Topology, differential geometry and physics

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Chapter 1

Introduction and Mathematical Backgrounds

1.1 Backgrounds on GENERAL TOPOLOGY

1.1.1 Open/Closed sets and Basic Point-set Topology

Definition 1.1.1. (**Topological space**) Let X be any set and $\mathscr{T} = \{U_i \,|i \in I\}$ denote a collection of subsets of X. The pair $(X, \mathscr{T})$ is called a topological space if

(i) $\emptyset, X \in \mathscr{T}$

(ii) The union of (arbitrary many) elements of $\mathscr{T}$ belongs to $\mathscr{T}$

$\Longleftrightarrow J$ is any (arbitrary many) sub-collection of I, the family $\{U_i \,|i \in J\}$ satisfies $\bigcup_{j \in J} U_j \in \mathscr{T}$

(iii) The intersection of a finite #s of elements of $\mathscr{T}$ belongs to $\mathscr{T}$

$\Longleftrightarrow K$ is any finite sub-collection of K, the family $\{U_i \,|i \in K\}$ satisfies $\bigcap_{k \in K} U_k \in \mathscr{T}$

$\mathscr{T}$ is called a **topology** on X and X is called a **topological space**. The U_i are called the **open set** of $\mathscr{T}$

Example 1.1.2. The topological space $(X, \mathscr{T})$ is called

(a)**discrete topology**: if $\mathscr{T}$ is the collection of all the subsets of X

(b)**trivial topology**: if $\mathscr{T} = \{\emptyset, X\}$

(c)**usual topology**: if $\mathscr{T}$ is all the open intervals and their union when $X = \mathbb{R}$

NOTE: If we allow infinite intersection in (iii), the usual topology in $\mathbb{R}$ reduces to the discrete topology and is thus not very interesting.

Proof: For $\forall x \in X = \mathbb{R}$, the set $\{x\} = \bigcap_{n=1}^{\infty}(x - \frac{1}{n}, x + \frac{1}{n})$, being a countable intersection of open sets, is open. Now take any subset $A \in X$, $A = \bigcup_{x \in A}\{x\}$ is a union of open sets thus A is open. Therefore the topology is discrete.

Definition 1.1.3. (**basis, open neighborhood, interior point**, **closed set**)

(a) A basis for the topology of $(X, \mathscr{T})$: a subset $\mathscr{B} \subset \mathscr{T}$ s.t. each $U_i \in \mathscr{T}$ is the union of elements of $\mathscr{B}$

(b) An open neighborhood of $p \in X$ is an element U_i s.t. $p \in U_i$

(c) $x \in S \subset X$ is an interior point of S if $\exists U_i \in \mathscr{T}$ s.t. $x \in U_i \subset S$

$\Longrightarrow Int(S) \coloneqq \{x | x \textit{ is interior point of } S\}$ which can also be defined as the largest open set of S.

$\Longrightarrow Ext(S) \coloneqq \{x | x \textit{ is interior point of } X \backslash S\}$

$\Longrightarrow \partial(S) \coloneqq X - Int(S) \bigcup Ext(S)\}$ which can also be defined as $S - Int(S)$.

(d) A subset $S \subset X$ is closed if $S \backslash X = X - S$ is open.

Example 1.1.4. (more about open set)

(a) Any closed interval is a closed set under usual topology.

(b) In $\mathbb{R}^n$, $\times_i [a_i, b_i]$ is a closed set under usual topology.

(c) Whether a set $S \subset X$ is open or closed depends on X. e.g. The interval $I = (a, b)$ is open under usual topology of $\mathbb{R}$ but is not open under usual topology of $\mathbb{R}^2$

Definition 1.1.5. (**Metric**)

A **metric** $d : X \times X \to \mathbb{R}$ is a function which satisfies

(i) Symmetry: $d(x, y) = d(y, x)$

(ii) Positive semi-definite: $d(x, y) \geq 0$ and the equality holds iff $x = y$

(iii) Triangle inequality: $d(x, y) + d(x, z) \geq d(y, z)$

Definition 1.1.6. (**Metric space, induced topology**)

(a) If X is endowed with a metric, $(X, \mathscr{T})$ is a metric space whose open sets is defined by $U_\epsilon(x) = \{y|d(x, y) \leq \epsilon, x, y \in X\}$

(b) $(S, \mathscr{T}_S)$ is called induced topology space of $(X, \mathscr{T})$ if one defines a topology $\mathscr{T}_S \coloneqq \{S \cap U_i|U_i \in \mathscr{T}\}$

Example: $S^1 = \{x \in \mathbb{R}^2| \, \|x - y\| = 1, y \in \mathbb{R}^2\}$ is not open under usual (or metric) topology $\left(\mathbb{R}^2, \mathscr{T}\right)$ but is open under $\left(S^1, \mathscr{T}_{S^1} = \{S^1 \cap U_i|U_i \in \mathscr{T}\}\right)$

Definition 1.1.7. (**Hausdorff spaces, Euclidean(standard) topology of** $\mathbb{R}^n$)

(a)A topology space $(X, \mathscr{T})$ is a Hausdorff space if , for $\forall x, x' \in X$, $\exists U_x, U_{x'}$ s.t. $U_x \cap U_{x'} = \emptyset$

(b)Euclidean topology of $\mathbb{R}^n$ is $(\mathbb{R}^n, \mathscr{T}_{\mathbb{R}^n})$ where $\mathscr{T}_{\mathbb{R}^n} = \emptyset \cup \{\cup_{i \in I} B_\delta(x_i)|\forall I\}$.

The open ball $B_\delta(x_i) \coloneqq \{x \in \mathbb{K}^n| \, \|x - x_i\| \leq \delta\}$ where $\mathbb{K} \coloneqq \mathbb{C}$ *or* $\mathbb{R}$

Example: $\mathbb{R}$ with usual topology is a Hausdorff space. Any metric space is a Hausdorff space.

Definition 1.1.8. (**Countable basis or Second-countable**)

$(X, \mathscr{T})$ has a countable basis (second-countable) if $\exists$*a countable set*$\mathscr{T}_0 \subset \mathscr{T}$ s.t. $\forall U_i \in \mathscr{T}$ is the union of elements of $\mathscr{T}_0$

1.1.2 Convergence and Continuity

Definition 1.1.9. Let$(X, \mathscr{T})$ be a topological space

(a) A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ converges to the **limit of the sequence** x if $lim_{n \to \infty} x_n = x$

(b) $x \subset X$ is an **accumulation point** of sequence $\{x_n\}_{n \in \mathbb{N}}$ if any open neighborhood of x contains infinite points of the sequence.

Definition 1.1.10. $f : X \to X'$ is a map from $(X, \mathscr{T})$ to $(X', \mathscr{T}')$

(a) f is **continuous** if $f^{-1}(U_i') \in \mathscr{T}$ for $\forall U_i' \in \mathscr{T}'$.

(b) f is **continuous at** $p \in X$ if any open neighborhood $U'_{f(p)}$ of $f(p)$, there $\exists U_p \ni p$ s.t. $f[U_p] \subset U'_{f(p)}$.

NOTE: This definition of continuity reduces to "$\epsilon - \delta$" definition when $X = \mathbb{R}^n or \mathbb{C}^n$ with standard topology.

1.1.3 Compactness

Definition 1.1.11. (**compactness**)Let$(X, \mathscr{T})$ be a topological space and $S \subset X$

(a) S is **compact** if any open covering of S has finite sub-covering: if $S \subset \cup_{i \in I} U_i$, $\exists J \subset I$ s.t. $S \subset \cup_{i \in J} U_i$

(b) S is **relative** compact if $\bar{S}$ is compact

(c) X is l**ocally compact** if $\forall p \in X$, $\exists U_i \ni p$ s.t. $\bar{U}_i$ is relative compact.

Theorem 1.1.12. *(Heine-Borel) If $\mathbb{R}^n$ is equipped with standard topology, $S \subset \mathbb{R}^n$ is compact iff S is simultaneously closed and bounded ($\exists x \in \mathbb{R}^n, \delta > 0,$ s.t. $S \subset B_\delta(x)$)*

Proof. df □

1.1.4 Connectedness

Definition 1.1.13. (**Connected , arc-wise (path) connected, loop, Simply connected**)

(a)A topological space X is **connected** if it cannot be written $X = X_1 \sqcup X_2$, where X_1 and X_2 are open.

(b)A topological space X is **path-connected** if for $\forall p, q \in X$, there $\exists$ continuous map $f : [0, 1] \to X$, s.t. $f(0) = p$ and $f(1) = q$

(c)A loop in topological space X is a continuous map $f : [0, 1] \to X$, s.t. $f(0) = f(1)$. X is **simply-connected** if any loop in X can be continuously shrunk to a point.$\iff$ For $\forall p, q \in X$ and any two continuous paths $f_i : [0, 1] \to X$, $i = 0, 1$, s.t. $f_i(0) = p$ and $f_i(1) = q$, there $\exists$ continuous map $f : [0, 1] \times [0, 1] \to X$ (**HOMOTOPY**), s.t. $f(s, 0) = p$, $f(s, 1) = q$ for $\forall s \in [0, 1]$ and $f(0, t) = f_0(t)$, $f(1, t) = f_1(t)$ for $\forall t \in [0, 1]$.

Example: Identify the connectedness type: $\mathbb{R}$, $\mathbb{R} - \{0\}$, $\mathbb{R}^n$, $\mathbb{R}^n - \{\mathbf{0}\}$, $\mathbb{R}^n - \mathbb{R}$, $S^n(n \leq 2, n > 2)$, $T^n = \times_{i=1}^n S^1$

1.1.5 Homeomorphism and topological invariants

Definition 1.1.14. $f : X \to X'$ is a map from $(X, \mathscr{T})$ to $(X', \mathscr{T}')$

f is a **homeomorphism** (i) f is continuous (ii) f is bijective (iii) f^{-1} is continuous.

$\Longleftrightarrow \exists f : X \to X', g : X' \to X$ s.t. $f \circ g = id_{X'}, g \circ f = id_X$

Note: This is a equivalence relation under which geometrical objects are classified according to whether an object can be deformed into other by continuous transformation.

Definition 1.1.15. (Homotopy type)

X and X' are of the same homotopy type if $\exists f : X \to X', g : X' \to X$ s.t. f and g are both continuous.

Example: (a)S^1 and a cylinder $S^1 \times \mathbb{R}$

(b) $D^2 = \{(x, y)|x^2 + y^2 < 1\}$ and a point

(c) $D^2 - \{0\}$ and S^1, $\mathbb{R}^2 - \{0\}$ and S^1 , $\mathbb{R}^3 - \{0\}$ and S^2,

Definition 1.1.16. (Euler chracteristic)

1.2 MANIFOLD

1.2.1 Definition

Definition 1.2.1. (MANIFOLD) Topological space $(M, \mathscr{T})$ is an m-dimensional differential manifold if $\exists\{(U_i, \varphi_i)\}$ s.t.

(i)$\{U_i\}$ is a open cover of M i.e. $\cup_i U_i = M$

(ii) φ_i is a homeomorphism :$U_i \to V_i$ (V_i is open in $\mathbb{R}^n$ with usual topology)

(iii) Compatibility: if $U_i \cap U_j \neq \emptyset$, then $\varphi_j \circ \varphi_i^{-1}$ is smooth (infinitely differentiable or C^∞)

(U_i, φ_i) is a chart. $\{(U_i, \varphi_i)\}$ is an atlas.

U_i is coordinate neighborhood. φ_i is coordinate function.

Exercise 1.2.2. (Miscellaneous)

(a) Show that $S^n = \{x \in \mathbb{R}^n | \sum_{i=1}^{n+1} (x^i)^2 = 1\} \subset \mathbb{R}^{n+1}$ with induced topology is a manifold

Proof: Define a open over $S^n = \bigcup_{i=1}^{n+1} U_i^\pm$ where $U_i^+ = \{x \in S^n | x^i > 0\}$ and $U_i^- = \{x \in S^n | x^i < 0\}$ and corresponding coordinate function $\varphi_i^\pm(x) = (x^1 ... x^i = 0, ... x^{n+1})$

(b) Show that if M is a manifold and U is a subset of M , then U with its induced topology is a manifold

1.2.2 C^r/C^∞ map between manifolds and diffeomorphism

Definition 1.2.3. (C^r/C^∞ map) $f : M \to M'$ is C^r/C^∞ map if $\forall p \in U_i \subset M$, $f(p) \in U'_j \subset M'$ s.t. n-dimensional function $\varphi'_j \circ f \circ \varphi_i^{-1}$ on $V_i \subset \mathbb{R}^n$ is C^r/C^∞

Denote $y = \varphi'_j \circ f \circ \varphi_i^{-1}(x)$ by $y = f(x)$, C^r means that $y^\alpha = f^\alpha(x^\mu)$ is C^r/C^∞ with respect to each x^μ

Note: The charts in the same atlas are compatible, thus the definition above is independent of the choice of charts

Definition 1.2.4. (Diffeomorphism)

Manifold M and M' are diffeomorphic to each other if there $\exists$*homeomorphism* $f : M \to M'$ s.t.

(i) f is bijective

(ii)f and f^{-1} is C^∞

Definition 1.2.5. (curve and function on manifold, reparametrization)

(a) A open curve is a continuous map $\gamma : (a, b) \to M$ where $(a, b) \subset \mathbb{R}, a < 0 < b$ why the origin is in this interval

(b) A function on manifold is a map $f \in \mathscr{F}_M : M \to \mathbb{R}$

(c) $\gamma' : I' \to M$ is the reparametrization of $\gamma : I \to M$, if $\exists$onto map $\alpha : I \to I'$ s.t. (a) $\gamma = \gamma' \circ \alpha$. (b) induced map $t' = \alpha(t)$ has non-vanishing derivative.

1.2.3 Vector Fields

Definition 1.2.6. (**Vector space**) Vector space over a field $\mathbb{K}$ is a set V equipped with 2 maps: addition $+ : V \times V \to V$ and scalar multiplication $: \mathbb{R} \times V \to V$ s.t.

(i) $(V, +)$ forms a group

(ii)Compatibility of scalar multiplication with field multiplication: $a(b\mathbf{v}) = ab\mathbf{v}$

(iii)Identity of scalar multiplication: $1\mathbf{v} = \mathbf{v}$

(iv)Distributivity of scalar multiplication over field addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

(v)Distributivity of scalar multiplication over vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

Claim 1.2.7. $\mathscr{F}_M = C^\infty(M)$ is an **algebra over** $\mathbb{R}$ (a vector space equipped with a bilinear product $\cdot : V \times V \to V$, meaning that it is closed under multiplication$\cdot : V \times V \to V$ and addition, as well as multiplication by a real number s.t.

(i) Properties of vector space

(ii)Right distributivity: $(f + g) \cdot h = f \cdot h + g \cdot h$

(iii)Left distributivity: $f \cdot (g + h) = f \cdot g + f \cdot h$

(iv)Compatibility with scalars: $af \cdot (bg) = (ab)(f \cdot g)$

(v)$f + g = g + f$

Note that it is a commutative algebra i.e. $f \cdot g = g \cdot f$

Definition 1.2.8. (**Tangent vector**)A vector at $p \in M$ is a map $X_p : C^\infty(M) \to \mathbb{R}$, s.t. for $\forall \alpha \in \mathbb{R}$ and $f, g \in C^\infty(M)$

(i) $X_p(f + g) = X_p(f) + X_p(g)$

(ii) $X_p(\alpha f) = \alpha X_p(f)$

(iii) Leibniz Law: $X_p(fg) = X_p(f)g + fX_p(g)$

Definition 1.2.9. (**Tangent vector of curves**) $\dot{\gamma}(t) : C^\infty(M) \to \mathbb{R}$ s.t. $\dot{\gamma}(t)[f] \coloneqq \frac{d}{dt} f(\gamma(t))$. Sometimes $\dot{\gamma}(t)[f]$ is written as $\frac{\partial}{\partial t}|_{\gamma(t)}(f)$

Theorem 1.2.10. *$\gamma' : I' \to M$ is the reparametrization of $\gamma : I \to M$, then the tangent vector $\dot{\gamma}(t)$ and $\dot{\gamma}'(t')$ satisfy $\dot{\gamma}(t) = \frac{dt'(t)}{dt} \dot{\gamma}'(t')$, or $\frac{\partial}{\partial t} = \frac{dt'}{dt} \frac{\partial}{\partial t'}$*

Proof. $\dot{\gamma}(t)[f] = \frac{dt'}{dt} \frac{df(\gamma(t))}{dt'} = \frac{dt'}{dt} \frac{df(\gamma'(t'))}{dt'} = \frac{dt'}{dt} \dot{\gamma}'(t')[f]$ □

Definition 1.2.11. A **vector field** on M is a map $X \in \mathscr{X}(M)$ *or* $Vect(M) : C^\infty(M) \to C^\infty(M)$, s.t. for $\forall \alpha \in \mathbb{R}$ and $f, g \in C^\infty(M)$

(i) $X(f + g) = X(f) + X(g)$

(ii) $X(\alpha f) = \alpha X(f)$

(iii) Leibniz Law: $X(fg) = X(f)g + fX(g)$

Definition 1.2.12. (**Addition and scalar product of vector fields**)

(a)$(X + Y)(f) = X(f) + Y(f)$

(b) $(gX)(f) = gX(f)$

Exercise 1.2.13. Show that

(a) $X + Y$ and gX are still Vector fields where $g \in C^\infty(M)$

(b) Check that $\dot{\gamma}(t) \in T_{\gamma(t)}M$ using definition

(c)Let $X, Y \in Vect(M)$. Show that $X = Y$ only if $X_p = Y_p$ for $\forall p \in M$

1.2.4 Differential Forms

ONE-FORM

Definition 1.2.14. (**One-form**) on manifold M is a linear map $\omega \in \Omega^1(M) : Vect(M) \to C^\infty(M)$, s.t. for $\forall g \in C^\infty(M)$

(i) $\omega(X+Y) = \omega(X) + \omega(Y)$

(ii) $\omega(gX) = g\omega(X)$

Action of one-form on vector field: Define the inner product $\langle \; , \; \rangle : \Omega^1(M) \times Vect(M) \to C^\infty(M)$, then the action of one-form on vector field is given by $\omega(X) = \langle \omega, X \rangle$

Example: (a) (exterior derivative of f) For any $f \in C^\infty(M)$, there is a one-form defined by $df(X) = X(f) \underset{coordinate}{\overset{local}{\Longrightarrow}} X^\mu \partial_\mu f$ (or $\langle df, X\rangle = X(f)$ expressed in terms of inner product).

(b) A special case when $df = dx^\mu$ and $X = \partial_\nu \Longrightarrow \langle dx^\mu, \partial_\nu \rangle = \delta^\mu_\nu$, then $\omega(X) = \langle \omega, X \rangle = \omega_\mu X^\nu \, \langle dx^\mu, \partial_\nu \rangle = \omega_\mu X^\mu$

Definition 1.2.15. (**Addition and scalar product of 1-form**)

(a)$(\omega + \mu)(f) = \omega(f) + \mu(f)$‘

(b) $(g\omega)(f) = g\omega(f)$

TENSOR AND TENSOR FIELD

Definition 1.2.16. (Tensor) A tensor $T \in \mathcal{T}^q_{r,p}(M)$ of type (q, r) is a multilinear map : $\otimes^q T^*_p(M) \otimes^r_p(M) \to \mathbb{R}$, which can be written in a given coordinate system as $T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_q}} dx^{\nu_1} \dots dx^{\nu_r}$. The action of T on $V_i = V^\mu_i \partial/\partial x^\mu (1 \le i \le r)$ and $\omega_i = \omega_{i\mu} dx^\mu (1 \le i \le q)$ yields $T(\omega_1 \dots \omega_q; V_1 \dots V_q) = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \omega_{\mu_1} \dots \omega_{\mu_q} V^{\nu_1} \dots V^{\nu_r}$

Definition 1.2.17. (Tensor field) A tensor field of type (q, r) is a smooth assignment of $\mathcal{T}^q_{r,p}(M)$ at each point $p \in M$

Claim: p-form $\mathcal{T}^0_p(M) = \Omega^p(M)$

INDUCED MAP

Definition 1.2.18. (**Pushforward**) Smooth map $\phi : M \to N$ naturally induces a map $\phi_* : T_pM \to T_{\phi(p)}N$ s.t. $(\phi_* V)(f) = V(f \circ \phi) = V(\phi^* f)$. Extend this definition on whole M: $(\phi_* V)_{\phi(p)} = \phi_* V_p$

Definition 1.2.19. (**Pullback**) is a map $\phi^* : T^*_{\phi(p)}N \to T^*_p M$ s.t. $(\phi^*\omega)(V) = \omega(\phi_* V)$. Extend this definition on whole M: $(\phi^*\omega)_p = \phi^*\omega_{\phi(p)}$

Theorem 1.2.20. *(Exterior derivative is compatible with pullback)* $\phi^* df = d(\phi^* f)$

Proof. We can show that $(\phi^* df)_p = (d(\phi^* f))_p$ for $\forall p \in M$. $(\phi^* df)_p(V) = (df)_{\phi(p)}(\phi_* V) = ((\phi_* V)f)(p) = (V(f \circ \phi))(p) = (V(\phi^* f))(p) = d(\phi^* f)_p(V)$ □

Exercise 1.2.21. Consider smooth map $\phi : M \to N$ and $\psi : N \to W$, $f \in C^\infty(N)$, x^μ, y^α are the local coordinates in M and N.

(a) $V = V^\mu \partial_\mu \in T_p(M)$, show that the pushforward of V by ϕ is $W^\mu \frac{\partial}{\partial y^\mu} = V^\nu \frac{\partial y^\mu(x)}{\partial x^\nu} \frac{\partial}{\partial y^\mu}$, where $y = \phi(x)$ (a sloppy way of $y = \varphi'_j(\phi(\varphi^{-1}_i(x))$)

(b) $\omega = \omega^\mu dy_\mu \in T^*_{\phi(p)}(N)$, show that the pullback of ω by ϕ is $\xi_\mu dx^\mu = \omega^\nu \frac{\partial y^\nu(x)}{\partial x^\mu} dx^\mu$, where $y = \phi(x)$

(c) $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ (not reverse)

(d) $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ (reverse)

Proof. (a) $(\phi_* V)(f) = V(f \circ \phi) \Longrightarrow W^\mu \frac{\partial}{\partial y^\mu} f = V^\nu \frac{\partial}{\partial x^\nu}(f \circ \phi)$. Take ϕ to be the coordinate function y^α, then $W^\alpha = V^\nu \frac{\partial y^\alpha(x)}{\partial x^\nu}$

(b)$(\phi^*\omega)(V) = \omega(\phi_* V) \Longrightarrow \xi_\mu dx^\mu (V^\nu \frac{\partial}{\partial x^\nu}) = \omega_\mu dy^\mu (V^\nu \frac{\partial y^\alpha(x)}{\partial x^\nu} \frac{\partial}{\partial y^\alpha}) \Longrightarrow \xi_\mu V^\mu = \omega_\mu V^\nu \frac{\partial y^\mu(x)}{\partial x^\nu} \Longrightarrow \xi_\nu = \omega_\mu \frac{\partial y^\mu(x)}{\partial x^\nu}$

(c)$((\psi \circ \phi)_* V)(f) = V(f \circ (\psi \circ \phi)) = ((\psi_* \circ \phi_*)V)(f)$

(d)$((\psi \circ \phi)^*\omega)(V) = \omega((\psi \circ \phi)_* V) = \omega((\psi_* \circ \phi_*)V) = ((\phi^* \circ \psi^*)\omega)(V)$ □