101 Basic

Exponent Rules

1. The Product Rule $x^n x^m = x^{n+m}$
2. The Quotient Rule $\frac{x^n}{x^m} = x^{n-m}$
3. The Power Rule $(x^n)^m = x^{n \cdot m}$
4. Power of Zero $x^0 = 1$ (as long as $x \neq 0$) 0^0 is undefined
5. Negative Exponents $x^{-n} = \frac{1}{x^n}$
6. Fractional Exponents $x^{1/n} = \sqrt[n]{x}$
7. Distribute an Exponent over a Product $(ab)^n = a^n \cdot b^n$
8. Distribute an Exponent over a Quotient $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example. Simplify the following expressions and write your answers without negative exponents:

(a) $\dfrac{3x^{-2}}{x^4}$ $= \dfrac{3 \cdot \frac{1}{x^2}}{x^4}$

$= \dfrac{\frac{3}{1} \cdot \frac{1}{x^2}}{x^4} = \dfrac{\frac{3 \cdot 1}{1 \cdot x^2}}{x^4} = \dfrac{\frac{3}{x^2}}{x^4}$

$= \dfrac{\frac{3}{x^2}}{\frac{x^4}{1}} = \dfrac{3}{x^2} \cdot \dfrac{1}{x^4} = \dfrac{3 \cdot 1}{x^2 \cdot x^4}$

$= \boxed{\dfrac{3}{x^6}}$

since $x^2 \cdot x^4 = x^{2+4} = x^6$

$\dfrac{3x^{-2}}{x^4} = 3 \cdot \dfrac{x^{-2}}{x^4} = 3 \cdot x^{-2-4}$

$= 3 \cdot x^{-6} = 3 \cdot \dfrac{1}{x^6} = \boxed{\dfrac{3}{x^6}}$

$\dfrac{3x^{-2}}{x^4} = \dfrac{3}{x^4 x^2} = \boxed{\dfrac{3}{x^6}}$

(b) $\dfrac{4y^3}{y^{-5}}$ $= \dfrac{4y^3}{\frac{1}{y^5}} = \dfrac{\frac{4y^3}{1}}{\frac{1}{y^5}}$

$= \dfrac{4y^3}{1} \cdot \dfrac{y^5}{1} = \dfrac{4y^3 y^5}{1}$

$= \dfrac{4y^8}{1} = \boxed{4y^8}$

$\dfrac{4y^3}{y^{-5}} = 4y^{3-(-5)} = \boxed{4y^8}$

* negative exponent in numerator $\longleftrightarrow$ pos exponent in denominator

negative exponent in denom $\longleftrightarrow$ pos exponent in numerator

you can "pass a factor across the fraction bar" by switching the sign of the exponent $\oplus \leftrightarrow \ominus$

$\dfrac{4y^3}{y^{-5}} = 4y^3 y^5 = \boxed{4y^8}$

Example. Simplify the expression and write your answer without negative exponents:

$$\left(\frac{25x^4y^{-5}}{x^{-6}y^3}\right)^{3/2} = \left(\frac{25x^4x^6}{y^3y^5}\right)^{3/2} = \left(\frac{25x^{10}}{y^8}\right)^{3/2}$$

$(ab)^n = a^n b^n$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$$= \frac{25^{3/2}\cdot(x^{10})^{3/2}}{(y^8)^{3/2}}$$

$$= \frac{25^{3/2}\cdot x^{10\cdot 3/2}}{y^{8\cdot 3/2}} = \frac{25^{3/2}\cdot x^{15}}{y^{12}}$$

$$= \boxed{\frac{125x^{15}}{y^{12}}}$$

$\frac{3}{2} = 3\cdot\frac{1}{2} = \frac{1}{2}\cdot 3$

$25^{3/2} = 25^{3\cdot 1/2} = 25^{1/2\cdot 3}$

$= (25^3)^{1/2} = (25^{1/2})^3$

$= (\sqrt{25})^3 = 5^3 = 125$

Rules for Radicals

- $\sqrt[n]{a \cdot b} = \sqrt[n]{a}\,\sqrt[n]{b}$

$*\quad (a\cdot b)^{1/n} = a^{1/n}\, b^{1/n}$

$$\sqrt{9\cdot 16} = \sqrt{9}\cdot\sqrt{16} = 12$$

- $\sqrt[n]{\frac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\left(\frac{a}{b}\right)^{1/n} = \dfrac{a^{1/n}}{b^{1/n}}$

$$\sqrt[3]{\frac{64}{8}} = \frac{\sqrt[3]{64}}{\sqrt[3]{8}} = 2$$

⚠ $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$

$\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$

Ex: $\sqrt{1+1} \neq \sqrt{1} + \sqrt{1}$

$\sqrt{2-1} \neq \sqrt{2} - \sqrt{1}$

- $\boxed{a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m}$

$(a^m)^{1/n}$ $\quad (a^{1/n})^m$

Example. Simplify. (Assume all variables represent positive numbers.)

pull as much as possible out of radical sign

$$\sqrt{60x^2y^6z^{-11}}$$

$$\sqrt{\frac{60x^2y^6}{z^{11}}}$$

$$\sqrt{\frac{2^2\cdot 3\cdot 5\ x^2y^6}{z^{11}}}$$

$$\sqrt{\frac{2^2\cdot 3\cdot 5\ x^2\ y^2\cdot y^2\cdot y^2}{z^2\cdot z^2\cdot z^2\cdot z^2\cdot z^2\cdot z}}$$

$$\frac{\sqrt{2^2}\ \sqrt{3\cdot 5}\ \sqrt{x^2}\ \sqrt{y^2}\ \sqrt{y^2}\ \sqrt{y^2}}{\sqrt{z^2}\ \sqrt{z^2}\ \sqrt{z^2}\ \sqrt{z^2}\ \sqrt{z^2}\ \sqrt{z}} = \frac{2\sqrt{3\cdot 5}\cdot x\cdot y\cdot y\cdot y}{z\cdot z\cdot z\cdot z\cdot z\cdot\sqrt{z}}$$

$$= \boxed{\frac{2\sqrt{15}\cdot x\cdot y^3}{z^5\sqrt{z}}}$$

Example. Rationalize the denominator. (Assume x represents a positive number.)

$\frac{3x}{\sqrt{x}}$

rewrite w/o $\sqrt{\ }$ in denom

$$\frac{3x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{3x\sqrt{x}}{(\sqrt{x})^2} = \frac{3\cancel{x}\sqrt{x}}{\cancel{x}} = \boxed{3\sqrt{x}}$$

1. **Pull out the greatest common factor** GCF : the largest thing that divides each term

Example. Factor $15 + 25x$

$GCF = 5$

$5(3 + 5x)$

One way to finde GCF (or ggt in germany) is to seach for the smallest power of it, here it is the x^2 and y

Example. Factor $x^2y + y^2x^3$

GCF : x^2y

$x^2y(1 + xy)$

$$\frac{y^2x^3}{x^2y} \rightsquigarrow \frac{y\,\cancel{y}\,x\,\cancel{x\,x}}{\cancel{x\,x}\cdot\cancel{y}}$$

check: $x^2y(1 + xy)$

$= x^2y + x^3y^2$ ✓

2. **Factor by grouping**

Example. Factor $\underbrace{x^3 + 3x^2} + \underbrace{4x + 12}$

look for
* 4 terms

$x^2(x+3)+4(x+3)$

$(x^2+4)(x+3)$

3. Factor quadratics

Example. Factor $x^2 - 6x + 8$

$(x \pm \quad)(x \pm \quad)$

$(x-2)(x-4)$

check: $x^2 - 4x - 2x + 8$

$x^2 - 6x + 8$ ✓

1	8
2	4
~~4~~	~~2~~
~~8~~	~~1~~
-1	-8
-2	-4

Example. Factor $10x^2 + 11x - 6$ use factoring by grouping

Find two numbers that multiply to -60 and add to 11.

-1	60
-2	30
-3	20
-4	15
-5	12
	⋮

$15 - 4 = 11$

$10x^2 - 4x + 15x - 6$

$2x(5x-2) + 3(5x-2)$

$(2x+3)(5x-2)$

4. **Difference of squares**

$a^2 - b^2 = (a+b)(a-b)$

check: $(a+b)(a-b) = a^2 - \cancel{ab} + \cancel{ba} - b^2$

$= a^2 - b^2$ ✓

Example. Factor $x^2 - 16$

$x^2 - 16 = x^2 - 4^2$

$(x+4)(x-4)$

Example. Factor $9p^2 - 1$

$(3p)^2 - 1^2 = (3p+1)(3p-1)$

⚠ sum of squares $x^2 + 4 = x^2 + 2^2$ Does Not factor

5. **Difference or sum of cubes**

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

Example. Factor $y^3 + 27$

$$y^3+3^3 = (y+3)(y^2-y\cdot 3+3^2)$$
$$= (y+3)(y^2-3y+9)$$

FACTORING

ADDITIONAL EXAMPLES

Question. Which of these expressions DOES NOT factor?

A. $x^2 + x$ pull out common factor

$x(x+1)$

B. $x^2 - 25$ $(x)^2 - (5)^2$ $a^2 - b^2 = (a+b)(a-b)$

$(x+5)(x-5)$ difference of squares

C. $x^2 + 4$ does not factor

D. $\underbrace{x^3 + 2x^2} + \underbrace{3x + 6}$ factor by grouping

$x^2(x+2) + 3(x+2) \rightarrow (x+2)(x^2+3)$

E. $5x^2 \underbrace{- 14x} + 8$ factor a quadratic using grouping $5 \cdot 8 = 40$

$\underbrace{5x^2 - 4x} \underbrace{-10x + 8}$

$x(5x-4) - 2(5x-4)$

$(5x-4)(x-2)$

check: $5x^2 - 10x - 4x + 8$

$5x^2 - 14x + 8$

40

-14

2 numbers that multiply to 40 & add to -14

40	
-1	-40
-2	-20
-4	-10

Question. Factoring by grouping is handy for factoring which of these expressions? How can you tell?

A. $x^2 + x$

B. $x^2 - 25$

C. $x^2 + 4$

D. $\underbrace{x^3 + 2x^2} + \underbrace{3x + 6}$ ← 4 terms

E. $5x^2 - 14x + 8$

after splitting the middle term

Question. What are some of the main techniques of factoring?

pull out common factors ← ⚠ pull out common factors first

diff of squares

factor by grouping

factor quadratics

factor sums and diffs of cubes

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

⚠ you may need to keep factoring, using several of these methods

Extra Example. Factor $2z^2 + 3z - 14$.

$$2z^2 - 4z + 7z - 14$$

$$2z(z-2) + 7(z-2)$$

$$(2z+7)(z-2)$$

-28

-4 7

3

-28	
-1	28
1	-28
-2	14
2	-14
-4	7
4	-7

Extra Example. Factor $-5v^2 - 45v + 50$ pull out common factor first

$$-5(v^2 + 9v - 10)$$

$$-5(v+10)(v-1)$$

Reduce to lowest terms

Example. Simplify $\frac{21}{45}$ by reducing to lowest terms.

$21 = 7 \cdot 3$

$45 = 5 \cdot 3 \cdot 3$

$$\frac{7 \cdot \cancel{3}}{5 \cdot 3 \cdot \cancel{3}} = \frac{7}{15}$$

Example. Simplify $\frac{3x + 6}{x^2 + 4x + 4}$ by reducing to lowest terms.

$$\frac{3\cancel{(x+2)}}{(x+2)\cancel{(x+2)}} = \frac{3}{x+2}$$

Multiplying and Dividing

Example. Compute

a) $\frac{4}{3} \cdot \frac{2}{5} = \frac{4 \cdot 2}{3 \cdot 5} = \frac{8}{15}$

b) $\dfrac{\frac{4}{5}}{\frac{2}{3}} = \frac{4}{5} \cdot \frac{3}{2} = \frac{12}{10} = \frac{6}{5}$

Example. Compute $\dfrac{\frac{x^2+x}{x+4}}{\frac{x+1}{x^2-16}} = \frac{x^2+x}{x+4} \cdot \frac{x^2-16}{x+1} = \frac{(x^2+x)(x^2-16)}{(x+4)(x+1)}$

$= \frac{x\cancel{(x+1)}\cancel{(x+4)}(x-4)}{\cancel{(x+4)}\cancel{(x+1)}} = \boxed{x(x-4)}$

Adding and Subracting

Example. Subtract $\frac{7}{6} - \frac{4}{15}$ $= \frac{7}{6}\cdot\frac{5}{5} - \frac{4}{15}\cdot\frac{2}{2}$

CD: $6\cdot 15 = 90$

LCD: $6 = 2\cdot 3$, $15 = 3\cdot 5$ $= \frac{35}{30} - \frac{8}{30}$

$2\cdot 3\cdot 5 = 30$ $= \frac{27}{30} = \frac{3^3}{2\cdot 3\cdot 5} = \frac{3^2}{2\cdot 5} = \frac{9}{10}$

LCD: smallest CD

Example. Add $\frac{3}{2x+2} + \frac{5}{x^2-1}$

LCD: $2x+2 = 2(x+1)$

$x^2-1 = (x+1)(x-1)$

$2(x+1)(x-1)$

$$\frac{3}{2(x+1)}\cdot\frac{(x-1)}{(x-1)} + \frac{5}{(x-1)(x+1)}\cdot\frac{2}{2} = \frac{3(x-1)}{2(x+1)(x-1)} + \frac{5\cdot 2}{2(x+1)(x-1)}$$

$$\frac{3(x-1)+10}{2(x+1)(x-1)} = \frac{3x-3+10}{2(x+1)(x-1)} = \boxed{\frac{3x+7}{2(x+1)(x-1)}}$$

Example. $w^2 = 121$

write in standard form

$$w^2 - 121 = 0$$

$a = 1 \quad b = 0 \quad c = -121$

$$aw^2 + bw + c = 0$$

factor

$$(w+11)(w-11) = 0$$

set factors $= 0$

$w + 11 = 0$ or $w - 11 = 0$

$w = -11$ or $w = 11$

$$w^2 = 121$$

$$w = \pm\sqrt{121}$$

$$w = \pm 11$$

Example. Find all real solutions for the equation $y^2 = 18 - 7y$

① write in standard form

$$y^2 - 18 + 7y = 0$$

$$y^2 + 7y - 18 = 0$$

② factor

$$(y+9)(y-2) = 0$$

③ set factors equal to 0 & solve

$$y + 9 = 0 \quad \text{or} \quad y - 2 = 0$$

$$\Rightarrow \quad y = -9 \quad \text{or} \quad y = 2$$

check:

$$(-9)^2 \stackrel{?}{=} 18 - 7(-9) \quad \checkmark$$

$$2^2 \stackrel{\checkmark}{=} 18 - 7(2) \quad \checkmark$$

Example. Find all real solutions for the equation $x(x+2) = 7$

write eqn in standard form

$$x^2 + 2x = 7$$

$$x^2 + 2x - 7 = 0$$

-1	7
7	-1

use quadratic eqn

$a = 1 \quad b = 2 \quad c = -7$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{\underbrace{\overbrace{2^2}^{4} - \overbrace{4 \cdot 1 \cdot (-7)}^{-28}}_{32}}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{\cancel{32}\,16 \cdot 2}}{2} = \frac{-2 \pm \sqrt{16}\sqrt{2}}{2} = \frac{-2 \pm 4\sqrt{2}}{2}$$

$$\frac{-2}{2} \pm \frac{4\sqrt{2}}{2} = -1 \pm 2\sqrt{2}$$

answers: $\boxed{\begin{array}{l} -1 + 2\sqrt{2} \\ -1 - 2\sqrt{2} \end{array}}$

Example. Find all real solutions for the equation $\frac{1}{2}y^2 = \frac{1}{3}y - 2$

standard form

$$\frac{1}{2}y^2 - \frac{1}{3}y + 2 = 0$$

$$6\left(\frac{1}{2}y^2 - \frac{1}{3}y + 2\right) = 6 \cdot 0$$

$$3y^2 - 2y + 12 = 0$$

quadratic formula

$$x = \frac{2 \pm \sqrt{\underbrace{(-2)^2}_{4} - \underbrace{4(3)(12)}_{144}}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{-140}}{6}$$

no real solutions

Example. Solve: $\dfrac{x}{x+3} = \dfrac{1}{1} + \dfrac{1}{x}$

① Find LCD

$(x+3)\cdot x$

② Clear the denominators

$$((x+3)\,x)\,\frac{x}{x+3} = ((x+3)(x))\left(1 + \frac{1}{x}\right)$$

$$\cancel{(x+3)}(x)\,\frac{x}{\cancel{x+3}} = (x+3)(x)\cdot 1 + (x+3)\cancel{(x)}\cdot\frac{1}{\cancel{x}}$$

$$x^2 = (x+3)(x)\cdot 1 + (x+3)$$

③ Simplify & solve

$$\cancel{x^2} = \cancel{x^2} + 3x + x + 3$$

$$0 = 4x + 3$$

$$4x = -3$$

$$\boxed{x = -\frac{3}{4}}$$

④ plug in to check answer

$$\frac{-\frac{3}{4}}{-\frac{3}{4}+3} \overset{?}{=} 1 + \frac{1}{-\frac{3}{4}}$$

$$\frac{-\frac{3}{4}}{\frac{9}{4}} \overset{?}{=} 1 - \frac{4}{3}$$

$$-\frac{3}{9} \overset{?}{=} -\frac{1}{3} \quad ✓$$

Example. Solve: $\dfrac{4c}{c-5} - \dfrac{1}{c+1} = \dfrac{3c^2+3}{c^2-4c-5}$

① Find LCD

$(c-5)$, $(c+1)$, $(c-5)(c+1)$

LCD: $(c-5)(c+1)$

② clear denom

$\cancel{(c-5)}(c+1)\cdot\dfrac{4c}{\cancel{c-5}} - (c-5)\cancel{(c+1)}\cdot\dfrac{1}{\cancel{c+1}} = \cancel{(c-5)}\cancel{(c+1)}\dfrac{(3c^2+3)}{\cancel{(c-5)}\cancel{(c+1)}}$

③ Simplify

$4c^2+4c - (c-5) = 3c^2+3$

$4c^2+4c-c+5 = 3c^2+3$

$c^2+3c+5=3 \Rightarrow c^2+3c+2=0$

$(c+1)(c+2)=0$

$c+1=0$ or $c+2=0$

$c=-1$ or $c=-2$

④ check answers: $\cancel{c=-1}$ is extraneous solution $c=-2$

Example. Find all real solutions for the equation $x + \sqrt{x} = 12$

① Isolate the $\sqrt{\ }$ term

$$x + \sqrt{x} = 12$$
$$-x \qquad -x$$
$$\sqrt{x} = 12 - x$$

② Get rid of $\sqrt{\ }$ by squaring both sides

$$(\sqrt{x})^2 = (12-x)^2$$
$$x = (12-x)(12-x)$$
$$x = 144 - 12x - 12x + x^2$$
$$x = 144 - 24x + x^2$$
$$0 = 144 - 25x + x^2$$
$$x^2 - 25x + 144 = 0$$

③ solve

144

-25

$$(x-9)(x-16) = 0$$
$$x - 9 = 0 \text{ or } x - 16 = 0$$
$$\boxed{x = 9} \text{ or } \cancel{x = 16}$$

-1	-144
-2	-72
-3	-48
-4	-36
-6	-24
-8	-18
-9	-16
-12	-12

④ check solution & eliminate extraneous solutions

$x = 9$

$$9 + \sqrt{9} \stackrel{?}{=} 12$$
$$9 + 3 \stackrel{?}{=} 12 \checkmark$$

$x = 16$

$$16 + \sqrt{16} \stackrel{?}{=} 12$$
$$16 + 4 \stackrel{?}{=} 12 \quad \times$$

Example. Find all real solutions for the equation $2p^{4/5} = \frac{1}{8}$

① Isolate the part of eqn that involves the fractional exponent

$$\frac{1}{2}\cdot 2p^{4/5} = \frac{1}{8}\cdot\frac{1}{2}$$
$$p^{4/5} = \frac{1}{16}$$

② Get rid of the fractional exponent

$$(p^{4/5})^5 = \left(\frac{1}{16}\right)^5$$
$$p^{4/5\cdot 5} = \left(\frac{1}{16}\right)^5$$
$$p^4 = \left(\frac{1}{16}\right)^5$$
$$(p^4)^{1/4} = \pm\left(\left(\frac{1}{16}\right)^5\right)^{1/4}$$
$$p^{4\cdot\frac{1}{4}}$$
$$p^1 = \pm\left(\left(\frac{1}{16}\right)^5\right)^{1/4}$$

$$(p^{4/5})^{5/4} = \pm\left(\frac{1}{16}\right)^{5/4}$$
$$p^{4/5\cdot 5/4} = \pm\left(\frac{1}{16}\right)^{5/4}$$
$$p^1 = \pm\left(\frac{1}{16}\right)^{5/4}$$

$x^2 = 4$ | $x^3 = -8$
$x = \pm\sqrt{4}$ | $x = \sqrt[3]{-8}$
$x = \pm 2$ | $x = -2$

$(-2)^2 = 4$
$(2)^2 = 4$

$\left.\begin{array}{l}\sqrt[\text{even}]{\ } \\ (\ )^{1/\text{even}}\end{array}\right\}$ include $\pm$

$$p = \pm\left(\frac{1}{16}\right)^{5/4}$$

③ solve using exponent rules

$$p = \pm\left(\sqrt[4]{\frac{1}{16}}\right)^5 = \pm\left(\frac{\sqrt[4]{1}}{\sqrt[4]{16}}\right)^5 = \pm\left(\frac{1}{2}\right)^5 = \pm\frac{1^5}{2^5} = \pm\frac{1}{32}$$

④ check answers

$$\boxed{p = \frac{1}{32} \qquad p = -\frac{1}{32}}$$

$$2\left(\frac{1}{32}\right)^{4/5} = 2\cdot\frac{1^{4/5}}{32^{4/5}} = 2\cdot\frac{1}{(\sqrt[5]{32})^4} = 2\cdot\frac{1}{16} = \frac{1}{8}$$

Example. Solve the equation $5|4p - 3| + 16 = 1$

① isolate abs value part

$$5|4p-3|+16=1$$

$$5|4p-3|=-15$$

$$|4p-3|=-3 \times$$

② Think about distance on number line

$\boxed{\text{no solutions}}$

102 Inequality

Example. Translate between inequality and interval notation.

	Inequality Notation	Interval Notation
A.	$-3 \le x < 1$	$[-3, 1)$
B.	$x < 5$	$(-\infty, 5)$
C.	$-15 < x$	$(-15, \infty)$
D.	$4 \ge x > 0$	$(0, 4]$

$x < 5$

$0 < x \le 4$

$< \le$

$\cancel{[4, 0)}$

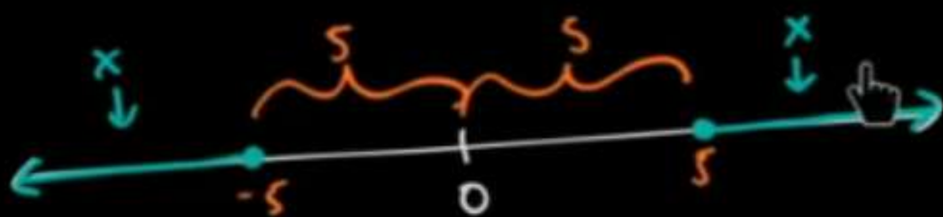
Note. In interval notation, the smaller number is **always** on the left side.

Example. What x-values satisfy $|x| < 5$?

$-5 < x < 5$

$(-5, 5)$

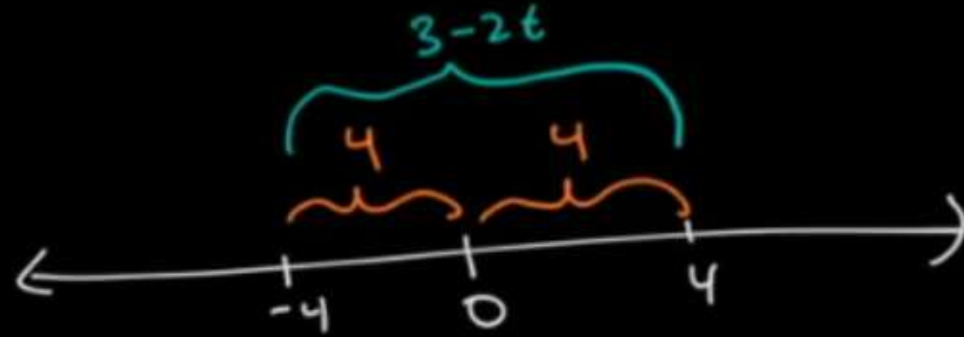
Example. What x-values satisfy $|x| \geq 5$?



$x \leq -5$ OR $x \geq 5$

$(-\infty, -5] \cup [5, \infty)$

Example. Solve $|3 - 2t| < 4$



$$-4 < 3-2t < 4$$

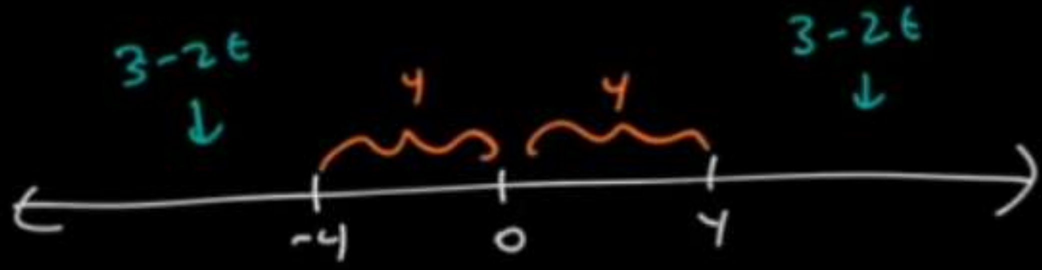
$$-7 < -2t < 1$$

$$\frac{-7}{-2} > t > \frac{1}{-2}$$

$$\frac{7}{2} > t > -\frac{1}{2}$$

$$\left(-\frac{1}{2}, \frac{7}{2}\right)$$

Example. Solve $|3 - 2t| > 4$



$3 - 2t < -4$ OR $3 - 2t > 4$

$-2t < -7$ OR $-2t > 1$

$t > \frac{7}{2}$ OR $t < -\frac{1}{2}$

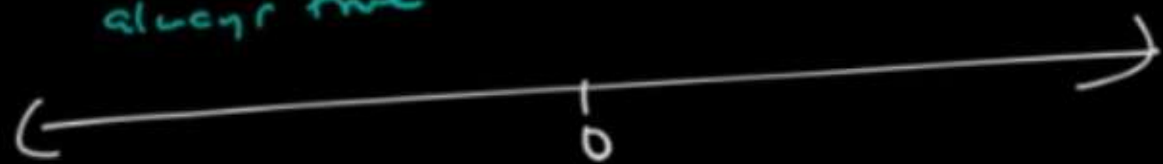
$(-\infty, -\frac{1}{2}) \cup (\frac{7}{2}, \infty)$

Example. Solve $2|4x+5|+7 \geq 1$

$$2|4x+5| \geq -6$$

$$|4x+5| \geq -3$$

always true



$(-\infty, \infty)$

all real numbers

Example. Solve: $-5(x+2)+3>8$

$$-5x-10+3>8$$

$$-5x-7>8$$

$$-5x>15$$

$$x<\frac{15}{-5}$$

$$\boxed{x<-3}$$

$(-\infty,-3)$

If you multiply or divide an inequality by a negative number, then you need to reverse the inequality.

$$-x<-5$$

$$x>5$$

Example. Solve: $3x - 4 \leq x - 8$ OR $6x + 1 > 10$

$3x - 4 \leq x - 8$

$3x \leq x - 4$

$2x \leq -4$

$x \leq -2$

OR

$6x + 1 > 10$

$6x > 9$

$x > \frac{9}{6}$

$x > \frac{3}{2}$

$(-\infty, -2] \cup (\frac{3}{2}, \infty)$

Solve: $-\frac{2}{3}y > -12$ AND $-4y + 2 > 5$

$y < -12 \cdot -\frac{3}{2}$

$y < 18$

$-4y > 3$

$y < \frac{3}{-4}$

AND $y < -\frac{3}{4}$

$\left(-\infty, -\frac{3}{4}\right)$

Example. Solve: $-3 \leq 6x - 2 < 10$

$-3 \leq 6x - 2$ AND $6x - 2 < 10$

$\vdots$

$-\frac{1}{6} \leq x$ AND $x < 2$

$-\frac{1}{6} \leq x < 2$

$$-3 \leq 6x - 2 < 10$$
$$-1 \leq 6x < 12$$
$$-\frac{1}{6} \leq x < 2$$

$[-\frac{1}{6}, 2)$

Example. Solve $x^2 < 4$

$\cancel{x < 2}$

$x^2 = 4$

$x = 2$

$x = -2$

Ex: $x = -10$ satisfies $x < 2$
but doesn't satisfy $x^2 < 4$ since $(-10)^2 = 100 \not< 4$

① Move all terms to one side, with 0 on other side

$x^2 < 4$

$x^2 - 4 < 0$

② Solve the equation

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x - 2 = 0 \qquad x + 2 = 0$

$x = 2 \qquad x = -2$

③ Plot solutions to equation.

$x = -3$: $(-3)^2 - 4 = 9 - 4 = 5$

$x = 0$: $0^2 - 4 = -4$

$x = 10$: $10^2 - 4$

$\boxed{-2 < x < 2}$ $\quad (-2, 2)$

Example. Solve $x^3 \geq 5x^2 + 6x$

① Move all terms to one side

$$x^3 - 5x^2 - 6x \geq 0$$

② Solve equation by factoring

$$x^3 - 5x^2 - 6x = 0$$

$$x(x^2 - 5x - 6) = 0$$

$$x(x-6)(x+1) = 0$$

$x = 0 \quad x = 6 \quad x = -1$

③ Write solution to eqn on number line

$x(x-6)(x+1)$

−	−	−	$x=-2$
−	−	+	$x=-\frac{1}{2}$
+	−	+	$x=1$
+	+	+	$x=100$

$[-1, 0] \cup [6, \infty)$

Example. Solve $\dfrac{x^2+6x+9}{x-1} \leq 0$

① Move all terms to one side ✓

② Solve equation

$$\frac{x^2+6x+9}{x-1} = 0$$

$$x^2+6x+9=0$$
$$(x+3)^2=0$$
$$x=-3$$

③ Find where rational expression DNE

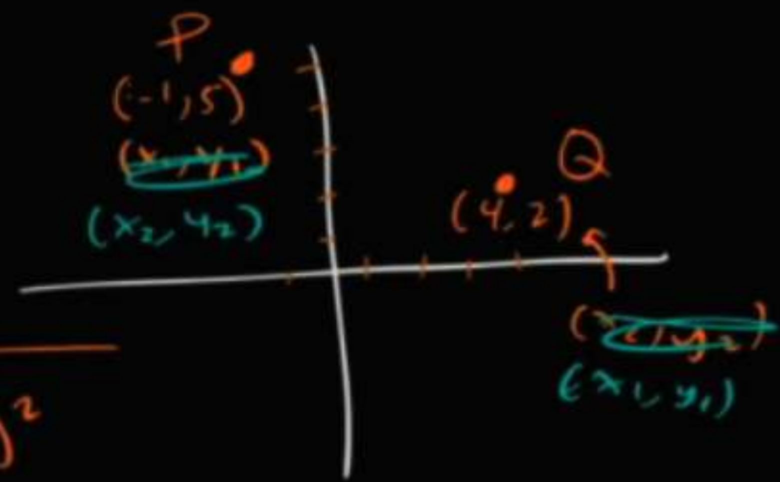
$x-1=0$ $\quad$ $x=1$

④ Put all these values on number line

$\boxed{x<1}$ $\quad$ $(-\infty, 1)$

103 Functions & Polynomials

Example. Find the distance between the points $P(-1,5)$ and $Q(4,2)$.



$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$d = \sqrt{(4-(-1))^2+(2-5)^2}$$

$$d = \sqrt{5^2+(-3)^2}$$

$$= \sqrt{25+9}$$

$$= \sqrt{34}$$

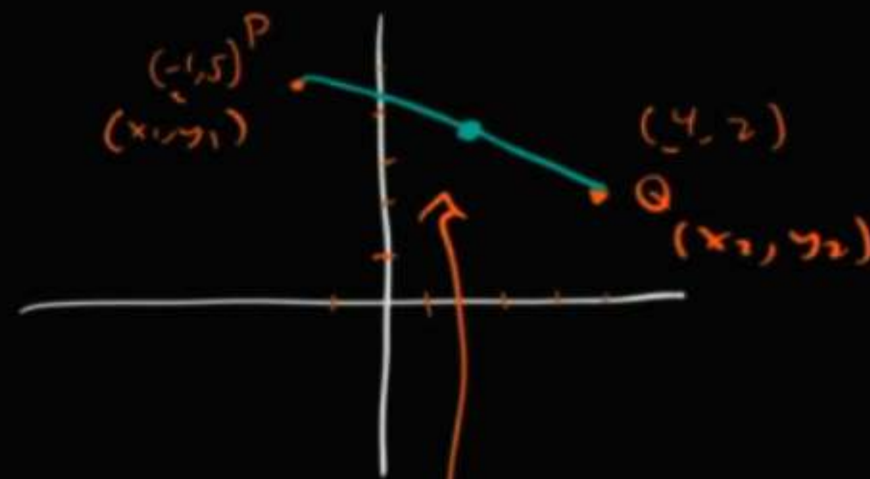
$$d = \sqrt{(-1-4)^2+(5-2)^2}$$

$$= \sqrt{(-5)^2+3^2}$$

$$= \sqrt{25+9}$$

$$= \sqrt{34}$$

Example. Find the midpoint of the segment between the points $P(-1,5)$ and $Q(4,2)$.



$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\left(\frac{-1+4}{2}, \frac{5+2}{2}\right)$$

$$\left(\frac{3}{2}, \frac{7}{2}\right)$$

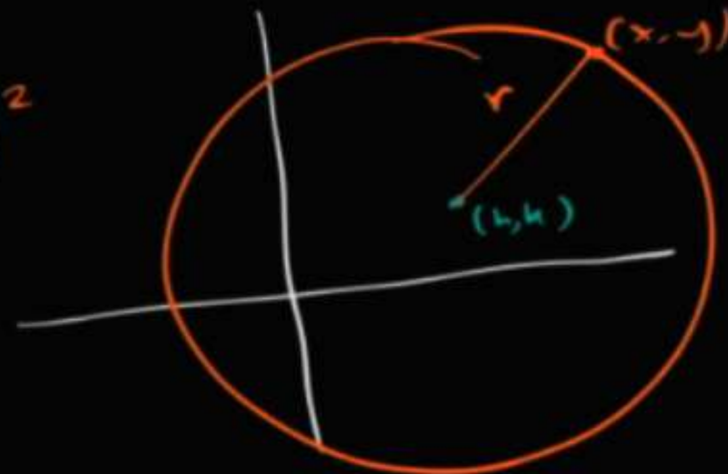
Note. The equation of a circle with radius r centered at the point (h,k) is given by:

$$\text{dist} = r = \sqrt{(x-h)^2+(y-k)^2}$$

$$r^2 = \left(\sqrt{(x-h)^2+(y-k)^2}\right)^2$$

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

radius r, center (h,k)



<u>Ex</u> Eqn for circle of radius 6, center $(0,-3)$

$r=6$ $\quad (h,k)$

$$(x-0)^2+(y-(-3))^2=6^2$$

$$\boxed{x^2+(y+3)^2=36}$$

Example. Does this equation represent a circle? If so, what is the center and what is the radius?

$$(x-5)^2 + (y+6)^2 = 5$$

$(x-h)^2 + (y-k)^2 = r^2$

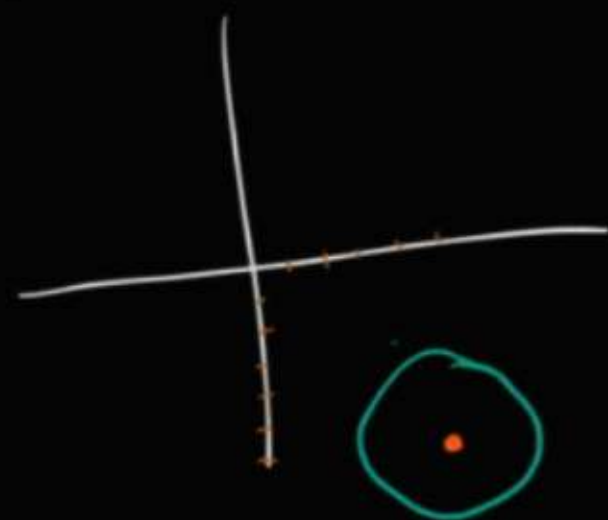
let $h = 5$ $y = -6$ $5 = r^2$

$\Rightarrow r = \sqrt{5}$

$(5, -6)$

center

radius



Example. Does this equation represent a circle? If so, what is the center and what is the radius?

$$9x^2 + 9y^2 + 72x - 18y + 36 = 0$$

$(x-h)^2 + (y-k)^2 = r^2$

$(x-h)^2 = (x-h)(x-h)$

$= x^2 - hx - hx + h^2$

$= \underbrace{x^2 - 2hx}{} + h^2$

$x^2 + y^2 + 8x - 2y + 4 = 0$

$x^2 + 8x \quad + y^2 - 2y = -4$

$\left(\frac{8}{2}\right)^2 = 4^2 = 16$

$x^2 + 8x + 16 + y^2 - 2y = -4 + 16$

$\left(\frac{-2}{2}\right)^2 = 1$

$\underbrace{x^2 + 8x + 16}{} + \underbrace{y^2 - 2y + 1}{} = -4 + 16 + 1$

$\boxed{(x+4)^2 + (y-1)^2 = 13}$

$\boxed{\text{center: } (-4, -1) \quad \text{radius: } \sqrt{13}}$

Completing the square

$Ax^2 + Ay^2 + Bx + Cy + D = 0$

$(x+4)(x+4)$

$= x^2 + 4x + 4x + 16$

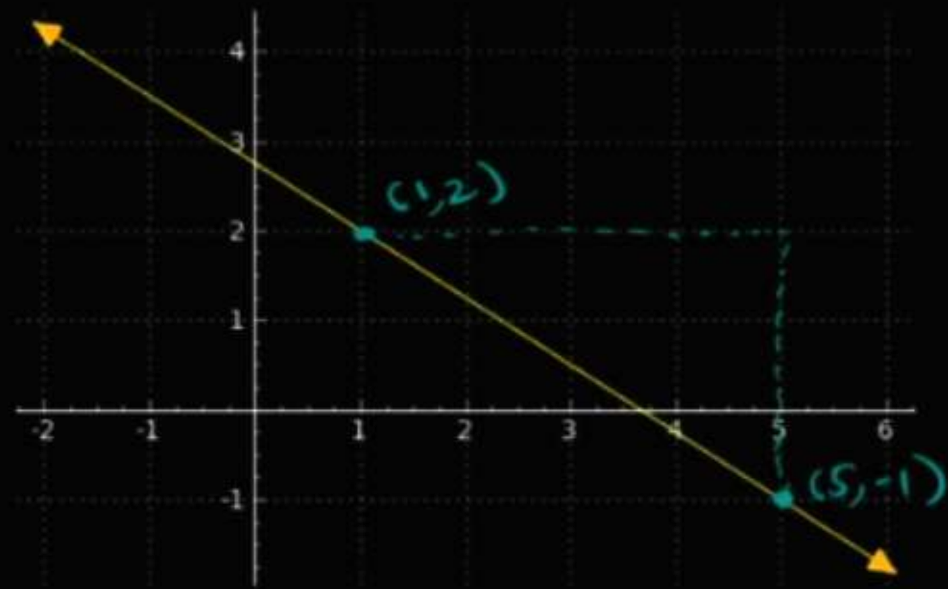
$= x^2 + 8x + 16$

$(y-1)(y-1)$

$y^2 - y - y + 1$

$y^2 - 2y + 1$

Example. Find the equation of this line.



$y = mx + b$

slope

y-intercept

$\text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

where (x_1, y_1) & (x_2, y_2) represent points on the line

$\text{slope} = \boxed{\dfrac{-3}{4}}$ OR $\dfrac{-1-2}{5-1} = \dfrac{-3}{4}$ $\quad m = -\dfrac{3}{4}$

$y = mx + b \Rightarrow y = -\frac{3}{4}x + b$

plug in pt: $(1,2)$ for x & y

$2 = -\frac{3}{4} \cdot 1 + b \Rightarrow 2 = -\frac{3}{4} + b \Rightarrow 2 + \frac{3}{4} = b$

$b = \frac{8}{4} + \frac{3}{4} = \frac{11}{4}$ $\quad \boxed{y = -\frac{3}{4}x + \frac{11}{4}}$

Example. Find the equation of the line through the points $(1, 2)$ and $(4, -3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{4-1} = \frac{-5}{3}$$

method 1)

$y = mx + b$ slope-intercept form

$y = -\frac{5}{3}x + b$ use one pt $(1, 2)$

$2 = -\frac{5}{3} \cdot 1 + b \Rightarrow b = 2 + \frac{5}{3} = \frac{6}{3} + \frac{5}{3} = \frac{11}{3}$

$y = -\frac{5}{3}x + \frac{11}{3}$

method 2

point slope form

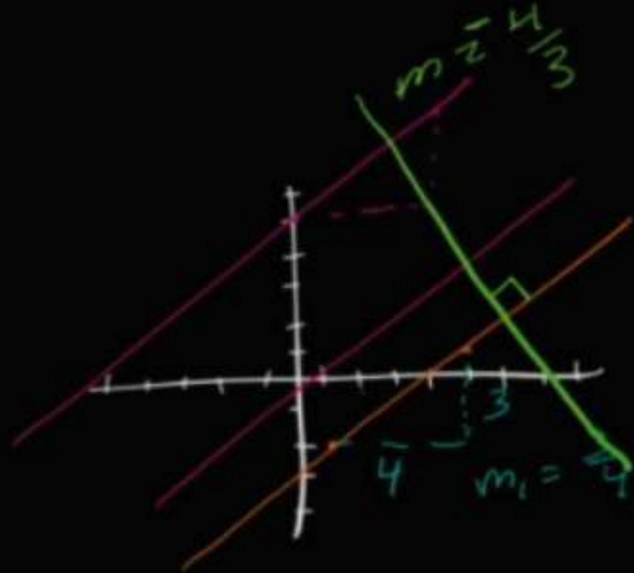
$y - y_0 = m(x - x_0)$ where (x_0, y_0) is a pt on line, m is slope

$m = \frac{-3-2}{4-1} = \frac{-5}{3}$ plug in pt $(1, 2)$

$y - 2 = -\frac{5}{3}(x - 1)$

$y - 2 = -\frac{5}{3}x + \frac{5}{3}$

$y = -\frac{5}{3} + 2 + \frac{5}{3} \Rightarrow \boxed{y = -\frac{5}{3}x + \frac{11}{3}}$



Parallel and Perpendicular Lines

parallel lines have the same slope

perpendicular lines have opposite reciprocal slopes

m_1	m_2 opposite reciprocal
2	$-\frac{1}{2}$
$-\frac{1}{3}$	3
$\frac{7}{2}$	$-\frac{2}{7}$

Example. Find the equation of a line that is parallel to the line $3y - 4x + 6 = 0$ and goes through the point $(-3, 2)$.

$3y - 4x + 6 = 0$

$3y = 4x - 6$

$y = \frac{4}{3}x - \frac{6}{3}$

$y = \frac{4}{3}x - 2$

$m_1 = \frac{4}{3}$

$m_2 = \frac{4}{3}$

new line

$y = \frac{4}{3}x + b$

$2 = \frac{4}{3}(-3) + b \Rightarrow 2 = -\frac{12}{3} + b \Rightarrow 2 = -4 + b$

$\Rightarrow b = 6$

$\boxed{y = \frac{4}{3}x + 6}$

Example. Find the equation of a line that is perpendicular to the line $6x + 3y = 4$ and goes through the point $(4, 1)$.

$$6x + 3y = 4$$
$$3y = -6x + 4$$
$$y = -\frac{6}{3}x + \frac{4}{3}$$
$$y = -2x + \frac{4}{3}$$

$m_1 = -2$

$m_2 = \frac{1}{2}$

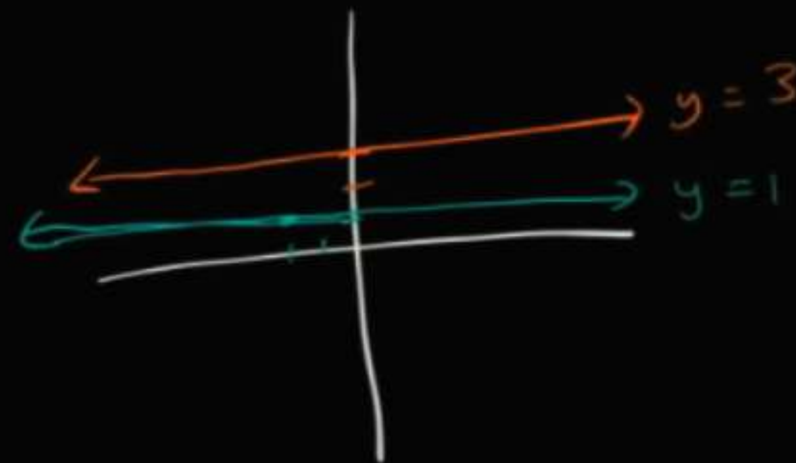
plug in $(4,1)$

new line

$$y = \frac{1}{2}x + b$$
$$1 = \frac{1}{2}\cdot 4 + b \Rightarrow 1 = 2 + b \Rightarrow b = -1$$
$$\boxed{y = \frac{1}{2}x - 1}$$

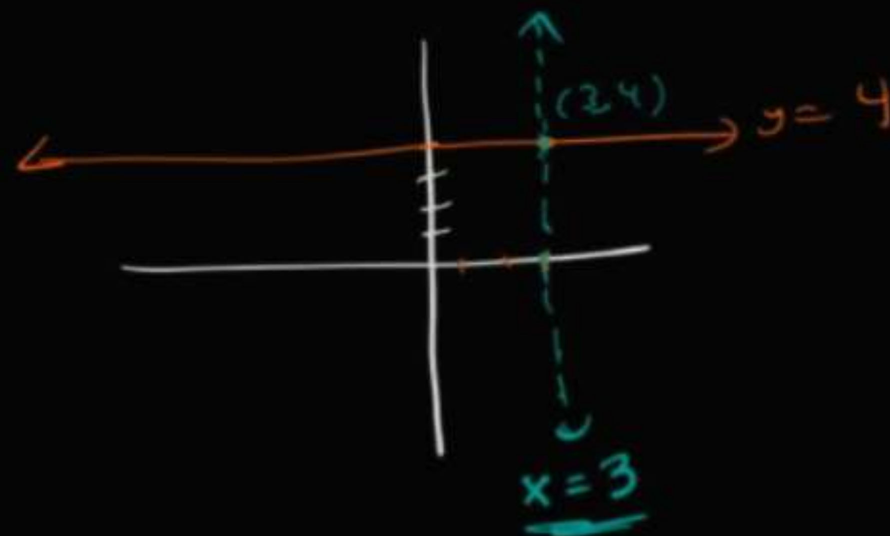
Example. Find the equation of the line that is parallel to $y = 3$ and goes through the point $(-2, 1)$.

$\boxed{y = 1}$

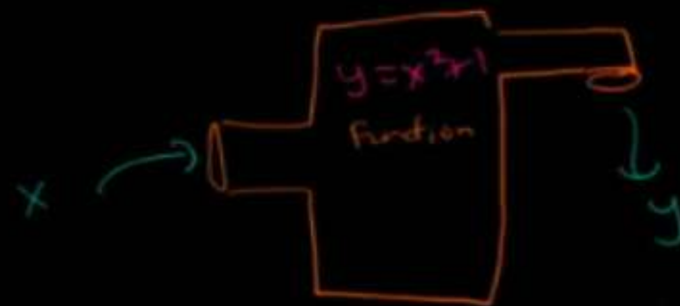


Example. Find the equation of the line that is perpendicular to $y = 4$ and goes through the point $(3, 4)$.

$\boxed{x = 3}$



Definition. A function is correspondence between input numbers (x-values) and output numbers (y-value) that sends each input number (x-value) to exactly one output number (y-value).



Sometimes, a function is described with an equation.

Example. $y = x^2 + 1$, which can also be written as $\underbrace{f(x)}_{y} = x^2 + 1$

f(x) is function notation NOT multiplication

What is $f(2)$? $f(5)$?

$f(2) = 2^2 + 1 = 5$ $f(5) = 5^2 + 1 = 26$

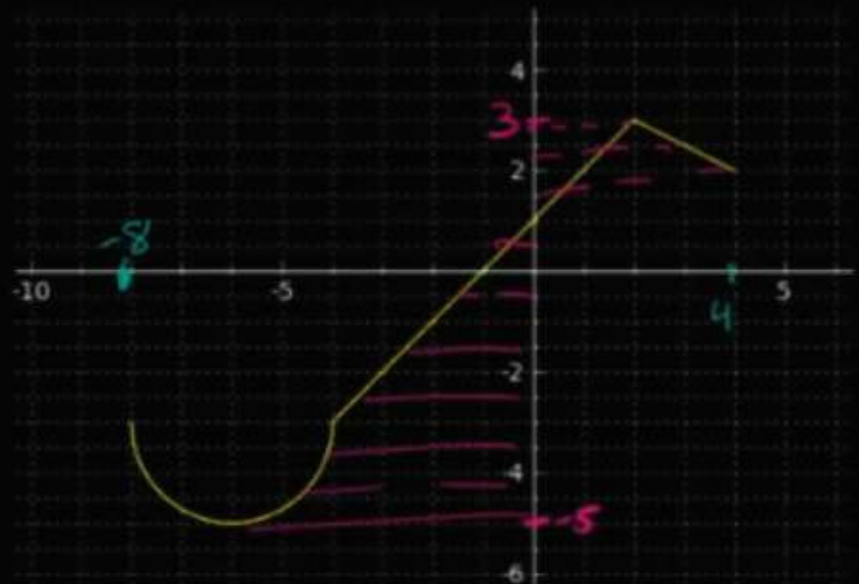
What is $f(a+3)$?

$f(a+3) = (a+3)^2 + 1 = a^2 + 6a + 9 + 1 = a^2 + 6a + 10$

$f(a+3) \neq a + 3^2 + 1$

Definition. The *domain* of a function is all possible x-values. The *range* is the y-values.

Example. What is the domain and range of the function $g(x)$ graphed below?



domain for $g(x)$

$-8 \leq x \leq 4$

$[-8, 4]$

range for $g(x)$

$-5 \leq y \leq 3$

$[-5, 3]$

Example. What are the domains of these functions?

A. $g(x) = \dfrac{x}{x^2 - 4x + 3}$

Need $x^2 - 4x + 3 \neq 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3$ or $x = 1$

exclude these values

domain:

To find domain:

1) Exclude x-values that make denominator 0

2) Exclude x-values that make an expression inside a $\sqrt{}$ negative

Domain: $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$

B. $f(x) = \sqrt{3 - 2x}$

exclude $3 - 2x < 0$

include $3 - 2x \geq 0$

$3 \geq 2x$

$x \leq \frac{3}{2}$

domain:

$(-\infty, \frac{3}{2}]$

C. $h(x) = \dfrac{\sqrt{3-2x}}{x^2-4x+3}$

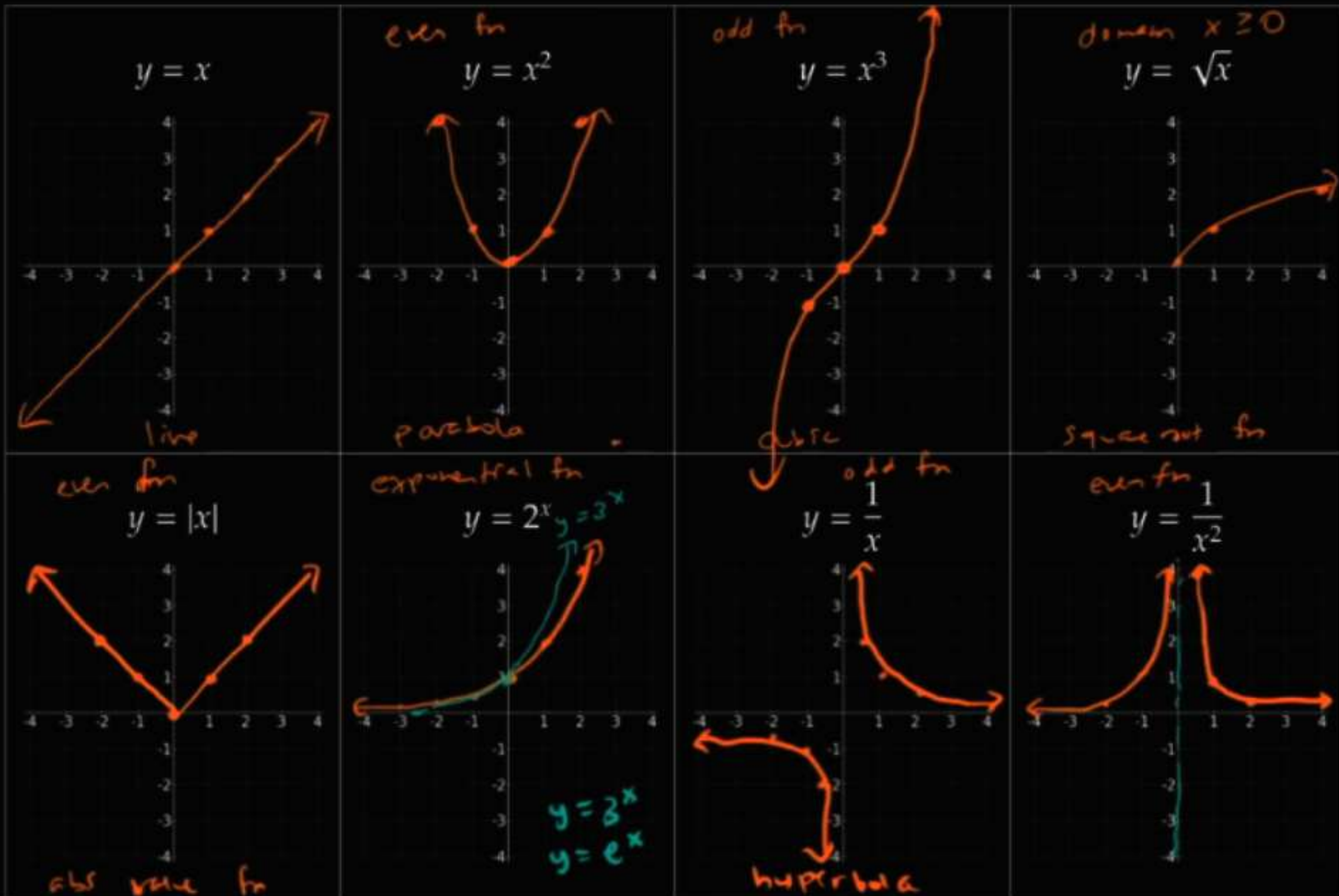
Need $x^2-4x+3 \neq 0 \iff x \neq 3$ and $x \neq 1$

AND

AND

Need $3-2x \geq 0 \iff x \leq \frac{3}{2}$

$(-\infty, 1) \cup (1, \frac{3}{2}]$



x	y
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
-1	$2^{-1} = \frac{1}{2}$

x	y
3	$2^3 = 8$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$\frac{1}{(-1)^2} = 1$ $\quad \frac{1}{(\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4$

$\frac{1}{(-2)^2} = \frac{1}{4}$

Review of Function Notation

Example. Rewrite the following, if $g(x) = \sqrt{x}$.

a) $g(x) - 2 =$ $\sqrt{x} - 2$ ← -2 is on outside of fn

b) $g(x-2) =$ $\sqrt{x-2}$ ← -2 is on inside of fn

c) $g(3x) =$ $\sqrt{3x}$ ← multiply by 3 inside of fn

d) $3g(x) =$ $3\sqrt{x}$ ← multiply by 3 on outside of fn

e) $g(-x) =$ $\sqrt{-x}$ ← negative sign on inside of fn

Example. Rewrite the following in terms of $g(x)$, if $g(x) = \sqrt{x}$.

f) $\sqrt{x} + 17 =$ $g(x) + 17$ ← add 17 on outside of fn

g) $\sqrt{x + 12} =$ $g(x+12)$ ← add 12 on inside of fn

h) $-36 \cdot \sqrt{x} =$ $-36g(x)$ ← multiply by -36 on outside of fn

i) $\sqrt{\frac{1}{4}x} =$ $g\left(\frac{1}{4}x\right)$ ← multiply by $\frac{1}{4}$ on the inside of fn

Example. Graph

- $y = \sqrt{x}$
- $y = \sqrt{x} - 2$ ← −2 on outside of fn
- $y = \sqrt{x - 2}$ ← −2 on inside of fn

x	$\sqrt{x}$	$\sqrt{x} - 2$
0	0	−2
1	1	−1
4	2	0

x	$x-2$	$y = \sqrt{x-2}$
2	0	0
3	1	1
6	4	2

Rules for transformations:

$y = \sqrt{x} - 2$

- Numbers on the *outside* of the function affect the y-values and result in vertical motions. These motions are in the direction you expect.

$y = \sqrt{x-2}$

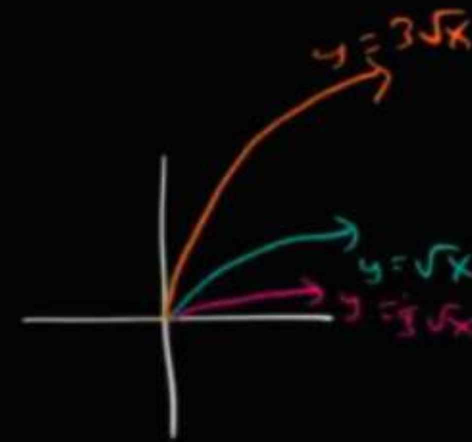
- Numbers on the *inside* of the function affect the x-values and result in horizontal motions. **These motions go in the opposite direction from what you expect.**

- Adding results in a shift (translations)

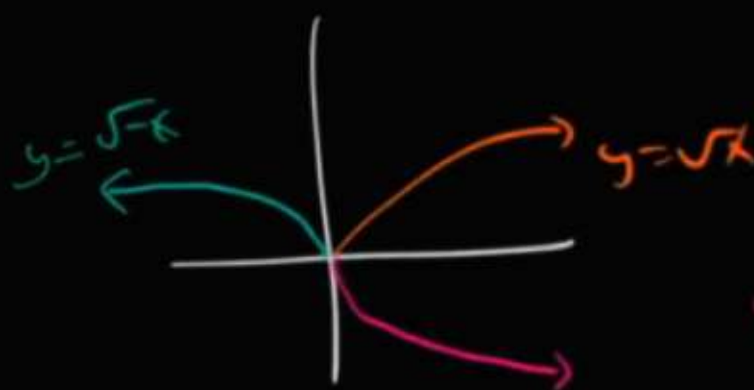
- Multiplying results in a stretch or shrink $y = 3\sqrt{x}$

  $y = \frac{1}{3}\sqrt{x}$

- A negative sign results in a reflection



$y = \sqrt{-x}$



$y = -\sqrt{x}$

Example. Consider $g(x) = \sqrt{x}$. How do the graphs of the following functions compare to the graph of $y = \sqrt{x}$?

a) $y = \sqrt{x} - 4$ move it down by 4 units

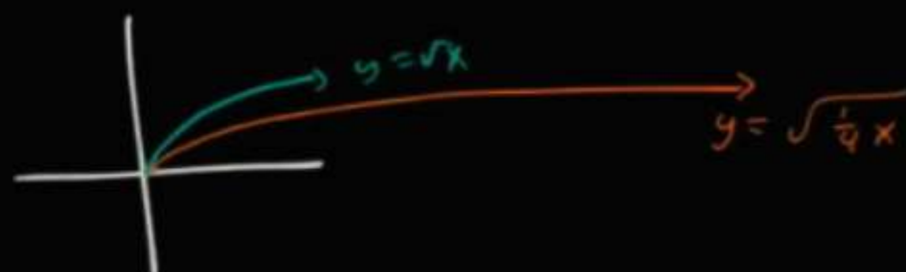
b) $y = \sqrt{x + 12}$ move graph left by 12 units

c) $y = -3 \cdot \sqrt{x}$

stretch vertically by 3 & reflect vertically (across x-axis)

d) $y = \sqrt{\frac{1}{4}x}$

stretch horizontally by factor of 4



$$y = \sqrt{\tfrac{1}{4}x}$$
$$= \sqrt{\tfrac{1}{4}}\sqrt{x}$$
$$y = \tfrac{1}{2}\sqrt{x}$$

$y = \sqrt{x}$

$y = x^2$

$y = |x|$

Transformations of Functions

- numbers on outside ⟷ y-values ⟷ vertical motions
- number on inside of fn ⟷ x-values ⟷ horizontal motions

- adding/subtraction ⟷ translations (shifts)
- multiplying/dividing ⟷ stretches + shrinks
- negative sign ⟷ reflection

$y = 3\sqrt{x} + 2$

Example. Which of these equations represent quadratic functions?

- $g(x) = -5x^2 + 10x + 3$

  yes

  $g(x) = ax^2 + bx + c$

  $a = -5 \quad b = 10 \quad c = 3$

- $f(x) = 1x^2 + 0 \cdot x + 0$

  yes

  $a = 1 \quad b = 0 \quad c = 0$

  $\boxed{\text{standard form}}$

  $y = ax^2 + bx + c$

- $y = 3x - 2$

  No, linear

  $\boxed{\text{vertex form}}$

  $y = a(x-h)^2 + k$

- $y = 2(x-3)^2 + 4$

  $y = 2(x-3)(x-3) + 4$

  $y = 2(x^2 - 3x - 3x + 9) + 4$

  $y = 2x^2 - 12x + 18 + 4$

  $y = 2x^2 - 12x + 22$

  yes

Example. Graph the following functions. For each graph label the vertex and the x-intercepts.

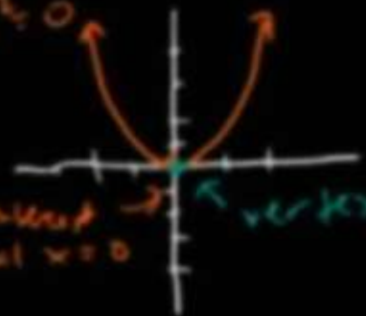
$y = ax^2 + bx + c$

If $a > 0$ parabola open upward

If $a < 0$ parabola opens down

A. $f(x) = x^2$

$f(0) = 0^2 = 0$



x-intercept at $x = 0$

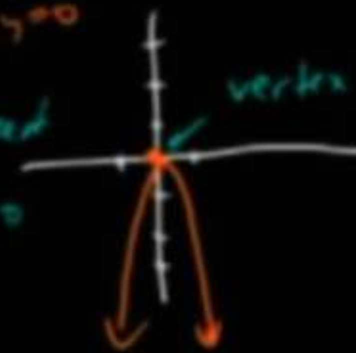
vertex $(0,0)$

$y = 0$

$\Rightarrow x^2 = 0 \quad x = 0$

B. $y = -3x^2$

$x = 0 \Rightarrow y = 0$



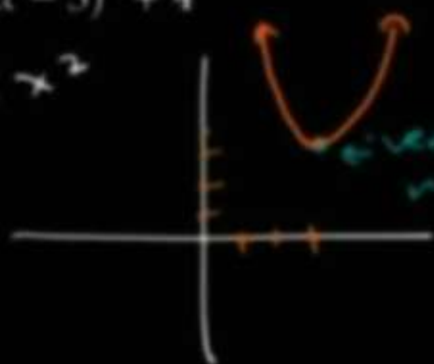
vertex $(0,0)$

x-intercept at $x = 0$

C. $y = 2(x-3)^2 + 4$

$y = 2x^2$

vertex at $(0,0)$



vertex $(3,4)$

no x-intercepts

move right by 3, up by 4

D. $g(x) = 5x^2 + 10x + 3$

$y = 5(-1)^2 + 10(-1) + 3 = -2$

$x = 0 \Rightarrow y = 3$

$y = 3$

$\boxed{y = ax^2 + bx + c}$

x-intercepts $\rightarrow$ set $y = 0$

$0 = 5x^2 + 10x + 3$

$x = \dfrac{-10 \pm \sqrt{10^2 - 4 \cdot 5 \cdot 3}}{2 \cdot 5}$

$x = \dfrac{-10 \pm \sqrt{40}}{10}$

$x = \dfrac{-10}{10} \pm \dfrac{\sqrt{40}}{10}$

$= -1 \pm \dfrac{2\sqrt{10}}{10}$

$= -1 \pm \dfrac{\sqrt{10}}{5}$

vertex has x-coord of $-\dfrac{b}{2a}$

$-\dfrac{10}{2 \cdot 5} = -1$

$y = a(x-h)^2 + k \leftarrow$ vertex form

vertex at (h,k)

Summary

To graph a quadratic function $f(x) = ax^2 + bx + c$

- The graph has the shape ... of a parabola
- The parabola opens up if ... $a > 0$ and down if ... $a < 0$
- To find the x-intercepts ... set $y = 0$ i.e. $f(x) = 0$ and solve for x
- To find the vertex ... read it off as h, k or use vertex formula
  $y = a(x-h)^2 + k$
  x-coord of vertex $= -\frac{b}{2a}$
- To find additional points on the graph ....
  plugging
  $y = ax^2 + bx + c$
  y-coord vertex, plug in the x-coord

Example. Convert this quadratic function to standard form: $f(x) = -4(x-3)^2 + 1$

$$f(x) = -4(x-3)(x-3)+1$$
$$= -4(x^2-6x+9)+1$$
$$= -4x^2+24x-36+1$$
$$\boxed{f(x) = -4x^2+24x-35}$$

ax^2+bx^1+c

Example. Convert this quadratic function to vertex form: $g(x) = 2x^2 + 8x + 6$

$g(x) = a(x-h)^2+k$

vertex: (h,k)

vertex formula:

x-coord. of vertex: $\frac{-b}{2a} = \frac{-8}{2(2)} = -2$

y-coord of vertex:

$g(-2) = 2(-2)^2+8(-2)+6 = -2$

vertex: $(-2,-2)$

$g(x) = a(x-(-2))^2+(-2)$

$g(x) = a(x+2)^2-2$

$$\boxed{g(x) = 2(x+2)^2-2}$$

$\rightsquigarrow g(x) = 2(x^2+4x+2)-2$

$= 2x^2+8x+6$

Example. Find the x-intercepts and the vertex for $y = 3x^2 + 7x - 5$.

x-intercepts

$$0 = 3x^2 + 7x - 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

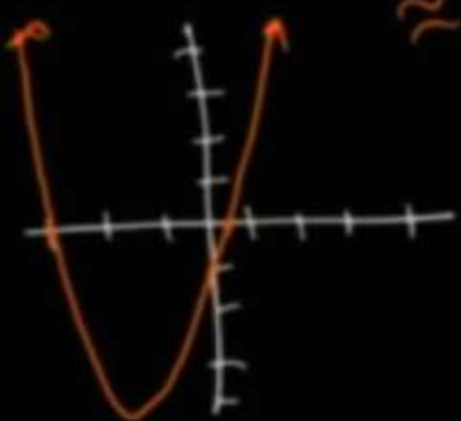
$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-5)}}{2(3)}$$

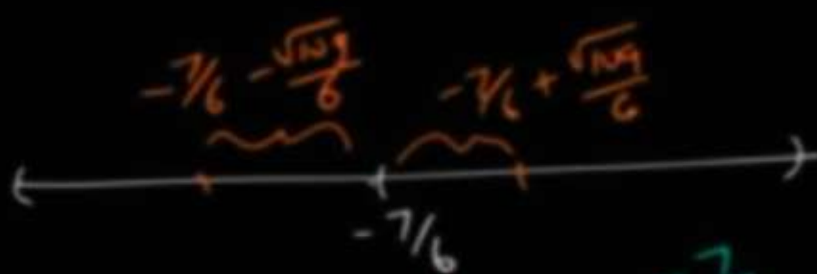
$$x = \frac{-7 \pm \sqrt{109}}{6}$$

$$x = -\frac{7}{6} + \frac{\sqrt{109}}{6}, \quad x = -\frac{7}{6} - \frac{\sqrt{109}}{6}$$

$$\approx -\frac{7}{6} + \frac{10}{6} \qquad \approx -\frac{7}{6} - \frac{10}{6}$$

$$\approx \frac{1}{2} \qquad \approx -3$$





x-coord of vertex: $-\frac{7}{6}$

Definition. The **degree** of the polynomial is the largest exponent.

$y = 5x^4 - 21x^3 - 17x + 2 \leftarrow$ degree is 4

The **leading term** is the term with the largest exponent

$y = 2 + 17x - 21x^3 + 5x^4$ leading term is $5x^4$

Definition. The **leading coefficient** is the number in the leading term

5 is leading coeff, since leading term is $5x^4$

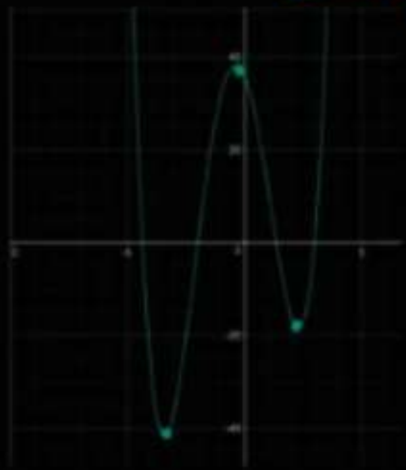
Definition. The **constant term** is the term with no x's in it

constant term is 2

Example. For $p(x) = 5x^3 - 3x^2 - 7x^4 + 2x + 18$, what is the

- degree? 4
- leading term? $-7x^4$
- leading coefficient? -7
- constant term? 18

Definition. In the graph of $f(x) = x^4 + 2x^3 - 15x^2 - 12x + 36$ below, the marked points are called ... turning points



OR

local extreme points

OR

local maximum and minimum points

degree 4
3 turning pts

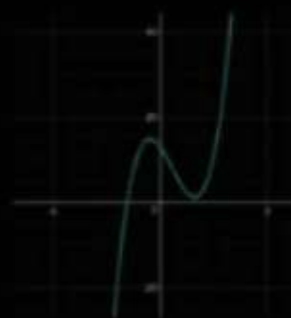
Compare the degrees of the polynomials to the number of turning points:

$f(x) = -2x^2 + 2x + 8$ $f(x) = 3x^3 - 5x^2 - 7x + 13$ $f(x) = x^4 + 6x^2 - 17$

Fact:

$\boxed{\#\text{ turning pts} \leq \text{degree} - 1}$

degree 2
1 turning pt



degree 3
2 turning pts

degree 4
1 turning pt

Definition. The **end behavior** of a function is how the "ends" of the function look as $x \to \infty$ and $x \to -\infty$.

Consider the end behavior for these polynomials:

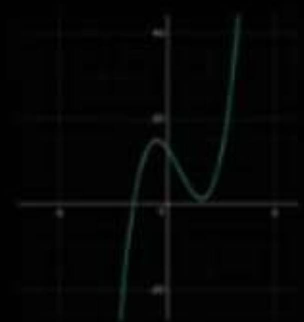
$f(x) = -2x^2 + 2x + 8$

↙ ↘

$f(x) = x^4 + 2x^3 - 15x^2 - 12x + 36$

↖ ↗

$f(x) = 3x^3 - 5x^2 - 7x + 13$



↙ ↗

$f(x) = -x^3 + 12x$

↖ ↘

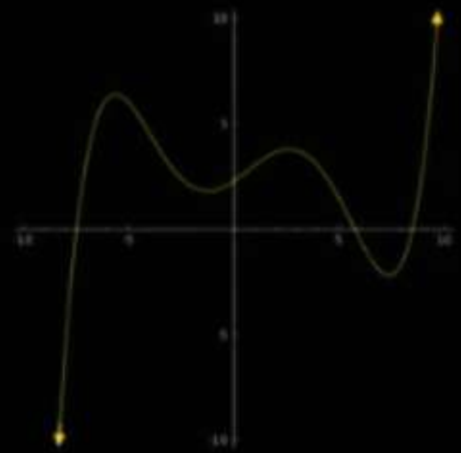
leading coeff

degree	positive	negative
even	↖ ↗	↙ ↘
odd	↙ ↗	↖ ↘

$y = x^2$ $y = -x^2$

$y = x^3$ $y = -x^3$

Example. What can you tell about the equation for the polynomial graphed below?



degree is odd

leading coeff is pos

degree ≥ 5

turning pts $\leq$ degree -1

$4 \leq$ degree -1

degree ≥ 5

degree could be 5, 7, 9, ...

Definition. An **exponential function** is a function that can be written in the form

$$f(x) = a \cdot b^x$$

variable is in the exponent

where a and b are real numbers, $a \neq 0$, and $b > 0$.

$f(x) = 3x^2$
variable in base
so NOT exponential fn

Note. We require that $a \neq 0$ because ...

otherwise, $f(x) = 0 \cdot b^x \Rightarrow f(x) = 0$ constant fn

Note. We require that $b > 0$ because ...

otherwise, for ex. if $b = -1$ $\quad f(x) = a \cdot (-1)^x$

$f(\frac{1}{2}) = a \cdot (-1)^{1/2} = a \cdot \sqrt{-1}$

not a real #

if $b = 0$ $\quad f(x) = a \cdot 0^x = 0$

The number a is called the ... initial value

$f(x) = a \cdot b^x$

$f(0) = a \cdot b^0 = a$

$f(0) = a$

start at when $x = 0$ we get $f(0) = a$

and the number b is called the ... base

Fact. In the graph of $y = a \cdot b^x$:

- The parameter a gives the y-intercept.
- The parameter b tells how the graph is increasing or decreasing.
- If $b > 1$, the graph is increasing.
- If $b < 1$, the graph is decreasing.
- The closer b is to the number 1, the flatter the graph.

So, for example, the graph of $y = 0.25^x$ is *(circle one)* flatter / more steep than the graph of $y = 0.4^x$.

Fact. The graph of an exponential function $y = a \cdot b^x$

- has a horizontal asymptote at the line $y = 0$.
- has domain $(-\infty, \infty)$.
- has range $(0, \infty)$ if $a > 0$
  $(-\infty, 0)$ if $a < 0$

Note. The most famous exponential function in the world is $f(x) = e^x$. This function is sometimes written as $f(x) = \exp(x)$. The number e is Euler's number, and is approximately 2.718281828459045 23...

Example. The United Nations estimated the world population in 2010 was 6.79 billion, growing at a rate of 1.1% per year. Assume that the growth rate stays the same. Write an equation for the population at t years after the year 2010.

$1.1\% \rightsquigarrow 0.011$

# years since 2010	world population in billions
0	6.79
1	$1 \cdot 6.79 + 0.011 \cdot 6.79 \to 6.79(1+0.011) = 6.79(1.011)$
2	$6.79 \cdot (1.011)^2$
$\vdots$	
t	$6.79 \cdot (1.011)^t$

initial population: 6.79

growth factor: 1.011

$$P(t) = 6.79(1.011)^t$$

P = world population in billions

t = time in years since 2010

the year 2050:

$$P(40) = 6.79 \cdot (1.011)^{40}$$
$$= 10.5 \text{ billion}$$

exponential growth $\leftrightarrow$ increasing

exponential decay $\leftrightarrow$ decreasing

Example. Seroquel is metabolized and eliminated from the body at a rate of 11% per hour. If 400 mg are given, how much remains in the body after 24 hours?

time in hours since dose	# mg of seroquel
0	400
1	$1\cdot 400 - 0.11\cdot 400 = 400(1-0.11) = 400\cdot 0.89$
2	$400\cdot 0.89\cdot 0.89 = 400\cdot 0.89^2$
⋮	
t	$400\cdot(0.89)^t$

"initial amount" (400); "growth factor" (0.89)

$$f(t) = 400\cdot(0.89)^t$$

$$f(24) = 400\cdot(0.89)^{24} = 24.4 \text{ mg}$$

$f(t)$ = # mg of seroquel in body

t = # hours since dose

To find the growth factor b,

start with % increase or decrease

↓

write it as a decimal

↓

add or subtract from 1

Applications of Exponential Functions

$f(t) = A \cdot b^t$

initial amount

growth factor

	%	r	b
+	3%	0.03	1+0.03=1.03
+	1.1%	0.011	1+0.011=1.011
−	11%	−0.11	1−0.11=0.89

$b = 1 + r$

% change written as decimal

if quantity is increasing, have $b > 1$, if quantity is decreasing, then $b < 1$

Example. An antique car is worth \$50,000 now and its value increases by 7% each year. Write an equation to model its value x years from now.

after 1 year, its value is $50{,}000 + 0.07 \cdot 50{,}000 = 50{,}000(1+0.07)$

$50{,}000(1.07)$

after 2 years, value is $50000(1+0.07)^2$

$50{,}000(1.07)^2$

multiply again by 1.07

after x years $\boxed{V(x) = 50000(1.07)^x}$

value

initial value

growth factor

$1 + 0.07$

initial value

$\boxed{V(x) = A \cdot b^x = A(1+r)^x}$

growth factor

% increase written as decimal

Example. My Toyota Prius is worth \$3,000 now and its value decreases by 5% each year. Write an equation to model its value x years in the future.

after 1 year, value $3{,}000 - 0.05 \cdot 3000 \rightarrow 3000(1-0.05)$

$3000(0.95)$

after 2 years value $3000(0.95)^2$

after x years, value $3000(0.95)^x$

$V(x) = 3000(0.95)^x$

$\boxed{V(x) = A \cdot b^x = A(1-r)^x}$

initial value

growth factor

$1 - 0.05$

0.05

% decrease written as decimal

Example. The number of bacteria in a petri dish is modeled by the equation

$$f(x) = 12 \cdot (1.45)^x$$

where $f(x)$ represents the number of bacteria in thousands, and x represents the number of hours since noon.

- What was the number of bacteria at noon? 12 thous.
- By what percent is the number of bacteria increasing every hour? 45 %

$f(x) = 12 \cdot (1.45)^x$

$f(x) = A \cdot b^x$

$f(x) = A \cdot (1+r)^x$

$A = 12$ ← 12 thousand

$b = 1.45$ ← growth factor

$r = 0.45$ ← 45% increase

Example. The population of salamanders $s(x)$ on an island is modeled by

$$s(x) = 3000 \cdot 0.78^x$$

where x is the number of years since 2015.

- What was the population in 2015? 3000
- By what percent is the population decreasing every year? 22%

$s(x) = A \cdot b^x$

$s(x) = A \cdot (1-r)^x$

initial

% decrease written as decimal

$A = 3000$

$b = 0.78 = 1 - r$

$r = 1 - 0.78$

$r = 0.22$

Example. You take out a loan for $1,200 at an annual interest rate of 6%, compounded *monthly*. If you pay back the loan with interest as a lump sum, how much will you owe after 3 years?

6% annual interest rate, compounded monthly

compounding 12 times per year

$\frac{6}{12}$% interest each time interest is added

0.5% interest, decimal 0.005

time in years	# months	money
0	0	1,200
1	12	$1,200 \cdot (1.005)^{12}$
2	24	$1,200(1.005)^{24}$
3	36	$1,200(1.005)^{36}$
t	12t	$1,200(1.005)^{12t}$

$$P(t) = 1200(1.005)^{12t}$$

t = # years

$$P(3) = 1200(1.005)^{12 \cdot 3}$$
$$= 1436.02$$

A = initial amount

r = annual interest rate compounded n times per year

$$P(t) = A\left(1 + \frac{r}{n}\right)^{nt}$$

compound interest formula

Example. You invest \$4000 in an account that gives 2.5% interest compounded *continuously*. How much money will you have after 5 years?

$P(t) = A \cdot e^{rt}$

r = annual interest rate, written as a decimal

$P(t)$ = money

A = initial amount

0.025

t = time in years

$e = 2.718\ldots$

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

$P(t) = 4000 \cdot e^{0.025t}$

$P(5) = 4000 \cdot e^{0.025 \cdot 5} = 4532.59$

$2\% \to 0.02$

Summary:

- Let r represent ...the annual interest rate, written as a decimal
- Let t represent ...the number of years
- Let A represent the initial amount of money ...

Annual interest:

$$P(t) = A \cdot (1+r)^t$$

Compound interest, compounded n times per year:

$$P(t) = A\left(1+\frac{r}{n}\right)^{nt}$$

Compound interest, compounded continuously.

$$P(t) = Ae^{rt}$$

Definition. $\log_a b = c$ means $a^c = b$.

base

$\log_a b = c$ means $a^c = b$

$\log_a b$ answ $\Rightarrow$ what power do you raise a to, to get b?

You can think of logarithms as exponents: $log_a b$ is the exponent (or "power") that you have to raise a to, in order to get b. The number a is called the *base* of the logarithm. The base is required to be a positive number.

Example.

$$\log_2 8 = 3 \text{ because } 2^3 = 8$$

$$\log_2 y = \square \text{ means } 2^{\square} = y$$

Example. Evaluate the following expressions by hand by rewriting them using exponents instead of logs:

a) $\log_2 16 =$ 4 $\qquad 2^{\boxed{4}} = 16$

b) $\log_2 2 =$ 1 $\qquad 2^{\boxed{1}} = 2$

c) $\log_2 \frac{1}{2} =$ -1 $\qquad 2^{\boxed{-1}} = \frac{1}{2}$

d) $\log_2 \frac{1}{8} =$ -3 $\qquad 2^{\boxed{-3}} = \frac{1}{8}$ $\qquad \frac{1}{8} = \frac{1}{2^3} = 2^{-3}$

e) $\log_2 1 =$ 0 $\qquad 2^{\boxed{0}} = 1$

Example. Evaluate the following expressions by hand by rewriting them using exponents instead of logs:

a) $\log_{10} 1,000,000 =$ 6 $\qquad 1,000,000 = 10^6 \qquad 10^{\square} = 10^6$

b) $\log_{10} 0.001 = \log_{10} 10^{-3} = -3 \qquad 10^{\square} = 10^{-3}$

c) $\log_{10} 0 =$ DNE $\qquad 10^{\square} = 0$

d) $\log_{10} -100 =$ DNE $\qquad 10^{\square} = -100$

Note. It is possible to take the log of numbers that are >0 but not of numbers that are ≤ 0. In other words, the domain of the function $f(x) = \log_a(x)$ is: $(0, \infty)$.

Note. $\ln x$ means $\log_e x$, and is called the **natural log**. $\qquad e \approx 2.718\ldots$

$\log x$, with no base, means $\log_{10} x$ and is called the **common log**.

You can find $\ln x$ and $\log x$ for various values of x using the buttons on your calculator.

Example. Rewrite using exponents.

a) $\log_3 \frac{1}{9} = -2$ $\qquad 3^{-2} = \frac{1}{9}$

b) $\log_{10} 13 = 1.11394$ $\qquad 10^{1.11394} = 13$

c) $\ln \frac{1}{e} = -1$

$\log_e \frac{1}{e} = -1$ $\qquad e^{-1} = \frac{1}{e}$

Example. Rewrite the following using logs. Do not solve for any variables.

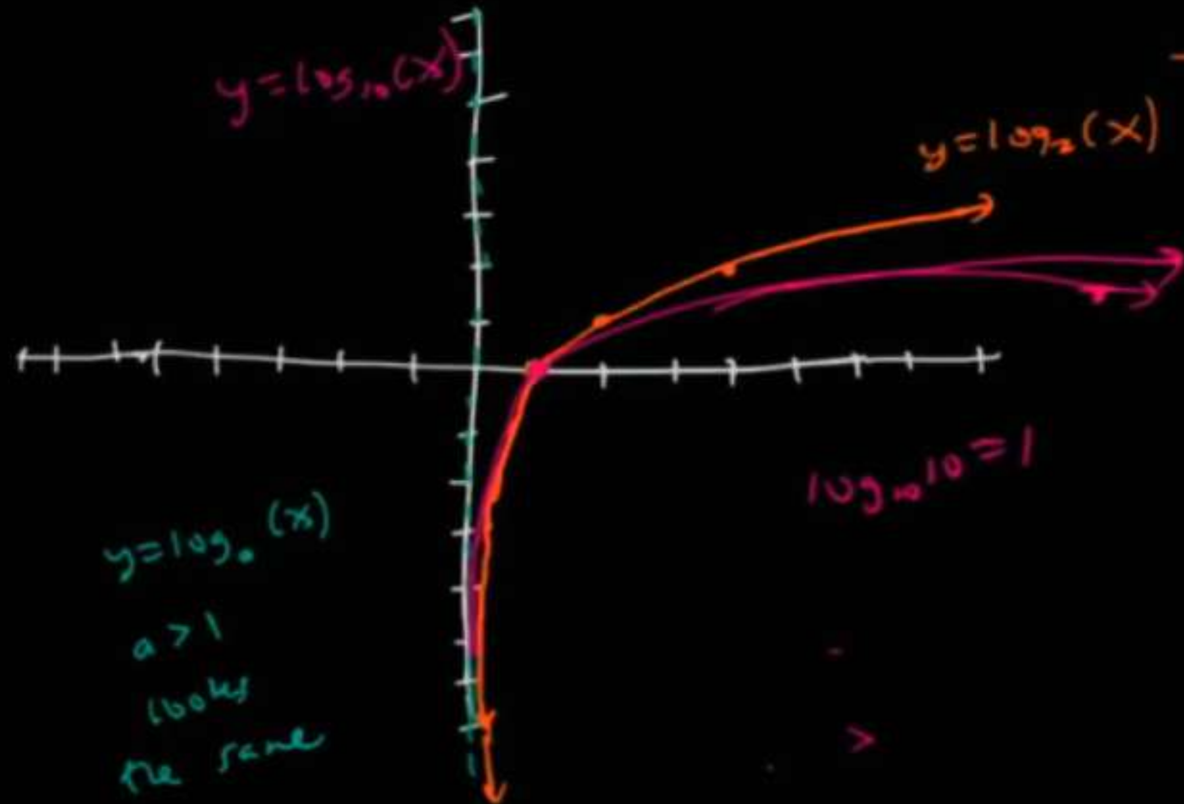
a) $3^u = 9.78$ $\qquad \log_3 \boxed{9.78} = \boxed{u}$ $\qquad \log_a b = c$

base $\qquad \Longleftrightarrow$

b) $e^{3x+7} = 4 - y$ $\qquad \log_e \boxed{4-y} = \boxed{3x+7}$ $\qquad a^c = b$

$\ln(4-y) = 3x+7$

Example. Graph $y = \log_2(x)$ by plotting points.



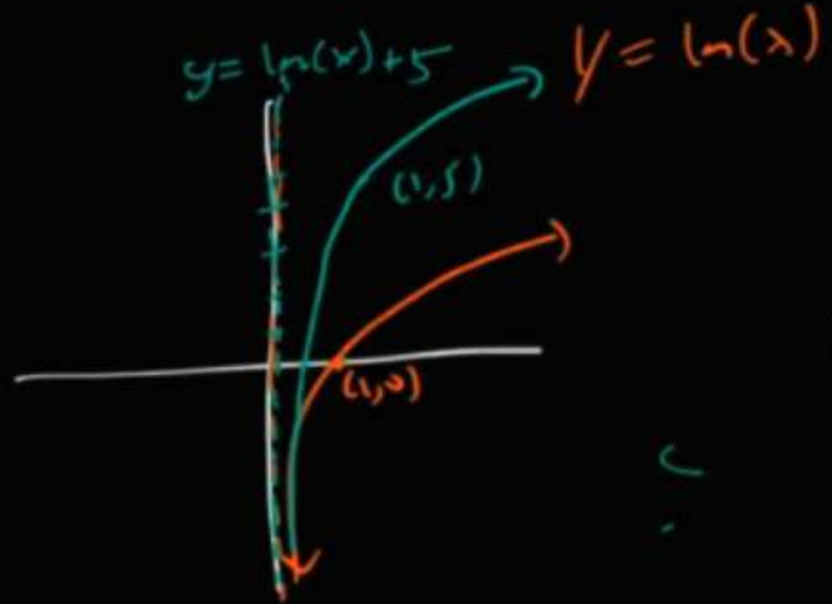
x	y
$\frac{1}{8}$	$\log_2(\frac{1}{8}) = -3$
$\frac{1}{4}$	$\log_2(\frac{1}{4}) = -2$
$\frac{1}{2}$	$\log_2(\frac{1}{2}) = -1$
1	$\log_2(1) = 0$
2	$\log_2(2) = 1$
4	$\log_2(4) = 2$
8	$\log_2(8) = 3$
16	$\log_2(16) = 4$
-4	~~$\log_2(-4)$~~ DNE
0	~~$\log_2(0)$~~ DNE

✓ 1) Domain: $x > 0$ $\quad (0, \infty)$

✓ 2) Range: all real #'s $\quad (-\infty, \infty)$

✓ 3) Vertical asymptote at y-axis, i.e. at the line $x = 0$

Example. Graph $y = \ln(x) + 5$. Find the domain, range, and asymptotes.



	$y = \ln(x)$	$y = \ln(x) + 5$
Domain	$(0, \infty)$	$(0, \infty)$
Range	$(-\infty, \infty)$	$(-\infty, \infty)$
V.A.	$x = 0$	$x = 0$

Find the domain of $f(x) = \ln(2 - 3x)$.

Need $2 - 3x > 0$

$2 > 3x$

$\frac{2}{3} > x$

$x < \frac{2}{3}$

Domain: $\left(-\infty, \frac{2}{3}\right)$

Example. Evaluate:

a) $\log_{10} 10^3 = 3$ $\quad \log(10^3)$

b) $\log_e e^{4.2} = 4.2$ $\quad \ln(e^{4.2})$

c) $10^{\log_{10} 1000} = 1000$

d) $e^{\log_e 9.6} = 9.6$

Log Rule: For any base a, $\log_a a^x = x$

What power do we raise a to, to get a^x $\quad a^{\boxed{x}} = a^x$

Log Rule: For any base a, $a^{\log_a x} = x$

$\log_a x$ means the power we raise a to, to get x. Raise a to that power, & get x.

An exponential fn and a log fn with the same base UNDO EACHOTHER.

Note. Recall some of the exponent rules.

1. $2^0 = 1$
2. Product rule: $2^m \cdot 2^n = 2^{m+n}$ If we multiply two numbers, we add the exponents.
3. Quotient rule: $\frac{2^m}{2^n} = 2^{m-n}$ If we divide two numbers, we subtract the exponents.
4. Power rule: $(2^m)^n = 2^{m \cdot n}$ If we take a power to a power, we multiply the exponents.

Note. The exponent rules hold for any base, not just base 2.

The corresponding logarithm rules are:

1. $\log_2 1 = 0$
2. Product rule: $\log_2(xy) = \log_2(x) + \log_2(y)$ The log of the product is the sum of the logs.
3. Quotient rule: $\log_2\left(\frac{x}{y}\right) = \log_2(x) - \log_2(y)$ The log of the quotient is the difference of the logs.
4. Power rule: $\log_2(x^n) = n \cdot \log_2(x)$
   $= \log_2(x) \cdot n$
   When you take the log of an expression with an exponent, you can bring down the exponent and multiply.

Note. The logarithm rules hold for any base, not just base 2.

Exponent Rule	Log Rule	Name of Log Rule
$a^0 = 1$	$\log_a 1 = 0$	–
$a^m \cdot a^n = a^{m+n}$	$\log_a(xy) = \log_a(x) + \log_a(y)$	Product Rule
$\frac{a^m}{a^n} = a^{m-n}$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	Quotient Rule
$(a^m)^n = a^{mn}$	$\log_a(x^n) = n \cdot \log_a(x)$	Power Rule

Example. Rewrite as a single log:

a) $\underbrace{\log_5 a - \log_5 b}$ $+ \log_5 c$

$= \log_5 \frac{a}{b} + \log_5 c$

$= \log_5 \left(\frac{a}{b} \cdot c\right)$

$= \log_5 \left(\frac{ac}{b}\right)$

b) $\ln(x+1) + \ln(x-1) - 2\ln(x^2-1)$

$\ln((x+1)(x-1)) - 2\ln(x^2-1)$

$\ln((x+1)(x-1)) - \ln(x^2-1)^2$

$\ln\left(\frac{(x+1)(x-1)}{(x^2-1)^2}\right)$

$\ln\left(\frac{\cancel{(x^2-1)}}{\cancel{(x^2-1)}^2}\right)$

$\ln\left(\frac{1}{x^2-1}\right)$

Log Rules

1) $\log_a 1 = 0$
2) $\log_a xy = \log_a x + \log_a y$
3) $\log_a \frac{x}{y} = \log_a x - \log_a y$
4) $\log_a x^m = m \log_a x$

⚠ There is no log rule for

$\log_a(x+y)$

$\log_a(x+y) \neq \log_a(x) + \log_a(y)$

no rule to rewrite this

$a^m + a^n$

Example. Solve: $2^{2x-3} = 5^{x-2}$

$$\log 2^{2x-3} = \log 5^{x-2}$$

$$(2x-3)\log 2 = (x-2)\log 5$$

$$2x\log 2 - 3\log 2 = x\log 5 - 2\log 5$$

$$2x\log 2 - x\log 5 = -2\log 5 + 3\log 2$$

$$x(2\log 2 - \log 5) = -2\log 5 + 3\log 2$$

$$x = \frac{(-2\log 5 + 3\log 2)}{(2\log 2 - \log 5)}$$

$$= 5.106$$

① Isolating tricky parts

② Take log of both sides

③ Use log rules to bring down the exponents

④ Distribute.

⑤ Group x terms on one side, other terms on the side

⑥ Isolate x by Factoring it out & dividing

Example. Solve: $5 \cdot e^{-0.05t} = 3 \cdot e^{0.2t}$

$$e^{-0.05t} = \frac{3}{5} e^{0.2t}$$

$$\ln e^{-0.05t} = \ln\left(\frac{3}{5} e^{0.2t}\right)$$

$$-0.05t \ln e = \ln \frac{3}{5} + \ln e^{0.2t}$$

$$-0.05t \ln e = \ln \frac{3}{5} + 0.2t \ln e$$

$$-0.05t = \ln \frac{3}{5} + 0.2t$$

$$-0.05t - 0.2t = \ln \frac{3}{5}$$

$$t(-0.05 - 0.2) = \ln \frac{3}{5}$$

$$t = \frac{\ln\left(\frac{3}{5}\right)}{-0.05 - 0.2}$$

$$t = 2.0433$$

1. Isolate tricky parts / simplify
2. Take ln of both sides
3. Use log rules to ultimately bring down exponents

$$\ln e = \log_e e = 1$$

4. Distribute (not needed)
5. Bring terms with t to one side, terms without t to other side
6. Isolate t by factoring out t & dividing

Example. Solve: $2\ln(2x+5) - 3 = 1$

$2\ln(2x+5) = 4$

$\ln(2x+5) = 2$

$e^{\ln(2x+5)} = e^2$

$2x+5 = e^2$

$2x = e^2 - 5$

$\boxed{x = \frac{e^2-5}{2}}$

check:

$2\ln(\cancel{2}(\frac{e^2-5}{\cancel{2}})+5) - 3 \stackrel{?}{=} 1$

$2\ln(e^2 \cancel{-5} \cancel{+5}) - 3 \stackrel{?}{=} 1$

$2\ln(e^2) - 3 \stackrel{?}{=} 1$

$2\log_e(e^2) - 3 \stackrel{?}{=} 1$

$2 \cdot 2 - 3 \stackrel{?}{=} 1$

$4 - 3 \stackrel{?}{=} 1$ ✓

① Isolate the tricky part

② Take e to the power of both sides

$e^{\ln A} = e^{\log_e A} = A$

③ Finish solving for x

④ Check answer! – may get extraneous solutions

Example. Solve: $\log(x+3)+\log(x)=1$

① Isolate the trick part (can't do it here)

② Take 10 to the power of both sides

③ Finish solving

④ check solutions

alternative method

use log rules

$\log((x+3)(x))=1$

$10^{\log((x+3)(x))}=10^1$

$(x+3)(x)=10$

$10^{\log(x+3)+\log(x)}=10^1$

$10^{\log(x+3)}\cdot 10^{\log(x)}=10$

$(x+3)(x)=10$

$x^2+3x=10$

$x^2+3x-10=0$

$(x+5)(x-2)=0$

$\cancel{x=-5}$ or $\boxed{x=2}$

$x=-5$ ← extraneous solution

$\log(-5+3)+\log(-5)\stackrel{?}{=}1$

$\log(-2)+\log(-5)\stackrel{?}{=}1$

DNE

$\boxed{x=2}$

$\log(2+3)+\log(2)\stackrel{?}{=}1 \Rightarrow \log 5+\log 2\stackrel{?}{=}1$

$\log(5\cdot 2)\stackrel{?}{=}1$

$\log_{10}10\stackrel{?}{=}1$ ✓

The general Rule of logs and exponents

key step:

use exponential fns to undo the log

i.e. take e to power of both sides to undo ln

take 10 to power of both sides to undo $\log_{10}$

Example. Suppose you invest \$1600 in a bank account that earns 6.5% annual interest, compounded annually. How many years will it take until the account has \$2000 in it, assuming you make no further deposits or withdrawals?

$f(t) = 1600 \cdot (1.065)^t$

where $f(t)$ is money after t years

$2000 = 1600 \cdot \underline{(1.065)^t}$

solve for t

$\frac{2000}{1600} = (1.065)^t$

$\frac{5}{4} = (1.065)^t$

$\ln\frac{5}{4} = \ln(1.065)^t$

$\ln\frac{5}{4} = t\ln(1.065)$

$\frac{\ln\frac{5}{4}}{\ln(1.065)} = t \Rightarrow t \approx 3.54$ years

Example. A population of bacteria contains 1.5 million bacteria and is growing by 12% per day. What is its *doubling time*?

means amount of time it takes for a quantity to double in size

$P(t)$ # bacteria in millions t = time in days

$P(t) = 1.5 \cdot (1.12)^t$

$3 = 1.5\,(1.12)^t$

$\frac{3}{1.5} = (1.12)^t$

$\ln\left(\frac{3}{1.5}\right) = \ln(1.12)^t$

$\ln\left(\frac{3}{1.5}\right) = t\ln(1.12)$

$t = \frac{\ln(2)}{\ln(1.12)} \approx 6.12$ days.

☆ Doubling time only depends on growth rate, not the initial population.

$P(t) = A \cdot (1.12)^t$

initial pop

To find doubling time:

$2\cancel{A} = \cancel{A} \cdot (1.12)^t$

$\ln 2 = \ln(1.12)^t$

$\ln 2 = t\ln(1.12)$

$t = \frac{\ln 2}{\ln(1.12)} \approx 6.12$

Example. Suppose a bacteria population doubles every 15 minutes. Write an equation for its growth using the exponential equation $y = a \cdot b^t$, where t represents time in minutes. Assume the initial population is 350 bacteria.

$\boxed{y = a \cdot b^t}$

$t = \#$ minutes

$y = \#$ bact

$a = 350$

$y = 350 \cdot b^t$

DT: when $t = 15$

$y = 700$

$700 = 350 \cdot b^{15}$

$\frac{700}{350} = b^{15}$

$2 = b^{15}$

$2^{1/15} = (b^{15})^{1/15}$

$2^{1/15} = b^1 \quad b = 2^{1/15} \approx 1.047294$

$y = 350 \cdot (2^{1/15})^t$

use $\boxed{y = a \cdot e^{rt}}$ ← continuous growth model

$y = 350 e^{rt}$

when $t = 15 \quad y = 700$

$700 = 350 e^{r \cdot 15}$ solve for r

$\frac{700}{350} = e^{r \cdot 15} \Rightarrow 2 = e^{r \cdot 15}$

$\ln 2 = \ln e^{r \cdot 15}$

$\ln 2 = r \cdot 15 \cdot \ln e \Rightarrow 1$

$15 r = \ln 2 \Rightarrow r = \frac{\ln 2}{15}$

$y = 350 e^{\frac{\ln 2}{15} t}$

$y = 350 \left(e^{\ln 2 / 15}\right)^t$

$y = 350 \left((e^{\ln 2})^{1/15}\right)^t = 350 (2^{1/15})^t$

Example. The half life of radioactive Carbon-14 is 5750 years. A sample of bone that originally contained 200 grams of C-14 now contains only 40 grams. How old is the sample?

Radioactivit = radioactive

half life = the amount of time it takes for a quantity to decrease to half as much

$$f(t) = a \cdot e^{rt}$$

amount of rad. C-14 · initial amount

$$\tfrac{1}{2}\cancel{a} = \cancel{a} \cdot e^{r5750}$$

$$\tfrac{1}{2} = e^{r5750}$$

$$\ln(\tfrac{1}{2}) = \ln(e^{r5750})$$

$$\ln(\tfrac{1}{2}) = r \cdot 5750$$

$$r = \frac{\ln(\frac{1}{2})}{5750}$$

~~$f(t) = a \cdot b^t$~~

HL: 5750

when $t = 5750$

$f(t) = \tfrac{1}{2} \cdot a$

$$f(t) = a \cdot e^{\left(\frac{\ln(\frac{1}{2})}{5750} t\right)}$$

$$40 = 200 \cdot e^{(\ln(\frac{1}{2})/5750\, t)}$$

$$\frac{40}{200} = e^{\ln(\frac{1}{2})/5750\, t}$$

$$\ln \tfrac{1}{5} = \ln e^{\ln(\frac{1}{2})/5750\, t}$$

$$\ln \tfrac{1}{5} = \frac{\ln \frac{1}{2}}{5750} t$$

$$t = \ln \tfrac{1}{5} \,/\, (\ln(\tfrac{1}{2})/5750) \approx 13,351$$

$$\begin{cases} 3x - 2y = 4 \\ 5x + 6y = 2 \end{cases}$$

Method 1: Substitution

isolate one variable using one eqn
substitute it into other eqn

1st eqn

$3x - 2y = 4$

$3x = 4 + 2y$

$x = \frac{4+2y}{3} \Rightarrow x = \frac{4}{3} + \frac{2}{3}y$

2nd eqn

$5x + 6y = 2$

$5\left(\frac{4}{3} + \frac{2}{3}y\right) + 6y = 2$

$\frac{20}{3} + \frac{10}{3}y + 6y = 2$

$\frac{10}{3}y + 6y = 2 - \frac{20}{3}$

$3\left(\frac{10}{3}y + 6y\right) = 3\left(2 - \frac{20}{3}\right)$

$10y + 18y = 6 - 20$

$28y = -14 \Rightarrow y = \frac{-14}{28} = -\frac{1}{2}$

$\boxed{y = -\frac{1}{2}}$

$\boxed{\left(1, -\frac{1}{2}\right)}$

$3x - 2y = 4$

$3x - 2\left(-\frac{1}{2}\right) = 4 \Rightarrow 3x + 1 = 4$

$3x = 3 \Rightarrow \boxed{x = 1}$

Method 2: Elimination

multiply each eqn by a constant to make coefficients of one variable match

$3x - 2y = 4$

$5x + 6y = 2$

$5(3x - 2y) = 5 \cdot 4$

$3(5x + 6y) = 3 \cdot 2$

$15x - 10y = 20$

$-(15x + 18y = 6)$

$-28y = 14$

$y = \frac{14}{-28} = -\frac{1}{2}$

$3x - 2y = 4$

$3x - 2\left(-\frac{1}{2}\right) = 4$

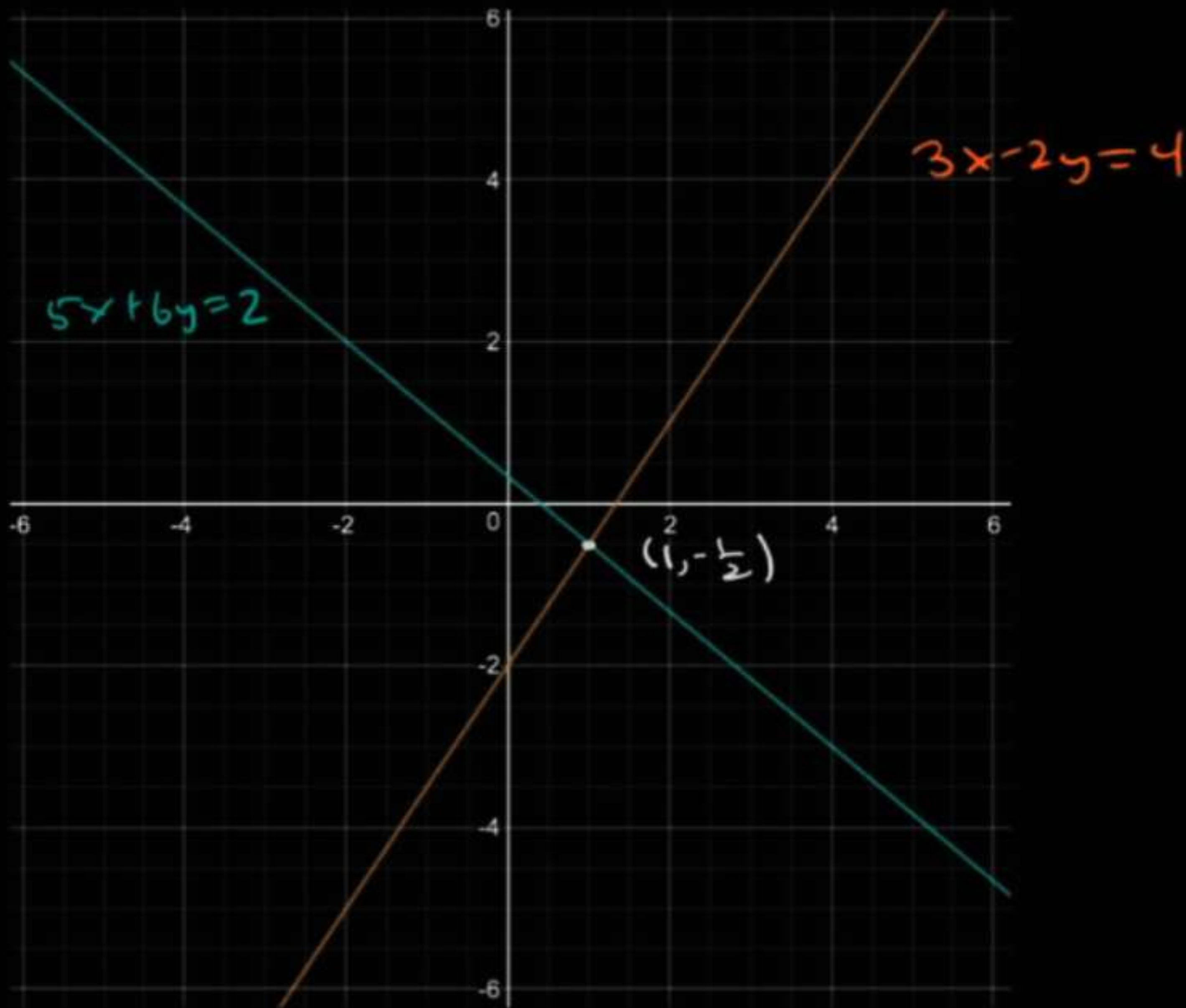
$3x + 1 = 4$

$3x = 3$

$x = 1$

$\boxed{x = 1, \; y = -\frac{1}{2}}$

$\left(1, -\frac{1}{2}\right)$



$5x+6y=2$
$3x-2y=4$
$(1,-\frac{1}{2})$
-6
-4
-2
0
2
4
6

Example. Solve the system of equations:

inconsistent system

$$8y = 1 + 4x$$
$$3x - 6y = 2$$

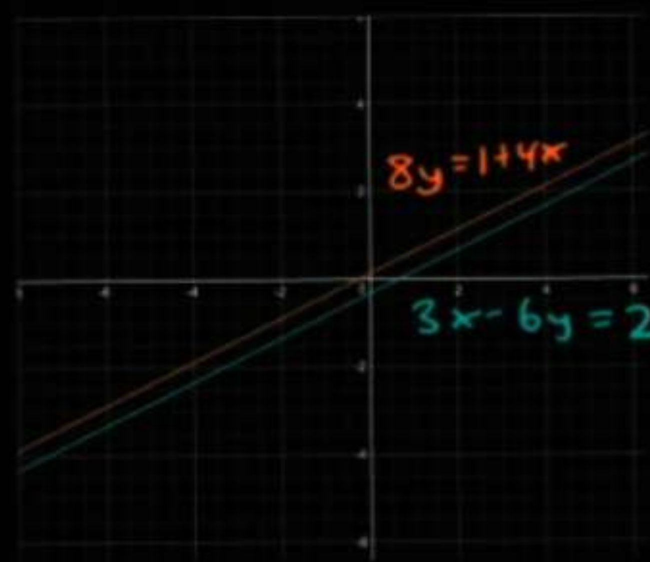
$-4x + 8y = 1$

$3x - 6y = 2$

$$\begin{array}{r} -12x + 24y = 3 \\ +\,(12x - 24y = 8) \\ \hline 0 + 0 = 11 \end{array}$$

$\Rightarrow\Leftarrow$

no solution



$y = \frac{1}{2}x + \frac{1}{8}$

$-6y = -3x + 2$

$y = \frac{1}{2}x + \left(-\frac{1}{3}\right)$

Example. Solve the system of equations: dependent system

$$\begin{cases} x + 5y = 6 \\ 3x + 15y = 18 \end{cases}$$

$x = 6 - 5y$

$3(6-5y) + 15y = 18$

$18 - 15y + 15y = 18$

$18 = 18$

infinitely many solutions

(x,y) where $x+5y=6$ i.e. $x = -5y+6$ will work

x	y
6	0
1	1
$1\frac{2}{3}$	$\frac{1}{3}$

$d = r \cdot t$ $\qquad r_1 + r_2 = r_{total}$

Example. Elsa's boat has a top speed of 6 miles per hour in still water. While traveling on a river at top speed, she went 10 miles upstream in the same amount of time she went 30 miles downstream. Find the rate of the river current.

	distance	rate	time
upstream	10	$6-r$	t
downstream	30	$6+r$	t

rate in still water $\qquad$ rate of current

t = time

r = speed of current

$$\boxed{\begin{aligned} 10 &= (6-r)t \\ 30 &= (6+r)t \end{aligned}} \implies \begin{aligned} \frac{10}{6-r} &= t \\ \frac{30}{6+r} &= t \end{aligned} \implies \frac{10}{6-r} = \frac{30}{6+r}$$

$$\cancel{(6-r)}(6+r)\frac{10}{\cancel{6-r}} = \frac{30}{\cancel{6+r}}(6-r)\cancel{(6+r)}$$

$$(6+r)10 = 30(6-r)$$

$$60 + 10r = 180 - 30r$$

Example. Household bleach contains 6% sodium hypochlorite. How much household bleach should be combined with 70 liters of a weaker 1% sodium hypochlorite solution to form a solution that is 2.5% sodium hypochlorite?

	volume of NaClO	volume of water	volume of solution
household bleach → 6% solution	$0.06 \cdot x$	$x - 0.06x$ $0.94x$	x
1% solution	$0.01 \cdot 70$ 0.7	$0.99 \cdot 70$ 69.3	70
2.5% solution	$0.025 \cdot (70+x)$	$0.975 \cdot (70+x)$	$70 + x$

NaClO before mixing = NaClO after mixing

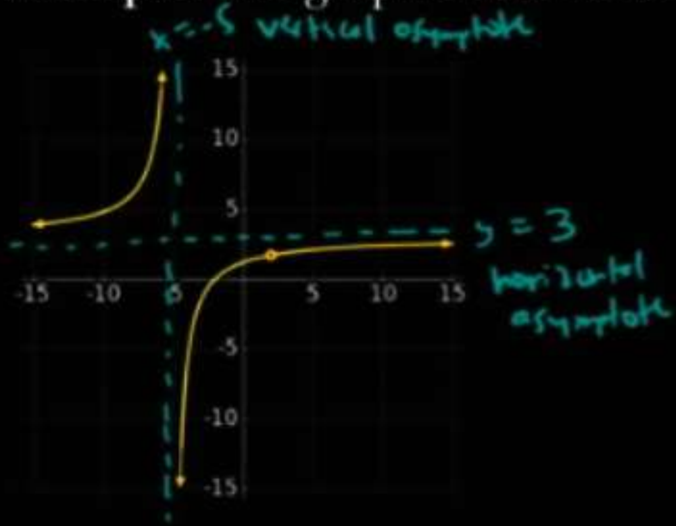
water before = water after

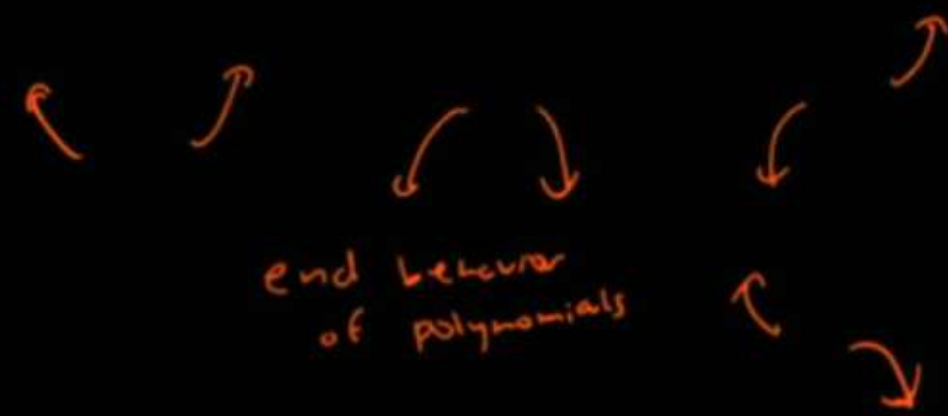
solution before = solution after

$$0.06x + 0.7 = 0.025(70+x)$$
$$1000\,(0.06x + 0.7) = 1000 \cdot 0.025(70+x)$$
$$60x + 700 = 25(70+x)$$
$$60x + 700 = 1750 + 25x$$
$$35x = 1050 \Rightarrow \boxed{x = 30 \text{ liters}}$$

x = volume of household bleach (6% solution)

Example. The graph of the function $h(x) = \dfrac{3x^2 - 12}{x^2 + 3x - 10}$ is shown below.





How is the graph of this function $h(x)$ different from the graph of a polynomial?

What is the end behavior of the graph? horizontal asymptote at $y = 3$

$$h(x) = \frac{3x^2 - 12}{x^2 + 3x - 10} \rightarrow \approx \frac{3x^2}{x^2} = 3$$

What is the behavior of the graph of this function $h(x)$ near $x = -5$?

vertical asymptote at $x = -5$

$$h(x) = \frac{3(x^2-4)}{(x+5)(x-2)} = \frac{3(x-2)(x+2)}{(x+5)(x-2)}$$

at $x = -5$, denominator is 0
but numerator is not 0

What is going on at $x = 2$? hole at $x = 2$

hole $(2, \frac{4}{3})$

$$h(x) = 3\frac{(x+2)}{(x+5)} \quad \text{as long as } x \neq 2$$

$$y = 3\frac{(2+2)}{2+7} = \frac{12}{9} = \frac{4}{3}$$

For a rational function

- find the vertical asymptotes by: setting denominator = 0
- find the holes by:

hole → x-values where numerator and denom are both zero and those factors cancel out

V.A → all other x-values where denom = 0

- find the horizontal asymptotes by:

look at highest power term in numerator and denominator

Example. What are the horizontal asymptotes for these functions?

degree of num < degree of denom

1. $f(x) = \dfrac{5x + 4}{3x^2 + 5x - 7}$

$\dfrac{5x}{3x^2} = \dfrac{5}{3x} \to 0$ H.A. $y = 0$

degree of num = degree of denom

2. $g(x) = \dfrac{2x^3 + 4}{3x^3 - 7x}$

$\dfrac{2x^3}{3x^3} = \dfrac{2}{3} \to \dfrac{2}{3}$ H.A. $y = \dfrac{2}{3}$

take ratio of leading coefficients

degree of num > degree of denom

3. $h(x) = \dfrac{x^2 + 4x - 5}{2x - 1}$

$\dfrac{x^2}{2x} = \dfrac{x}{2}$ $\to \infty$ when $x \to \infty$

$\to -\infty$ when $x \to -\infty$

no. H.A.

Example. Find the vertical asymptotes, horizontal asymptotes, and holes:

$$q(x) = \frac{3x^2 + 3x}{2x^3 + 5x^2 - 3x}$$

H.A. $\frac{3x^2}{2x^3} \to \frac{3}{2x} \to 0$ $\boxed{y = 0}$

V.A. & hole

set denom = 0

factor both num & denom

$q(x) = \frac{3x(x+1)}{x(2x^2+5x-3)}$

$q(x) = \frac{3x(x+1)}{x(2x-1)(x+3)}$

$(2x-1)(x+3)$

$2x^2+6x-x-3$ ✓

$2x^2+5x-3$

hole at $x=0$

y-value of hole $\frac{3(0+1)}{(2\cdot 0-1)(0+3)} = \frac{3}{-3} = -1$

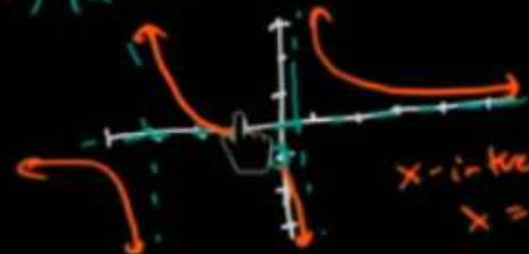
$\boxed{(0,-1)}$

$q(x) = \frac{3(x+1)}{(2x-1)(x+3)}$ when $x \neq 0$

V.A. when $(2x-1)(x+3) = 0$

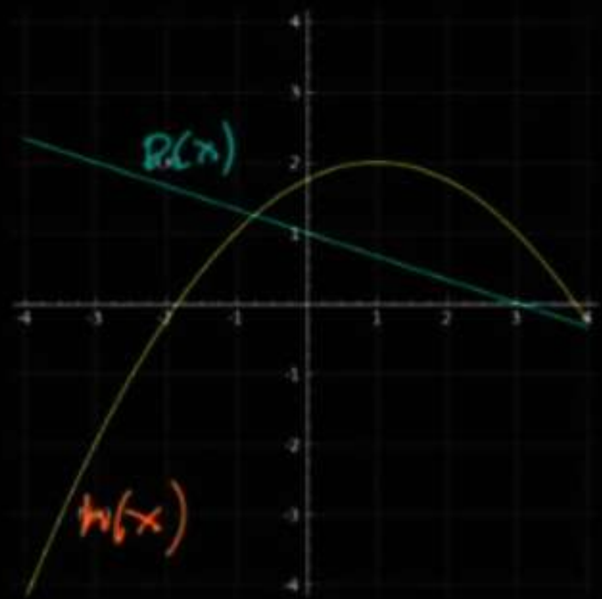
when $2x-1=0$ or $x+3=0$

$\boxed{x = \frac{1}{2} \text{ or } x = -3}$



x-intercept at $x=-1$

Example. The following graphs represent two functions h and p.



$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Find

$(h-p)(x) = h(x) - p(x)$

a) $(h-p)(0) = h(0) - p(0)$

$= 1.8 - 1 = 0.8$

b) $(ph)(-3) = p(-3) \cdot h(-3)$

$= 2 \cdot (-2) = -4$

Example. Let $p(x) = x^2 + x$. Let $q(x) = -2x$. Find:

a) $q \circ p(1) = q(p(1))$ $\quad p(1) = 1^2 + 1 = 2$

$= q(2)$ $\quad q(2) = -2 \cdot 2 = -4$

$= -4 \leftarrow$

b) $q \circ p(x) = q(\underbrace{p(x)}_{x^2+x}) = q(x^2+x) = -2(x^2+x)$

$\boxed{q \circ p(x) = -2x^2 - 2x}$

$q \circ p(1) = -2 \cdot 1^2 - 2 \cdot 1 = -4$

c) $p \circ q(x) = p(\underbrace{q(x)}_{-2x}) = p(-2x) = (-2x)^2 + (-2x)$

$\boxed{p \circ q(x) = 4x^2 - 2x}$ $\quad q \circ p \neq p \circ q$

d) $p \circ p(x) = p(\underbrace{p(x)}_{x^2+x}) = p(x^2+x) = (x^2+x)^2 + (x^2+x)$

$= x^4 + 2x^3 + x^2 + x^2 + x$

$= x^4 + 2x^3 + 2x^2 + x$

what happens to the box, that is $f(x)$, my second fn applied

Example. $h(x) = \sqrt{\boxed{x^2+7}}$. Find functions f and g so that $h(x) = f \circ g(x) = f(g(x))$

inside the box is $g(x)$, my first fn applied

$h = f \circ g$

$g(x) = x^2+7$

$f(x) = \sqrt{x}$

$f \circ g(x) = f(\underbrace{g(x)}_{x^2+7})$

$= f(x^2+7) = \sqrt{x^2+7}$ ✓

OR

$h(x) = \sqrt{\boxed{x^2}+7}$

first fn applied $g(x) = x^2$

second fn applied $f(x) = \sqrt{x+7}$

check: $f \circ g(x) = f(g(x)) = f(x^2) = \sqrt{x^2+7}$ ✓

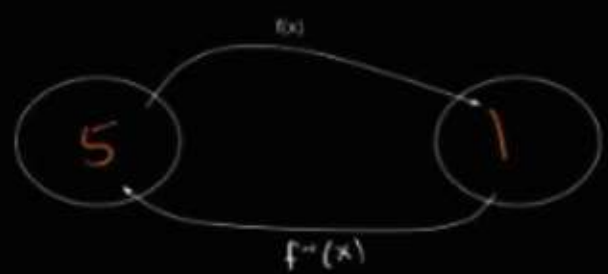
Example. Suppose $f(x)$ is the function defined by the chart below:

x	2	3	4	5
$y =$ $f(x)$	3	5	6	1

In other words,

- $f(2) = 3$
- $f(3) = 5$
- $f(4) = 6$
- $f(5) = 1$

Definition. The **inverse function** for f, written $f^{-1}(x)$, undoes what f does.

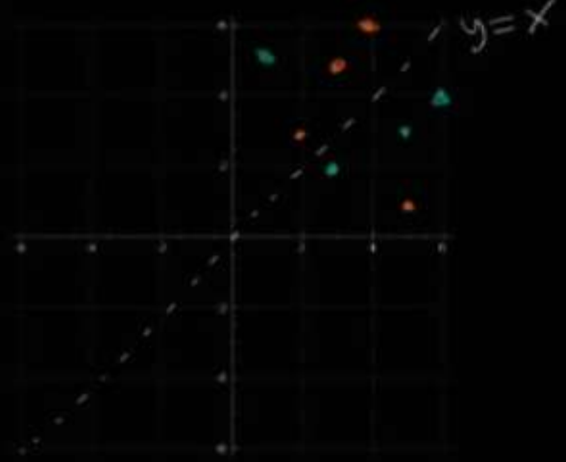


- $f^{-1}(\ 3\ ) = 2$
- $f^{-1}(5) = 3$
- $f^{-1}(6) = 4$
- $f^{-1}(1) = 5$

x	3	5	6	1
$y =$ $f^{-1}(x)$	2	3	4	5

Key Fact 1. Inverse functions reverse the roles of y and x.

Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same axes below. What do you notice about the points on the graph of $y = f(x)$ and the points on the graph of $y = f^{-1}$?



x	2	3	4	5
$y=f(x)$	3	5	6	1

x	3	5	6	1
$y=f^{-1}(x)$	2	3	4	5

Key Fact 2. The graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by reflecting over the line $y = x$.

x	2	3	4	5
$f(x)$	3	5	6	1

x	3	5	6	1
$f^{-1}(x)$	2	3	4	5

In our same example, compute:

$f^{-1} \circ f(2) = f^{-1}(f(2)) = f^{-1}(3) = 2$

$f^{-1} \circ f(3) = f^{-1}(f(3)) = f^{-1}(5) = 3$

$f^{-1} \circ f(4) = f^{-1}(f(4)) = f^{-1}(6) = 4$

$f^{-1} \circ f(5) = f^{-1}(f(5)) = f^{-1}(1) = 5$

$f \circ f^{-1}(3) = f(f^{-1}(3)) = f(2) = 3$

$f \circ f^{-1}(5) = f(f^{-1}(5)) = f(3) = 5$

$f \circ f^{-1}(6) = f(f^{-1}(6)) = f(4) = 6$

$f \circ f^{-1}(1) = f(f^{-1}(1)) = f(5) = 1$

Key Fact 3. $f^{-1} \circ f(x) =$ $\underline{\quad x \quad}$ and $f \circ f^{-1}(x) =$ $\underline{\quad x \quad}$. This is the mathematical way of saying that f and f^{-1} undo each other.

Example. $f(x) = x^3$. Guess what the inverse of f should be. Remember, f^{-1} undoes the work that f does.

$$f^{-1}(x) = \sqrt[3]{x}$$

$$f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

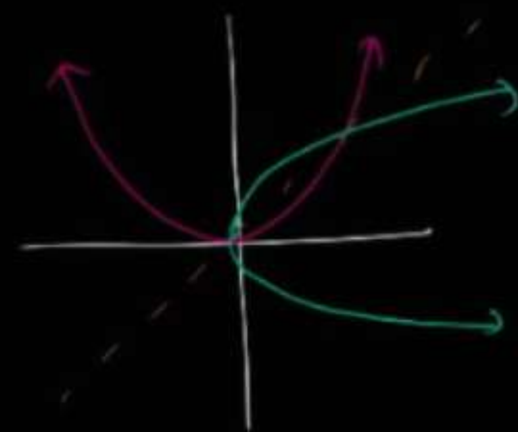
$$f^{-1}(f(x)) = \sqrt[3]{x^3} = x$$

Question. Do all functions have inverse functions? That is, for any function that you might encounter, is there always a **function** that is its inverse?

Try to find an example of a function that does **not** have an inverse **function**.

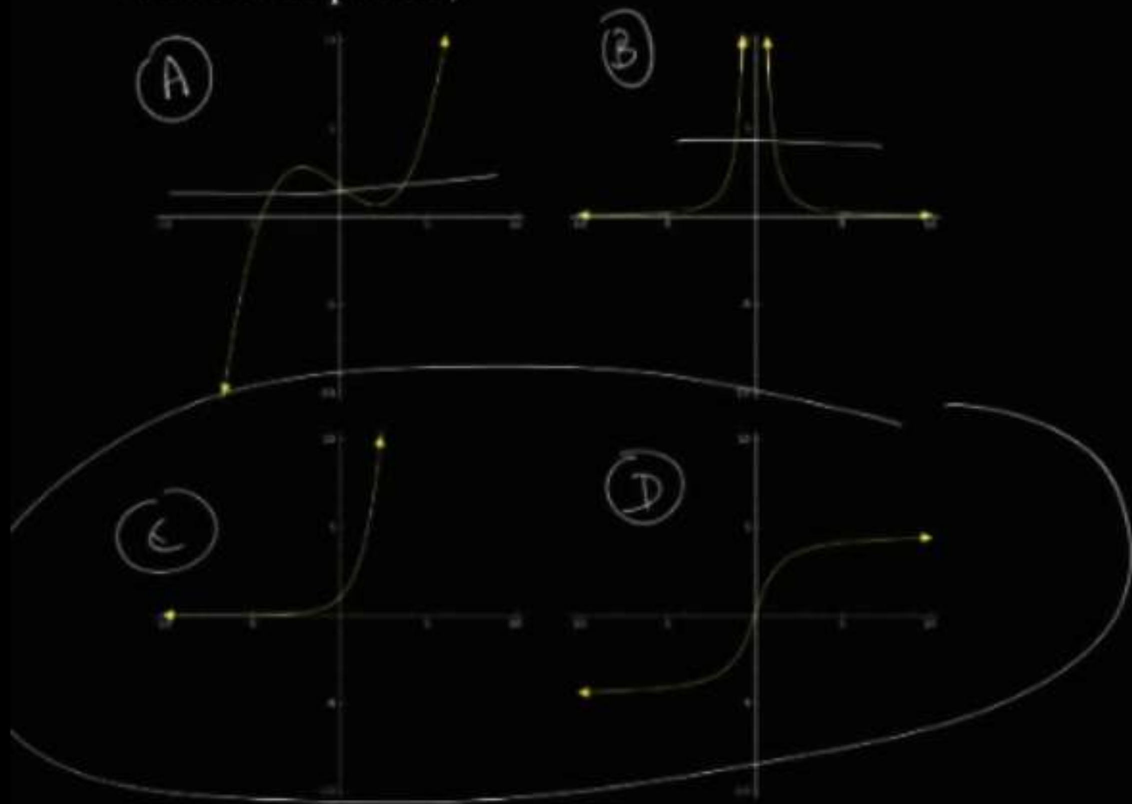
for each x in domain, there is only one corresponding y-value

$f(x) = x^2$



$f^{-1}(4) = 2, -2$

Key Fact 4. A function f has an inverse function if and only if the graph of f satisfies the **horizontal line test** (i.e. every horizontal line intersects the graph of $y = f(x)$ in at most one point.)



Satisfy horizontal line test

Definition. A function is **one-to-one** if it passes the horizontal line test. Equivalently, a function is one-to-one if for any two different x-values x_1 and x_2, $f(x_1)$ and $f(x_2)$ are different numbers. Sometimes, this is said: f is one-to-one if, whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$.

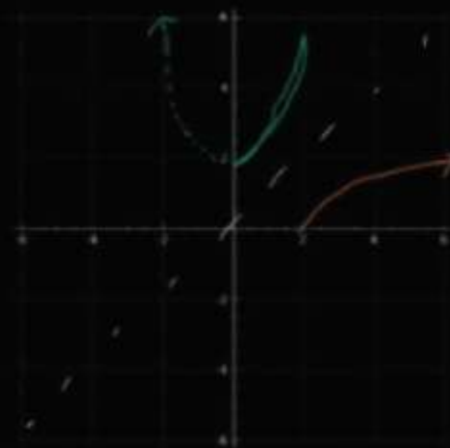
Example. (Tricky) Find $p^{-1}(x)$, where $p(x) = \sqrt{x-2}$ drawn. Graph $p^{-1}(x)$ on the same axes as $p(x)$.

$y = \sqrt{x-2}$ $\quad y \geq 0$

$x = \sqrt{y-2}$

$x^2 = y-2$

$y = x^2+2, \quad x \geq 0$



For the function $p(x) = \sqrt{x-2}$, what is:

- the domain of p? $x-2 \geq 0$, $x \geq 2$, $[2, \infty)$
- the range of p? $y \geq 0$ $\quad [0, \infty)$
- the domain of p^{-1}? $x \geq 0$ $\quad [0, \infty)$
- the range of p^{-1}? $y \geq 2$ $\quad [2, \infty)$

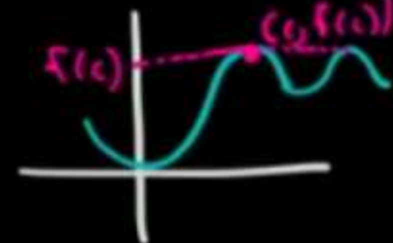
Key Fact 5. For any invertible function f, the domain of $f^{-1}(x)$ is the range of $f(x)$ and the range of $f^{-1}(x)$ is the domain of $f(x)$

Definition. A function $f(x)$ has an **absolute maximum** at $x = c$ if $f(c) \geq f(x)$ for all x-values in the domain of f

the y-value at $x=c$ is as big as it ever gets.

The y-value $f(c)$ is called the absolute maximum value of f.

and the point $(c, f(c))$ is called an absolute maximum point.

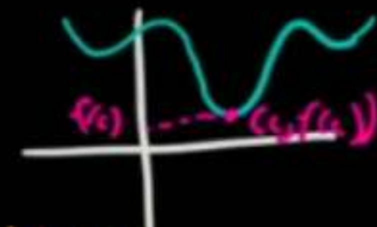


Definition. A function $f(x)$ has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for all x-values in the domain of f

the y-value at $x=c$ is as small as it ever gets.

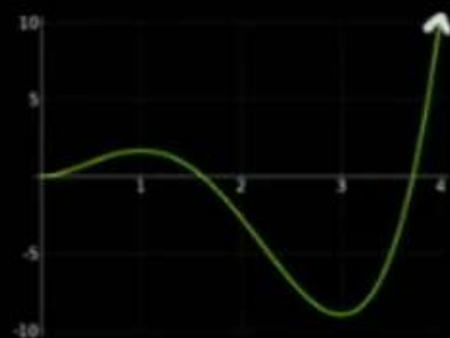
The y-value $f(c)$ is called the absolute minimum value of f

and the point $(c, f(c))$ is called an absolute minimum point



abs min value $= -8$

abs min point: $(3, -8)$



abs max value: 10

abs max point: $(4, 10)$

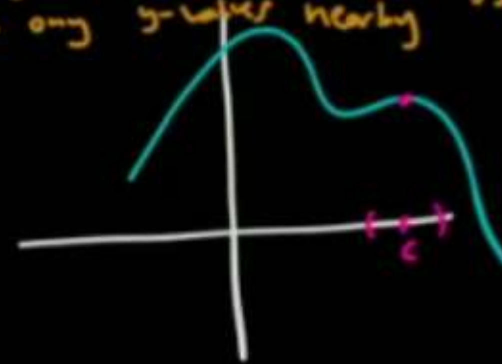
or no abs max value if it continues forever

Definition. Absolute maximum and minimum values can also be called

global maximum and minimum values.

Definition. A function $f(x)$ has an **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x-values in an open interval around c.

the y-value at $x=c$ is bigger than any y-values nearby



The y-value $f(c)$ is called a local maximum value for f

and the point $(c, f(c))$ is called a local maximum point

Definition. A function $f(x)$ has an **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all x-values in an open interval around c

The y-value $f(c)$ is called a local minimum value

and the point $(c, f(c))$ is called a local minimum point.

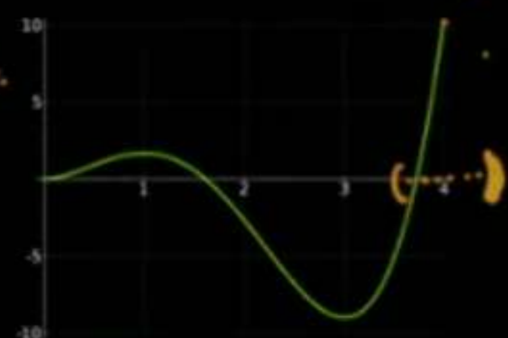
domain $[0,4]$

also abs min pt.

local min pt at $(3,-8)$

local max pt at $(1,2)$

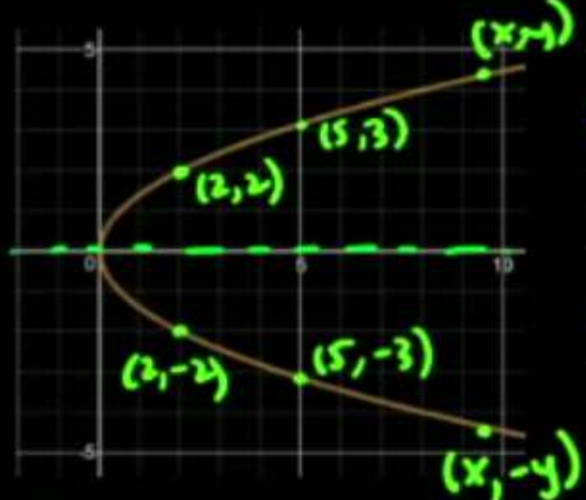
not abs max pt



Some sources do not consider $(4,10)$ to be a local max pt b/c f is not defined in an open interval around $x=4$.

Definition. Local maximum and minimum values can also be called relative maximum and minimum values.

Definition. A graph is *symmetric with respect to the x-axis* if ...it has mirror symmetry with x-axis as the mirror line.

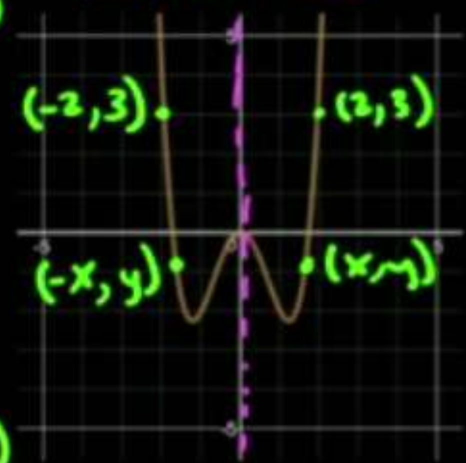


Whenever a point (x, y) is on the graph, the point

$(x, -y)$

is also on the graph.

Definition. A graph is *symmetric with respect to the y-axis* if ...it has mirror symmetry with the y-axis as the mirror line

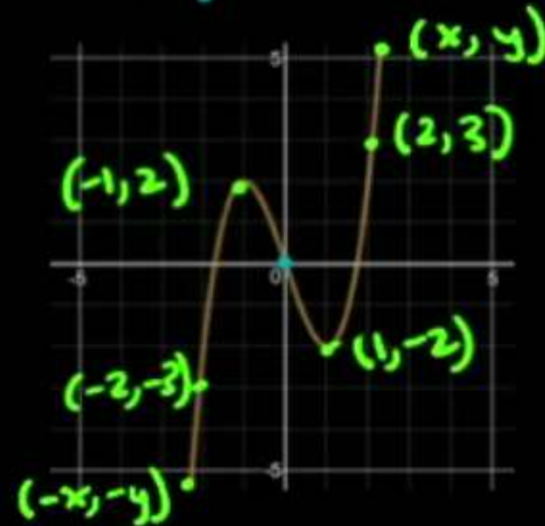


Whenever a point (x, y) is on the graph, the point

$(-x, y)$

is also on the graph.

Definition. A graph is *symmetric with respect to the origin* if ...it has 180° rotational symmetry around the origin



Whenever a point (x, y) is on the graph, the point

is also on the graph.

Example. The function f is defined as follows:

$$f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ -2x + 3 & \text{if } x \geq 1 \end{cases}$$

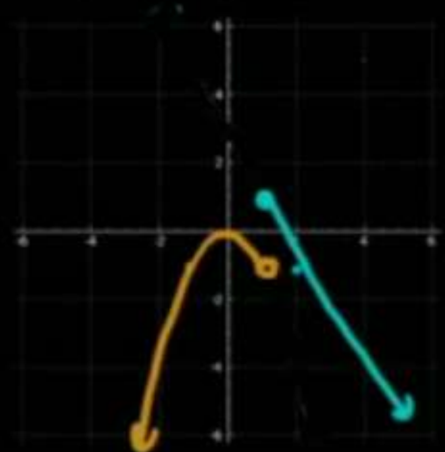
1. What is $f(-2)$? What is $f(1)$ What is $f(3)$?

$f(-2) = -(-2)^2 = -4$

$f(1) = -2(1) + 3 = 1$

$f(3) = -2(3) + 3 = -3$

2. Graph $y = f(x)$.



$$g(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ -2x+1 & \text{if } x \geq 1 \end{cases}$$

g(x) is continuous

3. Is $f(x)$ continuous? NO

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Note. To convert between degrees and radians, it is handy to use the fact that 180 degrees equals π radians.

Example. Convert -135° to radians.

positive angle

negative angle

$$-135\cancel{^\circ} \cdot \frac{\pi \text{ radians}}{180\cancel{^\circ}} = \frac{-135\pi}{180} \text{ radians} = -\frac{3\pi}{4} \text{ radians}$$

$$= -2.3562 \text{ radians}$$

Example. Convert $\frac{5\pi}{4}$ radians to degrees.

$$\frac{5\pi}{4} \cancel{\text{radians}} \cdot \frac{180^\circ}{\pi \cancel{\text{radians}}} = \frac{5\cancel{\pi} \cdot 180}{4\cancel{\pi}}{}^\circ = 225^\circ$$

Example. Convert 7 radians to degrees.

$$7 \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}} = \frac{7 \cdot 180^\circ}{\pi} = 401.07^\circ$$

Sometimes, angles are given in terms of degrees, minutes, and seconds, as in $32°17'25''$

degrees minutes seconds

A minute is ... $\frac{1}{60}$ degree

$\boxed{60 \text{ minutes} = 1 \text{ degree}}$

A second is ... $\frac{1}{60}$ minute $= \frac{1}{60} \cdot \frac{1}{60}$ degree $= \frac{1}{3600}$ degree

$\boxed{3600 \text{ seconds} = 1 \text{ degree}}$

Example. Convert $32°17'25''$ to a decimal number of degrees.

$$32° + 17' + 25''$$

$$32° + 17\cancel{'} \cdot \frac{1°}{60\cancel{'}} + 25\cancel{''} \cdot \frac{1°}{3600\cancel{''}}$$

$$32° + \frac{17}{60}° + \frac{25}{3600}° = 32.2903°$$

Example. Convert $247.3486°$ to degrees, minutes, and seconds.

$$247° + 0.3486°$$

$$0.3486\cancel{°} \cdot \frac{60'}{1\cancel{°}} = 20.916'$$

$$247° + 20.916'$$

$$0.916\cancel{'} \cdot \frac{60''}{1\cancel{'}} = 54.96''$$

$$247° + 20' + 54.96''$$

$$\boxed{247°20'55''}$$

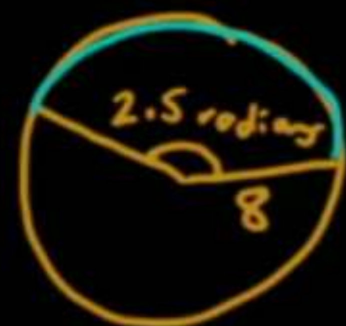
The circumference of a circle is given by the formula ... $C = 2\pi r$

circumference

radius

Example. A circular pool has a radius of 8 meters. Find the arclength spanned by a central angle of 2.5 radians.

$C = 2\pi \cdot 8 \text{ m}$



arclength = fraction of circle · circum

$$= \frac{2.5 \text{ rad}\cancel{\text{ians}}}{\cancel{2\pi} \text{ r}\cancel{\text{adians}}} \cdot \cancel{2}\pi \cdot 8 \text{ m}$$

$$= 2.5 \cdot 8 \text{ m} = 10 \text{ m}$$

The arclength is related to the angle it spans according to the formula ...

$$\text{arclength} = \frac{\theta}{\cancel{2}\pi} \cdot \cancel{2}\pi r \qquad \text{arclength} = \underbrace{\theta}_{\text{angle in radians}} \cdot \overbrace{r}^{\text{radius}}$$

The area of a circle is given by the formula ... $A = \pi r^2$

area ↗ A, radius ↖ r

Example. Find the area of a sector of a circle of radius 10 meters, that spans an angle of $\frac{\pi}{6}$ radians.

$$\text{area of circle} = \pi (10\,\text{m})^2 = 100\pi\ \text{m}^2$$

$$\text{area of sector} = \text{fraction of circle} \cdot \text{area of circle}$$

$$= \frac{\pi/6 \ \cancel{\text{radians}}}{2\cancel{\pi}\ \cancel{\text{radians}}} \cdot 100\cancel{\pi}\ \text{m}^2$$

$$= \frac{100\pi}{12}\ \text{m}^2 = \frac{25}{3}\pi\ \text{m}^2 = 26.18\ \text{m}^2$$

The area of a sector is related to the angle is spans according to the formula ...

angle in radians

$$\text{area of sector} = \frac{\theta}{2\cancel{\pi}} \cdot \cancel{\pi} r^2$$

radian in circle → 2π

angle of sector in radians

$$\text{area of sector} = \frac{\theta}{2} r^2$$

↖ radius

Consider a spinning wheel.

Definition. The *angular speed* is ... angle it goes through in a unit of time

angle/time

and has units of ... radians/sec or °/min, etc.

Definition. The *linear speed* is ... distance a point on rim of wheel travels in a unit of time $\rightsquigarrow$ dist/time

and has units of ... m/sec, ft/min, etc.

angle/time

Example. A ferris wheel with radius 20 m is making 1 revolution every 2 minutes. What is its angular speed? The linear speed of a point on its rim?

$$\frac{1 \text{ rev}}{2 \text{ min}} = \frac{1}{2} \frac{\text{rev}}{\text{min}}$$

$$\text{angular speed} = \frac{1}{2} \frac{\cancel{\text{rev}}}{\text{min}} \cdot \frac{2\pi \text{ radians}}{1 \cancel{\text{rev}}}$$

$$= \frac{1}{2} \cdot 2\pi \text{ radians}/\text{min} = \pi \frac{\text{radians}}{\text{min}}$$

$$\text{linear speed} = \pi\, \cancel{\text{radians}}/\text{min} \cdot \frac{\cancel{2\pi} \cdot 20 \text{ m}}{\cancel{2\pi} \cancel{\text{radians}}} = 20\pi \text{ m}/\text{min}$$

The linear speed is related to the angular speed by the formula ..

v ω

$$v = \omega \cdot \frac{\text{circum}}{2\pi}$$

$$v = \omega \cdot \frac{\cancel{2\pi} r}{\cancel{2\pi}}$$

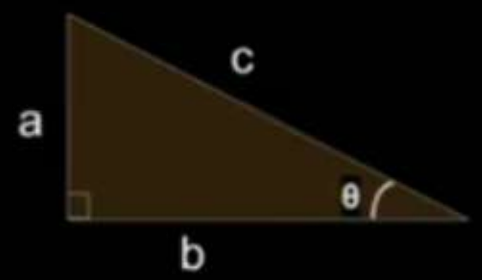
angular speed

$$v = \omega \cdot r$$

linear speed

radius

For a right triangle with sides a, b, and c and angle θ as drawn,



we define

$$\sin\theta = \frac{opp}{hyp} = \frac{a}{c}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{b}{c}$$

$$\tan\theta = \frac{opp}{adj} = \frac{a}{b}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{adj}{hyp}} = \frac{hyp}{adj} = \frac{c}{b}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{opp}{hyp}} = \frac{hyp}{opp} = \frac{c}{a}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{opp/adj} = \frac{adj}{opp} = \frac{b}{a}$$

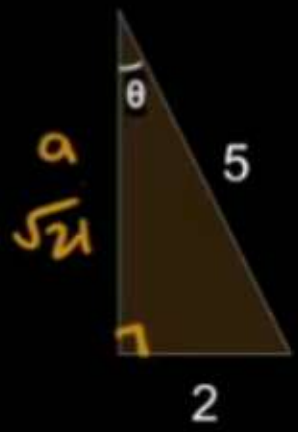
SOHCAHTOA

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{why?}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{opp}{hyp}}{\frac{adj}{hyp}} = \frac{opp}{\cancel{hyp}} \cdot \frac{\cancel{hyp}}{adj}$$

$$= \frac{opp}{adj} = \tan(\theta)$$

Example. Find the exact values of all six trig functions of angle θ in this right triangle.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2}{5} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{5}{2}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{21}}{5} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{21}}{5}} = \frac{5}{\sqrt{21}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{\sqrt{21}} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{21}}{2}$$

$$a^2 + b^2 = c^2$$
$$a^2 + 2^2 = 5^2$$
$$a^2 + 4 = 25$$
$$a^2 = 21$$
$$a = \sqrt{21}$$

Example. Without using a calculator, find sin(45°) and cos(45°) using a right triangle with hypotenuse 1.

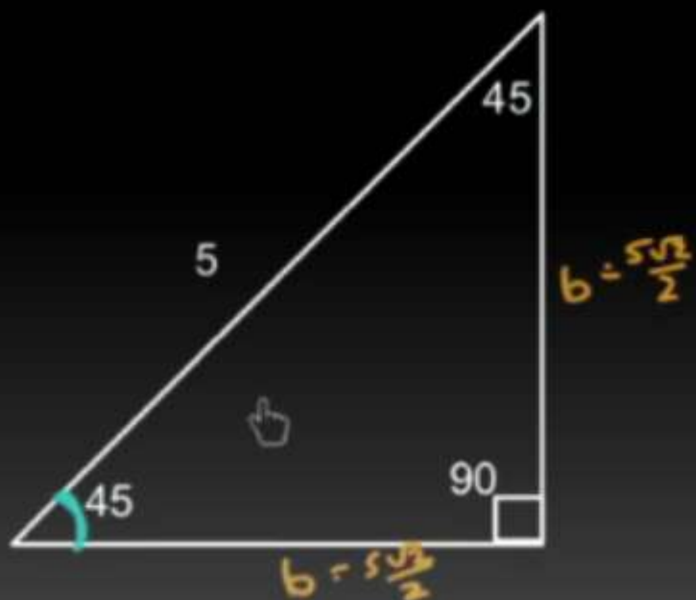
$$a^2+a^2=1^2$$
$$2a^2=1$$
$$a^2=\frac{1}{2}$$
$$a=\pm\sqrt{\frac{1}{2}}$$

$$a=\frac{\sqrt{1}}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{(\sqrt{2})^2}=\frac{\sqrt{2}}{2}$$

$$\sin(45^\circ)=\frac{\text{opp}}{\text{hyp}}=\frac{\frac{\sqrt{2}}{2}}{1}=\frac{\sqrt{2}}{2}$$

$$\cos(45^\circ)=\frac{\text{adj}}{\text{hyp}}=\frac{\frac{\sqrt{2}}{2}}{1}=\frac{\sqrt{2}}{2}$$

Example. Without using a calculator, find sin(45°) and cos(45°), and tan(45°) using a right triangle with hypotenuse 5.

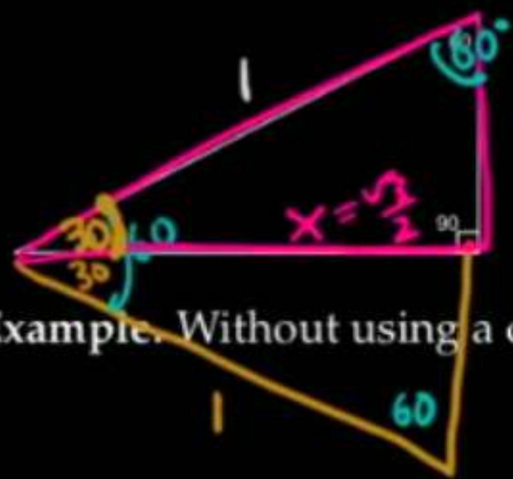


$$b^2+b^2=5^2$$
$$2b^2=25$$
$$b^2=\frac{25}{2}$$
$$b=\pm\sqrt{\frac{25}{2}}$$
$$b=\frac{\sqrt{25}}{\sqrt{2}}$$
$$b=\frac{5}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{5\sqrt{2}}{2}$$

$$\sin(45^\circ)=\frac{\text{opp}}{\text{hyp}}=\frac{\frac{5\sqrt{2}}{2}}{5}=\frac{\sqrt{2}}{2}$$

$$\cos(45^\circ)=\frac{\text{adj}}{\text{hyp}}=\frac{\frac{5\sqrt{2}}{2}}{5}=\frac{\sqrt{2}}{2}$$

Example. Without using a calculator, find sin(30°) and cos(30°).



$x^2 + \left(\frac{1}{2}\right)^2 = 1^2$

$x^2 + \frac{1}{4} = 1$

$x^2 = \frac{3}{4}$

$x = \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$

$\sin(30^\circ) = \frac{opp}{hyp} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

$\cos(30^\circ) = \frac{adj}{hyp} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

Example. Without using a calculator, find sin(60°) and cos(60°).

$\sin(60^\circ) = \frac{opp}{hyp} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

$\cos(60^\circ) = \frac{adj}{hyp} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

To summarize:

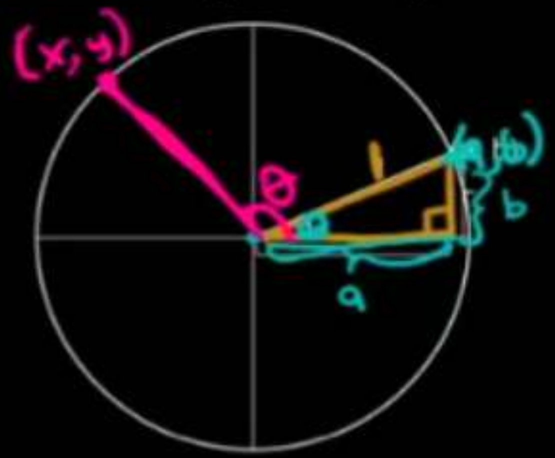
angle θ in degrees	angle θ in radians	$\cos(\theta)$	$\sin(\theta)$
30°	$\pi/6$	$\sqrt{3}/2$	$\frac{1}{2}$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
60°	$\pi/3$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

$\frac{1}{2}$ $\quad \frac{\sqrt{2}}{2}$ $\quad \frac{\sqrt{3}}{2}$

$30^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{6}$

In the figure, a right triangle is drawn in the unit circle. It's top vertex has coordinates (a, b). In terms of a and b,

- How long is the base of the triangle?
- How long is its height?



θ starts at the positive x-axis and goes counterclockwise.

Using the triangle definition of sine and cosine, what are $\sin(\theta)$ and $\cos(\theta)$, in terms of a and b?

$\cos(\theta) = a$ = x-coord of point on unit circle at angle θ

$\sin(\theta) = b$ = y-coord of point on unit circle at angle θ

$\tan(\theta) = \frac{b}{a} = \frac{\text{y-coord}}{\text{x-coord}}$

Definition. For angles θ that can't be part of a right triangle we find the point on the unit circle at angle θ and **define**

$\cos(\theta) =$ x-coord $\quad \sin(\theta) =$ y-coord $\quad \tan(\theta) = \frac{\text{y-coord}}{\text{x-coord}}$

Example. For the angle ϕ drawn,

$\sin(\phi) =$ 0.9397

$\cos(\phi) =$ -0.3420

$\tan(\phi) = \frac{0.9397}{-0.3420} = -0.3639$

Even Odd Property:

$\cos(-\theta) = \cos(\theta)$

even fn

$\sin(-\theta) = -\sin(\theta)$

odd fn

Example. Determine if $\tan(\theta)$ is an even or odd function.

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

$\tan(\theta)$ is odd

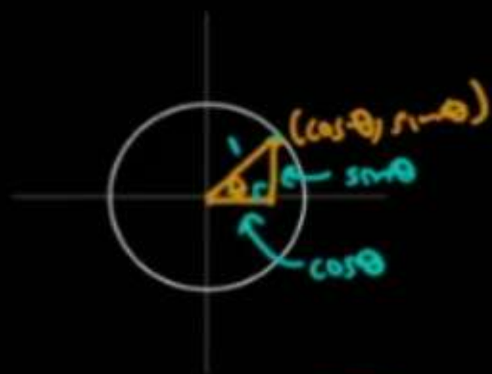
Pythagorean Property:

$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$\cos^2\theta$ means $(\cos(\theta))^2$

$\sin^2\theta$ means $(\sin(\theta))^2$



Example. If $\sin(t) = -\frac{2}{7}$, and t is angle that lies in quadrant III, find $\cos(t)$.

$$\cos^2 t + \sin^2 t = 1$$

$$(\cos t)^2 + \left(-\frac{2}{7}\right)^2 = 1$$

$$(\cos t)^2 + \frac{4}{49} = 1$$

$$(\cos t)^2 = 1 - \frac{4}{49} = \frac{45}{49}$$

$$\cos t = \pm\sqrt{\frac{45}{49}} = \pm\frac{\sqrt{45}}{7}$$

$$\cos t = -\frac{\sqrt{45}}{7}$$

$$a^2 + 2^2 = 7^2$$

$$a^2 + 4 = 49$$

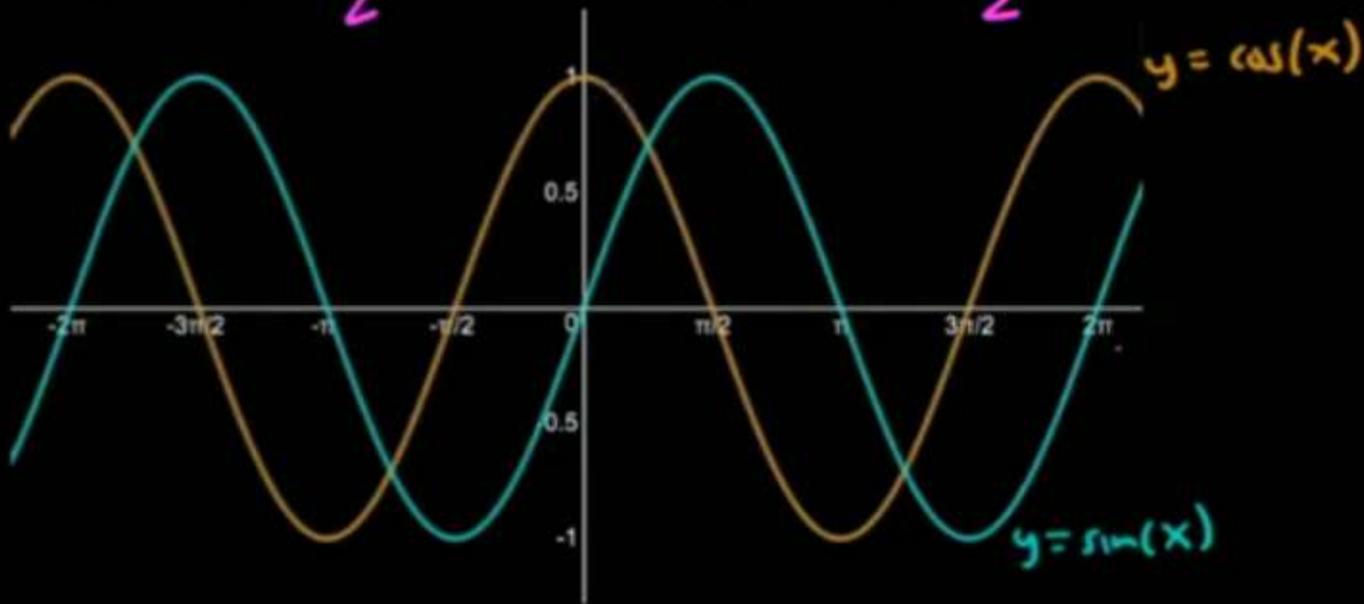
$$a^2 = 45$$

$$a = \pm\sqrt{45}$$

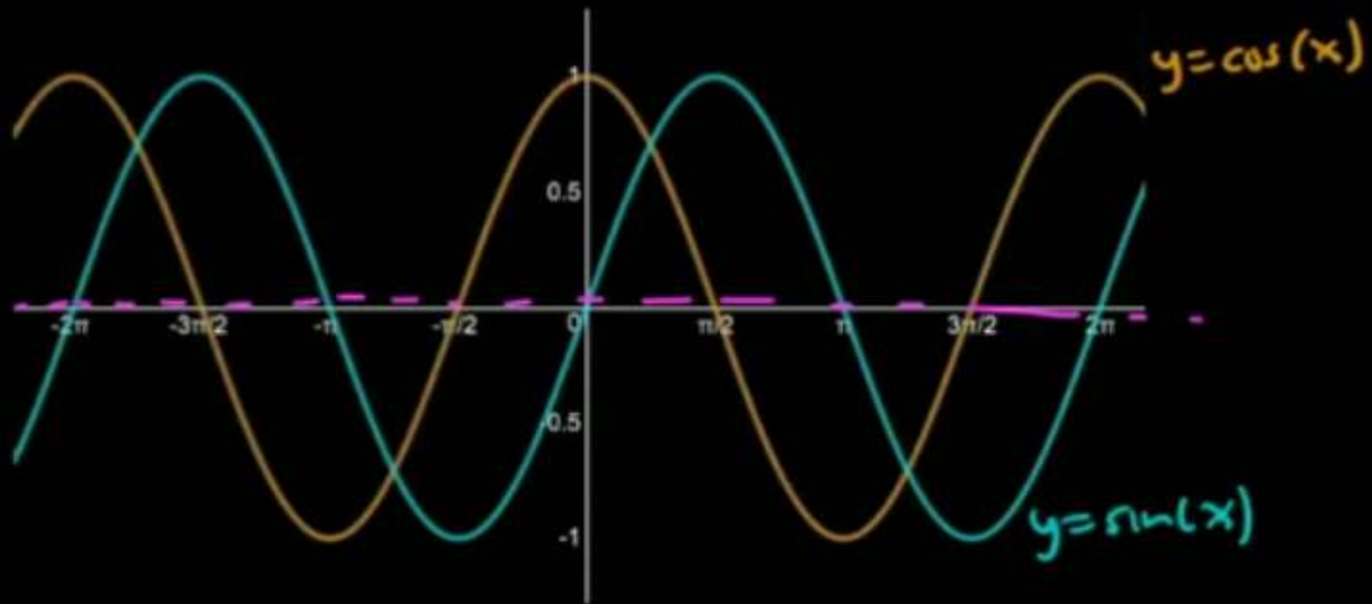
$$\cos(t) = \frac{\text{adj}}{\text{hyp}} = \frac{-\sqrt{45}}{7}$$

The graph of $y = \cos(x)$ is the same as the graph of $y = \sin(x)$ shifted horizontally by $\frac{\pi}{2}$ to the left

$\cos(x) = \sin(x + \frac{\pi}{2})$ $\qquad$ $\sin(x) = \cos(x - \frac{\pi}{2})$



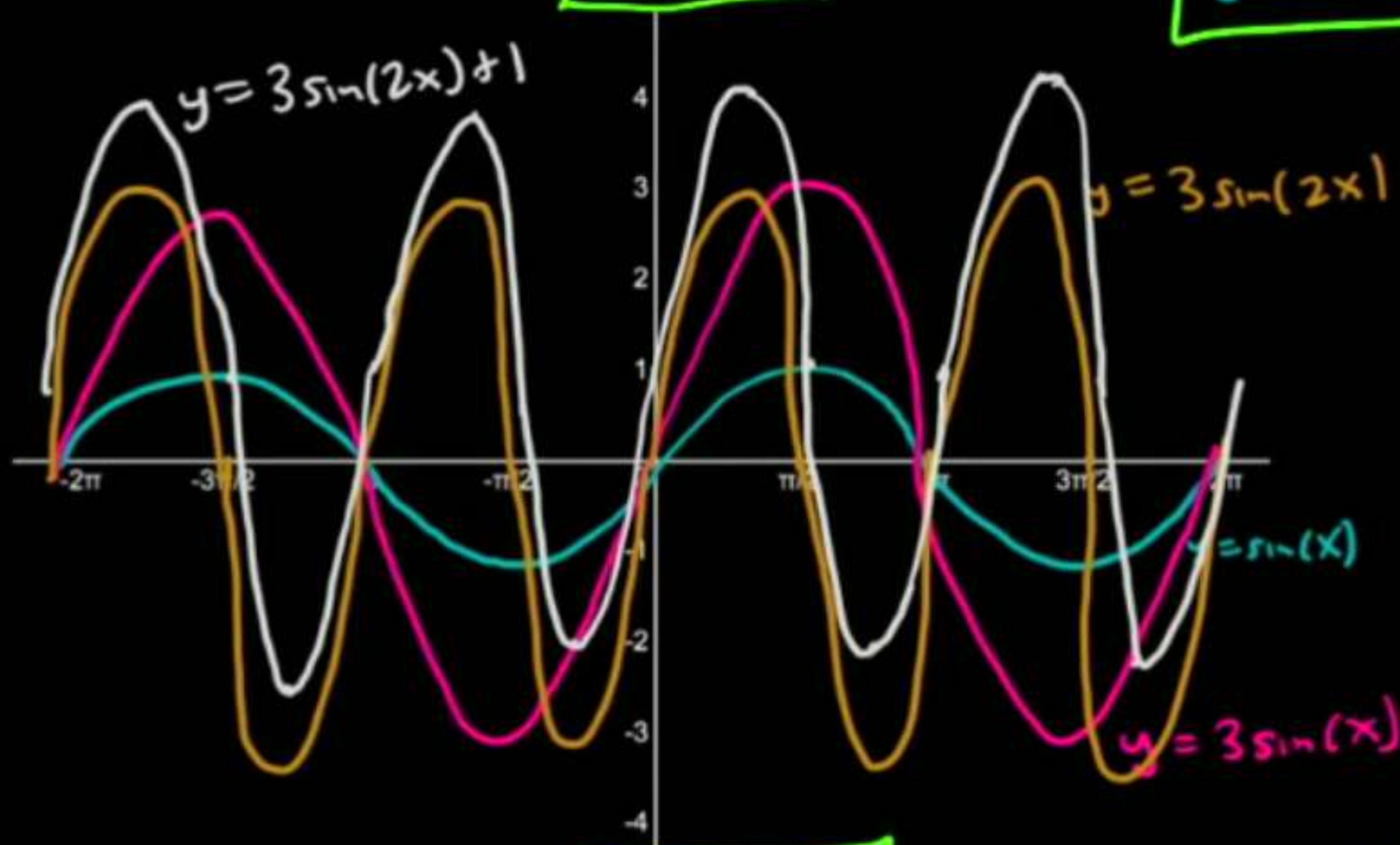
- Domain $(-\infty, \infty)$
- Range $[-1, 1]$
- Even / odd $y = \cos(x)$ is an even fn
  $y = \sin(x)$ is an odd fn
- Abs max and min values
  $\downarrow$ $y = 1$ $\qquad$ $\downarrow$ $y = -1$



- Midline is the horizontal line halfway in between max & min pts
  $y=0$
- Amplitude is the vertical distance between a max point and the midline
  amplitude is 1
- Period is the horizontal length of the smallest repeating unit
  period is 2π
  $\cos(x+2\pi) = \cos(x)$
  $\sin(x+2\pi) = \sin(x)$

Example. Graph the function $\boxed{y = 3\sin(2x)}$

$\boxed{y = \sin(x)}$



Example. Graph the function $\boxed{y = 3\sin(2x) + 1}$

	midline	amplitude	period
$y=\sin(x)$	$y=0$	1	2π
$y=3\sin(2x)$	$y=0$	3	$\frac{1}{2}\cdot 2\pi = \pi$
$y=3\sin(2x)+1$	$y=1$	3	π

$y = 3\sin(2x - \frac{\pi}{2})$

Example. Graph the function $y = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

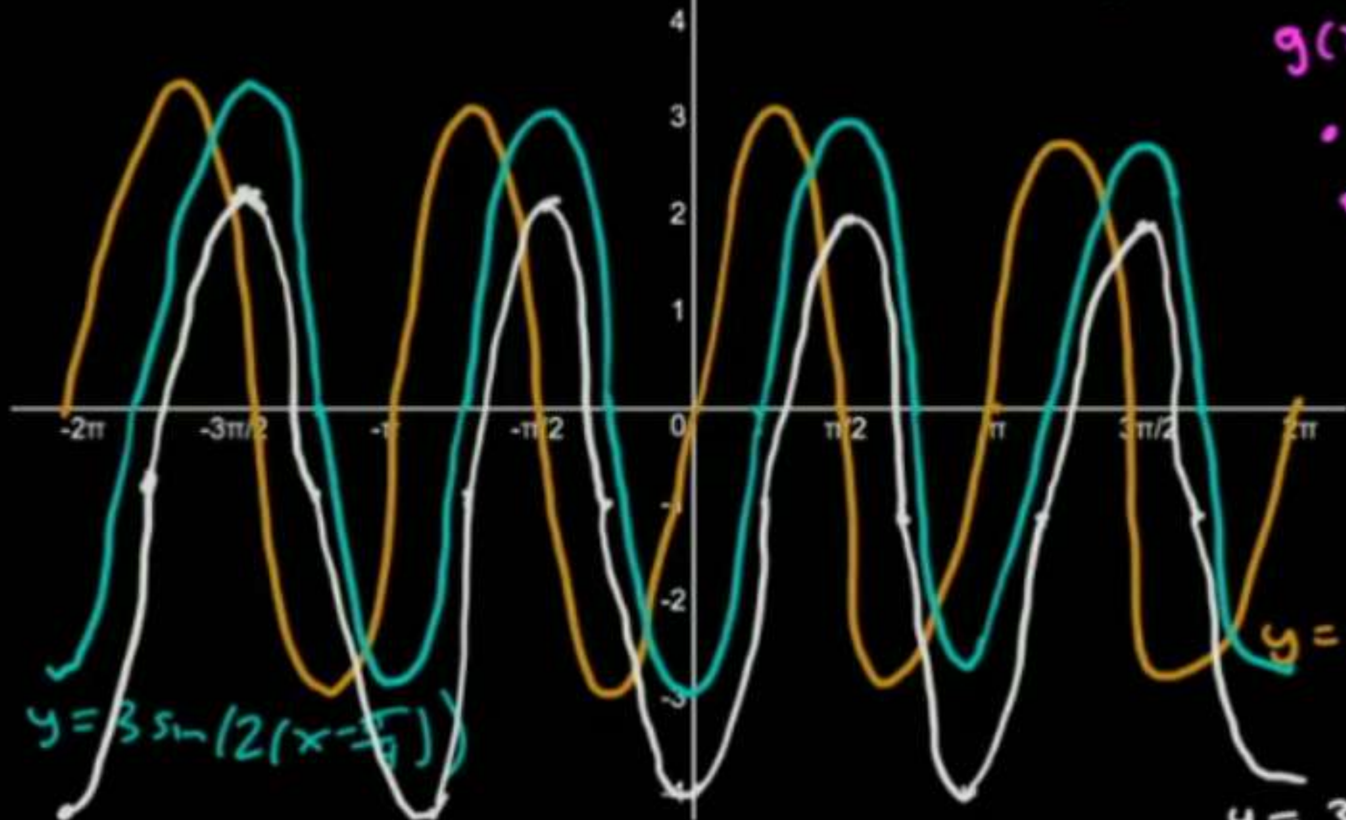
$y = 3\sin(2x)$

$f(x) = 3\sin(2x)$

$g(x) = f(x - \frac{\pi}{4})$

- Graph $f(x)$
- shift its graph right by $\frac{\pi}{4}$

$g(x) = 3\sin(2(x - \frac{\pi}{4}))$



$y = 3\sin(2(x - \frac{\pi}{4})) - 1$

Example. Graph the function $y = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right) - 1$

	midline	amplitude	period	phase shift / horiz shift
$y = \sin(x)$	$y = 0$	1	2π	
$y = 3\sin(2(x - \frac{\pi}{4})) - 1$	$y = -1$	3	$\frac{1}{2} \cdot 2\pi = \pi$	right by $\frac{\pi}{4}$

Summary

$y = 3\sin(2(x - \frac{\pi}{4})) - 1$

For the graphs of $y = A\cos(Bx - C) + D$ and $y = A\sin(Bx - C) + D$ with B positive.

$y = 3\sin(2x - \frac{\pi}{2}) - 1$

- midline

  $y = D$

- amplitude

  $|A|$

  $y = A\cos(Bx - C) + D$

  $y = A\cos(B(x - \frac{C}{B})) + D$

- period

  $\frac{1}{B} \cdot 2\pi \rightsquigarrow \frac{2\pi}{B}$

  $y = A\sin(B(x - \frac{C}{B})) + D$

- horizontal shift $\leftrightarrow$ phase

  $\frac{C}{B}$ to the right if $\frac{C}{B}$ is pos
  to the left if $\frac{C}{B}$ is neg

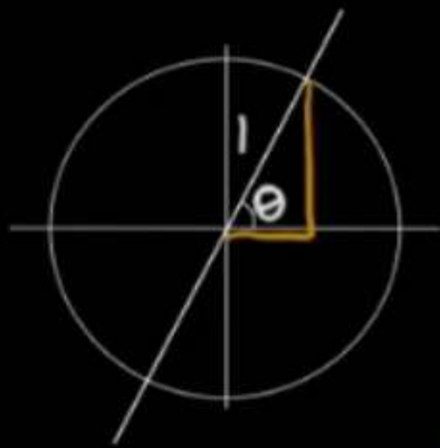
$y = \frac{1}{3}\cos(\frac{1}{2}x + 3) - 5 \rightsquigarrow y = \frac{1}{3}\cos(\frac{1}{2}(x + 6)) - 5$

midline: $y = -5$ ampl: $\frac{1}{3}$ period: $\frac{2\pi}{\frac{1}{2}} = 4\pi$

horiz shift: 6 units left

Graph $y = \tan(x)$

Example. What is the slope of the line at angle θ drawn below?



rise $= \sin\theta$

run $= \cos\theta$

slope $= \frac{\text{rise}}{\text{run}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$y = \tan(x)$

y: slope of the line at that angle

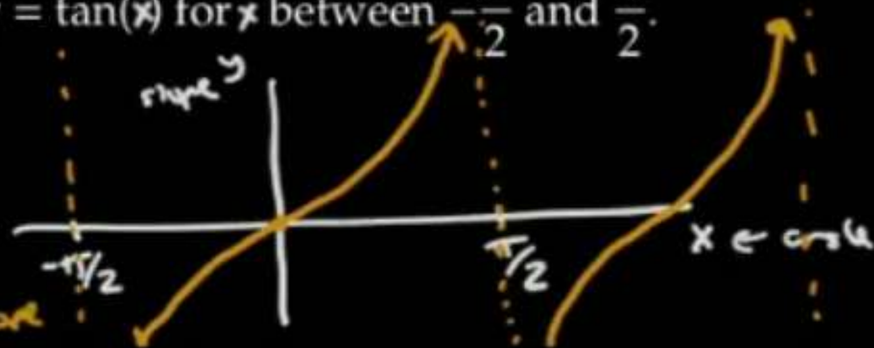
x: angle

- When the angle is zero, the slope is ... 0
- As the angle increases toward $\frac{\pi}{2}$, the slope goes to ... ∞
- As the angle goes from zero towards $-\frac{\pi}{2}$, the slope goes to ... $-\infty$
- At exactly $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, the slope is ... undefined

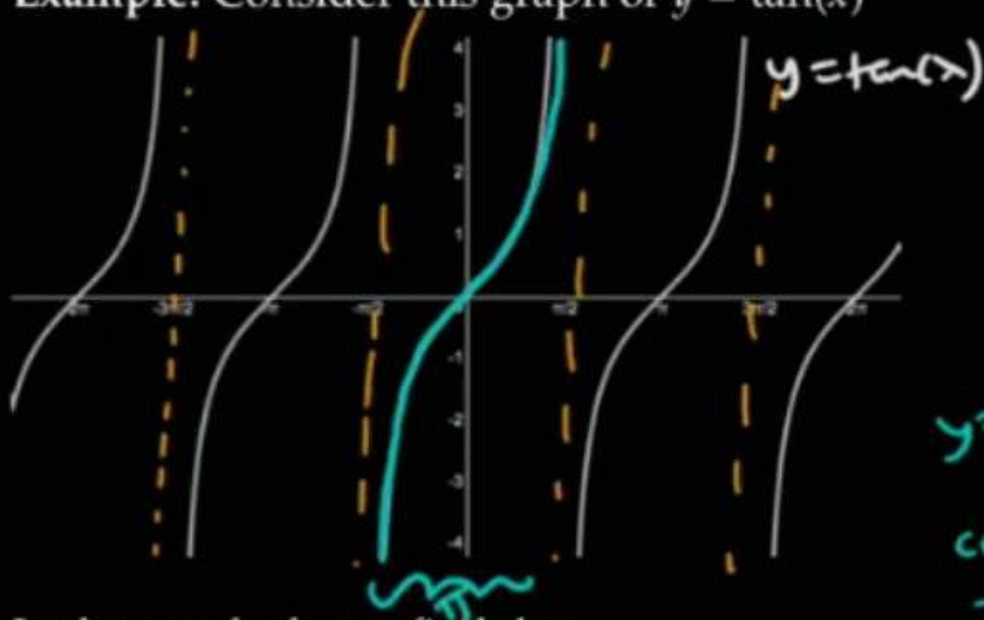
Sketch a rough graph of $y = \tan(x)$ for x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

periodic with period π

b/c rotating a line by π (which is 180°) gives the same line with the same slope



Example. Consider this graph of $y = \tan(x)$



$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$

sin(x) is 0 at 0, π, 2π, ...

cos(x) is 0 at −π/2, π/2, 3π/2 ...

In the graph above, find the:

x-intercepts: ... −2π, −π, 0, π, 2π, ... π·k where k is an integer

vertical asymptotes: ... $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ ... $\frac{\pi}{2}\cdot k$ where k is an odd integer

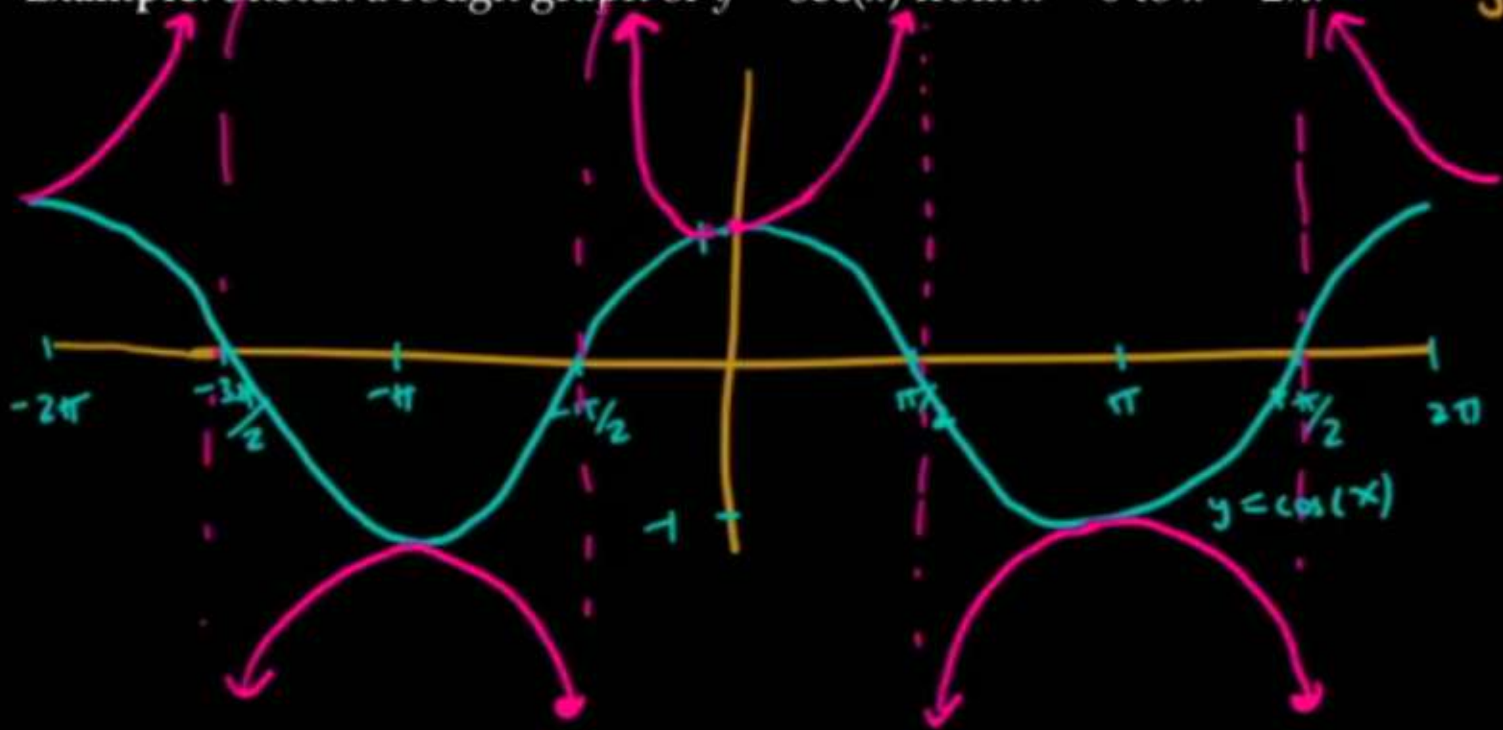
domain: $\{x \mid x \neq \frac{\pi}{2}\cdot k$ for k odd integer$\}$

range: $(-\infty, \infty)$

period: π

Example. Sketch a rough graph of $y = \sec(x)$ from $x = 0$ to $x = 2\pi$.

$y = \frac{1}{\cos(x)}$



x-intercepts: None

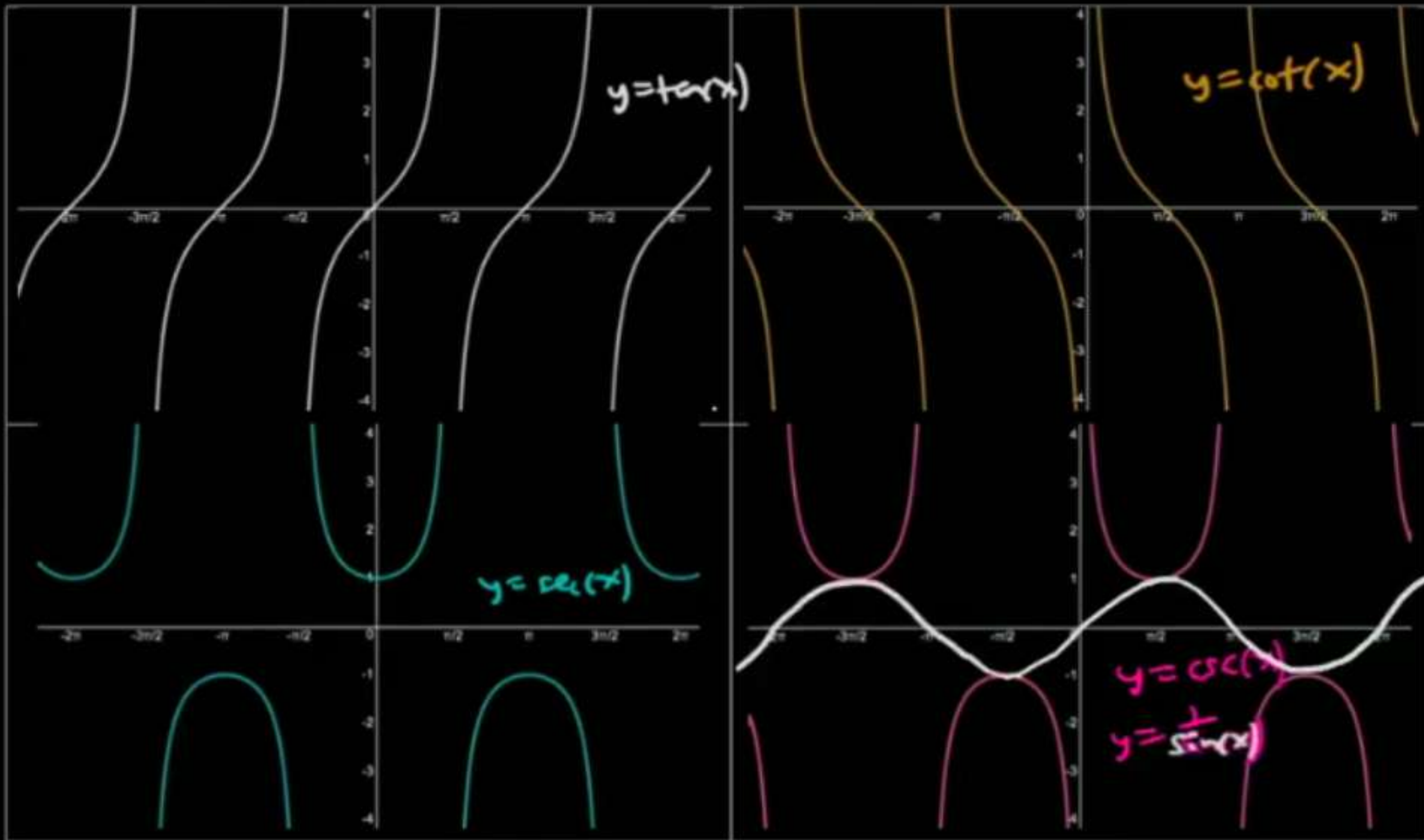
vertical asymptotes: ← when $\cos(x) = 0$; $\pi/2, 3\pi/2, \ldots$ $\frac{\pi}{2} \cdot k$ where k is an odd integer

domain: $\{x \mid x \neq \frac{\pi}{2}k$ for k odd integer$\}$

range: $(-\infty, -1] \cup [1, \infty)$

period: 2π

Here are graphs of $y = \tan(x)$, $y = \cot(x)$, $y = \sec(x)$ and $y = \csc(x)$.



Example. Sketch a graph of the function $y = 2\csc(\pi x + \frac{\pi}{2}) + 1$.

vertical stretch by 2

shrink horiz by $\frac{1}{\pi}$

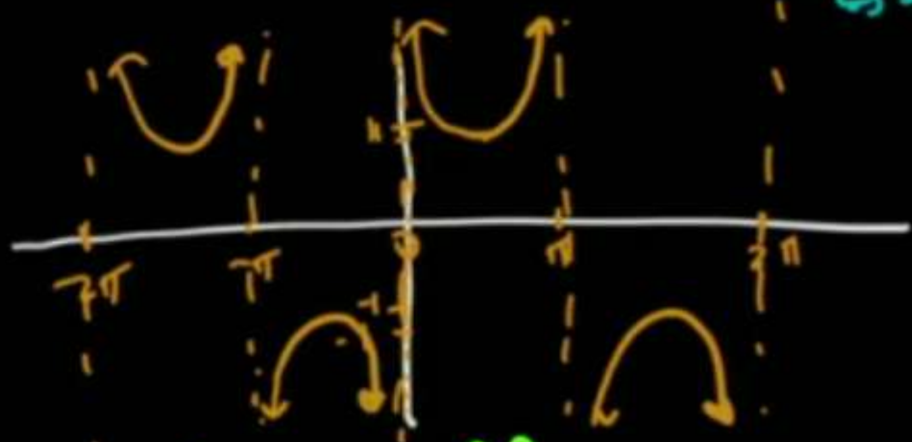
period $= \frac{2\pi}{\pi} = 2$

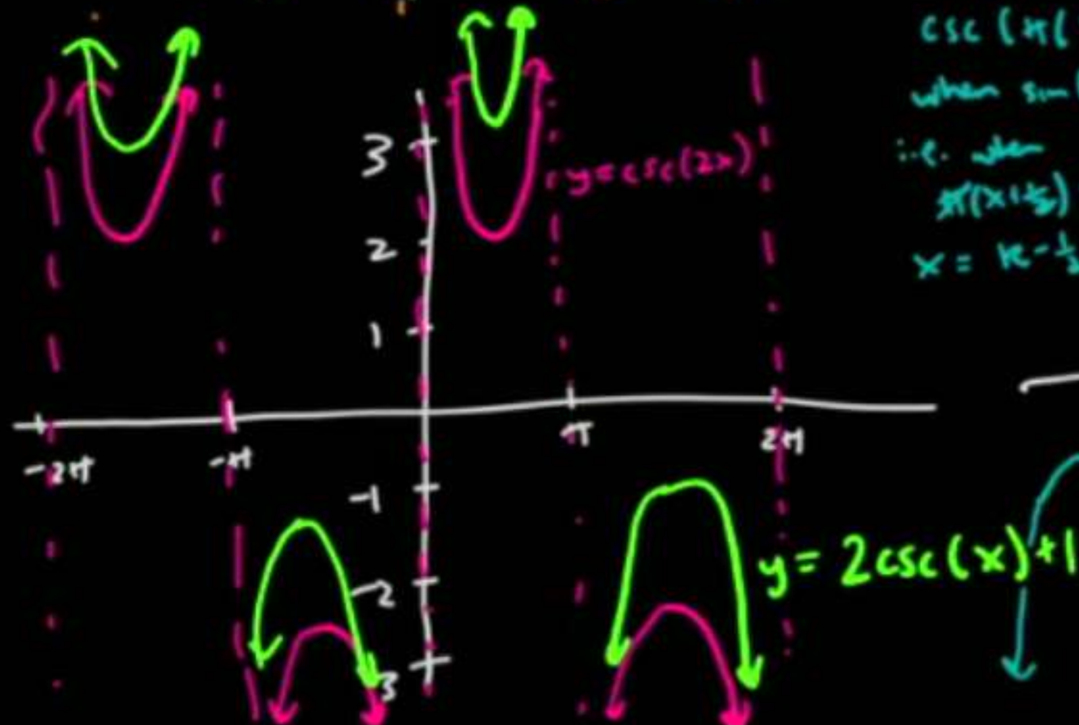
shifts up by 1

indeed horiz shift but NOT by $\frac{\pi}{2}$

horiz shift by $\frac{1}{2}$ to the left

$y = \csc(x) = \frac{1}{\sin(x)}$





$y = 2\csc(\pi(x+\frac{1}{2}))+1$

find V. A.

$\csc = \frac{1}{\sin}$

$\csc(\pi(x+\frac{1}{2}))$ DNE

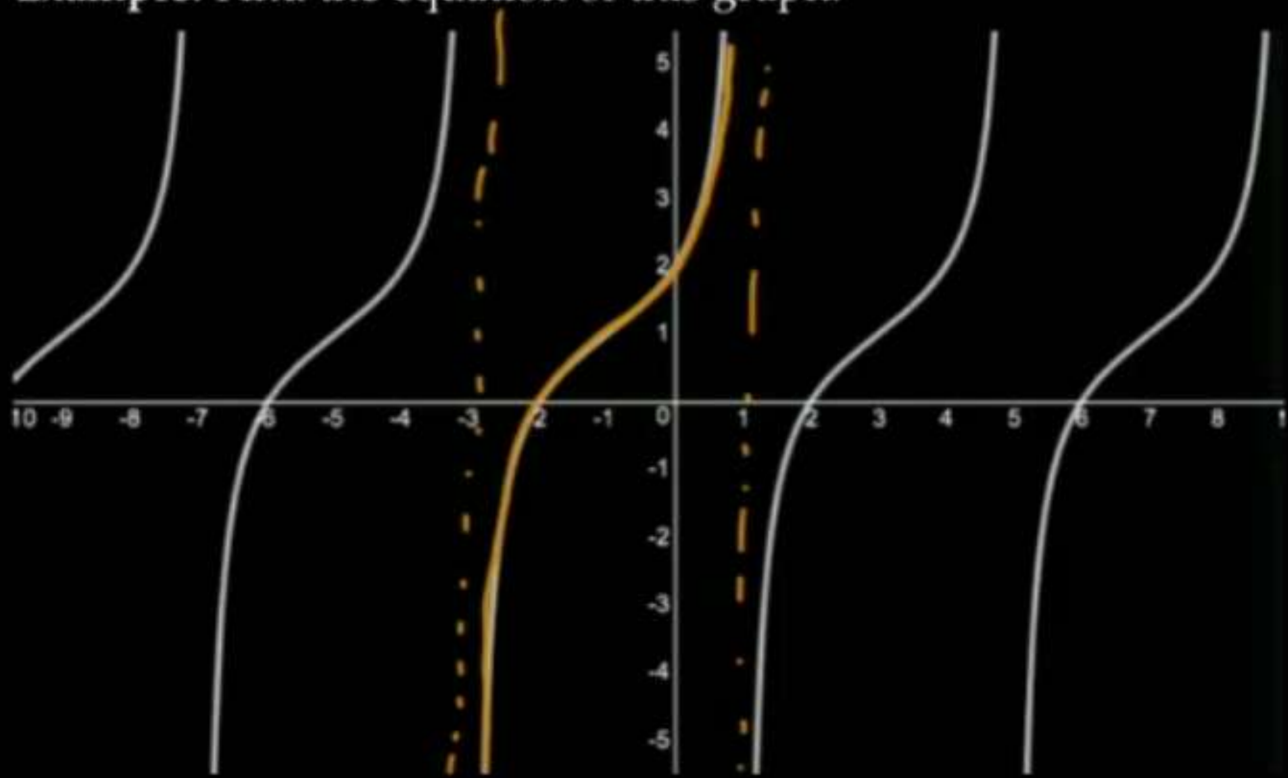
when $\sin(\pi(x+\frac{1}{2})) = 0$

i.e. when

$\pi(x+\frac{1}{2}) = k\pi$

$x = k - \frac{1}{2}$

Example. Find the equation of this graph.



$y = \tan(x)$
has period of π

our graph has a period of 4

$y = \tan(Bx)$

$\frac{\pi}{B} = 4$

$\Rightarrow \pi = 4B$

$\Rightarrow B = \frac{\pi}{4}$

$y = \tan\left(\frac{\pi}{4}x\right)$

$\boxed{y = \tan\left(\frac{\pi}{4}(x+1)\right) + 1}$

$y = A\tan\left(\frac{\pi}{4}(x+1)\right) + 1$

plug in $(0,2)$

$2 = A\tan\left(\frac{\pi}{4}(0+1)\right) + 1$

solve for A $\quad 2 = A \cdot 1 + 1 \Rightarrow A = 1$

* Assume $B > 0$

For the function $y = A\tan(Bx - C) + D$, how do the numbers A, B, C, and D affect the graph?

vertical stretch of $|A|$

changes period from π to $\frac{\pi}{B}$

shifts vertically by D

orig period of tan, cot is π

$$y = A\tan\left(B\left(x - \frac{C}{B}\right)\right) + D$$

phase shift right by $\frac{C}{B}$

same for $y = A\cot(Bx - C) + D$

* Assume $B > 0$

For the function $y = A\sec(Bx - C) + D$, how do the numbers A, B, C, and D affect the graph?

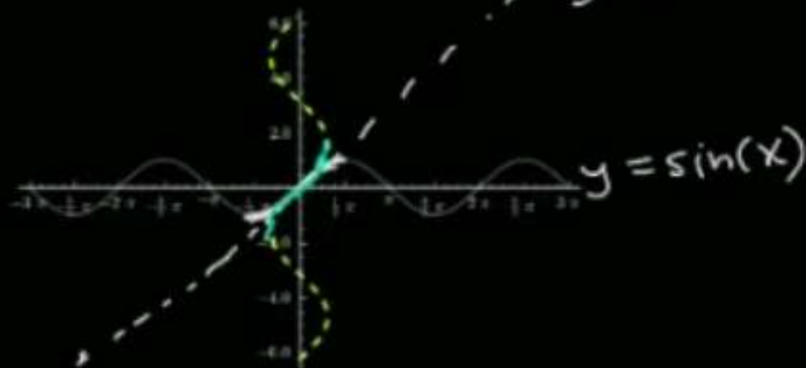
OR

$$y = A\csc(Bx - C) + D$$

orig period of sec, csc is 2π

B changes period from 2π to $\frac{2\pi}{B}$

Inverse Sine Function



$\pi/2 \xrightarrow{\sin} 1$

$\theta \xrightarrow{\sin} x$

angles $\xleftarrow{\arcsin}$ numbers

$\pi/2 \longleftarrow 1$

Restricted sin(x) **has** **Domain:** $[-\pi/2, \pi/2]$ **Range:** $[-1, 1]$

Inverse sine function
arcsin(x) has Domain: $[-1, 1]$ Range: $[-\pi/2, \pi/2]$

arcsin(x) is the (circle one) angle / number between $-\pi/2$ and $\pi/2$ whose ... sine is x.

y = arcsin(x) means: $x = \sin(y)$

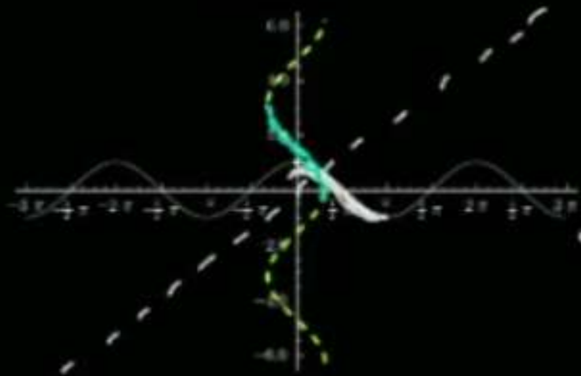
and $-\pi/2 \le y \le \pi/2$

Alternative Notation: $\sin^{-1}(x)$ means $\arcsin(x)$, the inverse sine function.

Warning: ⚠ $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$ $\frac{1}{\sin(x)} = \csc(x)$ $\sin^{-1}(x) = \arcsin(x)$ inverse sine fn

← not the same →

Inverse Cosine Function



$y = \cos(x)$

$\frac{\pi}{4} \xrightarrow{\text{cosine}} \frac{\sqrt{2}}{2}$

$\theta \rightarrow x$

angles $\xleftarrow{\text{arccos}}$ numbers

$\frac{\pi}{4} \leftarrow \frac{\sqrt{2}}{2}$

Restricted cos(x) **has** **Domain:** $[0, \pi]$ **Range:** $[-1, 1]$

arccos(x) has Domain: $[-1, 1]$ Range: $[0, \pi]$

arccos(x) is the (circle one) angle / number between 0 and π whose ... cosine is x

y = arccos(x) means: $x = \cos(y)$

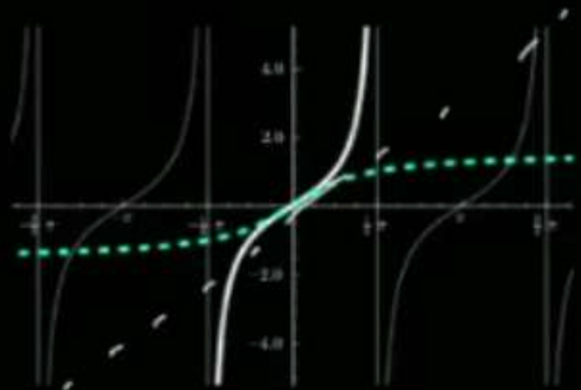
and $0 \leq y \leq \pi$

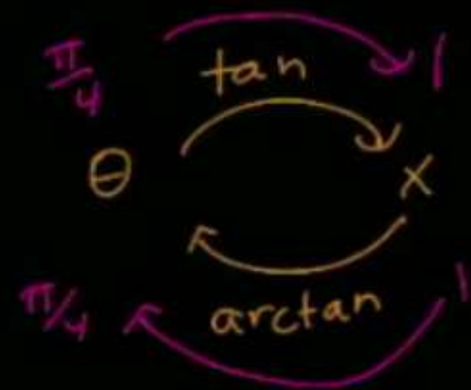
Alternative Notation: $\cos^{-1}(x)$ means $\arccos(x)$

Warning: ⚠ $\cos^{-1}(x) \neq \frac{1}{\cos(x)}$ $\frac{1}{\cos(x)} = \sec(x)$ $\cos^{-1}(x) = \arccos(x)$

not the same

Inverse Tangent Function





Restricted $\tan(x)$ **has** **Domain:** $(-\frac{\pi}{2}, \pi/2)$ **Range:** $(-\infty, \infty)$

arctan(x) has Domain: $(-\infty, \infty)$ Range: $(-\frac{\pi}{2}, \pi/2)$

arctan(x) is the (circle one) angle / number between $-\pi/2$ and $\frac{\pi}{2}$ whose ... tan is x.

$y = \arctan(x)$ means: $x = \tan(y)$ and $-\pi/2 < y < \pi/2$

Alternative Notation: $\tan^{-1}(x)$ means $\arctan(x)$,

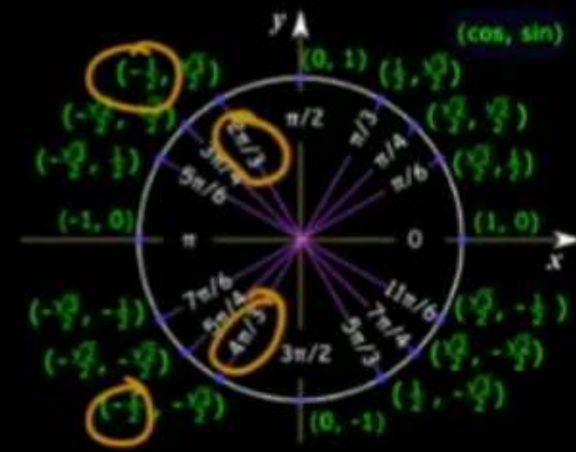
Warning: $\tan^{-1}(x)$ means the inverse trig fn $\arctan(x)$

not equal to $\frac{1}{\tan(x)}$ which is $\cot(x)$

Example. For the equation $2\cos(x) + 1 = 0$,

(a) Find the solutions in the interval $[0, 2\pi)$.

$$2\cos(x) = -1$$
$$\cos(x) = -\frac{1}{2}$$
$$\boxed{x = 2\pi/3,\ 4\pi/3}$$



(b) Give a general formula for ALL solutions (not just those in the interval $[0, 2\pi)$.

$$\left.\begin{array}{l} 2\pi/3 + 2\pi k \\ 4\pi/3 + 2\pi k \end{array}\right\}\text{ for } k \text{ any integer}$$

$$\boxed{x = \frac{2\pi}{3} + 2\pi k,\ \frac{4\pi}{3} + 2\pi k \quad \text{for } k \text{ any integer}}$$

Example. For the equation $2\cos(t) = 1 - \cos(t)$

(a) Find all solutions in the interval $[0, 2\pi)$ ∇ use radian mode

$3\cos(t) = 1$

$\cos(t) = \frac{1}{3}$ ← has ∞ many solutions

not quite the same

$\arccos(\cos(t)) = \arccos(\frac{1}{3})$

$t = \arccos(\frac{1}{3}) \approx 1.2310$

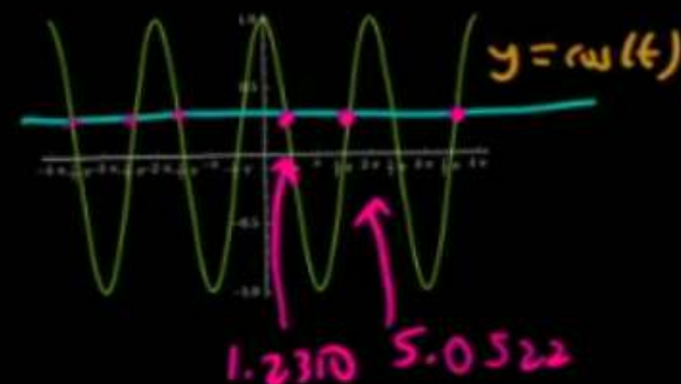
↖ has one solution lies between 0 and π

The other solution in $[0, 2\pi)$ is given by $2\pi - 1.2310 = 5.0522$

$x = 1.2310, 5.0522$ Find the second solution by 2π − (first solution)

(b) Find all solutions.

$x = 1.2310 + 2\pi k,\ 5.0522 + 2\pi k$



Example. For the equation $4\sin(x) - 1 = 2$

(a) Find all solutions in the interval $[0, 2\pi)$

$4\sin(x) = 3$

$\sin(x) = \frac{3}{4}$

$\arcsin(\sin(x)) = \arcsin\left(\frac{3}{4}\right)$

$x = \arcsin\left(\frac{3}{4}\right) = 0.8481$

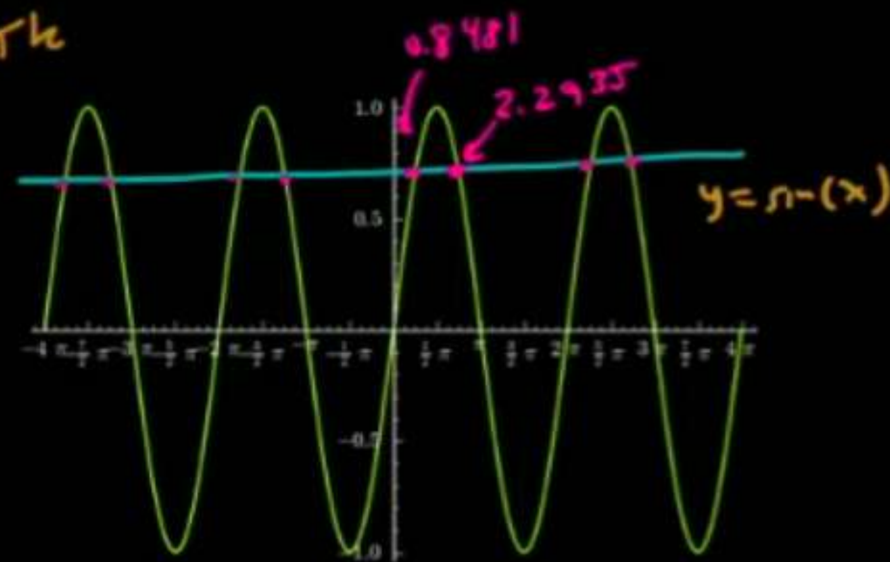
or $\pi - 0.8481 = 2.2935$

$\boxed{x = 0.8481,\ 2.2935}$

To get second angle, use $\pi -$ (first angle)

(b) Find all solutions.

$x = 0.8481 + 2\pi k,\ 2.2935 + 2\pi k$



Example. For the equation $\tan(x) = 4$ period is π

(a) Find all solutions in the interval $[0, 2\pi)$

$x = \tan^{-1}(4) = 1.3258$

or

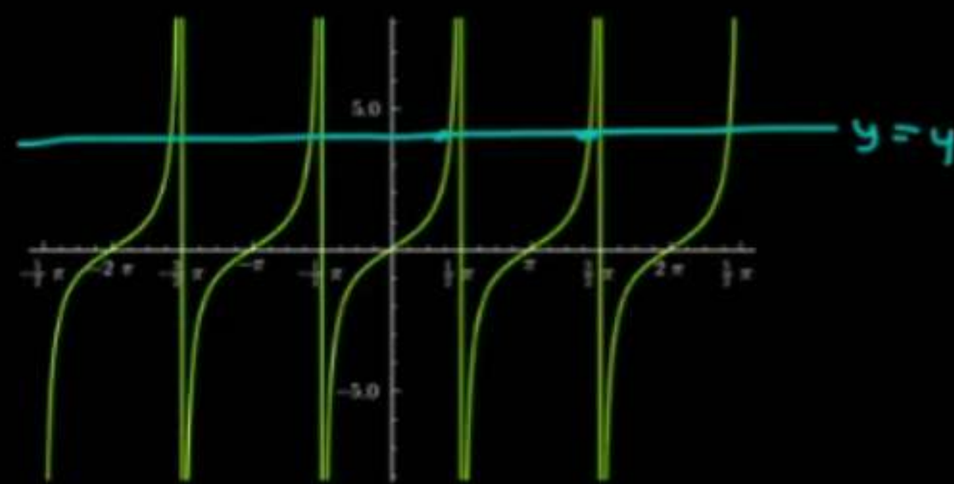
$\pi + 1.3258 = 4.4674$

$x = 1.3258, 4.4674$

(b) Find all solutions.

$x = 1.3258 + 2\pi k, 4.4674 + 2\pi k$

$x = 1.3258 + \pi k$



Summary:

(a) To find solutions to the equation $\cos(t) = M$

$$t = \boxed{\begin{array}{l} \cos^{-1}(M) + 2\pi k \\ 2\pi - \cos^{-1}(M) + 2\pi k \end{array}}$$

(b) To find solutions to the equation $\sin(t) = N$

$$t = \boxed{\begin{array}{l} \sin^{-1}(N) + 2\pi k \\ \pi - \sin^{-1}(N) + 2\pi k \end{array}}$$

(c) To find solutions to the equation $\tan(t) = P$

$$t = \begin{array}{l} \tan^{-1}(P) + 2\pi k \\ \tan^{-1}(P) + \pi + 2\pi k \end{array} \quad \text{OR} \quad t = \boxed{\tan^{-1}(P) + \pi k}$$

Example. Find solutions to the following equations:

a) $x^2 - 6x = 7$

$x^2 - 6x - 7 = 0$

$(x-7)(x+1) = 0$

$x - 7 = 0$ or $x + 1 = 0$

$x = 7$ or $x = -1$

Not a Identity, because it hold for some values and not for all

b) $x^2 - 6x = 7 + (x-7)(x+1)$

$x^2 - 6x = \cancel{7} + x^2 + x - 7x \cancel{- 7}$

$x^2 - 6x = x^2 - 6x$

solutions: all real #'s

Definition. The second equation is called an *identity* because ... it holds for all values of the variable.

Example. Decide which of the following equations are identities.

a) $\sin(2x) = 2\sin(x)$ Not an identity

$x = 0$

$\sin(2 \cdot 0) = 2\sin(0)$

$0 \quad 0$

$x = \frac{\pi}{2}$

$\sin(2 \cdot \pi/2) \neq 2\sin(\frac{\pi}{2})$

$0 \quad 2$

b) $\cos(\theta + \pi) = -\cos(\theta)$ is an identity

Evidence:

$\cos(0+\pi) = -\cos(0)$

$\cos(\frac{\pi}{6}+\pi) = -\cos(\frac{\pi}{6})$

more evidence

c) $\sec(x) - \sin(x)\tan(x) = \cos(x)$ is an identity

$$\sec(x) - \sin(x)\tan(x) = \frac{1}{\cos(x)} - \sin(x) \cdot \frac{\sin(x)}{\cos(x)}$$

$$= \frac{1}{\cos(x)} - \frac{\sin^2(x)}{\cos(x)}$$

$$= \frac{1-\sin^2(x)}{\cos(x)} = \frac{\cos^2(x)}{\cos(x)} = \cos(x)$$

$\cos^2(x) + \sin^2(x) = 1$

$1 - \sin^2(x) = \cos^2(x)$

To prove that an equation is an identity

- use algebra
- use other identities like the Pythag ident.

rewrite one side of eqn to get other side

To prove that an eqn is not an identity

- plug in numbers that "break it"

To decide if an eqn is an identity

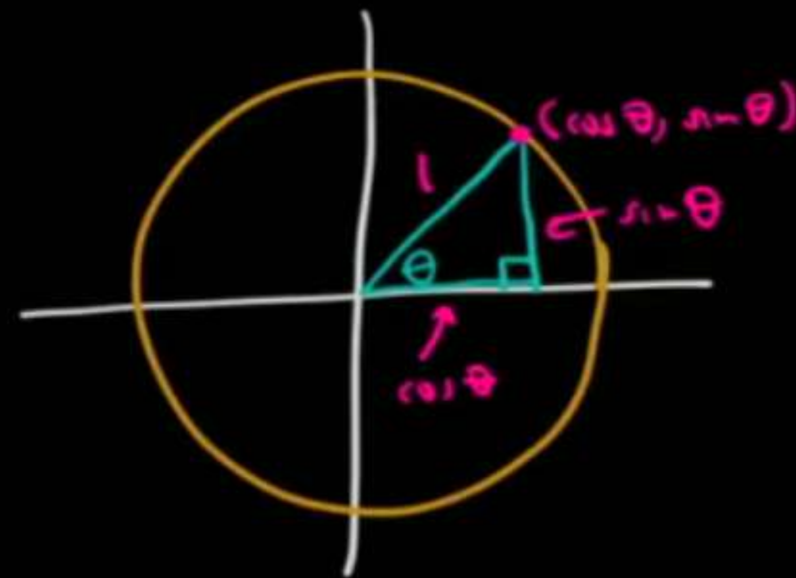
- plug in numbers
- graph left & right sides

$\cos^2(\theta) + \sin^2(\theta) = 1$

By Pythagorean Thm

$(\cos\theta)^2 + (\sin\theta)^2 = 1^2$

$\cos^2\theta + \sin^2\theta = 1$



$\tan^2(\theta) + 1 = \sec^2(\theta)$

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$1 + \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 = \left(\frac{1}{\cos(\theta)}\right)^2$$

$$1 + (\tan(\theta))^2 = (\sec(\theta))^2$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$\cot^2(\theta) + 1 = \csc^2(\theta)$

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2\theta}$$

$$\frac{\cos^2(\theta)}{\sin^2(\theta)} + \frac{\sin^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$\left(\frac{\cos(\theta)}{\sin(\theta)}\right)^2 + 1 = \left(\frac{1}{\sin(\theta)}\right)^2$$

$$(\cot(\theta))^2 + 1 = (\csc(\theta))^2$$

$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

Question. Is it true that $\sin(A+B) = \sin(A) + \sin(B)$? No

For example: $A = \pi/2 \quad B = \pi$

$$\sin\left(\frac{\pi}{2} + \pi\right) \neq \sin\left(\frac{\pi}{2}\right) + \sin(\pi)$$

$$\sin\left(\frac{3\pi}{2}\right) \qquad 1 + 0$$

$$-1 \neq 1$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

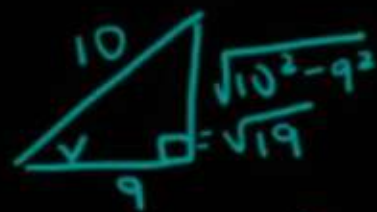
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

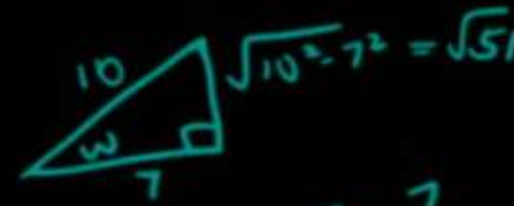
$$\begin{aligned}\sin(A-B) = \sin(A+(-B)) &= \sin(A)\cos(-B) + \cos(A)\sin(-B)\\ &= \sin(A)\cos(B) + \cos(A)(-\sin(B))\\ &= \sin(A)\cos(B) - \cos(A)\sin(B)\end{aligned}$$

$$\begin{aligned}\cos(A-B) = \cos(A+(-B)) &= \cos(A)\cos(-B) - \sin(A)\sin(-B)\\ &= \cos(A)\cos(B) + \sin(A)\sin(B)\end{aligned}$$

Example. If $\cos(v) = 0.9$ and $\cos(w) = 0.7$, find $\cos(v + w)$. Assume v and w are in the first quadrant.

$$\cos(v+w) = \cos(v)\cos(w) - \sin(v)\sin(w)$$





$$\cos(v) = 0.9 = \frac{9}{10} = \frac{\text{adj}}{\text{hyp}}$$

$$\cos(w) = 0.7 = \frac{7}{10}$$

$$\sin(v) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{19}}{10}$$

$$\sin(w) = \frac{\sqrt{51}}{10}$$

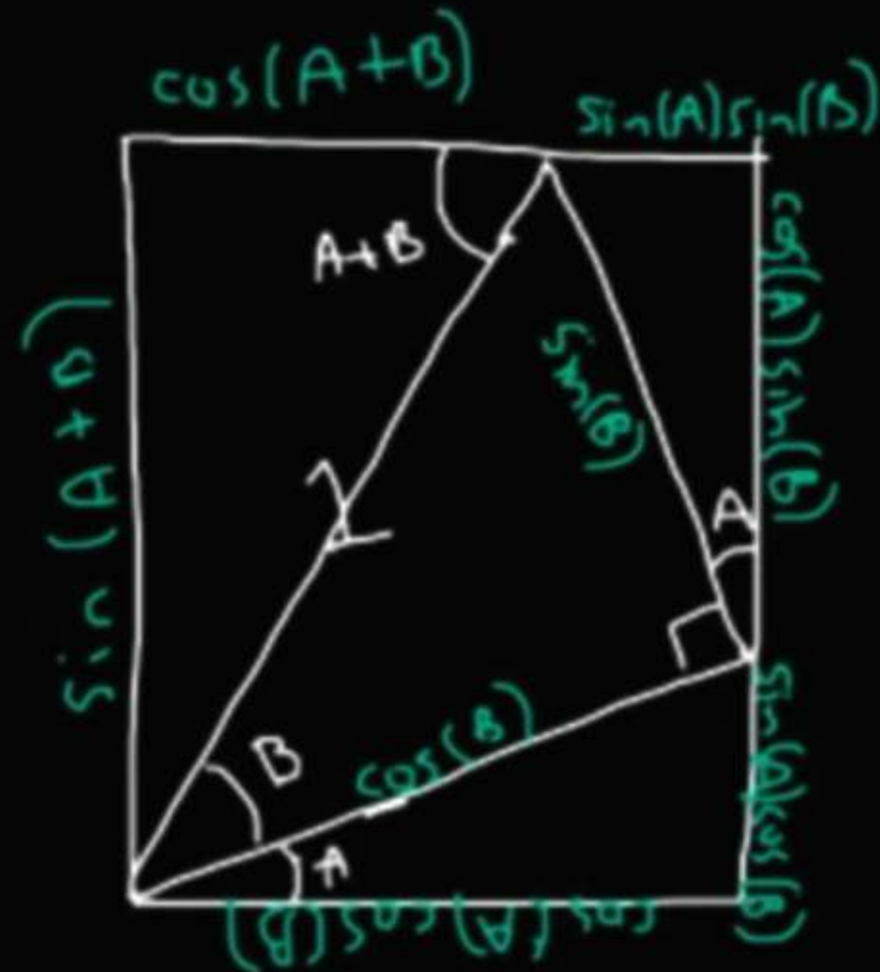
$$\cos(v+w) = 0.9 \cdot 0.7 - \frac{\sqrt{19}}{10} \cdot \frac{\sqrt{51}}{10}$$

$$= 0.3187$$

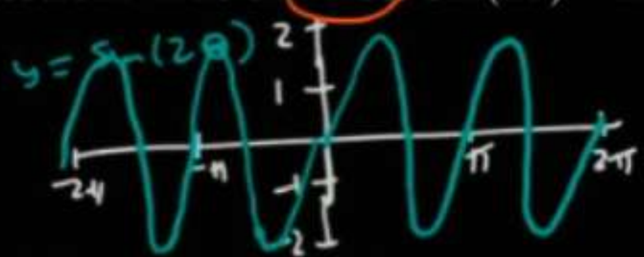
Proof of the angle sum formular:

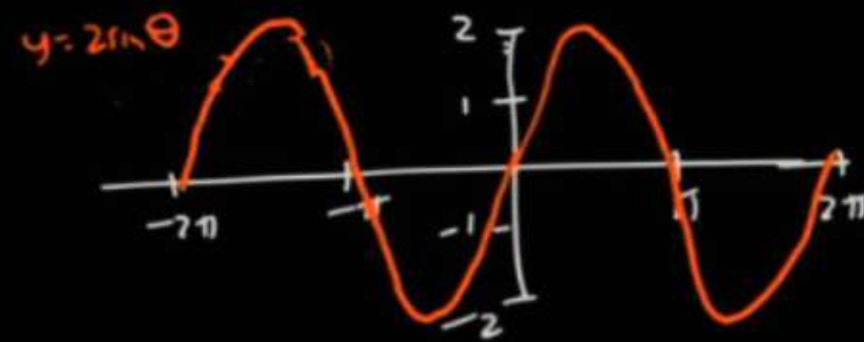
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$



Question. True or False: $\sin(2\theta) = 2\sin(\theta)$





Double Angle Formulas:

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

$$\boxed{\cos 2\theta = 1 - 2\sin^2\theta}$$

$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

$$\boxed{\cos 2\theta = 2\cos^2\theta - 1}$$

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta)$$
$$\cos 2\theta = 2\cos^2\theta - 1$$

Example. Find $\cos(2\theta)$ if $\cos(\theta) = -\dfrac{1}{\sqrt{10}}$ and θ terminates in quadrant III.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1}$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\cos(2\theta) = 2\left(-\frac{1}{\sqrt{10}}\right)^2 - 1 = \frac{2}{10} - 1 = -\frac{8}{10} = -\frac{4}{5}$$

Example. Solve the equation $2\cos(x) + \sin(2x) = 0$.

$$2\cos(x) + 2\sin(x)\cos(x) = 0$$

$$2\cos(x)\left(1 + \sin(x)\right) = 0$$

$2\cos(x) = 0$ or $1 + \sin(x) = 0$

$\cos(x) = 0$ or $\sin(x) = -1$

$$x = \frac{\pi}{2} + 2\pi k, \quad \frac{3\pi}{2} + 2\pi k$$

Half Angle Formulas:

$$\boxed{\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}}$$

$$\cos(2x) = \cancel{\cos^2(x) - \sin^2(x)}$$

① $\cos(2x) = 2\cos^2 x - 1$

$\theta = 2x \quad x = \frac{\theta}{2}$

$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$\cos\theta + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$

$\frac{\cos\theta + 1}{2} = \cos^2\left(\frac{\theta}{2}\right)$

$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta + 1}{2}}$

② $\cos(2x) = 1 - 2\sin^2 x$

$\theta = 2x \quad x = \frac{\theta}{2}$

$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

$\cos\theta + 2\sin^2\left(\frac{\theta}{2}\right) = 1$

$2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos\theta$

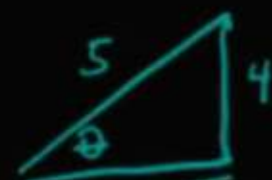
$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$

$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$

Example. Suppose that $\sin(\theta) = \frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$. Find the exact values of $\cos\left(\frac{\theta}{2}\right)$ ← positive and $\sin\left(\frac{\theta}{2}\right)$. ← positive

$$\cos\frac{\theta}{2} = + \sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = + \sqrt{\frac{1-\cos\theta}{2}}$$



$$\sqrt{5^2-4^2} = \sqrt{9} = 3$$

$$|\cos\theta| = \frac{3}{5} \quad \frac{\text{adj}}{\text{hyp}}$$

$$\cos\theta = -\frac{3}{5}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{2}{5}}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

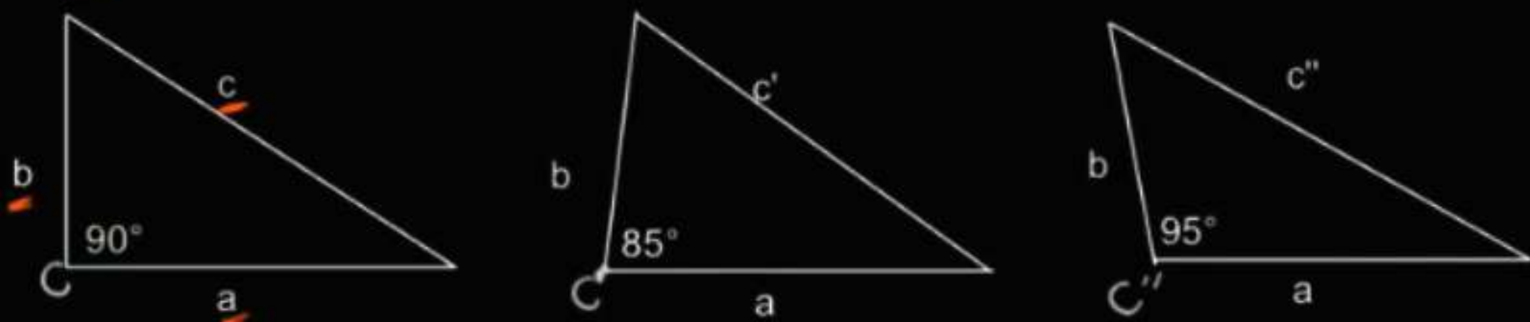
Pythagorean Theorem $\longleftrightarrow$ Law of cosines

right triangle

$c^2 = a^2 + b^2$

$c^2 = a^2 + b^2 +$ correction factor

triangles that are not necessarily right triangles



$c^2 = a^2 + b^2$

$(c')^2 < a^2 + b^2$

$(c')^2 = a^2 + b^2 - (\text{a bit})$

$(c'')^2 > a^2 + b^2$

$(c'')^2 = a^2 + b^2 + (\text{a bit})$

Theorem. *(The Law of Cosines) For any triangle with sides a, b, and c and angle C opposite side c,*

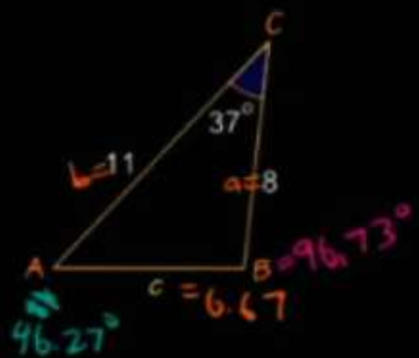
$$c^2 = a^2 + b^2 - 2ab\cos C$$

if $C = 90°$, then $\cos C = 0$ so $c^2 = a^2 + b^2$

if $C < 90°$, then $\cos C > 0$ so $2ab\cos C > 0$

if $C > 90°$, then $\cos C < 0$ so $2ab\cos C < 0$

Example. Find all side lengths and angles of this triangle. SAS



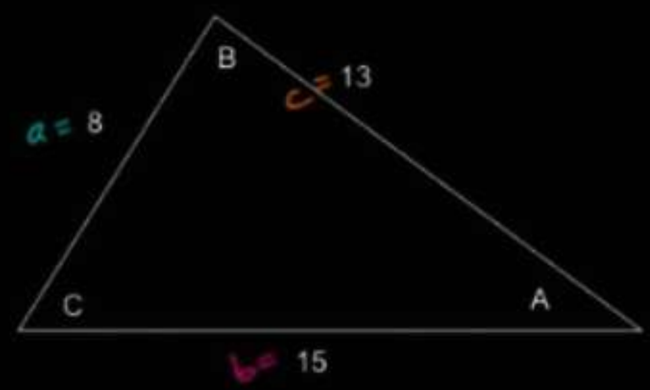
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 8^2 + 11^2 - 2\cdot 8\cdot 11\cdot \cos 37^\circ$$
$$c^2 = 44.44$$
$$c = \sqrt{44.44} = 6.67$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$11^2 = 8^2 + (6.67)^2 - 2(8)(6.67)\cos B$$
$$11^2 - 8^2 - (6.67)^2 = -2(8)(6.67)\cos B$$
$$\frac{11^2 - 8^2 - (6.67)^2}{-2(8)(6.67)} = \cos B$$

$$\cos B = -0.1172$$
$$B = \cos^{-1}(-0.1172) = 96.73^\circ$$

$$A + 37^\circ + 96.73^\circ = 180^\circ$$
$$A = 180^\circ - 37^\circ - 96.73^\circ$$
$$= 46.27^\circ$$

Example. Find the angles of this triangle. SSS



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$13^2 = 8^2 + 15^2 - 2(8)(15)\cos C$$

$$\frac{13^2 - 8^2 - 15^2}{-2(8)(15)} = \cos C \implies \cos C = 0.5$$

$$C = \cos^{-1}(0.5) = 60^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$15^2 = 8^2 + 13^2 - 2(8)(13)\cos B$$

$$\frac{15^2 - 8^2 - 13^2}{-2(8)(13)} = \cos B$$

$$\implies \cos B = 0.0385$$

$$B = \cos^{-1}(0.0385) = 87.79^\circ$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$8^2 = 15^2 + 13^2 - 2(15)(13)\cos A$$

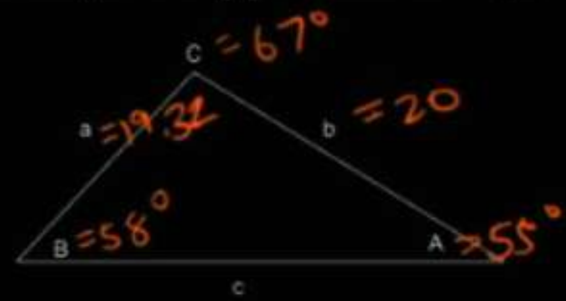
$$\frac{8^2 - 15^2 - 13^2}{-2(15)(13)} = \cos A$$

$$\implies \cos A = 0.8462$$

$$A = \cos^{-1}(0.8462) = 32.20^\circ$$

$$60^\circ + 32.20^\circ + 87.79^\circ \approx 180^\circ$$

Example. Suppose $A = 55^\circ$, $C = 67^\circ$, and $b = 20$. Solve the triangle.



$$B = 180^\circ - 67^\circ - 55^\circ = 58^\circ$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 55^\circ}{a} = \frac{\sin 58^\circ}{20}$$

$$\cancel{a} \cdot \frac{\sin 55^\circ}{\cancel{a}} = a \cdot \frac{\sin 58^\circ}{20}$$

$$\frac{20}{\sin 58^\circ} \sin 55^\circ = a \cdot \frac{\cancel{\sin 58^\circ}}{\cancel{20}} \cdot \frac{\cancel{20}}{\cancel{\sin 58^\circ}}$$

$$a = \frac{20 \sin 55^\circ}{\sin 58^\circ} = 19.32$$

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin 67^\circ}{c} = \frac{\sin 58^\circ}{20}$$

$$\sin 67^\circ = \frac{\sin 58^\circ}{20} \cdot c$$

$$\frac{\sin 67^\circ \cdot 20}{\sin 58^\circ} = c$$

$$c = 21.71$$

Example. (an ambiguous case) Suppose that $a = 8$ and $b = 7$, and $B = 40°$. Solve the triangle, finding all possible solutions.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Ex: if $\sin A = \frac{1}{2}$

then A could be $30°$ or $150°$

$$\frac{\sin A}{8} = \frac{\sin 40°}{7} = \frac{\sin C}{c}$$

$$\sin A = \frac{8 \cdot \sin 40°}{7} = 0.7346$$

$$A = \sin^{-1}(0.7346) = 47.27°$$

$$\text{or } 180° - 47.27° = 132.73°$$

①

$$C = 180° - 40° - 47.27° = 92.73°$$

$$\frac{\sin 40°}{7} = \frac{\sin 92.73°}{c} \Rightarrow c = \frac{\sin 92.73° \cdot 7}{\sin 40°}$$

$$c = 10.88$$

②

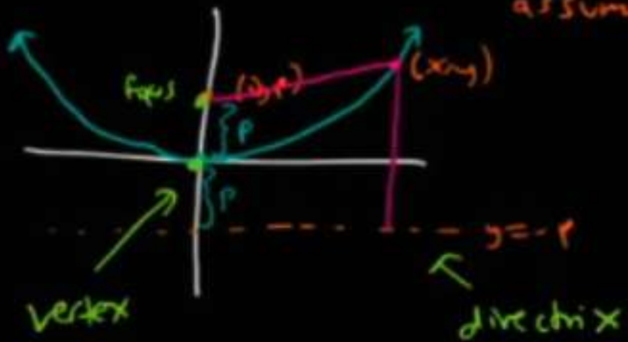
$$C = 180° - 40° - 132.73° = 7.27°$$

$$\frac{\sin 40°}{7} = \frac{\sin 7.27°}{c}$$

$$c = \frac{\sin 7.27° \cdot 7}{\sin 40°} = 1.38$$

Example. Find the equation of the points equidistant from the point $(0, p)$ and the horizontal line $y = -p$.

assume $p > 0$



point (x,y)

distance from $(0,p)$

$$\sqrt{(x-0)^2+(y-p)^2}$$

distance from $y=-p$

$$y-(-p)$$
$$y+p$$

assume $p<0$

$$\sqrt{(x-0)^2+(y-p)^2} = y+p$$

$$\left(\sqrt{(x-0)^2+(y-p)^2}\right)^2 = (y+p)^2$$

$$x^2+(y-p)^2 = (y+p)^2$$

$$x^2+\cancel{y^2}-2py+\cancel{p^2} = \cancel{y^2}+2py+\cancel{p^2}$$

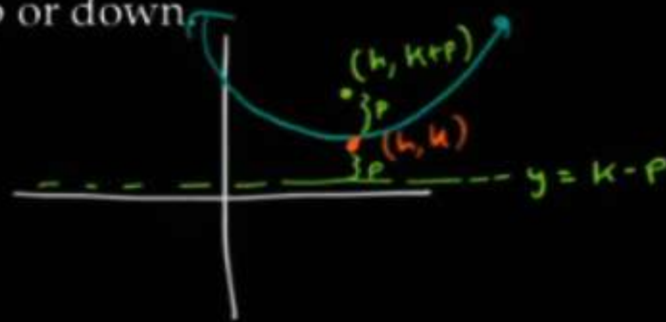
$$\boxed{x^2 = 4py} \rightsquigarrow \quad y = \left(\frac{1}{4p}\right)x^2$$

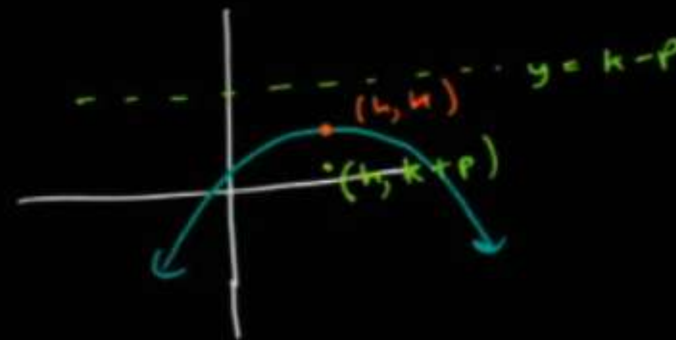
$$y = ax^2$$

$y = x^2$
vertical stretch or shrink to
$y = \frac{1}{4p}x^2$

p represents the distance between the vertex & the focus
also the distance between the vertex & the directrix

Example. Find the equation of a parabola with vertex at (h, k), assuming the parabola opens up or down.





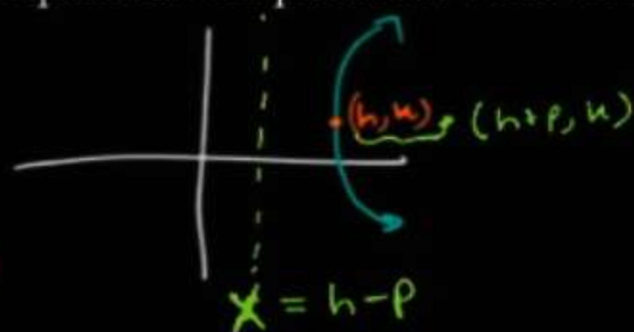
$$\left.\begin{aligned} x^2 &= 4py \\ y &= \frac{1}{4p}x^2 \end{aligned}\right\} \text{original vertex } (0,0)$$

$$\left.\begin{aligned} y &= \frac{1}{4p}(x-h)^2 + k \\ y - k &= \frac{1}{4p}(x-h)^2 \\ (x-h)^2 &= 4p(y-k) \end{aligned}\right\} \text{transformed vertex } (h,k)$$

Example. Find the equation of a parabola with vertex at (h, k), assuming the parabola left or right.

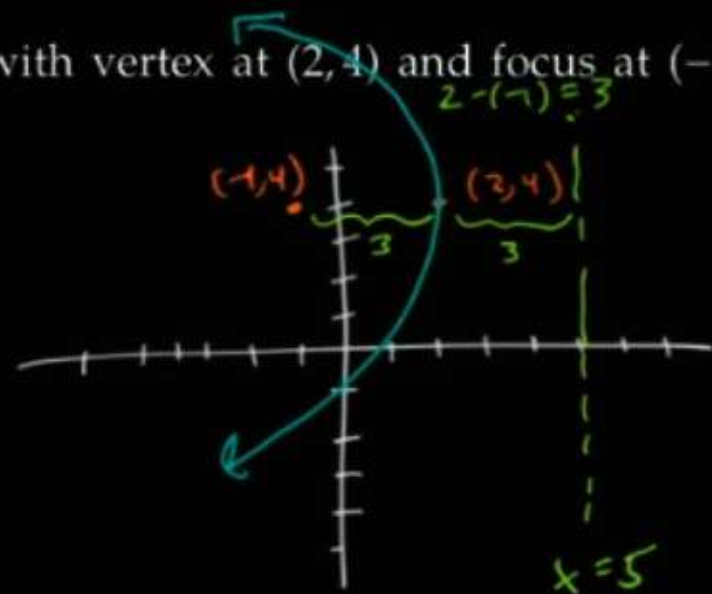
$y^2 = 4px$

$(y-k)^2 = 4p(x-h)$



Example. Find the equation of a parabola with vertex at $(2,4)$ and focus at $(-1,4)$. Graph the parabola.

$2-(-1)=3$



$$(y-k)^2 = 4p(x-h)$$

$$(h,k) = (2,4)$$

because parabola opens left, not right $\rightarrow$ $p<0$ $\quad p=-3$

$$(y-4)^2 = 4(-3)(x-2)$$

$$(y-4)^2 = -12(x-2)$$

Parabolas: Vertex, Focus, and [illegible]

$(y-k)^2 = 4p(x-h) \leftarrow$ open right if $p>0$, left if $p<0$

$(x-h)^2 = 4p(y-k) \leftarrow$ open up if $p>0$, down if $p<0$

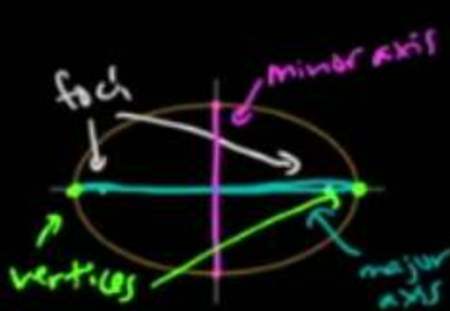
(h,k) vertex

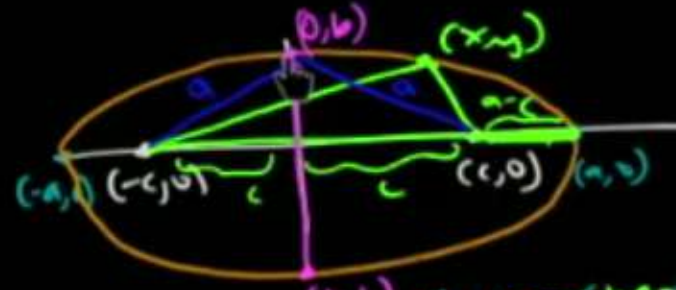
$|p|$ gives distance

btwn vertex & focus

btwn vertex & directrix

Features of an ellipse





$\cancel{c}+\cancel{c}+a-\cancel{c}+a-\cancel{c}=2a$

Example. Find the equation of an ellipse with foci at $(-c, 0)$ and $(c, 0)$ and vertices at $(-a, 0)$ and $(a, 0)$. Sum of distances is constant $= 2a$

$$\sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

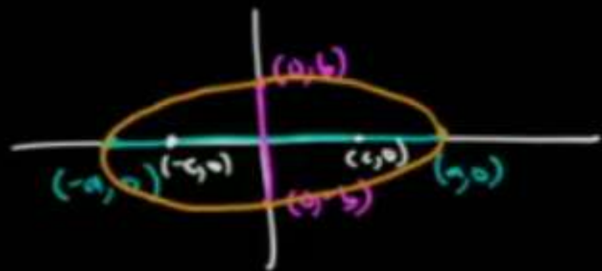
$$\vdots$$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2) \qquad b^2=a^2-c^2$$

$$\frac{b^2x^2+a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{Note } b<a$$

Summary: For $a > b$, the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ represents ... an ellipse, elongated in horizontal direction

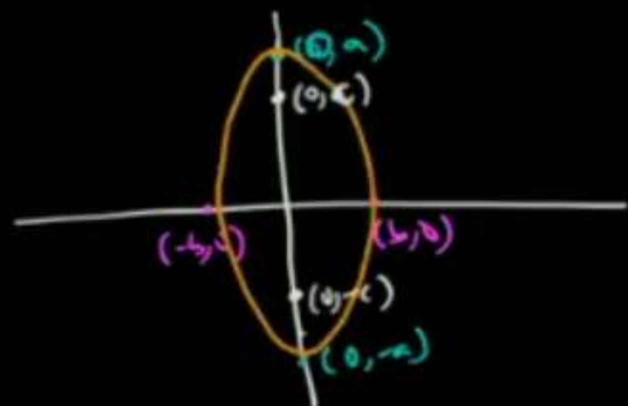


$b^2 = a^2 - c^2$

$b^2 + c^2 = a^2$

$c^2 = a^2 - b^2$

For $a > b$, the equation $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ represents ...



What about an ellipse centered at (h, k)?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \rightsquigarrow \quad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

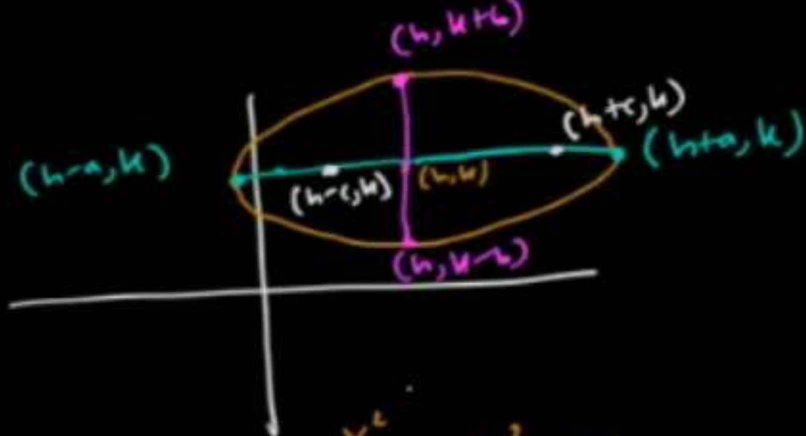
center at $(0,0)$

elongated in horiz direction $a > b$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

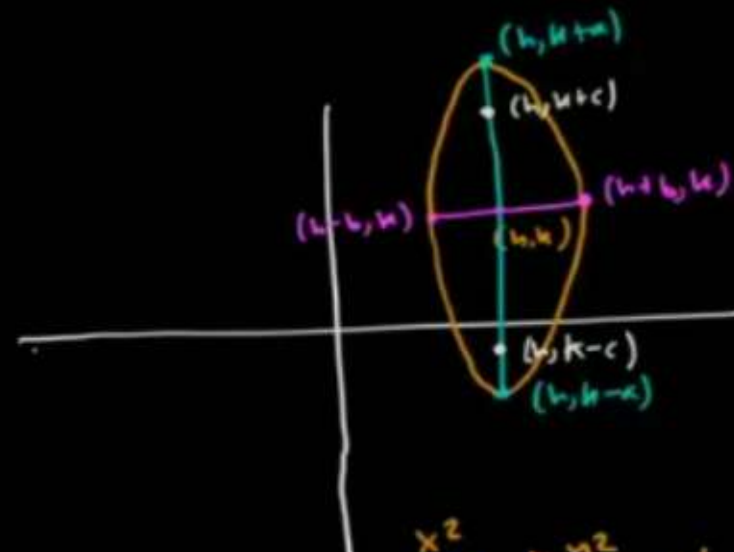
center at $(0,0)$

elongated in vertical direction $a > b$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

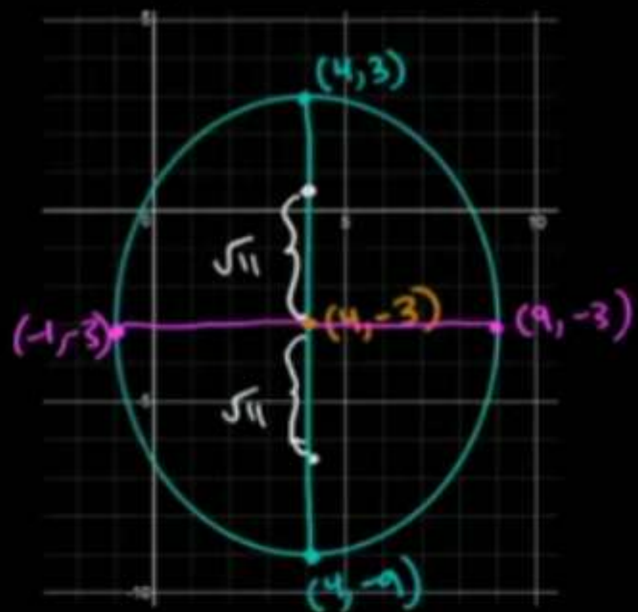
$a > b$



$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$a > b$

Example. Write the equation of the ellipse drawn. Find its foci.



$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \qquad a > b$$

center $(h,k) = (4,-3)$

$a = 6$

$b = 5$

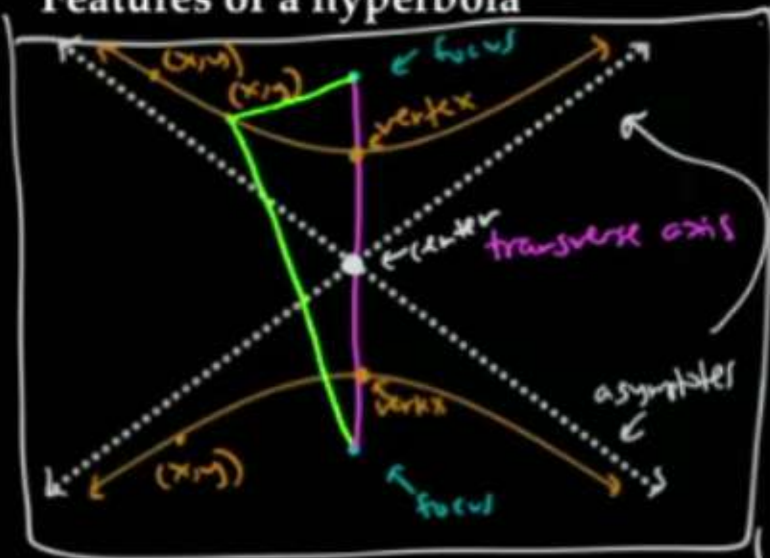
$$\frac{(x-4)^2}{5^2} + \frac{(y+3)^2}{6^2} = 1$$

$c^2 = a^2 - b^2 \Rightarrow c^2 = 6^2 - 5^2 = 11 \Rightarrow c = \sqrt{11}$

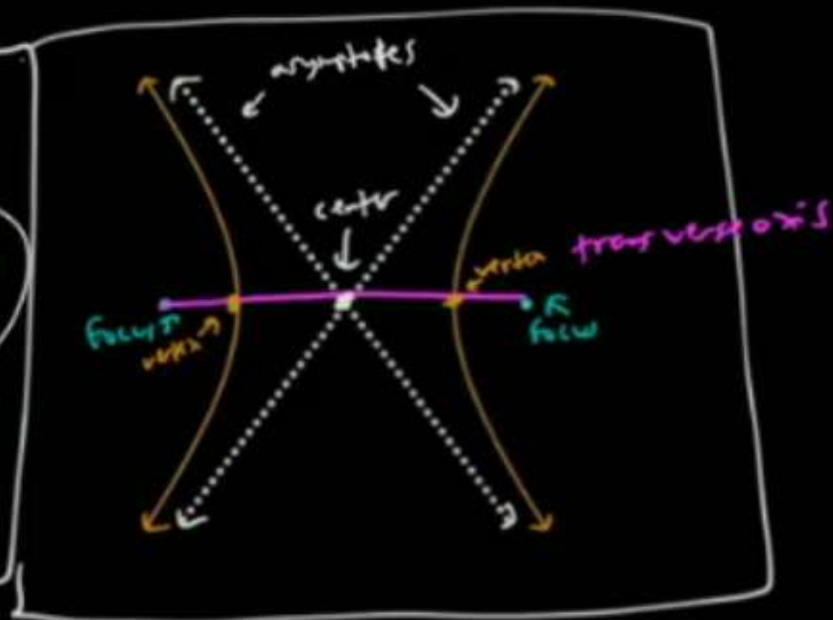
foci: $(4, -3+\sqrt{11})$ $\quad (4, -3-\sqrt{11})$

Definition. A hyperbola is the set of points (x, y) such that ... difference of the distances between (x,y) and each of two fixed points (the foci) is a constant.

Features of a hyperbola



vertical traverse axis



horizontal traverse axis

Example. Find the equation of an hyperbola with foci at $(-c,0)$ and $(c,0)$ and vertices at $(-a,0)$ and $(a,0)$.

difference in distances $= 2a$

$$\sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} = 2a$$

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a$$

$$\vdots$$

$$(c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2)$$

$$\frac{b^2x^2 - a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

let $b^2 = c^2 - a^2$

$c^2 = a^2 + b^2$

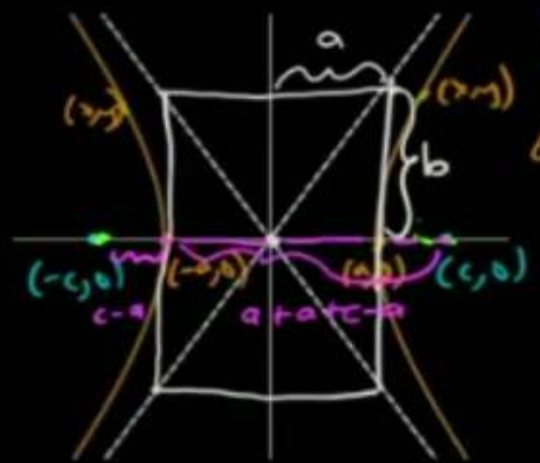
$a^2 = c^2 - b^2$

$a =$ distance from center to vertex

$c =$ distance from center to focus

$y = \frac{b}{a}x$, $y = -\frac{b}{a}x$

asymptotes



$$(a+\cancel{a}+c-\cancel{a}) - (c-a) = a+\cancel{c}-\cancel{c}+a$$
$$= 2a$$

Example. Find the equation of an hyperbola with foci at $(-c, 0)$ and $(c, 0)$ and vertices at $(-a, 0)$ and $(a, 0)$.

difference in distances = 2a

horiz transverse axis

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

$b^2 = c^2 - a^2$

$c^2 = a^2 + b^2$

$a^2 = c^2 - b^2$

a = distance from center to vertex

c = distance from center to focus

asymptotes:

$y = \frac{b}{a}x,\quad y = -\frac{b}{a}x$

a goes with positive term

b goes with negative term

vertical transverse axis

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

$b^2 = c^2 - a^2$

$c^2 = a^2 + b^2$

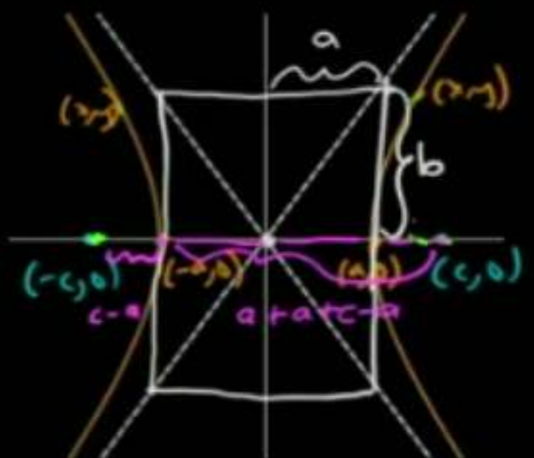
$a^2 = c^2 - b^2$

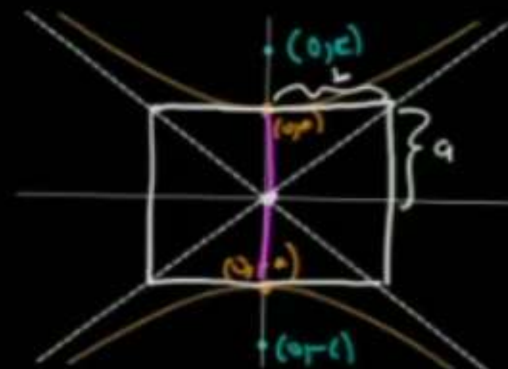
a = distance from center to vertex

c = distance from center to focus

asymptotes

$y = \frac{a}{b}x,\quad y = -\frac{a}{b}x$





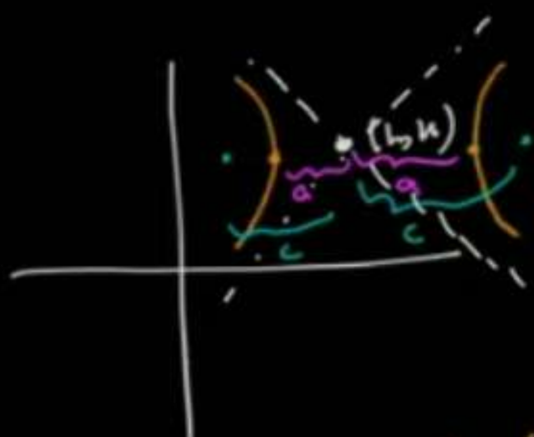
$$(a + \cancel{a} + c - \cancel{a}) - (c - a) = a + \cancel{c} - \cancel{c} + a$$
$$= 2a$$

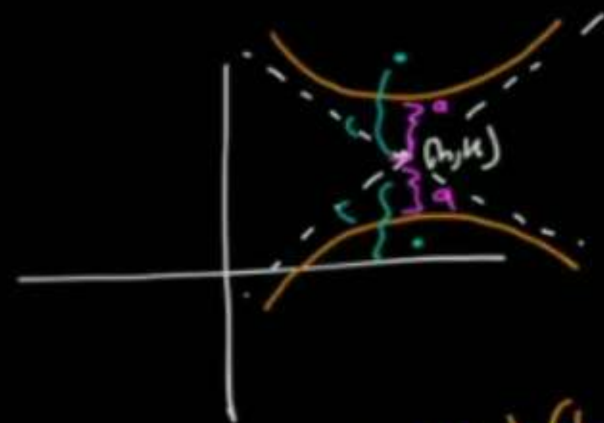
What about an hyperbola centered at (h, k)?

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ centered at origin $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

center at (h,k)

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$





vertices $(h+a,k)$ $(h-a,k)$

foci: $(h+c,k)$ $(h-c,k)$

$c^2 = a^2 + b^2$

asymptote slope is $\frac{b}{a}$, $-\frac{b}{a}$

$y-k = \frac{b}{a}(x-h)$ $y-k = -\frac{b}{a}(x-h)$

vertices $(h,k+a)$, $(h,k-a)$

foci: $(h,k+c)$, $(h,k-c)$

asymptote slope is $\frac{a}{b}$, $-\frac{a}{b}$

$y-k = \frac{a}{b}(x-h)$

$y-k = -\frac{a}{b}(x-h)$

Example. Graph the hyperbola $\dfrac{(x-6)^2}{4} - \dfrac{(y+3)^2}{25} = 1$.

Find its center, vertices, foci, and asymptotes.

center: $(6,-3)$

$a = \sqrt{4} = 2$

$b = \sqrt{25} = 5$

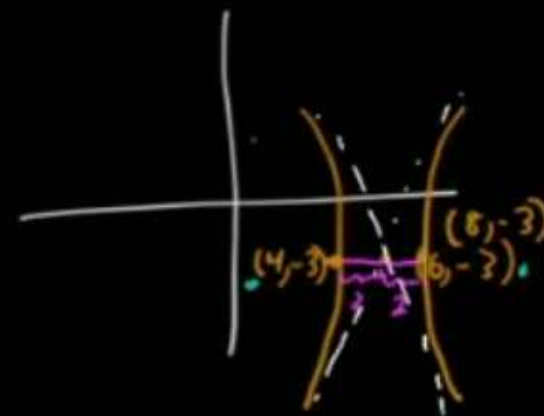
vertices $(4,-3)$ and $(8,-3)$

$c^2 = a^2 + b^2 = 4 + 25 = 29 \quad c = \sqrt{29}$

foci $(6-\sqrt{29},-3) \quad (6+\sqrt{29},-3)$

slope: $\frac{b}{a} = \frac{5}{2} \quad , -\frac{b}{a} = -\frac{5}{2}$

$y+3 = \frac{5}{2}(x-6) \;, \quad y+3 = -\frac{5}{2}(x-6)$



Example. Plot the points, given in polar coordinates.

1. $(8, -\frac{2\pi}{3})$

   r θ

2. $(5, 3\pi)$

   r θ

3. $(-12, \frac{\pi}{4})$

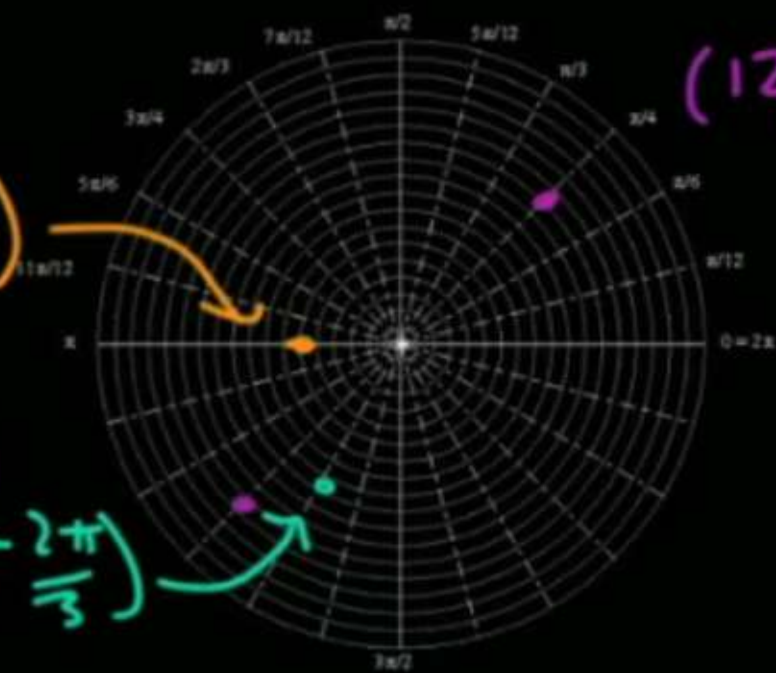
   r θ

$(5, 3\pi)$
$= (5, \pi)$

$(-12, \pi/4)$
$=$
$(12, \frac{5\pi}{4})$

$(8, -\frac{2\pi}{3})$

$(12, \pi/4)$

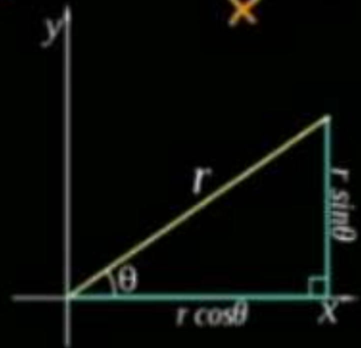


Note. A negative angle means to go clockwise from the positive x-axis. A negative radius means jump to the other side of the origin, that is, $(-r, \theta)$ means the same point as $(r, \theta + \pi)$

$(r, \theta) \quad (x, y)$

Note. To convert between polar and Cartesian coordinates, note that:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r^2 = x^2 + y^2 \qquad r = \pm\sqrt{x^2+y^2}$
- $\tan\theta = \dfrac{y}{x}$



$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}$

Example. Convert $(5, -\frac{\pi}{6})$ from polar to Cartesian coordinates.

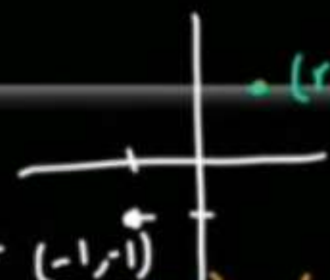
$x = r\cos\theta \qquad y = r\sin\theta$

$x = 5\cos(-\pi/6) = 5\frac{\sqrt{3}}{2} \qquad y = 5\sin(-\pi/6) = -\frac{5}{2}$

Example. Convert $(-1, -1)$ from Cartesian to polar coordinates.

$r^2 = x^2 + y^2 \Rightarrow r^2 = (-1)^2 + (-1)^2 = 2 \qquad r = \sqrt{2} \text{ or } -\sqrt{2}$

$\tan\theta = y/x = -1/-1 = 1 \qquad \theta = \pi/4, 5\pi/4 \pm 2\pi k$



$(r, \theta) = (\sqrt{2}, \pi/4)$

$(x,y) = (-1,-1)$

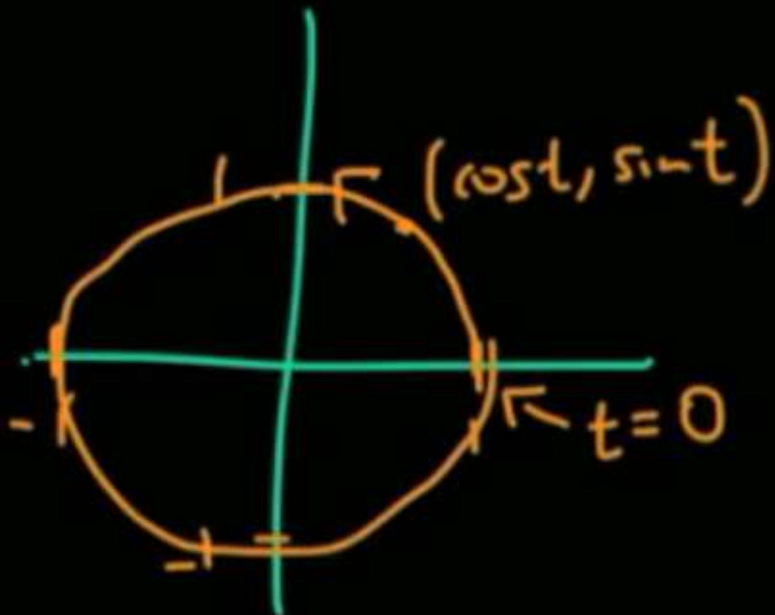
$(r,\theta) = (\sqrt{2}, 5\pi/4) = (-\sqrt{2}, \pi/4)$

Example. Plot each curve and find a Cartesian equation:

1. $x = \cos(t)$, $y = \sin(t)$, for $0 \le t \le 2\pi$
2. $x = \cos(-2t)$, $y = \sin(-2t)$, for $0 \le t \le 2\pi$
3. $x = \cos^2(t)$, $y = \cos(t)$

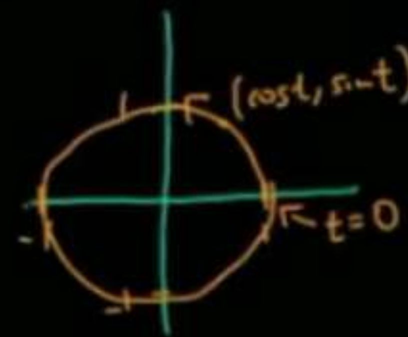
$x^2 + y^2 = 1$

$\cos^2 t + \sin^2 t = 1$



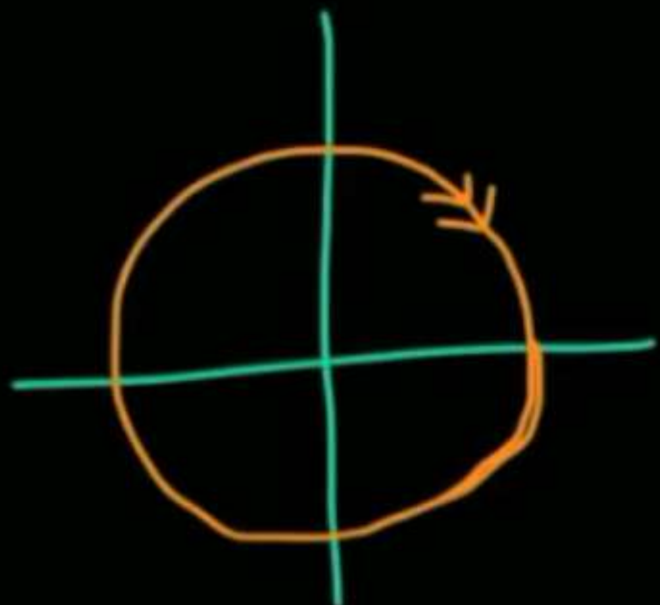
t	$x = \cos(t)$	$y = \sin(t)$
0	1	0
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$
π	-1	0
$5\pi/4$	$-\sqrt{2}/2$	$-\frac{\sqrt{2}}{2}$
$3\pi/2$	0	-1
$7\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$

Example. Plot each curve and find a Cartesian equation:



1. $x = \cos(t)$, $y = \sin(t)$, for $0 \leq t \leq 2\pi$ $\quad x^2+y^2=1$
2. $x = \cos(-2t)$, $y = \sin(-2t)$, for $0 \leq t \leq 2\pi$ $\quad x^2+y^2=1$
3. $x = \cos^2(t)$, $y = \cos(t)$

t	$x=\cos(-2t)$	$y=\sin(-2t)$
0	1	0
$\pi/8$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\pi/4$	0	-1
$\pi/2$	-1	0
$3\pi/4$	0	1
π	1	0
$\vdots$		
2π	1	0



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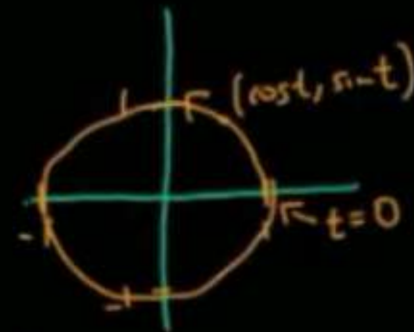
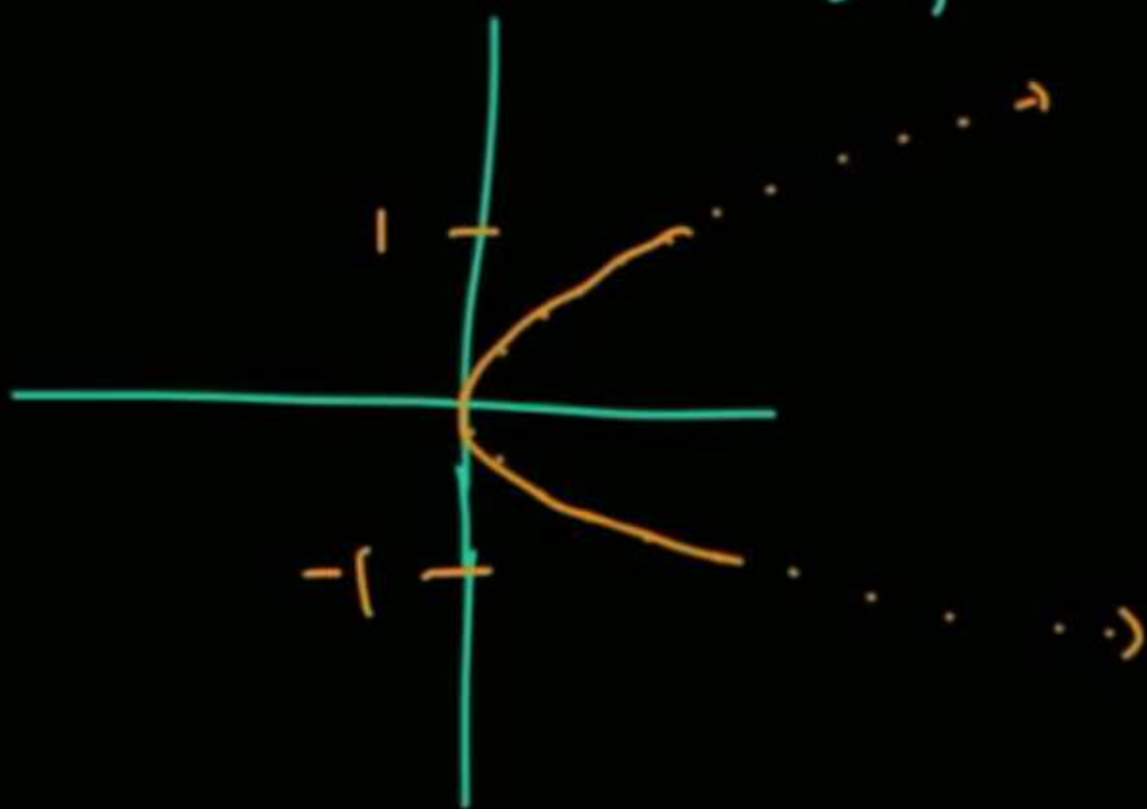
3. $x = \cos^2(t)$, $y = \cos(t)$ $\quad x=y^2, \; -1 \le y \le 1$





Example. Write the following in parametric equations:

1. $y = \sqrt{x^2 - x}$ for $x \leq 0$ and $x \geq 1$

$x = t$ for $t \leq 0, t \geq 1$

$y = \sqrt{t^2 - t}$

= copy cat parametrization''

2. $25x^2 + 36y^2 = 900$

$$\frac{25x^2}{900} + \frac{36y^2}{900} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

set $\frac{x}{6} = \cos t$

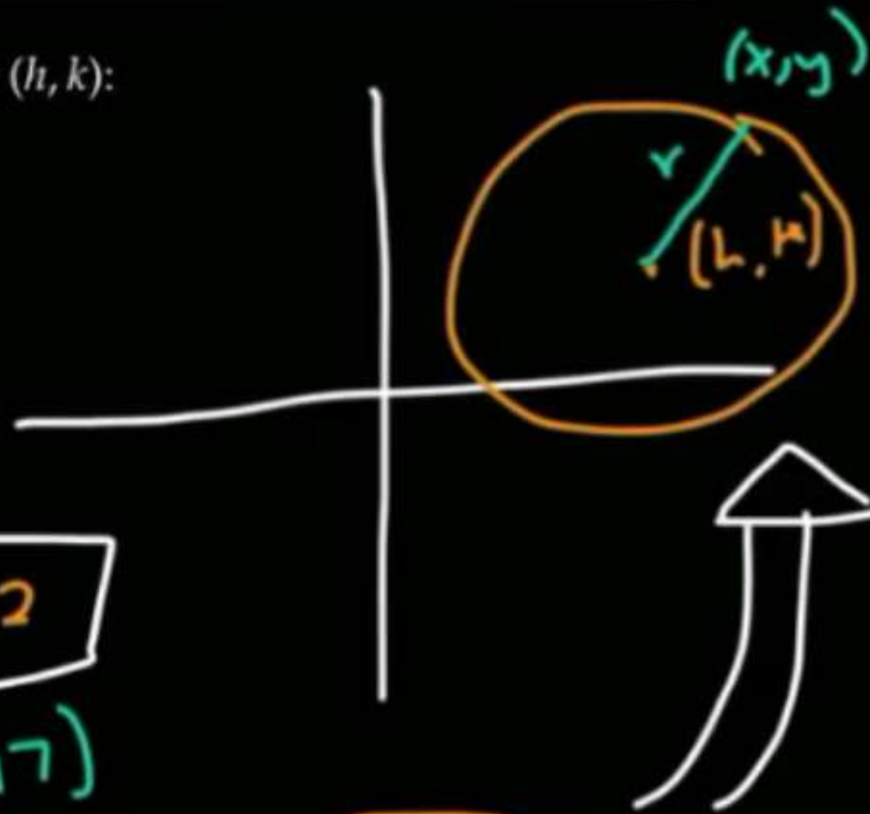
$\frac{y}{5} = \sin t$

$$\boxed{\begin{aligned} x &= 6\cos t \\ y &= 5\sin t \end{aligned}}$$

Example. Describe a circle with radius r and center (h, k):

a) with a Cartesian equation

b) with parametric equations



$$\sqrt{(x-h)^2+(y-k)^2} = r$$

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

Ex: $r=5$ center $(-3,17)$

$$(x+3)^2+(y-17)^2=25$$

$x=\cos t \quad y=\sin t$

$x=r\cos t \quad y=r\sin t$

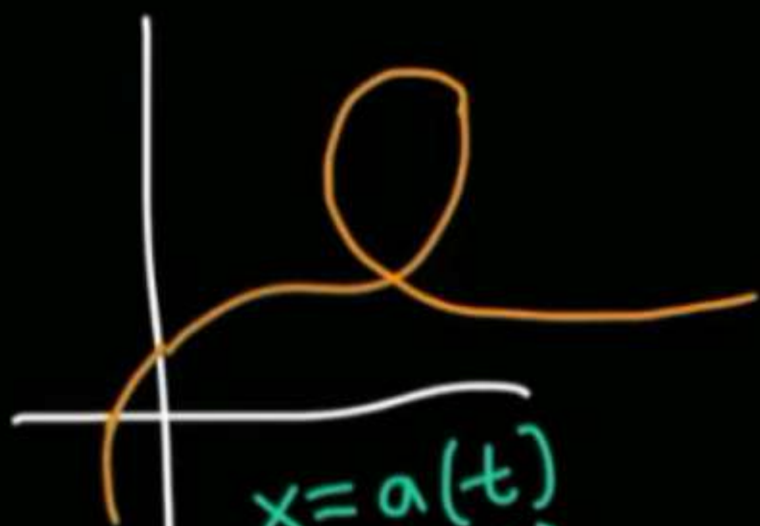
$$\boxed{x=r\cos t+h \qquad y=r\sin t+k}$$

$x=5\cos t-3, \quad y=5\sin t+17$

$y=f(x)$

$x=a(t)$
$y=b(t)$

§10.1 - Parametric Equations



$x=a(t)$
$y=b(t)$

$(x-h)^2+(y-k)^2=r^2$

$x=r\cos(t)+h \quad y=r\sin(t)+k$

Source:

https://www.youtube.com/playlist?list=PLWKjhJtqVAbl5SlE6aBHzUVZ1e6q1Wz0v
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