Dynamic Mean Field Theory of Random Recurrent Neural Networks

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1 MSRDJ Path Integral Formalism

Consider a stochastic differential equation of the form

$$\partial_t x = f(x) + \xi(t) \tag{1}$$

where $f(x)$ is a nonlinear function in the dynamical process, and $\xi(t)$ is noise. Using the Ito convention, the equation can be discretized as

$$\psi(x_t) \equiv x_t - \Big[x_{t-1} + f(x_{t-1})\Delta t + \xi_t \Delta t + x_0 \delta_{t0}\Big] = 0 \tag{2}$$

Our goal is to obtain the probability distribution of the dynamical path $P[x(t)]$, which in its discretized form can be written as

$$p(x_1, \cdots, x_N) = \prod_{t=1}^{N} \int \mathrm{d}\xi_t \; \rho(\xi_t)\, \delta\big(x_t - x_t^\star(x_{t-1}, \xi_t)\big) \tag{3}$$

where $x_t^\star(x_{t-1}, \xi_t)$ is the solution of equation (2), i.e., the zero of the function $\psi(x_t) = 0$, and is unique. Using the composite property of the Dirac δ function and its Fourier integral representation, we obtain

$$\delta\big(x_t - x_t^\star(x_{t-1}, \xi_t)\big) = |\psi'(x_t)|\, \delta\big(\psi(x_t)\big) = \delta\big(\psi(x_t)\big) = \int \frac{\mathrm{d}\tilde{x}_t}{2\pi i}\, e^{\tilde{x}_t \psi(x_t)} \tag{4}$$

Equation (2) can be rewritten as

$$\begin{aligned}
p(x_1, \cdots, x_N) &= \prod_t \int \mathrm{d}\xi_t \; \rho(\xi_t) \int \frac{\mathrm{d}\tilde{x}_t}{2\pi i}\, e^{\tilde{x}_t \psi(x_t)} && \text{(5a)}\\
&= \prod_t \int \frac{\mathrm{d}\tilde{x}_t}{2\pi i}\Big[\int \mathrm{d}\xi_t\, e^{-\tilde{x}_t \xi_t \Delta t} \rho(\xi_t)\Big] \exp\Big[\tilde{x}_t\big(x_t - x_{t-1} - f(x_{t-1})\Delta t - x_0 \delta_{t0}\big)\Big] && \text{(5b)}\\
&= \int \Big[\prod_t \frac{\mathrm{d}\tilde{x}_t}{2\pi i}\Big]\Big[\prod_t \int \mathrm{d}\xi_t\, e^{-\tilde{x}_t \xi_t \Delta t} \rho(\xi_t)\Big] \exp\Big[\sum_t \tilde{x}_t\Big(\frac{x_t - x_{t-1}}{\Delta t} - f(x_{t-1}) - x_0 \frac{\delta_{t0}}{\Delta t}\Big)\Delta t\Big] && \text{(5c)}
\end{aligned}$$

In the limit $N \to \infty$, $\Delta t \to 0$, using $\frac{x_t - x_{t-1}}{\Delta t} \to \partial_t x$, $\frac{\delta_{t0}}{\Delta t} \to \delta(t)$ and $\sum_{t=1}^{N} \Delta t \to \int \mathrm{d}t$, and defining the moment generating functional of the noise process

$$Z_\xi[-\tilde{x}(t)] = \lim_{N\to\infty} \prod_t \int \mathrm{d}\xi_t\, e^{-\tilde{x}_t \xi_t \Delta t} \rho(\xi_t) \tag{6}$$

$p(x_1, \cdots, x_N)$ can be rewritten in continuous form as

$$p[x(t)] = \int D\tilde{x}\; Z_\xi[-\tilde{x}(t)] \exp\Big[\int \mathrm{d}t\; \tilde{x}\Big(\partial_t x - f(x) - x_0 \delta(t)\Big)\Big] \tag{7}$$

To obtain the statistics in the probability distribution, we can define the moment generating function. However, since the probability distribution $p[x(t)]$ is a functional of the path $x(t)$, we need to define the so-called moment generating functional

$$\begin{aligned}
Z[j(t)] &= \int Dx\; e^{\int j(t)x(t)\mathrm{d}t}\, p[x(t)] && \text{(8a)}\\
&= \int Dx \int D\tilde{x}\; Z_\xi[-\tilde{x}(t)] \exp\Big\{\int \mathrm{d}t\; \Big[\tilde{x}\Big(\partial_t x - f(x) - x_0 \delta(t)\Big) + jx\Big]\Big\} && \text{(8b)}
\end{aligned}$$

where $j(t)$ is called the source field. More generally, we can apply a perturbation field $\tilde{j}(t)$ to the system (1), modifying the dynamical equation to

$$\partial_t x = f(x) - \tilde{j}(t) + \xi(t) \tag{9}$$

By replacing $f(x) \to f(x) - \tilde{j}(t)$, we obtain the moment generating functional with the perturbation field

$$Z[j,\tilde{j}] = \int Dx \int D\tilde{x}\; Z_\xi[-\tilde{x}] \exp\left\{ \int \mathrm{d}t \left[\tilde{x}\Big(\partial_t x - f(x) - x_0\delta(t)\Big) + jx + \tilde{j}\tilde{x} \right] \right\} \tag{10}$$

Defining the action related to the dynamical equation itself as

$$S[x,\tilde{x}] = \int \mathrm{d}t\; \tilde{x}\Big(\partial_t x - f(x) - x_0\delta(t)\Big) \tag{11}$$

and expressing the moment generating functional of the noise in terms of its cumulant generating functional

$$W_\xi[-\tilde{x}(t)] = \ln Z_\xi[-\tilde{x}(t)] = \lim_{N\to\infty} \ln \prod_t \int \mathrm{d}\xi_t\, e^{-\tilde{x}_t \xi_t \Delta t} \rho(\xi_t) \tag{12}$$

the generating functional can be written as

$$Z[j,\tilde{j}] = \int D[x,\tilde{x}]\; \exp\Big(S[x,\tilde{x}] + W_\xi[-\tilde{x}] + j^\top x + \tilde{j}^\top \tilde{x} \Big) \tag{13}$$

For the common case of Gaussian white noise, we generally write it in the form

$$\langle \xi(t) \rangle = 0 \qquad \Big\langle \xi(t)\xi(t') \Big\rangle = g^2 \delta(t-t') \tag{14}$$

Using the discretization scheme $\delta(t-t') \to \frac{1}{\Delta t}$, the variance of ξ_t is $g^2/\Delta t$, and the probability density is written as

$$\rho(\xi_t) = \frac{\sqrt{\Delta t}}{\sqrt{2\pi} g} \exp\left(-\frac{\xi_t^2 \Delta t}{2g^2} \right) \tag{15}$$

Thus, the moment generating functional of the noise is

$$\begin{aligned}
Z_\xi[-\tilde{x}] &= \lim_{N\to\infty} \prod_t \int \frac{\sqrt{\Delta t}}{\sqrt{2\pi} g} \mathrm{d}\xi_t\, e^{-\tilde{x}_t \xi_t \Delta t} \exp\left(-\frac{\xi_t^2 \Delta t}{2g^2} \right) && \text{(16a)} \\
&= \lim_{N\to\infty} \prod_t \int \frac{1}{\sqrt{2\pi} g} \mathrm{d}\big(\xi_t \sqrt{\Delta t}\big)\, e^{-\tilde{x}_t \xi_t \Delta t} \exp\left[-\frac{1}{2}\frac{1}{g^2}\big(\xi_t\sqrt{\Delta t}\big)^2 - \tilde{x}_t \sqrt{\Delta t}\big(\xi_t \sqrt{\Delta t}\big) \right] && \text{(16b)} \\
&= \lim_{N\to\infty} \prod_t \exp\left(\frac{1}{2} g^2 \tilde{x}_t^2 \Delta t \right) && \text{(16c)} \\
&= \exp\left(\frac{1}{2} g^2 \int \tilde{x}^2(t)\, \mathrm{d}t \right) && \text{(16d)}
\end{aligned}$$

The cumulant generating functional is

$$W_\xi[-\tilde{x}] = \frac{1}{2} g^2 \int \tilde{x}^2(t)\, \mathrm{d}t \tag{17}$$

2 DMFT of Random Recurrent Neural Networks

Consider a random recurrent neural network with the following dynamical equation:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -x_i + \sum_{j=1}^N J_{ij} \phi(x_j) + \xi_i \tag{18}$$

where the nonlinear function $\phi(x) = \tanh(x)$, J_{ij} represents the coupling strength between neurons, and ξ_i is Gaussian white noise with the following statistical properties:

$$J_{ii} = 0 \qquad \langle J_{ij} \rangle = 0 \qquad \langle J_{ij}^2 \rangle = (1-\delta_{ij})\frac{J^2}{N} \tag{19}$$

$$\langle \xi_i(t) \rangle = 0 \qquad \langle \xi_i(t)\xi_j(t') \rangle = g^2 \delta_{ij}\delta(t-t') \tag{20}$$

Given a coupling matrix J, the moment generating functional can be written as

$$Z[\boldsymbol{j},\tilde{\boldsymbol{j}}](\mathrm{J}) = \int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}] \ \exp\left\{\boldsymbol{S} + \boldsymbol{j}^\top \boldsymbol{x} + \tilde{\boldsymbol{j}}^\top \tilde{\boldsymbol{x}} - \sum_i \int \mathrm{d}t \ \tilde{x}_i(t) \sum_j J_{ij}\phi\big(x_j(t)\big)\right\} \tag{21}$$

where

$$\boldsymbol{S} = \sum_i \int \mathrm{d}t \Big[\tilde{x}_i(\partial_t + 1)x_i + \frac{1}{2}g^2\tilde{x}_i^2\Big] \tag{22}$$

Averaging the moment generating functional over the disorder J, we obtain

$$\overline{Z[\boldsymbol{j},\tilde{\boldsymbol{j}}]} \equiv \big\langle Z[\boldsymbol{j},\tilde{\boldsymbol{j}}](\mathrm{J}) \big\rangle_{\mathrm{J}} = \int \prod_{i,j} \mathrm{d}J_{ij}\, P(J_{ij})\, Z[\boldsymbol{j},\tilde{\boldsymbol{j}}](\mathrm{J}) \tag{23a}$$

$$= \int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}] \ \exp\Big(\boldsymbol{S} + \boldsymbol{j}^\top \boldsymbol{x} + \tilde{\boldsymbol{j}}^\top \tilde{\boldsymbol{x}}\Big) \underbrace{\int \prod_{i,j} \mathrm{d}J_{ij}\, e^{-\sum_{i,j} J_{ij} \int \mathrm{d}t\, \tilde{x}_i \phi(x_j)}\, P(J_{ij})}_{\equiv Z_0} \tag{23b}$$

where

$$Z_0 = \prod_{i \neq j} \int \mathrm{d}J_{ij} \ \exp\left\{-\frac{J_{ij}^2}{2J^2/N} - J_{ij}\int \mathrm{d}t\, \tilde{x}_i \phi(x_j)\right\} \tag{24a}$$

$$= \exp\left\{\frac{J^2}{2N}\sum_{i\neq j}\left(\int \mathrm{d}t\, \tilde{x}_i\phi(x_j)\right)^2\right\} \tag{24b}$$

$$= \exp\left\{\frac{J^2}{2N}\sum_{i\neq j}\iint \mathrm{d}t\mathrm{d}t' \ \tilde{x}_i^t\, \phi_j^t\, \tilde{x}_i^{t'}\phi_j^{t'}\right\} \tag{24c}$$

$$= \exp\left\{\frac{1}{2}\iint \mathrm{d}t\mathrm{d}t'\left[\left(\sum_i \tilde{x}_i^t\, \tilde{x}_i^{t'}\right)\left(\frac{J^2}{N}\sum_j \phi_i^t\, \phi_i^{t'}\right)\right]\right\} \tag{24d}$$

The integrand in equation (24c) can be decomposed into two terms using $\sum_{i\neq j} x_i y_j = \sum_{i,j} x_i y_j - \sum_i x_i y_i = \sum_i x_i \sum_j y_j - \sum_i x_i y_i$, where the second term is an $\mathcal{O}(1/N)$ small quantity and can be neglected. Therefore,

$$\overline{Z[\boldsymbol{j},\tilde{\boldsymbol{j}}]} = \int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}] \ \exp\left\{\boldsymbol{S} + \boldsymbol{j}^\top \boldsymbol{x} + \tilde{\boldsymbol{j}}^\top \tilde{\boldsymbol{x}} + \frac{1}{2}\iint \mathrm{d}t\mathrm{d}t'\left[\left(\sum_i \tilde{x}_i^t\, \tilde{x}_i^{t'}\right)\left(\frac{J^2}{N}\sum_j \phi_i^t\, \phi_i^{t'}\right)\right]\right\} \tag{25}$$

Introducing the order parameter

$$Q(t,t') = \frac{J^2}{N}\sum_j \phi_i^t\, \phi_i^{t'} \tag{26}$$

we obtain

$$\begin{aligned}\overline{Z[\boldsymbol{j},\tilde{\boldsymbol{j}}]} &= \int \mathcal{D}Q\;\delta\Big(-\frac{N}{J^2}Q(t,t')+\sum_j \phi_i^t\,\phi_i^{t'}\Big)\int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}]\;\exp\Big(\boldsymbol{S}+\boldsymbol{j}^\top\boldsymbol{x}+\tilde{\boldsymbol{j}}^\top\tilde{\boldsymbol{x}}\Big)\\ &\qquad\times\exp\left\{\frac{1}{2}\iint \mathrm{d}t\mathrm{d}t'\Big[\Big(\sum_i \tilde{x}_i^t\tilde{x}_i^{t'}\Big)\Big(\frac{J^2}{N}\sum_j \phi_i^t\phi_i^{t'}\Big)\Big]\right\} \qquad (27a)\\ &= \int \mathcal{D}[Q,\widehat{Q}]\;\exp\left\{\iint \mathrm{d}t\mathrm{d}t'\;\widehat{Q}(t,t')\Big[-\frac{N}{J^2}Q(t,t')+\sum_j \phi_i^t\,\phi_i^{t'}\Big]\right\}\\ &\qquad\times\int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}]\;\exp\left\{\boldsymbol{S}+\boldsymbol{j}^\top\boldsymbol{x}+\tilde{\boldsymbol{j}}^\top\tilde{\boldsymbol{x}}+\frac{1}{2}\iint \mathrm{d}t\mathrm{d}t'\;Q(t,t')\sum_i \tilde{x}_i^t\,\tilde{x}_i^{t'}\right\} \qquad (27b)\\ &= \int \mathcal{D}[Q,\widehat{Q}]\;\exp\Big(-\frac{N}{J^2}Q^\top\widehat{Q}\Big)\int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}]\;\exp\Big(\boldsymbol{S}+\boldsymbol{j}^\top\boldsymbol{x}+\tilde{\boldsymbol{j}}^\top\tilde{\boldsymbol{x}}+\frac{1}{2}(\tilde{\boldsymbol{x}}^t)^\top Q\,\tilde{\boldsymbol{x}}_{t'}+(\boldsymbol{\phi}^t)^\top\widehat{Q}\,\boldsymbol{\phi}^{t'}\Big) \qquad (27c)\end{aligned}$$

Noting that the second integral in equation (27c) has decoupled different neurons, it can be written as

$$N\ln Z[j,\tilde{j};Q,\widehat{Q}] = \sum_i \ln\int \mathcal{D}[\tilde{x}_i,x_i]\exp\Big(S_i+j_i^\top x_i+\tilde{j}_i^\top\tilde{x}_i+\frac{1}{2}(\tilde{x}_i^t)^\top Q\,\tilde{x}_i^{t'}+(\phi_i^t)^\top\widehat{Q}\,\phi_i^{t'}\Big) \qquad (28)$$

Thus, we have

$$\overline{Z[j,\tilde{j}]} = \int \mathcal{D}[Q,\widehat{Q}]\,e^{N\mathcal{L}[j,\tilde{j};Q,\widehat{Q}]} \qquad (29)$$

where

$$\mathcal{L}[j,\tilde{j};Q,\widehat{Q}] = -\frac{N}{J^2}Q^\top\widehat{Q}+\ln Z[j,\tilde{j};Q,\widehat{Q}] \qquad (30)$$

In the limit $N\to\infty$, the integral in equation (29) can be evaluated using the Laplace method:

$$\overline{Z[j,\tilde{j}]} = e^{N\mathcal{L}[j,\tilde{j};Q^\star,\widehat{Q}^\star]} \qquad (31)$$

where $Q^\star$ and $\widehat{Q}^\star$ satisfy the saddle point equations

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta Q}\Big|_{\widehat{Q}=\widehat{Q}^\star} &= -\frac{1}{J^2}\iint \mathrm{d}t\mathrm{d}t'\;\widehat{Q}^\star+\frac{1}{2}\iint \mathrm{d}t\mathrm{d}t'\Big\langle\tilde{x}(t)\tilde{x}(t')\Big\rangle_{\mathcal{L}} = 0 \qquad (32a)\\ \frac{\delta\mathcal{L}}{\delta\widehat{Q}}\Big|_{Q=Q^\star} &= -\frac{1}{J^2}\iint \mathrm{d}t\mathrm{d}t'\;Q^\star+\iint \mathrm{d}t\mathrm{d}t'\Big\langle\phi\big(x(t)\big)\phi\big(x(t')\big)\Big\rangle_{\mathcal{L}} = 0 \qquad (32b)\end{aligned}$$

Specifically,

$$\begin{aligned}Q^\star(t,t') &= J^2\Big\langle\phi\big(x(t)\big)\phi\big(x(t')\big)\Big\rangle_{\mathcal{L}} = J^2C(t,t') \qquad (33a)\\ \widehat{Q}^\star(t,t') &= \frac{J^2}{2}\Big\langle\tilde{x}(t)\tilde{x}(t')\Big\rangle_{\mathcal{L}} = 0 \qquad (33b)\end{aligned}$$

where $\langle\bullet\rangle_{\mathcal{L}}$ denotes the average over all trajectories of x and $\tilde{x}$ determined by the minimum of $\mathcal{L}$:

$$\langle\bullet\rangle_{\mathcal{L}} = \frac{1}{Z[j,\tilde{j};Q,\widehat{Q}]}\int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}]\bullet e^{\boldsymbol{S}+\boldsymbol{j}^\top\boldsymbol{x}+\tilde{\boldsymbol{j}}^\top\tilde{\boldsymbol{x}}+\frac{1}{2}(\tilde{\boldsymbol{x}}^t)^\top Q\,\tilde{\boldsymbol{x}}_{t'}+(\boldsymbol{\phi}^t)^\top\widehat{Q}\,\boldsymbol{\phi}^{t'}} \qquad (34)$$

Thus, the result of the integral is

$$\begin{aligned}\overline{Z[j,\tilde{j}]} &= \int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}]\;\exp\left\{-\frac{N}{J^2}Q^{\star\top}\widehat{Q}^\star+\boldsymbol{S}^\star+\boldsymbol{j}^\top\boldsymbol{x}+\tilde{\boldsymbol{j}}^\top\tilde{\boldsymbol{x}}+\frac{1}{2}(\tilde{\boldsymbol{x}}^t)^\top Q^\star\,\tilde{\boldsymbol{x}}_{t'}+(\boldsymbol{\phi}^t)^\top\widehat{Q}^\star\,\boldsymbol{\phi}^{t'}\right\} \qquad (35a)\\ &= \int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}]\;\exp\left\{\tilde{\boldsymbol{x}}^\top(\partial_t+1)\boldsymbol{x}+\frac{1}{2}g^2\tilde{\boldsymbol{x}}^\top\tilde{\boldsymbol{x}}+\frac{1}{2}J^2(\tilde{\boldsymbol{x}}^t)^\top C\,\tilde{\boldsymbol{x}}^t+\boldsymbol{j}^\top\boldsymbol{x}+\tilde{\boldsymbol{j}}^\top\tilde{\boldsymbol{x}}\right\} \qquad (35b)\\ &= \int \mathcal{D}[\boldsymbol{x},\tilde{\boldsymbol{x}}]\;\exp\left\{\sum_i\int \mathrm{d}t\Big[\tilde{x}_i(\partial_t+1)x_i+j_ix_i+\tilde{j}_i\tilde{x}_i\Big]+\frac{1}{2}\sum_i\iint \mathrm{d}t\mathrm{d}t'\;\Big(J^2C(t,t')+g\delta(t-t')\Big)\tilde{x}_i^t\tilde{x}_i^{t'}\right\} \qquad (35c)\end{aligned}$$

The first term in the exponent of equation (35c) corresponds to the dynamical equation $\partial_t x = -x$, the second term corresponds to the white noise term $\xi(t)$, and the third term can be regarded as another noise term $\eta(t)$, satisfying

$$\left\langle \eta(t)\eta(t') \right\rangle = J^2 C(t,t') \tag{36}$$

Therefore, the disorder-averaged $\overline{Z[j,\tilde{j}]}$ can be interpreted as the generating functional of the single-particle dynamical equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + \xi(t) + \eta(t) \tag{37}$$