The Replica Symmetric Solution of Sherrington Kirkpatrick Model

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1 Introduction

The Hamiltonian of the Sherrington-Kirkpatrick model reads

$$H = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i, \tag{1}$$

where $\sigma_i \in \{1,-1\}$ are Ising spins, the interaction J_{ij} between any two spins is a quenched variable with the Gaussian distribution $J_{ij} \sim \mathcal{N}(J_0/N, J^2/N)$. The mean and variance of this distribution are both proportional to $1/N$:

$$\left\langle J_{ij}\right\rangle = \frac{J_0}{N}, \quad \left\langle J_{ij}^2\right\rangle = \frac{J_0^2}{N^2} + \frac{J^2}{N}. \tag{2}$$

The probability of each configuration is given by the Gibbs-Boltzmann distribution $P(\boldsymbol{\sigma}) = \exp(-\beta H)/Z$, where Z is the partition function. We denote $\mathrm{Tr} = \sum_{\{\boldsymbol{\sigma}\}} = \sum_{\sigma_1=\pm 1}\cdots\sum_{\sigma_N=\pm 1}$, so that the partition function is expressed as $Z = \mathrm{Tr}\exp(-\beta H)$. The free energy F can be calculated by the partition function as

$$F = -\frac{1}{\beta}\log Z. \tag{3}$$

However, Eq. (3) is only the free energy for a fixed interaction $\boldsymbol{J}$ sampled from the distribution, which is also called quench. One not depended on any specific system sample can be obtain by the average over the distribution of $\boldsymbol{J}$, which is called the quenched average or disorder average or configurational average, and denoted by $\langle\cdot\rangle$ in this paper:

$$\langle F\rangle = -\frac{1}{\beta}\langle\log Z\rangle. \tag{4}$$

The dependence of $\langle\log Z\rangle$ on $\boldsymbol{J}$ is so complex that it cannot be solved directly, and this is where the replica method comes into play.

2 Replica Trick

The replica trick is a mathematical technique based on the application of the formula

$$\langle\log Z\rangle = \lim_{n\to 0}\frac{\langle Z^n\rangle - 1}{n}. \tag{5}$$

In this case, the replica average of the partition function can be written as

$$\langle Z^n\rangle = \int \mathcal{D}J \ \ \mathrm{Tr}\ \exp\left(\beta\sum_{i<j}J_{ij}\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha + \beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}S_i^\alpha\right), \tag{6}$$

where the explicit expression of the integral measure $\mathcal{D}J$ is given by the distribution of J_{ij}

$$\mathcal{D}J = \prod_{i<j}\left[\mathrm{d}J_{ij}\ P\left(J_{ij}\right)\right] \propto \prod_{i<j}\mathrm{d}J_{ij}\ \ \exp\left[\sum_{i<j} -\frac{N}{2J^2}\left(J_{ij} - \frac{J_0}{N}\right)^2\right]. \tag{7}$$

Then, Eq. (6) is calculated as

$$\langle Z^n\rangle \propto \exp\left[\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}S_i^\alpha\right]\ \mathrm{Tr}\ \prod_{i<j}\left\{\int \mathrm{d}J_{ij}\ \exp\left[-\frac{N}{2J^2}J_{ij}^2+\left(\frac{J_0}{J^2}+\beta\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha\right)J_{ij}\right]\right\} \tag{8a}$$

$$\propto \mathrm{Tr}\ \exp\left\{\frac{1}{N}\sum_{i<j}\left(\frac{1}{2}\beta^2J^2\sum_{\alpha,\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+\beta J_0\sum_{\alpha}S_i^\alpha S_j^\alpha\right)+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}S_i^\alpha\right\} \tag{8b}$$

$$= \mathrm{Tr}\ \exp\left\{\frac{1}{N}\sum_{i<j}\left(\frac{1}{2}\beta^2J^2\left(2\sum_{\alpha<\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+n\right)+\beta J_0\sum_{\alpha}S_i^\alpha S_j^\alpha\right)+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}S_i^\alpha\right\} \tag{8c}$$

$$= \mathrm{Tr}\ \exp\left\{\frac{1}{N}\sum_{i<j}\left[\beta^2J^2\sum_{\alpha<\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+\frac{1}{2}\beta^2J^2n+\beta J_0\sum_{\alpha}S_i^\alpha S_j^\alpha\right]+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}S_i^\alpha\right\} \tag{8d}$$

$$= \exp\left[\frac{(N-1)\beta^2J^2n}{4}\right]\ \mathrm{Tr}\ \exp\left\{\frac{1}{N}\sum_{i<j}\left[\beta^2J^2\sum_{\alpha<\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+\beta J_0\sum_{\alpha}S_i^\alpha S_j^\alpha\right]+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}S_i^\alpha\right\}, \tag{8e}$$

where the integral term in Eq. (8a) is calculated as

$$\mathcal{I} = \int \mathrm{d}J_{ij}\ \exp\left[-\frac{1}{2}\frac{N}{J^2}J_{ij}^2+\left(\frac{J_0}{J^2}+\beta\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha\right)J_{ij}\right] \tag{9a}$$

$$= \sqrt{\frac{4\pi J^2}{N}}\exp\left[\left(\frac{J_0^2}{J^4}+\frac{2J_0}{J^2}\beta\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha+\beta^2\sum_{\alpha,\beta}^{n}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta\right)\Big/\frac{2N}{J^2}\right] \tag{9b}$$

$$= \sqrt{\frac{4\pi J^2}{N}}\exp\left(\frac{J_0^2}{2NJ^2}\right)\exp\left[\frac{1}{N}\left(\beta J_0\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha+\frac{1}{2}\beta^2J^2\sum_{\alpha,\beta}^{n}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta\right)\right], \tag{9c}$$

and the following trick is used in Eq. (8c)

$$\sum_{\alpha,\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta = 2\sum_{\alpha<\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+\sum_{\alpha}\left(S_i^\alpha S_j^\alpha\right)^2 = 2\sum_{\alpha<\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+n\,. \tag{10}$$

Considering $\left(\sum_i A_i\right)^2=\sum_i A_i^2+\sum_{i\neq j}A_iA_j=\sum_i A_i^2+2\sum_{i<j}A_iA_j$, i.e.

$$\sum_{i<j}A_iA_j=\frac{1}{2}\left[\left(\sum_i A_i\right)^2-\sum_i A_i^2\right], \tag{11}$$

we have

$$\frac{\beta^2J^2}{N}\sum_{i<j}\sum_{\alpha<\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta=\frac{\beta^2J^2}{2N}\left[\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^2-\sum_i\sum_{\alpha<\beta}(S_i^\alpha)^2(S_i^\beta)^2\right]=\frac{\beta^2J^2}{2N}\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^2-\frac{\beta^2J^2n}{2}, \tag{12}$$

and

$$\frac{\beta J_0}{N}\sum_{i<j}\sum_{\alpha}S_i^\alpha S_j^\alpha=\frac{\beta J_0}{2N}\left[\sum_{\alpha}\left(\sum_i S_i^\alpha\right)^2-\sum_i\sum_{\alpha}(S_i^\alpha)^2\right]=\frac{\beta J_0}{2N}\sum_{\alpha}\left(\sum_i S_i^\alpha\right)^2-\frac{\beta J_0 n}{2}. \tag{13}$$

Thus Eq. (8e) is written as

$$\langle Z^n\rangle \propto \exp\left[\frac{N\beta^2J^2n}{4}\right]\mathrm{Tr}\ \exp\left[\frac{\beta^2J^2}{2N}\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^2+\frac{\beta J_0}{2N}\sum_{\alpha}\left(\sum_i S_i^\alpha\right)^2+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}S_i^\alpha\right], \tag{14}$$

where the following approximation in the large N limit is used

$$\exp\left[\frac{(N-3)\beta^2J^2n}{4}-\frac{\beta J_0 n}{2}\right]\approx\exp\left(\frac{N\beta^2J^2n}{4}\right). \tag{15}$$

In order to linearize the quadratic term on the exponential, it is useful to introduce the Hubbard-Stratonovich transform, an inverse application of the Gaussian integral, as follow

$$\exp\left(\frac{y^2}{2}\right)=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\exp(xy). \tag{16}$$

Let $x=\beta J\sqrt{N}q_{\alpha\beta}$, and $y=\beta J\sum_i S_i^\alpha S_i^\beta/\sqrt{N}$, then we have

$$\exp\frac{\beta^2J^2}{2N}\left(\sum_i S_i^\alpha S_i^\beta\right)^2=\beta J\sqrt{N}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_{\alpha\beta}}{\sqrt{2\pi}}\exp\left(-\beta^2J^2N\frac{q_{\alpha\beta}^2}{2}+\beta^2J^2q_{\alpha\beta}\sum_i S_i^\alpha S_i^\beta\right). \tag{17}$$

Let $x=\sqrt{\beta J N}m_\alpha$, and $y=\sqrt{\beta J/N}\sum_i S_i^\alpha$, then we have

$$\exp\frac{\beta J_0}{2N}\left(\sum_i S_i^\alpha\right)^2=\sqrt{\beta J N}\int_{-\infty}^{\infty}\frac{\mathrm{d}m_\alpha}{\sqrt{2\pi}}\exp\left(-\beta J_0 N m_\alpha^2+\beta J_0 m_\alpha\sum_i S_i^\alpha\right). \tag{18}$$

Then Eq. (14) can be written as

$$\begin{aligned}\langle Z^n\rangle \propto &\exp\left[\frac{N\beta^2J^2n}{4}\right]\int_{-\infty}^{\infty}\prod_{\alpha<\beta}\mathrm{d}q_{\alpha\beta}\prod_\alpha\mathrm{d}m_\alpha\ \mathrm{Tr}\ \exp\left[-\frac{\beta^2J^2N}{2}\sum_{\alpha<\beta}q_{\alpha\beta}^2-\frac{\beta J_0 N}{2}\sum_\alpha m_\alpha^2\right]\\ &\times\exp\left[\beta^2J^2\sum_{\alpha<\beta}q_{\alpha\beta}\sum_i S_i^\alpha S_i^\beta+\beta J_0\sum_\alpha m_\alpha\sum_i S_i^\alpha\right]\exp\left[\beta h\sum_{i=1}^N\sum_{\alpha=1}^n S_i^\alpha\right] \qquad (19a)\\ =&\exp\left[\frac{N\beta^2J^2n}{4}\right]\int_{-\infty}^{\infty}\prod_{\alpha<\beta}\mathrm{d}q_{\alpha\beta}\prod_\alpha\mathrm{d}m_\alpha\ \exp\left[-\frac{\beta^2J^2N}{2}\sum_{\alpha<\beta}q_{\alpha\beta}^2-\frac{\beta J_0 N}{2}\sum_\alpha m_\alpha^2\right]\\ &\times\mathrm{Tr}\ \exp\left[\beta^2J^2\sum_{\alpha<\beta}q_{\alpha\beta}\sum_i S_i^\alpha S_i^\beta+\beta\sum_\alpha(J_0m_\alpha+h)\sum_i S_i^\alpha\right] \qquad (19b)\\ =&\exp\left[\frac{N\beta^2J^2n}{4}\right]\int_{-\infty}^{\infty}\prod_{\alpha<\beta}\mathrm{d}q_{\alpha\beta}\prod_\alpha\mathrm{d}m_\alpha\exp\left[N\left(-\frac{\beta^2J^2}{2}\sum_{\alpha<\beta}q_{\alpha\beta}^2-\frac{\beta J_0}{2}\sum_\alpha m_\alpha^2+\log\mathrm{Tr}\,\mathrm{e}^{\mathcal{L}}\right)\right], \qquad (19c)\end{aligned}$$

where we define

$$\mathcal{L}\equiv\beta^2J^2\sum_{\alpha<\beta}q_{\alpha\beta}S_i^\alpha S_i^\beta+\beta\sum_\alpha(J_0m_\alpha+h)S_i^\alpha, \tag{20}$$

in Eq. (19b) and used

$$\prod_{i=1}^N\mathrm{Tr}\ \mathrm{e}^{\mathcal{L}}=\left(\mathrm{Tr}\ \mathrm{e}^{\mathcal{L}}\right)^N=\exp\left[N\log\left(\mathrm{Tr}\ \mathrm{e}^{\mathcal{L}}\right)\right]. \tag{21}$$

In large N limit, the integral in Eq. (19c) can be calculated with the Laplace approximation, also known as the saddle-point approximation, i.e.

$$\int\mathrm{d}m\ e^{N\mathcal{F}(m)}\xrightarrow{N\to\infty}e^{N\mathcal{F}_{\max}(m^*)}. \tag{22}$$

Let

$$\mathcal{F} = -\frac{\beta^2 J^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0}{2} \sum_{\alpha} m_\alpha^2 + \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}}, \tag{23}$$

and the result of the integral is

$$\langle Z^n \rangle \propto \exp\left[\frac{N\beta^2 J^2 n}{4} - \frac{\beta^2 J^2 N}{2} \sum_{\alpha<\beta} \left(q_{\alpha\beta}^\star\right)^2 - \frac{\beta J_0 N}{2} \sum_{\alpha} \left(m_\alpha^\star\right)^2 + N \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} \right] \tag{24a}$$

$$= \exp\left\{ Nn \left[\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n} \sum_{\alpha<\beta} \left(q_{\alpha\beta}^\star\right)^2 - \frac{\beta J_0}{2n} \sum_{\alpha} \left(m_\alpha^\star\right)^2 + \frac{1}{n} \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} \right] \right\} \tag{24b}$$

$$\approx 1 + Nn \left\{ \frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n} \sum_{\alpha<\beta} \left(q_{\alpha\beta}^\star\right)^2 - \frac{\beta J_0}{2n} \sum_{\alpha} \left(m_\alpha^\star\right)^2 + \frac{1}{n} \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} \right\} \tag{24c}$$

where we used Taylor expansion in Eq. (24c), and $q_{\alpha\beta}^\star, m_\alpha^\star = \arg\max_{\{q_{\alpha\beta}, m_\alpha\}} \mathcal{F}$. Through $\frac{\partial}{\partial q_{\alpha\beta}} \mathcal{F} = \frac{\partial}{\partial m_\alpha} \mathcal{F} = 0$, we arrive at

$$q_{\alpha\beta}^\star = \frac{1}{\beta^2 J^2} \frac{\partial}{\partial q_{\alpha\beta}} \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} = \frac{1}{\beta^2 J^2} \frac{\mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} \beta^2 J^2}{\mathrm{Tr}\, \mathrm{e}^{\mathcal{L}}} S^\alpha S^\beta = \frac{\mathrm{Tr}\, S^\alpha S^\beta \mathrm{e}^{\mathcal{L}}}{\mathrm{Tr}\, \mathrm{e}^{\mathcal{L}}}, \tag{25}$$

$$m_\alpha^\star = \frac{1}{\beta J_0} \frac{\partial}{\partial m_\alpha} \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} = \frac{1}{\beta J_0} \frac{\mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} \beta J_0}{\mathrm{Tr}\, \mathrm{e}^{\mathcal{L}}} S^\alpha = \frac{\mathrm{Tr}\, S^\alpha \mathrm{e}^{\mathcal{L}}}{\mathrm{Tr}\, \mathrm{e}^{\mathcal{L}}}. \tag{26}$$

The free energy density f is finally written as

$$f = -\frac{1}{N} \langle F \rangle = -\frac{1}{\beta} \lim_{n\to 0} \frac{\langle Z^n \rangle - 1}{Nn} = -\frac{1}{\beta} \lim_{n\to 0} \left\{ \frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n} \sum_{\alpha<\beta} \left(q_{\alpha\beta}^\star\right)^2 - \frac{\beta J_0}{2n} \sum_{\alpha} \left(m_\alpha^\star\right)^2 + \frac{1}{n} \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}} \right\}. \tag{27}$$

3 Replica Symmetry Ansatz

To continue solving Eq. (27), we need to consider the dependencies of $q_{\alpha\beta}$ and m^α for different replica index. A naive idea is that they are independent of index, i.e. $\forall \alpha, \beta$, $q_{\alpha\beta} = q$, $m_\alpha = m$, also called *replica symmetry ansatz*. The replica symmetric free energy is written as

$$f_{\mathrm{RS}} = -\frac{1}{\beta} \lim_{n\to 0} \left\{ \frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2 (n-1)}{4} q^2 - \frac{\beta J_0}{2} m^2 + \frac{1}{n} \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}^\star} \right\} \tag{28a}$$

$$= -\frac{1}{\beta} \left\{ \frac{\beta^2 J^2}{4} \left(1 + q^2\right) - \frac{\beta J_0}{2} m^2 + \lim_{n\to 0} \frac{1}{n} \log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}^\star} \right\}, \tag{28b}$$

where $\mathcal{L}^\star \equiv \beta^2 J^2 q \sum_{\alpha<\beta} S^\alpha S^\beta + \beta \left(J_0 m + h\right) \sum_\alpha S^\alpha$.

The final item is calculated as

$$\frac{1}{n}\log \mathrm{Tr}\, \mathrm{e}^{\mathcal{L}^\star} = \frac{1}{n}\log \mathrm{Tr}\;\exp\left\{\frac{1}{2}\beta^2 J^2 q\left(\sum_\alpha S^\alpha\right)^2 - \frac{1}{2}\beta^2 J^2 q n + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right\} \tag{29a}$$

$$= \frac{1}{n}\log\left\{\exp\left(-\frac{\beta^2 J^2 q n}{2}\right)\;\mathrm{Tr}\;\exp\left[\frac{1}{2}\beta^2 J^2 q\left(\sum_\alpha S^\alpha\right)^2 + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right]\right\} \tag{29b}$$

$$= \frac{1}{n}\log\left\{\exp\left(-\frac{\beta^2 J^2 q n}{2}\right)\;\mathrm{Tr}\int Dz\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right)\right\} \tag{29c}$$

$$= \frac{1}{n}\log\int Dz\;\exp\left\{n\log\left[2\cosh\left(\beta \hat{H}(z)\right)\right] - \frac{n}{2}\beta^2 J^2 q\right\} \tag{29d}$$

$$\approx \frac{1}{n}\log\left\{1 + n\int Dz\;\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right] - \frac{n}{2}\beta^2 J^2 q\int Dz\right\} \tag{29e}$$

$$\approx \int Dz\;\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right] - \frac{1}{2}\beta^2 J^2 q\,, \tag{29f}$$

where we used Hubbard-Stratonovich transform again in Eq. (29b) and reparameterized $\hat{z}$ by a standard Gaussian variable z, rewriting integral variables as Gaussian integral measures

$$\exp\left[\frac{1}{2}\beta^2 J^2 q\left(\sum_\alpha S^\alpha\right)^2\right] = \int \mathrm{d}\hat{z}\sqrt{\frac{\beta^2 J^2 q}{2\pi}}\exp\left(-\frac{\hat{z}^2}{2}\beta^2 J^2 q\right)\exp\left(\beta^2 J^2 q\hat{z}\sum_\alpha S^\alpha\right) \tag{30a}$$

$$= \int\frac{\mathrm{d}z}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha\right) \tag{30b}$$

$$= \int Dz\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha\right). \tag{30c}$$

The last item in Eq. (29c) is calculated as

$$\mathrm{Tr}\int Dz\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right) = \int Dz\;\mathrm{Tr}\;\exp\left[\sum_\alpha S^\alpha\left(\beta J\sqrt{q}z + \beta\left(J_0 m + h\right)\right)\right] \tag{31a}$$

$$= \int Dz\prod_{\alpha=1}^{n}\mathrm{Tr}\;\exp\left[S^\alpha\beta\hat{H}(z)\right] \tag{31b}$$

$$= \int Dz\left\{2\cosh\left(\beta\hat{H}(z)\right)\right\}^n \tag{31c}$$

$$= \int Dz\exp\left\{n\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right]\right\}, \tag{31d}$$

where we defined $\hat{H}(z) \equiv J\sqrt{q}z + (J_0 m + h)$.

Finally, the replica symmetric free energy is

$$f_{\mathrm{RS}} = -\frac{1}{\beta}\left\{\frac{\beta^2 J^2}{4}\left(1 + q^2\right) - \frac{\beta J_0}{2}m^2 + \lim_{n\to 0}\left\{\int Dz\;\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right] - \frac{1}{2}\beta^2 J^2 q\right\}\right\} \tag{32a}$$

$$= \frac{\beta J^2}{4}\left(q-1\right)^2 + \frac{J_0}{2}m^2 - \frac{1}{\beta}\int Dz\;\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right]. \tag{32b}$$

Through

$$\frac{\partial}{\partial m}f_{\mathrm{RS}} = -\beta J_0 m + \int \mathrm{D}z\;(\tanh\beta\hat{H}(z))\cdot\beta J_0 = 0\,, \tag{33}$$

$$\frac{\partial}{\partial q}f_{\mathrm{RS}} = \frac{\beta^2 J^2}{2}(q-1) + \int \mathrm{D}z\;(\tanh\beta\hat{H}(z))\cdot\frac{\beta J}{2\sqrt{q}}z = 0\,, \tag{34}$$

we obtain a set of closed equations, called saddle point equations

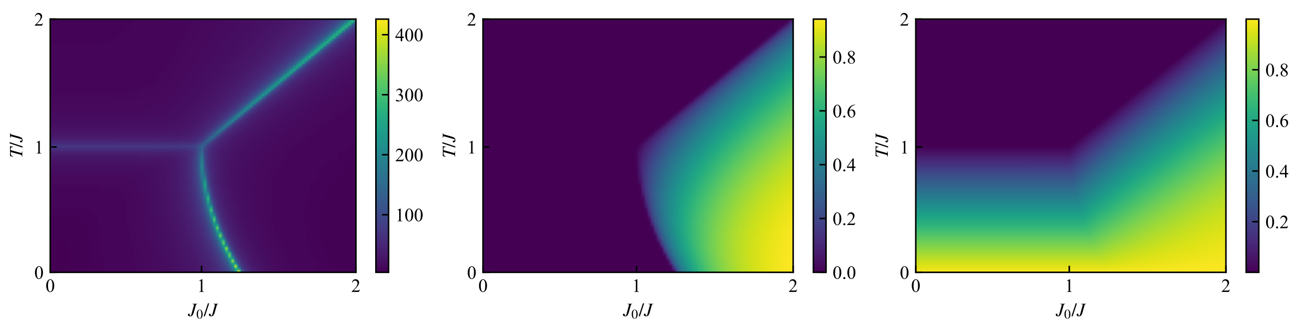
$$m = \int Dz \tanh \beta \hat{H}(z)\,, \tag{35}$$

$$q = 1 - \int Dz \,\mathrm{sech}^2 \beta \hat{H}(z) = \int Dz \tanh^2 \beta \hat{H}(z)\,. \tag{36}$$

4 Phase diagram

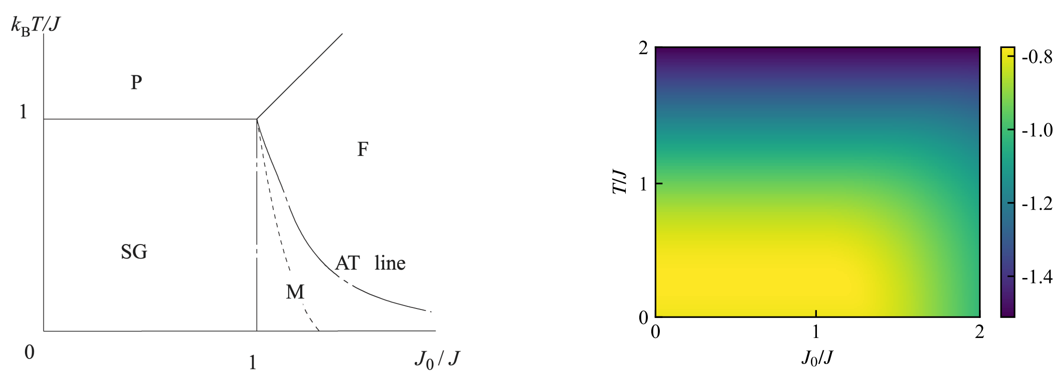
Considering a simple case where $h = 0$, we use numerical methods to iterate Eq. (35) and Eq. (36), and then calculate the free energy Eq. (32b) with the fixed points of m and q. The results of order parameters m and q are shown in Fig. 1, which (especially the interaction steps) recover the well-known phase diagram of the SK model as shown in Fig. 2(a)[1]. The free energy density with different J_0 and T is shown in Fig. 2(b).

Due to the Frustration, the spin in the SK model is frozen at low temperature, yet remains highly disordered, with the order parameter $m = 0$. But this is a phase different from the paramagnetic phase (also $m = 0$) and is called the spin glass phase. In short, $m \neq 0$ identifies the ferromagnetic phase, and the EA order parameter q is introduced to distinguish between the paramagnetic phase ($q = 0$) and the spin glass phase ($q \neq 0$).



(a) iteration steps (b) magnetization m (c) EA order parameter q

Figure 1: The results of the replica symmetric solution for the SK model by numerical iteration.



(a) the phase diagram of order parameters[1] (b) the phase diagram of free energy

Figure 2: The phase diagram of the SK model.

[1]Nishimori, Hidetoshi, *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford, 2001), p. 20