \to -4^+} \frac{5x}{x+4} \Rightarrow \frac{-}{+} \Rightarrow -$$

$$\lim_{x \to -4} \frac{5x}{|x+4|} = -\infty$$



Note. If $\lim_{x\to a} f(x) \neq 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$

could be $-\infty$
could be ∞
could just be DNE
if one sided limits are ∞, $-\infty$

Note. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$

could exist
could not exist

Limits of this form are called $\frac{0}{0}$ indeterminate form

Limit Law:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

as long as these limits exist

and provided that $\lim_{x\to a} g(x) \neq 0$

What if $\lim_{x\to a} g(x) = 0$?

① $\lim_{x\to a} f(x) \neq 0$

$\lim_{x\to a} g(x) = 0$

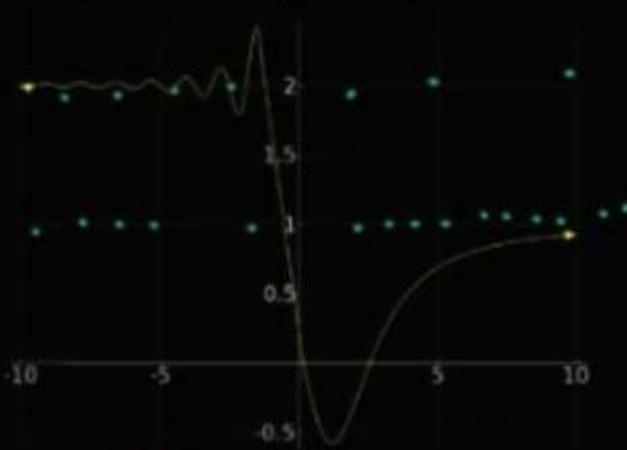
V.A.

$\infty, -\infty$

② $\lim_{x\to a} f(x) = 0$

$\lim_{x\to a} g(x) = 0$

Example. What happens to $f(x)$ as x goes through larger and larger positive numbers? Larger and larger negative numbers?



$$\lim_{x \to \infty} f(x) = 1$$

$$\lim_{x \to -\infty} f(x) = 2$$

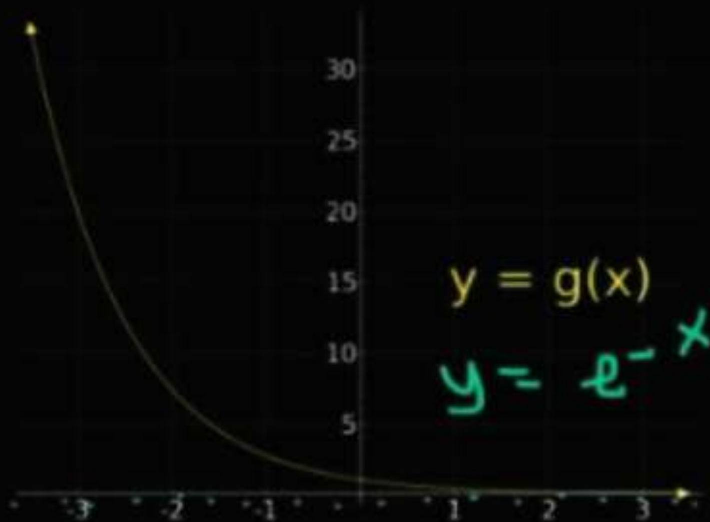
"limits at infinity" means $x \to \infty$ or $x \to -\infty$

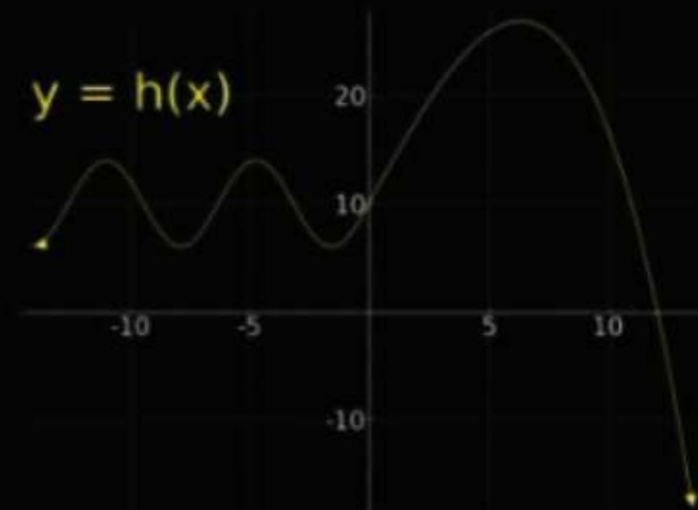
horizontal asymptotes

"infinite limits" means $f(x) \to \infty$ or $f(x) \to -\infty$

vertical asymptotes

Example. For the functions $g(x)$ and $h(x)$ drawn below, what are the limits at infinity?





$$\lim_{x \to \infty} g(x) = 0$$

$$\lim_{x \to -\infty} g(x) = \infty$$

$$\lim_{x \to \infty} h(x) = -\infty$$

$$\lim_{x \to -\infty} h(x) \text{ DNE}$$

Example. What are $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$?

x	$\frac{1}{x}$
10	$\frac{1}{10} = 0.1$
100	$\frac{1}{100} = 0.01$
1000	$\frac{1}{1000} = 0.001$

x	$\frac{1}{x}$
-10	$-\frac{1}{10} = -0.1$
-100	$-\frac{1}{100} = -0.01$
-1000	$-\frac{1}{1000} = -0.001$

$\lim_{x\to\infty} \frac{1}{x} = 0$ $\qquad$ $\lim_{x\to-\infty} \frac{1}{x} = 0$

$\lim_{x\to\infty} f(x)$

$\lim_{x\to-\infty} f(x)$

could be 0, any other number, ∞, $-\infty$, DNE

Example. What are $\lim_{x\to\infty} \frac{1}{x^3} = 0$ and $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

as $x \to \infty$ $x^3 \to \infty$ $\qquad$ as $x \to \infty$ $\sqrt{x} \to \infty$

so $\frac{1}{x^3} \to 0$ $\qquad$ so $\frac{1}{\sqrt{x}} \to 0$

$\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}$

$\lim_{x\to\infty} \frac{1}{x^r}$ $\quad r > 0$

Note. $\lim_{x\to\infty} \frac{1}{x^r} = 0$ whenever $r > 0$.

Also $\lim_{x\to-\infty} \frac{1}{x^r} = 0$ for $r > 0$ (avoid values of r like $\frac{1}{2}$ that don't make sense for $x < 0$)

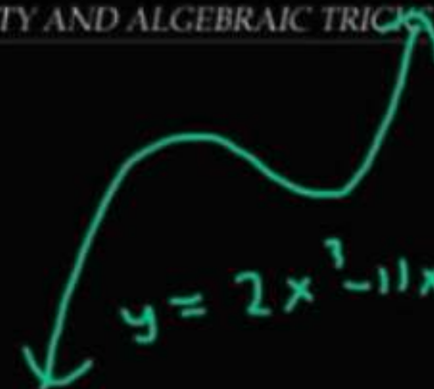
For $r < 0$ e.g. $r = -2$

$\frac{1}{x^{-2}} = x^2$ $\quad \lim_{x\to\infty} x^2 = \infty \neq 0$

① factor out highest power terms

② degree of num & denom, highest power terms

Example. Evaluate $\lim_{x\to\infty} \dfrac{5x^2 - 4x}{2x^3 - 11x^2 + 12x}$



$\frac{\infty}{\infty}$ indeterminate form

$$\lim_{x\to\infty} \frac{\cancel{x^2}\left(5 - \frac{4}{x}\right)}{\cancel{x^3}^{\,1}\left(2 - \frac{11}{x} + \frac{12}{x^2}\right)}$$

$$= \lim_{x\to\infty} \underbrace{\frac{1}{x}}_{\to 0} \cdot \frac{\left(5 - \underbrace{\cancel{\frac{4}{x}}}_{\to 0}\right)}{\left(2 - \underbrace{\cancel{\frac{11}{x}}}_{\to 0} + \underbrace{\cancel{\frac{12}{x^2}}}_{\to 0}\right)}$$

$$= 0 \cdot \frac{5}{2} = \boxed{0}$$

Example. Find $\lim_{x\to-\infty} \dfrac{3x^3+6x^2+10x+2}{2x^3+x^2+5}$

$$= \lim_{x\to-\infty} \frac{\cancel{x^3}\left(3+\frac{6}{x}+\frac{10}{x^2}+\frac{2}{x^3}\right)}{\cancel{x^3}\left(2+\frac{1}{x}+\frac{5}{x^3}\right)}$$

$\frac{6}{x}\to 0$, $\frac{10}{x^2}\to 0$, $\frac{2}{x^3}\to 0$, $\frac{1}{x}\to 0$, $\frac{5}{x^3}\to 0$

$$= \frac{3}{2}$$

Example. Find $\lim\limits_{x \to -\infty} \dfrac{x^4 - 3x^2 + 6}{-5x^2 + x + 2}$

$$= \lim_{x \to -\infty} \frac{x^4\left(1 - \frac{3}{x^2} + \frac{6}{x^4}\right)}{x^2\left(-5 + \frac{1}{x} + \frac{2}{x^2}\right)}$$

$$= \lim_{x \to -\infty} \frac{x^2}{1} \frac{\left(1 - \cancel{\frac{3}{x^2}}^{\to 0} + \cancel{\frac{6}{x^4}}^{\to 0}\right)}{\left(-5 + \underbrace{\cancel{\frac{1}{x}}}_{0} + \underbrace{\cancel{\frac{2}{x^2}}}_{0}\right)}$$

$$= \lim_{x \to -\infty} \underbrace{x^2}_{\infty} \underbrace{\left(\frac{1}{-5}\right)}_{-}$$

$$= -\infty$$

Recap:

$$\lim_{x\to\infty} \frac{5x^2 - 4x}{2x^3 - 11x^2 + 12x} = \lim_{x\to\infty} \frac{5x^2}{2x^3} = \lim_{x\to\infty} \frac{5}{2x} = 0$$

$$\lim_{x\to-\infty} \frac{3x^3 + 6x^2 + 10x + 2}{2x^3 + x^2 + 5} = \lim_{x\to-\infty} \frac{3x^3}{2x^3} = \lim_{x\to-\infty} \frac{3}{2} = \frac{3}{2}$$

$$\lim_{x\to-\infty} \frac{x^4 - 3x^2 + 6}{-5x^2 + x + 2} = \lim_{x\to-\infty} \frac{x^4}{-5x^2} = \lim_{x\to-\infty} \frac{x^2}{-5} = -\infty$$

① factor out highest power terms

② degree of num & denom, highest power terms

---

degree of numerator < degree of denom

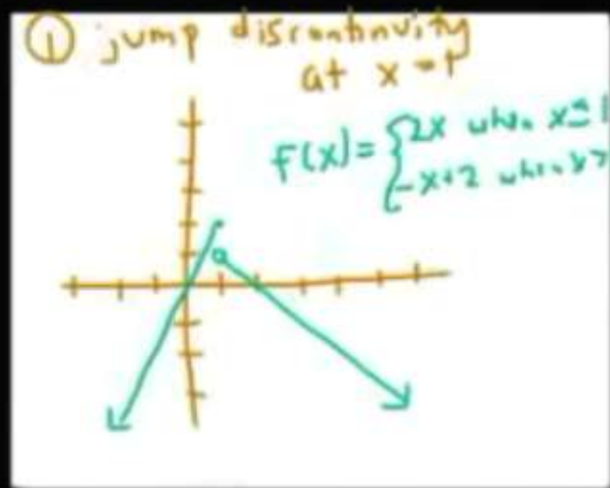
$\Rightarrow \lim\limits_{\substack{x\to\infty \\ -\infty}} = 0$

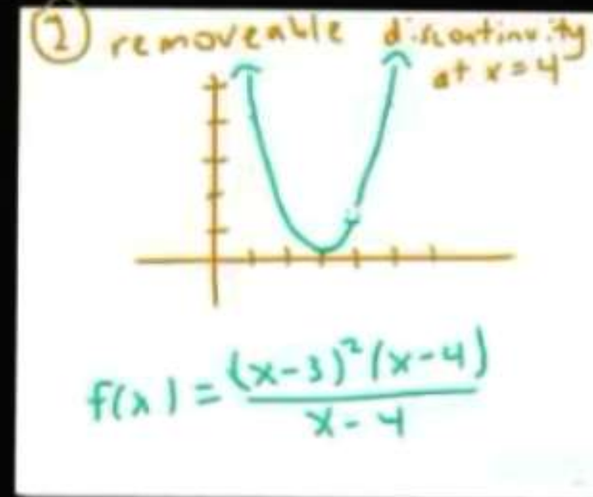
degree of numerator = degree of denom

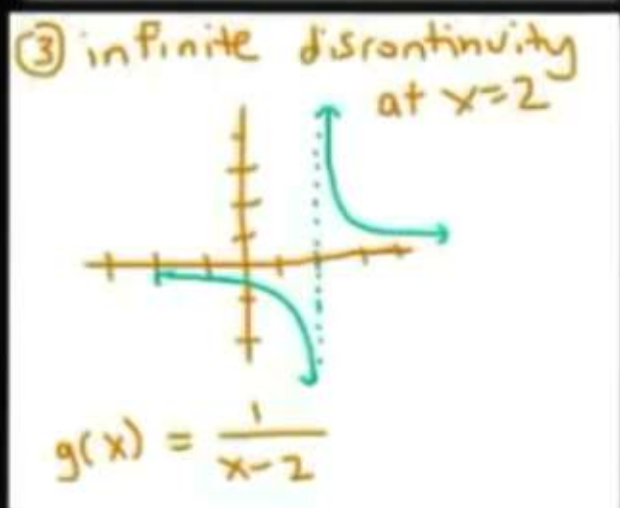
$\Rightarrow \lim\limits_{\substack{x\to\infty \\ -\infty}}$ is quotient of highest power terms

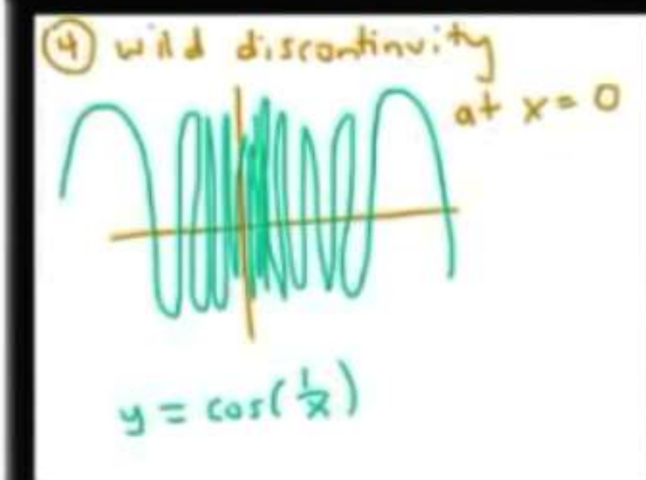
degree of numerator > degree of denom

$\Rightarrow \lim\limits_{\substack{x\to\infty \\ -\infty}}$ is ∞ or $-\infty$







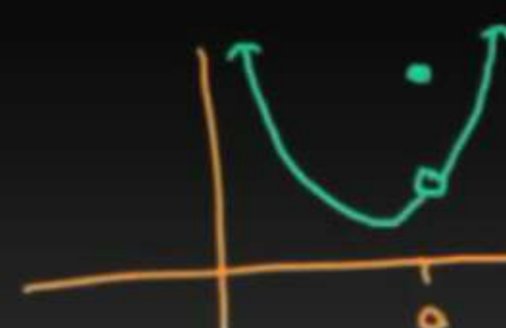


<u>Def</u> a function f is continuous at $x = a$ if

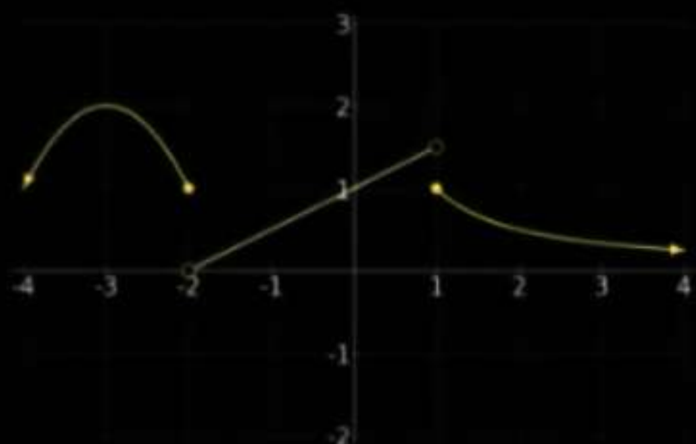
① $\lim_{x \to a} f(x)$ exists

② $f(a)$ exists

③ $\lim_{x \to a} f(x) = f(a)$



Example. Consider the function $f(x)$ at $x = -2$ and at $x = 1$.



$$f(-2) = \lim_{x \to -2^-} f(x)$$

f is continuous from the left at $x = -2$

$$f(1) = \lim_{x \to 1^+} f(x), \text{ so}$$

f is continuous from the right at $x = 1$

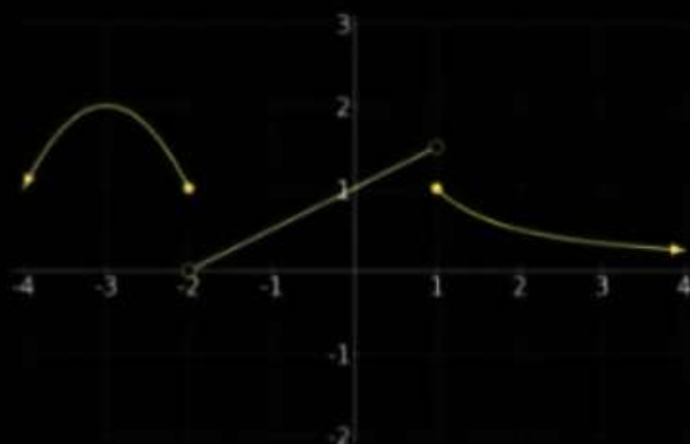
Definition. A function $f(x)$ is **continuous from the left** at $x = a$ if

$$\lim_{x \to a^-} f(x) = f(a)$$

A function $f(x)$ is **continuous from the right** at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

Example. Consider the function $f(x)$ at $x = -2$ and at $x = 1$.



$$f(-2) = \lim_{x \to -2^-} f(x)$$

f is continuous from the left at $x = -2$

$$f(1) = \lim_{x \to 1^+} f(x), \text{ so}$$

f is continuous from the right at $x = 1$

Definition. A function $f(x)$ is **continuous from the left** at $x = a$ if

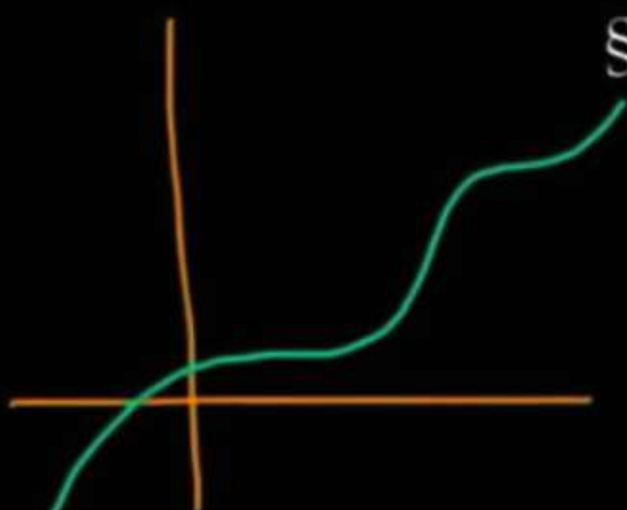
$$\lim_{x \to a^-} f(x) = f(a)$$

A function $f(x)$ is **continuous from the right** at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

§2.5.1 Continuity

Informal definition:



"A function is continuous if you can draw it without picking up your pencil."

Limit definition

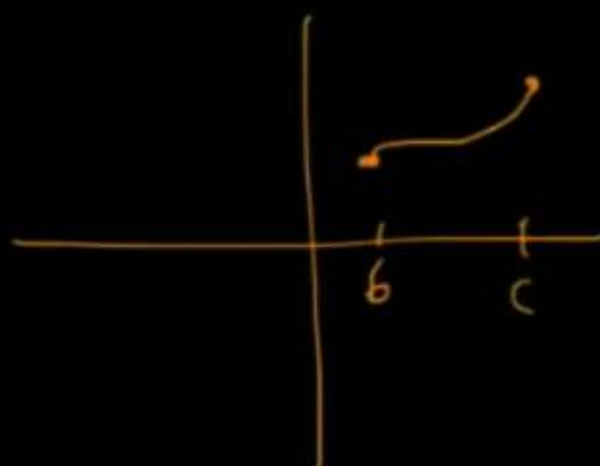
A fn $f(x)$ is continuous at $x = a$ if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Definition. We say that a function $f(x)$ is continuous on the interval (b, c) if

$f(x)$ is continuous at $x = a$ for every a in (b, c)

We say that $f(x)$ is continuous on an interval $[b, c]$ if



→ f is continuous at $x = a$ for every a in (b, c)
→ f is cont from the right at b
→ f is cont from the left at c

Question. What kinds of functions are continuous everywhere? $(-\infty, \infty)$

polynomials

$y = \sin(x)$

$y = \cos(x)$

$y = |x|$

Question. What kinds of functions are continuous on their domains?

polynomials

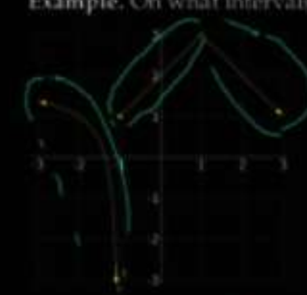
rational functions $f(x) = \dfrac{5x-2}{(x-3)^2(x+4)}$

trig fns, inverse trig fns,

$y = \ln(x)$, $y = e^x$

$(0, \infty)$

Example. On what intervals is $g(x)$ continuous?



$(-\infty, -1) \cup [-1, 1) \cup (1, \infty)$

sums, differences, products, and quotients of continuous fns are continuous on their domains

$$y = \sin(x) + \ln(x)$$

the compositions of continuous fns are continuous on their domains

$y = \ln(\sin(x))$ is cont where defined

$$\cdots \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$$

Example. Find $\lim_{x\to 2} \cos\left(\frac{x^2-4}{2x-4}\pi\right)$.

as $x \to 2$

$$\frac{x^2-4}{2x-4}\pi = \frac{\cancel{(x+2)}(\cancel{x-2})}{2(\cancel{x-2})}\pi$$

$$= \frac{x+2}{2}\pi \text{ for } x \neq 2$$

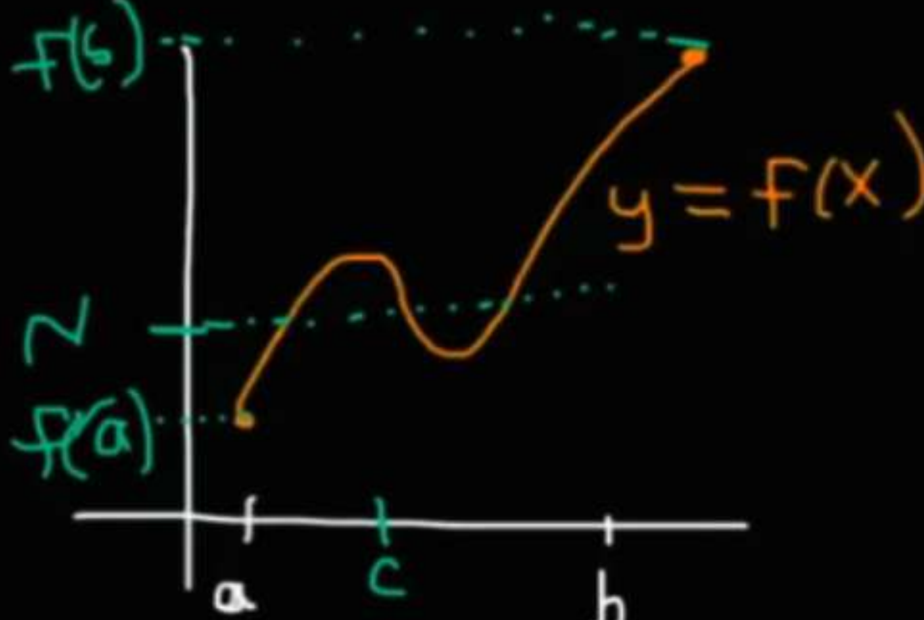
$$\longrightarrow \frac{2+2}{2}\pi = 2\pi$$

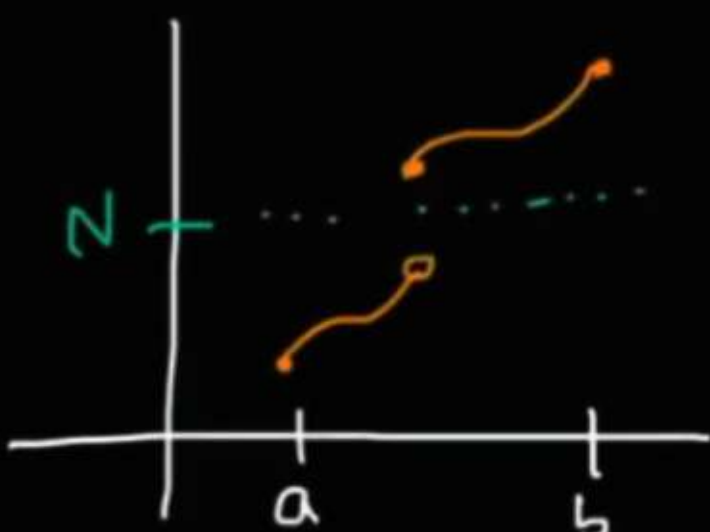
$$\lim_{x\to 2} \frac{x^2-4}{2x-4}\pi = 2\pi$$

$$\lim_{x\to 2} \cos\left(\frac{x^2-4}{2x-4}\pi\right) = \cos(2\pi) = 1$$

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$ if f is continuous.

Theorem. *(Intermediate Value Theorem)*





If f is continuous on $[a,b]$

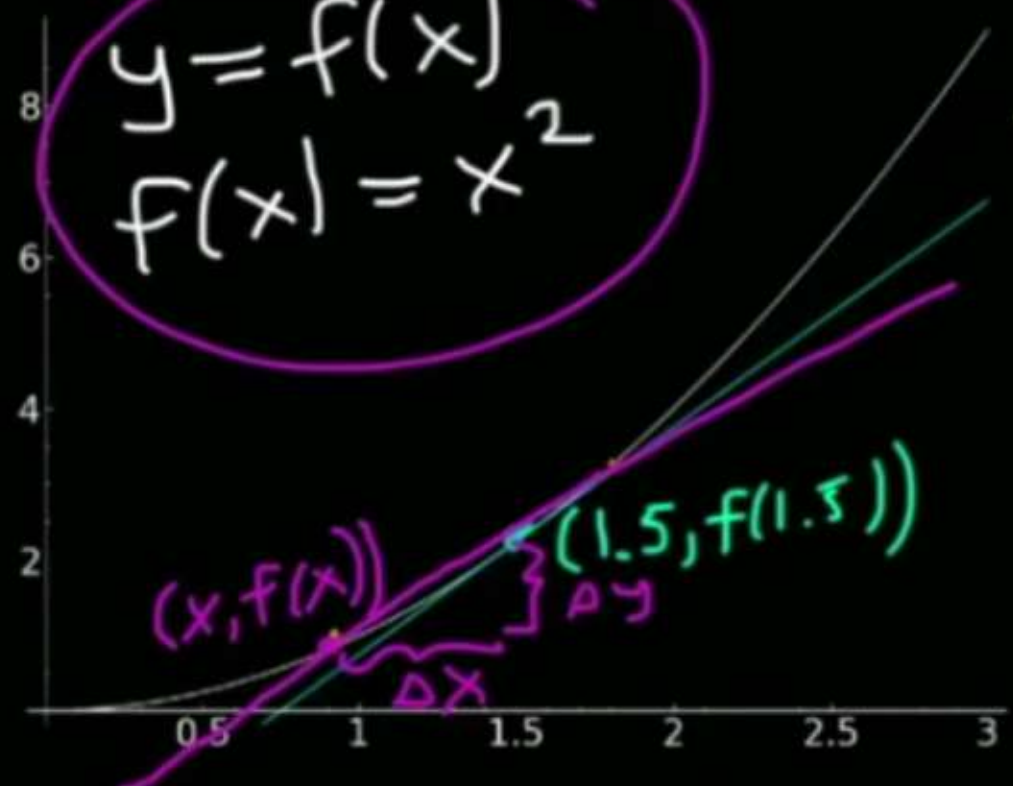
if N is a number between $f(a)$ and $f(b)$ then there has to be a number c in the interval (a,b) such that $f(c)=N$.

Work only on continuous func

Find the slope of the tangent line.

$y = f(x)$

$f(x) = x^2$



first point	point	of secant line
1.5	3	4.5
1.5	2	3.5
1.5	1.6	3.1
1.5	1.51	3.01
1.5	x	$\boxed{\dfrac{f(x)-f(1.5)}{x-1.5}}$
1.5	1	2.5
1.5	1.4	2.9
1.5	x	$\dfrac{f(1.5)-f(x)}{1.5-x}$

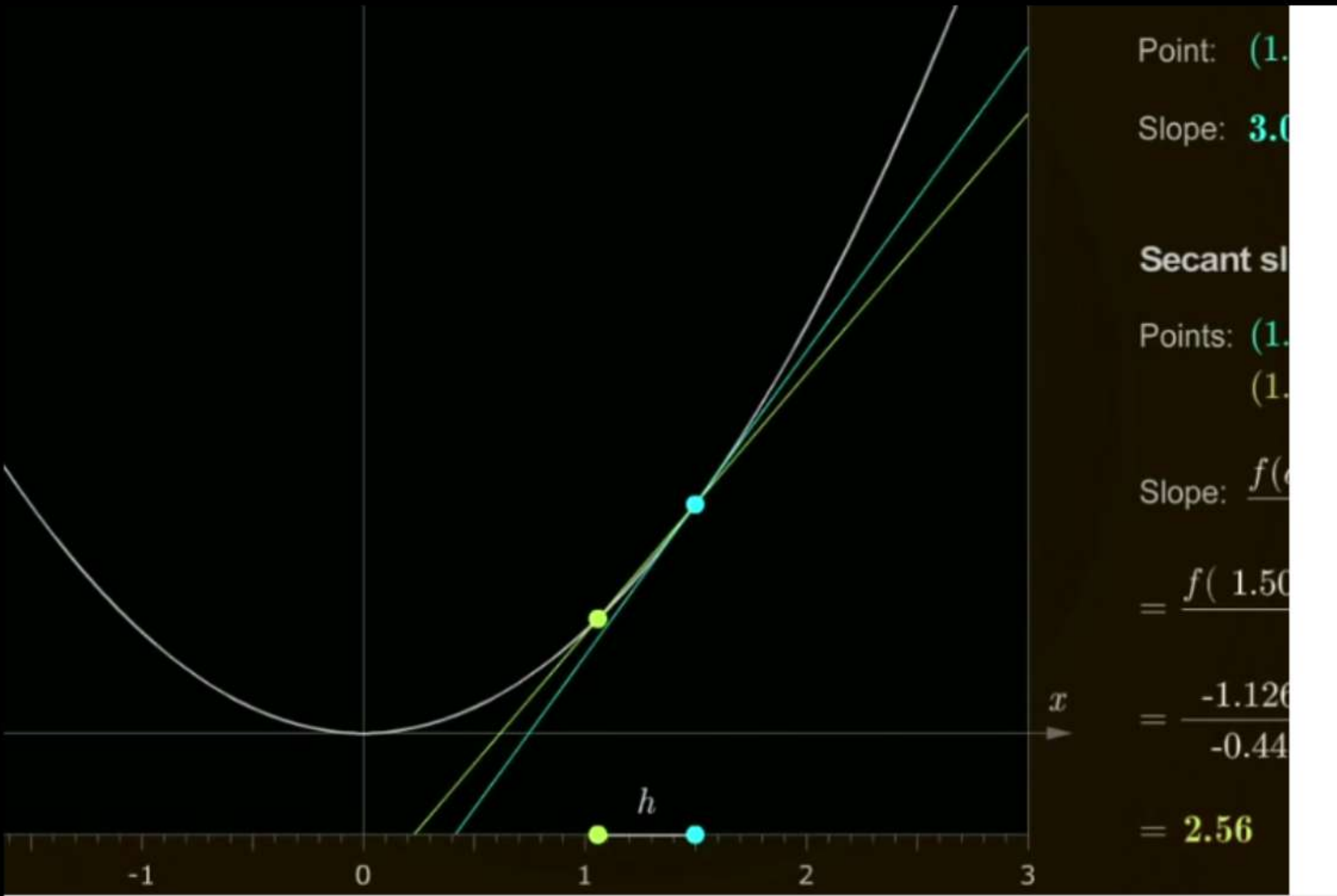
$$\boxed{\frac{f(x)-f(1.5)}{x-1.5}}$$

slope of tangent line

$$= \lim_{x \to 1.5} \frac{f(x)-f(1.5)}{x-1.5}$$

derivative of $f(x)$ at $x = 1.5$

$$f'(1.5) = \lim_{x \to 1.5} \frac{f(x)-f(1.5)}{x-1.5} = \boxed{3}$$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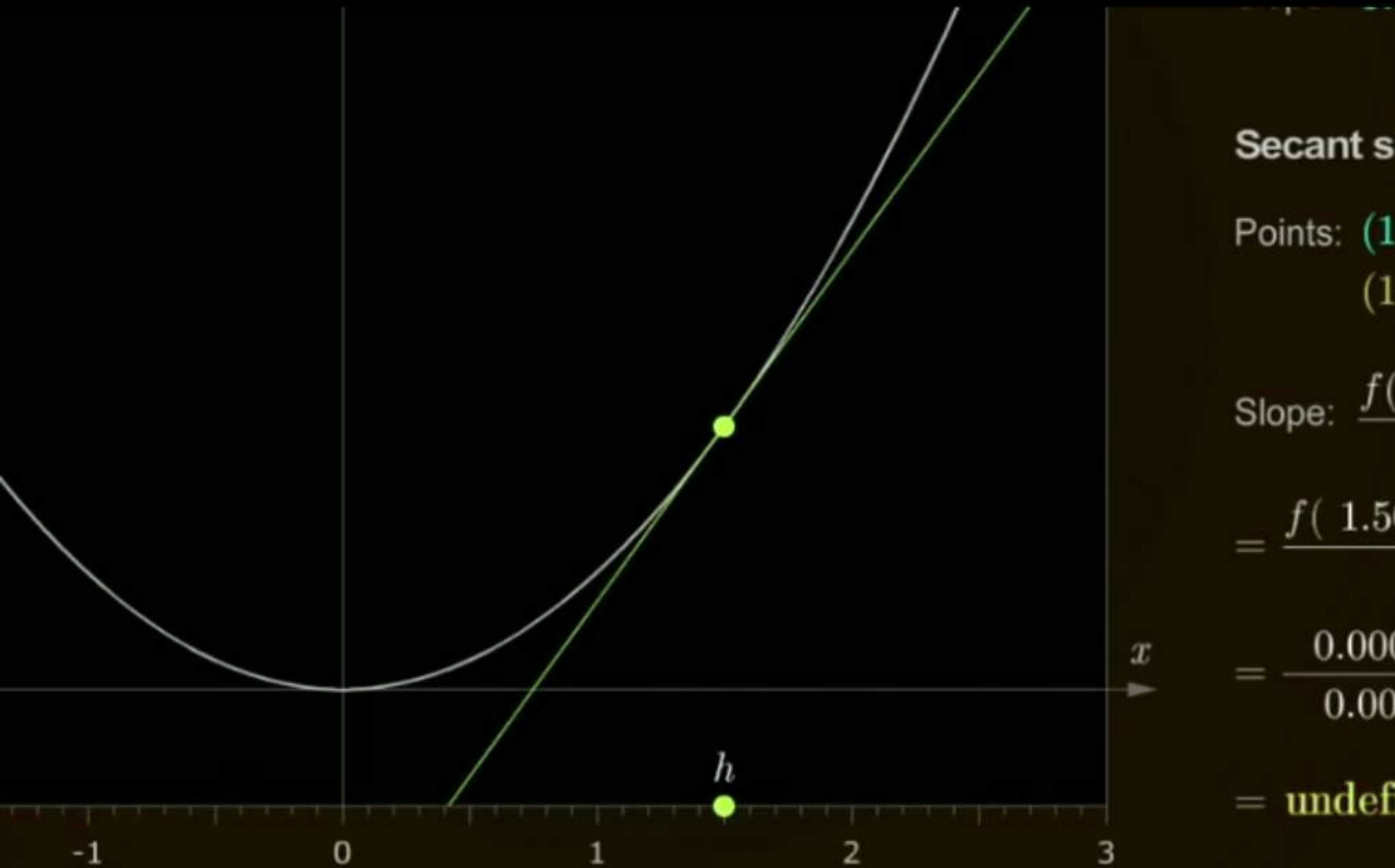

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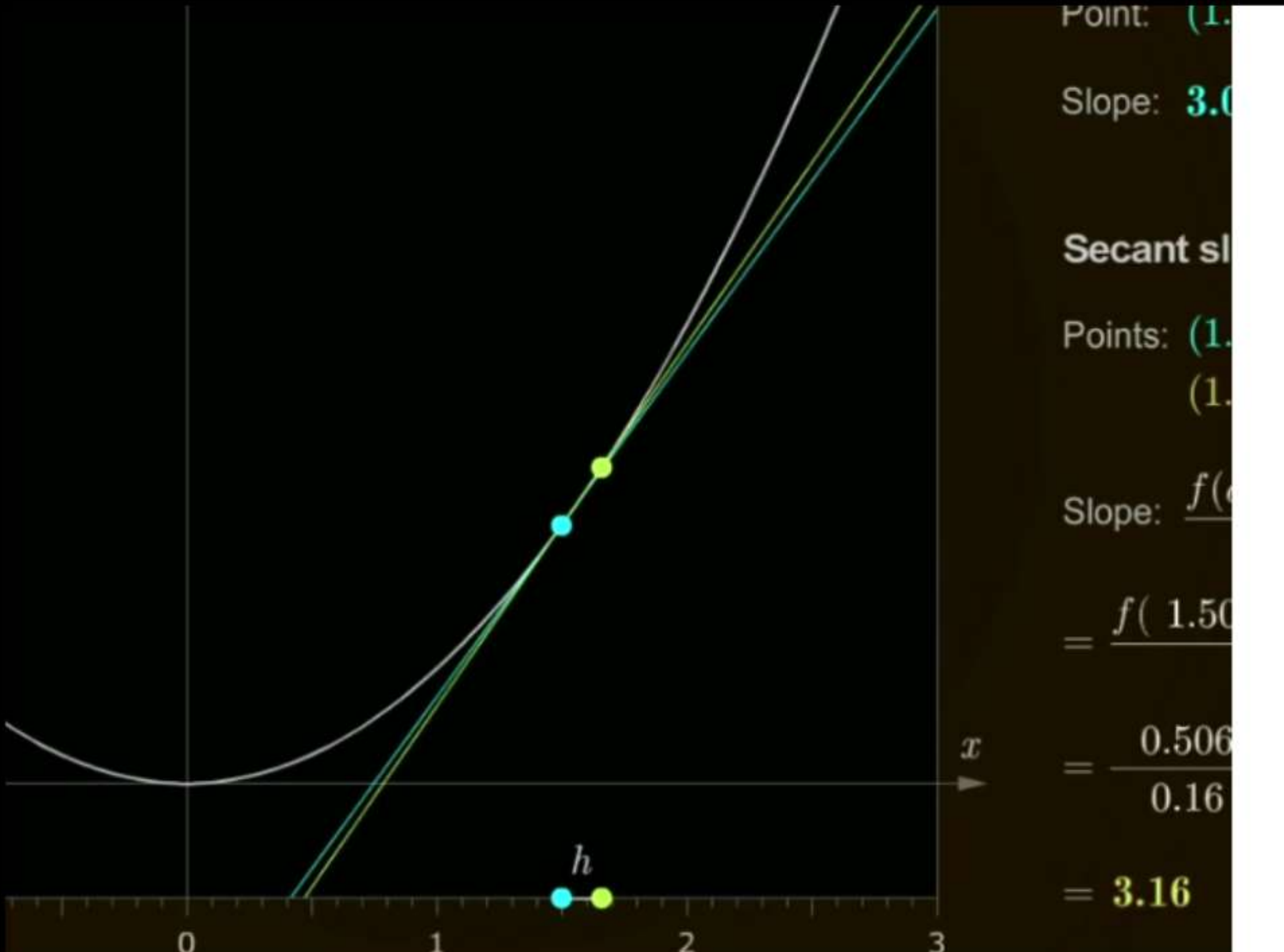
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3

$$f'(1.5) = \lim_{x \to 1.5} \frac{f(x) - f(1.5)}{x - 1.5}$$

Definition. The **derivative** of a function $y = f(x)$ at an x-value a is given by:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f(x)$ is differentiable at $x = a$ if this limit exists

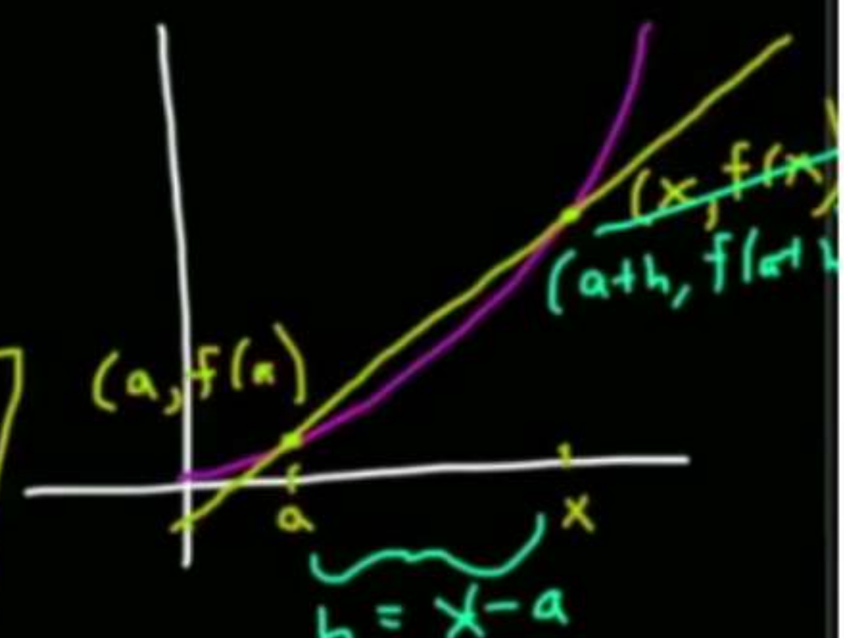
Another version of the definition of derivative

$$h = x - a$$

$$x = a + h$$

$$f'(a) = \lim_{x \to a} \frac{f(a+h) - f(a)}{h}$$

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$



The following expressions represent the derivative of some function at some value a. For each example, find the function and the value of a.

1. $\lim_{x \to -1} \frac{(x+5)^2 - 16}{x+1}$

$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$a = -1 \quad x+1 = x - (-1) \quad x - a \quad a = -1$

$f(x) = (x+5)^2 \quad f(-1) = (-1+5)^2 = 16$

2. $\lim_{h \to 0} \frac{3^{2+h} - 9}{h}$

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$f(x) = 3^x \quad 9 = 3^a \quad a = 2$

$f(a+h) = f(2+h) = 3^{2+h}$

Example. Find the derivative of $f(x) = \dfrac{1}{\sqrt{3-x}}$ at $x = -1$.

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{\sqrt{3-(-1+h)}} - \dfrac{1}{\sqrt{3-(-1)}}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{\sqrt{4-h}} - \dfrac{1}{\cancel{\sqrt{4}}\ 2}}{h} \qquad \frac{0}{0}$$

common denom

$$(\sqrt{4-h})(2)$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{\sqrt{4-h}} \cdot \dfrac{2}{2} - \dfrac{1}{2} \cdot \dfrac{\sqrt{4-h}}{\sqrt{4-h}}}{h}$$

!!!See how the common denominator i calcualted, no as you think!!

$$= \lim_{h \to 0} \frac{\dfrac{2 - \sqrt{4-h}}{\sqrt{4-h} \cdot 2}}{h}$$

$$= \lim_{h \to 0} \frac{2 - \sqrt{4-h}}{\sqrt{4-h} \cdot 2} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{2 - \sqrt{4-h}}{2h\sqrt{4-h}} \qquad \frac{0}{0}$$

2st part of equation

$$= \lim_{h \to 0} \frac{2-\sqrt{4-h}}{2h\sqrt{4-h}} \cdot \frac{(2+\sqrt{4-h})}{(2+\sqrt{4-h})}$$

$$= \lim_{h \to 0} \frac{4 + \cancel{2\sqrt{4-h}} - \cancel{2\sqrt{4-h}} - (\sqrt{4-h})^2}{2h\sqrt{4-h}\,(2+\sqrt{4-h})}$$

$$= \lim_{h \to 0} \frac{4 - (4-h)}{2h\sqrt{4-h}\,(2+\sqrt{4-h})}$$

$$= \lim_{h \to 0} \frac{\cancel{4} - \cancel{4} + \cancel{h}}{2\cancel{h}\sqrt{4-h}\,(2+\sqrt{4-h})}$$

$$= \lim_{h \to 0} \frac{1}{2\sqrt{4-h}\,(2+\sqrt{4-h})}$$

$$= \frac{1}{2\sqrt{4}\,(2+\sqrt{4})} = \boxed{\frac{1}{16}}$$

Example. Find the equation of the tangent line to $y = x^3 - 3x$ at $x = 2$.

$y = 2^3 - 3\cdot 2$
$= 2$

slope of tangent line $= f'(2)$

$\frac{0}{0}$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^3 - 3(2+h) - (2^3 - 3\cdot 2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2^3} + 3\cdot 2^2\cdot h + 3\cdot 2\cdot h^2 + h^3 - \cancel{3\cdot 2} - 3\cdot h - \cancel{2^3} + \cancel{3\cdot 2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(12 + 6h + h^2 - 3)}{\cancel{h}}$$

$$= \lim_{h\to 0} 12 + 6h + h^2 - 3 = \boxed{9}$$

$y = mx + b$ $\quad m = 9$ $\quad y = 9x + b$

$(2,2)$ $\quad 2 = 9\cdot 2 + b \Rightarrow b = -16$

§2.7.3 Computing Derivatives from the Definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

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tricks for evaluating limits algebraically

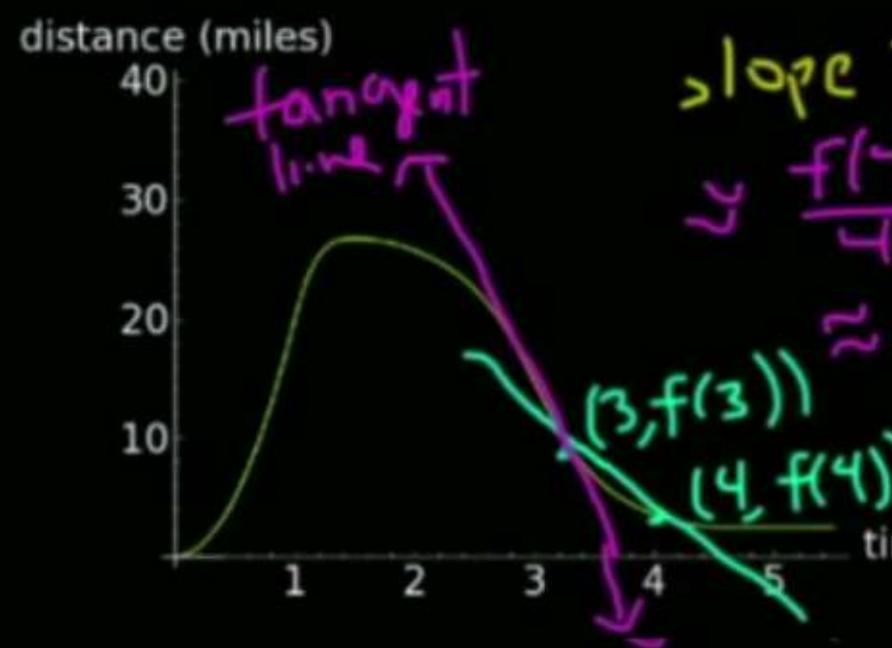
Yet later you will learn how to calculate those Derivatives using shourtcuts, for that keep this in mind

$f(x)$

Interpretation of Derivatives   forecast, its exk.

* slope of secant line $\frac{f(x) - f(a)}{x - a}$ is average rate of change
* slope of tangent line $f'(c)$ is instantaneous rate of change

Example. The graph of $y = f(x)$ represents my distance from campus on a bike ride heading due north.



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\text{change in dist}}{\text{change in time}} = \text{velocity}$$

$\downarrow$

speed in a direction

$$\approx \frac{f(4)-f(3)}{4-3} \approx \frac{4-12}{4-3} \approx -8 \text{ mph}$$

$$\text{speed} = |\text{velocity}|$$

- Interpret the slope of secant line through the points $(3, f(3))$ and $(4, f(4))$.
- Interpret the slope of the tangent line at $x = 3$.

slope of secant line gives average velocity over the interval $3 \leq x \leq 4$

derivative at $x=3$ = $f'(3)$ = slope of the tangent line at $x=3$

instantaneous velocity at $x = 3$

slope of tangent line at $x = 3$

= lim of slope of secant lines

$$= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

Example. Suppose $f(x)$ represents the temperature of your coffee in degrees Fahrenheit as a function of time in minutes x since you've set it on the counter. Interpret the following equations:

1. $f(0) = 140$

   At time 0, temp is $140°$

2. $f(10) - f(0) = -20$

   temp goes down by $20°$ as time goes from 0 to 10 minutes

3. $\dfrac{f(10) - f(0)}{10} = -2$

   temp is decreasing by an average of $2°$ per minute as time changes from 0 to 10 minutes

4. $f'(15) = -0.5$

   At exactly 15 min, the temperature is decreasing at a rate of $0.5°$ per minute.

Example. Suppose $g(x)$ represents the fuel efficiency of a Toyota Prius in miles/gallon as a function of x, the speed in miles/ hour that it is traveling. Interpret the following equations.

1. $g(45) = 52$ At 45 mph, the fuel efficiency is 52 mpg.

2. $g(40) - g(35) = 10$ As speed increases from 35 to 40 mph, fuel efficiency goes up by 10 mpg.

3. $\dfrac{g(40) - g(35)}{5} = 2$

   average rate of change of fuel efficiency is 2 mpg per mph as speed increases from 35 to 40 mph.

4. $g'(60) = -2$

   At Exactly 60 mph, fuel efficiency is decreasing at a rate of 2 mpg per mph.

Example. For the function $f(x) = 1/x$, find the derivative $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

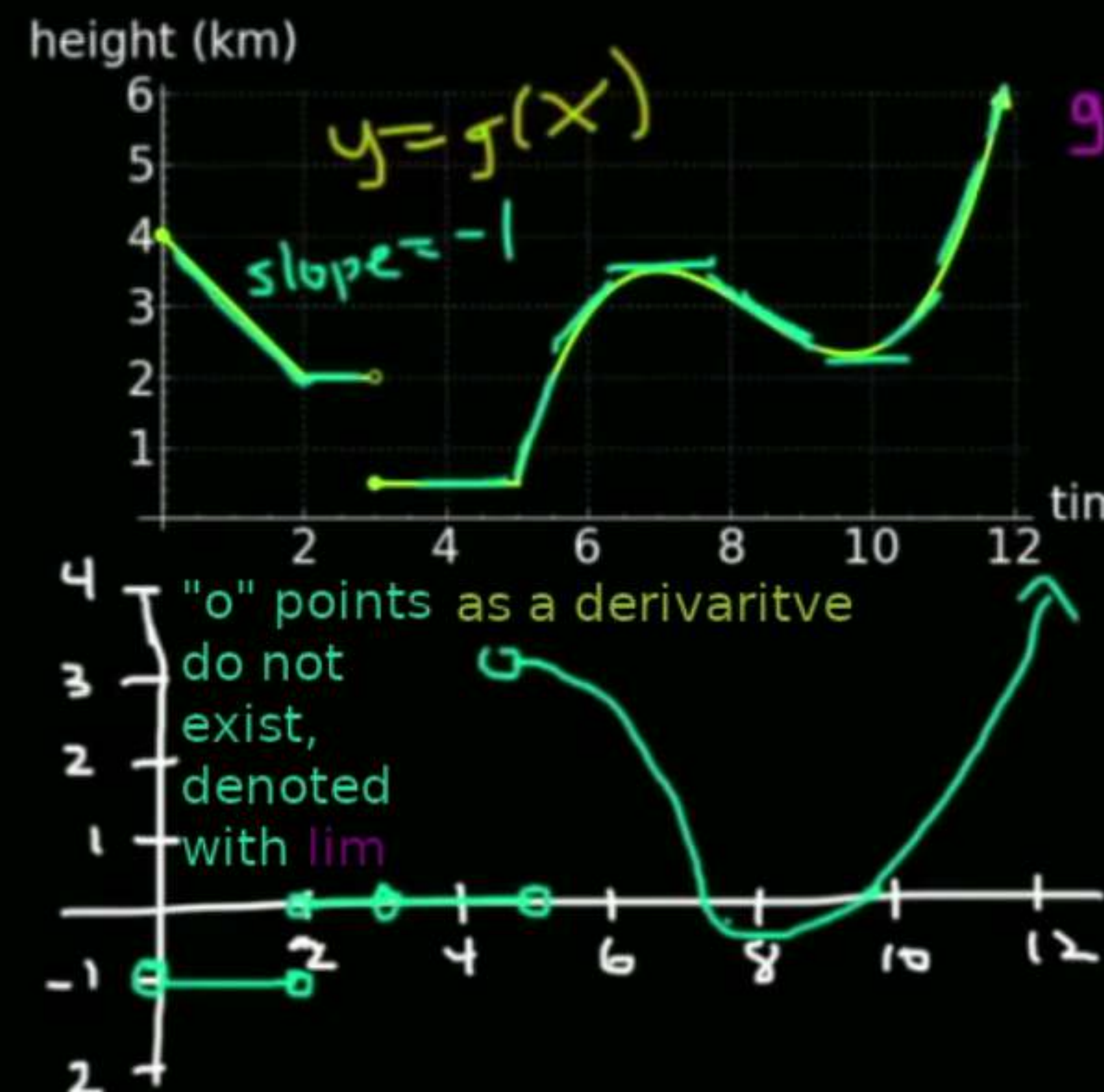
$$= \lim_{h \to 0} \frac{\frac{1}{x+h} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x+h}{x+h}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{(x+h)x} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - \cancel{h}(-1)}{(x+h)(x)(\cancel{h})} = \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$= \frac{-1}{(x+0)x} = \boxed{\frac{-1}{x^2}}$$

Example. The height of an alien spaceship above the earth's surface is graphed below. Graph the rate of change of the height as a function of time.



$$g'(3) = \lim_{h\to 0} \frac{g(3+h) - g(3)}{h}$$

$h>0 \quad g(3+h) = \frac{1}{2}$

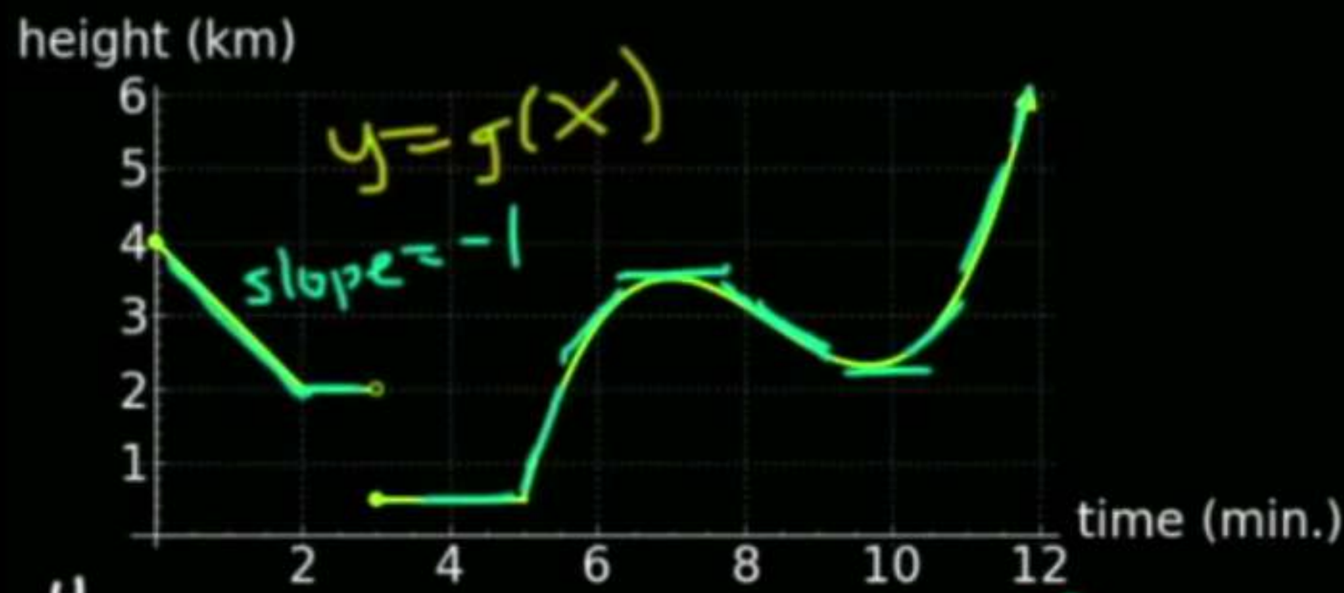
$h<0 \quad g(3+h) = 2$

$g(3) = \frac{1}{2}$

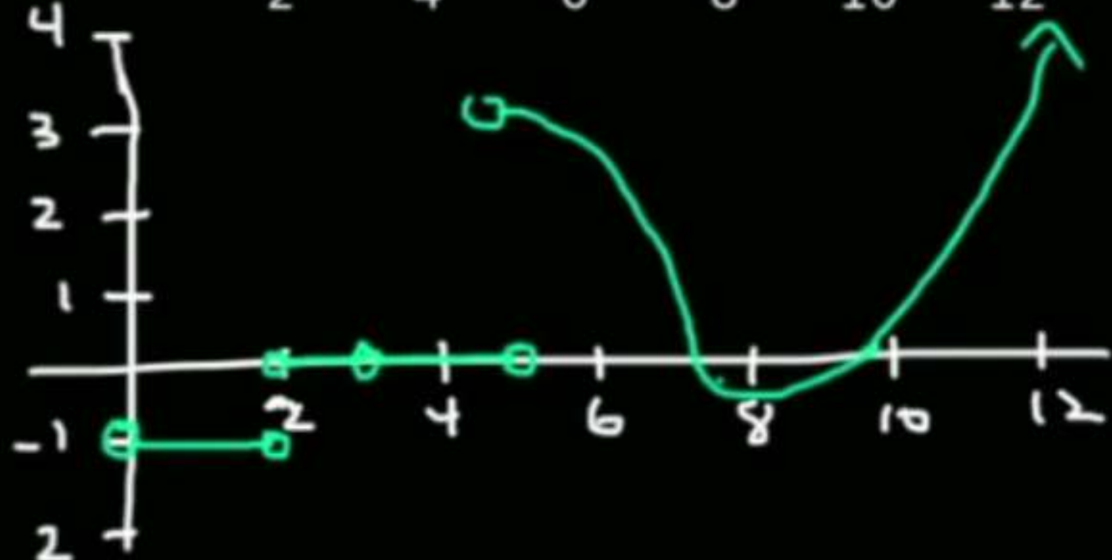
$$\lim_{h\to 0^+} \frac{g(3+h)-g(3)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{2} - \frac{1}{2}}{h} = 0$$

$$\lim_{h\to 0^-} \frac{g(3+h)-g(3)}{h} = \lim_{h\to 0^-} \frac{2-\frac{1}{2}}{h} = \lim_{h\to 0^-} \frac{\frac{3}{2}}{h} = -\infty$$

Example. The height of an alien spaceship above the earth's surface is graphed below. Graph the rate of change of the height as a function of time.



domain of g(x)
[0, ∞)



domain of g'(x)
(0,2) ∪ (2,3)
∪ (3,5) ∪ (5,∞)

1) $f(x)$ does not exist at $x=a$

Ways that a derivative can fail to exist at x=a.

2) f has a corner at $x=a$

$f(x)=|x|$

$f'(x)=-1$ when $x<0$

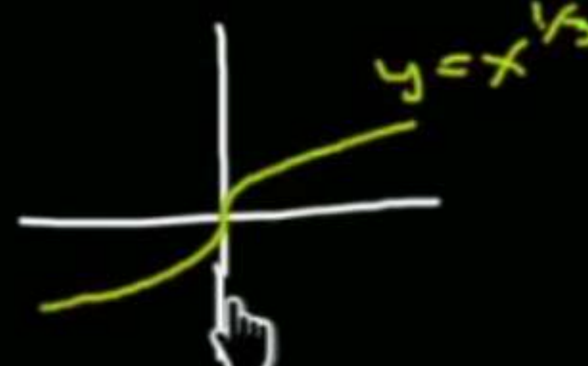
$f'(x)=1$ when $x>0$

$f'(0)$ DNE

f has a cusp at $x=a$

3) f has a discontinuity at $x=a$

4) $f(x)=x^{1/3}$



Definition. A function is **differentiable** at $x = a$ if ... $f'(a)$ exists.

Exk. Proof

$\lim_{x \to a} f(x) = f(a)$ so f is cont at $x = a$

Theorem. *If a function is differentiable at $x = a$, then it is continuous at $x = a$.*

Definition. A function is **differentiable** on an open interval (b, c) if ...

f is Differentiable at every point a in (b, c)

Theorem. *If $f(x)$ is not continuous at $x = a$, then ...* f is not differentiable at $x = a$

If $f(x)$ is differentiable at $x = a$, then ... f has to be continuous at $x = a$

If $f(x)$ is continuous at $x = a$, f may or may not be differentiable at $x = a$.

$f(x) = |x|$

§2.8.3 A Proof that Differentiable Functions are Continuous

If f is differentiable at $x=a$,
then f is continuous at $x=a$.

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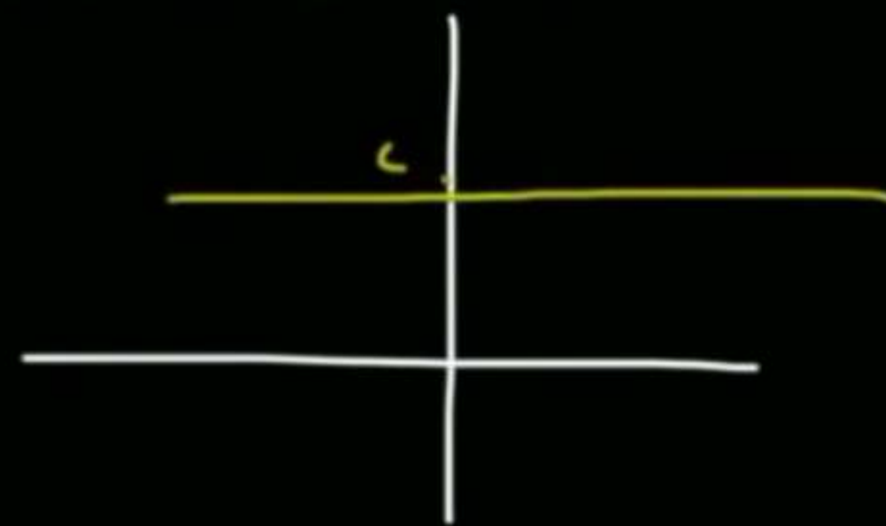
If f is not continuous at $x=a$,
then f is not differentiable at $x=a$.

1. Derivative of a constant c

$$f(x) = c$$

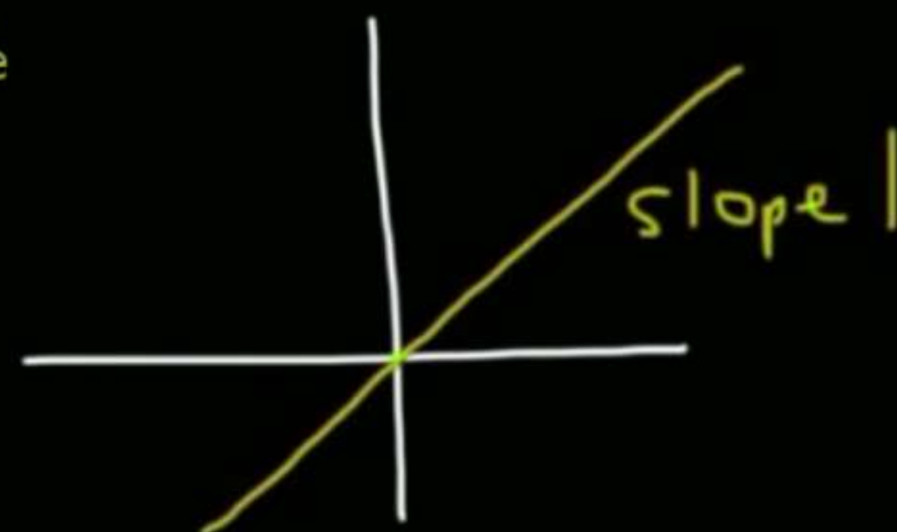


$$\frac{df}{dx} = 0$$

Tricks and Shortcut for examine the derivatives

2. Derivative of $f(x) = x$

$$f'(x) = 1$$



3. Power Rule $y = x^n$ where n is any real number

$$\frac{dy}{dx} = n \cdot x^{n-1}$$

Very useful Tricks and Shortcut for examine the derivatives dy/dx

Example. Find the derivatives of these functions:

1. $y = x^{15}$ $\quad \frac{dy}{dx} = 15 \cdot x^{14}$
2. $f(x) = \sqrt[3]{x}$ $\quad f(x) = x^{1/3}$ $\quad f'(x) = \frac{1}{3} \cdot x^{-2/3} = \frac{1}{3x^{2/3}}$
3. $g(x) = \frac{1}{x^{3.7}}$ $\quad g(x) = x^{-3.7}$ $\quad \frac{dy}{dx} = -3.7 \cdot x^{-4.7} = -\frac{3.7}{x^{4.7}}$

4. Derivative of a constant multiple

If c is a real number, and f is a differentiable function, then

$$\frac{d}{dx}\left(c \cdot f(x)\right) = c \frac{d}{dx} f(x)$$

Example. Find the derivative of $f(x) = 5x^3$.

$$\frac{d}{dx} 5x^3 = 5 \frac{d}{dx} x^3$$
$$= 5 \cdot 3x^2$$
$$= 15x^2$$

5. Derivative of a sum If f and g are differentiable functions,

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

6. Derivative of a difference If f and g are differentiable functions,

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Example. Find the derivative of $y = 7x^3 - 5x^2 + 4x - 2$.

$$\frac{dy}{dx} = \frac{d}{dx}7x^3 - \frac{d}{dx}5x^2 + \frac{d}{dx}4x - \frac{d}{dx}2$$

$$= 7\cdot 3x^2 - 5\cdot 2\cdot x + 4\cdot 1 - 0$$

$$= 21x^2 - 10x + 4$$

fn	deriv	2nd deriv	n^{th} deriv
$f(x)$	$f'(x)$	$f''(x)$	$f^{(n)}(x)$
y	y'	y''	$y^{(n)}$
	$\left.\frac{df}{dx}\right\vert_{x=3}$	$\frac{d}{dx}\left(\frac{df}{dx}\right)$	
	$\frac{d}{dx}f(x)$	$\frac{d^2f}{dx^2}$	$\frac{d^nf}{dx^n}$
	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\left.\frac{d^ny}{dx^n}\right\vert_{x=a}$
	Df		

e facts about e.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1$$

$$\underbrace{\lim_{h\to 0}\frac{e^{0+h}-e^0}{h}}_{\frac{d}{dx}e^x\big|_{x=0}} = 1 = 1 = e^0$$

$$\frac{d}{dx}e^x = e^x$$

Fact 2 $\Rightarrow$ Fact 3

Assume $\lim_{h\to 0}\frac{e^h-1}{h} = 1$

By def of deriv:

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$

$$= e^x\lim_{h\to 0}\frac{e^h-1}{h}$$

$$= e^x$$

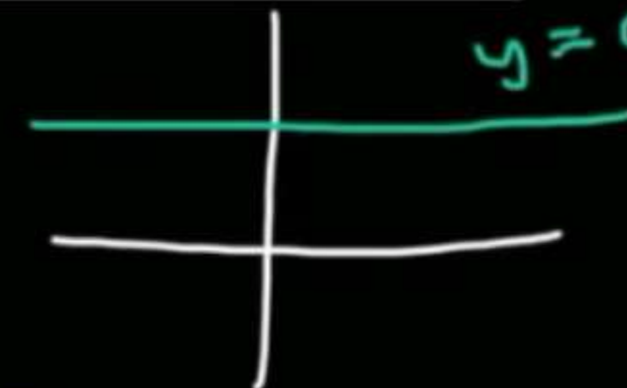
Example. Find the derivative of $g(x) = ex^2 + 2e^x + xe^2 + x^{e^2}$.

$$\frac{d}{dx}\left(ex^2 + 2e^x + xe^2 + x^{e^2}\right)$$

$$= \frac{d}{dx}\left(ex^2\right) + \frac{d}{dx}\left(2e^x\right) + \frac{d}{dx}\left(xe^2\right) + \frac{d}{dx}\left(x^{e^2}\right)$$

$$= \boxed{2ex + 2e^x + e^2 + e^2 \cdot x^{e^2-1}}$$

Note. For a constant c, $\frac{d}{dx}(c) = 0$.

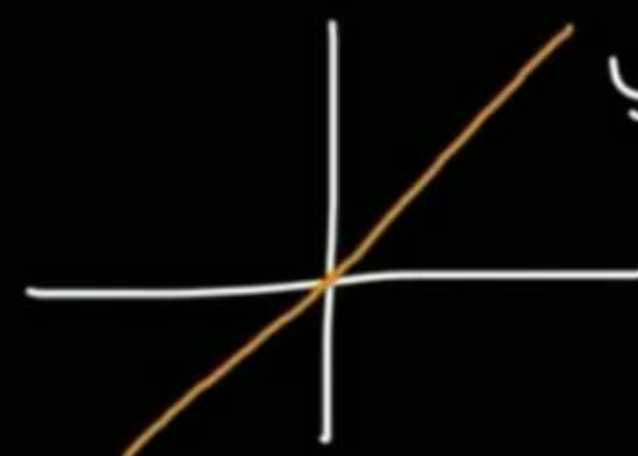


Proof. .

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Note. $\frac{d}{dx}(x) = 1$.



Proof. .

$$\frac{d}{dx} x = \lim_{h \to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Theorem. *(The Power Rule)* $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof. (Proof for n a positive integer only).

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2}\right)}{\cancel{h}}$$

$$= nx^{n-1}$$

Theorem. *(The Power Rule)* $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof. (Proof for n a positive integer only).

$$f'(a) = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{\cancel{x-a}}$$

$$= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1}$$

$$= na^{n-1} \qquad \blacksquare$$

Theorem. *(Constant Multiple Rule) If c is a constant, and f is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Proof. .

$$\frac{d}{dx}cf(x) = \lim_{h\to 0} \frac{c\,f(x+h) - c\,f(x)}{h}$$

$$= c \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= c \cdot \frac{d}{dx} f(x) \qquad \boxed{M}$$

Theorem. *(Sum Rule) If f and g are differentiable functions, then*

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof. .

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \qquad \boxed{M}$$

Theorem. *(Difference Rule) If f and g are differentiable functions, then*

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Proof. .

$$\frac{d}{dx}\left[f(x) + (-1)g(x)\right]$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}(-1)g(x)$$

$$= \frac{d}{dx}f(x) + (-1)\frac{d}{dx}g(x)$$

$$= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Recall Sum and Difference Rules: If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Is $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}f(x) \cdot \frac{d}{dx}g(x)$? **NO**

Ex $f(x) = x \quad g(x) = x^2$

$$\frac{d}{dx}[x \cdot x^2] = \frac{d}{dx}x^3 = 3x^2$$

$$\frac{d}{dx}x \cdot \frac{d}{dx}x^2 = 1 \cdot 2x = 2x$$

$$\neq$$

The Product Rule If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}(g(x)) + \frac{d}{dx}[f(x)]\cdot g(x)$$

Example. Find $\left(\sqrt{t}\cdot e^t\right)'$

$$\frac{d}{dt}\left(\sqrt{t}\cdot e^t\right) = \sqrt{t}\,\frac{d}{dt}(e^t) + \frac{d}{dt}(\sqrt{t})\cdot e^t$$

$$= \sqrt{t}\cdot e^t + \frac{d}{dt}(t^{1/2})\cdot e^t$$

$$= \sqrt{t}\cdot e^t + \frac{1}{2}t^{-1/2}\cdot e^t$$

$$= \sqrt{t}\cdot e^t + \frac{e^t}{2\sqrt{t}}$$

Example. Find $\frac{d}{dz}\left(\frac{z^2}{z^3+1}\right)$

$$= \frac{(z^3+1)2z - z^2(3z^2+0)}{(z^3+1)^2}$$

$$= \frac{2z^4+2z-3z^4}{(z^3+1)^2}$$

$$= \frac{2z-z^4}{(z^3+1)^2}$$

If f and g are differentiable functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$

Example. Find $\dfrac{d}{dz}\left(\dfrac{z^2}{z^3+1}\right)$

$$= \frac{(z^3+1)2z - z^2(3z^2+0)}{(z^3+1)^2}$$

$$= \frac{2z^4+2z-3z^4}{(z^3+1)^2}$$

$$= \frac{2z-z^4}{(z^3+1)^2}$$

$$\frac{a}{dx}[f(x)g(x)] = f(x)\frac{a}{dx}[g(x)] + \frac{a}{dx}[f(x)]\,g(x)$$

Proof

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)\left[f(x+h) - f(x)\right] + f(x)\left[g(x+h) - g(x)\right]}{h}$$

$$= \lim_{h\to 0} \left\{ g(x+h)\left[\frac{f(x+h) - f(x)}{h}\right] + f(x)\left[\frac{g(x+h) - g(x)}{h}\right] \right\}$$

$$= \lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right] + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$$

$$= g(x) \cdot \frac{d}{dx} f(x) + f(x) \cdot \frac{d}{dx} g(x)$$

Proof of Product Rule

Proof:

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{f(x+h)}\cdot\frac{f(x)}{f(x)} - \frac{1}{f(x)}\cdot\frac{f(x+h)}{f(x+h)}}{h}$$

$$= \lim \frac{f(x) - f(x+h)}{f(x+h)\,f(x)\cdot h}$$

$$= \lim_{h\to 0} \left(\frac{\;f(x)}{h}\right)\cdot\frac{1}{f(x+h)f(x)}$$

$$= -\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\; \lim_{h\to 0} \frac{1}{f(x+h)f(x)}$$

$$= -\frac{d}{dx}f(x)\cdot\frac{1}{f(x)f(x)} = \frac{-\frac{d}{dx}f(x)}{(f(x))^2} \quad \square$$

Exk.

The Reciprocal Rule If f is a differentiable function, then

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{\frac{d}{dx}(f(x))}{(f(x))^2}$$

The Quotient Rule If f and g are differentiable functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{(g(x))^2}$$

Proof:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right)$$

$$= f(x)\cdot\frac{d}{dx}\left(\frac{1}{g(x)}\right) + \frac{d}{dx}f(x)\cdot\frac{1}{g(x)}$$

$$= f(x)\cdot\frac{-\frac{d}{dx}g(x)}{(g(x))^2} + \frac{\frac{d}{dx}f(x)}{g(x)}\cdot\frac{g(x)}{g(x)}$$

$$= \frac{-f(x)\frac{d}{dx}g(x) + \frac{d}{dx}[f(x)]\,g(x)}{(g(x))^2}$$

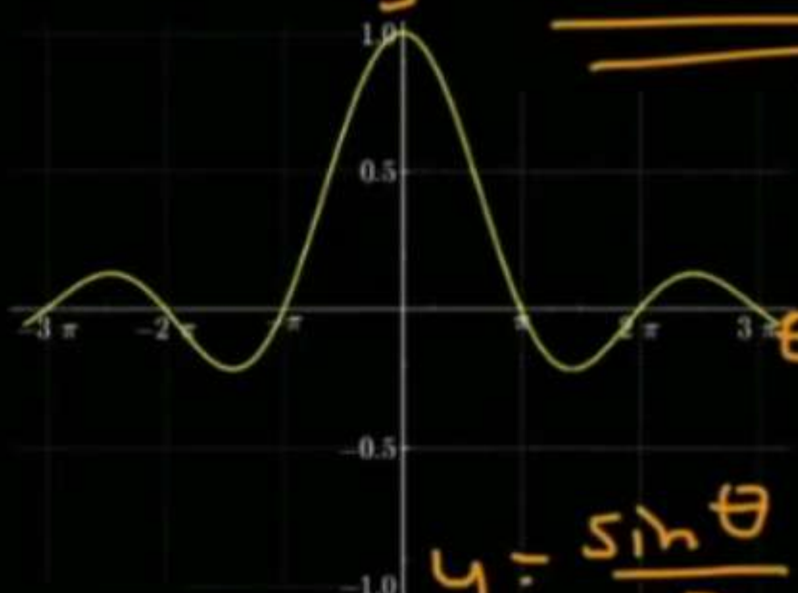
$\boxed{m}$

Special Limit #1: $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Special Limit #2: $\lim\limits_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$$

θ in radians



$y = \frac{\sin\theta}{\theta}$

$y = \frac{\cos\theta - 1}{\theta}$

$\frac{d}{dx}\sin x = \cos x$ $\qquad \frac{d}{dx}\cos x = -\sin x$

$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1} \Rightarrow \sin\theta \approx \theta$ when θ is near 0

Example. Estimate $\underbrace{\sin 0.01769}_{\approx 0.01769}$ without a calculator.

⚠ θ is in radians

check: $\sin 0.01769 \approx 0.0176890774$

Example. Find $\lim\limits_{x\to 0}\dfrac{\tan 7x}{\sin 4x}$.

$$= \lim_{x\to 0}\frac{\frac{\sin 7x}{\cos 7x}}{\frac{\sin 4x}{1}} = \lim_{x\to 0}\frac{\overbrace{\sin 7x}^{\approx 7x}}{\cos 7x}\cdot\frac{1}{\underbrace{\sin 4x}_{\approx 4x}}$$

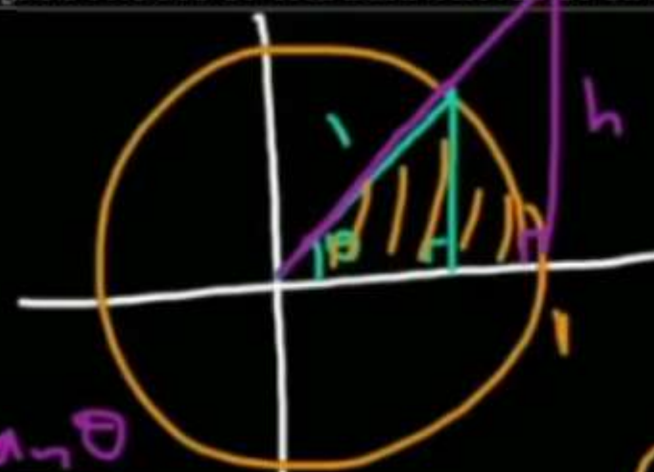
$$\approx \lim_{x\to 0}\frac{7x}{(\cos 7x)\cdot 4x} = \frac{7}{4}\lim_{x\to 0}\frac{1}{\cos 7x} = \frac{7}{4}$$

Rigorous solution:

$$\lim_{x\to 0}\frac{\sin 7x}{\cos 7x}\cdot\frac{7x}{7x}\cdot\frac{1}{\sin 4x}\cdot\frac{4x}{4x}$$

$$= \lim_{x\to 0}\underbrace{\frac{\sin 7x}{7x}}_{\to 1}\cdot\underbrace{\frac{1}{\cos 7x}}_{\to 1}\cdot\underbrace{\frac{4x}{\sin 4x}}_{\to 1}\cdot\frac{7\cancel{x}}{4\cancel{x}} = \boxed{\frac{7}{4}}$$

Proof that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\boxed{M}$



$$\frac{\cos\theta\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2}$$

$$\text{area} = \cancel{\pi}\cdot\frac{\theta}{2\cancel{\pi}} = \frac{\theta}{2}$$

$$\cos\theta\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta} \qquad \theta > 0$$

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

$$\text{area} = \frac{1}{2}\cdot b\cdot h = \frac{1}{2}\cos\theta\cdot\sin\theta$$

$$\lim_{\theta\to 0} \cos\theta \le \lim_{\theta\to 0} \frac{\theta}{\sin\theta} \le \lim_{\theta\to 0} \frac{1}{\cos\theta}$$

$\cos\theta \to 1$, $\frac{1}{\cos\theta} \to 1$

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$

$$\text{area} = \frac{1}{2}\cdot b\cdot h = \frac{1}{2}\cdot 1\cdot\tan\theta$$

Proof that $\lim_{\theta\to0}\dfrac{\cos\theta-1}{\theta}=0$

$$\lim_{\theta\to0}\frac{\cos\theta-1}{\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}=\lim_{\theta\to0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}$$

$$=\lim_{\theta\to0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$

$\sin^2\theta+\cos^2\theta=1$

$\cos^2\theta-1=-\sin^2\theta$

$$=\lim_{\theta\to0}\underbrace{\frac{-\sin\theta}{\theta}}_{\to -1}\cdot\underbrace{\frac{\sin\theta}{\cos\theta+1}}_{\to \frac{0}{1+1}=0}$$

$$=-1\cdot0=0$$

Proof that $\frac{d}{dx}(\sin x) = \cos x$

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right)$$

$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\,\frac{\sin h}{h}\right]$$

$$= \underbrace{\lim_{h\to 0}\sin x}_{\to \sin x}\ \underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{\to 0} + \underbrace{\lim_{h\to 0}\cos x}_{\to \cos x}\ \underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{\to 1}$$

$$= \cos x \quad \boxed{\text{M}}$$

Proof that $\frac{d}{dx}(\cos x) = -\sin x$

$$\frac{d}{dx}\cos x = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left(\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right)$$

$$= \lim_{h \to 0} \left(\underbrace{\cos x}_{\cos x} \cdot \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{0} - \underbrace{\sin x}_{\sin x} \cdot \underbrace{\frac{\sin h}{h}}_{1} \right)$$

$$= -\sin x$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x$$

Example. A particle moves up and down along a straight line. Its position in mm at time t seconds is given by the equation $s(t) = t^4 - \frac{16}{3}t^3 + 6t^2$.

Find $s'(t)$ and $s''(t)$. What do they represent?

$$s'(t) = 4t^3 - 16t^2 + 12t$$

$$s''(t) = 12t^2 - 32t + 12$$

$$s'(t) = \frac{ds}{dt} = \text{velocity} = v(t)$$

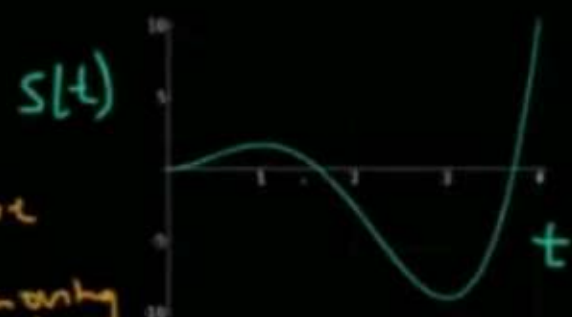
$$s''(t) = \frac{d^2s}{dt^2} = \frac{d}{dt}v(t) = \text{acceleration} = a(t)$$

$v(t) > 0$ particle is moving up
$v(t) < 0$ particle is moving down
$v(t) = 0$ particle is at rest

$F = m \cdot a$
$a(t) > 0 \Rightarrow$ particle is being pulled upwards
$a(t) < 0 \Rightarrow$ particle is being pulled down

Example. Use the table of values to describe the particle's motion at time 1.5 seconds and at time 2.5 seconds.

t	$s(t)$	$s'(t)$	$s''(t)$
1.5	0.5625	−4.5	−9
2.5	−6.77	−7.5	7



at $t = 1.5$ sec

$s(t) > 0 \Rightarrow$ particle above position 0

$s'(t) < 0 \Rightarrow$ particle is moving down

$s''(t) < 0 \Rightarrow$ velocity is decreasing (getting more and more negative)

$\Rightarrow$ particle is moving down faster and faster

speed $= |v(t)|$ is increasing

velocity and acceleration have the same sign

$\Rightarrow$ particle is speeding up

velocity and acceleration have opposite signs

$\Rightarrow$ particle is slowing down

at $t = 2.5$ sec

$s(2.5) < 0 \Rightarrow$ particle is below position 0

$s'(2.5) < 0 \Rightarrow$ particle is moving down

$s''(2.5) > 0 \Rightarrow$ velocity is increasing (getting less negative)

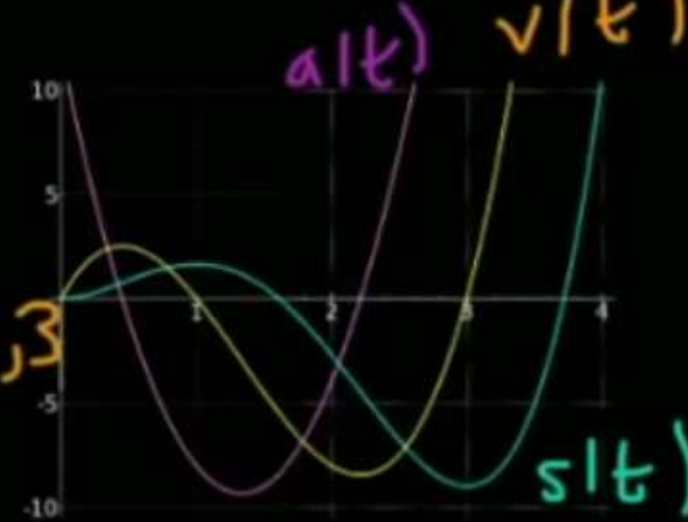
$\Rightarrow$ particle is slowing down

speed $= |v(t)|$ is decreasing

Example. A particle moves up and down along a straight line. Its position in mm at time t seconds is given by the equation $s(t) = t^4 - \frac{16}{3}t^3 + 6t^2$.

$s'(t) = 4t^3 - 16t^2 + 12t$

$s''(t) = 12t^2 - 32t + 12$



1. When is the particle at rest?

$s'(t) = 0 = 4t^3 - 16t^2 + 12t$

$0 = 4t(t^2 - 4t + 3)$

$0 = 4t(t-1)(t-3) \Rightarrow t = 0, 1, 3$

2. When is the particle moving up? moving down?

moving up: $v(t) > 0$, $(0,1) \cup (3, \infty)$

moving down: $v(t) < 0$, $(-\infty, 0) \cup (1, 3)$

$v(t)$: $-$ | 0 | $+$ | 1 | $-$ | 3 | $+$

3. When is the particle speeding up? slowing down?

speeding up: $v(t), a(t)$ both $+$ or both $-$

slowing down: $v(t), a(t)$ have opposite signs

$a(t) = 0 \qquad 0 = 12t^2 - 32t + 12 = 4(3t^2 - 8t + 3)$

$$t = \frac{8 \pm \sqrt{64-36}}{6} = \frac{4}{3} \pm \frac{\sqrt{7}}{3} \approx 0.45,\ 2.22$$

$v(t), a(t)$ both +

$(0, 0.45) \cup (3, \infty)$

$v(t), a(t)$ both −

$(1, 2.22)$

particle speeding up

$\left(0, \frac{4}{3} - \frac{\sqrt{7}}{3}\right) \cup \left(1, \frac{4}{3} + \frac{\sqrt{7}}{3}\right) \cup (3, \infty)$

$v(t), a(t)$ have opposite signs

$(-\infty, 0) \cup (0.45, 1) \cup (2.22, 3)$

particle slowing down

$(-\infty, 0) \cup \left(\frac{4}{3} - \frac{\sqrt{7}}{3}, 1\right) \cup \left(\frac{4}{3} + \frac{\sqrt{7}}{3}, 3\right)$

Example. A particle moves up and down along a straight line. Its position in mm at time t seconds is given by the equation $s(t) = t^4 - \frac{16}{3}t^3 + 6t^2$.

1. What is the net change in position for the particle between 1 and 4 seconds?

$t=1 \quad s(1) = 5/3 \simeq 1.67$ mm

$t=4 \quad s(4) = 32/3 \simeq 10.67$ mm

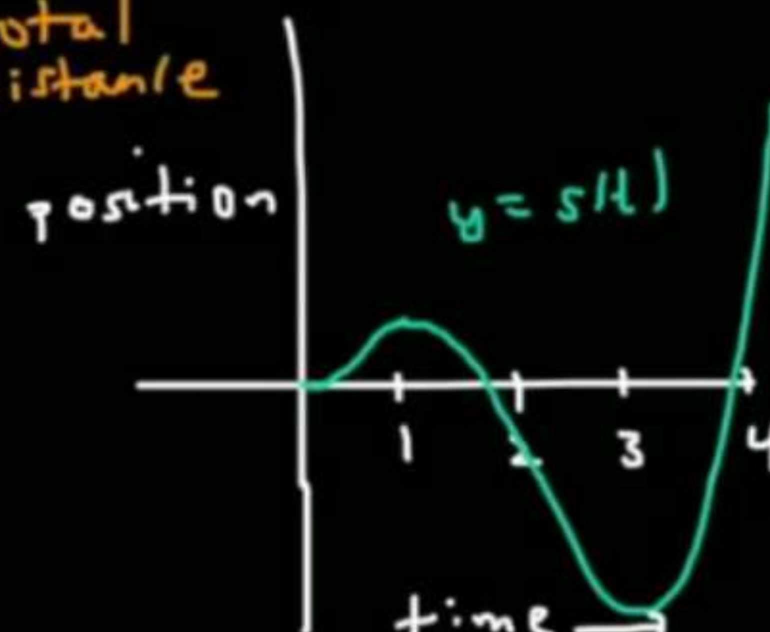
$s(4) - s(1) = 9$ mm

2. What is the total distance traveled by the particle between 1 and 4 seconds?

$|s(3)-s(1)| + |s(4)-s(3)|$ = total distance

$\left|-27-\frac{5}{3}\right| + \left|\frac{32}{3} - (-27)\right|$

$= \frac{199}{3} = 66.\overline{3}$ mm



Example. Suppose that the total cost of producing x tie-dyed T-shirts is $C(x)$.

Is $C(x)$ an increasing or decreasing function?

Is $C'(x)$ increasing or decreasing?

Interpret the following:

1. $C(204) - C(200)$

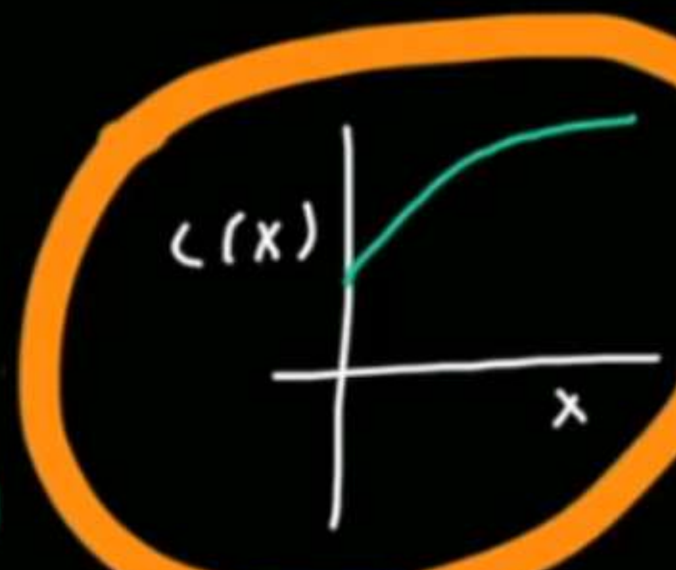
additional cost for making 204 instead of 200 t-shirts

2. $\dfrac{C(204) - C(200)}{4}$ = average rate of change of $C(x)$

$/t-shirt

3. $C'(200)$

instantaneous rate of change of $C(x)$ in $/t-shirt



$C(x)$ is called:

cost function

$C'(x)$ is called:

marginal cost

Example. Suppose the cost function for producing iPads is given by $C(x) = 500+300\sqrt{x}$.

$x = \#$ iPads

$C(x) =$ cost in \$

1. Find and interpret $C(401) - C(400)$.

$$C(401) - C(400) = (500 + 300\sqrt{401}) - (500 + 300\sqrt{400}) = \$7.50$$

It costs \$7.50 to go from producing 400 iPads to 401 iPads.

2. Find and interpret $C'(400)$. ← marginal cost

$$C'(x) = 300 \cdot \tfrac{1}{2} x^{-1/2}$$

$$C'(400) = 300 \cdot \tfrac{1}{2} \cdot 400^{-1/2} = \frac{300}{2\sqrt{400}} = \$7.50\ /\text{ipad}$$

$$C'(400) \approx \frac{C(401) - C(400)}{1}$$ ← average rate of change

Example. Write the following functions as a composition of functions:

1. $h(x) = \sqrt{\boxed{\sin(x)}}$

$h(x) = f(g(x))$

inner function: $g(x) = \sin(x)$

outer function: $f(u) = \sqrt{u}$

2. $k(x) = 5\boxed{(\tan x + \sec x)}^3$

inner fn: $g(x) = \tan(x) + \sec(x)$

outer fn: $f(u) = 5(u)^3$

3. $r(x) = e^{\boxed{\sin(\boxed{x^2})}}$

inner fn: $g(x) = x^2$

outer fn: $f(u) = e^{\sin(u)}$

OR

---

inner fn: $g(x) = \sin(x^2)$

outer fn: $f(u) = e^{(u)}$

OR

inner fn: $g(x) = x^2$

middle fn: $f(u) = \sin(u)$

outer fn: $h(v) = e^{v}$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}$$

$$u = g(x)$$
$$y = f(u)$$
$$= f(g(x))$$

$$\frac{du}{dx} = g'(x)$$

$$\frac{dy}{du} = f'(u)$$
$$= f'(g(x))$$

$$\frac{dy}{dx} = (f \circ g)'(x)$$

$$f'(x) = 3 * 2(x^2 - 5)^2 \underbrace{\frac{d}{dx}(x^2 - 5)}_{\text{chain rule}}$$

SIMPLIFY

$$f'(x) = 6\,(x^2 - 5)^2 (2X)$$

$$f'(x) = 6(x^2 - 5)^2(2X)$$

$$f'(x) = 12X(x^2 - 5)^2$$

Example. Find the derivative of $h(x) = \sqrt{\sin(x)}$.

$$h(x) = (\sin(x))^{1/2}$$

$$h(x) = (\sin(x))^{1/2}$$

inner fn: $g(x) = \sin(x)$

$g'(x) = \cos(x)$

outer fn: $f(u) = u^{1/2}$

$f'(u) = \frac{1}{2} u^{-1/2}$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$\boxed{h'(x) = \frac{1}{2}(\sin(x))^{-1/2} \cdot \cos(x)}$$

Example. Find the derivative of $k(x) = 5(\tan x + \sec x)^3$

inner fn: $g(x) = \tan(x) + \sec(x)$

outer fn: $f(u) = 5u^3$

$$K'(x) = 15(\tan(x) + \sec(x))^2 \cdot \frac{d}{dx}(\tan(x) + \sec(x))$$

$$= \boxed{15(\tan(x) + \sec(x))^2 \cdot (\sec^2(x) + \sec(x)\tan x)}$$

Example. Find the derivative of $r(x) = e^{\sin(x^2)}$

$$r(x) = e^{\sin(x^2)}$$

$$r'(x) = e^{\sin(x^2)} \cdot \frac{d}{dx}\left(\sin(x^2)\right)$$

$$= e^{\sin(x^2)} \cdot \cos(x^2) \cdot \frac{d}{dx} x^2$$

$$= e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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Example. Show that $(5^x)' = (\ln 5) \cdot 5^x$. Hint: rewrite 5^x as $e^{(\ln 5)x}$.

$$e^{\ln 5} = 5$$

$$\boxed{e^{\ln 5 \cdot x} = (e^{\ln 5})^x = 5^x}$$

$$\begin{aligned}
(5^x)' &= (e^{\ln 5 \cdot x})' \\
&= (e^{(\ln 5 \cdot x)})' \\
&= e^{(\ln 5 \cdot x)} \cdot \ln 5 \\
&= 5^x \cdot \ln 5 \\
&= \ln 5 \cdot 5^x \quad \boxed{m}
\end{aligned}$$

$$\boxed{\frac{d}{dx} a^x = \ln a \cdot a^x}$$

Example. Find the derivative of $y = \underbrace{\sin 5x}\,\underbrace{\sqrt{2^{\cos 5x}+1}}$.

$$\frac{dy}{dx} = \sin 5x \cdot \frac{d}{dx}\left(2^{\cos 5x}+1\right)^{1/2} + \frac{d}{dx}(\sin 5x)\cdot\left(2^{\cos 5x}+1\right)^{1/2}$$

$$= \sin 5x \cdot \frac{1}{2}\left(2^{\cos 5x}+1\right)^{-1/2}\cdot \frac{d}{dx}\left(2^{\cos 5x}+1\right) + \frac{d}{dx}(\sin 5x)\cdot\left(2^{\cos 5x}+1\right)^{1/2}$$

$$= \sin 5x\cdot\frac{1}{2}\left(2^{\cos 5x}+1\right)^{-1/2}\cdot\ln 2\cdot 2^{\cos 5x}\cdot\frac{d}{dx}\cos 5x + \frac{d}{dx}(\sin 5x)\cdot\left(2^{\cos 5x}+1\right)^{1/2}$$

$$= \sin 5x\cdot\frac{1}{2}\left(2^{\cos 5x}+1\right)^{-1/2}\cdot\ln 2\cdot 2^{\cos 5x}(-\sin 5x)\cdot 5 + (\cos 5x)\cdot 5\cdot\left(2^{\cos 5x}+1\right)^{1/2}$$

$$= -\frac{5}{2}\ln 2\cdot\sin^2 5x\cdot 2^{\cos 5x}\left(2^{\cos 5x}+1\right)^{-1/2} + 5\cos 5x\cdot\left(2^{\cos 5x}+1\right)^{1/2}$$

Example. Using the following table of values, find $\left.\frac{d}{dx}(f \circ g)\right|_{x=1}$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	9	-5
2	4	5	10	-3
3	1	3	2	-1
4	0	2	6	0

$$\left.\frac{d}{dx}(f \circ g)\right|_{x=1} = \left. f'(g(x)) \cdot g'(x) \right|_{x=1}$$
$$= f'(g(1)) \cdot g'(1)$$
$$= f'(2) \cdot g'(1)$$
$$= 10 \cdot (-5)$$
$$= \boxed{-50}$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

The justification of the Chain Rule

$$\frac{d}{dx} f \circ g(a) = \lim_{x \to a} \frac{f \circ g(x) - f \circ g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{g(x) - g(a)}{g(x) - g(a)}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

$$= \lim_{g(x) \to g(a)} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot g'(a)$$

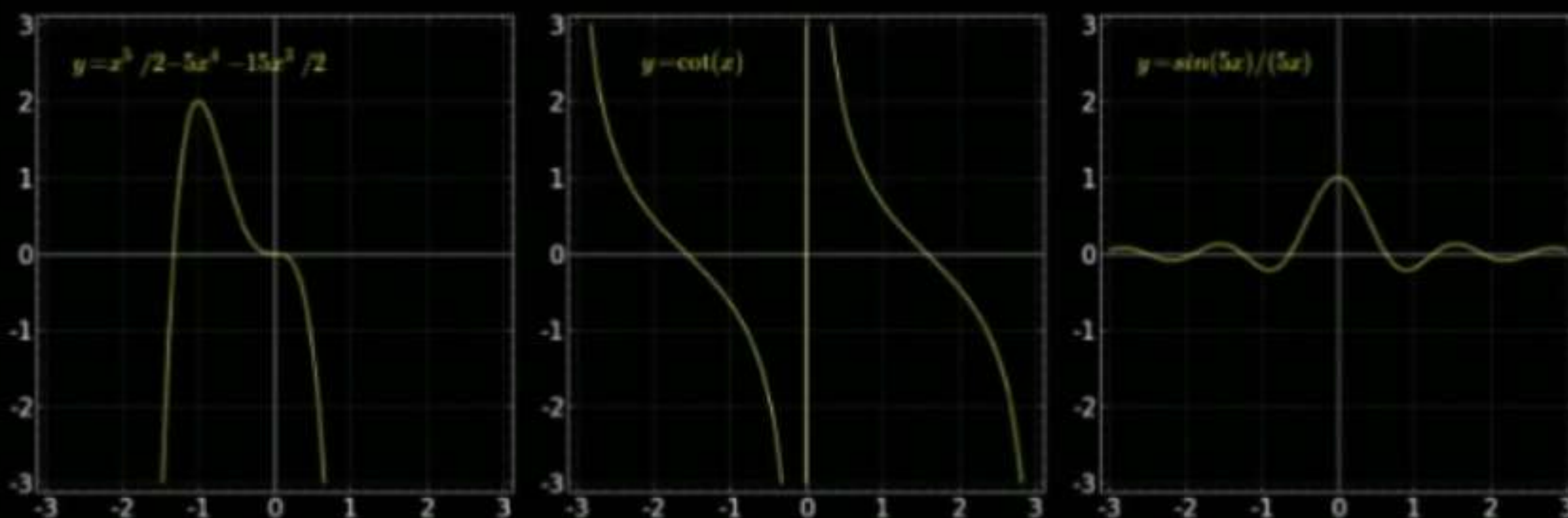
$$= \lim_{u \to g(a)} \frac{f(u) - f(g(a))}{u - g(a)} \cdot g'(a)$$

$u = g(x)$

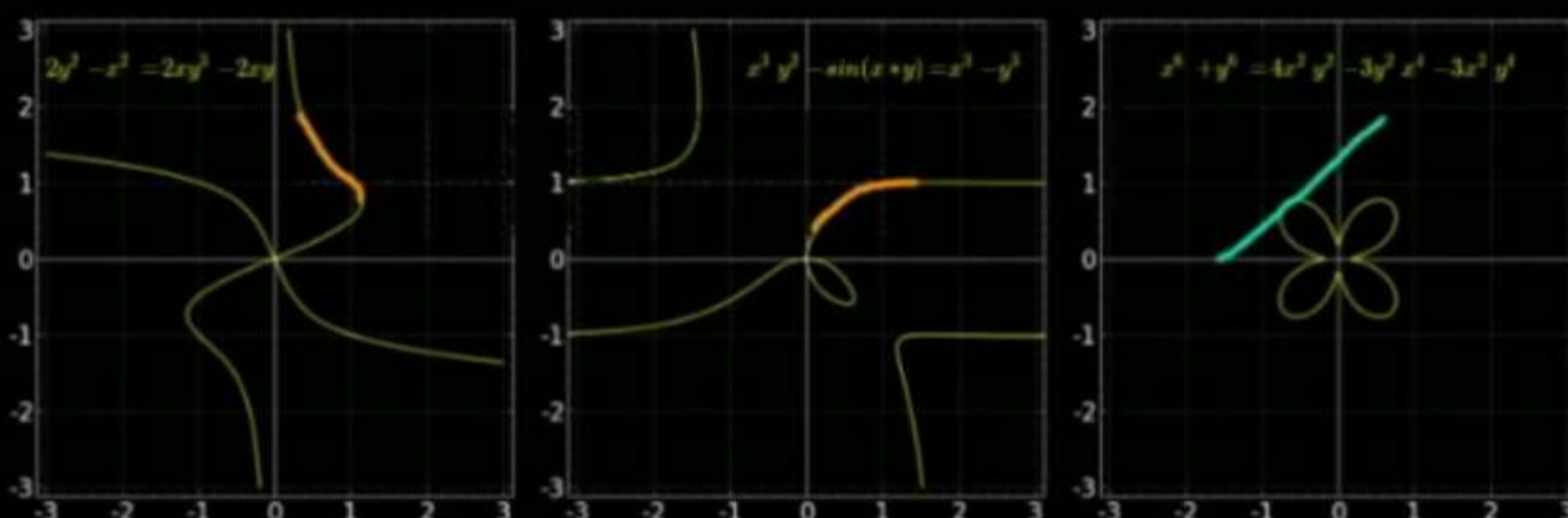
$$= f'(g(a)) \cdot g'(a)$$

⚠ $g(x) - g(a)$ might be zero so this isn't quite right

Explicitly defined functions



Implicitly defined curves



$\frac{dy}{dx}$

Example. Find the equation of the tangent line for $9x^2 + 4y^2 = 25$ at the point $(1, 2)$.

Method 1: solve for y

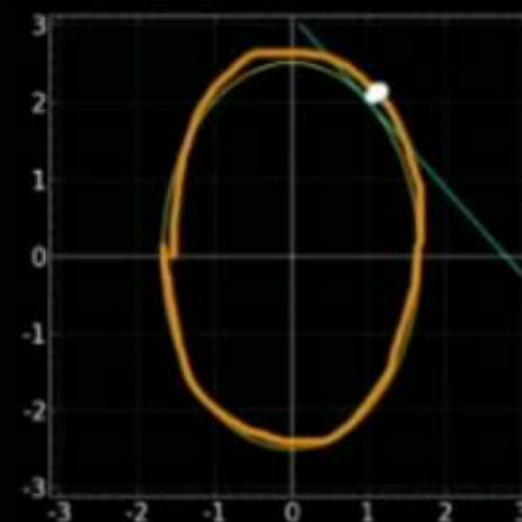
$$4y^2 = 25 - 9x^2$$

$$y^2 = \frac{25 - 9x^2}{4}$$

$$y = \pm\sqrt{\frac{25 - 9x^2}{4}}$$

$$y = \pm\frac{\sqrt{25 - 9x^2}}{2}$$

This eq work only if the given point is above 0, if under then, in the left eq you have to change from + to -



$$y = \frac{1}{2}(25 - 9x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{2}(25 - 9x^2)^{-1/2}\cdot(-18x)$$

$$\frac{dy}{dx} = \frac{-18x}{4(25 - 9x^2)^{1/2}}$$

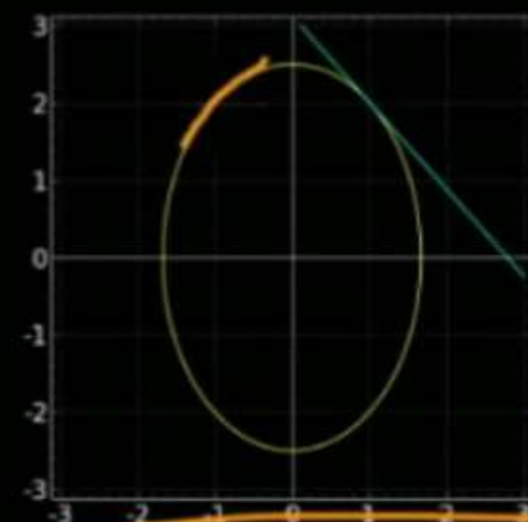
$$\frac{dy}{dx} = \frac{-9x}{2\sqrt{25 - 9x^2}}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{-9}{2\sqrt{25-9}} = \boxed{-\frac{9}{8}}$$

This method is more universal, then the "slove for y" method => common method to Use

Example. Find the equation of the tangent line for $9x^2 + 4y^2 = 25$ at the point $(1, 2)$.

Method 2: Implicit differentiation



$$\frac{d}{dx}(9x^2 + 4y^2) = \frac{d}{dx}(25)$$

$$9\frac{d}{dx}x^2 + 4\frac{d}{dx}y^2 = 0$$

$$9 \cdot 2x + 4 \cdot 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-18x}{8y} = -\frac{9}{4} \cdot \frac{x}{y}$$

$x = 1$

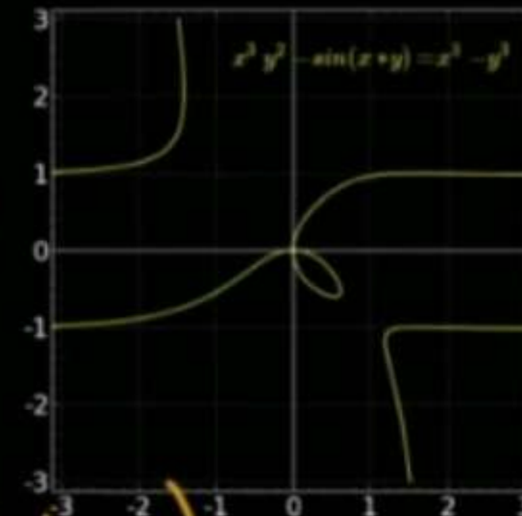
$y = 2$

$$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{9}{4} \cdot \frac{1}{2} = \boxed{-\frac{9}{8}}$$

$$\boxed{y = -\frac{9}{8}x + \frac{25}{8}}$$

Example. Find y' if $x^3y^2 - \sin(xy) = x^3 - y^3$

Note: on this eq, I make mistake, on the cos, whether is + or - so it may differ, if you recalcualted whole eq



$$\frac{d}{dx}\left(x^3y^2 - \sin(xy)\right) = \frac{d}{dx}\left(x^3 - y^3\right)$$

$$\frac{d}{dx}x^3y^2 - \frac{d}{dx}\sin(xy) = \frac{d}{dx}x^3 - \frac{d}{dx}y^3$$

$$x^3 2y\frac{dy}{dx} + 3x^2\cdot y^2 - \cos(xy)\left(x\frac{dy}{dx} + 1\cdot y\right) = 3x^2 - 3y^2\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$:

$$x^3 2y\frac{dy}{dx} + 3x^2y^2 + \cos(xy)\,x\frac{dy}{dx} + \cos(xy)\cdot y = 3x^2 - 3y^2\frac{dy}{dx}$$

$$x^3 2y\frac{dy}{dx} + \cos(xy)\,x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 3x^2 - 3x^2y^2 - \cos(xy)\,y$$

$$\frac{dy}{dx}\left(x^3 2y + \cos(xy)\,x + 3y^2\right) = 3x^2 - 3x^2y^2 - \cos(xy)\cdot y$$

$$\frac{dy}{dx} = \frac{3x^2 - 3x^2y^2 - y\cos(xy)}{2x^3y - x\cos(xy) + 3y^2}$$

Find $\frac{d}{dx}(5^x)$. Hint: rewrite 5 as e to a power.

$$5 = e^{\ln 5}$$

$\ln 5 = \log_e 5$ means: the power that we raise e to, to get 5.

$$5^x = (e^{\ln 5})^x = e^{\ln 5 \cdot x}$$

$$\frac{d}{dx}(5^x) = \frac{d}{dx}\left(e^{\underline{\ln 5 \cdot x}}\right)$$

$$= e^{\ln 5 \cdot x} \cdot \frac{d}{dx}(\ln 5 \cdot x)$$

$$= e^{\ln 5 \cdot x} \ln 5$$

$$= (e^{\ln 5})^x \ln 5$$

$$\frac{d}{dx}(5^x) = 5^x \cdot \ln 5$$

Note. variable is in the exponent

① $\frac{d}{dx}(a^x) = a^x \cdot \ln a$

② $\frac{d}{dx}(x^a) = a \cdot x^{a-1}$ using Power rule

variable is in the base

$$\frac{d}{dx}(e^x) = e^x \cdot \cancel{\ln e}^{\,1} = e^x$$

d/dx log_a(x)

$\frac{d}{dx}\log_a(x)$

$\frac{d}{dx}y$

$$y = \log_a x$$

$$a^y = x$$

$$\frac{d}{dx}a^y = \frac{d}{dx}x$$

$$\ln a \cdot a^y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\ln a \cdot a^y}$$

$$\frac{dy}{dx} = \frac{1}{(\ln a)x}$$

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}}$$

$$\frac{d}{dx}\ln x = \frac{1}{(\ln e)x}$$

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Find $\frac{d}{dx}\ln|x|$

$y = \ln|x|$ $\quad x \neq 0$

$y = \ln x$ $\quad x > 0$

$<x \quad x \geq 0$

$$\ln|x| = \begin{cases} \ln x & x \geq 0 \\ \ln(-x) & x < 0 \end{cases}$$

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{d}{dx}\ln x & x \geq 0 \\ \frac{d}{dx}\ln(-x) & x < 0 \end{cases}$$

Overview

$y = \ln x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$

$y = \log_a x \qquad \frac{d}{dx}\log_a x = \frac{1}{(\ln a)\,x}$

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$\left(\frac{1}{(-x)}\right)(-1) \quad x < 0$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

Example. Find $\frac{d}{dx}(x^x)$.

$y = x^x$ Find $\frac{dy}{dx}$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{d}{dx} \ln y = \frac{d}{dx}(x \ln x)$

$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = y(1+\ln x) \Rightarrow \frac{dy}{dx} = x^x(1+\ln x)$

For eq, when you have the variables in the base and the exponent

Example. Find $\frac{d}{dx}(\tan x)^{1/x}$.

$$y = (\tan x)^{1/x}$$

$$\ln y = \ln (\tan x)^{1/x}$$

$$\ln y = \frac{1}{x}\ln(\tan x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\frac{1}{x}\ln(\tan x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\,\frac{1}{\tan x}\cdot \sec^2 x + (-1\cdot x^{-2})\ln(\tan x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\cdot\frac{1}{\frac{\sin x}{\cos x}}\cdot\frac{1}{\cos^2 x} - \frac{\ln(\tan x)}{x^2}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\cdot\frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cos^{\cancel{2}} x} - \frac{\ln(\tan x)}{x^2}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\csc x \sec x - \frac{\ln(\tan x)}{x^2}$$

$$\frac{dy}{dx} = y\left(\frac{1}{x}\csc x \sec x - \frac{\ln(\tan x)}{x^2}\right)$$

$$\frac{dy}{dx} = (\tan x)^{1/x}\left(\frac{\csc x \sec x}{x} - \frac{\ln(\tan x)}{x^2}\right)$$

Example. Find the derivative of $y = \frac{x\cos(x)}{(x^2+x)^5}$

$$\ln y = \ln\left(\frac{x\cos(x)}{(x^2+x)^5}\right)$$

$$\ln y = \ln x + \ln(\cos(x)) - \ln(x^2+x)^5$$

$$\ln y = \ln x + \ln(\cos(x)) - 5\ln(x^2+x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\ln x + \ln(\cos(x)) - 5\ln(x^2+x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cos(x)}(-\sin x) - 5\frac{1}{x^2+x}(2x+1)$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} - \frac{\sin x}{\cos x} - \frac{5(2x+1)}{x^2+x}\right)$$

$$\frac{dy}{dx} = \frac{x\cos x}{(x^2+x)^5}\left(\frac{1}{x} - \tan x - \frac{5(2x+1)}{x^2+x}\right)$$

Logarithmic Differentiation

$$\frac{d}{dx}x^{a} = a \cdot x^{a-1}$$

$$\frac{d}{dx}a^{x} = (\ln a)a^{x}$$

$$\frac{d}{dx}x^{x} = ?$$

1) set $y =$ expression
2) take ln of both sides
3) find $\frac{d}{dx}$ of both sides
4) solve for $\frac{dy}{dx}$

$y = \sin^{-1}(x)$

Find $\frac{d}{dx}\sin^{-1}(x)$ $\quad \frac{d}{dx}y$ where $y = \sin^{-1}(x)$

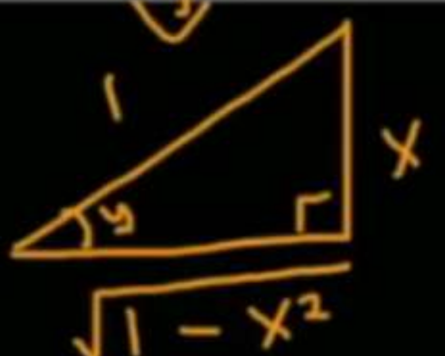
$x = \sin y$

$$\frac{d}{dx}x = \frac{d}{dx}\sin(y)$$

$$1 = \cos(y)\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\cancel{\frac{dy}{dx} = \frac{1}{\cos(\sin^{-1}(x))}}$$



$$\sin y = x$$

$$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}$$

Find $\frac{d}{dx}\arccos(x)$

$y = \arccos(x) \qquad x = \cos y$

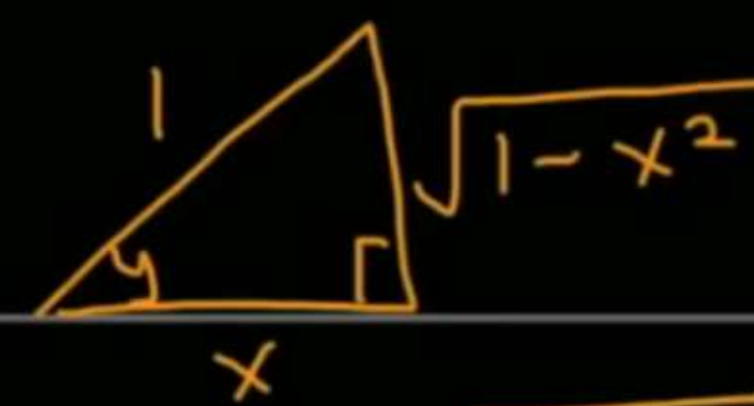
Find $\frac{dy}{dx}$

$x = \cos y$

$\frac{d}{dx}x = \frac{d}{dx}\cos y$

$1 = -\sin y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-1}{\sin y}$



$\sin(y) = \sqrt{1-x^2}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}}$$

Find $\frac{d}{dx}\arctan(x)$

$$y = \arctan(x)$$

$$x = \tan y$$

$$\frac{d}{dx}x = \frac{d}{dx}\tan y$$

$$1 = \sec^2 y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\sec y = \frac{\sqrt{1+x^2}}{1}$$

$$\sec^2 y = 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\boxed{\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$$

$\frac{d}{dx}\cot^{-1}(x)$ $\frac{d}{dx}\sec^{-1}(x)$ $\frac{d}{dx}\csc^{-1}(x)$

Summary

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$$

Example. Find the derivative of $y = \tan^{-1}\left(\frac{a+x}{a-x}\right)$.

$$\boxed{\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{a+x}{a-x}\right)^2} \cdot \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

$$= \frac{1}{1+\left(\frac{a+x}{a-x}\right)^2} \cdot \frac{(a-x)\cdot 1-(a+x)(-1)}{(a-x)^2}$$

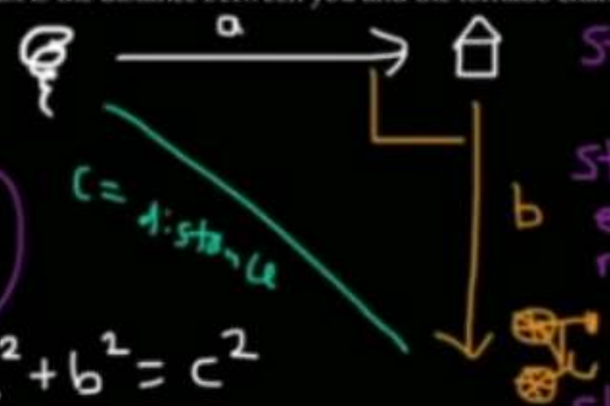
$$= \frac{1}{1+\left(\frac{a+x}{a-x}\right)^2} \cdot \frac{(a-x)\cdot 1-(a+x)(-1)}{(a-x)^2}$$

$$= \frac{a-\cancel{x}+a+\cancel{x}}{\left(1+\frac{(a+x)^2}{(a-x)^2}\right)(a-x)^2}$$

$$= \frac{2a}{(a-x)^2+(a+x)^2} = \frac{2a}{a^2-2ax+x^2+a^2+2ax+x^2}$$

$$= \frac{2a}{2a^2+2x^2} = \frac{a}{a^2+x^2}$$

Example. A tornado is 20 miles west of us, heading due east towards Phillips Hall at a rate of 40 mph. You hop on your bike and ride due south at a speed of 12 mph. How fast is the distance between you and the tornado changing after 15 minutes?



$\frac{dc}{dt}$

Step 1: draw a picture

Step 2: write down equation(s) that relate the quantities of interest

Step 3: derive both sides of the equation with respect to time t

Step 4: plug in numbers and solve for quantity of interest

$$a^2+b^2=c^2$$

$$\frac{d}{dt}(a^2+b^2)=\frac{d}{dt}c^2$$

$$2a\cdot\frac{da}{dt}+2b\frac{db}{dt}=2c\frac{dc}{dt}$$

$$\frac{da}{dt}=-40 \qquad \frac{db}{dt}=12$$

$t = 0.25$ hours

$a=10 \qquad b=3 \qquad c^2=10^2+3^2 \qquad c=\sqrt{109}$

$$2\cdot 10\cdot(-40)+2\cdot 3\cdot 12=2\cdot\sqrt{109}\cdot\frac{dc}{dt}$$

$$\frac{dc}{dt}=\frac{-800+72}{2\sqrt{109}}\approx -35 \text{ mph}$$

⚠ Don't plug in numbers until the end

⚠ Use negative numbers for quantities that are decreasing

Example. Water flows into a tank at a rate of 3 cubic meters per minute. The tank is shaped like a cone with a height of 4 meters and a radius of 5 meters at the top. Find the rate at which the water level is rising in the tank when the water height is 2 meters.

1) Draw picture.
2) Write down equations.
3) Derive both sides of equation with respect to t
4) Plug in numbers & solve

$\frac{dh}{dt}$

volume of cone $= \frac{1}{3}$(area of base) $\cdot$ height

$$V = \frac{1}{3} \cdot \pi r^2 \cdot h \qquad \frac{r}{h} = \frac{5}{4}$$

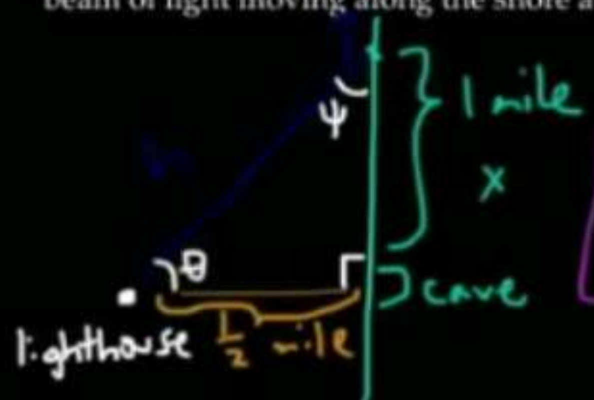
$$V = \frac{1}{3}\pi \left(\frac{5}{4}h\right)^2 \cdot h \qquad r = \frac{5}{4}h$$

$$V = \frac{25}{48}\pi h^3$$

$$\frac{dV}{dt} = \frac{25}{48}\pi \cdot 3h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\frac{25}{48}\pi \cdot 3 \cdot 2^2} = \frac{12}{25\pi} \text{ m/s} \simeq 0.15 \text{ m/s}$$

Example. A lighthouse that is half a mile west of shore has a rotating light that makes 2 revolutions per minute in the counterclockwise direction. The shoreline runs north-south, and there is a cave on the shore directly east of the lighthouse. How fast is the beam of light moving along the shore at a point 1 mile north of the cave?



① Draw picture

② Write down equations

$$\frac{2\text{ rev}}{\text{min}} \cdot \frac{2\pi\text{ radians}}{\text{rev}} = 4\pi\ \text{rad}/\text{min} = \frac{d\theta}{dt}$$

$$\tan\theta = \frac{x}{\frac{1}{2}} \Rightarrow \boxed{\tan\theta = 2x}$$

③ Derive with respect to t

④ Plug in numbers & solve

$$\sec^2\theta \cdot \frac{d\theta}{dt} = 2\frac{dx}{dt}$$

when $x = 1$

$$\frac{d\theta}{dt} = 4\pi, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \frac{h}{\frac{1}{2}} = \frac{\sqrt{1^2 + (\frac{1}{2})^2}}{\frac{1}{2}} = \frac{\sqrt{\frac{5}{4}}}{\frac{1}{2}} = \sqrt{5}$$

$$(\sqrt{5})^2 \cdot 4\pi = 2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{5 \cdot 4\pi}{2} = 10\pi\ \text{miles}/\text{min}$$

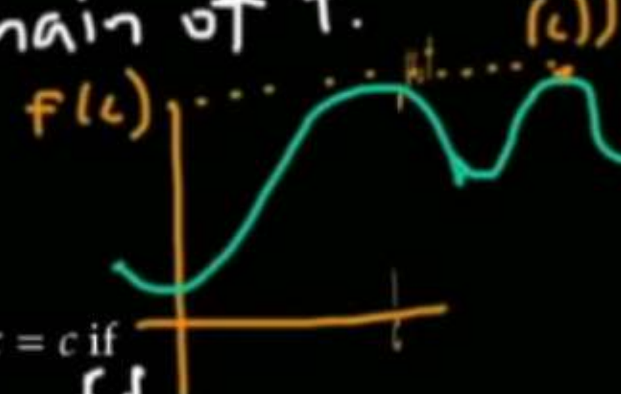
$$\frac{dx}{dt} = 10\pi\ \text{miles}/\text{min} \cdot 60\ \text{min}/\text{hr} = 600\pi\ \text{mph} \simeq 1885\ \text{mph}$$

Left is the conversion of eq

Definition 39.1. A function $f(x)$ has an **absolute maximum** at $x = c$ if
$f(c) \geq f(x)$ for all x in the domain of f.

The point $(c, f(c))$ is called an abs. max. point

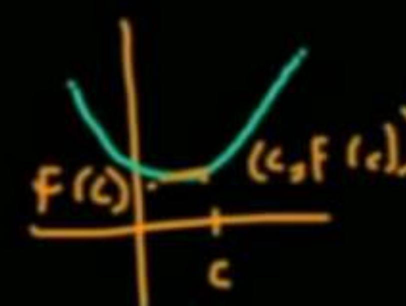
$f(c)$ is called the abs. max value.



Definition 39.2. A function $f(x)$ has an **absolute minimum** at $x = c$ if
$f(c) \leq f(x)$ for all x in the domain of f

The point $(c, f(c))$ is called an abs. min. point

$f(c)$ is called the abs. min value.



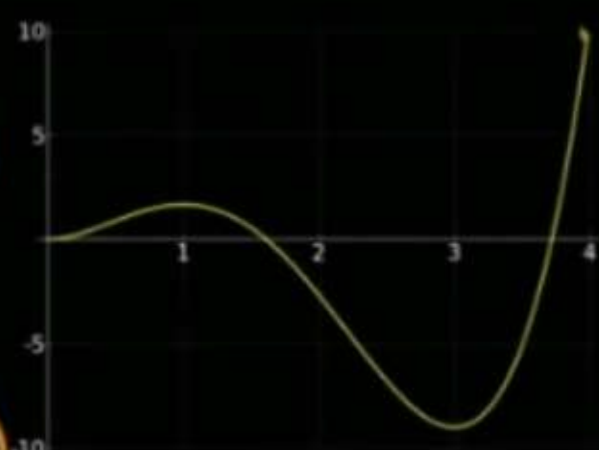
abs min value $= -8$

abs min pt: $(3, -8)$

domain $[0, 4]$

abs max value: 10

abs max point: $(4, 10)$



Definition 39.3. Absolute maximum and minimum values can also be called

Global max and min values.

Definition 39.4. A function $f(x)$ has an **local maximum** at $x = c$ if
$f(c) \geq f(x)$ for all x near c

The point $(c, f(c))$ is called a local max point

$f(c)$ is called a local max value.

Definition. A function $f(x)$ has an **local minimum** at $x = c$ if
$f(c) \leq f(x)$ for all x near c

The point $(c, f(c))$ is called a local min point

$f(c)$ is called a local min value

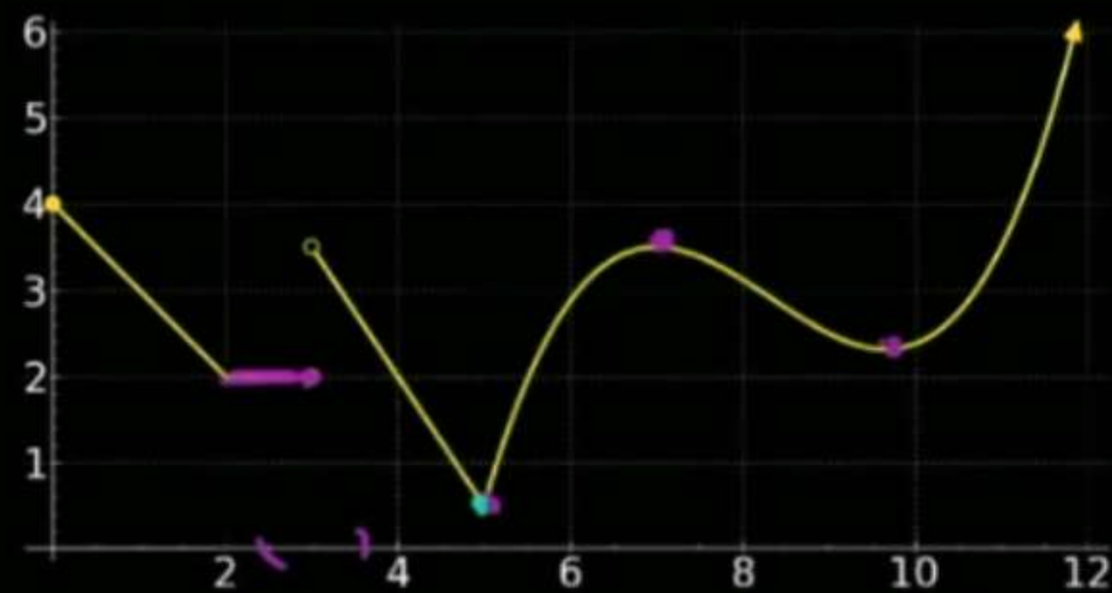
Domain [0,4]

local min pt at $(3,-8)$

local max pt at $(1,2)$

Note $(4,10)$ is not a local max because $f(x)$ is not defined on an open interval around 4.

Definition. Local maximum and minimum values can also be called relative max and min values



Example. 1. Mark all local maximum and minimum points. in green

2. Mark all global maximum and and minimum points. in red

3. What is the absolute maximum value of the function? What is the absolute minimum value?

there is none

≈ 0.5

Definition. A number c is **critical number** for a function f if:

$f'(c)$ DNE or $f'(c) = 0$

Note: If f has a local max or min at $x = c$
the c must be a critical number for f.
$(c, f(c))$ is a critical point for f

If c is a critical number,
then f may or may not
have a local max or min at c.

$f(x) = x^3$ at $c = 0$
$f'(x) = 3x^2$ $f'(0) = 0$
so 0 is a critical number

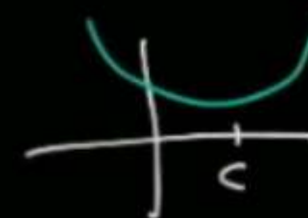
But f does NOT have a local max or min at $x = 0$.

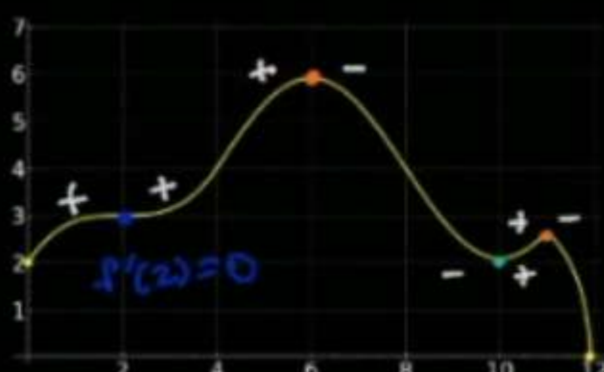
Definition. $f(x)$ has a local maximum at $x = c$ if:

$f(c) \geq f(x)$ for all x in an open interval around c.

Definition. $f(x)$ has a local minimum at $x = c$ if:

$f(c) \leq f(x)$ for all x in an open interval around c.





f has local max at $x=6$ and $x=11$

f has local min at $x=10$

If f has a local max or local min at $x=c$, then $\underbrace{f'(c)=0 \text{ or } f'(c) \text{ DNE}}_{\text{i.e. } c \text{ is a critical number}}$

⚠ It is possible for f to have a critical number at $x=c$ (ie. $f'(c)=0$ or DNE) but not have a local max or min at $x=c$.

First Derivative Test: If f is a cont fn near $x=c$ and if c is a critical number, then

$f'(x)$ for $x<c$	$f'(x)$ for $x>c$	extreme point at $x=c$?
$+$	$-$	local max
$-$	$+$	local min
$+$	$+$	no local extreme point
$-$	$-$	

Second Derivative Test: If f is cont near $x=c$, then

If $f'(c)=0$ and $f''(c)>0$, then f has a local min at $x=c$.

If $f'(c)=0$ and $f''(c)<0$, then f has a local max at $x=c$.

(Note that if $f''(c)=0$ or DNE, then Second Deriv Test is inconclusive.)

Example. Find the absolute maximum and minimum values for $g(x) = \dfrac{x-1}{x^2+x+2}$ on the interval $[0,4]$.

check critical numbers ✓

$g'(x) = 0$ or $g'(x)$ DNE

$$g'(x) = \frac{-x^2+2x+3}{(x^2+x+2)^2}$$

$g'(x)$ DNE where $(x^2+x+2)^2 = 0$

x in $[0,4]$ $\quad x^2+x+2 \geq 2$

so $(x^2+x+2)^2 \neq 0$

$g'(x) = 0 \Rightarrow -x^2+2x+3 = 0$

$\Rightarrow x^2-2x-3 = 0 \quad (x-3)(x+1) = 0$

$\boxed{x = 3}$ or $\cancel{x = -1}$

$$g(3) = \frac{3-1}{3^2+3+2} = \frac{2}{14} = \frac{1}{7}$$

check end points

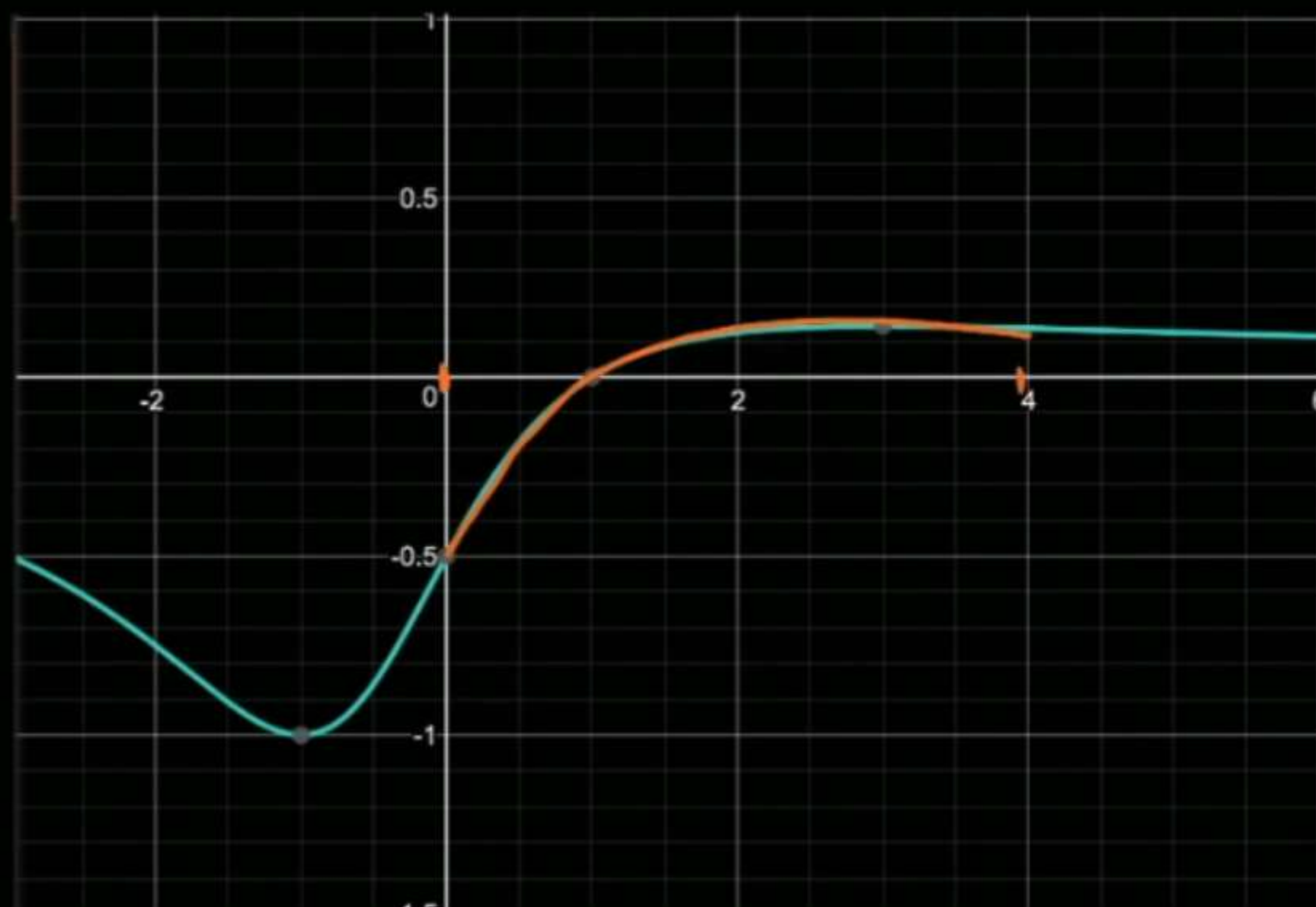
$x = 0 \quad x = 4$

$$g(0) = \frac{0-1}{0^2+0+2} = -\frac{1}{2}$$

$$g(4) = \frac{4-1}{4^2+4+2} = \frac{3}{22}$$

x	$g(x)$	
0	$\boxed{-\frac{1}{2}}$	⇐ abs min value
3	$\boxed{1/7}$	⇐ abs max value
4	$3/22$	

$\frac{1}{7} = \frac{3}{21} > \frac{3}{22}$


1
0.5
-2
0
2
4
-0.5
-1

Example. Find the absolute extreme values for $f(x) = |x| - x^2$ on the interval $[-2, 2]$.

check critical numbers

$$f(x) = \begin{cases} x - x^2 & \text{when } x \geq 0 \\ -x - x^2 & \text{when } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 - 2x & \text{when } x > 0 \\ -1 - 2x & \text{when } x < 0 \end{cases}$$

$f'(x) = 0$ or $f'(x)$ DNE

when $1 - 2x = 0$ for $x > 0$

where $-1 - 2x = 0$ for $x < 0$

$\Rightarrow \boxed{x = \frac{1}{2}}$ for $x > 0$

$\boxed{x = -1/2}$ for $x < 0$

when $\boxed{x = 0}$

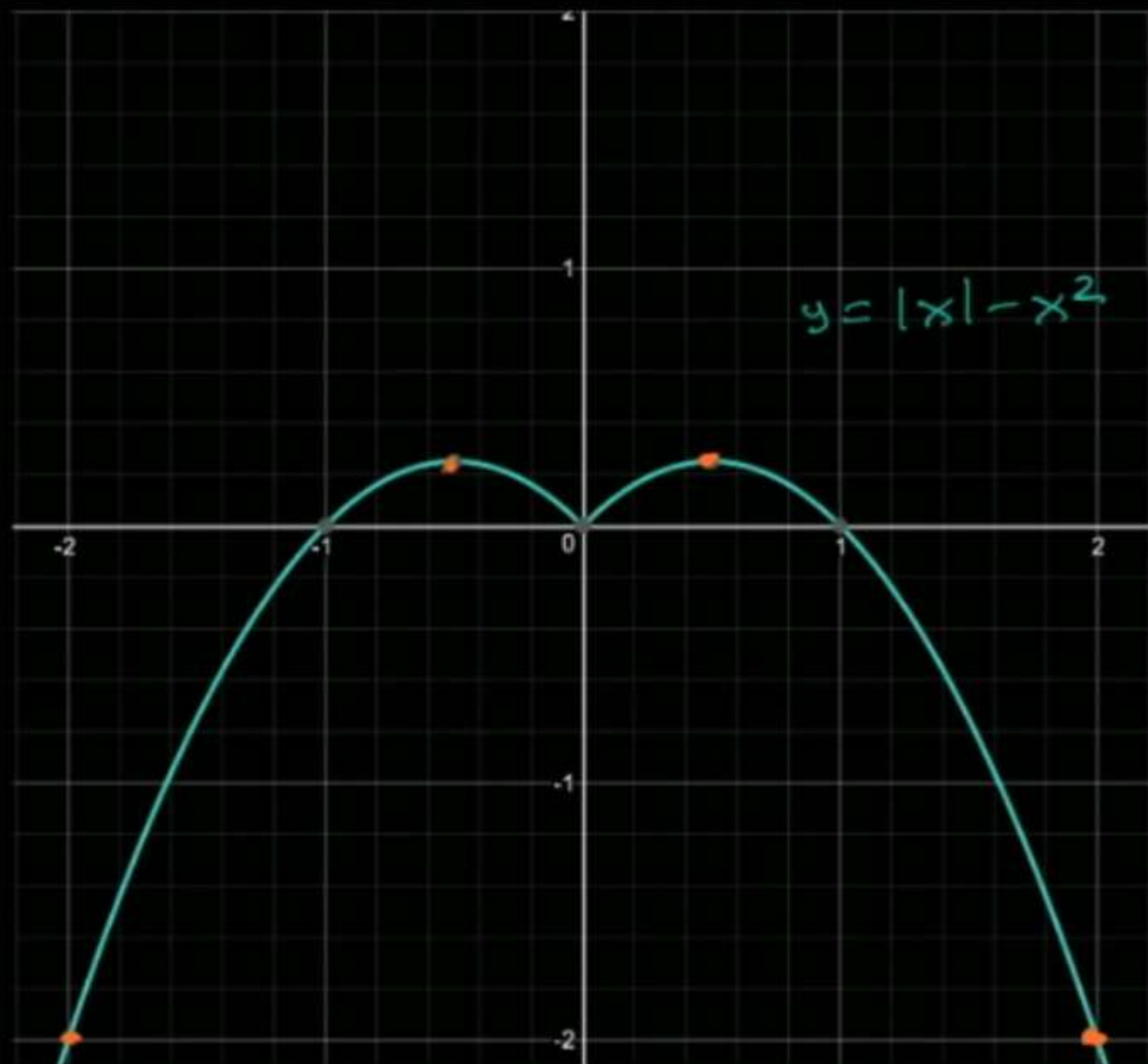
check endpts

$x = -2$, $x = 2$

x	$f(x)$
-2	$\lvert -2 \rvert - (-2)^2 = 2 - 4 = -2$
$-\frac{1}{2}$	$\lvert -\frac{1}{2} \rvert - (-\frac{1}{2})^2 = \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$
0	0
$\frac{1}{2}$	$\boxed{\frac{1}{4}}$
2	-2

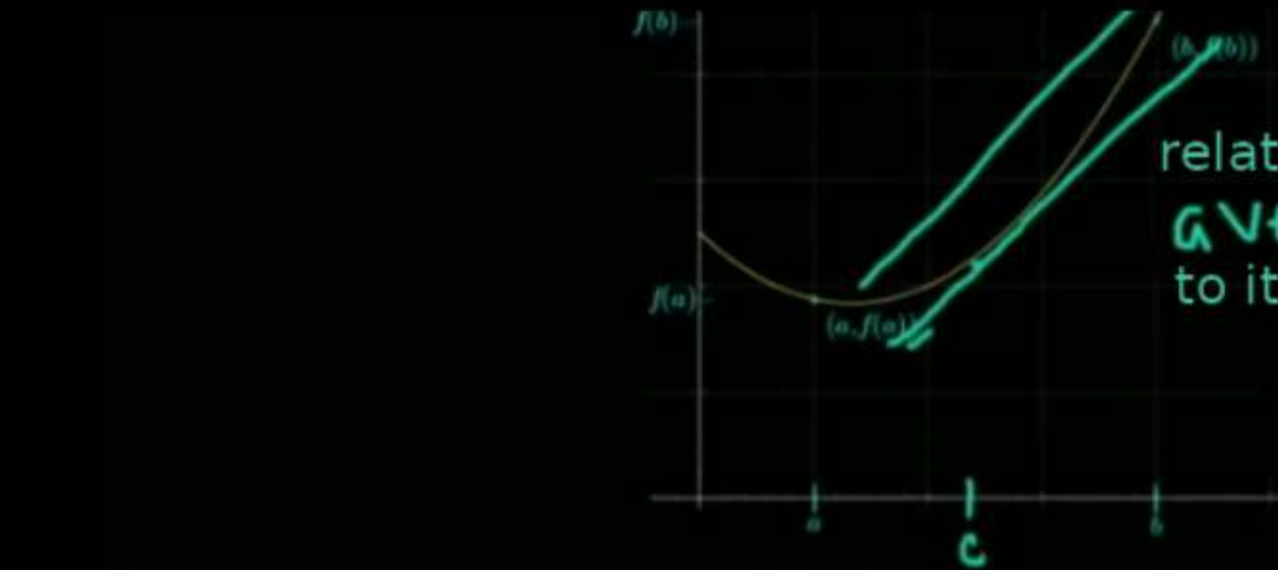
abs max value

abs min value



$y = |x| - x^2$
-2
-1
0
1
2

§4.2.1 The Mean Value Theorem



relate its **average rate of change** to its **instantaneous rate of change**

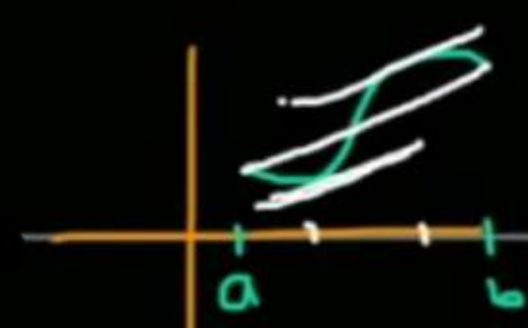
Theorem. *The Mean Value Theorem. Let $f(x)$ be a function defined on $[a,b]$ such that*

1. *$f(x)$ is continuous on $[a,b]$*
2. *$f(x)$ is differentiable on (a,b)*

then there is a number c in $[a,b]$ such that ... the average rate of change of f on $[a,b]$ is equal to the derivative of f at c. i.e. $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

Question. Is the number c unique?

No



Example. Verify the mean value theorem for $f(x) = 2x^3 - 8x + 1$ on the interval $[1,3]$.

1) f is cont on [1,3] ✓
2) f is diff'ble on (1,3) ✓ f is a polynomial

Find c in [1,3] s.t.

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

$$6c^2 - 8 = \frac{31-(-5)}{2}$$

$$6c^2 - 8 = 18$$

$$6c^2 = 24$$

$$c^2 = 4 \Rightarrow c = \pm 2$$

$$f'(x) = 6x^2 - 8$$

$$f'(c) = 6c^2 - 8$$

$$f(3) = 31$$

$$f(1) = -5$$

$$\boxed{c=2}$$

$$f'(2) = 18 = \frac{f(3)-f(1)}{3-1}$$

✓

Example. If f is a differentiable function and $f(1) = 7$ and $-3 \leq f'(x) \leq -2$ on the interval $[1, 6]$, then what is the biggest and smallest values that are possible for $f(6)$?

$$\frac{f(6)-f(1)}{6-1} = f'(c) \text{ for some } c \text{ in } [1,6]$$

$$-3 \leq f'(c) \leq -2$$

$$-3 \leq \frac{f(6)-\cancel{f(1)}^{7}}{6-1} \leq -2$$

$$-15 \leq f(6) - 7 \leq -10$$

smallest possible value $\rightarrow$ $-8 \leq f(6) \leq -3$ $\leftarrow$ largest possible value

Theorem 41.1. *Rolle's Theorem. If $f(x)$ is a function defined on $[a, b]$ and*

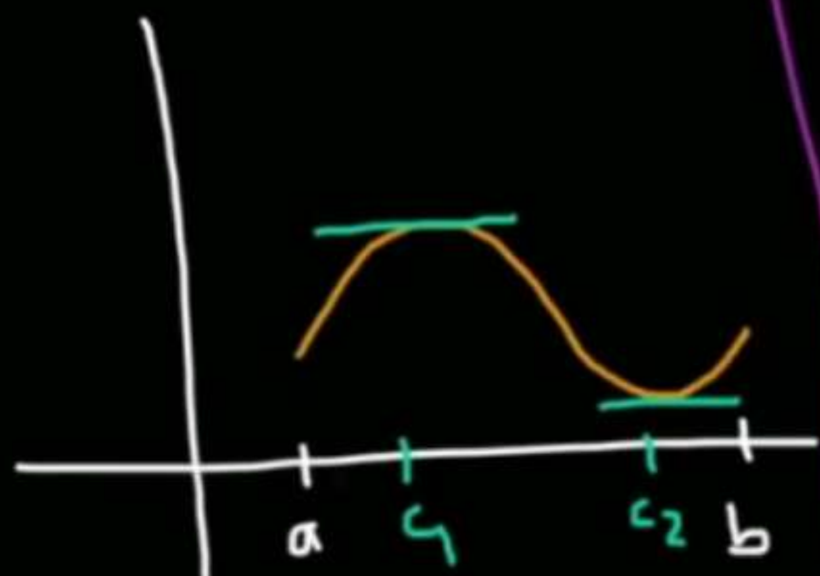
1. *$f(x)$ is continuous on $[a, b]$*
2. *$f(x)$ is differentiable on (a, b),*
3. *and* $f(a) = f(b)$

then there is a number c in $[a, b]$ such that .. $\boxed{f'(c) = 0.}$

Avg. rate chnage

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\boxed{f'(c) = 0}$$



Summary

§4.2.1 The Mean Value Theorem

average rate of change = instantaneous rate of change

$$\frac{f(b)-f(a)}{b-a} \qquad\qquad f'(c)$$

Theorem. *(Mean Value Theorem for Integrals) For a continuous function $f(x)$ on an interval $[a,b]$, there is a number c with $a \le c \le b$ such that* $f(c) = \dfrac{\int_a^b f(x)dx}{b-a}$.

f_{ave} = f average values

a is not var

Proof using the Intermediate Value Theorem:

If $f(x)$ is constant on $[a,b]$, then $f_{ave} = f(c)$ for all c.

Suppose f is not constant on $[a,b]$. Then f has a min value m and a max value M.

$$m \le f_{ave} \le M$$

because

$$m \le f(x) \le M$$

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\frac{m(b-a)}{b-a} \le \frac{\int_a^b f(x)\,dx}{b-a} \le \frac{M(b-a)}{b-a}$$

$$m \le f_{ave} \le M$$

by IVT, $f_{ave} = f(c)$ for some c in $[a,b]$

IVT: For a continuous fn $f(x)$ defined on $[x_1, x_2]$, if L between $f(x_1)$ & $f(x_2)$ then for some c between x_1 & x_2,

$f(c) = L$.

Theorem. *(Mean Value Theorem for Integrals) For a continuous function $f(x)$ on an interval $[a,b]$, there is a number c with $a \leq c \leq b$ such that* $f(c) = \dfrac{\int_a^b f(x)dx}{b-a}$.

Proof using the Mean Value Value Theorem for Functions:

Let $g(x) = \int_a^x f(t)\,dt$

Note $g(a) = \int_a^a f(t)\,dt = 0$ *

$g(b) = \int_a^b f(t)\,dt$ *

Note: $g'(x) = f(x)$ * by Fund. Thm of Calc.

by MVT for fns:

$g'(c) = \dfrac{g(b) - g(a)}{b-a}$ for some c in (a,b)

$f(c) = \dfrac{\int_a^b f(t)\,dt - 0}{b-a}$

$\boxed{\text{///}}$

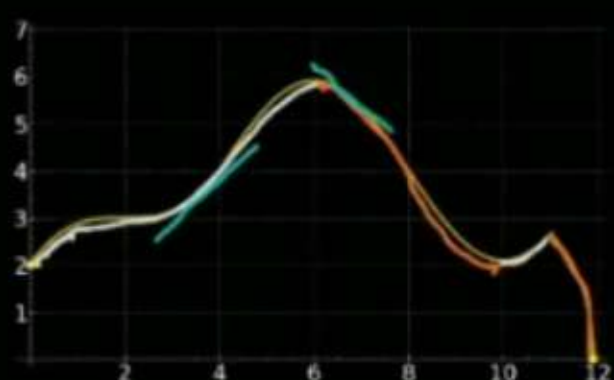
MVT for functions

If $g(x)$ is cont. on $[a,b]$, and differentiable on (a,b), then there is some c in (a,b) such that

$g'(c) = \dfrac{g(b) - g(a)}{b-a}$

Definition. The function $f(x)$ is *increasing* if:
if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$

Definition. The function $f(x)$ is *decreasing* if:
if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$



f is increasing on $(0,6) \cup (10,11)$

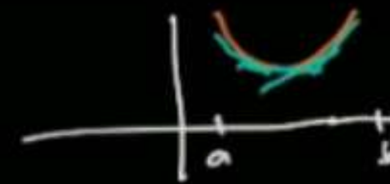
f is decreasing on $(6,10) \cup (11,12)$

Increasing / Decreasing Test:

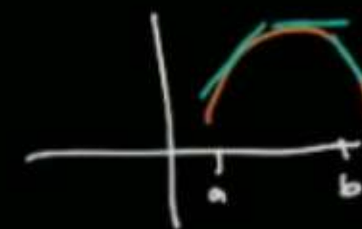
If $f'(x) > 0$ for all x on an interval, then: f is increasing on this interval

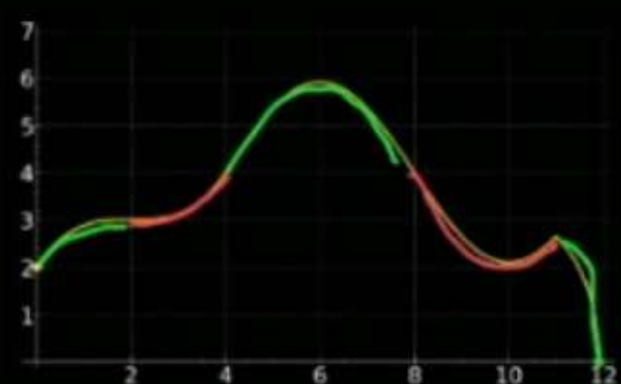
If $f'(x) < 0$ for all x on an interval, then: f is decreasing on this interval

Definition. $f(x)$ is concave up on an interval (a, b) if:
the tangent lines lie below the graph of the function



Definition. $f(x)$ is concave down on an interval (a, b) if:
the tangent lines lie above the graph of the function





f is concave up on $(2,4) \cup (8,11)$
f is concave down on $(0,2) \cup (4,8) \cup (11,12)$

Concavity Test:

If $f''(x) > 0$ for all x on an interval, then: f is concave up on that interval.

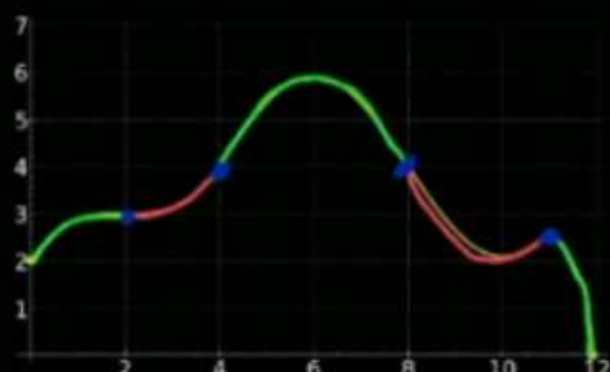
+ 

If $f''(x) < 0$ for all x on an interval, then: f is concave down on that interval.

- 

Definition. $f(x)$ has an *inflection point* at $x = c$ if: f(x) is continuous at c and it changes concavity at c.

i.e. f changes from concave up to concave down at $x=c$
or f changes from concave down to concave up at $x=c$



f has inflection points at
$x=2$, $x=4$, $x=8$, $x=11$

Summary

wheather
- is increasing and decreasing $\Leftarrow f'$

wheather
- is concave up and down $\Leftarrow f''$

wheather
- has inflection points $\Leftarrow f''$

answer Q?

Inflection Point Test: If $f''(x)$ changes sign at $x=c$, then f has an inflection point at $x=c$.

⚠ $f''(c) = 0$ or DNE does not guarantee an inflection point

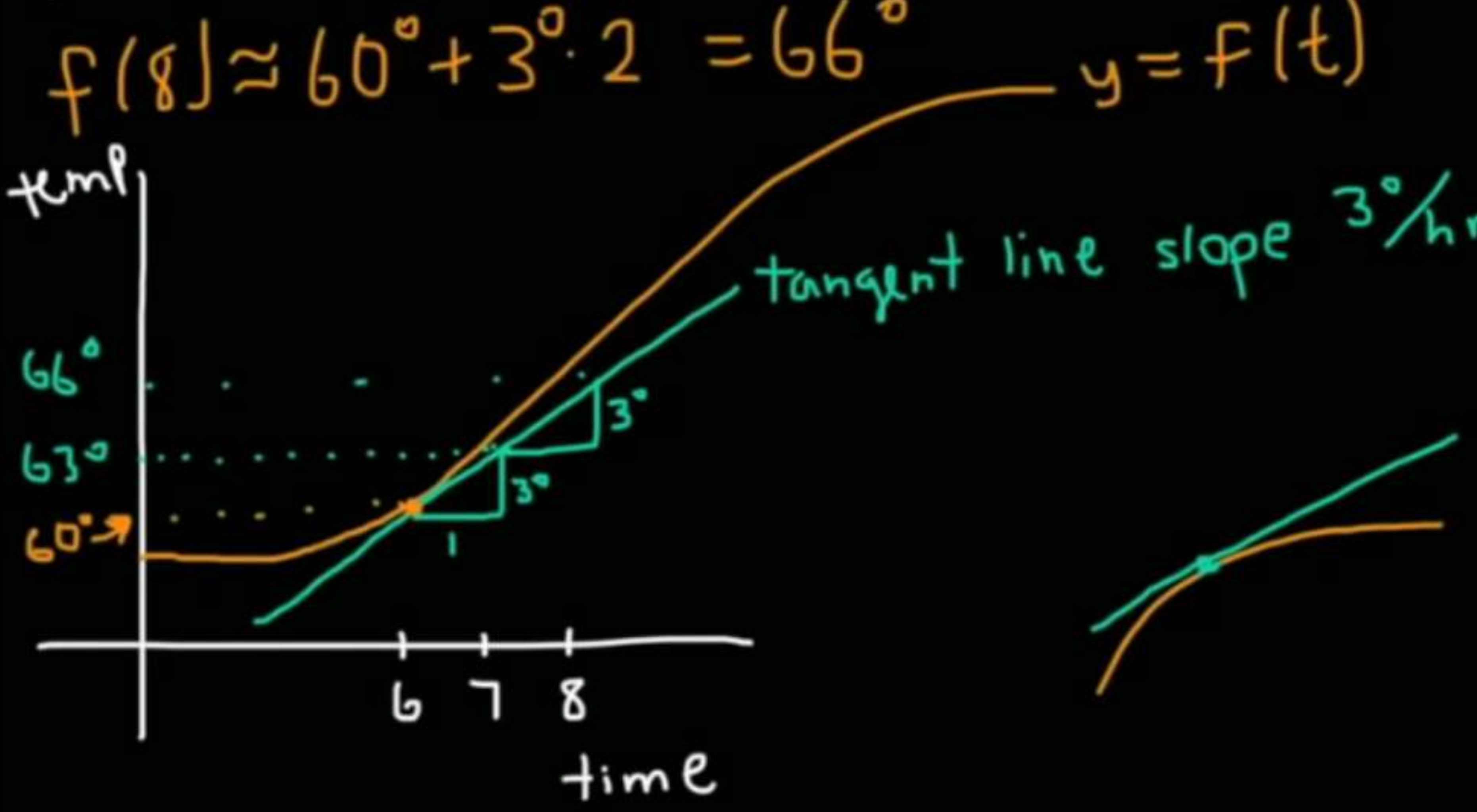
Ex: $f(x) = x^4$ $\qquad f'(x) = 4x^3 \qquad f''(x) = 12x^2$

$f''(0) = 0$ but there is no inflection point at $x=0$

Example. Suppose $f(t)$ is the temperature in degrees Fahrenheit at time t (measured in hours), where $t = 0$ represents midnight. Suppose that $f(6) = 60°$ and $f'(6) = 3°/\text{hr}$. What is your best estimate for the temperature at 7:00 am? 8:00 am?

$$f(7) \approx 60° + 3° = 63°$$

$$f(8) \approx 60° + 3° \cdot 2 = 66°$$



Linear Approximation

tangent line at $x = a$
slope $= f'(a)$

$\Delta x = x - a$

Approximation Principle:

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$

Linear Approximation:

$$f(x) \approx \underbrace{f(a) + f'(a)\cdot(x-a)}_{L(x)}$$

equation of tangent line at $x = a$

$y - y_0 = m(x - x_0)$

$y - f(a) = m(x - a)$

$y - f(a) = f'(a)(x - a)$

$\boxed{y = f(a) + f'(a)(x-a)}$

Linearization of f at a:

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

$f(x) \approx L(x)$

Example. Use the approximation principle to estimate $\sqrt{59}$ without a calculator.

$$\boxed{f(a+\Delta x) \simeq f(a) + f'(a)\cdot \Delta x}$$

$$f(x) = \sqrt{x}$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}$$

$$f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{16}$$

$$a = 64$$

$$a + \Delta x = 59$$

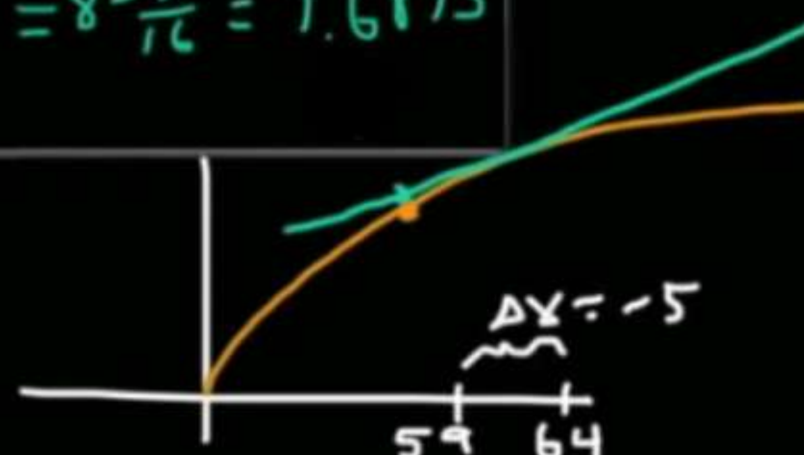
$$64 + \Delta x = 59$$

$$\Delta x = -5$$

$$f(59) \simeq f(64) + f'(64)\cdot(-5)$$

$$\sqrt{59} \simeq \sqrt{64} + \frac{1}{16}(-5) = 8 - \frac{5}{16} = 7.6875$$

$$\sqrt{59} = 7.6811457\text{5}$$



Example. Use a linearization of $y = \sin(x)$ to estimate $\sin(33°)$ without a calculator.

$$f(x) \approx \boxed{L(x) = f(a) + f'(a)(x-a)}$$

$$f(x) = \sin(x) \qquad \frac{d}{dx}\sin x = \cos x$$

$$a = \cancel{30°} = \pi/6$$

$$x = 33° \cdot \frac{\pi}{180°} = \frac{11\pi}{60}$$

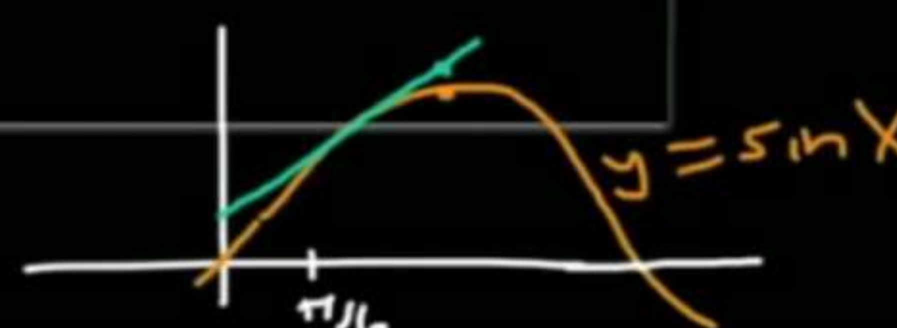
$$L(x) = \sin(\pi/6) + \sin'(\pi/6)(x - \pi/6)$$

$$L(x) = \frac{1}{2} + \cos(\pi/6)(x - \pi/6)$$

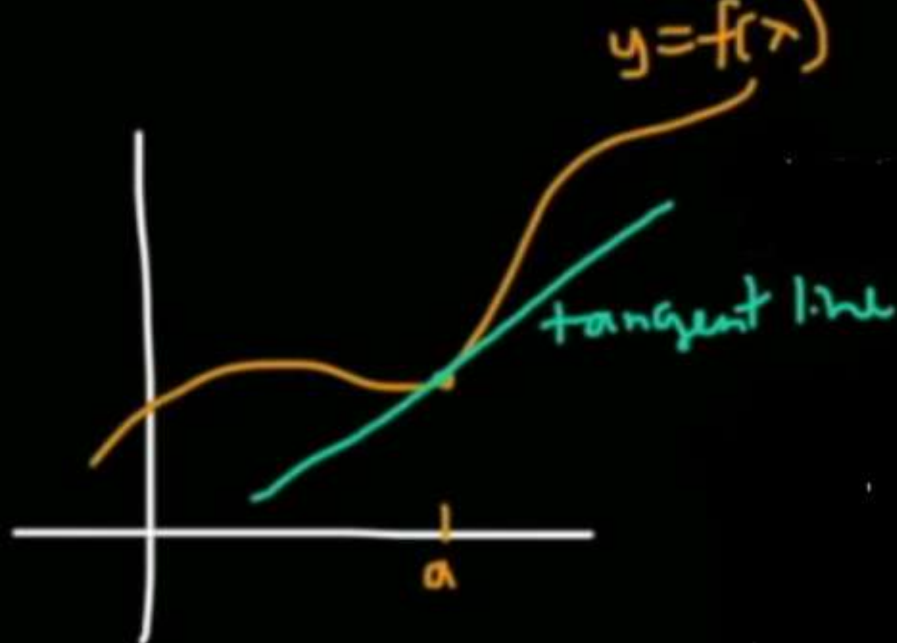
$$\sin(x) \approx L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) \leftarrow \text{linearization of } \sin(x) \text{ at } \pi/6$$

$$\sin\left(\frac{11\pi}{60}\right) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{11\pi}{60} - \pi/6\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{60} \approx 0.5453$$

$$\sin(33°) = \sin(11\pi/60) = 0.5446$$



§3.10.1 Linear Approximation



approximation principle

$$f(a+\Delta x) \approx f(a) + f'(a)\cdot \Delta x$$

linear approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

linearization

$$L(x) = f(a) + f'(a)(x-a)$$

Definitions and Terminology

tangent line to $f(x)$ at $x=a$

slope $= f'(a)$

$f(a) + f'(a)\cdot\Delta x$

$f(a+\Delta x) \approx f(a) + f'(a)\cdot\Delta x$

$f(a+\Delta x) - f(a) \approx f'(a)\cdot\Delta x$

$\Delta f \approx df$

$\Delta y \approx dy$

Definitions:

The differential $dx = \Delta x$

The differential $df = f'(a)\,dx = f'(a)\Delta x$

The differential $dy = df$

The change in f $\Delta f = f(a+\Delta x) - f(a)$

The change in y $\Delta y = \Delta f$

Example. For $f(x) = x\ln x$, find:

$f'(x) = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$= 1 + \ln x$

1. $df = f'(x)dx$

$df = (1+\ln x)dx$

2. df when $x = 2$ and $\Delta x = -0.3$ $dx = -0.3$

$df = (1+\ln 2)(-0.3) \simeq -0.5079$

3. Δf when $x = 2$ and $\Delta x = -0.3$

$\Delta f = f(x+\Delta x) - f(x)$

$= (x+\Delta x)\ln(x+\Delta x) - x\ln x$

$\Delta f = (2-0.3)\ln(2-0.3) - 2\ln 2 \simeq -0.4842$

Theorem. *L'Hospital's Rule* Suppose f and g are differentiable and $g'(x) \neq 0$ in an open interval around a (except possibly at a). If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate form, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

or is ∞ or $-\infty$.

Example. $\lim_{x\to 0} \frac{\sin(x) - x}{(\sin x)^3}$ $\quad \frac{0}{0}$

$$= \lim_{x\to 0} \frac{\cos(x) - 1}{3(\sin x)^2 \cos x} \quad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos(x) - 1}{3(\sin x)^2} \cdot \frac{1}{\cos x}$$

$$= \lim_{x\to 0} \frac{\cos(x) - 1}{3(\sin x)^2} \cdot \lim_{x\to 0} \frac{1}{\cos x} \to 1$$

$$= \lim_{x\to 0} \frac{-\cancel{\sin(x)}}{6\cancel{\sin x}\cos(x)}$$

$$= \lim_{x\to 0} \frac{-1}{6\cos(x)} = \boxed{-\frac{1}{6}}$$

Example. $\lim_{x\to +\infty} \frac{x}{3^x}$ $\quad \frac{\infty}{\infty}$

$$\lim_{x\to \infty} \frac{1}{\ln 3 \cdot 3^x} = 0$$

Example. $\lim_{x\to 0^+} \underbrace{\sin x}_{\to 0} \underbrace{\ln x}_{\to -\infty}$

$0 \cdot \infty$ indeterminate form

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \quad \to \quad \frac{\infty}{\infty}$$

or

$$\lim_{x\to 0^+} \frac{\sin x}{\frac{1}{\ln x}} \quad \to \quad \frac{0}{0}$$

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)}}$$

$$= \lim_{x\to 0^+} \frac{1}{x} \cdot \frac{\sin^2 x}{(-\cos x)} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x \underbrace{\cos x}_{\to 1}}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x} \cdot 1 \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{-2\sin x \cos x}{1} = \frac{-2\cdot 0 \cdot 1}{1} = \boxed{0}$$

Example. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

1^∞ $\quad 1^{\square}=1 \quad \square^\infty=\infty$

$$y=\left(1+\frac{1}{x}\right)^x$$

$$\ln y=\ln\left(1+\frac{1}{x}\right)^x$$

$$\ln y=x\ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\overset{\nearrow\infty}{x}\,\ln\overset{\nearrow 0}{\left(1+\frac{1}{x}\right)}$$

$\infty\cdot 0$

$\frac{\infty}{\infty}$ or $\frac{0}{0}$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \frac{0}{0}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\cancel{\frac{1}{x^2}}\right)}{-\cancel{\frac{1}{x^2}}}=\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}=1$$

$$\lim_{x\to\infty}\ln y=1$$

$$\lim_{x\to\infty}y=\lim_{x\to\infty}e^{\overset{\nearrow 1}{\ln y}}=e^1=e$$

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e}$$

§4.4.2 L'Hospital's Rule - Additional Indeterminate Forms

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$0 \cdot \infty \longrightarrow \frac{0}{0} \quad \frac{\infty}{\infty}$

$f(x) \cdot g(x) \longrightarrow \dfrac{f(x)}{\frac{1}{g(x)}} \qquad \dfrac{g(x)}{\frac{1}{f(x)}}$

$\left.\begin{array}{l} \infty^0 \\ 0^0 \\ 1^\infty \end{array}\right\} \ln y \qquad y = f(x)^{g(x)}$

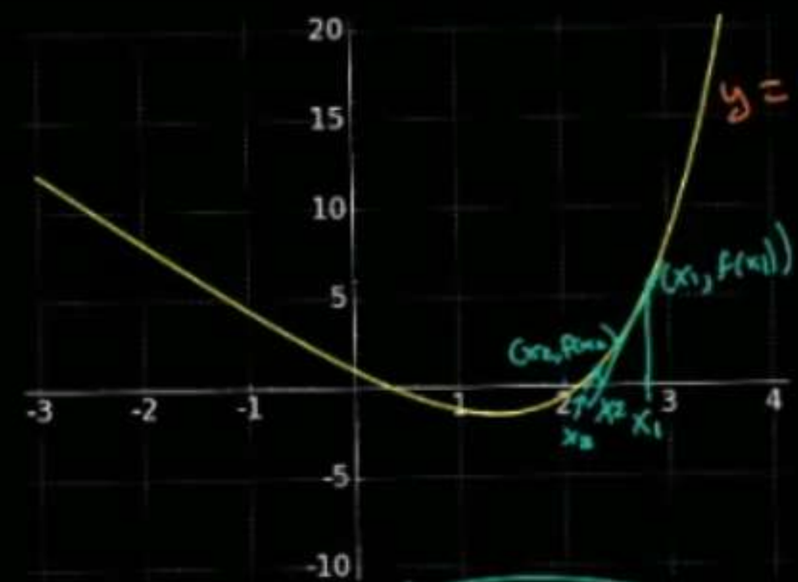
$\boxed{1^\infty} \to \ln 1^\infty \to \infty \ln 1 \to \infty \cdot 0$

$\boxed{\infty^0} \to \ln \infty^0 \to 0 \ln \infty \to 0 \cdot \infty$

$\boxed{0^0} \to \ln 0^0 \to 0 \ln 0 \to 0 \cdot (-\infty)$

L'Hospital's Rule

Example. Find a zero for the function $f(x) = e^x - 4x$.



① Make a guess

② Find the tangent line

③ Find x-intercept for tangent line

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Initial guess: x_1

Tangent line through $(x_1, f(x_1))$

$y = f(x_1) + f'(x_1)(x - x_1)$

$y - y_1 = m(x - x_1)$

$m = f'(x_1)$

$y_1 = f(x_1)$

$y - f(x_1) = f'(x_1)(x - x_1)$

$y = f(x_1) + f'(x_1)(x - x_1)$

Find x-intercept of tangent line

$0 = f(x_1) + f'(x_1)(x - x_1)$

$-f(x_1) = f'(x_1)(x - x_1)$

$-\dfrac{f(x_1)}{f'(x_1)} = x - x_1$

$x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

x_2

Tangent line through $(x_2, f(x_2))$

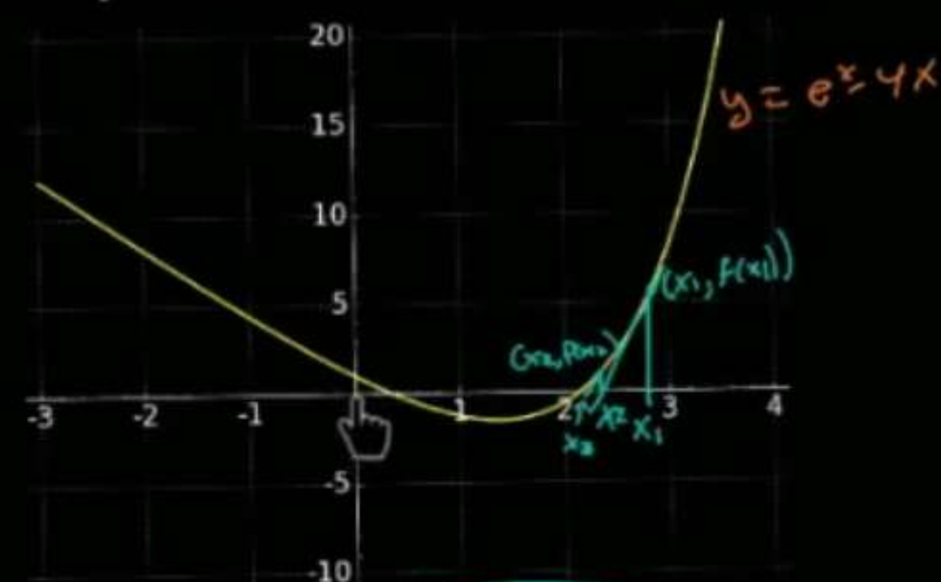
$y = f(x_2) + f'(x_2)(x - x_2)$

Find x-intercept of tangent line

⋮

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Example. Find a zero for the function $f(x) = e^x - 4x$.



① Make a guess
② Find the tangent line
③ Find x-intercept for tangent line

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$$f(x) = e^x - 4x$$
$$f'(x) = e^x - 4$$

$$x_{n+1} = x_n - \frac{e^{x_n} - 4x_n}{e^{x_n} - 4}$$

$$x_1 = 3$$
$$x_2 = 3 - \frac{e^3 - 4\cdot 3}{e^3 - 4}$$
$$x_2 = 2.49734119$$
$$x_3 = 2.49 - \frac{e^{2.49} - 4\cdot 2.49}{e^{2.49} - 4}$$
$$x_3 = 2.2322194$$
$$x_4 = 2.15860801$$
$$x_5 = 2.15331858$$
$$x_6 = 2.15329236$$
$$x_7 \approx 2.15329236$$

Example. If $g'(x) = 3x^2$, what could $g(x)$ be?

$g(x) = x^3$

OR

$g(x) = x^3 + 7$

OR

$g(x) = x^3 + C$ for some constant C

Definition. A function $F(x)$ is called an antiderivative of $f(x)$ on an interval (a, b) if

$F'(x) = f(x)$ on (a,b).

x^3 is an antiderivative of $3x^2$

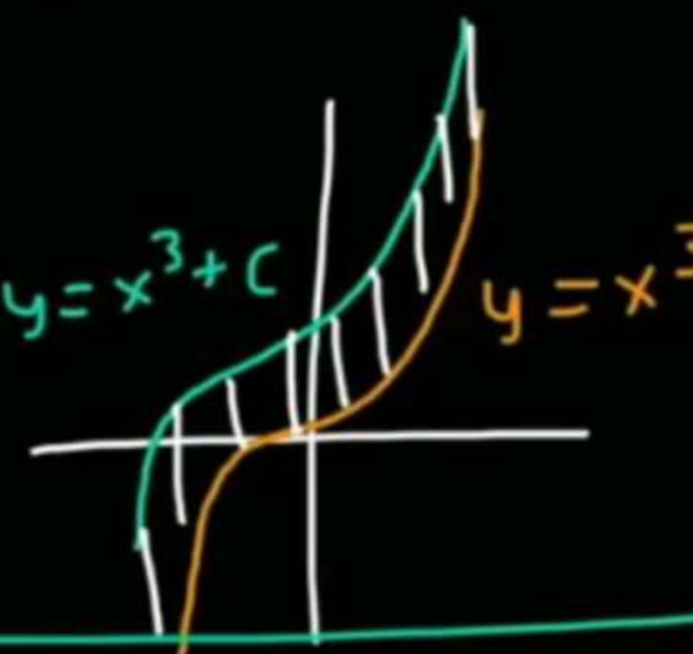
$\boxed{x^3 + C}$ is an antiderivative of $3x^2$

a general antiderivative

Question. What are all the antiderivatives of $f(x) = 3x^2$?

$$F(x) = x^3 + \boxed{C}$$

could there be other antiderivatives? NO



If $F(x)$ is an antiderivative for $f(x)$ then all other antiderivatives for $f(x)$ can be written in the form $F(x) + C$

Table of Antiderivatives:

Function $f(x)$	Antiderivative $F(x)$	Function $f(x)$	Antiderivative $F(x)$
1	$x + C$	$\sec^2(x)$	$\tan(x) + C$
x	$\frac{x^2}{2} + C$	$\sec(x)\tan(x)$	$\sec(x) + C$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$	$\frac{1}{1+x^2}$	$\arctan(x) + C$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
e^x	$e^x + C$	$a \cdot x^n$	$a \cdot \frac{x^{n+1}}{n+1} + C$
$\sin(x)$	$-\cos(x) + C$	$a \cdot f(x)$	$a \cdot F(x) + C$
$\cos(x)$	$\sin(x) + C$	$f(x) + g(x)$	$F(x) + G(x) + C$

$$\frac{d}{dx}\frac{x^2}{2} = \frac{\cancel{2}x}{\cancel{2}} = x \qquad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{\cancel{(n+1)}x^n}{\cancel{n+1}} = x^n$$

Example. Find the general antiderivative for $f(x) = \frac{5}{1+x^2} - \frac{1}{2\sqrt{x}}$

$$f(x) = 5 \cdot \frac{1}{1+x^2} - \frac{1}{2} \cdot x^{-1/2}$$

$$F(x) = 5 \cdot \arctan(x) - \cancel{\tfrac{1}{2}} \frac{x^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + C$$

$$\boxed{F(x) = 5 \cdot \arctan(x) - \sqrt{x} + C}$$

Example. If $g'(x) = e^x - 3\sin(x)$, and $g(2\pi) = 5$, find $g(x)$.

$$g(x) = e^x + 3\cos(x) \quad + C$$

$$g(2\pi) = e^{2\pi} + 3\underbrace{\cos(2\pi)}_{1} + C = 5$$

$$e^{2\pi} + 3 + c = 5$$

$$c = 2 - e^{2\pi}$$

$$\boxed{g(x) = e^x + 3\cos(x) + 2 - e^{2\pi}}$$

Example. $f''(x) = \sqrt{x}(x - \frac{1}{x})$. Find an equation for $f(x)$, if $\boxed{f(1) = 0 \text{ and } f(0) = 2.}$

7=7
1=7
1=1
if
st
won't
MATCH
UP
In
EQ

$$f''(x) = \sqrt{x} \cdot x - \frac{\sqrt{x}}{x}$$

$$f''(x) = x^{3/2} - x^{-1/2}$$

$$f'(x) = \frac{x^{5/2}}{5/2} - \frac{x^{1/2}}{1/2} + C$$

$$f'(x) = \frac{2}{5}x^{5/2} - 2x^{1/2} + C$$

$$f(x) = \frac{2}{5} \cdot \frac{x^{7/2}}{7/2} - 2\frac{x^{3/2}}{3/2} + Cx + D$$

$$f(x) = \frac{4}{35}x^{7/2} - \frac{4}{3}x^{3/2} + Cx + D$$

$$f(0) = D = 2$$

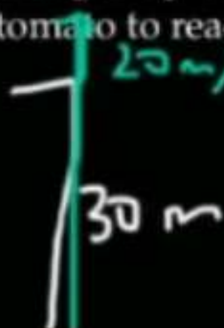
$$f(x) = \frac{4}{35}x^{7/2} - \frac{4}{3}x^{3/2} + Cx + 2$$

$$f(1) = \frac{4}{35} - \frac{4}{3} + C + 2 = 0$$

$$C = -2 - \frac{4}{35} + \frac{4}{3} = \frac{-82}{105}$$

$$\boxed{f(x) = \frac{4}{35}x^{7/2} - \frac{4}{3}x^{3/2} - \frac{82}{105}x + 2}$$

Example. You stand at the edge of a cliff at height 30 meters. You throw a tomato straight up in the air at a speed of 20 meters per second. How long does it take the tomato to reach the ground? What is its velocity at impact?



$a(t) = -9.8 \ m/s^2$ $\quad a(t) = -32 \ ft/s^2$

$v(0) = 20 \ m/s$

$s(0) = 30 \ m$

$s''(t) = -9.8$

$s'(t) = -9.8t + c_1$

$s'(0) = 20$

$s'(0) = -9.8 \cdot 0 + c_1 = 20$

$c_1 = 20$

$s'(t) = -9.8t + 20$

$s(t) = -9.8\dfrac{t^2}{2} + 20t + c_2$

$s(0) = 30$

$s(0) = -9.8 \cdot \dfrac{0^2}{2} + 20 \cdot 0 + c_2 = 30$

$c_2 = 30$

$$\boxed{s(t) = -9.8\frac{t^2}{2} + 20t + 30}$$

$s(t) = 0$

$0 = \dfrac{-9.8t^2}{2} + 20t + 30$

Solve with quadratic func for t

$t = \cancel{-1.17}, \ 5.25 \ s$

$v(5.25)$

$= s'(5.25)$

$= -9.8 \cdot 5.25 + 20$

$-31.45 \ m/s$

Fact: *If $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative of $f(x)$ can be written in the form $F(x) + C$ for some constant C.*

Note. If $g'(x) = 0$ on the interval (a, b), then $g(x) = C$ for some constant C.

Proof: MVT: for any x_1, x_2 in (a,b)

$$\frac{g(x_2)-g(x_1)}{x_2-x_1} = g'(x_3) = 0$$ for some x_3 between x_1 and x_2

$g(x_2)-g(x_1)=0$

$\Rightarrow g(x_2)=g(x_1) \Rightarrow g(x)$ is a constant

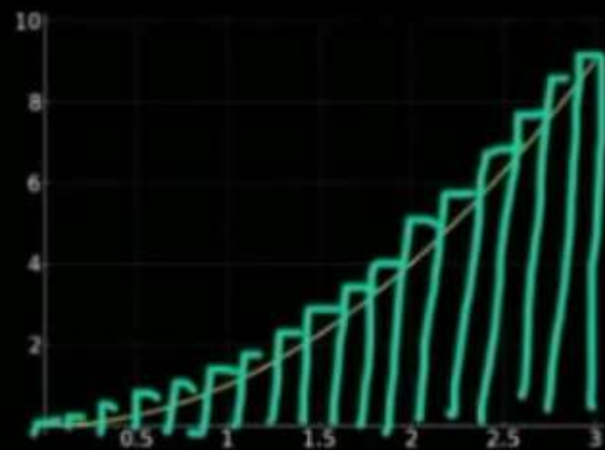
Note. If $g_1(x)$ and $g_2(x)$ are two functions defined on (a, b) and $g_1'(x) = g_2'(x)$ on (a, b), then $g_1(x) = g_2(x) + C$ for some constant C.

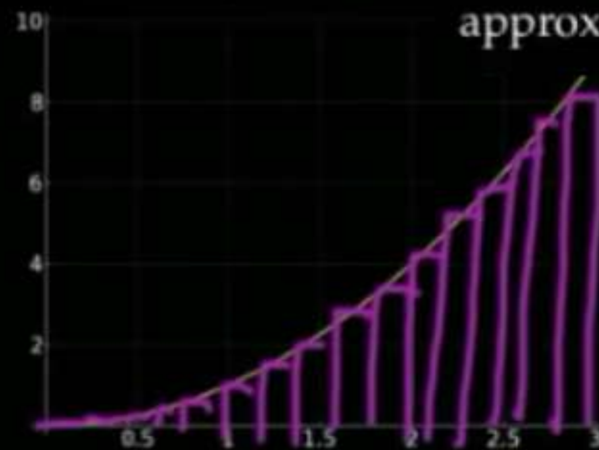
Proof: $g_1'(x) = g_2'(x) \Rightarrow g_1'(x) - g_2'(x) = 0$

$(g_1(x) - g_2(x))' = 0 \Rightarrow g_1(x) - g_2(x) = C$

$\Rightarrow g_1(x) = g_2(x) + C$

Example. Estimate the area under the curve $y = x^2$ between $x = 0$ and $x = 3$ by approximating it with ~~100~~ n rectangles.





base $= \frac{3}{n} = \Delta x$

right endpt $x_i = \frac{3}{n} \cdot i$

height $h_i = \left(\frac{3}{n} i\right)^2$

base $= \frac{3}{n} = \Delta x$

left endpoint $x_i = \frac{3}{n}(i-1)$

height $h_i = \left(\frac{3}{n}(i-1)\right)^2$

Estimate area using n rectangles:

$$\sum_{i=1}^{n} \frac{3}{n} \cdot \left(\frac{3i}{n}\right)^2 \qquad \sum_{i=1}^{n} \frac{3}{n} \left(\frac{3(i-1)}{n}\right)^2$$

The exact area is given by the limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\frac{3i}{n}\right)^2 \quad \text{OR} \quad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\frac{3(i-1)}{n}\right)^2$$

Example. Compute the exact area under the curve $y = x^2$ between $x = 0$ and $x = 3$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\frac{3i}{n}\right)^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\cdot\frac{3^2}{n^2}\cdot i^2$$

$$= \lim_{n\to\infty} \frac{3}{n}\cdot\frac{3^2}{n^2} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{9}{2}\,\frac{(n+1)(2n+1)}{n^2}$$

$$= \frac{9}{2}\lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} = \frac{9}{2}\cdot 2 = 9$$

Fact:

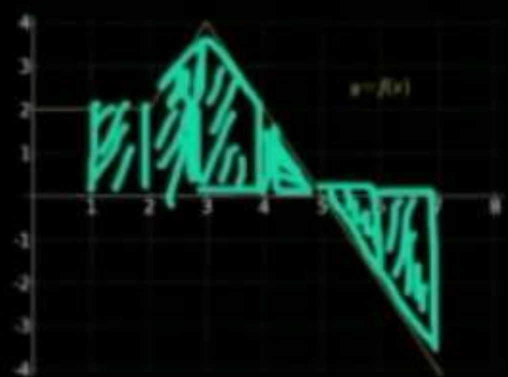
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2 + 2^2 = 5 = \frac{2(2+1)(4+1)}{6}$$

area under curve $= 9$

accumulated area function

Example. Suppose $f(x)$ has the graph shown, and let $g(x) = \int_1^x f(t)\,dt$.



Find:

$g(1) = \int_1^1 f(t)\,dt = 0$

$g(2) = \int_1^2 f(t)\,dt = 2$

$g(3) = \int_1^3 f(t)\,dt = 5$

$g(4) = \int_1^4 f(t)\,dt = 8$

$g(5) = \int_1^5 f(t)\,dt = 9$

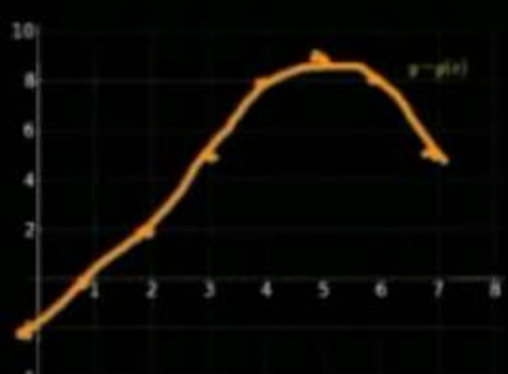
$g(6) = \int_1^6 f(t)\,dt = 8$

$g(7) = \int_1^7 f(t)\,dt = 5$

$g(0) = \int_1^0 f(t)\,dt$

$= -\int_0^1 f(t)\,dt$

$= -2$



$g'(x) > 0$ when $f(x) > 0$

$g'(x) < 0$ when $f(x) < 0$

$g'(x) = 0$ where $f(x) = 0$

$g'(x) = f(x)$

Theorem. *(Fundamental Theorem of Calculus, Part 1) If $f(x)$ is continuous on $[a,b]$ then for $a \leq x \leq b$ the function*

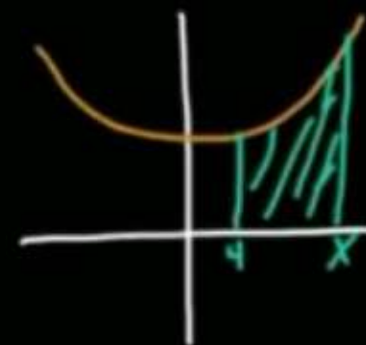
$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b) and

$$g'(x) = f(x)$$

Example. Find

1. $\frac{d}{dx}\left(\int_5^x \sqrt{t^2+3}\,dt\right) = \sqrt{x^2+3}$

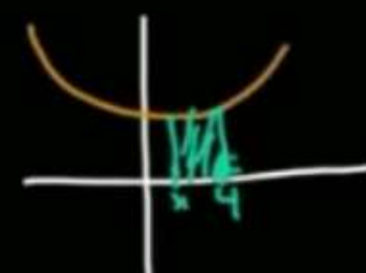


2. $\frac{d}{dx}\int_4^x \sqrt{t^2+3}\,dt = \sqrt{x^2+3}$

3. $\frac{d}{dx}\int_x^4 \sqrt{t^2+3}\,dt = \frac{d}{dx}\left(-\int_4^x \sqrt{t^2+3}\,dt\right)$

$= -\frac{d}{dx}\int_4^x \sqrt{t^2+3}\,dt$

$= -\sqrt{x^2+3}$



4. $\frac{d}{dx}\int_4^{\sin(x)} \sqrt{t^2+3}\,dt$

$\frac{d}{dx} f(u(x)) = \frac{df(u)}{du}\cdot\frac{du(x)}{dx}$ $u(x)=\sin(x)$

$$\frac{d}{dx}\int_4^{\sin(x)} \sqrt{t^2+3}\,dt = \frac{d}{du}\int_4^u \sqrt{t^2+3}\,dt \cdot \frac{d\sin(x)}{dx}$$

$$= \sqrt{u^2+3}\cdot\cos(x)$$

$$= \sqrt{(\sin(x))^2+3}\cdot\cos(x)$$

$$= \sqrt{\sin^2(x)+3}\cdot\cos(x)$$

Theorem. *(Fundamental Theorem of Calculus, Part 2) If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is <u>any antiderivative</u> of f, that is F is any function such that $F' = f$.

$$f(x) = F'(x)$$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

= "The integral of the derivative is the original function."

Suppose $G(x)$ is another antideriv for $f(x)$.

$G(x) = F(x) + C$

So $G(b) - G(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a)$

Example. Find $\int_{-1}^{-5} 3x^2 - \frac{4}{x}\,dx$ $= x^3 - 4\cdot\ln|x|\Big|_{-1}^{-5}$

$F(x)\Big|_a^b = F(b) - F(a)$

$= \left((-5)^3 - 4\cdot\ln|-5|\right) - \left((-1)^3 - 4\ln|-1|\right)$

$= -125 - 4\ln 5 - (-1) + 4\ln 1$

$= -124 - 4\ln 5 \approx -130.438$

Example. Find $\int_1^4 \frac{y^2 - y + 1}{\sqrt{y}}\,dy$

$= \int_1^4 \frac{y^2 - y + 1}{y^{1/2}}\,dy = \int_1^4 (y^2 - y + 1)\,y^{-1/2}\,dy$

$= \int_1^4 y^{3/2} - y^{1/2} + y^{-1/2}\,dy$

$= \frac{y^{5/2}}{5/2} - \frac{y^{3/2}}{3/2} + \frac{y^{1/2}}{1/2}\Big|_1^4$

$= \frac{2}{5}y^{5/2} - \frac{2}{3}y^{3/2} + 2y^{1/2}\Big|_1^4$

$= \left(\frac{2}{5}4^{5/2} - \frac{2}{3}\cdot 4^{3/2} + 2\cdot 4^{1/2}\right) - \left(\frac{2}{5}\cdot 1^{5/2} - \frac{2}{3}\cdot 1^{3/2} + 2\,1^{1/2}\right)$

$(4^{1/2})^5$

$= \frac{2}{5}\cdot 32 - \frac{2}{3}\cdot 8 + 2\cdot 2) - \left(\frac{2}{5} - \frac{2}{3} + 2\right)$

$= 146/15$

Summary

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

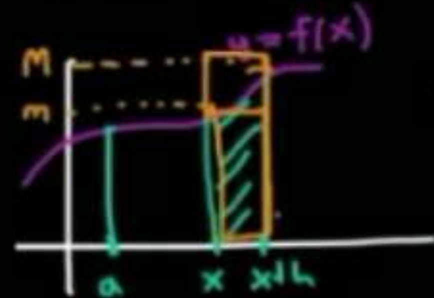
Theorem. *(Fundamental Theorem of Calculus, Part 1) If $f(x)$ is continuous on $[a,b]$ then for $a \le x \le b$ the function*

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b) and

$$g'(x) = f(x)$$

Proof: $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h}$



$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

M = max value of $f(x)$ on $[x, x+h]$

m = min value of $f(x)$ on $[x, x+h]$

$$= \lim_{h\to 0} f(c) \quad \text{for some } c \text{ between } x \text{ \& } x+h$$

$$= f(x) \quad \blacksquare$$

$$m\cdot h \le \int_x^{x+h} f(t)\,dt \le M\cdot h$$

$$m \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le M$$

$f(c)$ for some c in $[x, x+h]$ by IVT

Theorem. *(Fundamental Theorem of Calculus, Part 2) If f is continuous on $[a,b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

Proof: $G(x) = \int_a^x f(t)\,dt$

$\Rightarrow G'(x) = f(x) \Rightarrow G(x)$ is an antideriv for $f(x)$

$G(b) - G(a) = \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{0}$

Let $F(x)$ be any antideriv of $f(x)$.

$F(x) = G(x) + C$

$F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) - G(a) = \int_a^b f(t)\,dt$

$\blacksquare$

Example. Find $\int 2x\sin(x^2)\,dx$

$u = x^2 \qquad du = 2x\,dx$

$$\int \sin(u)\,du$$

$$= -\cos(u) + C$$

$$= \boxed{-\cos(x^2) + C}$$

check:

$$\frac{d}{dx}\left(-\cos(x^2) + c\right)$$

$$= \sin(x^2) \cdot 2x$$

Example. Evaluate $\int \frac{x}{1+3x^2}\, dx$

$u = 1+3x^2 \qquad du = 6x\,dx$

$\frac{1}{6}du = x\,dx$

$$\int \frac{\frac{1}{6}du}{u} = \frac{1}{6}\int \frac{1}{u}\,du$$

$$= \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln|1+3x^2| + C$$

Example. $\int e^{7x}\, dx$

$u = 7x \qquad du = 7\,dx$

$\frac{1}{7}du = dx$

$$= \int e^u \cdot \frac{1}{7}\,du$$

$$= \frac{1}{7}e^u + C = \frac{1}{7}e^{7x} + C$$

Substitution with Definite Integrals

Example. $\int_e^{e^2} \frac{\ln(x)}{x}\, dx$

$u = \ln(x) \quad du = \frac{1}{x}\, dx$

Worry about bounds of integration now:

when $x = e$ $\quad u = \ln(e) = 1$

when $x = e^2$ $\quad u = \ln(e^2) = 2$

$$\int_1^2 u\, du$$

$$= \frac{u^2}{2}\Big|_{u=1}^{u=2} = \frac{2^2}{2} - \frac{1^2}{2} = \frac{1}{2}$$

Worry about bounds of integration later.

$u = \ln(x) \quad du = \frac{1}{x}\, dx$

$$\int_e^{e^2} \frac{\ln(x)}{x}\, dx$$

$$= \int \frac{\ln x}{x}\, dx = \int u \cdot du = \frac{u^2}{2}$$

$$= \frac{(\ln(x))^2}{2}\Big|_{x=e}^{x=e^2}$$

$$= \frac{(\ln(e^2))^2}{2} - \frac{(\ln(e))^2}{2} = \frac{2^2}{2} - \frac{1}{2} = \frac{1}{2}$$

Why does the substitution method work?

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x)) \cdot g'(x)$$

$$\int F'(g(x)) \cdot g'(x)\,dx = \int \frac{d}{dx} F(g(x))\,dx$$

$$= F(g(x)) + C$$

$$\int F'(g(x))\, g'(x)\,dx$$

$$u = g(x) \quad du = g'(x)\,dx$$

$$= \int F'(u)\,du = F(u) + C = F(g(x)) + C$$

The average of a list of numbers $q_1, q_2, q_3, \ldots, q_n$ is:

$$\text{average} = \frac{q_1 + q_2 + \cdots + q_n}{n} = \frac{\sum_{i=1}^{n} q_i}{n}$$

For a continuous function $f(x)$ on an interval $[a, b]$, we could estimate the average value of the function by sampling it at a bunch of x-values: $x_1, x_2, \ldots x_n$

spaced Δx apart

$$\text{approx ave value} = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

The approximation gets better as the number of sample points gets larger, so we can define

$$\text{average} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} \cdot \frac{\Delta x}{\Delta x}$$

note: $n \cdot \Delta x = b - a$

$$= \lim_{\Delta x \to 0} \frac{\sum_{i=1}^{n} f(x_i)\cdot \Delta x}{b-a} = \boxed{\frac{\int_a^b f(x)\,dx}{b-a}}$$

The average of a list of numbers $q_1, q_2, q_3, \ldots, q_n$ is:

$$\text{average} = \frac{q_1 + q_2 + \cdots + q_n}{n} = \frac{\sum_{i=1}^{n} q_i}{n}$$

For a continuous function $f(x)$ on an interval $[a, b]$, we could estimate the average value of the function by sampling it at a bunch of x-values: $x_1, x_2, \cdots x_n$

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$$\text{approx ave value} = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

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$$= \lim_{\Delta x \to 0} \frac{\sum_{i=1}^{n} f(x_i) \cdot \Delta x}{b - a} = \boxed{\frac{\int_a^b f(x)\,dx}{b-a}}$$

Example. Find the average value of the function $g(x) = \frac{1}{1-5x}$ on the interval $[2,5]$.

$$g_{ave} = \frac{\int_2^5 \frac{1}{1-5x}\,dx}{5-2}$$

$u = 1-5x$ $\quad x=2 \quad u=-9$

$du = -5\,dx$ $\quad x=5 \quad u=-24$

$-\frac{1}{5}du = dx$

$$= \frac{\int_{-9}^{-24} \frac{1}{u}\left(-\frac{1}{5}\right)du}{3} = \frac{1}{3}\left(-\frac{1}{5}\right)\ln|u|\Big|_{-9}^{-24}$$

$$= -\frac{1}{15}(\ln 24 - \ln 9) = -\frac{1}{15}\ln\frac{24}{9} = -\frac{1}{15}\ln\frac{8}{3} \approx -0.0654$$

Question. Is there a number c in the interval $[2,5]$ for which $g(c)$ equals its average value? If so, find all such numbers c. If not, explain why not.

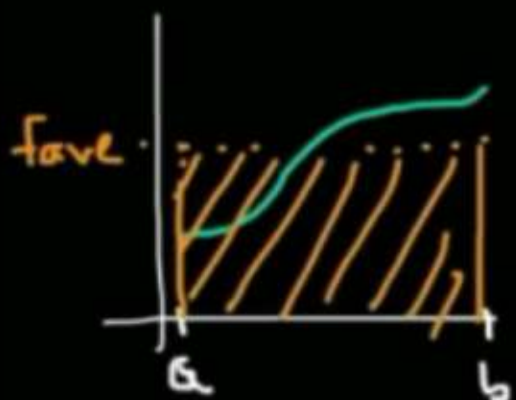
$$g(c) = g_{ave} \Rightarrow \frac{1}{1-5c} = -\frac{1}{15}\ln\frac{8}{3} \Rightarrow 1-5c = \frac{-15}{\ln\frac{8}{3}}$$

$$-5c = \frac{-15}{\ln\frac{8}{3}} - 1 \Rightarrow c = \frac{\frac{-15}{\ln 8/3} - 1}{-5} = \frac{3}{\ln 8/3} + \frac{1}{5}$$

$$\approx 3.25$$

Theorem. *(Mean Value Theorem for Integrals) For a continuous function $f(x)$ on an interval $[a,b]$, there is a number c with $a \le c \le b$ such that ...*

$$f(c) = f_{ave}$$



$$f(c) = \frac{\int_a^b f(x)\,dx}{b}$$

$$(b-a) f_{ave} = \int_a^b f(x)\,dx$$

$$f_{ave} = \frac{\int_a^b f(x)\,dx}{b-a}$$

MVT for integrals

Theorem. *(Mean Value Theorem for Integrals) For a continuous function $f(x)$ on an interval $[a,b]$, there is a number c with $a \le c \le b$ such that* $f(c) = \dfrac{\int_a^b f(x)\,dx}{b-a}$. ← f_{ave}

Proof using the Intermediate Value Theorem:

If $f(x)$ is constant on $[a,b]$, then $f_{ave} = f(c)$ for all c.

Suppose f is not constant on $[a,b]$. Then f has a min value m and a max value M.

$$m \le f_{ave} \le M$$

because

$$m \le f(x) \le M$$

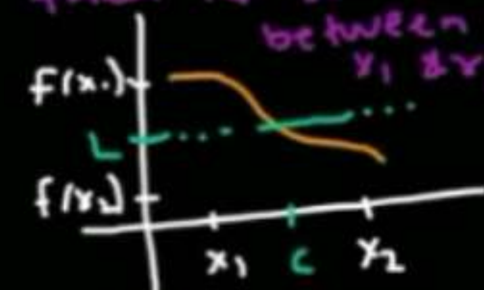
$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\frac{m(b-a)}{b-a} \le \frac{\int_a^b f(x)\,dx}{b-a} \le \frac{M(b-a)}{b-a}$$

$$m \le f_{ave} \le M$$

by IVT, $f_{ave} = f(c)$ for some c in $[a,b]$

IVT: For a cont fn $f(x)$ defined on $[x_1, x_2]$, if L between $f(x_1)$ & $f(x_2)$ then for some c between x_1 & x_2,



$f(c) = L$.

Theorem. *(Mean Value Theorem for Integrals) For a continuous function $f(x)$ on an interval $[a,b]$, there is a number c with $a \le c \le b$ such that* $f(c) = \dfrac{\int_a^b f(x)dx}{b-a}$.

Proof using the Mean Value Value Theorem for Functions:

MVT for functions

If $g(x)$ is cont. on $[a,b]$, and differentiable on (a,b), then there is some c in (a,b) such that

$$g'(c) = \frac{g(b)-g(a)}{b-a}$$

Let $g(x) = \int_a^x f(t)\,dt$.

Note $g(a) = \int_a^a f(t)\,dt = 0$ *

$g(b) = \int_a^b f(t)\,dt$ *

Note: $g'(x) = f(x)$ * by Fund. Thm of Calc.

by MVT for fns:

$$g'(c) = \frac{g(b)-g(a)}{b-a} \text{ for some } c \text{ in } (a,b)$$

$$f(c) = \frac{\int_a^b f(t)\,dt - 0}{b-a}$$

$\boxed{M}$